A Dynamic Analysis of Variable Annuities and Guaranteed Minimum Benefits

Jin Gao
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A DYNAMIC ANALYSIS OF VARIABLE ANNUITIES AND
GUARANTEED MINIMUM BENIFITS

BY

JIN GAO

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree
Of
Doctor of Philosophy
in the Robinson College of Business
Of
Georgia State University

GEORGIA STATE UNIVERSITY
ROBINSON COLLEGE OF BUSINESS
2010
ACCEPTANCE

This dissertation was prepared under the direction of JIN GAO’s Dissertation Committee. It has been approved and accepted by all members of that committee, and it has been accepted in partial fulfillment of the requirements for the degree of Doctor in Philosophy in Business Administration in the Robinson College of Business of Georgia State University.

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ABSTRACT

A DYNAMIC ANALYSIS OF VARIABLE ANNUITIES AND GUARANTEED MINIMUM BENEFITS

By

JIN GAO

Dec 2010

Committee Chair: Dr. Eric R. Ulm

Major Department: Risk Management and Insurance

We determine the optimal allocation of funds between the fixed and variable sub-accounts in a variable annuity with a GMDB (Guaranteed Minimum Death Benefit) clause featuring partial withdrawals by using a utility-based approach. In section two, the Merton method is applied by assuming that individuals allocate funds optimally in order to maximize the expected utility of lifetime consumption. It also reflects bequest motives by including the recipient's utility in terms of the policyholder's guaranteed death benefits. We derive the optimal transfer choice by the insured, and furthermore price the GMDB through maximizing the discounted expected utility of the policyholders and beneficiaries by investing dynamically in the fixed account and the variable fund and withdrawing optimally. In section three, we add fixed and stochastic income to the model and find that
both human capital and the GMDB will influence the insured's allocation and withdrawal decisions. Section four explores the GMDB effects if there is also a term life policy available in the market. Our work suggests that if term life insurance is available and is continuously adjustable, fairly priced GMDBs may not be useful investments and the existence of GMDBs does not affect term life policy demand significantly.
1 Introduction and Motivation

There are a number of risks (mortality risk, longevity risk, inflation risk and investment risk) for an individual approaching retirement age. Here, mortality risk refers to the loss of human capital in the event of an individual’s premature death. The traditional means for handling mortality risk is life insurance because life insurance generates an immediate bequest and provides protection against the effects of premature death, especially if the event occurs before individual wealth can be accumulated.

Longevity risk is the risk that a retiree’s savings might not be sufficient to support him due to a long life. There is available literature that tries to model investment and pension decisions with longevity risk in mind, for example Bengen (2001); Ameriks, Veres, and Warshawsky (2001); Milevsky and Robinson (2005); Milevsky, Moore, and Young (2006). Many of them refer to the use of annuities as a solution. Annuities are the opposite of life insurance, because annuities are defined as periodic payments that continue for a fixed period or for the remaining time of a designated life. Therefore, annuities protect against the longevity risk by providing an income that is not outlivable. Insurers sell several types of annuities, such as fixed annuities, variable annuities and equity-indexed annuities. Variable annuities can also be considered as an efficient means of investment to hedge against inflation risk.

1.1 History of Variable Annuities

In the early part of the 20th century, people regarded the fixed deferred annuity as an investment product for accumulating and safeguarding wealth to provide for their economic needs during retirement. Fixed deferred annuities provide a long term, low-risk investment but conservative rate of return. By using the fixed annuity as a retirement nestegg, people needed to pay a large principal to offset the annuity’s declining purchasing power due to inflation. On the other hand, a variable annuity also provides a lifetime income for retiree, but the income payments vary depending on stock market movements. Beside providing a lifetime
income, a variable annuity also protects against inflation by keeping the real purchasing power of the periodic payments during retirement given positive correlation between living cost and common stock prices over the long run. The first variable annuity contract was issued by the Teachers Insurance and Annuity Association (TIAA) - College Retirement Equity Fund in 1952 (Poterba 1997). This fund was established to provide variable annuity coverage within the retirement income program of TIAA. In 1959, the US Supreme Court ruled that the VAs fell under the joint jurisdiction of the Securities and Exchange Commission (SEC) and the state level insurance regulations department. Since the early 1990s, there has been a rapid growth of the variable annuity market. By 2000, annual variable annuity sales had reached a peak of $138 billion, more than twice the level of fixed annuity sales (Table 1 shows the growth rates of fixed and variable annuities). Condron (2008) reports that “more than $1.35 trillion was invested in variable annuities in the United States.”

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Data from Towers Perrin VALUE Survey and LIMRA data

In the United States, the funds within a variable annuity are held in subaccounts which are kept independent from other insurance company assets. Their benefits are based on the performance of the underlying bond or equity portfolio. Individuals buy variable annuities for many reasons: they are tax-deferred while protecting the policyholder from outliving their assets during retirement. In addition, insurers often offer various forms of option-like guarantees that insure against the negative risks inherent to subaccounts. However, variable annuity guarantees are different from regular options because they contain insurance characteristics that are life contingent. To protect against negative equity market movements, an essential aspect of a VA is the design of their guaranteed minimum benefits. Consequently,
variable annuity policyholders are able to maintain a greater weight in the equity portfolio.

According to Mueller (2009), despite the severe financial crisis, the prospective VA sales globally are still good, due to the following reasons:

- the US is an aging society, and a growing number of individuals are reaching retirement age;
- there is a growth in the size of retirement assets;
- only life insurers can offer lifetime guarantees;
- and, there has been a shift in the retirement savings responsibility from employers to employees.

Therefore, variable annuities can still be viewed as an important investment tool to provide retirement age income.

1.2 Guaranteed Minimum Benefits

Stone (2003) and Hardy (2003) give us an overview of many variable annuity guarantees. Variable annuity guarantees include Guaranteed Minimum Living Benefits (GMLB) and the Guaranteed Minimum Death Benefits (GMDB). Guaranteed Minimum Living Benefits (GMLB) include the following:

- GMIB: a Guaranteed Minimum Income Benefit is offered as a guaranteed minimum level of annuity payments upon annuitization, regardless of the performance of your annuity.

- GMAB: a Guaranteed Minimum Accumulation Benefit is offered as a one time “top up” of account value after the accumulation period, or a set period of time, e.g., after 15 years.

- GMMB: a Guaranteed Minimum Maturity Benefit is offered as a guaranteed minimum amount at the maturity of the contract.
• GMSB: a Guaranteed Minimum Surrender Benefit is offered “as a variation of the guaranteed minimum maturity benefit. Beyond some fixed date the cash value of the contract, payable on surrender, is guaranteed.” (Hardy 2003)

• GMWB: a Guaranteed Minimum Withdrawal Benefit is offered as guaranteed amounts via optional annual withdrawals. Higher guaranteed amounts are offered if the policyholder defers the initial withdrawal. Also, the guaranteed amounts may increase upon attaining certain age thresholds.

Guaranteed Minimum Death Benefits (GMDB) have been available in variable annuities since the 1990s, and provide the beneficiary a lump sum amount upon the insured’s death. A GMDB is an example of an option-like feature. It can be viewed as a put option with a random exercise time at the moment of death. This rider helps protect the policy’s beneficiary from negative market movements.

Many papers discuss GMDB riders. Milevsky and Posner (2001) apply risk-neutral option pricing methods to value GMDB riders embedded in annuity contracts. Milevsky and Salisbury (2001) notice that when the embedded options are out of the money, policyholders have a real option to lapse their policy and simultaneously repurchase the investment with higher death benefit. They assume that policyholders exercise this option optimally so that the lapse decision can be formulated as an optimal stopping problem. Based on this assumption, they calculate the surrender charge for the lapse to compensate the income loss of the insurance company. This surrender charge is derived by making the policyholder indifferent between keeping and lapsing the policy.

Another important option that is frequently available in these contracts is the option to transfer funds between a variable account and an attached fixed account that promises a fixed rate. The policyholders have two options for the allocation of their funds: one option is to leave the funds in the variable account, where the performance of the account will follow the market fluctuations; the other option would be to move the funds to the fixed account and forgo the market swings. Ulm (2006) discusses the effect of the real option to transfer funds between fixed and variable accounts. He uses the no-arbitrage pricing methodology
and gets the boundary between the area where all money is invested in the variable account and the area where all money is invested in the fixed account. He shows analytically that the option to transfer to the fixed fund has no value and will never be used unless the fixed growth rate is larger than the risk free rate less any asset fees taken off the variable account. If the fixed growth rate is less than this, the value of the option can be calculated and the approximate location of the optimal exercise boundary can be determined. Ulm (2010) models real policyholders transfer behavior. He uses data from Morningstar and NAIC annual statements to develop an empirical model. He compares his model with two other practical strategies: constant percentage rebalancing and buy-and-hold, and finds a model based on recent fund performance has a better fit. He concludes that the GMDB options will be overvalued and over-hedged if the policyholder’s empirical transfer choices are not taken into account.

Some GMDB contracts also contain a feature allowing the policyholder to withdraw from the invested capital at any time prior to the maturity of the contract. Bauer, Kling and Russ (2008) suggest a general solution to the GMDB with optimal partial withdrawals at discrete time horizons in a Black-Scholes option pricing model. Belanger, Forsyth and Labahn (2008) develop a pricing model from the issuer’s perspective based on partial differential equations to determine the no-arbitrage insurance charge for contracts with a GMDB clause featuring partial withdrawals. They demonstrate that higher fees are required for GMDB contracts with a partial withdrawal option.

1.3 Motivation

There are basically three ways one can think about pricing and hedging GMDBs. The first is Risk Neutral Optimality (Ulm 2006) that presupposes the worst case from the insurance company’s perspective. Risk neutral optimality assumes no arbitrage or the law of one price in a complete market in which GMDBs can be replicated and hedged by market traded instruments. From the insurer’s perspective, GMDBs are able to be traded and hedged.
Therefore, the worst thing is that policyholders behave optimally, and make the GMDB prices the largest under the risk neutral measure. The positive side of the risk neutral method is that the insurers are able to protect themselves from insolvency by presupposing the worst case and seldom underestimating the value of the option. The negative side is that the insurers will be over-hedged. The policyholders want to maximize the utility not the GMDB prices, so the worst case under risk averse optimality is not the worst case under risk neutral optimality. Under the assumption of complete markets, the risk neutral optimality is suitable as the GMDB would be tradable and the policyholders can always sell the GMDB for money. What the insurers have to do is to maximize the assets they are holding. Because transaction fees are incurred, the insurers will pay more cost to buy or sell more products. Also it is risky to get over-hedged, because the insurers cannot totally eliminate the risks.

The second is determining policyholder behavior through empirical analysis (Ulm 2010). By looking at what the policyholders really did, the insurance company can price the GMDB. The positive side is that it would have worked well in the past and the insurer would have minimized the variance of their total income stream. This method is less expensive than risk neutral optimality, so the insurance company can charge better (less) premiums. The negative side is if policyholders suddenly “wise up” on you, the insurer may be under-hedged and got in trouble.

There are disagreements and tradeoffs between risk neutral optimality and empirical analysis: the first one is doing something expensive and from the perspective of the worst case; the second one is doing something less expensive but is vulnerable if the policyholders suddenly behave rationally.

The third is utility based optimality assuming the policyholder’s optimal allocation and consumption choices given their preferences. This is the focus of this paper. It is potentially more realistic especially if the option is neither tradable nor hedgeable. Variable annuity markets are not complete, as it is usually not possible to sell your annuity to a third party and there may be barriers to surrendering it. Utility based models are a theoretically defensible way of treating the products with such restrictions (see Shreve (2003) pg. 70). Leung
and Sircar (2009) take this approach to employee stock options which have similar trading restrictions. Milevsky (2001a) first applies a utility based model in annuity analysis to choose if and when to annuitize, but the annuitization decision is irreversible which is different from our research in this paper. Charupat and Milevsky (2002) derive the optimal utility maximizing asset allocation between fixed and variable subaccounts within a variable annuity. However, they do not model the guarantee options in their research. In this thesis, we show that the guarantee options may change the insured’s allocation decision. Our paper is a good supplement to their research.

The equity markets slide in 2008 implied that there is a huge need for guarantee and option products. In the past, the separate fund guarantees, which are rarely in the money, resulted in a more lax approach to policy design, pricing, and risk management (see Hardy (2003) pg. 310). It showed the vulnerability of the insurers who craft these offerings. And it raised questions about just how well insurers understand the risk of dealing with complex financial instruments – such as the ones used to guarantee investments (see WSJ (May 4, 2009)).

WSJ (May 4, 2009) said: “Recently, variable annuity issuers raised prices and reduced benefits in effort to restore profits. Many insurers are reacting to steep losses they suffered from late 2008, as stock markets slid and the gap widened between their promises and the value of customers’ fund accounts. All told, the roughly two dozen insurers who dominate the guarantee business boosted their reserves and capital last year by more than 15 billion dollars to show regulators they can make good on the promises, according to ratings firms and consultants. Meanwhile, since last year, the insurers have faced sharply higher costs to buy financial hedges to offload the guarantees’ market risk. The costs soared as volatility spiked and interest rates fell – soaring to the point that ‘very few companies’ selling guarantees of minimum lifetime withdrawals ‘are actually pricing and designing products with sufficient margins to fully hedge the guarantees,’ barring a strong long-term recovery of stocks. What is needed is ‘more innovative’ product design.”

With continuous-time diffusion processes, any strategy that involves continuous readjust-
ment of a state variable when there are fixed costs would become infinitely expensive and could not be optimal. Instead, we believe that the optimal strategy is to change the state instantly in discrete amounts, thereby incurring the fixed costs only at isolated points in time.

This thesis contributes in several important ways. First, to date no one has examined optimal behavior in a utility-based incomplete market framework. Second, there has been a debate recently between financial advisors and insurance companies regarding the suitability of GMDBs given the existence of term life insurance. We contribute to this discussion by evaluating consumers’ willingness to pay for GMDB protection in utility-framework with a term insurance policy available.

The remainder of this paper is organized as follows: Section two introduces the “no labor income” model and numerical analysis. In this section, we determine the optimal allocation of funds between the fixed and variable subaccounts in a variable annuity using a utility based approach. This paper differs from Ulm (2006) in several ways. First, we assume the insureds are risk averse, so partial transfers between variable and fixed accounts could be optimal. More precisely, we apply the Merton (1969) method by assuming that individuals allocate funds in order to maximize the expected utility of lifetime consumption. In this model, the insured gets utility from consumption and has bequest motives. We include the effect on asset allocation from dissavings (consumption). We also reflect bequest motives by including the utility of the recipient of the policyholders guaranteed death benefits. When we derive the optimal transfer choice by the insured, we find the GMDB will increase the risky allocation in the VA account by incurring an “argument” between the policyholder and his beneficiary, especially especially when the guarantees are at-the-money. Furthermore we price the GMDB through maximizing the discounted expected utility of the policyholders and beneficiaries by investing dynamically in the fixed account and variable fund and withdrawing optimally.

In section three, we apply the idea of human capital (we assume labor income) to our model in addition to the variable annuity withdrawals. Human capital is the present value of
individual’s remaining lifetime labor income, and it will influence individual’s asset allocation choice. “The risks you can afford to take depend on your total financial situation, including the types and sources of your income exclusive of investment income” (see Malkiel (2004) pg. 342). Hanna and Chen (1997) study the optimal asset allocation by considering human capital. They conclude that the investors who have long investment horizons should apply all equity portfolio strategy. Bodie, Merton and Samuelson (1992) study the investment strategy given labor income. They find that younger investors should put more money in risky assets than should older investors. Chen, Ibbotson, Milevsky and Zhu (2006) take human capital into account, and argue that human capital affects asset allocation. There are roughly three stages of a person’s life\(^1\): the first stage is the growing up and getting educated stage; the second stage is the accumulation stage, in which people work and accumulate wealth; the third stage is the retirement/payout stage. The human capital generates significant amount of earnings during the accumulation stage. As individuals save and invest, human capital is transferred to financial capital. Chen et al. (2006) provide an approach to making the individuals’ financial decisions in purchasing life insurance, purchasing annuity products and allocating assets between stocks and bonds. Campbell and Viceira (2002) make some conclusions: 1. investors with “safe” labor income prefer investing more of their financial capital into equities; 2. if investors’ labor income is highly positively correlated with stock markets, they should choose an investment allocation with less equity exposures; 3. high labor flexibility tends to increase the proportion of allocation to equities. In our work, we discuss the policyholder’s decision with safe (fixed) and stochastic labor income. Our focus is on how human capital interacts with financial assets in the variable annuity account, and how the interaction changes the VA policyholder’s behaviors, including asset allocation and consumption choices. We find that there exists a human capital effect, and the “argument” incurred by GMDB does have an impact on the insured’s optimal decision. We provide the models that enable policyholders to customize their allocation and withdrawal decisions based on their own typical circumstances.

\(^1\)This paper focuses on the accumulation stage.
In section four, we bring term life policy into our model and check if the guarantee options add value to the contract even if the term life policy is available. Many papers study the life insurance demand. Campbell (1980) derives solutions to optimal life insurance demand on mortality risk. Lin and Grace (2005) examine the life cycle demand for different types of life insurance by using the Survey of Consumer Finances. They find the relationship between financial vulnerability and term life insurance demand, and that older people demand less term life insurance. We find a few papers studying the joint demand of term life insurance and annuities. Hong and Rios-Rull (2007) use an overlapping generation model of multiperson households to analyze social security, life insurance and annuities for families. Purcal and Piggott (2008) use an optimizing lifetime financial planning model to explore optimal life insurance purchase and annuity choices. Their model incorporates the consumption and bequests in an individual’s utility function. Policyholders’ needs for life insurance and annuities varied across different risk aversions and different bequest motives. We derive the insured’s optimal decisions in purchasing the term life policy, allocating and withdrawing assets in his VA account. We price the GMDB from the insurer’s perspective by incorporating the insured’s choices in a risk neutral model.

Finally, section five concludes the paper with some general remarks and directions for further research.

2 Optimal Consumption and Allocation in Variable Annuities with Guaranteed Minimum Death Benefits

There are several features a Guaranteed Minimum Death Benefit (GMDB) may comprise:

1. Roll-ups

If the account value goes down after the purchase of the variable annuity contract, the beneficiary will receive either the initial premium that was put in accumulated at some roll-
up rate or the account value accumulated. The roll-up rates vary between 0% and 7%. The table below illustrates how the GMDB is computed due to the roll-up rate $r_p$ for the first few years. $a_i$ is the account value at the beginning of $i$th year, for $i = 1, 2, 3, \cdots$.

Table 2: Computation of GMDB due to roll-up rate $r_p$

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<tr>
<th>Dates</th>
<th>Roll-up values</th>
</tr>
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<tbody>
<tr>
<td>1/1/2005</td>
<td>$a_0$</td>
</tr>
<tr>
<td>1/1/2006</td>
<td>$a_0 \times (1 + r_p)$</td>
</tr>
<tr>
<td>1/1/2007</td>
<td>$a_0 \times (1 + r_p)^2$</td>
</tr>
<tr>
<td>1/1/2008</td>
<td>$a_0 \times (1 + r_p)^3$</td>
</tr>
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</table>

Let us put forward an example to see how the roll-up benefit can protect the beneficiary from the equity market’s downside risk. In January 1998, an individual put $100 in a VA account and the closing date of the account was at the end of 2009. This person invested the entire amount in the S&P500 index.

Figure 1: Account Value v.s. roll-up level from 1998-2009
Figure 1 shows that if there was no GMDB option, the account value followed the index oscillations. If the insured died at a “bad” time, the bequest amount to the beneficiary would be low. Meanwhile, if there was a GMDB roll-up option (assume the annual roll-up rate $r_p = 2\%$), which increased the principal 2% annually, the bequest amount to the beneficiary was protected (the downside risk was eliminated), and the account value to the beneficiary was equal to the maximum of account value and the roll-up level (Figure 2).

2. Reset option

Here, the death benefit guarantee can be adjusted (moved up or down) at the beginning of every reset period. The frequency of the resetting interval ranges from once a year to once every five years. Table 3 below illustrates how the death guarantee due to the reset feature is computed for the first few years.
Table 3: Computation of GMDB due to the resetting option

<table>
<thead>
<tr>
<th>Dates</th>
<th>Reset values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2005</td>
<td>$a_0$</td>
</tr>
<tr>
<td>1/1/2006</td>
<td>$\max(a_0, a_1)$</td>
</tr>
<tr>
<td>1/1/2007</td>
<td>$\max(a_0, a_2)$</td>
</tr>
<tr>
<td>1/1/2008</td>
<td>$\max(a_0, a_3)$</td>
</tr>
</tbody>
</table>

If the policyholder in the above example bought the GMDB with resetting option in 1998, the bequest amount would be described in Figure 3.

3. Ratchet option

This is essentially a discrete lookback option – the death benefit equals to the larger of the amount invested or the ratcheted account value. More precisely, the death benefit guarantee only moves up at the beginning of every ratchet period. Table 4 shows how the death guarantee due to the ratchet feature is computed for the first few year.
Table 4: Computation of GMDB due to the ratchet option

<table>
<thead>
<tr>
<th>Dates</th>
<th>Ratchet values</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/1/2005</td>
<td>$a_0$</td>
</tr>
<tr>
<td>1/1/2006</td>
<td>$\max(a_0, a_1)$</td>
</tr>
<tr>
<td>1/1/2007</td>
<td>$\max(a_0, a_1, a_2)$</td>
</tr>
<tr>
<td>1/1/2008</td>
<td>$\max(a_0, a_1, a_2, a_3)$</td>
</tr>
</tbody>
</table>

If the policyholder in the above example bought the GMDB with ratchet option in 1998, the bequest amount would be described in Figure 4.

![Figure 4: Bequest Amount with monthly ratchet option from 1998-2009](image)

The aforementioned features are provided to the policyholders with an extra premium as riders to a base death benefit (which just contains return of the premium).

2.1 Models

In the model, we treat only return of premium and roll-up benefits. An individual
purchases a variable annuity product and makes a lump sum deposit. We restrict ourselves to insurance contracts with GMDB options only. There are two subaccounts in the VA account. One subaccount is a fixed account, which provides a fixed interest return \( g_t \), and the other subaccount is a variable account, which provides a return related to the stock market performance, with guaranteed minimum death benefit, i.e.

\[
(1) \quad k_t = a_0 \prod_{i=1}^{t} \left\{ (1 + r_{p_i}) \frac{a_i - c_i}{a_i} \right\},
\]

\[
(2) \quad b_t = \max(k_t, a_t - c_t) = \max(k_{t-1}(1 + r_{p_i}) \frac{a_t - c_t}{a_t}, a_t - c_t),
\]

where \( a_t \) is the total account value at time \( t \) in both the fixed account \((F)\) and the variable account \((S)\), i.e. \( a_t = F_t + S_t \); \( a_0 \) is the initial wealth; \( r_{p_i} \) is the guaranteed rate for GMDB at time \( t \); \( k_t \) is the guaranteed payment in the GMDB; \( b_t \) is investment guaranteed amount; and \( a_0 = b_0 = k_0 \); \( c_t \) is the withdrawal at the beginning of time \( t \), and the insured consumes \( c_t \) immediately. \( c_t \) is non-negative which means that deposits are not allowed in our model.

The money in the VA account is partitioned between these two sub-accounts. \( dS_t = \alpha_t S_t dt + \sigma_t S_t dB_t \), \( dF_t = g_t F_t dt \), where \( g_t \) is the risk-free rate and the fixed account grows at a rate \( g_t \); \( B_t \) is a standard Brownian motion. We denote by \( \omega_t \) the percentage of wealth held in the variable subaccount and \( 1 - \omega_t \) the proportion of wealth allocated in the fixed rate subaccount. The amount of withdrawals is \( c_t \), and it may vary with time.

\[
(3) \quad da_t = a_t [\omega_t r_t + (1 - \omega_t) g_t] dt + \omega_t \sigma_t a_t dB_t - c_t,
\]

We assume that the insureds have options to transfer money in between fixed and variable accounts. To be more realistic, we assume \( 0 \leq \omega_t \leq 1 \), which means that there are no short sales.

We assume the insured and the beneficiary are risk averse with the same utility function.
We apply a constant relative risk averse (CRRA) type utility which has a functional form

\[ U(c) = \begin{cases} 
\frac{c^{1-\gamma}}{1-\gamma}, & \gamma > 0, \gamma \neq 1, \\
\ln(x), & \gamma = 1.
\end{cases} \]

This utility has some special properties:

• it is a homogeneous function of degree 1 – γ for γ ≠ 1;

• γ is the coefficient of relative risky aversion; 1/γ is the intertemporal substitution elasticity between consumption in any two periods, i.e., it measures the willingness to substitute consumption between different periods.

If there is no possibility of death and no partial withdrawals in the accumulation stage and in the absence of a GMDB, the individual maximizes the expected utility at retirement date T. According to Charupat and Milevsky (2002), the objective function is

\[ \max_{\omega_t} E \left[ \frac{1}{1-\gamma} a_T^{1-\gamma} \right]. \tag{4} \]

The solution to the objective function is equal to

\[ \omega^* = \min \left[ \frac{r - g}{\gamma \sigma^2}, 1 \right], \tag{5} \]

where r is the risky asset’s expected rate of return; σ is the volatility of risky return; g is the risk-free asset’s rate of return; γ is the coefficient of relative risk aversion.

During the term of the contract, there are several possible types of events: the insured can

• transfer the funds between these two subaccounts;

• perform a partial surrender;

• completely surrender the contract;

• or pass away.
We incorporate these events into our “without consumption” and “with consumption” models.

2.1.1 “WITHOUT CONSUMPTION” CASE

In the first step, let us assume there are no surrenders. If we only consider the insured and beneficiary utility without consumption, we can get the objective function as

\[
\max_{\omega_t} E \left[ \sum_{t=1}^{T} \beta^t (\prod_{i=1}^{t-1} \phi_i) (1 - \phi_t) \zeta v_B(b_t) + \beta^T (\prod_{i=1}^{T} \phi_i) V_T+1(a_{T+1}) \right].
\]

The insured retires at the end of time \( T \). \( \phi_t \) is the survival rate at time \( t \). \( \zeta \) denotes the strength of the bequest motive and ranges from 0 to 1. If \( \zeta = 0 \), the insured has no bequest motive and does not want to leave a bequest to his beneficiary; if \( \zeta = 1 \), the insured has the strongest bequest motive and will treat his beneficiary like himself. \( v_B \) is the beneficiary’s value function. If the insured dies before retirement, the beneficiary will get the larger of the account value or the guaranteed amount. Once the beneficiary gets the money, the objective function of the beneficiary is

\[
\max_{\omega_B^t} E \left[ \beta^{T_B-t} (\prod_{i=t}^{T_B} \phi_i) v_{T_B+1}(b_{T_B+1}) \right].
\]

When the insured purchases the VA product, the beneficiary has \( T_B \) years until retirement age. If the insured dies at time \( t \), the beneficiary will receive the bequest and has \( T_B - t \) years until retirement age. She will optimally allocate the amount between risky and risk-free investments, and periodically consume the amount after her retirement. However, the beneficiary’s investment is not protected by the GMDB and we assume she has no bequest motive. If the insured survives until his retirement age, at the end of the policy period, he will get the entire account value without GMDB protection and annuitize it for his retirement life.
We get the Bellman equation for the insured:

\[(8) \quad V_t(a_t, b_t) = (1 - \phi_t)\zeta v_B(b_t) + \max_{\omega} \beta \phi_t E[V_{t+1}(a_{t+1}, b_{t+1}) | a_t],\]

subject to

\[(9) \quad V_{T+1}(a_{T+1}) = \sum_{t=T+1}^{T_{\max}} \beta^{t-(T+1)}(\prod_{i=T+1}^{t-1} \phi_i)u(\bar{c}),\]

\[(10) \quad \bar{c} = \frac{a_{T+1}}{\sum_{t=T+1}^{T_{\max}} \prod_{i=T+1}^{t-1} \phi_i (1 + r_f)^{T+1-t}},\]

\[a_{t+1} = a_t(\omega_t(1 + r_{t+1}) + (1 - \omega_t)(1 + g_t)),\]

\[0 \leq \omega_t \leq 1,\]

\[k_{t+1} = k_t(1 + r_{pt}),\]

\[b_{t+1} = \max(a_1 \prod_{i=1}^t (1 + r_{pt}), a_{t+1}),\]

\[= \max(k_{t+1}, a_{t+1}),\]

where \(r_t\) is the expected risky rate of return at time \(t\), \(r_f\) is the risk free rate of return, and \(\bar{c}\) is annuitization amount after retirement. In our model, the retired insured will receive a life time pay-out annuity\(^2\), and the monthly payout is \(\bar{c}\). The insured consumes \(\bar{c}\) and gets the utility.

2.1.2 “WITH CONSUMPTION” CASE

For simplicity, we assume that the events (the consumption and the allocation) can only occur at a discrete time. Therefore, state variables only change at integer time points \(t = 1, 2, \cdots, T\). The consumption, \(c_t \in [0, a_t]\), is taken out from the two subaccounts in the same ratio as the existing account value and are consumed immediately. We can get the

\(^2\)Life time payout annuity is an insurance product that converts an accumulated investment into income that the insurance company pays out over the life of the investor (Chen, et al. (2006)). Many papers study life time payout annuities on pricing of these products, and how much and when to annuitize. The literature includes Yaari (1965); Richard (1975); Milevsky and Young (2002); Brown (2001); Poterba (1997), Mitchell, Poterba, Warshawsky, and Brown (1999); Brown and Poterba (2000); Brown and Warshawsky (2001); Kapur and Orszag (1999); Blake, Cairns, and Dowd (2000), and Milevsky (2001).
objective function as

\[
\max_{\omega_t, c_t} E \left[ \sum_{t=1}^{T} \beta^t (\prod_{i=1}^{t-1} \phi_i) u(c_t) + \beta^T (\prod_{i=1}^{T} \phi_i) V_{T+1}(a_{T+1}) + \sum_{t=1}^{T} \zeta \beta^t (\prod_{i=1}^{t-1} \phi_i)(1-\phi_i)v_B(b_t) \right].
\]

Once the beneficiary receives the bequest \( b_t \), which is protected by the GMDB, the objective function for the beneficiary is

\[
\max_{\omega_t^B, c_t^B} E \left[ \sum_{t_B=t}^{T_B} \beta^{t_B-t} (\prod_{i=t}^{t_B-1} \phi_i) u(c_t^B) + \beta^{T_B-t} (\prod_{i=t}^{T_B} \phi_i) v_{T_B+1}(b_{T_B+1}) \right].
\]

The beneficiary will maximize her own utility by optimally choosing her own consumption \( c_t^B \) and investment allocation \( \omega_t^B \). As in the “without consumption” case, the beneficiary’s investment is not protected by the GMDB and she has no bequest motive.

The derived Bellman equation for the insured is

\[
V_t(a_t, b_t) = \max_{\omega_t, c_t} \left\{ u_t(c_t) + \beta \phi_t E[V_{t+1}(a_{t+1}, b_{t+1}) | a_t] + \zeta (1-\phi_t)v_t(b_t) \right\},
\]

subject to

\[
V_{T+1}(a_{T+1}) = \sum_{t=T+1}^{T_{\max}} \beta^{t-(T+1)} (\prod_{i=T+1}^{t-1} \phi_i) u(\bar{c}),
\]

\[0 \leq \omega_t \leq 1,\]

\[0 < c_t \leq a_t,\]

\[a_{t+1} = (\omega_t(1+r_{t+1}) + (1-\omega_t)(1+g_t))(a_t - c_t),\]

\[\bar{c} = \frac{a_{T+1}}{\sum_{t=T+1}^{T_{\max}} \prod_{i=T+1}^{t-1} \phi_i (1+r_f)^{T+1-t}},\]

\[k_{t+1} = k_t (1+r_{p_t}) \frac{a_{t+1} - c_{t+1}}{a_{t+1}},\]

\[b_{t+1} = \max(k_{t+1}, a_{t+1}).\]
Following Hardy (2003), all state variables are denoted as $(\cdot)_t^-$, $(\cdot)_t^+$, i.e. the value immediately before and after the transactions at the discrete time $t$, respectively. Withdrawals and consumptions occur at $t^-$, then at $t^+$, which is still at time $t$ but after withdrawal, the insured decides the amount to transfer between the fixed and the variable subaccounts. We also assume that the beneficiary gets the bequest immediately at $t^+$ just after the insured dies at $t^+$. Therefore, the Bellman equation will have 2 stages: at the 1st stage from $t^-$ to $t^+$, the insured gets the utility from consumption of withdrawal.

\begin{align}
V_{t^-}(a_{t^-}, b_{t^-}) &= \max_{c_t} \{ u(c_t) \} + \{ V_{t^+}(a_{t^+}, b_{t^+}) \}, \\
\implies V_{t^-}(a_{t^-}, m_{t}) &= \max_{c_t} \{ u(c_t) \} + \{ V_{t^+}(a_{t^+}, m_{t}) \}, \\
\text{where } & \quad m_{t} = \frac{a_{t^-}}{b_{t^-}}, \\
& \quad a_{t^+} = a_{t^-} - c_t = \left(1 - \frac{c_t}{a_{t^-}}\right) a_{t^-}, \\
& \quad \text{and } u(c_t) + V_{t^+}(a_{t^+}, m_{t}) = \frac{c_t^{1-\gamma}}{1-\gamma} + \left(1 - \frac{c_t}{a_{t^-}}\right)^{1-\gamma} V_{t^+}(a_{t^-}, m_{t}). 
\end{align}

It is easy to see that $m_{t}$ is same at $t^-$ and $t^+$. Because

\begin{align*}
b_{t^+} &= \frac{a_{t^+}}{a_{t^-}} b_{t^-} = \left(1 - \frac{c_t}{a_{t^-}}\right) b_{t^-}, \\
\frac{a_{t^+}}{b_{t^+}} &= \frac{1 - \frac{c_t}{a_{t^-}}}{1 - \frac{c_t}{a_{t^-} b_{t^-}}} = \frac{a_{t^-}}{b_{t^-}} = m_{t^-}.
\end{align*}

At the second stage from $t^+$ to $(t+h)^-$, the insured chooses a proportion $\omega$ to invest in the variable account,

\begin{align}
V_{t^+}(a_{t^+}, b_t) &= (1 - \phi_t) \zeta v(\max(a_{t^+}, b_t)) + \max_{t^+} \{ \phi_t \beta^h EV_{t+h^-}(a_{t+h^-}, b_{t+h^-}) \}, \\
\implies V_{t^+}(a_{t^+}, m_t) &= (1 - \phi_t) \zeta v(\max(a_{t^+}, m_t)) + \max_{t^+} \{ \phi_t \beta^h EV_{t+h^-}(a_{t+h^-}, m_{t+h^-}) \}, \\
\text{where } & \quad a_{t+h^-} = (\omega e^{rh} + (1 - \omega) e^{gh}) a_{t^+},
\end{align}
then we can get

\[
V_{t+}(a_{t+}, m_t) = (1 - \phi_t)\zeta v(\max(a_{t+}, b_{t+})) + \\
+ \max_{\omega_t} \left\{ \phi_t^2 \beta h EV_{t+h} \left( a_{t+}((\omega e^{rh} + (1 - \omega)e^{gh})) \right) \right\}.
\]

2.2 Numerical Methodology

Let us first assume that the insured buys the variable annuity with GMDB options in a lump sum at age 35. The insured can transfer and withdraw money every month. At age 65, the insured retires and annuitizes the variable annuity to support his retirement life. Let the expected risky rate of return \( r = 0.07 \); volatility of risky rate of return \( \sigma = 0.15 \); growth rate in fixed account \( g = 0.04 \); inflation rate \( r_f = 0.03 \); coefficient of relative risk aversion \( \gamma = 1.8 \). By Charupat and Milevsky (2002), the optimal allocation to the risky asset is \( \omega^* = 74\% \) at any asset level in each time period \( t \) with the survival rate \( \phi = 1 \) and guaranteed rate \( r_p = 0 \).

We will apply a trinomial lattice model in both “without” and “with consumption” cases.

2.2.1 “WITHOUT CONSUMPTION” CASE

Based on Hull (1997), we use a trinomial lattice to do the calculation. Assume the move-up factor is \( u = e^{\sigma \sqrt{3\Delta t}} \); the move-down factor \( d = 1/u \); the mean value in the variable account \( (S = \omega a) \); the mean of the continuous log-normal distribution \( E[S] = \omega ae^{rh} \) (assume \( h = \Delta t \)), and the variance is \( \text{Var}[S] = \omega^2 a^2 e^{2rh}[e^{\sigma^2h} - 1] \); the mean value in the fixed account \( (F = (1 - \omega)a) \) is \( E[F] = (1 - \omega)e^{gh} \), and variance is \( \text{Var}[F] = 0 \); and the covariance \( \text{Cov}[F, S] = 0 \).

According to Boyle (1988), there are three conditions to apply to the trinomial lattice:
1. the probabilities sum to one;
2. the mean of the discrete distribution is equal to the mean of the continuous log-normal distribution;
3. the variance of the discrete distribution is equal to the variance of the continuous distri-
bution.

The above three conditions are,

\begin{align}
(19) & \quad p_u + p_m + p_d = 1, \\
(20) & \quad p_u a_u + p_m a + p_d a_d = \alpha [\omega e^{rh} + (1 - \omega) e^{gh}], \\
(21) & \quad p_u a^2 u^2 + p_m a^2 + p_d a_d^2 - (\omega e^{rh} + (1 - \omega)e^{gh})^2 = \omega^2 a^2 e^{2rh}[\sigma^2 - 1].
\end{align}

By some algebraic transformation, we can get

\begin{align}
(22) & \quad p_u = \frac{A \omega^2 + B \omega + C}{(u - 1)(u - d)}, \\
(23) & \quad p_d = \frac{\omega(e^{rh} - e^{gh}) + e^{gh} - 1}{d - 1} = \frac{A \omega^2 + B \omega + C}{(d - 1)(u - d)}, \\
(24) & \quad p_m = 1 - \frac{A \omega^2 + B \omega + C}{(u - 1)(u - d)} - \frac{\omega(e^{rh} - e^{gh}) + e^{gh} - 1}{d - 1} + \frac{A \omega^2 + B \omega + C}{(d - 1)(u - d)},
\end{align}

where

\begin{align*}
A &= e^{(2r + \sigma^2)h} - 2e^{(r + g)h} + e^{2gh}, \\
B &= (e^{rh} - e^{gh})(2e^{gh} - d - 1), \\
C &= (e^{gh} - 1)(e^{gh} - d).
\end{align*}

Since

\begin{align}
(25) & \quad V_{t,j}(\omega) = (1 - \phi)\zeta v(b_{t,j}) + \beta\phi(p_u V_{t+1,j+1} + p_m V_{t+1,j} + p_d V_{t+1,j-1}),
\end{align}

to maximize $V_{t,j}$, we take the first derivative on $\omega$, and we get the optimal $\omega$:

\begin{align}
(26) & \quad \omega^* = -\frac{(d - 1)BV_{t+1,j} - V_{t+1,j-1} + (u - 1)(V_{t+1,j}V_{t+1,j+1})[(u - d)(e^{rh} - e^{gh}) - B]}{2A[(d - 1)(V_{t+1,j} - V_{t+1,j-1}) + (u - 1)(V_{t+1,j} - V_{t+1,j+1})]}.
\end{align}

By the no short-selling restriction, we know that $\omega \in [0, 1]$, so we only need to check 3 possible values of $\omega$: 0, $\omega^*$, 1, and if $\omega^* < 0$ or $> 1$, we only need to check 0 or 1.
We assume the insured can adjust his allocation at the beginning of each month, and starting wealth level at time 0 is 1. The algorithm to do the numerical values can be done as follows:

1. Initialize account value at time 1: \( a_1 = 1 \), and other parameter values;
2. Calculate the jump sizes \( u = e^{\sigma \sqrt{3} \Delta t}, d = \frac{1}{u} \) and \( m = 1 \);
3. Build the tree for account value \( a \) by using jump sizes until age 65;
4. Set terminal value \( V_{T+1}(a_{T+1}) \) by using equation (9), (10);
5. For \( t = T \) to 1, at each time period, use backward induction to maximize the insured’s utility:
   5.1 Calculate the optimal allocation \( \omega^*_t \) by (26);
   5.2 Calculate the transition probabilities \( p_u, p_d \) and \( p_m \) by plugging \( \omega^*_t \) into (22), (23) and (24);
   5.3 Derive \( V_t \) by (25) until \( t = 1 \).

Let us first assume the base case is \( r = 0.07, g = 0.04, r_f = 0.03, \beta = 0.97, r_p = 0, \sigma = 0.15, \phi = 0.99, \gamma = 1.8, \zeta = 0.5 \). Then, we will check the changes of allocation by giving some shocks: (1) \( r = 0.055 \); (2) \( \sigma = 0.25 \); (3) \( \gamma = 2.5 \); (4) \( \zeta = 0.2 \); (5) \( p = 0.03 \).

Figure 5 (age 45 allocation under “without consumption” case) shows that the amount of money allocated to the variable account at age 45 when the option is at-the-money. An “argument” between the beneficiary and the insured is a helpful way of looking at the results. The insured prefers the allocation determined by Merton (1969) at all times and benefit levels. At all stock-to-strike levels, the beneficiary prefers a more aggressive allocation than the insured, as he is protected against downside risk. This effect is most pronounced when the account is at-the-money. When the account is significantly out-of-the-money, the downside protection is not very valuable and the beneficiary prefers an allocation near the Merton level. Therefore, there is no argument. When the account is significantly in-the-money, the beneficiary does not have a strong preference as he receives the strike in (nearly) every case. Again, there is no argument as the beneficiary is (nearly) indifferent. It is only
near the at-the-money level where the beneficiary has a strong and aggressive preference.

![Figure 5: Age 45 allocation under without consumption case](image)

The effect of the argument is clearly seen in this figure by the bumps around the at-the-money area, which are in the middle, for all parameter levels. Parameter changes primarily affect the level of the insured's preferred allocation rather than the size of the bump.

As the risky rate of return \( r \) decreases from 7% to 5.5%, the variable subaccount allocation reduces from 73% to around 37%. For \( r = 7\% \), as the asset level goes up, the allocation increases from 73%, which is a Merton allocation, to 77% (goes up 4%) around the at-the-money area, and then goes down to Merton allocation again; for \( r = 5.5\% \), the allocation increases from 36.7%, to 39% (goes up 2.3%), then goes down to 36.7%. As the risky rate of return decreases, the risky subaccount will lose some attraction to both insured and beneficiary.

As the stock market volatility increases, i.e. \( \sigma \) increases from 15% to 25%, the variable subaccount allocation reduces from 73% to around 28%. For \( \sigma = 25\% \), as the asset level goes up, the allocation increases from 26.6%, which is Merton allocation, to 28.2% (goes up 1.6%, compare to 4% at \( \sigma = 15\% \)) around at the money area, and then goes down to Merton
allocation again. As the volatility increases, the risky subaccount will not be as attractive as in low volatility case to both insured and beneficiary. Consistent to Merton (1969) and Charupat and Milevsky (2002), identical Sharpe Ratios produce identical results.

As insured’s risk aversion level increases, i.e. the coefficient of relative risk aversion $\gamma$ increases from 1.8 to 2.5, the allocation at all asset levels decreases about 27%, because the more risk averse the policyholder is, the more conservative allocation decision they would make. As $\gamma = 2.5$, the allocation increases from 52.7% to 55% (goes up 2.3%) around at the money area.

When the roll-up rate $r_p$ increases from 0% to 3%, the bump area will move to a higher asset level. At age 45, the highest “argument” point moves from $a = 1$ to 1.35, which is a significant increase. The increase of the strike price of the GMDB option is the reason for this move.

Decreasing the bequest motive from $\zeta = 0.5$ to 0.2 will keep the allocation level the same at almost all asset levels, but reducing the bequest motive makes insured care less about beneficiary. The hump level around at the money area is reduced to 1.7%.

![Figure 6: At the Money Allocation under without consumption case](image)

25
Figure 6 plots at the money allocation at all ages. It shows that at any parameter level, as the insured gets older, he will care more about himself. As the policyholder ages, he has less and less probability to die before retirement age. It is more likely that he, rather than his beneficiary, will consume the assets. He will be more concerned about his post-retirement life and keeping himself from outliving the assets during retirement. As a result, the “argument” moves in favor of the insured, and the amount in the risky asset decreases. As the risky rate of return \( r \) decreases from 7% to 5.5%, the risky account will be less attractive, as a result the insured puts less money in the risky subaccount. As the stock market volatility increases, i.e. \( \sigma \) increases from 15% to 25%, a risk averse insured will take less risk by transferring money from risky subaccount to risk free account. As the insured’s risk aversion level increases, i.e. the coefficient of relative risk aversion \( \gamma \) increases from 1.8 to 2.5, the insured is more concerned about the safety of the investment and will allocate less into risky subaccount. The bequest motive \( \zeta \) also has an effect, as the bequest motive decreases, the risky account allocation will slightly decrease, which also means the “argument” between the policyholder and the beneficiary decreases at all ages. Roll-up rate \( r_p \) case is somewhat more complicated: Increasing the roll-up rate \( r_p \) increases the at the money allocation when the policyholder is at a younger age; as the insured ages, the roll-up rate becomes less and less useful to protect the beneficiary, and as a result, the risky asset allocation drops more quickly than the base case, and finally converges to the base case allocation. Compared with all the parameter shocks, changes in roll-up rate \( r_p \) and bequest motive \( \zeta \) will only change the allocation at younger ages; as the insured ages, the allocation strategies converge. Changes in the risky rate of return \( r \), coefficient of relative risk aversion or volatility of stock market cause nearly parallel shifts, which change not only at younger ages, but also at older ones.

2.2.2 “WITH CONSUMPTION” CASE

From \( t^- \) to \( t^+ \), there is consumption incurred. We take the first order condition on the
first stage, and we will get

\[
\frac{\partial V}{\partial c_t} = c_t - \frac{(1 - \gamma) \left(1 - \frac{c_t}{a_{t-}}\right)^{-\gamma} V_{t+}(a_{t-},m_t)}{a_{t-}}.
\]

Let \( D = \left( \frac{1}{(1 - \gamma) V_{t+}(1,m_t)} \right)^{\frac{1}{\gamma}} \), we can get

\[
\frac{c_t}{a_{t-}} = \frac{D}{1 + D}.
\]

To match the second stage, we need to modify the expression of \( D \):

\[
D = \left( \frac{1}{(1 - \gamma) V_{t+}(1,m_t)} \right)^{\frac{1}{\gamma}} = \left( \frac{(m_t e^{r_p})^{1 - \gamma}}{(1 - \gamma) V_{t+}(m_t e^{r_p},m_t)} \right)^{\frac{1}{\gamma}}.
\]

At the second stage, immediately after the consumption, the insured may be dead and the insurer will pay the GMDB amount to the beneficiary at \( t^+ \). If the insured still survives, he will choose allocation to the separate subaccounts. This will be the same procedure as in the “without consumption” case.

Table 5: Common Parameters in the Base case

<table>
<thead>
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<th>Parameter</th>
<th>Value</th>
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<tr>
<td>Strength of Bequest Motive</td>
<td>( \zeta ) 0.5</td>
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<tr>
<td>Subjective Discount Rate</td>
<td>( \beta ) 0.97</td>
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<tr>
<td>Risk Free Rate</td>
<td>( r_f ) 3%</td>
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<tr>
<td>Coefficient of Relative Risk Aversion</td>
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<tr>
<td>Growth Rate of Fixed Subaccount</td>
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<tr>
<td>Expected Return of Risky Asset</td>
<td>( r ) 7%</td>
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<tr>
<td>Volatility of Risky Return</td>
<td>( \sigma ) 15%</td>
</tr>
<tr>
<td>GMDB roll-up rate</td>
<td>( r_p ) 0</td>
</tr>
<tr>
<td>Annual Survival Rate</td>
<td>( \phi ) 0.99</td>
</tr>
<tr>
<td>Annual Mortality Rate</td>
<td>( \mu ) 0.01</td>
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</tbody>
</table>
With the partial withdrawal option, one will see the behavior of the insureds change from the previous case. To do the sensitivity tests, the values of base parameters are set as in table 5.

a. Risk Aversion Sensitivity

![Figure 7: At the Money Allocation with different $\gamma$ under consumption case](image)

The purpose of this sensitivity test is to discover the optimal choices for the policyholder if all but one coefficient, relative risk aversion ($\gamma$), were kept constant. Figure 7 shows that as the risk aversion level increases, the policyholder will less likely to invest money into the variable subaccount. For all $\gamma > 1$, the proportion in the variable subaccount will decrease as the insureds age. As $\gamma$ increases, the proportion in the variable subaccount will decrease. All of these are consistent as in the “without consumption” case. As $\gamma < 1$, since the insured is not very risk averse, he will put all the money into his variable subaccount.

Figure 8 shows that the pre-retirement consumption ratio changes for different levels of risk aversion. As risk aversion level increases, the consumption ratio in each period will
increase, because people will be increasingly concerned about the investment risk in the future. A bird in the hand is worth two in the bush, so they will prefer consuming now rather than investing for the future. Also as the insured ages, he will more strongly consider for himself than his beneficiary. As in the “without consumption” case, the “argument” moves in favor of the insured. Therefore, the consumption ratios increase under different risk aversion levels. One can also find that the consumption ratios converge under different risk aversion levels at Merton level once the policyholder reaches his retirement age.

Figure 8: At the Money Withdrawal ratio with different $\gamma$ under consumption case

Figure 9 shows, at age 45, the proportion of consumption to account value will correspond given different levels of risk aversion. The proportion of funds consumed is roughly constant when the GMDB is in the far out-of-the-money area, and the value of the consumption proportion is consistent with the value derived in the Merton model. As $\gamma < 1$, consumption ratio decreases as the GMDB becomes more deeply in the money; while as $\gamma > 1$, the consumption level increases as the asset level decreases (GMDB goes deeper and deeper in-the-money). The first result seems intuitive, as an in-the-money withdrawal costs the beneficiary
far more than the insured by roll-up property and even a mild bequest motive will cause the insured to maintain the account.

Figure 9: Age 45 proportion of consumption with different $\gamma$ under consumption case

The reason for the counterintuitive $\gamma > 1$ result in figure 9 is that we assume withdrawals from the variable annuity are the only source of consumption. For most currently available variable annuities, the GMDB strike is reduced proportionally with the reduction of the VA account value. When $\gamma = 1$, utility is logarithmic. A proportional reduction in the higher strike value reduces the beneficiary’s utility by exactly the same amount as a reduction of the lower account value harms the insured’s utility. When $\gamma < 1$, the beneficiary’s utility is reduced by more than the insured’s utility. When $\gamma > 1$, the beneficiary’s utility is actually reduced less than the insured’s utility is. In the most extreme case, the insured’s utility can approach $-\infty$. The insured sells his beneficiary up the river for a loaf of bread. A more realistic model would include an outside consumption source in addition to the partial withdrawals from the variable annuity account. This has been done in the next section.

Figure 10 shows the optimal allocation choice at age 45 under different risk aversion
levels. If the insured is less risk averse ($\gamma < 1$ or around 1), he will put the entire fund in the variable subaccount. At $\gamma > 1$, if the account value is small and less than the at-the-money level, the beneficiary does not care about the allocation because she has downside protection, and the insured still allocates the wealth using Merton’s method. If the account value is greater than the guaranteed value, the beneficiary and the insured agree on the allocation. But around at-the-money area, there are arguments between them, bequest motive will take effect and more money will be put in the risky account. As the risk aversion gets larger, the insureds will care more for themselves. Therefore, the bump level decreases as risk aversion level increases.

![Figure 10: Age 45 allocation with different $\gamma$ under consumption case](image)

$b$. Bequest Motive Sensitivity

The purpose of this sensitivity test was to discover the optimal choices for the policyholder if all but one parameter, bequest motive ($\zeta$), were kept constant. Figure 11 shows the at-the-money asset allocation for different bequest motives with $\gamma = 2$. When the bequest motive $\zeta = 0$, the asset allocation level agrees with the Merton result at all ages. As the bequest
motive increases, the proportion in the risky account will increase. But at any non-zero level of policyholder's bequest motive, the allocation to the risky account decreases as age grows until converges to the allocation at ζ = 0 (the Merton's allocation) at age 65.

Figure 11: At the Money Allocation with different ζ under consumption case

As the bequest motive level ζ increases, the pre-retirement consumption ratio in each period will decrease as shown in figure 12. With a larger bequest motive, the insured prefers to reduce the consumption to leave more money to his beneficiary. The consumption (partial withdrawal) will increase as the insured ages under all bequest motive levels. One can see that the higher the bequest motive is, the lower the consumption ratio is. The insured cares more for himself as he grows older. One can also observe that the consumption levels converge as the insured grows older and consumption ratios become the same at retirement age. The reason is that after the insured retires, the wealth in VA account will become a lifetime payout annuity, and there is no bequest motive any more.
Figure 12 shows the consumption ratio vs. account value at age 45 with different levels of bequest motive. As the bequest motive increases, the proportion of funds consumed will decrease. As the GMDB goes deeper in-the-money, the consumption ratios at all non-zero bequest motives converge to the zero bequest level. In addition, consumption ratios stay constant as the GMDB goes out-of-the-money. This is also counter-intuitive, and the reason is the same as in the risk aversion sensitivity test: we assume that withdrawals from the variable annuity are the only source of the policyholder’s consumption. In this test, we set $\gamma = 2$ in the $\zeta$ test, the beneficiary’s utility is actually reduced less than the insured’s utility is, so the insured becomes selfish. The problem should disappear if an outside income is included in addition to the partial withdrawals from the variable annuity account.
Figure 13: Age 45 proportion of consumption with different $\zeta$ under consumption case

Figure 14: Age 45 allocation with different $\zeta$ under consumption case

Figure 14 shows the proportion allocated to the risky asset vs. account value at age 45.
with different bequest motives. As $\zeta = 0$, the insured has no bequest motive and he only cares about himself, and his allocation decision follows the Merton rule. As the insured has the bequest motive and wants to leave a bequest to his beneficiary, he starts to make choices to maximize the joint utility. Therefore, around the at-the-money area, the insured breaks the Merton’s rule and tries to take more risk to help the beneficiary accumulate more benefits. One can observe the spikes around at-the-money area corresponding to different levels of bequest motive: the stronger the bequest motive is, the higher the spike (risky allocation) is.

c. Roll-up Rate Sensitivity

![Figure 15: At the Money Allocation with different $r_p$ under consumption case when $\gamma = 2$](image)

The purpose of the roll-up rate sensitivity test was to discover the optimal choices for the policyholder if only one variable, roll-up rate, was changed. Figure 15 shows at-the-money asset allocations under different roll-up rates. At any level of roll-up rate, the insured reduces the proportion of the amount in the variable subaccount during the whole examined period.
(age 35 to 65), which is consistent to the “without consumption” case. As the insured ages, the insured has more probability to get the wealth himself. Therefore, he will care more about his retirement life and put more money into riskless account. As the insured is young (at the beginning of the contract), higher roll-up rate \( r_p \) encourage more risky allocation. As the insured ages, roll-up protection weakens, because the beneficiary has less chance to get the GMDB protection.

Figure 15 is somewhat counter-intuitive: At 45 years old, the risky allocation for \( r_p = 0\% \) is highest one, although the \( r_p = 0\% \) case provides least downside protection. This is because for \( r_p = 1\% \) and \( r_p = 3\% \), the level of return of premium and roll-up benefits is increasing period by period. Therefore \( a = 1 \) is not a real “at-the-money” area for \( r_p > 0 \), and the argument area for \( r_p > 0 \) will go higher as the insured ages, but as usual the hump level will decrease.

![Figure 16: At the Money proportion of consumption with different \( r_p \) under consumption case when \( \gamma = 2 \)](image)

Figure 16 (at-the-money consumption ratios under different roll-up rates with \( \gamma = 2 \)) and 17 (at-the-money consumption ratios under different roll-up rates with \( \gamma = 0.5 \)) show the
consumption ratios under different roll-up rates. As $\gamma > 1$ (Figure 16), the consumption ratio with lower roll-up rate is higher than that of higher roll-up rate. A higher roll-up rate gives the insured more incentive to keep the money for the beneficiary, because GMDB strike price goes up, which makes the withdrawal hurt the beneficiary much more than the policyholder himself. Since the policyholder has bequest motives, he will reduce the consumption for his beneficiary. As the roll-up rate goes high enough (e.g. $r_p = 3\%$), the consumption ratio decreases for the whole pre-retirement period. While as $\gamma < 1$ (Figure 17) the consumption ratio with lower roll-up rate is lower than that of higher roll-up rate.

![Figure 17: At the Money proportion of consumption with different $r_p$ under consumption case when $\gamma = 0.5$](image)

Figure 18 shows the asset allocation at age 45 for different roll-up rates. A higher roll-up rate moves the at-the-money area to a higher asset level. Correspondingly, the “argument” between the insured and the beneficiary follows the up-move of the at-the-money level to a higher asset level. One can also observe that the size of the GMDB effect (the hump level) on the allocation increases proportionally with the roll-up rate.

Figure 19 shows the proportion of funds consumed vs. account value at age 45 for dif-
different roll-up rates. The insured consumes less when the roll-up rate is higher, because the consumption under the higher roll-up rate case will hurt the beneficiary much more.

Figure 18: Age 45 allocation with different $r_p$ under consumption case

Figure 19: Age 45 proportion of consumption with different $r_p$ under consumption case
**d. Volatility Sensitivity**

The purpose of the volatility sensitivity test was to discover the optimal choices for the policyholder if only one variable, equity market risk, was changed. As in the “without consumption” case, higher stock market volatility will make the insured more conservative and invest less money into the variable subaccount (Figure 20) during the whole test period. Greater volatility also reduces the policyholder’s consumption ratio (Figure 21 and Figure 22). Also as a result of higher risk, less “arguments” will be incurred between the insured and beneficiary, and you will see a lower hump level (Figure 23). In a word, if the future is very uncertain, the policyholder will prefer to spend less and save more in risk free account to avoid depleting the retirement fund.

![Figure 20: At the Money Allocation with different σ under consumption case](image-url)
Figure 21: At the Money proportion of consumption with different $\sigma$ under consumption case

Figure 22: Age 45 proportion of consumption with different $\sigma$ under consumption case
Figure 23: Age 45 allocation with different $\sigma$ under consumption case

Figure 24: At the Money Allocation with different $r$ under consumption case
Figure 25: At the Money proportion of consumption with different $r$ under consumption case

Figure 26: Age 45 proportion of consumption with different $r$ under consumption case
e. Risky Rate of Return Sensitivity

The purpose of the risky rate of return sensitivity test was to discover the optimal choices for the policyholder if all but one parameter, risky rate of return ($r$), were kept constant. Figure 24 shows at-the-money asset allocation with different risky rate of returns. As one expects, the $\omega$, the percentage of wealth held in the variable subaccount, decreases as the insured ages.

At any given age, for example at age 45, the allocation choice will obey the Merton rule except around the at-the-money area (Figure 27), which is consistent to previous discussions. We also see that a lower expected risky rate of return will have lower hump level, because both the insured and his beneficiary lose interests in investing money into risky assets with a small risk premium.

With a higher expected risky rate of return, the insured will get more returns from the variable subaccount, and also will have more money to consume. Therefore the at-the-money
consumption ratio is higher under the higher risky asset return case (Figure 25) during the whole test period. At any given age, for example at age 45, one can find that the consumption ratio will be higher under the higher risky rate of return \( (r = 7\%) \) at all asset levels (Figure 26).

2.3 COMPARISON BETWEEN “WITH” AND “WITHOUT” CONSUMPTION

In comparison to the “without consumption” case, the bump level (asset allocation \( \omega \)) of the “with consumption” at each age decreases. We believe the reason for this change is that the insured has one more choice, i.e. in addition to choosing an allocation, they can consume (partial withdraw) the money in the VA account to maximize the utility.

2.4 GMDB Pricing & Delta Ratio

2.4.1 Fees and Expenses of GMDB contracts

Like other investments, fees and expenses are incurred in variable annuities:

1. Investment Management Fee is a payment to the management company for the services and investment portfolio recommendations. These fees will vary depending on the various subaccount options within the annuity.

2. Administrative charges cover the paperwork, record keeping, and periodic reports to the annuity policyholders

3. Surrender charges are charged if withdrawals occur before a specified period of time. It is usually a percentage of the account value and declines over time. Most variable annuities permit partial withdrawals each year without a surrender charge.

4. Mortality and Expense Charges are also called “M & E” fee. They are used to pay: (1) the mortality risk related with the guaranteed death benefit; (2) a guarantee that annual expenses will not exceed a certain percentage of assets; and (3) an allowance
for profit.

Except surrender charges, the other fees and expenses, which reduce the rate of return, are implicitly included in our model, and we will focus on GMDB charges.

### 2.4.2 GMDB Pricing & Delta Ratio

In the “with consumption” case, optimal allocation $\omega$ and withdrawal level $c$ are derived. By assuming the insured and beneficiary are the same type of person, and both of them make (from their perspective) optimal choices, the insurer can use the strategy of applying the insured’s optimal allocation $\omega$ and lapse (partial withdrawal) $c$ into a risk neutral model to price the GMDB options and implement a delta hedging strategy.

There are several findings in pricing the GMDB: first, as the risk aversion level $\gamma$ decreases, the insured will increase the allocation of money in variable subaccount, and the GMDB will become more valuable in protecting downside risk and delta ratio rises correspondingly. Second, the Sharpe Ratio matters in deciding the GMDB prices and delta ratios: identical Sharpe Ratios produce identical results. Given the expected risky rate of return $r$ and other parameters fixed, the GMDB price and delta ratio go down as the stock market volatility $\sigma$ increases. One would normally expect that a riskier asset would produce a higher GMDB price, but instead the value decreases because a higher equity market volatility will lower the amount invested in the variable subaccount, and a less risky investment will need less GMDB protection. Equivalently, given equity market volatility and other parameters fixed, the GMDB price and delta ratio go down as the expected risky rate of return $r$ goes down. The roll-up rate $r_p$ is also a major factor to the GMDB price and the delta ratio. Higher roll-up rate means the beneficiary can keep more money in bad market states, and it encourages the “argument” between the insured and his beneficiary to make the insured take more aggressive allocation at the money. Therefore the GMDB price and delta ratio will go up as we increase the roll-up rate. Bequest motive is also very important in determining the GMDB price: a higher bequest motive increases the allocation in the variable subaccount.
and discourages the withdrawal from the VA account. As a result, a higher bequest motive makes the GMDB more valuable.

Table 6: GMDB Price and Delta ratio $\delta$ with different risk aversion level $\gamma$ under $\sigma = 15$

<table>
<thead>
<tr>
<th>Delta ratio</th>
<th>GMDB price</th>
<th>Risk Aversion</th>
<th>Bequest</th>
</tr>
</thead>
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Other parameter values: annual risky rate of return $r = 7\%$, annual volatility $\sigma = 15\%$, annual fixed growth rate $g = 4\%$, annual discount rate $\beta = 0.97$, annual mortality rate $\mu = 1\%$, annual roll-up rate $r_p = 0\%$. 

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Table 7: GMDB Price and Delta ratio $\delta$ with different risk aversion level $\gamma$ under $\sigma = 25$

<table>
<thead>
<tr>
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</tr>
<tr>
<td>-0.026224</td>
<td>0.01036</td>
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</tr>
<tr>
<td>-0.026927</td>
<td>0.01101</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>-0.027458</td>
<td>0.01136</td>
<td>0.5</td>
<td>0.8</td>
</tr>
<tr>
<td>-0.027790</td>
<td>0.01156</td>
<td>0.5</td>
<td>1</td>
</tr>
</tbody>
</table>

Other parameter values: annual risky rate of return $r = 7\%$, annual fixed growth rate $g = 4\%$, annual discount rate $\beta = 0.97$, mortality rate $\mu = 1\%$, annual roll-up rate $r_p = 0\%$. 
Table 8: GMDB Price and Delta ratio $\delta$ with different risk aversion level $\gamma$ under $\sigma = 35\%$

<table>
<thead>
<tr>
<th>Delta ratio</th>
<th>GMDB price</th>
<th>Risk Aversion</th>
<th>Bequest</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.004289</td>
<td>8.93E-05</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>-0.004450</td>
<td>9.58E-05</td>
<td>2</td>
<td>0.2</td>
</tr>
<tr>
<td>-0.004696</td>
<td>1.06E-04</td>
<td>2</td>
<td>0.5</td>
</tr>
<tr>
<td>-0.004938</td>
<td>1.17E-04</td>
<td>2</td>
<td>0.8</td>
</tr>
<tr>
<td>-0.005096</td>
<td>1.24E-04</td>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>-0.005113</td>
<td>1.29E-04</td>
<td>1.8</td>
<td>0</td>
</tr>
<tr>
<td>-0.005321</td>
<td>1.39E-04</td>
<td>1.8</td>
<td>0.2</td>
</tr>
<tr>
<td>-0.005636</td>
<td>1.56E-04</td>
<td>1.8</td>
<td>0.5</td>
</tr>
<tr>
<td>-0.005951</td>
<td>1.72E-04</td>
<td>1.8</td>
<td>0.8</td>
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<tr>
<td>-0.006156</td>
<td>1.84E-04</td>
<td>1.8</td>
<td>1</td>
</tr>
<tr>
<td>-0.006833</td>
<td>2.42E-04</td>
<td>1.5</td>
<td>0</td>
</tr>
<tr>
<td>-0.007144</td>
<td>2.63E-04</td>
<td>1.5</td>
<td>0.2</td>
</tr>
<tr>
<td>-0.007609</td>
<td>2.97E-04</td>
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<td>0.5</td>
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<td>3.57E-04</td>
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<td>1</td>
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<td>-0.009384</td>
<td>4.98E-04</td>
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<td>0</td>
</tr>
<tr>
<td>-0.009855</td>
<td>5.46E-04</td>
<td>1.2</td>
<td>0.2</td>
</tr>
<tr>
<td>-0.010569</td>
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<td>1.2</td>
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</tr>
<tr>
<td>-0.011283</td>
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<tr>
<td>-0.025725</td>
<td>0.006907698</td>
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<td>0.5</td>
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<tr>
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<td>0.8</td>
</tr>
<tr>
<td>-0.028511</td>
<td>0.008623802</td>
<td>0.5</td>
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</tbody>
</table>

Other parameter values: annual risky rate of return $r = 7\%$, annual fixed growth rate $g = 4\%$, annual discount rate $\beta = 0.97$, annual mortality rate $\mu = 1\%$, annual roll-up rate $r_p = 0\%$. 
Table 9: GMDB price and Delta ratio $\delta$ with different roll-up rates $r_p$ and different volatilities $\sigma$

<table>
<thead>
<tr>
<th>Delta ratio</th>
<th>GMDB price</th>
<th>Bequest $\zeta$</th>
<th>Roll-up Rate</th>
<th>Volatility $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.010303</td>
<td>0.00083</td>
<td>0</td>
<td>0%</td>
<td>15%</td>
</tr>
<tr>
<td>-0.010517</td>
<td>0.00086</td>
<td>0.2</td>
<td>0%</td>
<td>15%</td>
</tr>
<tr>
<td>-0.010829</td>
<td>0.00091</td>
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<td>0%</td>
<td>15%</td>
</tr>
<tr>
<td>-0.011126</td>
<td>0.00095</td>
<td>0.8</td>
<td>0%</td>
<td>15%</td>
</tr>
<tr>
<td>-0.011312</td>
<td>0.00099</td>
<td>1</td>
<td>0%</td>
<td>15%</td>
</tr>
<tr>
<td>-0.013839</td>
<td>0.00131</td>
<td>0</td>
<td>1%</td>
<td>15%</td>
</tr>
<tr>
<td>-0.014119</td>
<td>0.00136</td>
<td>0.2</td>
<td>1%</td>
<td>15%</td>
</tr>
<tr>
<td>-0.014525</td>
<td>0.00144</td>
<td>0.5</td>
<td>1%</td>
<td>15%</td>
</tr>
<tr>
<td>-0.014910</td>
<td>0.00152</td>
<td>0.8</td>
<td>1%</td>
<td>15%</td>
</tr>
<tr>
<td>-0.015158</td>
<td>0.00157</td>
<td>1</td>
<td>1%</td>
<td>15%</td>
</tr>
<tr>
<td>-0.028398</td>
<td>0.00404</td>
<td>0</td>
<td>3%</td>
<td>15%</td>
</tr>
<tr>
<td>-0.028752</td>
<td>0.00419</td>
<td>0.2</td>
<td>3%</td>
<td>15%</td>
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<td>-0.029258</td>
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</tr>
<tr>
<td>-0.029746</td>
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<td>3%</td>
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</tr>
<tr>
<td>-0.030048</td>
<td>0.00474</td>
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</tr>
<tr>
<td>-0.006736</td>
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<td>0</td>
<td>0%</td>
<td>25%</td>
</tr>
<tr>
<td>-0.006954</td>
<td>0.00026</td>
<td>0.2</td>
<td>0%</td>
<td>25%</td>
</tr>
<tr>
<td>-0.007280</td>
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<td>0%</td>
<td>25%</td>
</tr>
<tr>
<td>-0.007595</td>
<td>0.00031</td>
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<td>0%</td>
<td>25%</td>
</tr>
<tr>
<td>-0.007802</td>
<td>0.00033</td>
<td>1</td>
<td>0%</td>
<td>25%</td>
</tr>
<tr>
<td>-0.010153</td>
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<td>0</td>
<td>1%</td>
<td>25%</td>
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<tr>
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<tr>
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</tr>
<tr>
<td>-0.029865</td>
<td>0.00255</td>
<td>0</td>
<td>3%</td>
<td>25%</td>
</tr>
<tr>
<td>-0.030531</td>
<td>0.00272</td>
<td>0.2</td>
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<td>25%</td>
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<tr>
<td>-0.031501</td>
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<td>3%</td>
<td>25%</td>
</tr>
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<td>3%</td>
<td>25%</td>
</tr>
<tr>
<td>-0.033029</td>
<td>0.00338</td>
<td>1</td>
<td>3%</td>
<td>25%</td>
</tr>
</tbody>
</table>

Other parameter values: annual risky rate of return $r = 7\%$, annual fixed growth rate $g = 4\%$, annual discount rate $\beta = 0.97$, annual mortality rate $\mu = 1\%$, risk aversion level $\gamma = 2$. 
3 Optimal Allocation and Consumption with Guaranteed Minimum Death Benefits with Labor Income

In this section, we extend the model by incorporating labor income and unemployment risk. In the model, we assume there are other income sources, e.g. labor income (human capital), besides the withdrawals from the variable annuity.

3.1 Models

An individual purchases a variable annuity contract with GMDB options and makes a lump sum deposit to the variable annuity account. Once the insured receives labor income at the beginning of period $t$, he will make his consumption decision. If his labor income is not enough to support his consumption, he will make a decision to withdraw. Simultaneously, the GMDB level will be reduced proportionally with the withdrawal ratio. In our model, there are no deposits, which means all the periodic income will be consumed at the current time $t$. After the consumption and withdrawal decision, still at time $t$, the policyholder will decide the allocation between fixed and variable subaccounts in the VA account. If the policyholder dies at time $t$, the amount in the VA account, which is protected by the GMDB, will be inherited by his beneficiary. The beneficiary gets the bequest and maximizes her own utility by optimal allocation and withdrawal. The policyholder makes all these decisions to maximize the joint utility of his beneficiary and of himself.

Again, we assume CRRA utility function

$$U(c) = \begin{cases} \frac{c^{1-\gamma}}{1-\gamma}, & \gamma > 0, \gamma \neq 1, \\ \ln(c), & \gamma = 1, \end{cases}$$

where $\gamma$ is the relative risk aversion level; $c$ is the consumption. The policyholder and
beneficiary get utility through consumption.

If the insured has labor income, the objective function is

\[
(28) \quad \max_{\omega_t, c_t} \sum_{t=1}^{T} \beta_t \left( \prod_{i=1}^{t} \phi_i \right) u(c_t) + \beta^T \left( \prod_{i=1}^{T} \phi_i \right) V_{T+1}(a_{T+1}) + \sum_{t=1}^{T} \beta^t \left( \prod_{i=1}^{t-1} \phi_i \right) (1 - \phi_t) \zeta v_B(b_t).
\]

Now we incorporate outside income \( y_t \), and we assume \( d_t \) is the withdrawal amount from the VA account. The consumption amount for the insured is the sum of withdrawal and labor income \( d_t + y_t \).

We can also derive the beneficiary’s objective function as follows,

\[
(29) \quad \max_{\omega_B, c_B} E \left[ \sum_{t_B = t}^{T_B} \beta^{t_B - t} \left( \prod_{i=t}^{t_B-1} \phi_i \right) u(c_B^t) + \beta^{T_B - t} \left( \prod_{i=t}^{T_B} \phi_i \right) V_{T_B+1}(b_{T_B+1}) \right].
\]

When the insured deposits the VA account, the beneficiary has \( T_B \) years until retirement age. If the insured dies at time \( t \), the beneficiary will receive the bequest and has \( T_B - t \) years until her retirement age. She will withdraw optimally to consume and will allocate the amount between risky and risk-free accounts, and will transform the money to a lifetime payout annuity after her retirement. However, the beneficiary’s investment is not protected by the GMDB and we also assume she has no bequest motive. If the insured survives until his retirement age, at the end of the policy period, he will get the entire account value and annuitize it for his retirement life.

By the objective function of the insured, we can derive the Bellman equation of the insured,

\[
(30) \quad V_t(a_t, b_t) = \max_{\omega_t, d_t} \left\{ u_t(c_t) + (1 - \phi_t) \zeta v_B(b_t) + \beta \phi_t E[V_{t+1}(a_{t+1}, b_{t+1}) | a_t, r_t] \right\},
\]

subject to

\[
\begin{align*}
    a_1 &= b_1, \\
    c_t &= y_t + d_t, \quad 0 \leq d_t \leq a_t,
\end{align*}
\]
$a_{t+1} = (\omega_t(1 + r_{t+1}) + (1 - \omega_t)(1 + g_t))(a_t - d_t), \quad 0 \leq \omega_t \leq 1,$

$k_{t+1} = k_t(1 + r_p)\frac{a_{t+1} - d_{t+1}}{a_{t+1}},$

$b_{t+1} = \max(k_{t+1}, a_{t+1}),$

$V_{T+1}(a_{T+1}) = \sum_{t=T+1}^{T_{\max}} \beta^{t-(T+1)}(\prod_{i=T+1}^{t-1} \phi_t)u(\bar{c}).$

Most notation is the same as the previous section. The return of premium and roll-up benefit level $k_t$ is reduced proportionally with the withdrawal amount $d_t$. $\bar{c}$ is the periodic consumption after retirement, and we assume it is the level payment from the variable annuity account. $\bar{c}$ can be derived from the terminal account value as follows,

$$a_{T+1} = (\omega_T(1 + r_{T+1}) + (1 - \omega_T)(1 + r_f))(a_T - d_T)$$

$$= \bar{c} \sum_{t=T+1}^{T_{\max}} \prod_{i=T+1}^{t-1} \phi_t(1 + r_f)^{T+1-t},$$

$$\Rightarrow \bar{c} = \frac{a_{T+1}}{\sum_{t=T+1}^{T_{\max}} \prod_{i=T+1}^{t-1} \phi_t(1 + g_T)^{T+1-t}}.$$

The beneficiary’s constant consumption level after her retirement can be derived in the same way.

The first order conditions on $\omega_t$ and $d_t$ are

\begin{align*}
(d) & : 0 = u'(y_t + d_t) - \beta \phi_t \left\{ E \frac{\partial V_{t+1}(a_{t+1}, b_{t+1}, r_{t+1})}{\partial a_{t+1}} [\omega_t(1 + r_{t+1}) + (1 - \omega_t)(1 + g_t)] ight. \\
& + \left. \frac{E \partial V_{t+1}(a_{t+1}, b_{t+1}, r_{t+1})}{\partial b_{t+1}} \left( \frac{b_t(1 + r_p)}{a_t} I_{k_{t+1} > a_{t+1}} + (\omega_t(1 + r_{t+1}) + (1 - \omega_t)(1 + g_t))I_{k_{t+1} \leq a_{t+1}} \right) \right\}
\end{align*}

\begin{align*}
& - \beta(1 - \phi_t) \left\{ E \frac{\partial V_{t+1}(a_{t+1}, b_{t+1}, r_{t+1})}{\partial a_{t+1}} \left[ \omega(1 + r_{t+1}) + (1 - \omega_t)(1 + g_t) \right] I_{k_{t+1} \leq a_{t+1}} ight. \\
& + \left. E \frac{\partial V_{t+1}(a_{t+1}, b_{t+1}, r_{t+1})}{\partial b_{t+1}} \left( \frac{b_t(1 + r_p)}{a_t} I_{k_{t+1} > a_{t+1}} \right) \right\},
\end{align*}

\begin{align*}
(\omega) & : 0 = \beta \phi_t \left\{ E \frac{\partial V_{t+1}(a_{t+1}, b_{t+1}, r_{t+1})}{\partial a_{t+1}} (r_{t+1} - g_t)(a_t - d_t) \right\} \\
& + \beta(1 - \phi_t) \left\{ E \frac{\partial V_{t+1}(a_{t+1}, b_{t+1}, r_{t+1})}{\partial a_{t+1}} (r_{t+1} - g_t)(a_t - d_t) I_{k_{t+1} \leq a_{t+1}} \right\}.
\end{align*}
Following Hardy (2003), all state variables are denoted as $(\cdot)_{t^-}$, $(\cdot)_{t^+}$, i.e. the value immediately before and after the transactions at the discrete time $t$, respectively. We assume the policyholder is employed and receives labor income $y_t$ at $t^-$. Withdrawal and consumption also occur at $t^-$. Then the insured determines the amount to transfer between the fixed and the variable subaccounts at $t^+$, which is still at time $t$ but after the receipt of income, the decisions of withdrawal and consumption. We also assume that the beneficiary gets the bequest immediately at $t^+$ just after the insured dies at $t^+$. Therefore, we transform the above Bellman equation to two stages. At the 1st stage from $t^-$ to $t^+$, the insured gets the utility from consumption which is equal to the sum of optimal withdrawal from the variable annuity account and labor income.

\begin{equation}
V_{t^-}(a_{t^-}, b_{t^-}) = \max_{d_t} \left\{ u(c_t) + (1 - \phi_t)\zeta v_B(b_{t^+}) + V_{t^+}(a_{t^+}, b_{t^+}) \right\}.
\end{equation}

Since there is mortality risk and bequest motives, if the policyholder dies before retirement, the beneficiary will get utility from consumption of the bequest which is protected by the GMDB. The GMDB level will be reduced proportionally with the insured’s withdrawal ratio at $t^-$ from the variable annuity account. We transform the above equation by replacing all variables subscripted by $t^+$ with the corresponding expressions subscripted by $t^-$. 

\begin{equation}
V_{t^-}(a_{t^-}, b_{t^-}) = \max_{d_t} \left\{ u(y_t + d_t) + (1 - \phi_t)\zeta v_B\left(b_t - \frac{a_{t^-} - d_t}{a_{t^-}}\right) + V_{t^+}\left(a_{t^+} - d_t, b_t - \frac{a_{t^-} - d_t}{a_{t^-}}\right) \right\}.
\end{equation}

Under CRRA assumption, we can derive a constant factor\footnote{It can be generated numerically.} $\psi_t$ at period $t$ to make 

\begin{equation}
v_B\left(b_t - \frac{a_{t^-} - d_t}{a_{t^-}}\right) = \psi_t\left(b_t - \frac{a_{t^-} - d_t}{a_{t^-}}\right)^{1-\gamma}, \text{ given } \psi_t < 0 \text{ if } \gamma > 1, \text{ and } \psi_t > 0 \text{ if } \gamma < 1. \end{equation}

And Equation (32) can be rewritten as

\begin{equation}
V_{t^-}(a_{t^-}, b_{t^-}) = \max_{d_t} \left\{ u(y_t + d_t) + (1 - \phi_t)\zeta \psi_t\left(b_t - \frac{a_{t^-} - d_t}{a_{t^-}}\right)^{1-\gamma} + V_{t^+}\left(a_{t^+} - d_t, b_t - \frac{a_{t^-} - d_t}{a_{t^-}}\right) \right\}.
\end{equation}
At the 2nd stage from $t^+$ to $(t+1)^-$, the insured gets the utility from optimally allocating the amount in between two subaccounts. $V_{t^+}(a_{t^+}, b_{t^+})$ will be the contingent discounted value of $EV_{t+1^-}(a_{t+1^-}, b_{t+1^-})$.

$$V_{t^+}(a_{t^+}, b_{t^+}) = \max_{\omega_t} \{ \beta \phi_t EV_{t+1^-}(a_{t+1^-}, b_{t+1^-}) \}.$$  \(34\)

### 3.2 Numerical Methodology

There is no closed form solution to our model, so we will apply numerical analysis. We continue to apply a trinomial lattice to solve the optimal allocation from $t^+$ to $(t+1)^-$. Since an outside income is added, the one-dimensional lattice in section two does not work for deriving the optimal withdrawal choice from $t^-$ to $t^+$. We need to apply a two-Dimensional lattice in this section.

We solve the policyholder’s utility optimization problem by backward induction from the retirement age $t = T$ (at the beginning of age 65, $T = 360$) to $t = 1$ (at the beginning of age 35) by discretizing the beginning-of-period fund value $A = [0, a_{\text{max}}]$, into 51 nodes, $\hat{A} = \{a_1, a_2, \cdots, a_{51}\}$, and Guaranteed Minimum Death Benefit level $B = [0, b_{\text{max}}]$, into 51 nodes, $\hat{B} = \{b_1, b_2, \cdots, b_{51}\}$. Therefore, at any given period $t$, we will have a $51 \times 51$ space (51 asset levels by 51 GMDB levels) and the total state space for the whole time period of a policyholder is $51 \times 51 \times 360$ (51 asset levels by 51 GMDB levels by 360 time periods, and 360 time periods correspond to 360 months from age 35 to 65).

All state variables are denoted as $\cdot_{t^-}$, $\cdot_{t^+}$. We assume the terminal value at time $(T + 1)^-$ by using

$$\bar{c} = \frac{a_{T+1^-}}{\sum_{i=T+1}^{T_{\text{max}}} a_i} \prod_{i=T+1}^{t-1} \phi_i (1 + r_f)^{T+1-t}.$$  \(35\)

\(^4\)Let $a_1 = (1/u)^{25}$, where $u = e^{\sigma \sqrt{3\Delta t}}$ is the jump size, and $a_i = a_1 \times u^{i-1}$, for $i = 2, 3, \cdots, 51$. If annual volatility $\sigma = 15\%$, then the monthly jump size $u \approx 1.07788$. The lowest asset level is $a_1 = 0.1534$, the middle node is $a_{26} = 1$, and the largest node is $a_{51} = 6.52$

\(^5\)We assume the value of every node in $\hat{A}$ is equal to the value of every node in $\hat{B}$, i.e. $a^i = b^i$ for $i = 1, 2, \cdots, 51$. 

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\[(36) \quad V_{T+1}(a_{T+1}) = \sum_{t=T+1}^{T_{\text{max}}} \beta^{t-(T+1)} \prod_{i=T+1}^{t-1} \phi_i u(\bar{c}).\]

After we get the terminal values, we can maximize the insured’s utility backward from \(T^+\) to \(1^-\).

\textbf{a. Transition from \((t + 1)^-\) to \(t^+\)}

As indicated before, the insured decides the allocation between the fixed and variable subaccounts in this time interval. We apply the trinomial tree to solve for the optimal allocation. Given GMDB level \(b_i\), let

\[(37) \quad a_{t+1^{-}} = a_{t^+} \times \begin{pmatrix} 1/u & 1 & u \end{pmatrix}\]

for all nodes on \(a_{t^+}\). Since \(a_{t+1^{-}}\) might not be necessary on those 51 nodes, we use cubic spline interpolation to get the values of \(V_{t+1^{-}}(a_{j+1}, b_i)\), \(V_{t+1^{-}}(a_j, b_i)\), and \(V_{t+1^{-}}(a_{j-1}, b_i)\).

The probabilities \(p_u, p_d,\) and \(p_m\) have been derived in equations (22), (23) and (24). Then at any given GMDB level \(b_i\), we derive

\[(38) \quad V_{t^+}(a_j, b_i) = \beta \phi (p_u V_{t+1^{-}}(a_{j+1}, b_i) + p_m V_{t+1^{-}}(a_j, b_i) + p_d V_{t+1^{-}}(a_{j-1}, b_i))\]

for \(i = 1, 2, \cdots, 51\) and \(j = 1, 2, \cdots, 51\), where \(V_{t^+}(\cdot, \cdot)\) is a \(51 \times 51\) matrix at time \(t^+\).

\textbf{b. Transition from \(t^+\) to \(t^-\)}

The insured needs to determine the optimal withdrawal \(d_t\). At any given GMDB level \(b_i\):

1. initialize the withdrawal amount \(d_{t,k} = a_j - a_{j-k}\) for all \(k < j\);

2. for all \(k\)'s, we derive

\[(39) \quad V_{t^-}^k(a_j, b_i) = u_t(d_{t,k} + y_t) + (1 - \phi_t) \zeta \psi_t \max(b_{i-k}, a_{t-k}))^{1-\gamma} + V_{t^+}(a_{j-k}, b_{i-k}).\]

From the above equation, we see that as the policyholder withdraws money from the account,
the GMDB level will also reduce proportionally. Numerically, we reduce the $V_{t^-}$ diagonally;

3. let $V_{t^-}(a_{j-k^*}, b_{i-k^*}) = \max(V_{t^-}^1(b_{i-1}, a_{i-1}), \cdots, V_{t^-}(b_{i-k}, a_{i-k}))$, then we can locate the position of the maximum $V_{t^-}$ in the $51 \times 51$ space. The maximum $V_{t^-}$ is not necessary on the matrix nodes, but it must be between the diagonal points $V_{t^-}(a_{j-k^*+1}, b_{i-k^*+1})$ and $V_{t^-}(a_{j-k^*+1}, b_{i-k^*+1})$;

4. choose optimal withdrawal $d_t$ to maximize $V_{t^-}$ by using quadratic interpolation,

$$
\begin{pmatrix}
 a_{j-k^*+1} & a_{j-k^*} & 1 \\
 a_{j-k^*+1} & a_{j-k^*} & 1 \\
 a_{j-k^*+1} & a_{j-k^*} & 1 \\
\end{pmatrix}
\begin{pmatrix}
 \alpha_1 \\
 \alpha_2 \\
 \alpha_3 \\
\end{pmatrix}
= 
\begin{pmatrix}
 V_{t^-}(a_{j-k^*+1}) \\
 V_{t^-}(a_{j-k^*}) \\
 V_{t^-}(a_{j-k^*+1}) \\
\end{pmatrix}
$$

then we can derive the value of the parameters $\alpha_1$, $\alpha_2$ and $\alpha_3$ which are used to estimate $V_{t^-}$,

$$
\begin{pmatrix}
 \alpha_1 \\
 \alpha_2 \\
 \alpha_3 \\
\end{pmatrix}
= 
\begin{pmatrix}
 a_{j-k^*+1} & a_{j-k^*} & 1 \\
 a_{j-k^*+1} & a_{j-k^*} & 1 \\
 a_{j-k^*+1} & a_{j-k^*} & 1 \\
\end{pmatrix}^{-1}
\begin{pmatrix}
 V_{t^-}(a_{j-k^*+1}) \\
 V_{t^-}(a_{j-k^*}) \\
 V_{t^-}(a_{j-k^*+1}) \\
\end{pmatrix}
$$

By using the calculated values of $\alpha_1$, $\alpha_2$ and $\alpha_3$, we can solve the optimal withdrawal amount $d_t$ by maximizing the following equation.

$$
\max_{d_t} \{u_t(d_t + y_t) + \zeta(1 - \phi_t)\psi_t(b_t \frac{a_j - d_t}{a_j})^{1-\gamma} + \alpha_1(a_j - d_t)^2 + \alpha_2(a_j - d_t) + \alpha_3 \}
$$

s.t. $0 \leq d_t \leq a_j$.

We repeat the “transition from $(t+1)^-$ to $t^+$” and the “transition from $t^+$ to $t^-$” until $t = 1^-$. Therefore, we can get optimal asset allocation $\omega_t$ and optimal withdrawal amount $d_t$.

In the remaining part of this section, we use the following parameter values as a base
case to do the sensitivity tests.

Table 10: Common Parameters in the Base case

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Strength of Bequest Motive</td>
<td>ζ</td>
<td>0.5</td>
</tr>
<tr>
<td>Subjective Discount Rate</td>
<td>β</td>
<td>0.97</td>
</tr>
<tr>
<td>Risk Free Rate</td>
<td>r_f</td>
<td>3%</td>
</tr>
<tr>
<td>Coefficient of Relative Risk Aversion</td>
<td>γ</td>
<td>2</td>
</tr>
<tr>
<td>Growth Rate of Fixed Subaccount</td>
<td>r_g</td>
<td>4%</td>
</tr>
<tr>
<td>Expected Return of Risky Asset</td>
<td>r</td>
<td>7%</td>
</tr>
<tr>
<td>Volatility of Risky Return</td>
<td>σ</td>
<td>15%</td>
</tr>
<tr>
<td>GMDB roll-up rate</td>
<td>r_p</td>
<td>0</td>
</tr>
<tr>
<td>Annual Survival Rate</td>
<td>φ</td>
<td>0.99</td>
</tr>
<tr>
<td>Annual Mortality Rate</td>
<td>μ</td>
<td>0.01</td>
</tr>
<tr>
<td>Monthly Income</td>
<td>y</td>
<td>0.01</td>
</tr>
</tbody>
</table>

3.2.1 Risk Aversion Sensitivity

The purpose of this test was to discover the optimal choices for the policyholder if all but one coefficient, relative risk aversion (γ), were kept constant. Figure 28 shows that as the risk aversion increases, the money in the variable subaccount will decrease. This is just as in the “no labor income” case, because when \( γ < 1 \), (i.e. the insured is not very risk averse) and the policyholder will put all of his money into the variable subaccount between the ages of 35 and 65; when \( γ > 1 \), the proportion in the variable subaccount conversely decreases as the insured ages. Figure 7 on risk aversion sensitivity partially explains the reason for this relationship: as the insured cares more for himself he is less likely to make risky investments. Furthermore, the probability that his beneficiary will get the Guaranteed Minimum Death Benefit’s protection also decreases. It was also found that a person’s risk aversion level is proportional to their investment amount, this is to say that the higher the risk aversion level means that less will be invested into the variable subaccount. In comparison to the “no labor income” case, the allocation into the variable subaccount is much larger because of the “human capital” effect. The logic is as follows: the traditional professional advice on asset allocation for individuals is that the appropriate strategic asset allocation varies with age or
According to theories, asset allocation crucially depends on whether labor income and human capital are taken into consideration. If human capital is ignored then individuals should optimally maintain constant portfolio weights throughout their lives given certain assumptions, including the investor’s risk aversion (Merton 1969, Charupat and Milevsky 2002). When we take labor income into account, individuals appear to optimally change their asset allocations in ways related to their life cycles and characteristics of their incomes. In this section, fixed labor income (“safe” human capital) is assumed, and due to this, the experiment’s policyholder, who has a “safe” labor income, will prefer investing more financial capital into equity as he expects a large present value for future labor income, which is comparable to riskless investment in bonds (Campbell and Viceira 2002). Therefore, the policyholder’s preference to take more risk in his VA account (as opposed to what was previously observed in the “no labor income” case) is a condition of whether or not he has an outside income. As the insured ages, his human capital, the ability to secure that outside income, declines, leaving him more dependent than ever on financial assets in the VA account. It follows that because the insured will be more dependent on the VA account, over time, he will choose to make less riskier investments so that his income is not disrupted in retirement. Therefore, the decrease of risky asset allocation as the insured ages should be explained by both human capital and GMDB effects.

The risky account allocation at the policyholder’s retirement age produce a convergence at Merton’s allocation point, which explains why, in comparing figures 28 and 7, the cases of risky asset allocations made “with income” and “without income” are the same at retirement age given any risk aversion levels.
Figure 28: At the Money Allocation with different $\gamma$ when $y = 0.01$

Figure 29: Age 45 allocation with different $\gamma$ when $y = 0.01$
Figure 29 shows, at age 45, the allocation of funds given different levels of risk aversion. For example, when $\gamma = 0.5$, the policyholder is not being very risk averse and therefore gambles with his wealth by investing it into the variable subaccount. When $\gamma > 1$, the policyholder is shown to have taken an aggressive allocation strategy with the exception of when the asset level is very low or high. The hump shape in the middle is created by the joint effects of human capital and GMDB. When $\gamma = 3$, two humps appear. The first hump signifies an "argument" between the policyholder and his beneficiary over the policy’s Guaranteed Minimal Death Benefit (for obvious reasons), and appears near the “at-the-money” area. Upon resolution of the “argument” the area of the first hump will be smoothed and only the second hump shall remain, which is due to the human capital effect.

According to the CRRA utility function property, the optimal asset allocation will not change according to the amount of wealth in the VA account ($\zeta = 0$ in “no labor income” case as in figure 10). However, the wealth referred to in this particular experiment includes both financial wealth, in the VA account, and human capital. In such a case as this, human capital is less risky to rely upon than investing into the variable subaccount. When asset level
is low, human capital dominates total wealth and asset allocation, because it is conditionally more reliable. An increase in asset level in the VA account not only increases total wealth but also reduces the percentage of total wealth represented by human capital. As a result, it should be expected that a risk-averse policyholder should put most of his account value into a risky account because the asset level is low and therefore of little consequence. Meanwhile, optimal asset allocation in the VA account becomes more and more conservative as the asset level increases.

Figure 29, is rather counter-intuitive. The allocation is low when the asset level is low (it cannot be ignored if the asset level quickly becomes very low but then goes up abruptly due to low risk asset allocation once more) and risky investment proportionally increases quickly to a very high point (for $\gamma = 1.5$ or $2$, $\omega$ goes to 100%; for $\gamma = 3$, $\omega$ goes to around 85%) before gradually declining even as asset level keeps increasing. The reason for this counter-intuitive shifting is that in our model we assume that the policyholder will consume all the monthly income during the current period. The “rational” policyholder “expects” that he is “over consuming” at the low asset level and has had no chance to transfer his human capital to financial capital in anticipation of his retirement. Therefore, he will act conservatively because the asset level is low and he wants to save some money for retirement. If the account value is too low, the insured will also take risk and increase the risky allocation, because there is very little to lose, but perhaps a lot to gain. The “over consumption” problem can be fixed if we assume alternative investment vehicles to save income (transferring human capital to financial capital), which will be an extension to this model in future studies. Therefore, the bump level in the figure can be regarded as the joint effect of the GMDB “argument” from earlier and human capital.

Figure 30 shows, at age 45, the optimal withdrawal choices given different levels of risk aversion. As the risk aversion increases, the policyholder will consume more because he will be more concerned about the investment risk.

This figure also fixes the counter-intuitive outcome in the “no income” case in the previous section (Figure 9). In the “no income” case, as $\gamma > 1$, the consumption ratio increases as VA
asset level decreases (GMDB goes deeper in the money). As outside income is introduced in this section, low asset levels, where GMDB is in-the-money, have low consumption ratios while high consumption ratios occur at high asset level areas, where GMDB is out-of-the-money.

3.2.2 Bequest Motive Sensitivity

In this part, a bequest motive sensitivity test was done. The purpose of this test was to discover the optimal choices for the policyholder if all but one coefficient, bequest motive factor ($\zeta$), were kept constant.

Figure 31 shows at-the-money asset allocation for different bequest motive levels. Just like in the “no labor income” case, as the bequest motive $\zeta$ increases, the proportion of the risky investment will increase. At any levels of bequest motive, the allocation to the variable subaccount decreases as age increases and finally converges at the Merton level at age 65. In the “no labor income” case, as $\zeta = 0$ (i.e. no bequest motive, no GMDB effects), the insured only cares about himself and the asset allocation level agrees with the Merton result (keep at a constant rate) between the ages of 35 and 65. Once outside income is introduced into the model, the Merton rule is broken. Even without bequest motive ($\zeta = 0$, which implies no GMDB effect), the insured takes a very aggressive investment strategy at younger age. The reason is that when the insured is young, he expects to have a long accumulation stage in front of him, and the present value of his expected future earnings is often substantial, which can be regarded as his risk free bond investment at a large amount. As a result, the human capital effect pushes him to choose the aggressive allocation to the variable subaccount.

For $\zeta > 0$, the aggressive allocation can be regarded as the joint effects from both “human capital” and GMDB “argument”. As the $\zeta = 0$ allocation starts to decrease from 100% risky allocation, $\zeta > 0$ risky allocations still keep at 100% because of the GMDB effect. Therefore, the higher the bequest motive is, the longer the risky allocation strategy applies.
Figure 31: At the Money Allocation with varied $\zeta$ when $y = 0.01$

Figure 32: At the Money Withdrawal with varied $\zeta$ when $y = 0.01$
Figure 33: Age 45 withdrawal proportion with varied $\zeta$ when $y = 0.01$

Figure 32 shows at-the-money proportion of withdrawal. A higher level of bequest motive decreases the withdrawal ratio. As $\zeta \geq 0.5$, i.e. the policyholder has a very strong bequest motive, he is willing to reduce consumption and save more money for his beneficiary, and therefore the withdrawal $d_t$ decreases to zero. As the policyholder ages, he will consider not only the beneficiary but also himself, and he will put more and more attention to himself. Therefore he will prefer consuming less and saving more for himself, and the withdrawal ratio will go to zero and the labor income will become the only source of consumption as he grows older.

Figure 33 plots how at age 45, given different levels of bequest motive, the withdrawal ratio will be. A lower level of bequest motive incurs the withdrawal choice at a lower asset level; while a higher level of bequest motive encourages the policyholder to save more for his beneficiary. As the asset level is low, especially when the GMDB is in-the-money, the insured will hurt his beneficiary far more than the benefit he gets. The insured with a higher
bequest motive starts to withdraw the wealth from the VA account at a higher asset level. The difference of withdrawal ratios between $\zeta = 0$ and $\zeta > 0$ is caused by the GMDB effect.

### 3.2.3 Volatility Sensitivity

The purpose of the volatility sensitivity test was to discover the optimal choices for the policyholder if all but one coefficient, equity market risk, were kept constant. Figure 34 shows that the insured will decrease the risky investment and put more money into the riskless account when the equity market volatility is high. At any volatility levels, as the insured ages, the risky asset allocation will be reduced due to less human capital expectation and weakening of the GMDB effect. At the retirement age, the risky asset allocation converges at the Merton’s point.

![Figure 34: At the Money Allocation with different Volatilities when $y = 0.01$](image)

Figure 34: At the Money Allocation with different Volatilities when $y = 0.01$
Figure 35 and 36 show, at age 45, the policyholder’s response in the allocation and withdrawal choices given different equity market risks. If the expected risk on the market
increases (annual volatility changes from $\sigma = 15\%$ to $25\%$), the insured fears for the investment risks, and he would like to decrease risky account allocation to keep the asset safe. At the same time he would like to increase withdrawal ratio to consume more at the current period. As volatility becomes larger, one can also observe the spike levels are lower, which implies the risk averse beneficiary does not have much incentive to “argue” with the insured, and the human capital effect also becomes smaller.

3.2.4 Rate of Return Sensitivity

The purpose of the rate of return sensitivity test was to discover the optimal choices for the policyholder if only one coefficient, rate of return on risky asset, was changed. Figure 37 shows at-the-money asset allocation with different risky rate of return. When the risky rate of return is low, the policyholder will have no incentive to do risky investments. Therefore, the percentage of wealth held in the variable subaccount will be low. Given any levels of risky rate of return, the allocation to risky account decreases as the insured ages from 35 to 65.

If one study the insured’s allocation choice facing different risky rate of return at age 45 (figure 38), one can find that the insured would like to transfer more assets to the variable subaccount with higher expectation on equity market return. From this figure, one can also observe two hump shapes incurred by the joint effects on allocation choices, i.e. the GMDB effect (the hump around at-the-money area) and human capital effects (the hump at higher asset level), which are consistent with the results derived in the risk aversion sensitivity test example.

With a higher risky rate of return, the insured will also have more money to withdraw from the account and consume to increase his living standards (Figure 39).
Figure 37: At the Money Allocation with different $r$ when $y = 0.01$

Figure 38: Age 45 allocation with different $r$ when $y = 0.01$
3.2.5 Unemployment Risk

Now the assumption of unemployment risk is added to our model. The policyholder is assumed to face unemployment risk. Therefore, there are two states of labor income: state 1. \( y = \tilde{y} \), the policyholder is employed and receives a positive compensation \( \tilde{y} > 0 \); state 2. \( y = 0 \), the policyholder is unemployed and receives no income.

We assume the Markov transition probability matrix for wage shock is as follows

\[
\Pi(y_{t+1}|y_t) = \begin{pmatrix}
0.95 & 0.05 \\
0.3 & 0.7 \\
\end{pmatrix}
\]

If the policyholder is employed at time \( t \), he has a 95% chance to keep employed and a 5% chance to get unemployed at time \( t + 1 \); if the policyholder is unemployed at time \( t \), his probability to find a job at \( t + 1 \) is 30% and his probability of remaining unemployed at \( t + 1 \) is 70%. The steady state probability of employment is \( \pi_1 = \frac{6}{7} \) and the steady state
The probability of unemployment is \( \pi_2 = \frac{1}{7} \).

Figure 40 (age 45 allocation with \( y = 0.01 \) and \( y = 0.02 \)) details the optimal allocation choices for the insured given an earnings risk. Since the insured is rational, he will take the unemployment risk into account when he allocates the wealth to maximize his utility. Acknowledgment of the unemployment risk by the insured ensures that he has even less expectation on his human capital than if he were in a “fixed income” situation. Therefore, there is a smaller human capital effect on the insured’s allocation of funds with the acknowledgment. Corresponding to the smaller effect is that allocation to the risky account is less than it would be in a “fixed income” situation.

A further observation is that there is not a great difference between allocations made whether employed or unemployed for either \( y = 0.01 \) or \( y = 0.02 \). Given the Markov transition matrix above, the unemployed policyholder is expected to wait only 3 months (1/0.3) for a new job. Since three months is a negligible amount of time overall, a rational policyholder will not reduce the risky allocation by much for this reason. Consistent with the previous analyses, at any income level two hump shapes are seen: the hump shape around the “at-the-money” area is caused by the GMDB “arguments” between the insured and his beneficiary, and the other hump shape is from human capital effects.

In figure 40, as the stock to strike level is low (as \( a/b < 3 \)), the allocation by the policyholder with low income (\( y = 0.01 \)) is greater than the allocation would be if his income were higher (\( y = 0.02 \)). The occurrence is counter-intuitive, because the insured with higher income means he has higher human capital and he needs to put more money in the variable subaccount. The counter-intuitive result is due perhaps to model limitations, because the model assumes that the policyholder consumes all periodic income in the current period. This induces an “over consumption” problem and distorts the results when the asset level is low. The distortion caused by the “over consumption”, or over spending, is redressed once the policyholder withdraws funds in order to cover the additional living costs.

Figures 43 (Age 45 withdrawal proportion when \( y = 0.01 \)) and 44 (Age 45 withdrawal proportion when \( y = 0.02 \)) below shows the effect the policyholder has once he begins the
withdrawal of funds from the VA account.

Figure 40: Age 45 allocation with earning risk when $y = 0.01 \& 0.02$

Figure 41: At the Money withdrawal proportion with earning risk when $y = 0.01$
Figure 42: At the Money withdrawal proportion with earning risk when $y = 0.02$

Figure 43: Age 45 withdrawal proportion with earning risk when $y = 0.01$
Figure 41 and 42 show at-the-money pre-retirement (from age 35 to 65) withdrawal proportion with $y = 0.01$ and $y = 0.02$. The insured in the employed state will not withdraw at all, because he can live well by his outside income; the unemployed insured has to withdraw money from the VA account to support his living, but he withdraws at a decreasing ratio as he ages because he has to lower his living standards and consume less to save more for his retirement costs.

When the insured is unemployed, he has to withdraw from the VA account to support his life. From figure 43 and figure 44, age 45 withdrawal proportion with unemployment risk, one can observe that the more money the unemployed policyholder has, the smaller ratio he withdraws. It can be explained from several aspects. First, “diminishing marginal utility of consumption” is at work here. As the asset level is low, the insured has to withdraw a large proportion to cover the living expense. As the asset level increases, the absolute value of consumption is still increasing, but the consumption ratio is decreasing. Also because of the concavity of the utility function, the insured’s satisfaction does not increase as much as his consumption, therefore the consumption ratio will not proportionally increase as asset
level increases. Second, because of the GMDB proportional reduction property, the insured’s withdrawal will hurt his beneficiary more than his gain from consumption. Therefore, any insured with bequest motives will withdraw and spend less as the asset level increases. Finally, the policyholder is still looking for a new job and there is no outside income source at the current period. As a result, he wants to spend less to accumulate more financial assets for his retirement.

It is also helpful to draw a comparison between unemployment cases in figure 43 and figure 44 with the “no labor income” case in the previous section (Figure 9 or Figure 13) because it shows how low asset levels correspond to high amounts of asset withdrawal from the VA account. In the “no income” case, as \( \gamma > 1 \) (in base case of that section \( \gamma = 2 \)), the withdrawal proportion increases as GMDB asset levels decrease, which is the same situation in this model. It is expected that the unemployed person with no income should lower their living standards accordingly, but they still need to buy necessities. As the asset level goes down, the cost of necessities goes down much more slowly and then stops below this point, the policyholder cannot reduce consumption any more. As a result, the ratio of withdrawal to asset level is increasing as asset level goes down. The difference between “no income” case and “unemployment” case here is that as asset level goes up, in the former case the withdrawal ratio keeps at a constant level while the withdrawal ratio goes down in the latter case. The real withdrawal ratio number is different: the withdrawal ratio is around 0.004 in the “no income” case; while with the “unemployment” case the withdrawal ratios are much higher (for \( y = 0.01 \), it ranges from 0.0047 to 0.007, and for \( y = 0.02 \), it ranges from 0.0053 to 0.0072). The reason is that in the unemployment case, the policyholder expects to get employed and receive labor income after several months, therefore he might spend more.

When the insured is employed (Figure 43, 44), he will not withdraw from the VA account as the asset level is low, because the periodic income is enough to cover his living costs. The insured will start to withdraw as the asset level increases to a certain level. As people get richer, their living standards rise as well. When the labor income cannot support their higher living standards, they will start to withdraw from their savings. One can observe that the
employed policyholder starts to withdraw when the amount in VA account is around 140 times of his monthly income. Also the consumption ratios for the steady state are derived, which reflect the long run trend of the policyholder’s behavior.

Let us assume the above “unemployment risk” case is the same as the base case and do the sensitivity tests for the “unemployment risk” case. First, let us assume that the employed insured is not likely to get unemployed, so that the new Markov transition probability matrix for wage shock is as follows

$$\Pi(y_{t+1}|y_t) = \begin{pmatrix} 0.995 & 0.005 \\ 0.3 & 0.7 \end{pmatrix}$$

If the policyholder is employed at time $t$, he has a 99.5% chance to stay employed and a 0.5% chance to become unemployed at time $t+1$; if the policyholder is unemployed at time $t$, his probability of finding a job at $t+1$ is 30% and his probability of remaining unemployed at $t+1$ is 70%. The steady state probability of employment is $\pi_1 \approx 98.36\%$ and the steady state probability of unemployment is $\pi_2 \approx 1.64\%$.

In comparison to figure 40, figure 45 shows that the policyholder makes more aggressive allocations in both states. According to the new Markov transition probability above, once the insured becomes employed, the unemployed risk is very close to zero. The employed insured believes his job position is safe, and will regard his human capital as a much safer investment than in the base case. The unemployed insured expects that after 3 months (the same mean waiting time as in the base case), he will get a job and will have very small probability of becoming unemployed again. Therefore, the unemployed insured will also make a more aggressive allocation than the base case.

Figures 46 and 47 shows the policyholder’s withdrawal proportion at age 45 when $y = 0.01$ and $y = 0.02$ respectively. In both figures, one can observe that the policyholder withdraws more in both states than in the base case. When the policyholder is employed, he will expect a stable income source and be willing to consume. When the policyholder is unemployed, he expects to get a job in several months and then will have very small probability of losing
his job. Therefore he will also increase the withdrawal ratio.

Figure 45: Age 45 allocation with earning risk when \( y = 0.01 \& 0.02 \) in a good job market

Figure 46: Age 45 withdrawal ratio with earning risk when \( y = 0.01 \) in a good job market
If the job market is bad, the probability for an unemployed individual to find a job in the next time period is only 10%. The Markov transition probability matrix for wage shock has been changed to

\[
\Pi(y_{t+1}|y_t) = \begin{pmatrix}
0.95 & 0.05 \\
0.1 & 0.9
\end{pmatrix}
\]

The steady state probability of employment is now \( \pi_1 = \frac{2}{3} \) and the steady state probability of unemployment is \( \pi_2 = \frac{1}{3} \).

In comparison to the base case, figure 48 shows that the policyholder makes more conservative allocation in both states. Once the insured becomes unemployed, there is a very small probability of finding a new job, and he expects to use 10 months looking for a new job. The employed insured does not feel safe, because his chance of keeping his job does not increase, but once he loses his job, it will be hard to find a new one. Therefore his human capital is a riskier investment than in the base case. The insured will make a more conservative
allocation in both states to save more for his retirement.

Figure 48: Age 45 allocation with earning risk when $y = 0.01&0.02$ in a bad job market

Figure 49: Age 45 withdrawal ratio with earning risk when $y = 0.01$ in a bad job market
Figures 49 and 50 shows the policyholder’s withdrawal proportion at age 45 when \( y = 0.01 \) and \( y = 0.02 \) respectively. In both figures, one can observe that the policyholder withdraws less in both states than in the base case. When the policyholder is employed, he will be more concerned about unemployment, because the job market is bad and it is hard to get reemployed. This will result in a smaller withdrawal ratio. When the policyholder is unemployed, he expects to wait more periods to get a new job than in the base case. Therefore he will also decrease the withdrawal ratio.

If the job market is weak, which means that the probability for an unemployed individual to find a job in the next time period is small, i.e. 10%, while it is hard for an employed individual to lose his job, i.e. 0.005%. Then the Markov transition probability matrix for wage shock will become

\[
\Pi(y_{t+1}|y_t) = \begin{pmatrix}
0.995 & 0.005 \\
0.1 & 0.9
\end{pmatrix}
\]
The steady state probability of employment is \( \pi_1 \approx 0.9524 \) and the steady state probability of unemployment is \( \pi_2 \approx 0.0476 \).

In comparison to the base case, figure 51 shows that the risky asset allocations are greater than in the base case. If the policyholder is employed, he will find his job position is safer than in the base case and will prefer a riskier allocation. If the policyholder is unemployed, he will expect to take a longer time looking for a job, but once he finds one, he will less likely to be fired. He may also take a more aggressive allocation than when he is unemployed in the base case. Moreover, one can observe that the difference between the allocation choices in the employed and unemployed states are greater than in the base case, especially near the second hump (the human capital hump). The larger difference is caused by the difficulty in finding a new job and the stability in the current job position.
Figures 52 and 53 shows the policyholder’s withdrawal proportion at age 45 when $y = 0.01$ and $y = 0.02$ respectively. In both figures, one can observe that the policyholder withdraws
more in both states than in the base case. Once the policyholder is placed in a job, he will have safe long term safe human capital, and this will increase his withdrawal ratio as in the “fixed income” case. Also, an unemployed policyholder requires a longer period to get a job, but once he gets it, he will lock in “safe” human capital.

4 Optimal Allocation and Consumption with Guaranteed Minimum Death Benefits with Labor Income and Term Life Insurance

In this section, the model of the previous sections is extended by incorporating optimal term life insurance demand. Because human capital is often the largest asset an investor possesses when he is young, protecting human capital from potential risks also should be considered as a part of overall investment advice. The risk of the loss of the policyholder’s human capital – the mortality risk – to the household can be partially hedged by a term life insurance policy. Since a GMDB can also help policyholders hedge the risk of the loss of human capital, GMDB options and term life insurance can be considered as substitute goods. However, they are not perfect substitute goods. GMDB and term life have their own properties: Term life insurance has no correlation with equity markets, and it is purely regarded as a protection for human capital; the variable annuity products follow the performance of equity markets, and the GMDB is a protection against downside risks on equity markets. In the following section, we are trying to check if the GMDB options add value to the VA contract even if a term life policy is available.

4.1 Models

Let us first take a look at “without GMDB” case: an individual purchases a variable annuity contract without GMDB option and makes a lump sum deposit to the variable annuity account. Once the insured receives labor income at the beginning of period $t$, he will
make his consumption decision. If labor income is not enough to support his consumption, he will make a decision to withdraw from the VA account. At the same time, he will also decide whether he needs a term life policy to help his beneficiary in case of his own premature death. After consumption, withdrawal and term life purchase decisions, still at time $t$, the policyholder will decide the optimal allocation between the fixed and variable subaccounts in the VA account. If the policyholder dies at time $t$, the amount in the VA account $a_t$, will be inherited by his beneficiary. In addition to the bequest from the policyholder’s VA account, the beneficiary also gets the term life policy payment $F_t^c$. If the insured survives until his retirement age, at the end of the policy period, he will get the entire account value and annuitize it for his retirement life.

The insured’s objective function in the “without GMDB” case can be written as

\[
\max_{\omega_t, d_t, P_t} E \left[ \sum_{t=1}^{T} \beta^t (\prod_{i=1}^{t} \phi_i) u(c_t) + \beta^T (\prod_{i=1}^{T} \phi_i) V_{T+1}(a_{T+1}) + \sum_{t=1}^{T} \beta^t (\prod_{i=1}^{t-1} \phi_i) (1 - \phi_t) \zeta_v(a_t + F_t^c) \right],
\]

where $P$ is the premium of the term-life policy; $F^c$ is the face amount of the policy; $d_t$ is the amount of the withdrawal from the policy. $v_B$ is the beneficiary’s value function, and it depends on the policyholder’s bequest motive $\zeta$ which measures the importance of the beneficiary’s benefits to the policyholder. Once the beneficiary gets the bequest, she will maximize her own utility by optimal allocations and withdrawals. The beneficiary’s objective function can be written as

\[
\max_{\omega^B, c^B_t} E \left[ \sum_{t_B=t}^{T_B} \beta^{t_B-t} (\prod_{i=t}^{t_B-1} \phi_i) u(c^B_t) + \beta^{T_B-t} (\prod_{i=t}^{T_B} \phi_i) v_{T_B+1}(a^{B}_{T_B+1}) \right].
\]

In this case, if the insured dies at time $t$, the bequest amount received by the beneficiary is $a^B_t = a_t + F^c_t$. When the insured deposits in the VA account, the beneficiary has $T_B$ years until her retirement age. When the insured dies, the beneficiary will receive the bequest and has $T_B - t$ years until retirement age and will transform the money into a lifetime payout.
annuity after she retires. In these $T_B - t$ years, she will optimally withdraw to consume and will optimally allocate the amount between risky and risk-free investments. She will get the terminal value $a_{T_B+1}^B$ at the beginning of $T_B + 1$st year. However, we do not assume a bequest motive for the beneficiary.

From the insured’s objective function, we can derive his Bellman equation as follows

$$V_t(a_t) = \max_{\omega_t, d_t, P_t} \left\{ u_t(c_t) + (1 - \phi_t)\zeta v(a_t + F_t^c) + \beta \phi_t E[V_{t+1}(a_{t+1}) | a_t, r_t] \right\}$$

subject to

$$c_t \equiv y_t + d_t, \quad 0 \leq d_t \leq a_t,$$

$$a_{t+1} = (\omega_t(1 + r_{t+1}) + (1 - \omega_t)(1 + g_t))(a_t - d_t), \quad 0 \leq \omega_t \leq 1,$$

$$a_{T+1} = (\omega_T(1 + r_{T+1}) + (1 - \omega_T)(1 + g_T))(a_T - d_T),$$

$$\bar{c} = \frac{a_{T+1}}{\sum_{t=T+1}^{T_{\max}} \prod_{i=T+1}^{t-1} \phi_i (1 + r_f)^{T+1-t}},$$

$$V_{T+1}(a_{T+1}) = \sum_{t=T+1}^{T_{\max}} \beta^{t-(T+1)}(\prod_{i=T+1}^{t-1} \phi_i)u(\bar{c}),$$

$$F_t^c = \frac{P_t}{(1 - \phi_t)(1 + \eta)}, \quad 0 \leq P_t < d_t + y_t,$$

$\eta$ is the loading element, which refers to the amount that must be added to the pure premium to cover other expenses, profit, and a margin for contingencies. The other notation is the same as the previous section. $\bar{c}$ is the periodic consumption after retirement, and we assume it is the level periodic payment from the variable annuity account. $\bar{c}$ can be derived from the terminal account value as follows,

$$a_{T+1} = (\omega_T(1 + r_{T+1}) + (1 - \omega_T)(1 + g_T))(a_T - d_T)$$

$$= \bar{c} \sum_{t=T+1}^{T_{\max}} \prod_{i=T+1}^{t-1} \phi_i (1 + r_f)^{T+1-t},$$

$$\bar{c} = \frac{a_{T+1}}{\sum_{t=T+1}^{T_{\max}} \prod_{i=T+1}^{t-1} \phi_i (1 + r_f)^{T+1-t}}.$$
Then we consider the “with GMDB” case: an individual purchases a variable annuity contract with GMDB options and makes a lump sum deposit to the variable annuity account. Once the insured receives labor income at the beginning of period $t$, he will make the consumption decision. If labor income is not enough to support his consumption, he will make a decision to withdraw from the VA account. Simultaneously, the GMDB level will be reduced proportionally with the withdrawal ratio. At the same time, he will decide if he needs a term life policy to help his beneficiary after his own premature death. After the consumption, withdrawal and term life purchase decision, still at time $t$, the policyholder will decide the optimal allocation choice between the fixed and variable subaccounts in the VA account. If the policyholder dies at time $t$, the amount in the VA account, which is protected by the GMDB ($b_t$), will be inherited by his beneficiary. In addition to the bequest from the policyholder’s VA account, the beneficiary also gets the term life policy payment ($F^c_t$).

If the insured has labor income and is holding a term-life insurance policy, the objective function is

$$\max_{\omega_t, d_t, P_t} E \left[ \sum_{t=1}^{T} \beta^t (\prod_{i=1}^{t} \phi_i) u(c_t) + \beta^T (\prod_{i=1}^{T} \phi_i) V_{T+1}(a_{T+1}) + \sum_{t=1}^{T} \beta^t (\prod_{i=1}^{t-1} \phi_i) (1-\phi_t) v_B(b_t+F^c_t) \right],$$

where $v_B$ is the beneficiary’s value function. The beneficiary gets the bequest ($F^c_t + b_t$) and maximizes her own utility by optimal allocations and withdrawals. The objective function for the beneficiary is as follows,

$$\max_{\omega_B^t, c_B^t} E \left[ \sum_{t_B=t}^{T_B} \beta^{t_B-t} (\prod_{i=t}^{t_B-1} \phi_i) u(c_B^t) + \beta^{T_B-t} (\prod_{i=t}^{T_B} \phi_i) V_{T_B+1}(b_{T_B+1}) \right].$$

If the insured dies at time $t$, the bequest amount received by the beneficiary is $b_B^t = b_t + F^c_t$. She will get the terminal value $b_{T_B+1}$ at the beginning of $T_B + 1$st year.
From the insured’s objective function, we can derive the insured’s Bellman equation

\begin{equation}
V_t(a_t, b_t) = \max_{\omega_t, d_t, P_t} \left\{ u_t(c_t) + (1 - \phi_t)\zeta v_B(b_t + F^c_t) + \beta \phi_t E[V_{t+1}(a_{t+1}, b_{t+1}) \mid a_t, r_t] \right\}
\end{equation}

subject to

\begin{align*}
a_1 &= b_1, \\
c_t &\equiv y_t + d_t, \quad 0 \leq d_t \leq a_t, \\
a_{t+1} &= (\omega_t(1 + r_{t+1}) + (1 - \omega_t)(1 + g_t))(a_t - d_t), \quad 0 \leq \omega_t \leq 1, \\
k_{t+1} &= k_t(1 + r_f)\frac{a_{t+1} - d_{t+1}}{a_{t+1}}, \\
b_{t+1} &= \max(k_{t+1}, a_{t+1}), \\
F^c_t &= \frac{P_t}{(1 - \phi_t)(1 + \eta)}, \quad 0 \leq P_t < d_t + y_t, \\
V_{T+1}(a_{T+1}) &= \sum_{t=T+1}^{T_{max}} \beta^{t-(T+1)} \left( \prod_{i=T+1}^{t-1} \phi_i \right) u(\bar{c}).
\end{align*}

\(\bar{c}\) is the insured’s periodic consumption after retirement, and we assume it is the level periodic payment from the variable annuity account. \(\bar{c}\) can be derived from the terminal account value as in the “no GMDB” case.

We can see that if the policyholder buys a VA product with GMDB options and term life policy, the beneficiary is protected against the loss of the insured’s human capital and investment risk simultaneously. Any policyholder with bequest motive will be better off purchasing both. Because these two products are partial substitutes for each other, the policyholder needs to consider the costs and benefits of having both products.

As before, we denote state variables as \((\cdot)_{t-}, (\cdot)_{t+}\) to solve the maximization problem, i.e. the value immediately before and after the transactions at the discrete time \(t\), respectively. We assume the policyholder is employed and receives labor income \(y_t\) at \(t^-\). Withdrawal, consumption and term life purchase also occur at \(t^-\). Then the insured decides the amount to transfer between the fixed and the variable subaccounts at \(t^+\), which is still at time \(t\)
but after the receipt of income, withdrawal, consumption and term life purchase. We also assume that the beneficiary gets the bequest immediately at \( t^+ \) just after the insured dies at \( t^+ \). Therefore, the insured’s Bellman equation can be expressed by two-stage Bellman equations. At the 1st stage from \( t^- \) to \( t^+ \), the insured gets the utility from consumption which is equal to the sum of optimal withdrawal from the variable annuity account and labor income minus the term life premium. Since there is mortality risk and bequest motives, if the policyholder dies before retirement, the beneficiary will get the utility from the bequest at \( t^+ \). At the 2nd stage from \( t^+ \) to \( t + 1^- \), the insured maximizes his utility by allocating optimally the amount between two subaccounts.

In the “no GMDB” case, the bequest amount is equal to the account value \( a_t \) at the moment of insured’s death. The insured’s two-stage bellman equations are as follows

1. From \( t^- \) to \( t^+ \)

\[
V_{t^-}(a_{t^-}) = \max_{d_t, P_t} \left\{ u(c_t) + \zeta (1 - \phi_t) v_B(a_{t^-} - d_t + F_t^c) + V_{t^+}(a_{t^+}) \right\}
\]

\[
= \max_{d_t, P_t} \left\{ u(y_t + d_t - P_t) + \zeta (1 - \phi_t) v_B(a_{t^-} - d_t + F_t^c) + V_{t^+}(a_t - d_t) \right\}
\]

\[
= \max_{d_t, P_t} \left\{ u(y_t + d_t - P_t) + \zeta (1 - \phi_t) \psi_t (a_{t^-} - d_t + F_t^c)^{1-\gamma} + V_{t^+}(a_{t^-} - d_t) \right\}.
\]

Under the CRRA assumption, we derive a constant factor\(^6\) \( \psi_t \) at period \( t \) to make \( v_B(a_{t^-} - d_t + F_t^c) = \psi_t (a_{t^-} - d_t + F_t^c)^{1-\gamma} \), given \( \psi_t < 0 \) if \( \gamma > 1 \) and \( \psi_t > 0 \) if \( \gamma < 1 \).

2. From \( t^+ \) to \( (t + 1^-) \)

\[
V_{t^+}(a_{t^+}) = \max_{\omega_t} \{ \beta \phi_t EV_{t+1^-}(a_{t+1^-}) \}.
\]

By taking the first order condition of \( d_t \) and \( P_t \) on \( V_{t^-}(a_{t^-}) \), we get

\[
P_t = \frac{1 + A_1}{1 + A_1} d_t + \frac{y_t - A_1 a_t}{1 + A_1} \quad \text{where} \quad A_1 = \left[ \frac{\zeta \psi_t (1 - \gamma)}{1 + \eta} \right]^{1/\gamma}.
\]

\(^6\)Same as in Section 3
From the above equation, we can see that the term life premium $P_t$ is linearly related to withdrawal amount $d_t$ from the account, given $0 \leq P_t \leq d_t + y_t$. Since $A_1$ is positive, $P_t$ grows in pace with the withdrawal amount $d_t$ and income $y_t$, and is negatively correlated with wealth level $a_t$. If the insured withdraws from the VA account, the beneficiary will get a lower bequest from the VA account, and the insured is willing to buy more term life insurance to transfer his wealth to his beneficiary. If the account value $a_t$ is high, term life is not very important any more.

If the variable annuity account contains a GMDB option, the insured’s two-stage Bellman equations are as follows,

1. From $t^-$ to $t^+$

\[
V_{t^-}(a_t, b_t) = \max_{d_t, P_t} \left\{ u(c_t) + \zeta(1 - \phi_t)v_B(b_t + F_t^c) + V_{t^+}(a_{t^+}) \right\} \\
= \max_{d_t, P_t} \left\{ u(c_t) + \zeta(1 - \phi_t)v_B(b_t \frac{a_t - d_t}{a_t} + F_t^c) + V_{t^+}(a_{t^+}) \right\},
\]

\[
\implies V_{t^-}(a_t, b_t) = \max_{d_t, P_t} \left\{ u(y + d_t - P_t) + \zeta(1 - \phi)v_B(b_t \frac{a_t - d_t}{a_t} + F_t) + V_{t^+}(a_t - d_t) \right\} \\
= \max_{d_t, P_t} \left\{ u(y + d_t - P_t) + \zeta(1 - \phi)v_B(b_t \frac{a_t - d_t}{a_t} + F_t)^{1 - \gamma} + V_{t^+}(a_t - d_t) \right\}.
\]

2. From $t^+$ to $(t + 1)^-$

\[
V_{t^+}(a_t, b_t) = \max_{\omega \phi} \left\{ \beta \phi EV_{t+1^-}(a_{t+1^-}, b_{t+1^-}) \right\}.
\]

By taking the first order condition of $d_t$ and $P_t$ on $V_{t^-}$, we get

\[
P_t = \frac{1 + \frac{A_1 b_t}{A_2}}{1 + \frac{A_1}{(1 - \phi)(1 + \eta)}} d_t + \frac{y - A_1 b_t}{1 + \frac{A_1}{(1 - \phi)(1 + \eta)}} \quad \text{where} \quad A_1 = \left[ \frac{\zeta \psi_t (1 - \gamma)}{1 + \eta} \right]^{1/\gamma}.
\]

With GMDB, the term life premium is also linearly related to the withdrawal amount $d_t$ from the account. As in the “no GMDB” case, $P_t$ grows in pace with the withdrawal amount $d_t$ and the income $y_t$. If there is no withdrawal, i.e. $d_t = 0$, $P_t$ is negatively correlated with
the GMDB level $b_t$, because the term life is partially substituted for by the GMDB protection. If $d_t > 0$, we find that a higher asset level $a_t$ reduces the term life demand. The GMDB level $b_t$ also negatively correlates with the term life demand.

### 4.2 Numerical Methodology

In the case of “with both GMDB and term life insurance”, we use two-stage Bellman equations to get the numerical results and apply a 2-Dimensional lattice. We solve the policyholder’s utility optimization problem by backward induction (month by month) from the retirement age $t = T$ (at the beginning of age 65, $T = 360$) to $t = 1$ (at the beginning of age 35). We use the same settings of $\hat{A}$ and $\hat{B}$ in Section 3\(^7\).

All state variables are denoted as $(\cdot)_{t-}$, $(\cdot)_{t+}$. We assume the terminal value at time $(T + 1)^-$ by using

\begin{equation}
\bar{c} = \frac{a_{T+1-}}{\sum_{t=T+1}^{T_{\text{max}}} \prod_{i=T+1}^{t-1} \phi_i (1 + r_f)^{t+1-i}},
\end{equation}

\begin{equation}
V_{T+1-}(a_{T+1-}) = \sum_{t=T+1}^{T_{\text{max}}} \beta^{t-(T+1)} (\prod_{i=T+1}^{t-1} \phi_i) u(\bar{c}).
\end{equation}

After we get the terminal values, we can maximize the insured’s utility backward from $T^+$ to $1^-$.

a. **Transition from $(t + 1)^-$ to $t^+$**

As indicated before, the insured decides the allocation between the fixed and variable subaccounts in this time interval. We apply the trinomial tree to solve for the optimal allocation. Given GMDB level $b_t$, let

\begin{equation}
a_{t+1-} = a_{t+} \times \begin{pmatrix} 1/u & 1 & u \end{pmatrix},
\end{equation}

for all nodes on $a_{t+}$. Since $a_{t+1-}$ might not be always on those 51 nodes we defined in advance,
we need to use cubic spline interpolation to get the values of \( V_{t+1}^{+}(a_{j+1}, b_i), V_{t+1}^{+}(a_j, b_i), \) and \( V_{t+1}^{+}(a_{j-1}, b_i) \).

The probabilities \( p_u, p_d, \) and \( p_m \) have been derived in equations (34), (35) and (36). Then at any given GMDB level \( b_i \), we derive

\[
V_{t+1}^{+}(a_j, b_i) = \beta \phi(p_u V_{t+1}^{+}(a_{j+1}, b_i) + p_m V_{t+1}^{+}(a_j, b_i) + p_d V_{t+1}^{+}(a_{j-1}, b_i)),
\]

for \( i = 1, 2, \cdots, 51 \) and \( j = 1, 2, \cdots, 51 \), where \( V_{t+}^{+} \) is a \( 51 \times 51 \) matrix at time \( t^+ \).

b. Transition from \( t^+ \) to \( t^- \)

The insured needs to determine the optimal withdrawal \( d_t \) and the term life premium \( P_t \).

At any given GMDB level \( b_i \):

1. Initialize the withdrawal amount \( d_{t,k} = a_j - a_{j-k} \) for all \( k < j \); the term life premium \( P_t \) can be obtained from equation (57) corresponding to different \( d_{t,k} \), given \( 0 \leq P_t \leq d_{t,k} + y_t \);

2. For all \( k \)'s, we derive

\[
V_{t-}^{k}(a_j, b_i) = u_t(d_{t,k} + y_t) + (1 - \phi_t) \psi_t(\max(b_{i-k}, a_{i-k}) + F^c_t)^{1-\gamma} + V_{t+}(a_j, b_{i-k})
\]

where \( F^c_t = \frac{P_t}{(1 - \phi_t)(1 + \eta)} \).

From the above equation, one can see that as the policyholder withdraws money from the account, the GMDB level will also be reduced proportionally. Numerically, we need to reduce the \( V_{t-}^{+} \) diagonally;

3. Let \( V_{t-}(a_{j-k}, b_{i-k}) = \max(V_{t-}(b_{i-1}, a_{i-1}), \cdots, V_{t-}(b_{i-k}, a_{i-k})) \), so we can locate the position of the maximum \( V_{t-}^{+} \) in the \( 51 \times 51 \) space. The maximum \( V_{t-}^{+} \) is not necessarily on the matrix nodes, but it must be between the diagonal points \( V_{t-}(a_{j-k+1}, b_{i-k+1}) \) and \( V_{t-}(a_{j-k}, b_{i-k-1}) \);
4. Choose the optimal withdrawal \( d_t \) to maximize \( V_t^- \) by using quadratic interpolation,

\[
\begin{pmatrix}
    a_{j-k^*-1}^2 & a_{j-k^*-1} & 1 \\
    a_{j-k^*}^2 & a_{j-k^*} & 1 \\
    a_{j-k^*+1}^2 & a_{j-k^*+1} & 1
\end{pmatrix}
\begin{pmatrix}
    \alpha_1 \\
    \alpha_2 \\
    \alpha_3
\end{pmatrix} =
\begin{pmatrix}
    V_t^-(a_{j-k^*-1}) \\
    V_t^-(a_{j-k^*}) \\
    V_t^-(a_{j-k^*+1})
\end{pmatrix}
\]

then we can derive the value of the parameters \( \alpha_1, \alpha_2 \) and \( \alpha_3 \) which are used to estimate \( V_t^- \),

\[
\begin{pmatrix}
    \alpha_1 \\
    \alpha_2 \\
    \alpha_3
\end{pmatrix} =
\begin{pmatrix}
    a_{j-k^*-1}^2 & a_{j-k^*-1} & 1 \\
    a_{j-k^*}^2 & a_{j-k^*} & 1 \\
    a_{j-k^*+1}^2 & a_{j-k^*+1} & 1
\end{pmatrix}^{-1}
\begin{pmatrix}
    V_t^-(a_{j-k^*-1}) \\
    V_t^-(a_{j-k^*}) \\
    V_t^-(a_{j-k^*+1})
\end{pmatrix}
\]

By using the calculated values of \( \alpha_1, \alpha_2 \) and \( \alpha_3 \), we can solve for the optimal withdrawal amount \( d_t \) and the optimal term life policy \( F_t^c \) by maximizing the following equation.

\[
\max_{d_t} \{ u_t(d_t + y_t) + \zeta(1 - \phi_t)\psi_t(b_t \cdot \frac{a_j - d_t}{a_j} + F_t^c)^{1-\gamma} + \alpha_1(a_j - d_t)^2 + \alpha_2(a_j - d_t)^2 + \alpha_3 \}
\]

subject to

\[
0 \leq d_t \leq a_j,
\]

\[
A_1 = \left[ \frac{\zeta \psi_t(1 - \gamma)}{1 + \eta} \right]^{-1/\gamma} P_t = \frac{1 + \frac{A_1 b_t}{a_j}}{1 + \frac{A_1}{(1-\phi)(1+\eta)}} d_t + \frac{y - A_1 b_t}{1 + \frac{A_1}{(1-\phi)(1+\eta)}},
\]

\[
0 \leq P_t \leq d_{t,k} + y_t,
\]

\[
F_t^c = \frac{P_t}{(1 - \phi_t)(1 + \eta)}.
\]

We repeat the “transition from \((t+1)^-\) to \(t^+\)” and the “transition from \(t^+\) to \(t^-\)” until \( t = 1^- \). We can get the optimal asset allocation \( \omega_t \) and optimal withdrawal amount \( d_t \) from age 65 to age 35.

In the remainder of this section, some numerical sensitivity tests have been done. We
assume the base case parameter values are as follows,

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loading element of Term Life</td>
<td>$\eta$</td>
</tr>
<tr>
<td>Strength of Bequest Motive</td>
<td>$\zeta$</td>
</tr>
<tr>
<td>Subjective Discount Rate</td>
<td>$\beta$</td>
</tr>
<tr>
<td>Risk Free Rate</td>
<td>$r_f$</td>
</tr>
<tr>
<td>Coefficient of Relative Risk Aversion</td>
<td>$\gamma$</td>
</tr>
<tr>
<td>Growth Rate of Fixed Subaccount</td>
<td>$r_g$</td>
</tr>
<tr>
<td>Return of Risky Asset</td>
<td>$r$</td>
</tr>
<tr>
<td>Volatility of Risky Return</td>
<td>$\sigma$</td>
</tr>
<tr>
<td>GMDB roll-up rate</td>
<td>$r_p$</td>
</tr>
<tr>
<td>Annual Survival Rate</td>
<td>$\phi$</td>
</tr>
<tr>
<td>Annual Mortality Rate</td>
<td>$\mu$</td>
</tr>
<tr>
<td>Labor Income</td>
<td>$y$</td>
</tr>
</tbody>
</table>

In the following analyses, all adjustments are made monthly. This is a little unrealistic, because the term life insurance demand $F_t^c$ might not be able to be adjusted monthly in real life. Results might change if $F_t^c$ is not so easily adjustable. In figure 54 and figure 55, the life insurance demand at age 35 are checked, given different at-the-money asset levels\(^8\), and we find that as the at-the-money wealth level decreases, the demand for the term life insurance increases at any risk aversion levels and any given insurance loadings. In the comparison of the term life insurance demand with $\gamma = 2$ and $\gamma = 3$ (Figure 56), one can see that when the asset level is low, people are willing to buy a term life insurance policy for their beneficiaries. From the figure, as $\eta = 0$, there is no difference in determining the term life insurance demands for poor people in the $\gamma = 2$ case and the $\gamma = 3$ case; as $\eta = 0.5$, the difference in the term life insurance demand between different risk aversion levels is also very small when the wealth level is low. The difference expands as the asset level rises, because rich insureds have alternative way to give bequests to their beneficiaries. Therefore people’s incentives to buy term life insurance will decrease as their wealth level increases. One can also find that risk aversion levels matter in determining the term life insurance demand: the

\(^8\)At any asset level, we use $a_{35} = b_{35}$
higher the risk aversion is, the higher the term life demand is.

Figure 54: Age 35 Term Life Insurance Demand with $\gamma = 3$

Figure 55: Age 35 Term Life Insurance Demand with $\gamma = 2$
The basis points which the insured is willing to pay for the GMDB at the beginning of the contract are derived in Table 12. In doing that, the quadratic interpolation method is applied to estimate the basis points of GMDB options. We know that for an identical expected rate of return $r_1$, any policyholder with a bequest motive will be better off in a VA account with GMDB than in a VA account without GMDB protection. This implies that $r_1$ in an with GMDB account is equal to $r_1 + \epsilon$ in an without GMDB account for some $\epsilon > 0$.

Given three expected rates of return $r_1 < r_2 < r_3$, we get the policyholder’s utility at age 35 in the “no GMDB” account; the utility the policyholder can get in a VA account with GMDB should be between the utilities of no GMDB accounts with $r_1$ and $r_3$ (given $r_3$ is large enough).

We assume $\bar{v}$ is the utility of the GMDB account with $r_1$ at age 35; $v_1$, $v_2$ and $v_3$ are the utilities of the no GMDB accounts with expected rate of return $r_1$, $r_2$ and $r_3$\(^9\) respectively. $v_1$, $v_2$ and $v_3$ can be expressed by the quadratic forms of $r_1$, $r_2$ and $r_3$ as follows,

\(^9\)We assume $r_1 = 7\%$, $r_2 = 7.2\%$ and $r_3 = 7.5\%$. 

Figure 56: Term Life Insurance Demand comparison with different $\gamma$
We can derive the parameter values of $\theta_1$, $\theta_2$ and $\theta_3$ by

\[
\begin{pmatrix}
\theta_1 \\
\theta_2 \\
\theta_3
\end{pmatrix} = \begin{pmatrix}
  r_1 & 1 \\
  r_2 & 1 \\
  r_3 & 1
\end{pmatrix}^{-1} \begin{pmatrix}
  v_1 \\
  v_2 \\
  v_3
\end{pmatrix}
\]

By using the parameters, we can derive

\[
\bar{v} = \theta_1 (r_1 + \epsilon)^2 + \theta_2 (r_1 + \epsilon) + \theta_3.
\]

We solve the above equation (68) for $\epsilon$, and then 10000$\epsilon$ is denoted as the basis points that the insured is willing to pay. From table 12, one can find that at any given risk aversion level, the basis points decrease as at-the-money asset level decreases. The less risk averse the policyholder is, the smaller the basis points he is willing to pay. This means less risk averse policyholders are reluctant to pay costs for the GMDB protection. This can be partially explained by the fact that our model does not allow policyholders to save their labor income, and the only way for poor people to leave bequest is to buy a term life policy. The poor insureds may regard the term life insurance as an efficient tool to transfer the wealth to their beneficiary. Therefore, a poor insured is willing to pay less for the GMDB and buy more term life insurance. Life insurance has some substitute effects on the GMDB. Also as $\eta$ increases, the policyholder will have to pay more premium for the optimal term life benefit. Therefore, the attractiveness of term life will decrease, and the policyholders are willing to pay a little more for GMDB and buy less term life coverage. It may also be explained by the fact that the term insurance is adjustable monthly in this model while the
GMDB is locked-in at $t = 0$.

By incorporating optimal choices of the insured, the insurance company can price the GMDB in a risk neutral way (Table 13). From that table, one can see that the gap between the price from the insurer’s perspective and the price evaluated from the insured’s perspective is very large, even when the term life loading factor is increased to $\eta = 0.5$. The reason there is little influence from the term life loading factor increase is that the increase of $\eta$ does not change the insured’s allocation and withdrawal choices significantly, but the insurer prices the GMDB by considering the insured’s allocation and lapse choices. Therefore the GMDB price is still much higher than the amount the insured is willing to pay.

Table 12: GMDB at the money basis points at age 35 from insured’s perspective

<table>
<thead>
<tr>
<th>Account Value</th>
<th>$\gamma$</th>
<th>$\eta = 0$</th>
<th>$\eta = 0.1$</th>
<th>$\eta = 0.5$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>2.177285617</td>
<td>2.400562312</td>
<td>3.293030008</td>
</tr>
<tr>
<td>0.927743486</td>
<td>3</td>
<td>1.966135023</td>
<td>2.169414838</td>
<td>2.983092028</td>
</tr>
<tr>
<td>0.860707976</td>
<td>3</td>
<td>1.746936916</td>
<td>1.929426976</td>
<td>2.660911555</td>
</tr>
<tr>
<td>0.798516219</td>
<td>3</td>
<td>1.520668824</td>
<td>1.681812085</td>
<td>2.328635989</td>
</tr>
<tr>
<td>0.740818221</td>
<td>3</td>
<td>1.287852328</td>
<td>1.427278767</td>
<td>1.988576075</td>
</tr>
<tr>
<td>0.687289279</td>
<td>3</td>
<td>1.049560852</td>
<td>1.166851376</td>
<td>1.641915008</td>
</tr>
<tr>
<td>0.637628152</td>
<td>3</td>
<td>0.810416601</td>
<td>0.905675891</td>
<td>1.295146776</td>
</tr>
<tr>
<td>0.637628152</td>
<td>2</td>
<td>1.676527952</td>
<td>1.839097421</td>
<td>2.477152309</td>
</tr>
<tr>
<td>0.927743486</td>
<td>2</td>
<td>1.558313657</td>
<td>1.708742323</td>
<td>2.298101408</td>
</tr>
<tr>
<td>0.860707976</td>
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<td>1.438205208</td>
<td>1.576889229</td>
<td>2.118201469</td>
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<tr>
<td>0.798516219</td>
<td>2</td>
<td>1.31733512</td>
<td>1.444628011</td>
<td>1.942116934</td>
</tr>
<tr>
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<td>1.198946465</td>
<td>1.315253524</td>
<td>1.770765382</td>
</tr>
<tr>
<td>0.687289279</td>
<td>2</td>
<td>1.083839266</td>
<td>1.189776982</td>
<td>1.605799822</td>
</tr>
<tr>
<td>0.637628152</td>
<td>2</td>
<td>0.973051224</td>
<td>1.069179023</td>
<td>1.447886546</td>
</tr>
<tr>
<td>1</td>
<td>0.5</td>
<td>0.713440359</td>
<td>0.778438667</td>
<td>1.023804069</td>
</tr>
<tr>
<td>0.927743486</td>
<td>0.5</td>
<td>0.686895282</td>
<td>0.74987376</td>
<td>0.987981559</td>
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<tr>
<td>0.860707976</td>
<td>0.5</td>
<td>0.661334275</td>
<td>0.722357689</td>
<td>0.953425041</td>
</tr>
<tr>
<td>0.798516219</td>
<td>0.5</td>
<td>0.636698439</td>
<td>0.695828493</td>
<td>0.920064632</td>
</tr>
<tr>
<td>0.740818221</td>
<td>0.5</td>
<td>0.612932947</td>
<td>0.670228353</td>
<td>0.887834322</td>
</tr>
<tr>
<td>0.687289279</td>
<td>0.5</td>
<td>0.589987089</td>
<td>0.645503645</td>
<td>0.85667218</td>
</tr>
<tr>
<td>0.637628152</td>
<td>0.5</td>
<td>0.567814305</td>
<td>0.621604932</td>
<td>0.826520452</td>
</tr>
</tbody>
</table>
Table 13: GMDB fair price VS. insured’s expected price at age 35

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\eta$</th>
<th>GMDB fair price</th>
<th>Insured’s willingness to pay</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>0</td>
<td>0.002592358</td>
<td>0.000218</td>
</tr>
<tr>
<td>3</td>
<td>0.1</td>
<td>0.0026</td>
<td>0.000240</td>
</tr>
<tr>
<td>3</td>
<td>0.5</td>
<td>0.002622991</td>
<td>0.000329</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>0.002978453</td>
<td>0.000168</td>
</tr>
<tr>
<td>2</td>
<td>0.1</td>
<td>0.002985554</td>
<td>0.000184</td>
</tr>
<tr>
<td>2</td>
<td>0.5</td>
<td>0.003012131</td>
<td>0.000248</td>
</tr>
<tr>
<td>0.5</td>
<td>0</td>
<td>0.003169028</td>
<td>0.000071</td>
</tr>
<tr>
<td>0.5</td>
<td>0.1</td>
<td>0.003169035</td>
<td>0.000078</td>
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<tr>
<td>0.5</td>
<td>0.5</td>
<td>0.003169039</td>
<td>0.000102</td>
</tr>
</tbody>
</table>

At-the-money account value is $a = b = 1$

5 Conclusions

In this paper, we apply a dynamic utility-based model to derive optimal transfer and withdrawal choices for insureds who have variable annuity accounts with GMDBs.

In section two “Optimal Consumption and Allocation in Variable Annuities with Guaranteed Minimum Death Benefits”, we discuss “without consumption” and “with consumption (partial withdrawal)” cases are explored in-depth.

In the “without consumption” case, as the insured’s age increases, so too does the concern for himself, reflected in an increase in money transfers from the variable subaccount to the fixed subaccount. At any given age, as the coefficient of risk aversion $\gamma$ or market volatility $\sigma$ go up, or the risky asset rate of return $r$ goes down, the allocation to the variable account will go down; while if the bequest motive $\zeta$ goes up, the allocation to the variable subaccount will go up. The findings with regard to the roll-up rate $r_p$ were complicated: increasing the roll-up rate $r_p$ increases the at-the-money allocation when the insured is young, but allocation decreases very fast as age increases, before converging to the Merton allocation point.

In the “with consumption (partial withdrawal)” case, we derive similar results for the
insured’s allocation choice as in the “without consumption” case, and some exciting results for optimal withdrawal decisions. As the insured ages, he will reduce the wealth in the variable subaccount for any parameter values. The consumption rate for each period is positively correlated with the coefficient of risk aversion $\gamma$; the higher the bequest motive, the lower the consumption rate; if $\gamma < 1$, a higher roll-up rate $r_p$ increases the consumption rate, while if $\gamma > 1$, a higher roll-up rate $r_p$ decreases the consumption rate; greater volatility $\sigma$ decreases the consumption rate; a higher risky rate of return $r$ results in a higher consumption rate. At any given age (e.g. at age 45), the coefficient of risk aversion $\gamma$ reduces the risk taking incentive for the insured, i.e. reduces the risky allocation around the at-the-money area; a higher bequest motive $\zeta$ pushes the insured to take risky decisions around the at-the-money area; if the insured is promised some roll-up rate $r_p$, he will transfer more money to the variable subaccount, at the same time we will see the trigger point to take risky decisions move up; a higher expected risky rate of return $r$ also gives insured incentive to take more risk.

In both cases, insureds will apply a more aggressive investment strategy by transferring more money into variable subaccount when they have a GMDB rider, especially when the asset value is around the at-the-money level. If the insured puts more amounts into the variable subaccount, the GMDB is more and more valuable in protecting downside risk.

In section three, we incorporate the insured’s periodic labor income in our model. We explore two scenarios, i.e., a “fixed labor income” case and “stochastic labor income” case. If the insured is assumed to get fixed labor income, he will have a stable and substantial human capital, which is similar to a risk-free bond investment. The insured’s allocation choice will be much more aggressive than in the “no income” case we discuss in section two. In the “stochastic labor income” case, since the insured faces earnings risk, the allocation choice is less aggressive than the “fixed labor income” case. In both cases, at any given age, we will observe two hump shapes for an insured with a bequest motive. One is incurred by the GMDB argument around stock to strike area; the other is incurred by the human capital effect.
In section four, we add the term life insurance policy to check the GMDB value. By considering the optimal choice of allocation, withdrawal and term life policy purchase, the GMDB option is priced from the insurer’s perspective by plugging the optimal choice into the risk-neutral model. By comparing the basis points derived from both the insured’s and the insurer’s perspectives, we find that the GMDB is too expensive for the insured at any given risk aversion level, while the demand of term life policy does not change noticeably regardless of the availability of GMDB. Therefore, our work suggests that fairly priced GMDBs may not be good investments if term life insurance is an available option. It is possible that the results may change if the term insurance face amount is not continuously adjustable or an external savings vehicle is available.

6 Future Extensions

There are several extensions we plan to do in the future. So far, we assume the variable annuity with GMDB is the only investment vehicle in the model. According to our assumption, people purchase the GMDB contract in a lump sum and use up the labor income in each period. With these assumptions, the policyholder will incur “over consumption” when his total wealth becomes low. In future trials, one or more dimensions will be added to the current model, allowing for investment into other vehicles where the policyholder can choose to invest wealth in riskless and/or risky accounts. In this work, the policyholder’s labor income is not correlated with the investment market, however, if we assume that the policyholder’s labor income is correlated with the equity market, then the risk/return profile of policyholder’s human capital should be quite different. Concurrently, the policyholder’s choice should vary from the obtained results.

The return of premiums and roll-up benefits were considered in our paper. There are other features of GMDBs, e.g. Ratchet and Resetting GMDB. We may extend our model to study these two features. Furthermore, the current study focuses on the demand side, but in future research, this can be extended to discuss the interaction between demand (insureds)
and supply (insurers) sides.

7 References


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