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Promoting Mathematical Understanding through Open-Ended Tasks; Experiences of an Eighth-Grade Gifted Geometry Class

Carol H. Taylor

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__________________________________
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Taylor, C. (1993) *Problem solving and writing in mathematics*. Workshop conducted at NCTM Southeast Regional Meeting, Columbus, GA.
ABSTRACT
PROMOTING MATHEMATICAL UNDERSTANDING THROUGH OPEN-ENDED TASKS: EXPERIENCES OF AN EIGHTH-GRADE GIFTED GEOMETRY CLASS
by
Carol H. Taylor

Gifted students of mathematics served through acceleration often lack the opportunities to engage in challenging, complex investigations involving higher-level thinking. This purpose of this study was to examine the ways mathematically gifted students think about and do mathematics creatively as indicators of deep understanding through collaborative work on four open-ended tasks with high-level cognitive demand. The study focused on the mathematical thinking involved in students’ construction of mathematical understanding through the social interaction of group problem solving.

This case study used ethnographic methodology within a social constructivist frame with gifted education and sociocultural contextual influences. Participants were 15 gifted students in an 8th-grade gifted geometry class. Data collection included field notes, student artifacts, student journal entries, audio recordings, and reflections. Transcribed audio recordings were segmented (Tesch, 1990) into phases of interaction, coded by function, then coded by levels of exhibited mathematical thinking from observable cognitive actions (Dreyfus, Hershkowitz, & Schwarz, 2001; Williams, 2000; Wood, Williams, & McNeal, 2006), and analyzed for maintenance or decline of high-level cognitive demand (Stein, Smith, Henningsen, & Silver, 2000). Interpretive data analysis was connected to data analysis of transcribed recordings.
Results indicated social interaction among students enabled them to talk through the mathematics to understand mathematical concepts and relationships, to construct more complex meaning, and exhibit mathematical creativity, inventiveness, flexibility, and originality. Students consistently exhibited these characteristics indicating mathematical thinking at the levels of building-with analyzing, building-with synthetic-analyzing, building-with evaluative-analyzing, constructing synthesizing, and occasionally constructing evaluating (Dreyfus et al., 2001; Williams, 2000; Wood et al., 2006).

The results of the study support the claim of a relationship between mathematical giftedness and the ability to abstract and generalize (Sriraman, 2003), provide evidence that given the opportunity, students can construct deep mathematical understanding, and indicate the importance of social interaction in the construction of knowledge. This study adds to the body of knowledge needed in research on gifted education, problem solving, small-group interaction, mathematical thinking, and mathematical understanding, through empirically assessed classroom practice (Friedman-Nima et al., 2005; Good, Mulryan, & McCaslin, 1992; Hiebert & Carpenter, 1992; Lester & Kehle, 2003; Phillipson, 2007; Wood, Williams, & McNeal, 2006).
PROMOTING MATHEMATICAL UNDERSTANDING THROUGH OPEN-ENDED TASKS: EXPERIENCES OF AN EIGHTH-GRADE GIFTED GEOMETRY CLASS

By

Carol H. Taylor

A Dissertation

Presented in Partial Fulfillment of Requirements for the Degree of Doctor of Philosophy in Teaching and Learning in the Department of Middle-Secondary Education and Instructional Technology in the College of Education

Georgia State University

Atlanta, GA
2008
ACKNOWLEDGMENTS

Now faith is the substance of things hoped for, the evidence of things not seen.
Hebrews 11:11 KJV

First, I thank my husband who encouraged my dreams, pushed me to achieve them, and supported my efforts throughout these years beyond measure. Your unfailing faith in me, your abiding love, and your graciousness in doing everything for me sustained me through this process. You will always be the love of my life. Because of you, I celebrate achievement of a dream.

To my dear children, Jimmy and Michelle, thank you for your love, encouragement, understanding, patience, and your contributions to my successful completion of this goal. I’m proud to be your mother. To my grandchildren Alex and Cody, my sister, Candy, my niece and nephew, Chrissy Dale and Barry Jr., my mother-in-law and father-in-law, Nora and Marvin, thank you for your understanding when my work consumed all my time and your willingness to help out as necessary. I love each and every one of you.

Although my mother, Dale Henson, passed away during this journey, I am eternally grateful to her for instilling in me a passion for learning, for continually making sacrifices so I could learn more, for taking an expressed interest in my every endeavor, and loving me enormously. Undoubtedly, there will be an announcement posted somewhere in heaven regarding my accomplishment as mother, my greatest publicist, always ennobled others.

I extend an enormous thanks to my committee members for your guidance and expertise. Thank you Dr. McNeal for your willingness to sit on my committee and for the positive influence, constructive feedback, and insight you provided in my work. Thank you Dr. Stinson, for challenging my thinking and exposing me to work of exemplary scholars. You contributed immensely to the quality of my scholarship. I extend my appreciation to Dr. Thomas for assuming the responsibility of my chair and seeing me through to the end. It has been a pleasure working with you all these years. I especially wish to thank Dr. Karen Schultz, my former committee chair, current committee member, and esteemed mentor who consistently set standards for me to follow. Your accomplished teaching, scholarship, and high standards directed my career path to mathematics education, influenced my pedagogy, and encouraged my continued scholarship. Your integrity represents the cornerstone of a life of influence.

Thank you to my former gifted geometry students for your enthusiastic participation. Your willingness to accept challenge made my work pleasurable.

To my Lord who is always able, thank you for bountiful blessings.
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<tr>
<td>AERA</td>
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<td>ECER</td>
<td>Exceptional Children Educational Resources</td>
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<td>ERIC</td>
<td>Educational Resources Information Center</td>
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<td>PISA</td>
<td>Program for International Student Assessment</td>
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<td>MTF</td>
<td>Mathematical Tasks Framework</td>
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<td>NAGC</td>
<td>National Association for Gifted Children</td>
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<td>NCLB</td>
<td>No Child Left Behind</td>
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<td>National Research Council</td>
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<td>NMSA</td>
<td>National Middle School Association</td>
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<td>TIMSS</td>
<td>Third International Mathematics and Science Study</td>
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<td>UCSMP</td>
<td>University of Chicago School Mathematics Project</td>
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<td>ZPD</td>
<td>Zone of Proximal Development</td>
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CHAPTER 1
INTRODUCTION

Some believe that instead of selecting mathematically able pupils we should undertake an investigation of the possibilities for the maximal mathematical development of all pupils. But the one will always complement the other, since even with perfect teaching methods individual differences in mathematical abilities will occur - some will be more able, others less. Equality will never be achieved in this respect. Consequently, mathematics teachers should work systematically at developing the mathematical abilities of all pupils, at cultivating their interests in and inclinations for mathematics, and at the same time should give special attention to pupils who show above-average abilities in mathematics by organizing special work with them to develop these abilities further. (Krutetskii, 1976, 6-7)

Krutetskii (1976) recommended the development of mathematical abilities for all students, while also meeting the needs of students with above-average mathematical abilities, because a focus on one would strengthen the other. Despite recommendations for development of mathematical abilities for all students, implementation continues to fall short, and there has continued to be little focus on students with above-average mathematical ability (House, 1999). Mathematics education in the United States followed a behaviorist model of learning for most of the 20th Century (Palincsar, 1998). Students simply reproduced what the teacher said or what the teacher modeled in class. Growing dissatisfaction with limited development of students’ mathematical understanding resulted in calls for reform (Hiebert & Carpenter, 1992). Despite reform efforts in the field of mathematics education to promote learning for understanding, fulfillment of the goal in most mathematics classrooms remains elusive (Hiebert, 2003). Results of the
Program for International Student Assessment 2003 (PISA 2003) showed U.S. students performed below the average of participating countries in mathematical literacy and problem solving (Ferrini-Mundy & Schmidt, 2005). The reluctance to shift from a behaviorist perspective to teaching for understanding affects all learners, but especially impedes the development of the mathematically promising student because the qualities that represent mathematical promise are largely ignored (Graffam, 2003; Sheffield, 1999; Usiskin, 1999). Key terms applicable to this study are defined or described following the summary of this chapter.

The Problem and the Purpose of the Study

According to House (1999), one area of educational concern for the gifted and talented has been the ability versus equity dilemma. Some view gifted students as valuable resources, while others view provisions for these students as elitist (House). In 1980, amidst a continued focus on procedural knowledge, the National Council of Teachers of Mathematics (NCTM) stated in *An Agenda for Action*, “The student most neglected, in terms of realizing full potential, is the gifted student of mathematics. Outstanding mathematical ability is a precious societal resource, sorely needed to maintain leadership in a technological world” (p. 18).

The NCTM published recommendations for serving the mathematically gifted in 1987. The focus shifted from the needs of mathematically gifted students to meeting the needs of all students in 1989, when NCTM published the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989). These standards recommended a shift in thinking about mathematics teaching from a behaviorist perspective toward teaching for understanding, from a focus on procedures and correct answers to mathematical
reasoning, conjecturing, inventing, and problem solving. In 1994, NCTM appointed a task force to consider the needs of mathematically promising. Students with mathematical promise included students with high ability as well as those defined as gifted and talented (Sheffield, 1999). The recommendations of the task force were published in 1995. Many of the recommendations have yet to be implemented due to time and money constraints (Sheffield, 1999).

While serving as president of the NCTM, Cathy Seeley (2005) stated, “students deserve, and society demands, that we also support and advance our most able students” (p. 3). Seeley’s message on untapped potential called for examination of the needs of mathematically gifted students, how they are currently served, and the possibility of intervention in elementary school. Years earlier, Sheffield (1999) had already suggested the most prevalent gifted models of enrichment and acceleration lacked opportunities for students to think “deeply about a wide range of original, open-ended, or complex problems that encourage them to respond creatively in ways that are original, fluent, flexible, and elegant” (p. 46). The enrichment model used in elementary schools often limits the study of mathematics to weekly activities without specific mathematical objectives (Sheffield). The acceleration model commonly used in middle school and high school focuses on curriculum that is “a mile wide and an inch deep” (Schmidt, McKnight, & Raizen, 1996) only students move at a faster pace. Meeting the needs of the gifted and talented suffer further due to No Child Left Behind (NCLB) mandates as resources for the gifted are considered expendable (O’Neil, 2006). Consequently, many mathematically promising students languish under traditional curricula while others are often met with
more problems rather than enriching problems that allow for a deeper study of mathematical concepts (Greenes & Mode, 1999).

Jacobs et al. (2006) analyzed how the teaching in two 8th-grade U.S. classrooms from the Video Studies of the Third International Mathematics and Science Study 1995 (TIMSS) and TIMSS 1999 align with the NCTM 2000 Principles and Standards. Results were reported in terms of the NCTM process standards, which included problem solving, reasoning and proof, connections, communication, and representation. Less than 10% and 5% respectively, of classroom time was spent collaboratively, indicating limited communication between students; 75% of instructional time involved repetitive procedures with students working on 33 problems per class in 1995 and 32 problems per class in 1999 involving low complexity. No evidence of lessons involving deductive reasoning was found in the 1995 study, and only two lessons from the 1999 study involved deductive reasoning. More recent results of both the TIMSS 2003 and the PISA 2003 indicated performance of U.S. students continues to be dismal in most areas. While the TIMSS 2003 focused more on procedural knowledge in problem solving, the PISA 2003 focused on “interpretive and application outcomes” (Dossey, McCrone, & O'Sullivan, 2006, p. 38), and U.S. students ranked 24 out of 29 participating countries. The lower than average performance of U.S. students in application and use of mathematical ideas causes concern about the effectiveness of implementation of the NCTM Standards (Ferrini-Mundy & Schmidt, 2005) in U.S. schools. Gieger and Kilpatrick (1999) indicated earlier, that drawing conclusions about the population of mathematical promising students is difficult because of the cross section of students tested.
Rather than focusing on the trends of international comparisons, the focus should be on how mathematically promising students “are thinking about and doing mathematics” (Gieger & Kilpatrick, 1999, p. 38). Also concerned with the quality of students’ mathematical thinking, the American Educational Research Association (AERA) suggested raising the bar for all students by increasing the level of cognitive demand of instructional tasks, maintaining the level of cognitive demand, and increasing the opportunity for students to take higher-level mathematics (AERA, 2006). When instructional tasks are set up and implemented at a higher level of cognitive demand, students do better at reasoning and problem solving (AERA; Stein, Smith, Henningsen, & Silver, 2000).

Each year I, the researcher and teacher, receive a new class of 8th-grade geometry students who are good “technicians who can follow rules and apply those rules to routine exercises” (Sheffield, 1999, p. 45). Many are conditioned through previous experience in mathematics (Bishop, 1988) to determine what it takes to get the good grade (Wheatley, 1999) rather than seeking to understand. My students have lacked the opportunity to become mathematical thinkers. Hiebert (2003) proposed a conditional: “If students have more opportunity to construct mathematical understandings, they will construct them more often and more deeply” (p. 16). Although these students are being accelerated, the problem is they have lacked the opportunities to realize their full potential by engaging in challenging complex investigations, collaborative problem-solving experiences, and higher-level mathematical thinking. The purpose of this case study was to examine the ways mathematically gifted students think about and do mathematics creatively while working collaboratively on open-ended tasks with high-level cognitive demand.
Theory

Sfard (2003) suggested that theoretical perspectives are not mutually exclusive and “should be viewed as either complementary– that is, concerned with different aspects of the same phenomena – or incommensurable – that is speaking different languages rather than really conflicting each other” (p. 355). The metaphors acquisitionist and participationist can be used to describe the constructivists learning theories of Vygotsky and Piaget (Sfard, 1998). Acquisitionist represented the traditional view of learning within the cognitive domain of the individual while participationist represented views of learning within a community. Tenets of Vygotsky’s theory assumed a Piagetian stance. This blending of metaphors represented by Ernest’s (1998b) claim that the mind is the individual within the social, acquisitionist, and that mathematical learning occurs in a community of practice, participationist, was foundational in the work of this study.

A learning community focused on inquiry mathematics as the intersection of social constructivist theory, and gifted education within a sociocultural context was relevant to this study. The theory of social constructivism framed the study. Social constructivism is a theory that suggests the individual comes to know (acquisition) by using newly constructed knowledge gained through social interaction (participation) to amend, refine, or add to existing knowledge (Cobb & Yackel, 1996; Ernest, 1998a). The learning community was the union of individual mathematically gifted students and “the rich interconnections between cultural institutions, social practices, semiotic mediation, interpersonal relationships, and the developing mind” (Minick, Stone, & Forman, 1993, p. 6).
The social constructivist theoretical views that framed this study include the following (I extended the overarching views applicable to my study by adding the term gifted):

1. The *gifted* learner is both acquisionist and participationist in the process of coming to know (Cobb & Yackel, 1996; Ernest, 1998b; Jaworski, 1996; Palincsar, 1998).

2. Learning occurs when new knowledge is integrated with previous knowledge (Ernest, 1998b; Jaworski, 1996).

3. Social interaction within the *gifted* learning community is essential (Cobb & Yackel, 1996; Vygotsky, 1978).

Theoretically, I believe mathematical gifted ability must be developed and is not fixed at birth (Clark, 2002). I am the teacher of a group of students mostly identified as gifted in elementary school. Within the educational system these students are viewed as globally gifted (Winner, 1996). Globally gifted means once a child is identified as gifted in some academic domain according to specific criteria of the school district, the child is considered gifted in all areas (Winner). I do not share the view of global giftedness either. As a teacher of the gifted, I am aware that every pedagogical decision I make holds theoretical assumptions (White, 1999). My goal is not to argue a definition of giftedness, a theory of giftedness, or advocate specific programming because there are no absolutes in gifted education. My theoretical perspective centers on helping students actualize their mathematical potential when grouped as a gifted class through social constructivism. I have had no influence in getting these gifted students to a geometry class in the 8th-grade. My influence starts here. Geometry could be just another mathematics class for the
students, or it can be the gateway to a pursuit of mathematics. I believe the social setting of my classroom constitutes a culture where “there is prior (native) cultural knowledge held by each of the various actors, the action itself, and the stabilizing rules, expectations, and some understandings that are tacit” (Spindler & Spindler, 1992, p. 84). Within the public culture of my classroom, given the opportunity, high-level mathematical thinking can occur.

Research Questions

The following questions guided the study: How is the mathematical understanding of 8th-grade gifted geometry students elicited through exploration using open-ended problems? What levels of mathematical thinking do 8th-grade gifted geometry students demonstrate when engaged in collaborative problem solving on tasks with high-level cognitive demand?

Significance

The results of this study are significant for several reasons. First, this study supported by scholarly research, adds to the body of knowledge needed in areas of research on gifted education, problem solving, small-group interaction, mathematical thinking, and mathematical understanding through empirically assessed classroom practice (Friedman-Nima et al., 2005; Good, Mulryan, & McCaslin, 1992; Hiebert & Carpenter, 1992; Lester & Kehle, 2003; Phillipson, 2007; Wood, Williams, & McNeal, 2006). Hiebert (1992) indicated the extent of our explanations regarding students’ understanding influences the collective knowledge about teaching and learning by linking the results of individual studies.
The results of this study on the mathematical thinking of gifted students working within small groups on open-ended tasks link previous research in gifted education, problem solving, small-group interaction, mathematical thinking, and mathematical understanding together to provide a better understanding of the teaching and learning that occurs when these domains are combined.

Second, the results of this study serve as an example of “measuring the processes of mathematical thought” (Krutetskii, 1976, p. xvi) based on observable actions of students at work on problem-solving tasks (Dreyfus, Hershkowitz, & Schwarz, 2001; Williams, 2000; Wood et al., 2006). Krutetskii argued that identification of mathematically gifted students should occur through observation of students at work on problem-solving tasks based on the mathematical abilities he identified, rather than test scores alone. The National Association for Gifted Children (NAGC) and the National Middle School Association (NMSA) advocate the use of multiple approaches for identification of gifted students from minority and low economic groups (NAGC, 2007). While acknowledging the difficulty of measurement of abilities argued by Krutetskii (Wertheimer, 1999), the results of this study offer an example of how a conceptual framework used by Wood et al. based on the categories of mathematical thinking (Dreyfus, Hershkowitz, & Schwarz, 2001; Williams, 2000) might be used as an alternative means of identifying gifted students marginalized by psychometric identification only.

Third, the results of this study contribute to the literature on effective implementation of the NCTM (2000) process standards: problem solving, reasoning and proof, connections, communication, and representation. These standards recommended a
shift in thinking about mathematics teaching from the behaviorist perspective of traditional programs toward teaching for understanding (Hiebert, 2003), from a focus on procedures and correct answers to mathematical reasoning, conjecturing, inventing, and problem solving. NCTM (2000) suggested the social interaction involved in problem solving can contribute to the development of understanding. The results of this study substantiate the possibility of a true hypothesis and true conclusion to Hiebert’s conditional, “If students have more opportunity to construct mathematical understandings, they will construct them more often and more deeply” (p.16).

Background

In 1990, I was asked to participate in a mathematics project headed by a group of mathematics educators attempting to affect change in teacher practices by implementing the new NCTM Standards. Project participants embraced a constructivist perspective of learning and advocated reflective teaching. Consequently, my teaching pedagogy focuses on teaching for understanding according to recommendations of the NCTM Standards.

My county adopted the University of Chicago School Mathematics Project (UCSMP) curricula materials in 1991 and implemented an advanced mathematics program to correspond to the higher levels of mathematics presented in UCSMP. The focus of UCSMP was to provide opportunities for a deeper understanding of mathematics through problem solving and application. This program enabled 6th-grade students who scored at or above 93% on the Iowa Test of Basic Skills in fifth grade, to take UCSMP pre-algebra. This criterion included students who showed mathematical talent, but had not been identified as gifted (not domain specific) through formal testing. A sequential course was added each year. Students on this accelerated track could take geometry, a
high school course, in the eighth grade and could complete an advanced placement (AP) calculus class as a senior in high school. Two AP calculus courses are available to senior accelerated students: AP Calculus consists of a full year of high school work representative of the rigor of a first semester college calculus class and Calculus BC offers students an opportunity to receive college credit a course beyond AP Calculus, also referred to as Calculus AB. Although, the UCSMP series was discontinued after the next mathematics textbook adoption cycle, the advanced program established remains in place.

I worked with my first group of 8th-grade geometry students in 1993 using *USCMP Geometry* (Coxford, Usiskin, & Hirschhorn, 1993). Most students were either identified as gifted learners, or mathematically talented. To better meet the needs of these students, I returned to graduate school to add the gifted endorsement to my teaching certificate. I have worked with gifted or talented geometry students for 15 years. In my dual role as a teacher of mathematics and teacher of the gifted, I discovered my students were reluctant to apply previous knowledge to acquire a deeper understanding of mathematical concepts. This discovery led me to question what encourages mathematically gifted students to think about and do mathematics “creatively in ways that are original, fluent, flexible, and elegant” (Sheffield, 1999, p. 46) as indicators of deep understanding.

**Summary**

Implementation of standards to improve the mathematical abilities for all students has fallen short and there has been little focus on gifted students of mathematics (House, 1999). The reluctance to teach for understanding impedes the development of gifted
students of mathematics (Graffam, 2003; Sheffield, 1999; Usiskin, 1999). Gifted students of mathematics often just do more problems at a faster rate without opportunities to construct mathematical understandings (Sheffield). As a result, gifted students come to 8th-grade geometry relying mostly on procedural knowledge. They have lacked the opportunities to engage in challenging investigations, collaborative problem-solving experiences, and higher-level mathematical thinking. This case study focused on students’ construction mathematical understanding through the social interaction of collaborative problem solving requiring a high-level of cognitive demand. The purpose of this study was to examine the ways mathematically gifted students think about and do mathematics creatively as indicators of deep understanding while working collaboratively on open-ended tasks with high-level cognitive demand.

Definitions of Key Terms

The following key terms are defined or described as used in this study. Although many definitions or descriptions abound in the literature for some of the terms, I referred only to the work of scholars that I referenced in the study to better provide a coherent whole in the research process (Pirie, 1998).

State criteria for gifted students. Students identified as gifted must meet state criteria in one of two options. Students identified as gifted in kindergarten through grade two must score 99% on a composite or full standardized test of mental ability and either score 90% or better in math, reading or total on a standardized achievement test or 90% or better on a creativity test. Students identified in grade 3 through grade 12 must meet criteria in 3 of 4 areas. Students must score 96% or better on a standardized test of mental ability, score 90% or better in mathematics, reading or total on a standardized
achievement test, score 90% or better on a test of creativity, or score 90% or better on a motivation rating scale. Continuation in the program is contingent on maintaining eligibility based on academic grades. All of the students participating in this study with the exception of one student were identified as gifted students in grades K-5 according to these state criteria.

Mathematically promising student. The NCTM Task Force on Mathematically Promising Students established in 1995 (Wertheimer, 1999), described mathematical promise as a function of ability, motivation, belief, and experience, or opportunity for students to become future leaders and problem solvers. Mathematical promise represented a composite of characteristics, rather than numerical scores based on an achievement tests alone. This view assumed students of mathematical promise could be influenced through educational opportunities (Ernest, 1991; Krutetskii, 1976). The expanded description encouraged recognition of the multidimensionality of intelligence (House, 1999).

Mathematical Tasks Framework. Stein, Smith, Henningsen, and Silver (2000) developed the framework that matches tasks with goals for student learning according to the level of cognitive demand in three phases; selection, set-up and implementation. Cognitive demand referred to the “kind of thinking and level of thinking required of students” (Stein et al., p. 11) when working on a mathematical task. Levels of cognitive demand within the Mathematical Tasks Framework (MTF) included doing mathematics, procedures with connections, procedures without connections, or memorization, respectively from the highest level of cognitive demand to the lowest level. Doing mathematics was described as the active process of exploring situations, problems or
tasks, searching for patterns, conjecturing, defending, debating, justifying, generalizing or abstracting mathematics (Teppo, 1998, Stein et al.). Stein et al. referred to doing mathematics, related to mathematical tasks, as thinking which requires the highest level of cognitive demand due to the unpredictable nature of the task. Procedures with connections represented student thinking that drew on previous knowledge of procedures to construct a deeper understanding of mathematical concepts related to a task (Stein et al.). Procedures without connections were mathematical tasks that required little mathematical thinking by the student (Stein et al.). Memorization relied on recall only.

**Ability.** Krutetskii (1976) described ability as “a personal trait that enables one to perform a given task rapidly and well, in contrast to a habit or skill” (p. xiii). Krutetskii studied Soviet students ages six to sixteen and characterized the abilities of students who showed mathematical promise. According to Krutetskii, mathematically gifted students can formally grasp mathematical information, process the information logically, draw generalizations, think flexibly, change directions in processing, and curtail, or shorten mathematical thinking. Krutetskii’s structure of abilities also included the ability to retain information and to think mathematically about most situations. Usiskin (1999) summarized the abilities Krutetskii characterized, as flexibility, curtailment, logical thought, and formalization. Usiskin suggested that the ability to retain information and think in terms of mathematics, are more a result of the other abilities.

**Mathematical thinking.** Wood et al. (2006) described mathematical thinking as the “mental activity involved in the abstraction and generalization of mathematical ideas” (p. 226). Their description of mathematical thinking was based on the work of Dreyfus, Hershkowitz, and Schwarz (2001); Krutetskii (1976); and Williams (2000). Williams first
created a framework to classify cognitive activities of students during problem solving based on Krutetskii’s (1976) work and Bloom’s (1956) taxonomy. Williams then integrated the cognitive taxonomy with the three observable actions that Dreyfus et al., (2001) claimed occur during the cognitive activities of abstraction or generalization. The observable cognitive activities included recognizing, building-with, and construction (Dreyfus et al.). These categories were further subdivided into recognizing comprehending, recognizing applying, building-with analyzing, building-with synthetic-analyzing, building-with evaluative-analyzing, constructing synthesizing and constructing evaluating (Williams). The resulting framework provided a way of categorizing mathematical thinking with observable cognitive activity. Wood et al. studied the mathematical thinking of students as a result of the social interaction involved in group problem solving. The results of their study indicated social interaction was a component for the construction of mathematical knowledge. I used the categories of mathematical thinking and observable cognitive activities used by Wood et al. based on the work of Dreyfus et al. (2001) and Williams (2000) for data analysis.

Collaborative groups. Two terms, collaborative groups and cooperative groups are commonly used to describe the instructional strategy in which students work interactively in small groups organized to support interdependence for achieving a common goal (Johnson & Johnson, 2004). Earlier, Bruffee (1993) argued there was a difference in the two terms; Collaborative learning is grounded in social constructivism, involves higher order knowledge, and the teacher becomes part of the community of learners in the construction of knowledge. The distinguishing characteristic between the two terms according to Bruffee, was in the structure of the groups. Cooperative groups,
usually designed and implemented by the teacher, are more structured than collaborative
groups. Students also have more latitude in collaborative learning environments students.
Ernest (1998a) claimed social interaction within the learning community is an essential
component in social constructivism learning theory. Based on this key tenet and the
indication that high ability mathematics students can achieve significantly more in small-
group instruction (Peterson, Janiack, & Swing, 1981), I used the research on collaborative
groups to guide the use of groups in my study. Common elements for collaborative or
cooperative grouping included positive interdependence, promotive interaction,
individual and group accountability, social skills, and group processing (Johnson &
Johnson, 2004).

Mathematical Understanding. One of the six principles for school mathematics is
“Students must learn mathematics with understanding, actively building new knowledge
from experience and prior knowledge” (NCTM, 2000, p. 11). Achievement of the goal
remains as elusive as a specific definition (Hiebert & Carpenter, 1992). Numerous
theories both in mathematics education and in other fields have developed frameworks
for defining and assessing student understanding. Years earlier, the National Research
Council (1989) stated research provides evidence that students develop deep
mathematical understanding when actively involved in their own construction of
knowledge through group problem solving experiences allowing investigation and
communication. Next, I briefly describe several noted scholars’ view of mathematical
understanding that informed this work.

Sfard (2003) indicated that many thinkers, including Piaget and Vygotsky,
connected understanding to activity with a concept. Sfard suggested that the circularity
over which occurs first, understanding of a concept in order to use the concept, or use of
the concept to achieve an understanding of the concept actually adds to the growth in
mathematical learning. “The sense of understanding a concept and the ability to apply it
are like two legs that make forward movement possible thanks to the fact that they are
never in exactly the same place” (Sfard, 2003, p. 359).

Skemp (1987) included instrumental understanding, relational understanding, and
logical understanding in his framework for a “new model of intelligence” (p. 164). Instrumental understanding referred to “rules without reasons” (p. 153), relational understanding referred to “knowing what to do and why” (p. 153), and logical understanding referred to connecting “mathematical symbolism and notation with relevant mathematical ideas and to combine these ideas into chains of logical reasoning” (p. 166). According to Skemp (1976), deep understanding occurs through relational understanding. Sfard (1991) added another intuitive understanding.

Hiebert and Carpenter (1992) described mathematical understanding as the connection between mathematical ideas and concepts, to existing knowledge. Hiebert and Carpenter (1992) argued understanding increases as mathematical connections increase, or are strengthened through reorganization of a network of representations. In a recent study exploring mathematical understanding, Dosemagen (2004) indicated “experience, concepts, symbols, connections, reflection, and communication are all elements in the equation for mathematical understanding” (p. 45). Assuming Dosemagen’s claim is true, what is the evidence of mathematical understanding? Hiebert and Carpenter suggested evidence of students’ understanding occurs through their explanations and their
understanding is inferred by the measurer from their explanations. This premise is foundational to the work in this study.
CHAPTER 2
LITERATURE

Considerable debate continues over the synonymous use of gifted and talented and the lack of a clear definition of giftedness (Gagne, 2005). In this section, I discuss the evolving definition of giftedness to provide evidence of a shift in thinking about the concept of intelligence, the differing views of ability and mathematics in five educational ideologies, and how these influenced this research. I then discuss the reason for the limited availability of gifted research similar to my study. Following this discussion, I review the literature with similar elements contained in my research, and discuss the implications of specific studies relevant to my research intermittently in the review. Next, I include a review of literature concerning collaborative learning, a key construct of social constructivism. I conclude with a summary of how the literature related to my study.

Paradigm Shifts: Changes and Challenges

Although instances of education of the gifted and talented can be traced to the 1860s, the concept of giftedness began in the early 1900s (Delisle, 1997a). Educational programs in public institutions emerged in the 1950s. Until 1972, gifted identification was a result of a high score on a psychometric test. Since then, a shift in thinking about definition has gradually occurred (Gallagher, 2003). In the 1972 Marland Report to Congress, the U.S. Office of Education first described gifted and talented children as those capable of high performance in general intellectual ability, specific academic
aptitude, creative or productive thinking, leadership ability, visual or performing arts, or psychomotor ability. Psychomotor ability was dropped from the definition in 1978. Passage of U.S. Javits Gifted and Talented Students Education Act of 1988 (PL 100-297) was intended to set standards of learning higher for all students to include those who had previously been marginalized from gifted programs (Ross, 1997) by providing small amounts of money for research in these areas. The concerns for the gifted and talented were addressed again in a national report, National Excellence: A Case for Developing America’s Talent (1993) which stated:

The United States is squandering one of its most precious resources—the gifts and talents, and high interests of many of its students. In a broad range of intellectual and artistic endeavors, these youngsters are not challenged to do their best work. This problem is especially severe among economically disadvantaged and minority students, who have access to fewer advanced educational opportunities and whose talents often go unnoticed. (p.1)

In response, NCTM created a task force to address the concerns raised in the national report. The NCTM Task Force on Mathematically Promising Students established in 1995 (Wertheimer, 1999), described mathematical promise as a function of ability, motivation, belief, and experience, or opportunity for students to become future leaders and problem solvers. The NCTM definition evolved using the term promising as opposed to gifted and talented which precluded many disadvantaged students with high mathematical ability. Although the task force tried to move away from the narrow definition often accompanied by the use of the terms gifted and talented, these terms continue to be used.

As indicated in the No Child Left Behind Act, the terms gifted and talented continue to be used to describe students with high ability. The terms gifted and talented
were both used in the definition of gifted in The No Child Left Behind Act of 2001 (PL 107-110):

The term “gifted and talented”, when used with respect to students, children, or youth, means students, children, or youth who give evidence of high achievement capability in areas such as intellectual, creative, artistic, or leadership capacity, or in specific academic fields, and who need services or activities not ordinarily provided by the school in order to fully develop those capabilities (Title IX, Part A, Section 9101(22), p. 544).

The difficulty with consistent terminology and definition of gifted learners occurs in other areas in the field of gifted education as well. Wertheimer (1999) categorized programs for the gifted using Ernest’s educational ideologies although he indicated rarely does a program fit neatly within one perspective. Like Wertheimer, I used Ernest’s (1991) five ideologies of mathematics as a framework to further discuss how gifted education was viewed through the perspectives and the shifts in thinking about gifted education that have occurred more recently through these ideologies.

Ernest (1991) described five evolving educational ideologies of mathematics: The Industrial Trainer, the Technological Pragmatist, The Old Humanist, the Progressive Educator, and the Public Educator. The ideologies were presented at two levels in a British context, but transcended boundaries of nationality. The first level dealt with epistemology, a philosophy of mathematics, and moral values. These were discussed through the theory of the child and theory of society. The second level dealt with educational issues: political ideology, view of mathematics, ability, mathematical aims, learning, teaching mathematics, resources, mathematical assessment, and social diversity. These ideologies were not time periods, but mathematical perspectives based on beliefs and goals. The first three perspectives are more historical. Gallagher (2003) suggested
new knowledge in the field of gifted education along with rethinking of existing knowledge contributed to paradigm shifts in five areas pertaining to gifted education. These areas included the concept of intelligence, identification of gifted students, curriculum differentiation, the equity versus excellence dilemma, and the impact of technology on the roles of teacher and student.

The Industrial Trainer and Technological Pragmists represented utilitarian ideologies focused on maintaining the status quo of society that served special interest groups. The Industrial Trainer related “Victorian values and a Protestant work ethic” (Ernest, 1991, p. 141) and represented utilitarian education focused on trade and maintaining a social hierarchy. In this perspective, ability was fixed, mathematics a set of rules to be followed through skill and drill without discussion or cooperation, and without regard to any social issues such as multiculturalism (Ernest). The Technological Pragmists represented a more current version of the Industrial Trainer focused on social progress in terms of technology. Mathematics education influenced technological viability which impacted social progress. The view of mathematics included two parts, pure and applied, but only in relation to “short term payoffs” (p. 165). In this perspective, ability was deemed inherited, potential reached through good teaching, and learning only good for practical purposes related to employment. Individual interests were not recognized.

The Old Humanist, sometimes referred to as the Mathematician’s ideology, focused on pure mathematics. In this perspective, education and knowledge were the end products for the individual, mathematical ability believed inherited and hierarchical, and social concerns deemed unrelated to the study of mathematics. It was an elitist view.
Mathematical application was reserved for lower ability students and deemed not real mathematics (Ernest, 1991). Conversely, the Progressive Educator ideology focused on mathematics \textit{purely} in development of the individual child. The view of mathematics included pure and applied mathematics. Ability was believed to be inherited, but developed through innate stages based on mathematical experience. The teacher facilitated development of the individual with appropriate mathematical experiences. Social diversity was addressed only as needed to meet the needs of the individual student. Consequently, the social status quo remained unchanged.

Through a Progressive Educator perspective, teachers needed a way to identify and serve students (Ernest, 1991). Ability as measured by intelligence quotient (IQ) tests provided an easy means for identification. Terman (1925) helped set the standard for measuring giftedness through his work on intelligence. According to Terman, gifted students scored in the top 2% on the Stanford-Binet Intelligence Scale. Later, other psychometric means for identification such as Stanley’s Study for Mathematically Precocious Youth looked at high SAT scores at early ages (Wertheimer, 1999). In these models, high ability was assumed to transfer to other content areas. In Renzulli’s Revolving Door Model, a broader psychometric measure of up to 20% on IQ tests was initially used for screening (Wertheimer). Programs were then matched to domain specific abilities. The new paradigm suggests ability is developed through experience, ability can be domain specific, and these abilities should be measured through performance tasks (Matthews & Foster, 2006). Unfortunately, the Progressive Educator perspective still operates today. Heward (2000) reported that 73% of U.S. school systems continue to rely on psychometric measures of cognitive ability for means of identifying
gifted students. The challenge of this perspective is to shift means of identification to match the shift in thinking about who is gifted.

The Mathematician’s perspective focuses on pure mathematics (Ernest, 1991). Identification of gifted students for possible careers in mathematics occurs through acceleration and competition (Wertheimer, 1999). Students usually begin acceleration in middle school, take calculus in high school, and can earn college credit for AP classes. Intelligence is often assumed to be inherited. Longitudinal studies indicated accelerated students often pursue careers in mathematics (Swiatek & Benbow, 1991). School competitions such as the American High School Mathematics Examination, USA Mathematics Talent Search, and MATHCOUNTS filter out a small group of mathematically gifted students who may be offered apprenticeships working with practicing mathematicians. The changing paradigm of who is gifted and equity versus excellence challenges this perspective. The goal is to include more underrepresented populations with mathematical ability unrecognized due to the means of measurement early on.

The Industrial Trainer and the Technological Pragmatist in more recent years have continued to focus on the study of mathematics for social progress. Some reasons for identification of the gifted included developing an informed citizenry, developing world leaders, and preparing students to be competitive at the university level (Sheffield, 1999). Usiskin and Dossey (2004) reported 4.5% of freshman entering college in 1966 majored in mathematics, and this percent steadily declined to 0.7% of college freshmen in 2001 entering with a major in mathematics. Oakland and Rossen (2005) suggested that gifted programs supported by NCLB are needed to develop talent in order to maintain our
competing advantages in technology and education. They further asserted that the U.S. needed to be competitive educationally to benefit cities and states in attracting industry to prevent foreign outsourcing of jobs. Development of the mathematically gifted for socioeconomic purposes has contributed to the underrepresentation of minorities in gifted education (Ambrose, 2002).

Public Educator ideology focused on the philosophy of social constructivism. The Public Educator perspective viewed ability as fluid and influenced by experience (Krutetskii, 1976) in the social context, and that accessibility of knowledge to all learners reflected the denial of ownership of mathematics to an elite group. Like the Progressive Educator, the focus was on the individual, but focused on the individual as a contributing member of society, a society committed to social justice (Ernest, 1991). A Public Educator focused on teaching methods that included student to student discussion, student to teacher discussion, group work and problem solving for engagement, mastery, creativity, critical thinking, conflict, and social relevance for empowerment (Ernest). Unlike the Progressive Educator ideology, the Public Educator ideology challenged the social status quo (Freire, 1972). Often social activism was the focus. The Algebra Project and The Escalante Mathematics Project both focused on work with Hispanic or African American students espousing the view that all students have mathematical promise (Wertheimer, 1999). Statistics from the National Research Council (2002) indicated approximate 3% and 3.5% of Black and Hispanic students, respectively, were identified as gifted compared to 7.5% White and 10% Asian students. Traditional means of identifying and serving gifted students do not consider the social and cultural contexts of the classroom for students from diverse backgrounds (Kitano, 2007). Socioeconomic
underrepresentation was the focus of the NAGC recent publication *Overlooked Gems: A National Perspective on Low-Income Promising Learners* (2007). This perspective reflects the paradigm shift in definition, identification, instructional strategies, and equity versus excellence. The paradigm shift in instructional strategies is most apparent in this ideology. Rather than teach processes of thinking, the focus is on inquiry based learning.

As a teacher of the gifted in mathematics it is important to understand the view of each perspective, the paradigm shifts that impact these perspectives, and consequently the role they play in shaping my teaching strategies. The students in this study were identified by means typical in a Progressive Educator’s perspective. I view development of mathematical ability through a Public Educator’s perspective, as a product of the culture of the classroom. My insistence on opportunities for students of mathematical promise to use higher-order thinking skills could be considered social activism because of the contradictory nature of the status quo; Critical thinking versus mathematical indoctrination and individual empowerment versus test score success (Ernest, 1991). The opportunities typical for students from the Mathematician’s perspective are important to me especially in the sense of appreciation of the beauty in mathematics, but not in an elitist sense. My view of giftedness also includes a blend of the Technological Pragmatist in that students should have a willingness to give back to society because they are able, not because it is demanded to benefit special interest. I remind the reader, the focus of this research was on my gifted geometry class as a group of students with mathematical promise and not as an elitist view that these students were the only students with mathematical promise.
The Department of Education of each state defines and establishes identification criteria for gifted and talented students, while implementation and programming decisions are made at the district level (Evans, 1996). Usiskin and Dossey (2004) reported there are approximately 15,000 school districts in the United States. Consequently, systematically serving mathematically gifted students has been difficult due to lack of consistency in definition, identification, and programming. Also, two significant problems continue to occur when students are identified as gifted learners in elementary school: Students are characterized as globally gifted (Winner, 1996), and the assumption one program fits all learners (Matthews & Foster, 2006). This problem is characteristic of the old paradigm of the concept of intelligence.

Matthews and Foster (2006) offered a means for moving beyond the shifts in thinking to shifts in instructional strategies by narrowing giftedness to specific domains. Matthews and Foster (2006)\(^1\) compared the mastery model for educating gifted students to the mystery model of educating gifted students in terms of evidence-based programming versus belief-based programming. Their mastery model focused on Vygotsky’s (1978) zone of proximal development (ZPD); the zone in a student’s learning that is challenging, but attainable with facilitation, as the norm while seeking those that are working above the ZPD. The needs of these learners then would be matched with learning opportunities. Thinking about giftedness in terms of mastery is more inclusive of students who have been marginalized through identification of psychometric means only. The focus of this research represented the new paradigm on instructional strategies of student inquiry within specific content.

\(^1\) See Matthews and Foster (2006) for their ten point comparison of the mastery model versus mystery models for educating gifted students.
Gifted Research

In this section, I first discussed reasons for limited scholarly gifted literature. Next, I discussed gifted educational studies with similar elements to this research. The implications of specific studies on my research are addressed intermittently in the review of literature.

Albert (1969) surveyed the professional literature dealing with all forms of genius, creativity, and giftedness from 1927 to 1969 to determine conceptual shifts in gifted terminology. Freidman-Nima, O’Brien, and Frey (2005) extended the work of Albert by investigating the conceptual changes in the professional literature from 1965 to 2000. Over 28,000 articles from three data bases were identified based on a search of the terms gifted, creative, talent, gifted and disabled, gifted and disadvantaged, as well as lexicographic relatives. The three databases used were Educational Resources Information Center (ERIC) for academic educational literature, PsycINFO for academic psychological literature, and Exceptional Children Educational Resources (ECER) for literature on exceptionality. EndNote was used to categorize by year based on four search patterns, and then a Chi-square analysis was used. To determine content themes, the middle years of each decade were selected, alphabetized, and then every 10th abstract was sorted by content subcategories and reviewed by two judges. The conceptual categories most represented in the literature were educational training and creativity. A surprising finding was only 160 of the 723, or less than 25% of articles related to gifted education included supporting scholarly research. Friedman-Nima et al. (2005) concluded very few of the studies in gifted education add to the body of knowledge through empirically assessed practice. They argued that including teachers in the research
community could “inform the theory-practice cycle” (p. 52) adding to the body of knowledge on the effects of classroom practice.

Searches of more recent research based gifted literature to support this study in mathematics, or other academic areas, yielded studies focusing mostly on the individual gifted student, classification of the gifted, and attitudes and beliefs of the gifted. Literature results also included literature on experiences in honors, accelerated, or advanced placement mathematics classes, but were limited to the individual learner. Broadening the search to include literature on mathematical thinking and problem solving involving groups not limited to gifted students, only one study of mathematical thinking within a group of students while engaged in problem solving was found. The aspects of the Wood, Williams, and McNeal (2006) study are discussed in the section on collaborative learning and in the methodology section. Clearly, there is a lack of research specific to the mathematical thinking of gifted students engaged in small group problem solving. Hekimoglu (2004) indicated the need for more research on ways to provide mathematically gifted students opportunities to realize their potential.

Sriraman (2003) examined the experiences of nine students in an accelerated ninth-grade algebra class working five, increasingly complex, combinatorial problems over the course of 3 months. The structure of the problems provided a means of drawing a generality using the pigeon-hole principle. The pigeon-hole principle deals with counting arguments. A simple example would be, given 5 pigeons and only 4 pigeon-holes, one pigeon-hole will contain more than one pigeon. The research was conducted based on the conjecture that the more mathematically talented students would be able to
recognize the general principle through their solutions. Students had one week to solve each problem. Students worked individually, and showed all work in journals.

Sriraman (2003) read student journal entries then composed questions for an individual interview to elicit verbal explanation of student’s strategies. Piaget’s (1975) clinical interview technique was used to guide the interviews. Journal writing and interview transcriptions were coded according to orientation, organization, execution, and verification based on Lester’s (1985) problem-solving method for constant comparison of the student’s problem-solving behaviors. Sriraman stated generalization, reflection, and affect were patterns of behavior that emerged. Based on similarities and differences of the nine students’ solutions, three subsets of students were used to compare results in problem-solving behaviors, generalization and reflection, and the affective domain. The subsets included students who could abstract and generalize, students who relied on algebraic skills, and students who just used examples.

After data analysis, Sriraman (2003) accessed student records and found four of the nine students had been identified as mathematically gifted in elementary school. These four students could abstract and generalize. Sriraman then focused on the experiences of these four gifted students to show how these experiences support Krutetskii’s (1976) conclusions that generalization occurs through abstraction involving specific content and recognition of similar structure. Sriraman also suggested that his findings supported the conjectures of Piaget (1971) and Dubinsky (1991) that generalization is a function of reflective abstraction.

Although qualitative analysis and descriptive reporting was used in this study, the focus of Sriraman’s (2003) study was to validate his hypothesis that the more
mathematically able student will be able to generalize. Predicted outcomes appeared to drive the research. The problems were developed as pathways to abstraction and generalization. After verifying records, he then described the experiences of the four individual students mostly in terms of problem solving to provide evidence of a relationship between mathematical giftedness and the ability to abstract and generalize.

Hekimoglu (2004) used teaching experiment methodology\textsuperscript{2} to investigate differences in abstract reasoning of a gifted student and an average student. The teaching experiments consisted of three 70-minute sessions in which students worked on a mathematical task while the researcher asked for explanations for their reasoning. In the first two sessions, both students were present, but worked individually on a problem. When necessary the researcher used guiding questions to facilitate students’ transfer of previous knowledge to new situations involved in solving the problem. During the third session the researcher also inquired about the students’ perceptions of mathematics.

Hekimoglu (2004) relied on interpretative data analysis to draw conclusions based on actions involved, interactions between the students and the research, and the students’ mathematical talk. Recursive analysis was used to guide the next session of data collection. Recording and transcription of the sessions is indicated, but there is no indication of whether the sessions were videotaped, or audio recorded. Results of the research indicated that self-efficacy can impact mathematics performance and that the gifted student exhibited mathematical creativity, inventiveness, flexibility, and originality in solving problems. These findings support the need to provide mathematically gifted students opportunities to think about and do mathematics in creative ways.

\textsuperscript{2} Teaching experiment methodology is recording and analysis of episodic learning to structure the next teaching episode and originated for use in mathematics education (Steffe and Thompson, 2000).
Sriraman (2003) and Hekimoglu (2004) both focused on the individual gifted learner. Sriraman used the clinical interview to investigate gifted students' ability to abstract and generalize, and Hekimoglu used the teaching experiment. Both researchers were investigating the mathematical thinking of students. Both used interpretive data analysis, although I would argue that Sriraman's outcomes were more directed due to a predictive hypothesis. My research also examined the ways gifted students think about and do mathematics, but included the component of social interaction in the construction of knowledge.

Other studies I examined (Dosemagen, 2004; Neustadt, 2005) focused on the student's perceptions of their mathematical understanding. While focusing on students' perceptions did not directly impact my research in terms of methodology, the voice represented by these students speaks loudly about ways to meet the learner's needs and emphasized the need for research on opportunities for development of student understanding. I summarize the findings of Dosemagen's study that impacted my research.

Dosemagen (2004) conducted an action research study on how her Advance Placement (AP) Calculus students viewed their mathematical understanding and how the students thought that understanding developed. Rowan's (2001) action research model guided her research. Dosemagen began the study using Hiebert and Carpenter's (1992) representational thinking as the conceptual framework. As the study progressed, Dosemagen added metacognition (Flavell, 1985) after realizing the content of her students' reflections was more about their own thinking in regard to representations than

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3 Rowan’s (2001) model involves determining the problem, thinking about the problem, formulating an action plan, implementing the plan, making sense of the data, and communicating the results.
the representations themselves. She ended the study using the Cognitively Guided Instructional Research Model (Fennema, Carpenter, & Peterson, 1989) as the conceptual framework because the model included the teacher knowledge, teacher beliefs, and teacher decisions as influences in students’ understanding.

Dosemagen (2004), a National Board Certified Teacher (NBCT), examined how students could get to AP Calculus without conceptual understanding through a two-part action research study. Her senior AP students were participants. Sixteen of the 24 participants were in the top 10% of the senior class. Students were not specifically identified as gifted or talented. Dosemagen used a web-based forum for students to report and reflect to the teacher and other students. Students first responded to a prompt. The second response was a reaction to peer comments and reports. The prompts helped students make generalizations, describe processes of problem solving, and reflect on their learning. Students posted 20 responses over a 3-month period. The second part of Dosemagen’s action research involved ethnographic (Spradley, 1979) survey interviews (Goetz & LeCompte, 1984). Six of the 24 participants selected through maximum variation sampling (Lincoln & Guba, 1985) were interviewed according to Spradley’s interview protocol. Grounded theory methodology (Glaser & Strauss, 1967) was used to derive theory using interpretive data analysis.

Four themes emerged during the first phase of Dosemagen’s (2004) research based on student perceptions. These themes included learning strategies stressing visualization, connections to concepts, the importance of modeling application of concepts, and technological concerns. Students deemed visualization, connections among concepts, and the application of concepts important to their understanding of
mathematics. Findings from Dosemagen also included evidence that students were employing the use of multiple representation, an NCTM process standard, and metacognition. In the second phase of the research, results of the interviews indicated that students felt the classroom environment contributed to their own mathematical understanding.

Two specific findings in Dosemagen’s study impacted my study. The first impact was the focus of the tasks with high-level cognitive demand used in this study to target those things that Dosemagen’s students felt were important in understanding mathematics. The tasks provided students opportunities to apply what they know, to visualize, and to connect previous knowledge to new concepts. The second finding that impacted this study was the students’ perception of the importance of the classroom environment on individual mathematical understanding. This finding supports my theoretical perspective of social constructivism and the instructional strategy of small group problem solving.

This study built on the work of Sriraman (2003) and Hekimoglu (2004) regarding individual mathematical thinking and gifted students’ ability to generalize, by adding the dimension of social interaction to the study of students’ mathematical thinking. The investigation moved beyond the individual interview to investigating group thinking as students clarified, justified, and validated their own understanding. Wood et al. (2006) indicated the interaction pattern of argument increased student synthetic-analyzing and evaluative-analyzing, both higher levels of mathematical thinking. Whereas Sriraman and Hekimoglu developed interview questions to elicit students thinking, in this study students elicited other students’ thinking.
Collaborative Learning

Ernest (1998a) claimed social interaction within the learning community is an essential component in social constructivism learning theory. Based on this key tenet and the indication that high ability mathematics students can achieve significantly more in small-group instruction (Peterson, Janiack, & Swing, 1981), I used the research on collaborative groups to guide the use of groups in my study.

The use of small groups has been widely researched (Barkley, Cross, & Major, 2005). Early research by social theorists (Allport, 1924; Shaw, 1932; Watson, 1928) focused on individuals working in groups. Deutsch (1949) was the first to study the cohesiveness and motivation of learners in a cooperative group versus competitive learning. After a silent period in research on groups during the next few decades as the focus turned to the individual, renewed interest in the study of groups occurred during the 1970s (Gillies & Ashman, 2003).

Johnson, Maruyame, Johnson, Nelson, and Skon (1981) conducted a meta-analysis of 122 studies and found cooperative learning promotes higher achievement and motivation to learn over competitive and individual learning. A follow up meta-analysis (Johnson, Johnson, & Maruyame, 1983) of studies of homogeneously and heterogeneously grouped students indicated there is greater interpersonal attraction within homogenously grouped students. These two studies led to further investigation by Johnson and Johnson (1985) of variables that impact the relationship between cooperation, productivity, and attraction (Gillies & Ashman, 2003). Eleven variables were identified and grouped into three clusters associated with cognitive processes, social processes, and instructional processes as interlinking processes.
The use of the term cooperative learning became popular in the 1970s (Gillies & Ashman, 2003). Although many terms describe small group instruction, collaborative learning and cooperative learning are the two terms most often used interchangeably. Bruffee (1993) argued there is a difference between collaborative learning and cooperative learning. According to Bruffee, collaborative learning is grounded in social constructivism, involves higher order knowledge, and the teacher becomes part of the community of learners in the construction of knowledge. The distinguishing characteristic between the two terms according to Bruffee, was in the structure of the groups. Cooperative groups, usually designed and implemented by the teacher, are more structured than collaborative groups. Students also have more latitude in collaborative learning environments students. Johnson and Johnson (2004) viewed cooperative learning as an instructional strategy in which students work in small groups to enhance the learning of all members involved. Common elements for collaborative or cooperative grouping included positive interdependence, promotive interaction, individual and group accountability, social skills, and group processing (Johnson & Johnson, 2004). Like Sfard’s (1998) blend of acquisitionist and participationist metaphors, I viewed small group learning as a blend of cooperative and collaborative elements.

Johnson and Johnson (2004) described three types of cooperative learning groups as base, formal, and informal groups. In base groups, students work together for longer periods of time such as a semester. Formal cooperative groups work together for a class period to several weeks. In formal groups, teachers make instructional decisions, structure the task and positive interdependence, monitor and intervene when necessary, and assess learning and interaction. Informal groups work together for a few minutes to a
class period in discussions related to a lesson. The type of group used depends on curricular goals. I used a blend of formal groups and base groups in this study. I formed the groups based on results from a previous study, which are discussed later, and the students worked together for an extended period of time.

Grouping strategies included random selection, stratified random selection, teacher selection, and support groups for isolated students (Johnson & Johnson, 2004). Within each of these strategies were various methods for achieving the desired grouping. Mandel (2003) suggested grouping students heterogeneously by reading level and academic achievement. Then variables of gender, race, ethnicity, social considerations, multiple intelligences and leadership must be considered. In regard to race and ethnicity, Stinson (2006) and Walker (2006) argued knowledge of African American and Latino students’ intellectual communities is a necessary consideration to structure groups for peer support and encouragement of mathematical academic success.

In a recent study involving collaborative problem solving, Wood et al. (2006) explored the relationship between social interaction and students’ mathematical thinking in a classroom using traditional instruction and four reform classrooms. The instruction in the reform classrooms reflected the use of NCTM standards for teaching and learning. Students in the reform classrooms engaged in collaborative problem solving. The research was based on Bruner’s (1996) view that development of shared meaning and thinking is dependent on social interaction. The study focused on interaction patterns and mathematical thinking within a classroom culture. Information regarding the formation of the groups was not included.
Wood et al. (2006) screened 30 lessons in both the traditional classes, meaning instructional strategies utilizing traditional curricula and pedagogy (Hiebert, 2003) and reform classes. Rather than reform classes, Hiebert referred to nontraditional classrooms as “alternative programs” (2003, p. 16). Eight lessons were identified as representative of the culture of the classroom in the reform classrooms. Then, five lessons from these classrooms were selected for analysis. The three remaining lessons were used to confirm the results of the analysis for the reform classrooms. In the traditional classroom, 10 lessons were selected: eight lessons combined problem solving and use of the textbook and two lessons relied on use of the textbook only. Five of these lessons were selected for analysis, and the remaining five lessons were used to confirm interaction patterns and mathematical thinking.

Data analysis consisted of both quantitative and qualitative methods. Transcribed videotaped lessons were sectioned by interaction patterns, coded, and sectioned again by interaction type based on the perceived function. Some of the interaction patterns represented interaction patterns used in previous literature such as IRE (Hoetker & Ahlbrandt, 1969) and funnel (Bauersfeld, 1980), while others observed did not. Then, interaction patterns were counted within each classroom.

Wood et al. (2006) coded the same transcribed videotapes to analyze students’ mathematical thinking according to dimensions of mathematical thinking. Each line was coded for mathematical thinking according to the integrated categories of the cognitive taxonomy (Williams, 2000) with the three observable actions from Dreyfus, Hershkowitz, and Schwarz (2001).
Results from the study by Wood et al. (2006) indicated that within the reform classroom, interaction patterns during the greatest participation were related to higher levels of mathematical thinking. Wood et al. acknowledged the limitation of the small sample of reform classes based on only one aspect of reform. They attempted to offset this limitation with comparison to a traditional classroom. Analysis also included an interpretative analysis. Wood et al. concluded, “It is the social cognitive processes of joint attention and understanding of others’ communicative intentionality that is the medium by which mathematical thought develops through meaning making with others” (p. 250).

Summary

The connection between students’ understanding and the classroom environment revealed in the second phase of Dosemagen’s (2004) research supported the social theoretical perspective of my research to encourage and promote mathematical understanding. The learning strategies students perceived influenced their understanding of making connections between previously learned concepts to new concepts applied to my research. Tasks with higher-level cognitive demand made these connections and applications possible. The social aspect necessary for knowledge construction by the individual was also supported by Dosemagen’s findings. The studies of Sriraman (2003) and Hekimoglu (2004) indicated gifted students exhibited the capacity to think and do mathematics in ways that are creative and inventive, and that gifted students can reason abstractly and generalize. The capacity of gifted students to use higher-order thinking skills was supported both by Sriraman and Hekimoglu. The research of Wood et al. (2006) of public mathematical thinking versus individual thinking indicated the
importance of examining students’ mathematical thinking within the context of the classroom amidst all the interaction of the students. They argued that the social interaction of the classroom influences students’ construction of knowledge. The strengths of each of these research studies justified examining higher-level mathematical thinking shared among gifted students within small group learning communities and consequently strengthened this study.
CHAPTER 3
THEORY AND METHODOLOGY

The purpose of this study was to examine the ways mathematically gifted students think about and do mathematics creatively as indicators of deep understanding while working collaboratively on open-ended tasks with a high-level cognitive demand. The questions that guided this study are: How is mathematical understanding of 8th-grade gifted geometry students elicited through exploration using open-ended problems? What levels of mathematical thinking do 8th-grade gifted geometry students demonstrate when engaged in collaborative problem-solving tasks with high-level cognitive demand? I begin this chapter by justifying the use of case study within an ethnographic methodology and how this fits the theoretical framework of social constructivism. I then discuss the results of two pilot studies and how they impacted this study. Following this discussion is a description of the methodological design.

Rationale for Methodology Selection

Ernest (1998b) described educational research methodology as a blend of the theoretical framework and the assumptions of how the world is viewed. First, I briefly describe the philosophical connections between the theoretical framework and the methodology selected. In other words, I share how I view the world in the context of my classroom and knowledge construction. I then discuss the characteristics of natural inquiry that justified my use of ethnographic case study to better understand how students demonstrate mathematical understanding.
Social constructivism, a theory that suggests the individual comes to know by using newly constructed knowledge gained through social interaction to amend, refine or add to existing knowledge (Cobb & Yackel, 1996; Ernest, 1998b), framed this research. The sociocultural environment of my classroom, meaning all the influences of the culture of school, social practices, classroom norms, and the students operating together, was the context for knowledge construction for both students and me, the researcher (Cobb & Yackel). In terms of the research process, multiple constructed realities occurred as a result of the interaction between the students and me as they worked together in this study yielding further questions. Also as the researcher, I came to understand the knowledge of my students through my interpretations of their discussions, actions, and work filtered through my own sense of reality and represented a constructive process. Another consideration of this constructive process was that my preconceptions of possible outcomes, theoretical perspectives, and personal values may have influenced my observations. Construction occurs through acknowledging that our research is value-laden (Lather, 1986) and the constant filtering of our interpretations with those of our participants.

According to Pirie (1998), the goal of qualitative research in mathematics education is to add to the body of knowledge or advance understanding in some particular mathematical context. Achieving this goal depends on the selection of methodology that most effectively addresses the research questions. In this research, I examined how student understanding was affected by working within small groups on open-ended problems and described the levels of mathematical thinking involved. Because the focus was on the interaction of the students through my observations in my classroom, a
sociocultural environment, ethnography was an appropriate methodological approach. In this research, the ethnographic methods of data collection and analysis focused on small groups of students within the larger group. The explanatory power of an ethnographic perspective can add to the understanding of how to help gifted students realize their potential (Lundsteen, 1999), helped me come to know (Teppo, 1998) the mathematical thinking of my students and their interactions, and allowed me to construct knowledge of investigative teaching (Jaworski, 1998) giving me voice (Hertz, 1997) and power to effect change (Gitlin, 1990).

Ethnography, as a methodology, is often confused with ethnographic techniques. Ethnographic techniques of data collection and written accounts are often used without conducting an ethnographic study. Wolcott (1999) maintained the distinguishing feature between ethnography as a methodology and ethnographic techniques is first hand experience within a social or cultural context. The techniques common to ethnography include participant observation and inductive data analysis. In keeping with Wolcott’s distinction, the natural setting of my classroom provided the social context for this research. Erickson (1984b) described the context for school ethnography based on Malinowski’s (as cited in Erickson) categories of activity for society that included social organization, economics, and belief systems. He cautioned transference is not possible due to the “partial community” (Erickson, p. 54) of a classroom meaning observation in a classroom is limited. For the purposes of this study, the classroom was substituted for school. Social organization referred to the “statuses and roles that exist for persons in the school, and the networks of rights and obligations that link various statuses together” (Erickson, p. 54). The economics of the classroom referred to the social behavior that
could be traded in some form. The belief system involves “terms of definition, principles of valuation, rules of logic, methods of explanation for cause, and forms of predictive statements” (Erickson, p. 55).

The primary means of data collection was through participant observation (Merriam, 1998). Inductive data analysis was used to gain insight as soon as data collection began (Goetz & LeCompte, 1984; Merriam). The design evolved as a result of an iterative process. Understanding was constantly verified by the sources of data, in my case, the students (Lincoln & Guba, 1985; Merriam).

Similar to the ambiguity in the use of ethnography, Merriam (1998) stated there is confusion related to the view of case study as a process or outcome. The geometry class used in this study was bounded by time and activity, which represented a bounded unit, a characteristic of a case study (Stake, 1995). Miles and Huberman’s (1998) graphic representation provided the best justification for characterizing the research as case study. Consider a circle with a heart in the center. The heart represented the focus of the study, the interaction among my students as they worked collaboratively on open-ended tasks. Outside the heart, but bounded by the circle were those things typically operating within the classroom that I did not study. By this I mean all that operated within the mathematics classroom such as school and classroom norms, curricula goals, instructional goals and a host of others. Although Merriam suggested case study can be combined with ethnography, based on Miles and Huberman, I viewed case study more as a subsystem of ethnography. Shaw (1978) described one possible focal point of case study as how subjects will approach problems. My gifted geometry students were the focal point for this research. The collaborative small groups were sub-cases embedded within the larger
case. Ernest (1998b) described case study as “illustrative and generative” (p. 34). This case study of my geometry class, specifically the students working in small collaborative groups, illustrated through the rich descriptive experiences, the ways students think mathematically while working with open-ended tasks.

Pilot Studies

Pilot Study I

The purpose of Pilot Study I was to construct knowledge of investigative teaching needed in an ethnographic study. One can be an astute student of ethnography, but actual learning occurs through experience (Ball, 1990). Results of the pilot study were used to make changes in implementation of the mathematical tasks, the interaction of the researcher with the students, in data collection, and data analysis. One area of weakness revealed in the first pilot study concerned grouping practices. I later conducted a second pilot study to determine the best grouping strategy to use in this study.

Participants. The pilot study was conducted in the second semester of 2004. Eleven identified gifted students and one highly mathematically talented student enrolled in my 8th-grade gifted geometry class served as data sources. The class was comprised of four white females and eight white males. No ethnic minorities were represented.

Instrumentation. As a participant observer, I served as a research instrument in data collection and analysis (Merriam, 1998). Data collection was ongoing and occurred through participant observation, field notes, audio recordings of group interactions, informal interviewing, and student work.

Procedures and Data Analysis. Students worked in small cooperative groups on mathematical tasks with high-level cognitive demand related to four content areas.
Research was guided by the question: How does use of open-ended mathematical tasks impact students’ mathematical understanding of key concepts?

I observed student groups working on five different tasks. I used the MTF of Stein et al. (2000) to analyze the task, set up the task, and implement the task to maintain high-level cognitive demand. An audio recorder was used for each group to capture some of the rich mathematical discussions. As a participant observer, trying to observe three or four groups simultaneously proved to be difficult. I used jot notes during observation and, as soon as possible, expanded the notes using detailed description. Student work related to the task was also used as data.

Coding for themes also proved difficult and I returned to the heart of case study and chose to report through explanation of the implementation phase of each task. I used rich thick description (Geertz, 1973), explanation, and interpretation of each group’s experiences and then compared group analyses. The multiple sources of data collection, observations, audio recordings, and student artifacts enhanced the validity and the reliability of this study (Lincoln & Guba, 1985). In addition, to this triangulation (Lincoln & Guba), long-term observation, and peer examination was used to ensure trustworthiness.

Findings and Conclusions. The results of the pilot study indicated students engaged in high-level cognitive thinking through procedures with connections and doing mathematics. Student discourse allowed students to share ideas, to question, and conjecture. Each task built on prior knowledge, provided a means for students to monitor their own thinking, increased opportunities for more capable students to model high-level performance, and provided opportunities for me to draw on these conceptual connections.
Implications for the main study included an expanded role for me as a participant observer. I was intimidated by the Institutional Review Board (IRB) process and legal issues. Consequently, I was too reserved in my own interactions with the students. I did not press students for justifications for their conjectures and conclusions for fear of stepping out of bounds of the research proposal. This reluctance to intervene lowered the level of cognitive demand on one task. Other problems in maintaining high-level cognitive demand included time and consistency across groups. Time allowed for tasks requiring higher-level thinking can maintain or decline the level of cognitive demand (Stein et al., 2000). One class period was not adequate for the completion of most tasks and modifications were made for the main study.

Another change that occurred in this study as a result of the pilot study was a decrease in the number of tasks selected for use. Implementation of state mandated end-of-course testing for geometry beginning in 2006 required all course objectives to be covered by the third week in April. Consequently, data collection had to occur after end of course testing and before preparation for final exams in May. Time considerations necessitated a decrease in the number of tasks used to four including the introductory task. In addition to a change in the number of tasks used, I made changes in both data collection and data analysis. I discuss these changes in subsequent sections.

Pilot Study II

A second pilot study was conducted during the fall semester of 2006 with the geometry students who served as participants in this study as a result of the grouping problems encountered in the first pilot study. I investigated the efficacy of grouping strategies within my homogeneously grouped geometry class on achievement and
cohesiveness among the students. The resulting grouping decisions were used in the main study.

Participants. At the time of the second pilot study, there were 17 gifted geometry students in my class. The class was comprised of nine males and eight females between the ages of 13 and 14 years of age. Three of the students were of Asian decent, and the remaining students were White. Two of the students were also identified as special education students.

Data Collection. I used numerous strategies for data collection including participant observation, field notes during participant observation, teacher journal, student work, group tasks, student feedback, and student surveys. I did not have students keep a journal of their experiences during this study because this study involved investigation of an instructional strategy.

Procedures and Data Analysis. I first grouped students using random stratification of high, medium, and low based on performance in my class during the first 6 weeks of school. Next, I grouped students according to teacher selection for support of isolated students (Johnson & Johnson, 2004). The third grouping was based on reading and mathematics achievement on a standardized test as suggested by Mandel (2003). Some minor changes occurred due to student absenteeism. Analysis was based on my observation of cohesiveness among group members, student surveys, and scores on collaborative tasks.

Findings and Conclusions. The grouping strategy that worked best in terms of student achievement and cohesiveness among the students was random stratification based on my observations of performance during the first 6 weeks of school. Success was
measured by evidence indicating students worked effectively, performed high-level tasks, and began to form bonds linking the members. As the group formations changed, I observed less cohesiveness among the students. The strategy least successful in both achievement and cohesiveness was grouping according to mathematics and reading scores on a standardized test as suggested by Mandel (2003). Gifted adolescent students may score similarly on standardized tests, but each embraces their own interests which often are not shared by others (Delisle, 1997b). In one group, two outstanding readers were grouped together. In reality, a serious student with literary interests and an equally talented vivacious drama student did not mix well.

Through this pilot study, I discovered the grouping strategy that works best in terms of cohesiveness and student achievement. Cohesiveness among group members contributed significantly to the social interaction within the community of learners in this study. Although the range of scores was narrow and students were truly more homogeneously grouped, the small stratification seemed to work. I employed this same strategy in the main study.

The Study

In this section, I first describe the context of the study followed by a description of the students who served as data sources. As a participant observer (Lincoln & Guba, 1985; Merriam, 1998), I also served as a data source. I provide background regarding my experience as a teacher of mathematics and other information that influenced my role as a data source and research instrument. Next, I discuss the instrumentation and procedures used to collect data. This discussion is followed by a detailed description of the decisions and methods of data analysis through all phases of the research.
Setting

This study was conducted at a middle school in a suburban county outside a large city in the South. The middle school, referred to as Maple Street Middle School, is one of three schools located on the same complex serving students grades pre-kindergarten through twelfth grade. The high school and middle school are actually connected and have been in existence for 10 years. The school complex is located in an affluent pocket of the county. The county average household income for 2006 was $101,472 and the median household income was $79,558. Most students live in surrounding neighborhoods of middle to upper income homes. The median home cost for 2006 was $262,600. The entire area including the school complex is connected by an extensive system of golf cart paths. Students attending the middle school are predominately white with small populations of African American, Asian American, and Hispanic American students. The school has been designated as a model school for other middle schools to watch based on innovative educational practices and programs.

Students in middle school are separated by grade level and then divided into teams of students. Teams vary in size from about 85 to 120 students depending on the grade level population. Generally, all the students on a team have the same teacher of mathematics. One team’s mathematics teacher did not teach geometry, therefore geometry students on that team were combined with geometry students on my team for geometry instruction. Unlike high school, students in eighth grade, study a full year of geometry with the same teacher.

The geometry class met for 52 minutes per day for mathematics instruction and was scheduled during the sixth period of seven class periods. In addition to scheduled...
class time, students could visit my class from 8:00 am until 8:20 am each morning.

Students could come in for questions or help during a designated 20 minute period of


time at the end of each day when students returned to their homeroom. The time was

allocated for students to get organized to go home, to make up assignments, or to seek

help. Even though my homeroom students were present, geometry students could come in

individually to ask questions as needed.

Data Sources

Fifteen of the original 17 gifted students enrolled in my geometry class served as

the participants. This number of students differed from the number of students in Pilot

Study II conducted in the first semester because one student moved. Another student,

whose parents both work in research at a major research university, was not allowed to

participate in the study. No reason was given for denying participation. Six females and

nine males participated in the study. Two of the students were Asian. Eleven of the

students cross teamed for geometry instruction. Of the 15 participants, three of the boys

qualified for the gifted program in kindergarten, one female qualified in first grade, two

females and two males qualified in second grade, one female and two males qualified in

third grade, one female qualified in fourth grade, and one male qualified in sixth grade.

The male who qualified in the sixth grade was new to the state and may have been

identified earlier, but this state does not recognize gifted qualification from other states.

One male also qualified as a special education student in the third grade.

Nine of the 15 students’ fathers are pilots for a major airline and have military

backgrounds. One student’s father is a doctor. The other five are successful professionals.

Most of the students’ mothers do not work outside the home. The mothers who do work
outside the home are also professionals. Students are well traveled and have been exposed to various enriching experiences. Two students participate in summer enrichment programs in specific domains offered at major universities throughout the country. Success is expected of them in all of their endeavors both at school and outside of school. Most of the students are highly motivated and academically successful according to grades and standard measures.

The students in gifted geometry maintained a rigorous schedule of classes. Most took a full year of Spanish, German, or French in addition to a gifted English, or gifted science. In addition, students were involved in extracurricular activities both in school and outside of school. These include Beta Club, Academic Bowl, Science Olympiad, Math Team, Student Council, Symphonic Band, and Orchestra. Some of the students played football, basketball, softball, baseball, soccer, and volleyball. Specifically, one male was president of the student council and sat with a major city orchestra. One male won 2nd place at the Science Fair competition. Another male helped our school team take first place in the Academic Bowl regional competition, and he was also involved in Science Olympiad. The president of the student council and two different males represented our school and placed first in a regional math competition. All of the females except two have participated in basketball, softball, or cheerleading. Several were on the volleyball team. Four students were in the school symphonic band. The school symphonic band was the only middle school band in the country invited to a recent major national competition.

As the teacher in this course, I also was a participant. I began this study in my 17th year of teaching mathematics and my 14th year of teaching mathematically gifted
students in geometry in eighth grade. Ten years ago, serving as the school mathematics department chairperson, I helped open the school in which the study took place. I also served as the MATHCOUNTS coach all but 2 years since the school opened, placing at regional competition every year and moving to state competition four of those years. I earned a Bachelor of Science degree in 1990 with a major in middle school education with a concentration in mathematics and science. I received teacher of the gifted certification in 1996 and completed a Master of Education degree in 1998. I engaged in the voluntary process of National Board Certification and in November of 2001 was certified by the National Board for Professional Teaching Standards (NPBTS) in the area of Early Adolescence Mathematics. The requirements involved in the NPBTS process and the experience gained from conducting two qualitative research pilot studies validate my ability to collect, analyze, and interpret the data in this study.

I live in the same county as my students, but outside of the neighborhoods that comprise the school district. I have lived in this county for the last 22 years. My own children attended school in this county. Prior to my family’s return to this area, I lived in New England, up and down the eastern seaboard, on the gulf coast, and in the Pacific Northwest. My family moved 14 times in 10 years. These moves afforded me the opportunity to experience first hand the cultural diversity among different geographic regions of our country. The understanding I acquired of cultural differences among Americans has better equipped me for teaching students from all walks of life and from all areas. These experiences increased my awareness of the various forms of capital (Bourdieu, 1991) operating in the social action of society, the school system, and the classroom.
I am a baby boomer and grew up in an era in which opportunities for women were limited. I have continuously established goals for myself and strived to achieve them. I approach my students with the attitude that if you can dream it, you can do it. I believe in being a role model for my students. I am a wife, a mother, a grandmother, yet also a private pilot of single engine aircraft, an advanced certified diver, windsurfer, marathon runner, student, Sunday School teacher, and piano and flute player. Collectively, these various roles influence my teaching and interaction with the students within the social culture of the classroom.

Instrumentation

In the dual role as teacher and researcher, I was the main research instrument (Merriam, 1998; Lincoln & Guba, 1985). In this capacity, I was central to the research in that I was responsible for constantly checking the significance of observations, interpretations, analysis of data collected reflexively in relation to the goals of my research (Jaworski, 1998), and by verification of my interpretations with my students (Lincoln & Guba). Intersubjectivity refers to shared meanings constructed between the researcher and my students. According to Ball (1990), intersubjectivity adds to the rigor (Denzin, 1978) of the research.

As a human instrument, mistakes and missed opportunities occurred, and my bias influenced the data. As a research instrument, one has the ability to “explore the atypical or idiosyncratic responses to achieve a higher level of understanding” (Lincoln & Guba, 1985, p. 195). In addition, as a human instrument, three characteristics were considered to be advantageous to conduct qualitative research. I dealt with the ambiguity in the research process while being sensitive to each facet of the process by constant
communication with my students (Merriam, 1998). These are entailed in
intersubjectivity. As the classroom teacher, I had an advantage of an established rapport
with my students. Of course, I am aware as the human instrument that the rapport also
was a limitation in that I could have missed something that the relationship assumed.

Instrumentation included the way I collected data. I used a tape recorder for each
group to record their mathematics discussions. The audio recording captured the
mathematical discussion. However, the talk could never capture the equally important
body language, innuendo, mathematical work, calculations, frustration, satisfaction,
confidence, and pride as students contributed, conjectured, defended, debated, and
justified. Meaning often lay in the actual interaction of the students and was recorded by
the participant observer using field notes. For field notes, I used a form I created based on
the idea of a double journal (Bernard, 1995). The form was divided into two columns,
one for descriptive notes during observation, and the other column for reflection on the
observations. These columns also included rows creating large rectangles. This enabled
me to keep the notes for each group separate. I used numbers for the groups for
consistency in note taking and identification. In addition to the reflection section of the
field notes, I used reflection throughout the study to record my own perceptions about
what occurred, or those things that I thought were significant. These took the form of
typed notes that I shared with my students.

Instrumentation also included the written tasks that the students worked on in
small groups. Each of the tasks used were evaluated to require a high-level of cognitive
demand using the MTF (Stein et al., 2000). Tasks with high-level cognitive demand
provide opportunities for students to think and do mathematics in ways that are creative,
in ways that are inventive, and to reason abstractly and generalize (Hekimoglu, 2004; Sriraman, 2003; Stein et al). An introductory task (Appendix C) was used to acquaint students to the audio recorder. The introductory task was: Four goats are tied at the corners of a square field 100 meters on each side. The rope allows each goat to graze an area with a 50 m radius. When three of the goats are removed, the rope tying the fourth goat is lengthened to allow the goat to graze an area equal to the combined area of the four goats. Three additional tasks were used for actual data collection.

The first task referred to as basketball court renovation, a modified version of a similar task, required students to apply area formulas previously studied. The level of cognitive demand of the task according to the mathematical task analysis was procedures with connections (Stein et al., 2000). The task was: The school basketball court needed to be refinished after ten years of use. The students had to devise and implement a plan to determine the area to be painted using school colors and the areas to receive a hardwood finish. Then, students had to assume the court has been refinished and the first ballgame is about to start. Students had to determine the arc length available for each player around the center circle for a jump ball.

The second mathematical task once again involved a fictional remodeling of the school. This time the task represented a mural for the wall in a newly designated math lab. The mural was a circle containing shaded geometric figures representing hair, eyebrows, eyes, nose, and mouth. The context of the task was adapted to students’ previous use of Geometer’s Sketchpad in the study of polygons and circles and our discussion of a much needed math lab of our own. In keeping with the color scheme of school colors, the students had to determine how much blue paint was needed for the
shaded regions and how much black paint was needed for the unshaded regions. The task represented an open-ended task with no explicit pathways for solving. The level of cognitive demand was doing mathematics (Stein et al., 2000).

The third mathematical task used was called Julie’s Wheel (nzmaths, n.d.). The task was: Julie has three bicycle wheels she stacked against a shed. Interestingly, Julie noticed each wheel neatly fits together. Although not included in the problem, I informed the students that the larger wheel was tangent to the middle-sized wheel and the smaller wheel was positioned between the larger and middle-sized wheel so that it is tangent to both wheels. The problem indicated the radius of the largest wheel and the radius of the middle-sized wheel only. Students had to find the radius of the small wheel. Julie’s Wheel represented an open-ended task with no explicit or implicit pathways for solving. The level of cognitive demand was doing mathematics (Stein et al., 2000).

Individual students within the group were given a notebook and an ink pen. Sheffield (2000) suggested gifted students should express what they are thinking even when they think they are wrong. If an idea does not work, they can draw a line to indicate they are starting over. I discussed this process with the students prior to problem solving. The focus of the study was on mathematical thinking and I did not want recording work requirements to obscure mathematical discussion (Sheffield, 1999), or lower the level of cognitive demand (Stein et al., 2000). The notebooks simply allowed students a place to work. The notebook also later served as a journal for students to respond to my observations and thoughts related to the task.
Procedures

Students were informed of the purpose of the research both verbally and in writing. In order to participate in this study, each student had to assent to participate (see Appendix A) and have parental permission (see Appendix B). These documents included a copy of approved research protocol from the Institutional Review Board of the university, permission from the assistant superintendent of schools, and permission from the school principal. A description of data collection activities was included in parental permission and student assent forms. The students’ anonymity is protected by using pseudonyms. Students have access to all transcriptions and written interpretations, and have a voice in reporting data which served as member checking (Lincoln & Guba, 1985).

The research protocol for this study is an updated existing protocol from the pilot study conducted in 2004. The changes to the existing protocol were approved April 6, 2007. Permission was granted to conduct the study with my geometry students by both the assistant superintendent and the school principal in March, 2007. Approved parental permission forms and students’ assent forms were collected by April 19, 2007. The introductory task to get students acquainted with audio recording was implemented on April 20, 2007. Data collection and analysis from implementation of the other three tasks, field notes, student artifacts, student journals, and reflections began on April 25, 2007 and continued through May 18, 2007. I listened to the audio recordings as soon as possible after data collection to correlate with my field notes. The audio recordings were not transcribed and analyzed until after data collection was complete.
In addition to the description of data collection activities included in parental permission and student assent forms, I discussed the research with my students prior to data collection. The students were interested in the research process. What did I mean by high-level cognitive demand? How would the data be analyzed? I explained how I selected the tasks based on the work of Stein et al. (2000) and discussed how the level of cognitive demand of a task can be maintained or can decline. I also shared with the students the work of Wood et al. (2006) and how the categories of mathematical thinking (Dreyfus et al., 2001; Williams, 2000, 2002) were utilized that I planned to use for analysis. Sharing how I planned to use the work of Stein et al. and Wood et al. with my students helped their understanding of the importance of the research and their roles in the research process.

Data collection occurred through participant observation (Merriam, 1998) using field notes, audio recording of group interactions, student artifacts, my notes, and student’s notebook entries in response to my notes. I observed students working on the introductory task and three additional tasks. Each task required students to use the high-level cognitive demand of doing mathematics or procedures with connections as evaluated using the MTF (Stein et al., 2000).

For each of the mathematical tasks, I planned for students to work together in the same collaborative groups established earlier in the year. One student’s parent did not grant permission for participation which necessitated a change. The change caused two close friends to be associated during the introductory task and another change had to be made for the remaining three tasks. The composition of two groups remained intact. The grouping change balanced the remaining two groups with three of the original
participants grouped according to random stratification based on my observations of student performance during the first 6 weeks of school. The students sat in groups of four as indicated in the model of the gifted geometry classroom in Figure 1.

Figure 1: Gifted Geometry Classroom.

An audio recorder was used in each of the four groups to record the mathematical discussions related to the task. I purchased two new tape recorders for the study. I numbered the recorders according to group number. One of the newer recorders was assigned to Group 1 and worked fine for the introductory task and the basketball court task. The audio tape recording for the third task was blank. I assumed the group forgot to turn the recorder on. For the final task, group participants made sure the indicator light was on while discussing the task. Shortly into their discussion, the participants realized the light went off. They reset the recorder and the same problem occurred again. The batteries were replaced and the recorder still did not record. At this point the students and I realized there was a problem with the recorder. Not anticipating a problem, I sent the
students to the library for a tape recorder. Unfortunately, the library recording equipment is antiquated and although the rather large machine appeared to be recording, that tape was blank also. The recording of the mathematical discussion on one entire task and most of the last task for this group was lost.

In addition to the audio recordings, I circulated around the room listening, observing, and interacting when I pressed for justification of student’s conjectures. There were four groups of students simultaneously working on tasks as I took field notes. I watched and listened as intently as possible. I tried to observe from a central vantage point when possible, but often moved to participate in the interaction of the groups.

I listened to the tape recordings as soon as possible after completion of the task and correlated my field notes with the audio recordings. I typed up notes that represented things I noticed in the observation and from listening to the recording for each group. I then gave each student a copy and had them discuss the notes in the same groups. After they had time to discuss my notes, any shared meanings, and their impressions within the group, I had each student write his or her interpretation of the group’s work on the task and their thoughts regarding any shared meanings from group discussion in the notebook. Sharing my interpretations with the students and getting their feedback on those interpretations represented member checking (Lincoln & Guba, 1985). I use the term notebook rather than journal as the notebook took on some significance after one group made a connection to a movie in which the richness of an individual’s life is revealed from a notebook of reflections. The significance of the notebook connection was also shared with the students in class by other students. The audio recordings were not transcribed and analyzed until after data collection was completed.
Time is one factor that can cause the level of cognitive demand to decline (Stein et al., 2000). Based on the pilot study discussed earlier, I allowed two class periods for students to work on each task. Unlike the students in Pilot Study I, the students in most groups did not need the extra day except for Julie’s Wheel. The difference in responses of the students in this study could have been because students were not required to show all of their work as in the pilot study. The focus of this study was on higher-level thinking and not procedures. Showing work, the menial task of recording each step, for gifted students often counters the goal of thinking deeply about the mathematics involved in problem solving and can stifle the creativeness involved in problem solving (Sheffield, 1999). The second day was used for the students and me, the researcher, to share discoveries, conjectures, and interpretations. Students also had an opportunity to discuss my notes on observations and audio recordings in their groups, compare their interpretations and shared meanings. Students then had an opportunity to respond in the notebook. I read and responded to student entries in the notebooks as soon as possible. Students had an opportunity to react to my responses after the next task although they read my responses prior to beginning the next task.

Data Analysis

Constas (1992) argued that as qualitative researchers it is through our eyes that we “make public that which was previously maintained as private in the cognitive, social, and educational lives of the individuals studied” (p. 254) yet we often fail to clearly justify qualitative analysis through public disclosure of the methods used. Anfara, Jr., Brown, and Mangione (2002) also argued the need to publicly disclose methodological rigor and analytical defensibility by opening “the mind of the investigator to his or her
reader” (p.29). During data analysis, I experienced the overwhelming feeling Patton (1990) described involved in the process of making sense of enormous amounts data. I needed to open my mind to the reader. Peshkin (2000) described this process as reflective awareness and suggested the results strengthen the quality of our interpretations.

Data analysis occurred simultaneously with data collected through my observations, student artifacts, and student notebooks (Merriam, 1998). Member checking (Lincoln & Guba, 1985), a method used to ensure trustworthiness of the data, occurred when preliminary interpretations from the audio recordings and observations were correlated, typed up, and shared with my students and students had an opportunity to provide their input regarding those interpretations in their notebooks. I then read each student’s input and responded to each individual in the notebook. These written dialogues between the researcher and the students yielded more data for analysis. I discuss these results in the next chapter.

After data collection, I was faced with the question of how I would deal with the transcription of the audio recordings, an aspect of the research process often overlooked (Tilley & Powick, 2002). The focus of the research was on mathematical thinking so the transcription would have to be precise. Each tape recording captured the discussion that occurred within the small group. This often meant two, three, or four students talking at the same time in addition to background noise of the other groups discussing the same problem. An added dimension of the difficulty in transcription is the diverse ways gifted students process their thinking (Lovecky, 1994). Some students processed through silence while others veered off taking on some character temporarily, or multi-tasked in some other way only to rejoin the mathematical discussion once some idea was resolved.
(Hoyles, 1985). Tilley and Powick argued that transcription is often more translation than transference of talk on tape to text and involves both interpretation and analysis. As the researcher familiar with the participants and having observed the context of the recorded discussions, I chose to transcribe the tapes myself to capture the reality of the recorded discussions as closely as possible. Completing the transcription of the 16 tape recordings took over two months. The investment of time strengthened my awareness of the data and added to the trustworthiness of the data.

Data analysis for the Pilot Study I was purely explanatory related to my students’ work on each mathematical task as outlined by the MTF (Stein et al., 2000). In other words, I told my students’ story through each of the mathematical tasks used to elicit mathematical thinking. For this study, I chose to provide further evidence of understanding from the mathematical thinking generated by students while working on the mathematical tasks in addition to the richness of a descriptive account. A conceptual framework utilized by Wood et al. (2006) in their research on children’s mathematical thinking within a group appeared to match my needs, and the theoretical assumptions underlying their research fit identically with the social constructivist perspective of this study. As an added attraction, the framework was derived in part from the work of Krutetskii (1976).

Wood et al. (2006) studied mathematical thinking that occurs as a result of social interaction in both traditional classrooms and what they termed reform classrooms. Instructional practices within the reform classrooms were based on NCTM recommendations. The reform classrooms were further subdivided into strategy reporting or inquiry/argument with the differentiation suggested in the descriptions. Students in the
strategy reporting classrooms simply reported their solutions to problems. The focus of the inquiry/argument classroom was on understanding through interactive participation and discussion. My research was similar to the inquiry/argument classroom because the focus was on mathematical understanding elicited through interaction as students worked together on open-ended mathematical tasks.

The genesis of the conceptual framework used by Wood et al. (2006) to categorize mathematical thinking started with the work of Williams (2000). Williams first created a framework to classify cognitive activities of students during problem solving based on Krutetskii’s (1976) work and Bloom’s (1956) taxonomy.

Williams then integrated the cognitive taxonomy within the three observable epistemic actions that Dreyfus et al., (2001) claim occur during the cognitive activities of abstraction or generalization. The three observable cognitive activities included recognizing, building-with, and construction. These categories were further subdivided into recognizing comprehending, recognizing applying, building-with analyzing, building-with synthetic-analyzing, building-with evaluative-analyzing, constructing synthesizing and constructing evaluating (Williams, 2002). The resulting framework provided a way of categorizing mathematical thinking with observable cognitive activity. Williams (2002) suggested a more appropriate description of the categories was “nested categories of increasing intellectually complex activity” (p. 2). Earlier, Hershkowitz, Schwarz, and Dreyfus (2001) referred to the essential components in their “model for the genesis of abstraction” (p. 195) as the three “dynamically nested epistemic actions” of constructing, recognizing, and building-with.
Wood et al. (2006) coded transcribed videotapes to analyze students’ mathematical thinking according to the dimensions of mathematical thinking. Connections between the categories of thinking, examples of cognitive activity, and examples of students’ discussion were presented in a chart\(^4\). The first column included the integrated categories of the cognitive taxonomy with the observable actions from Dreyfus et al. (2001). The second column provided examples of cognitive activities associated with mathematical thinking as categorized by Dreyfus et al.; Krutetskii (1976); and Williams, (2000). The third column included examples of mathematical thinking taken from the transcribed lessons. I planned to modify Wood, Williams and McNeal’s categories of mathematical thinking to include characteristics of the MTF (Stein et al., 2000) for each level of cognitive demand and use a similar coding scheme. Coding did not go according to the plan.

Constas (1992) argued that documenting category development procedures adds to the integrity of the research. Documenting category development in essence provides an audit trail (Lincoln & Guba, 1985). The coding process came to a halt as quickly as it began. Coding just the mathematical thinking lacked a “logical connectedness” (Constas, 1992). My research differed from the inquiry/argument classroom of Wood et al. (2006) in that groups of gifted students involved in my research were problem solving without specified roles for individuals, rules for speaking, or expectations of reporting. There was no movement from small groups to whole group and I interacted with the students in

\(^4\) Wood, Williams, and McNeal (2006) presented the categories of mathematical thinking, examples of associated cognitive activity, and corresponding examples of student’s mathematical discussion in Figure 1, pp. 231-232. I used the first two columns, categories of mathematical thinking and examples of cognitive activity, in the fourth reanalysis of my data to code my students’ mathematical discussions in addition to the examples of cognitive activity with a list of descriptors I created from the work of Hershkowitz, Schwarz, and Dreyfus (2001).
the role of participant observer, not authoritarian teacher. Goetz and LeCompte (1984) described the essence of ethnographic research as the “holistic depiction of uncontrived group interaction” (p. 51). I selected the categories of mathematical thinking used by Wood et al (2006), a working model based on prior qualitative research (Dreyfus et al., 2001; Williams, 2000), but had to consider the functional relevance, the relationship of the parts to the whole, of what was occurring in the problem-solving discussions (Erickson, 1977).

Few of the interaction patterns described by Wood et al. (2006) could be used to describe the interaction I observed, or the interaction captured through the audio recordings of my students working together. I turned to the work of Pirie (1998) and Pirie and Schwarzenberger (1988) related to mathematical discussion and mathematical understanding between students. Using discourse analysis (Sinclair & Coulthard, 1975) Pirie and Schwarzenberger allocated the episodes of talk of students to three categories: talk clearly related to mathematics, verbal exchanges that were incomprehensible, and social chat. Next they used a constant comparative method of analysis (Glaser & Strauss, 1967), “repeated reanalysis of existing data” (p. 90) to refine the categories. Pirie’s use of repeated reanalysis of existing data was attractive because they continued to return to the data after some insight. I was concerned about constant comparative method of analysis as Glaser and Strauss intended because I was not attempting to derive theory. Lincoln and Guba (1985) suggested the researcher must be cognizant of this difference before adapting the constant comparative method for data analysis in a naturalistic study. With this in mind, I chose to implement the same process as Pirie and Schwarzenberger.
In the first coding, I separated the discussions into three categories: social chat, mathematical talk of the students, and my interaction with the students. Like Pirie and Schwarzenberger (1988), I recognized the social chat in the discussions does not inhibit mathematical discussion and in some ways was necessary for students’ continued engagement with the task. Saul (1999) suggested “students with high mathematical ability take in stimuli, react to them, adjust their reactions, and find places to rest and enjoy what they have thought about” (p. 83). Erickson (1977) suggested that making sense of the sometimes outrageous behavior is the “tour de force” (p. 61) of the ethnographer.

I returned to the data the second time to determine categories related to the interactions involved in problem solving, in other words, the context of the discussions. I made a notation for each line of the discussion such as understanding the task, discussing strategies, implementing the strategy, exploring, explaining, arguing, agreeing, justifying, inquiring, sharing, checking and others. I then made a bulleted list of each comment on three self-stick poster sized post-it sheets and put them on the wall for studying. I reduced (Marshall & Rossman, 1989) the bulleted lists to one page with four overarching categories and two related categories. Within these categories I made bulleted lists of the function of each statement or question. I realized the students’ powerful use of inquiry for clarification, justification, verification, and understanding yielded powerful statements of explanation, argument, challenge, agreement, transition, exploration, extension, understanding, and verbalizing the math in response to the inquiry. The overarching categories were understand the task, strategy, doing the math, checking, and two sub-categories of processing and decline. The sub-category of decline was only applicable for the last task, Julie’s Wheel. Processing appeared to occur through silence, social chat, and
sometimes students taking on a character. Hoyles (1985) indicated the cognitive function of talk includes the period of silence as students process the thinking of others in terms of their own thinking. Throughout the study, I was overwhelmed by the constant use of what Pirie (1998) termed “collaborative checking” (p. 93) in all phases of the discussion.

I returned to the data for a third time using the reduced list. From the reduction of the data and reflection about my interpretations of the purpose of students’ inquiry, statements and constant collaborative checking, more changes were necessary. This represented the repeated reanalysis of existing data (Pirie, 1998). As the students worked on the tasks, they moved back and forth from collaboratively understanding the task (UT), collaboratively discussing strategy or implementing strategy (S), and collaboratively doing the math (DM). I recognized the same types of statements and questions were used throughout each phase as a means of collaborative checking as well as to move the work forward. Students argued, agreed, clarified, explained, justified, verified, challenged, inquired, corrected, verbalized doing the math, extended ideas, and gave answers. I visualized this process as an equilateral triangle inscribed in a circle where the vertices represented the phases of understanding the task, discussing strategy, and doing the math and the circle represented collaborative checking, the means by which students verified processes and solutions as well as moved work back and forth (see Figure 2). After the third reanalysis, the data was now segmented (Tesch, 1990) into phases of interaction and each line of the discussion coded according to its function.

During the fourth reanalysis, the levels of mathematical thinking exhibited by my students were coded (see Figure 3) using the integrated categories of cognitive activities with the three epistemic actions involved in abstraction (Dreyfus et al., 2001) and used in
Doing the Math

Understanding the Task

Collaborative Checking

Strategy

Figure 2: Phases of interaction during group problem solving.

the work of Wood et al., (2006). I used the associated cognitive activities used by Wood et al. (2006) as indicators of specific levels of mathematical thinking (Dreyfus et al., 2001; Williams, 2000) supplemented by a list I created from the work of Hershkowitz, Schwarz, and Dreyfus (2001). I created a list of characteristics, or observable actions, of recognizing, building with, and constructing which students may exhibit when problem solving from the descriptions used in their work. This additional list of descriptions (see Table 1) helped me differentiate which levels from the framework utilized by Wood et al. (2006) correlated with the verbal expressions of my students. I also noted in each segment whether the level of cognitive was maintained or declined according to factors from Stein et al. (2000).
Table 1

*Mathematical Thinking Descriptions*

<table>
<thead>
<tr>
<th>Mathematical Thinking</th>
<th>Descriptions</th>
</tr>
</thead>
</table>
| Recognizing          | Students realize what they know is part of the task.  
                       | Students simply adapt what they know to the task.  
                       | Students may describe, explain, interpret, compare, report,  
                       | or classify. |
| Building-With        | Recognizing is nested in building-with. Students use what  
                       | they know in different ways to reach a goal. Occurs when  
                       | engaged in problem solving especially understanding the  
                       | task and determining strategy. Building-with involves  
                       | application and may include the use of rules and theorems  
                       | as well as the use of previously constructed artifacts.  
                       | Students may build-with when the teacher provides a hint.  
                       | Building-with may be used incorrectly. |
| Constructing         | Recognizing and building-with are nested in constructing.  
                       | When students use what they know to build more complex  
                       | meanings. Constructing involves reorganizing, integration,  
                       | and refinement and may be nested over several activities. |

Note. The three observable epistemic actions and descriptions of recognizing, building-with, and constructing were drawn from the work of Hershkowitz, Schwarz, and Dreyfus (2001) on abstraction as an activity undertaken by a group in context.
<table>
<thead>
<tr>
<th>Mathematical Thinking</th>
<th>Code</th>
<th>Examples from problem solving discussions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recognizing</td>
<td>RC</td>
<td>Task: Smiley</td>
</tr>
<tr>
<td>comprehending</td>
<td></td>
<td>Chad: So we determine the paint in the shaded areas? Terry: We have to find how much paint is going to be for each area.</td>
</tr>
<tr>
<td>Recognizing</td>
<td>RA</td>
<td>Sally: If we find the area of the big circle, then we could subtract to find these.</td>
</tr>
<tr>
<td>applying</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Building-with</td>
<td>BWA</td>
<td>Terry: Guys so if the radius is 8 and this is 8, then that means these two sides are 8 so the big triangle is equilateral. Chad: So that means all of these triangles (6 embedded triangles) are congruent.</td>
</tr>
<tr>
<td>analyzing</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Building-with</td>
<td>BWSA</td>
<td>Terry: If you split this right here, then you can use the Pythagorean Theorem. It’s split in half these two ways too which means these lines are each 6.9 long. Chad: Each of these triangles are the same. Terry: If you flip this, you have three. Chad: You draw a line straight down the center you get all the right triangles that are given are equal. The white triangles given are all the same. Terry: This line splits them in half. So this is a right triangle that goes to that corner, so all six triangles have to be the same.</td>
</tr>
<tr>
<td>synthetic-analyzing</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Building-with</td>
<td>BWEA</td>
<td>Introductory Task</td>
</tr>
<tr>
<td>evaluative-analyzing</td>
<td></td>
<td>Chad: It’s 100 exactly.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Sally: How did you get it? Explain.</td>
</tr>
<tr>
<td>Constructing</td>
<td>CS</td>
<td>Chad: I plugged in numbers until I got it right. OK, watch. 100 squared times pi, then divide by 4. What number is that?</td>
</tr>
<tr>
<td>Synthesizing</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Nested)</td>
<td>BWEA</td>
<td>Sally: How do you know its going to be ¼ of the circle?</td>
</tr>
<tr>
<td></td>
<td>CS</td>
<td>Chad: Because this is 90 right there, then you have another 90, another 90, and another 90. That’s 360.</td>
</tr>
<tr>
<td>(Nested)</td>
<td>BWEA</td>
<td>Terry: Then it would be. Yeah. Sarah: (doing the math) So the rope will be 100 meters.</td>
</tr>
<tr>
<td>Constructing</td>
<td>CE</td>
<td>Terry: That would be the radius, then times 2, all that divided by 4. Would that be the formula? Always? I know how he got it. I found the formula.</td>
</tr>
<tr>
<td>evaluating</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Note: Column 1 represents the integration of the three epistemic actions of abstraction from Dreyfus, Hershkowitz, and Schwarz (2001) and Williams’ (2000) categories of cognitive activities used in the work of Wood et al. (2006) and referred to later as “nested categories of increasingly intellectually complex activity” (Williams, 2002, p. 2). Abbreviations in column 2 represent codes used in the fourth reanalysis of data. Examples of problem solving discussions in column 3 correspond to the code from column 2 used in data analysis.

*Figure 3.* Codes and examples for levels of mathematical thinking.
The analysis using the framework utilized by Wood et al. (2006) could have added a quantitative element to the analysis as occurrences of mathematical thinking in each category could be counted and expressed as percentages as in their study. Utilizing quantitative methods within a qualitative research methodology is perfectly acceptable (Ernest, 1998b). In this study, to do so would have removed the data from the social context that it was taken.

After the analysis of the transcribed audio recordings was complete, I analyzed the 75 student artifacts using characteristics of the MTF (Stein et al., 2000) to determine if high-level cognitive demand was sustained. I used interpretative analysis for observation field notes and typed notes presented to students. I connected the interpretative analyses with the analysis of the audio recording. Lastly, I used my students’ reflections to compare my interpretations of the social interaction during problem solving and the resulting mathematical thinking and impact on student understanding.

Trustworthiness of the Data

Lather (1986) suggested that value free research is impossible and the researcher should openly admit the researcher’s values influence the research. Lather (2001) advocated “work that attests to the possibilities of its time yet, in the very telling, registers the limits of itself as a vehicle for claiming truth” (p. 486). The interpretations and conclusions drawn from my study offer the reader an opportunity to step inside my classroom, sit among my students, listen to their discussions, and draw their own conclusions regarding my research. While the researcher may not be neutral, the data must be neutral to protect the evidence through a system of credibility checks (Lather,
1986). To ensure the trustworthiness of the data, Lincoln and Guba (1985) list four criteria for evaluating the trustworthiness of a qualitative study: credibility, transferability, dependability, and confirmability. A study deemed credible, is also dependable (Lincoln & Guba). Lather emphasized face validity and catalytic validity in addition to triangulation and reflexivity.

Credibility correlates to internal validity (Lincoln & Guba, 1985). Several actions within the research process strengthened the internal validity of this study. As a research instrument and participant observer, I was immersed in the context of the classroom for long periods of time (Lincoln & Guba, 1985; Merriam, 1998) observing and recording data in field notes. Credibility was also increased by triangulating (Lincoln & Guba; Lather, 1986) the data reflected in the mathematical thinking framework, the MTF, field notes, reflective notes, student reflections, and student work to “build coherent justifications” (Merriam, p. 196) for the interpretations and conclusions of the study. Lather (1986) stated triangulation includes “multiple data sources, methods, and theoretical schemes” (p. 15). Triangulation was also criteria for ensuring confirmability.

Member checking entails verifying accuracy of conclusions with participants (Lincoln & Guba, 1985; Merriam, 1998). Interpretations and conclusions were shared with my students for their feedback and impressions throughout data collection and analysis. Lather (1986) referred to returning to the participants with “tentative results” (p.16), acquiring their feedback to amend or refine your interpretations, as face validity. Lastly, a colleague and former team member familiar with my teaching context with an Ed.D., and with whom I have a rapport established through simultaneous National Board certification process and doctoral programs, served as a peer examiner (Lincoln & Guba).
Transferability correlates to external validity. I collected detailed descriptive data. I used rich, thick description (Geertz, 1973; Merriam, 1998) to provide “illustrative and generative” (Ernest, 1998b, p. 34) experiences of my students allowing readers to live vicariously thus judging the transferability. In addition to rich descriptions of the experiences of my students, I included descriptions of the focus of the study, the background of the participants, the school they attend, the teams they are on, the activities in which they are involved, and the community in which they live to provide the context for judging compatibility. I included my role as a participant observer, and the relationship to my students, as well as the overall operation within the context of my classroom. All the data collected, 75 student artifacts, numerous student reflections, field notes, reflective notes, 16 audio tapes, and data generated as a result of data analysis, transcriptions, category development notes and posters were maintained in their original form. Maintaining all data collected in all phases of the research along with the detailed steps taken during data analysis, or audit trail, strengthens the dependability of this study. The audit trail ensured confirmability and facilitates potential replication of this study.

Catalytic validity refers to the transformative nature of the research process for both the researcher and participants (Lather, 1986). This study was illuminating and transformative for me, the researcher and teacher, in that through the research process I became acutely aware that “we often do not know what we are seeing, how much we are missing, what we are not understanding or even how to locate those lacks” (Lather, 2001, p. 486). In this respect, the research process was a self-awakening process for the researcher and teacher. This reflexivity also added to the credibility of the study. The participants, as evidenced from their reflections were transformed in the respect that they
enjoyed the challenging experiences and prefer this type of learning, but are too young and too accustomed to the status quo to advocate learning strategies that may help them realize their potential in the future.

Summary

In this chapter, I first provided a rationale for selecting ethnographic case study as the methodology for this study and then discussed the philosophical connections between the theoretical framework and the methodology selected. Ernest (1998b) described educational research methodology as a blend of the theoretical framework and the assumptions of how the world is viewed. Social constructivism, a theory that suggests the individual comes to know by using newly constructed knowledge gained through social interaction to amend, refine, or add to existing knowledge (Cobb & Yackel, 1996; Ernest, 1998b), framed this research. I examined how student understanding was affected by working within small groups on open-ended problems and described the levels of mathematical thinking involved. Because the focus was on the interaction of the students through my observations in my classroom, a sociocultural environment, ethnography was an appropriate methodological approach.

Wolcott (1999) maintained the distinguishing feature between ethnography as a methodology and ethnographic techniques is first hand experience within a social or cultural context. In keeping with Wolcott’s distinction, the natural setting of my classroom provided the social context for this research. Ethnography represents the holistic depiction of uncontrived group interaction. Next, I discussed the characteristics of natural inquiry that justified my use of ethnographic case study to better understand how students demonstrate mathematical understanding.
The geometry class used in this study was bounded by time and activity, which represented a bounded unit, a characteristic of a case study (Stake, 1995). I viewed case study more as a subsystem of ethnography. Shaw (1978) described one possible focal point of case study as how subjects will approach problems. My gifted geometry students were the focal point for this research. The collaborative small groups were sub-cases embedded within the larger case.

The discussion of two previous pilot studies included the purpose, participants, instrumentation, procedures, data analysis, findings and conclusions, and how the results impacted this study. The purpose of Pilot Study I was to construct knowledge of investigative teaching needed in an ethnographic study. Results of the pilot study were used to make changes in implementation of the mathematical tasks, the interaction of the researcher with the students, in data collection, and data analysis. One area of weakness revealed in the first pilot study concerned grouping practices. A second pilot study was conducted during the fall semester of 2006 with the geometry students who served as participants in this study as a result of the grouping problems encountered in the first pilot study. I investigated the efficacy of grouping strategies within my homogeneously grouped geometry class on achievement and cohesiveness among the students. The resulting grouping decisions were used in the main study.

Next, I discussed the setting of the study followed by the data sources. The study was conducted at a middle school in an affluent suburban county outside a large city in the South. Students attending the middle school are predominantly white with small populations of African American, Asian American, and Hispanic American students. Students in middle school are separated by grade level and then divided into teams of
students. Unlike high school, students in eighth grade, study a full year of geometry with the same teacher. Fifteen of the original 17 gifted students enrolled in my geometry class served as the participants. As a participant observer, I also served as a data source.

I provided background regarding my 15 years of experience as a teacher of gifted geometry as well as other information that influenced my role as a research instrument. Next, I discussed the instrumentation and procedures used to collect data. I used a tape recorder for each group to record their mathematics discussions. For field notes, I used a form I created based on the idea of a double journal (Bernard, 1995). My reflections took the form of typed notes that I shared with my students.

Students worked in small groups established earlier in the year through random stratification. The random stratification was based on my observations of performance during the first 6 weeks of school. Each of the tasks required high-level cognitive demand as evaluated by the Mathematical Tasks Framework (Stein et al., 2000). An introductory task (see Appendix C) was used to acquaint students to the audio recorder. The first task was referred to as basketball court renovation (see Appendix D). The level of cognitive demand of the task according to the mathematical task analysis was procedures with connections (Stein et al., 2000). After ten years of use, the court needed to be refinished. Students had to determine the area to be painted using school colors and the areas to receive a hardwood finish. The second mathematical task once again involved a fictional remodeling of the school (see Appendix E). This time the task represented a mural for the wall in a newly designated math lab created by a geometry program. The mural was a circle containing shaded geometric figures representing hair, eyebrows, eyes, nose, and mouth, referred to as smiley. The students had to determine
how much blue paint was needed for each feature and how much black paint was needed for the remaining face. The task represented an open-ended task with no explicit pathways for solving. The level of cognitive demand was doing mathematics (Stein et al., 2000).

The third mathematical task used was called Julie’s Wheel (nzmaths, n.d.). The task (see Appendix F) was: Julie has three bicycle wheels she stacked against a shed. Interestingly, Julie noticed each wheel neatly fits together. Students had to find the radius of the small wheel. Julie’s Wheel represented an open-ended task with no explicit or implicit pathways for solving. The level of cognitive demand was doing mathematics (Stein et al., 2000).

Next, I described data collection including the problems encountered. Constas (1992) argued for public disclosure of methods used. I took the reader through the account of how my plan for data analysis changed and the subsequent transcription and coding decisions, and how they were implemented. I first drew on the data analysis methods in the work of Pirie and Schwarzenberger (1988) on mathematical discussion for category development. On the fourth reanalysis, I drew on the work of Dreyfus et al. (2001), Hershkowitz et al., 2001, Williams (2000), and Wood et al. (2006) for categorizing levels of mathematical thinking based on observable cognitive activity. I also coded maintenance or decline of cognitive demand (Stein et al., 2000).

In the last section, I described the four criteria for trustworthiness (Lincoln & Guba, 1985) of a qualitative study and the steps used to ensure the trustworthiness of the data. In addition to these four criteria, I discussed face validity, catalytic validity, and reflexivity (Lather, 1986).
CHAPTER 4
DATA REPORTING

The purpose of this study was to examine the ways mathematically gifted students think about and do mathematics creatively as indicators of deep understanding while working collaboratively on open-ended tasks with a high-level cognitive demand. The questions that guided this study were: How is mathematical understanding of 8th-grade gifted geometry students elicited through exploration using open-ended problems? What levels of mathematical thinking do 8th-grade gifted geometry students demonstrate when engaged in collaborative problem-solving tasks with high-level cognitive demand? I begin this chapter with a brief overview of issues pertinent to understanding the results of this study. These include the diversity of gifted students and mathematical discussion, the Mathematical Tasks Framework (MTF), and mathematical thinking. Next, I provide a description of how the data is reported within the ethnographic methodology to best represent the “holistic depiction of uncontrived group interaction” (Goetz & LeCompte, 1984, p. 51) operating within the classroom. Following the method of reporting is an introduction to the group participants. I then describe each of the group’s problem-solving experiences within each task. I conclude the discussion of each task by comparing and contrasting the group experiences within the task. I close the chapter with a summary of the data reported.
Overview of Issues Pertinent to the Results

Diversity and Mathematical Discussion

Although students were grouped according to results from Pilot Study II on the efficacy of grouping strategies on achievement and cohesiveness, the reader must keep in mind that the groups are composed of individual gifted students and how vastly different gifted students can approach the same problem (Span & Overtoom-Corsmit, 1986).

Generally, once an idea was verbalized, the students would pursue that idea. Pirie and Schwarzenberger (1988) termed the mathematical talk that occurs when another student picks up an idea put forth by another student as interaction. Pirie and Schwarzenberger included interaction as a component in their definition of mathematical discussion. I used the phrase “picked up” rather than interaction. The phrase implies “critical listening has taken place” (Pirie & Schwarzenberger, p. 461). As the students worked, their interactive talk often allowed them to complete each other’s statements and functioned as collaborative checking (Pirie, 1998).

Mathematical Tasks Framework (MTF)

The MTF includes three phases. The first phase is analyzing the level of cognitive demand of the task as it appears in instructional materials. Tasks with lower-level cognitive demand involve memorization and procedures without connections to meaning. Tasks with higher-level cognitive demand include procedures with connections to conceptual ideas or doing mathematics (Stein et al., 2000). The second phase involved the set-up of the task by the teacher followed by third phase, implementation by the students. The tasks as instructional materials for this study required higher-level cognitive demand.
**Mathematical Thinking**

Williams (2000) first created a framework to classify cognitive activities of students during problem solving based on Krutetskii’s (1976) work and Bloom’s (1956) taxonomy. Williams integrated the cognitive taxonomy within the three observable actions that Dreyfus et al. (2001) claimed occurs during the cognitive activities of abstraction or generalization. The observable cognitive activities included recognizing, building-with, and construction (Dreyfus et al.). These categories were further subdivided into recognizing comprehending, recognizing applying, building-with analyzing, building-with synthetic-analyzing, building-with evaluative-analyzing, constructing synthesizing, and constructing evaluating (Dreyfus et al., 2001; Williams, 2000; Wood et al., 2006). Williams (2002) later suggested a more appropriate description of the categories was “nested categories of increasing intellectually complex activity” (p. 2). The resulting framework provided a way of categorizing mathematical thinking with observable cognitive activity and was used in a study by Wood et al. (2006). I drew on the work of Dreyfus et al., Hershkowitz et al. (2001), Williams, and Wood et al. to categorize the levels of mathematical thinking exhibited by my students through their discussion.

**Method of Reporting**

The categories that emerged through data analysis of the transcribed mathematical verbal exchanges included understanding the task, strategy, and doing the math. While one of these categories was apparent in most segments, the students sometimes exhibited an immediate understanding (Krutetskii, 1976), or intuitiveness (Sfard, 1991) in obtaining and processing mathematical information. Collaborative checking (Pirie, 1998)
permeated the mathematical discussion through all phases of problem solving. Although the categories that emerged from the data resemble Polya’s (1945) problem-solving heuristic and Lester’s (1985) modified model of Polya’s work which includes orientation, organization, execution, and verification, the categories in this study were not sequential phases. Data analysis of the verbal exchanges indicated students moved back and forth through phases of problem solving as needed to understand the task, and to refine, or amend their previous knowledge (Cobb & Yackel, 1996) to construct new knowledge. Consequently, the categories often overlapped.

Delineation of discussion of field notes, my notes to students, student artifacts, student responses, and the mathematical discussions would counter the ethnographic methodology. Goetz & LeCompte (1984) described the essence of ethnographic research as the “holistic depiction of uncontrived group interaction” (p. 51). Reporting follows the segments of discussion derived from the data analysis of the each group’s transcribed audio tapes in order to preserve the functional relevance, the relationship of the parts to the whole, of what was occurring in the problem-solving discussions (Erickson, 1977). The discussion was segmented according to the phases of interaction. In the descriptions of the experiences, I often used the terminology of the indicators that provided the evidence of specific levels of mathematical thinking from previous research. The citation behind the phrase indicates the source of that indicator. I weaved the analysis and interpretation of my field notes, student work, and my reflections, or notes to the students into the reporting of the transcribed segments to better depict the whole. After a review of the context of the mathematical task, I presented each group’s results nested within that task. Conducting qualitative research, like problem solving, is reflexive. I concluded the
presentation of the results for each group related to the basketball court task and the smiley task with a look back, using the students’ reflections. The lack of student reflection for the introductory task and Julie’s Wheel is explained in the presentation of the results. I then compared and contrasted the groups’ experiences within each task.

Meet the Group Participants

**Group 1: Ethan, Kate, Mike, and Tom**

Ethan joined Group 1 after the introductory task as a result of the group changes discussed earlier. Ethan was well rounded academically, athletically, and socially. He played football on the 8th-grade team and was on the championship wrestling team. Kate was more intuitive, but sometimes deferred to others. Her grades were not always indicative of her ability. She cheered for the 8th-grade basketball team and pursued other extra-curricular activities outside school. Mike was very literal and analytical. He persisted until he understood a concept. Mike was a competitive swimmer with the goal of receiving a swim scholarship to put him through engineering school. He aspires to become an engineer and design roller coasters. Mike received the principal’s award in the 8th-grade for his commitment to academic excellence, his exemplary character, and for his community service. Tom was a phenomenal reader and writer in addition to his mathematical ability. Highly motivated, and diligent in his work ethic, Tom earned high averages in every class. As a result, Tom often won grade-level academic awards. Tom recently earned the honor of Eagle Scout. Sadly, Tom’s father died during his 7th-grade year and Tom was an only child. His mother, a Ph.D., worked in the airline industry. She and Tom frequented Europe. Tom was the only student in this group who qualified for the gifted program in kindergarten.
Group 2: April, Chad, Sally, and Terry

April was highly motivated, meticulous in her work, and enjoyed the challenge of learning new things. She played 8th-grade basketball and volleyball. April cross teamed for geometry and was a leader on her team. Although Chad was gifted, he also received services from special education to assist organization and processing problems. Chad was intuitive, insightful, and possessed a keen sense of humor. Chad also exhibited characteristics of the absent minded professor without organization. If he did homework, he usually could not find it. His handwriting was poor and his spelling worse. These trappings did not impact his grade. Cognizant of the dimensions of Chad’s giftedness, I only assessed Chad on mathematical knowledge. We mutually termed his insight, “a Chad thing.” His peers often saw him as the absent minded professor or the nutty professor and could be in part because his insight did not align with their more logical sequence of thinking. His trademark was wearing tee-shirts with a pun related to intelligence. Chronologically, he was the youngest student in the class. Sally like April, was highly motivated and a diligent worker. She set extremely high goals for herself and would do what it took to achieve them. Sally was a phenomenal reader and often volunteered to read in class. She cheered on the 8th-grade squad football squad, played in the band, and pursued extra-curricular activities such as karate; Sally was a black belt. Sally also was an academic award winner. Terry was a fraternal twin. His twin was his only sibling. Terry attended literary camps during his summers at a major northeastern university. Excellence was expected of him. He won academic awards throughout middle school. During this study he was notified he was selected nationally as a Promising Young Writing by the National Council of Teachers of English. In addition to his
academic success, Terry sang in the chorus and was an All-State Chorus participant. Terry was the only student in this group who qualified for the gifted program in kindergarten.

*Group 3: Amy, Bruce, Joey, and Karol (Aka: Asian Sensation and Two White Kids)*

Amy, one of the Asian students, named the group. Amy was bright, creative, and hard working. She maintained a full schedule. She cheered during football season, played basketball during the winter, and played volleyball in the spring. Her demanding schedule included chorus, student government, and high involvement in Spanish activities. Amy’s vivaciousness offset her sometimes domineering attitude in the group. She loved challenging tasks. Bruce could be described as the quiet thinker. He quietly pondered, questioned when he needed information, and often conjectured. Bruce helped our school Academic Bowl team achieve the regional championship. He also participated in Science Olympiad at the state and national levels. He orchestrated the behind-the-scenes technology for the morning broadcast. Bruce moved into the area during his 6th-grade year and qualified in the gifted program during that year. Joey was also Asian. He served as president of student government, participated in Cold Pizza, a Christian organization, and often appeared on the morning broadcast. Joey was Daren’s (group 4) counterpart in the secret service routine as well as other routines. Joey also represented our school at regional mathematics competition. Outside school, he played the violin and sat with a major symphony orchestra. Karol was extremely bright, quiet most times, and meticulous in her work. She possessed a quiet laugh that punctuated most statements and questions. Karol willingly helped others when asked for help or for an explanation. Others viewed her as an authority for answers to homework. She sang in the school chorus.
Group 4: Bob, Daren, and Rita

Bob played on the 8th-grade football team and wrestled. He also participated in student government and Cold Pizza. Bob had an extremely high intellect, but achievement was not his greatest concern. Bob represented the school in regional mathematics competition. He was well rounded academically, athletically, and socially. Bob routinely portrayed some character and performed quite animatedly. Through all of his bravado, Bob showed great compassion to others. He possessed a keen desire to understand, not to achieve grades, but to add to his repertoire of knowledge. In addition, Bob was perceptive and insightful. He was the athletic, social version of Chad from Group 2. Rita was the only girl in this group. She previously attended private school and was new to our school and public education. Rita was hardworking, a deep thinker, and would not be intimidated by males. She took an active role in Spanish and sang in the chorus. Daren was the quarterback for the 8th-grade football team. He served as an officer in student government, participated in Cold Pizza, represented the school on the mathematics team, and regularly appeared on the morning broadcast. Daren also participated in the drama club. Like Bob, he arrived to class most days as a different character and sometimes in the literal sense. One day Daren showed up in a black suit complete with the ear piece and wire for his secret service routine. He too had a keen sense of humor. Daren qualified for the gifted program in kindergarten in problem solving and critical thought.
Mathematical Tasks: Group Problem-Solving Experiences

*Introductory Task*

The introductory task involved four goats tied at the corners of a square field 100 meters on each side. The rope allowed each goat to graze an area with a 50 m radius. When three of the goats were removed, the rope tying the fourth goat was lengthened to allow the goat to graze an area equal to the combined area of the four goats. The level of cognitive demand according to the MTF (Stein et al., 2000) was doing mathematics.

The introductory task was intended only to acquaint students with the recording process. Consequently, I did not summarize my observations, thoughts, or make notes on the audio recording for students as outlined for data collection with the other three tasks. The levels of mathematical thinking evident through the transcribed audio recording, student artifacts, and my field notes related to this task revealed creative thinking that was “original, fluent, flexible, and elegant” (Sheffield, 1999). The purpose of this case study was to examine the ways mathematically gifted students think about and do mathematics creatively, so I decided these results warranted inclusion in the discussion. I present these finding first to preserve the relationship of the parts to the whole picture of my students at work. Reporting follows the same method as discussed earlier with the exception of the conclusion.

*Group 1.* The composition of Group 1 for the introductory task included only three students rather than the originally planned group of four students due to the nonparticipation in the study by one student. The original students in this group were Kate, Mike, and Tom.
Kate read the problem to the group. The first segment of the discussion focused on understanding the task. Mike initiated and led this first segment by stating what they know about the problem interspersed with statements followed by a single word of inquiry like “right?” Tom and Kate either agreed or continued to understand the problem with similar statements or inquiry that served as a collaborative check (Pirie, 1998). The 12 short verbal exchanges involved what they know, inquiry, agreement, self-correction, and insight. After this brief discussion, Kate demonstrated her insight by stating the goats could go out 100 m. Mike verified his understanding of what Kate said with a drawing and stated, “So they can go like that.” The levels of mathematical thinking during this segment were recognizing comprehending, building-with analyzing and building-with synthetic-analyzing. Kate used building-with analyzing when she applied previous knowledge of mathematical procedures (Dreyfus et al., 2001). Mike’s thinking exhibited building-with synthetic-analyzing while discussing his use of a pictorial representation (Williams, 2000). A short period of processing through silence followed this first segment. The level of cognitive demand of doing mathematics (Stein et al., 2000) was maintained.

Leading the second segment, Kate suggested a strategy, but immediately moved to doing the math. Kate told the group to find the area of a circle with a radius of 50 m. Tom thought they needed to find the area of the square first. Mike did the math to find the area of a circle with a radius of 50 m and Kate concluded that was how much area the goats could cover, meaning how far they could graze. Although Tom continued to discuss the dimensions of the square causing Mike to question what they were trying to find, the
level of cognitive demand was not lowered. A brief silence occurred following this exchange while the students processed their thinking.

The second segment involved 12 short verbal exchanges consisting of inquiry, giving answers, and verbalizing what they did. The level of mathematical thinking exhibited in this segment was building-with analyzing. Students applied what they knew to the new situation (Dreyfus et al., 2001). The level of cognitive demand was maintained. Mike’s recognition that his drawing may not be correct indicated his level of mathematical thinking was building-with evaluative-analyzing. Coupled with making the simple complex, some gifted students need precision (Lovecky, 1994). Considering Mike’s drawing, Kate remarked, “There is an ungrazed portion in the center of the field. No goat can go there.” At this point there was no indication that Mike actually changed his thinking about how the goats were tied. This brief exchange served as a transition between phases of interaction.

During the third segment, Tom picked up on Mike’s idea (Pirie & Schwarzenberger, 1988) and moved the group into discussing strategy and doing the math. Tom said they needed to divide by 2. There were eight verbal exchanges in this short segment consisting of clarification, explanation, inquiry, and verbalizing doing the math. Mike said they should divide by 4 and both Mike and Tom did the math. Tom verbalized everything he did. In this situation his verbalizing did not serve as a collaborative check, but appeared to be verbalizing for approval (Pirie, 1998). Mike stated his answer and immediately questioned whether they really should have divided by 4. Mike monitored his own thinking and led the students to evaluate the reasonableness
of their method (Williams, 2000). This level of thinking was building-with evaluative-analyzing. The level of cognitive demand was maintained.

Mike’s inquiry moved the group back to understanding the task representing the fourth segment. Both Kate and Tom reread the problem. Kate used inquiry to clarify what she thought the problem was asking and Tom responded with agreement accordingly. The students collaboratively checked what they know. A brief silence followed this sequence and possibly served as processing. Afterward, Mike again inquired if his picture accurately represented the situation. Mike then slowly reread the problem. The students recognized several possibilities and wanted to know precisely (Lovecky, 1994) how the ropes were connected. Tom started to respond when Kate interrupted him, “So you find the total area that they can graze. So 7853.98 do times 4.” Once again, Kate clearly demonstrated insight in understanding what should be done. The 14 verbal exchanges involved rereading the problem, clarification, explanation, agreement, and inquiry. All of the discussion was recognizing comprehending except for two instances. Mike’s thinking exhibited building-with synthetic-analyzing when he considered the accuracy of his drawing and Kate demonstrated constructing-synthesizing by integrating previous knowledge into new insight (Krutetskii, 1976).

Kate’s insight moved the thinking to strategy. Mike and Kate did not have the opportunity to pursue her insight though because Tom argued they should multiply 1963.5 times 4 instead. Kate did not press her thinking and instead picked up Tom’s idea (Pirie & Schwarzenberger, 1988). Kate did the math and indicated they are back to 7853.98. Mike tried to move back to finding the length of the rope, but Tom interrupted suggesting they subtract 783.98 from the area of the square. Again Kate did the math.
Tom restated what they know and inquired “then what?” Mike tried to refocus the group and asked, “How do you find the length of the rope?” Although Tom has just led the group in a loop, he insisted he knew how to find the length of the rope. The levels of mathematical thinking during this segment are building with analyzing and building-with evaluative-analyzing. The following discussion demonstrates how the students in Group 1 patiently picked up Tom’s idea, but later showed frustration as they worked. The dialogue includes identification of the task followed by the group number, number of the line in the transcription, and the speaker’s name.

Intro G1 76 Tom: I know, OK
Intro G1 77 Mike: The length of the rope is 50 m
Intro G1 78 Tom: You can split it up into triangles like find the area of this is 2146.02 one half of 1963.5
Intro G1 79 Kate: One half of 1963.5 divided by 2 is equal to 981.75
Intro G1 80 Tom: 981.75 plus 981.75
Intro G1 81 Kate: You add it
Intro G1 82 Tom: Yeah
Intro G1 83 Kate: 1963.5. It’s the same thing!
Intro G1 84 Tom: OK, subtract 1963.5 from 2146.02. 981.75
Intro G1 85 Kate: Say that again, 2146.02 minus 1963.5 equal 182.53.
Intro G1 86 Tom: Alright, so the area of this triangle right here is 182.53 after you takeout the circle parts
Intro G1 87 Mike: Aren’t you trying to find the
Intro G1 88 Tom: You’re trying to find this part right here. And you know this right 90 here is 50. So if you find this
Intro G1 89 Kate: Because that area is
Intro G1 90 Mike: How do you know the ropes go to the center?
Intro G1 91 Tom: Because they are all tied together.
Intro G1 92 Kate: But how do you know they aren’t tied like this?
Intro G1 93 Mike: Yeah
Intro G1 94 Tom: Who ties their goats together anyway?
Intro G1 95 Kate: What if it was like that and that and that and that
Intro G1 96 Mike: I don’t think this word problem is worded right.
Intro G1 97 Tom: Let’s just go with this, this sounds good.
Intro G1 98 Mike: We need more information. This word problem is not cool.
Intro G1 99 Tom: Alright, let’s alright, so how do we find the area of a triangle, OK, so the area of a triangle is b times h divided by 2 right?
Intro G1 100 Mike: If they are tied like this, how do they have a 50 m radius?
In Intro G1 94 and Intro G1 101, Tom was clearly frustrated that his thinking was challenged. Mike also expressed his frustration in Intro G1 96 and Intro G1 98 by asserting the problem lacked the information needed for solving. In Intro G1 97, Tom was more concerned that the group sound like they were using higher-level thinking. This supports my interpretation that Tom sometimes verbalized for approval (Pirie, 1998). These statements signaled not only their frustration, but as gifted students their embarrassment they had not come to a quick solution. I remind the reader the group was not working in isolation, rather within the context of my classroom with three other groups working in close proximity who could easily have overheard their discussion. The students were cognizant of other students listening, observing, and possibly reacting to their input (Gillies & Ashman, 2003). This social capital (Bourdieu, 1991) operating within the group and the classroom can influence the individual student’s contribution to success or failure in achieving a goal. The level of cognitive demand could have declined due to Mike and Kate’s frustration, but Mike and Kate continued to pursue suggested strategies.

In Intro G1 102, Kate clarified how the goats were tied. Once again, Kate provided the information needed to move forward. Tom integrated this information with his thinking and said 50 m is the base of the triangle. The group pursued this line of thinking until Mike suggested he had something. He reviewed what they knew about the area of the square and the area of one circle using inquiry, “right?” as a collaborative check. Mike suggested they subtract the area of the four circles from the area of the square. Tom inquired if they have already done this. Kate exclaimed, “We already have
that. We have all these numbers! So how do you figure out the length of the rope?”

Kate’s frustration was obvious.

The fifth segment involved 67 verbal exchanges related to strategy and doing the math interspersed with collaborative checking. The verbal exchanges consisted of explanation, clarification, argument, challenge, inquiry, and giving answers. According to Hershkowitz et al. (2001), when students are engaged in the problem-solving process of explaining and reflecting on the process, their thinking represents recognizing nested within building-with. Mike evaluated the reasonableness of their method (Williams, 2000) when he challenged Tom’s thinking about the ropes going to the center and when he offered a different possibility. This thinking exhibited building-with evaluative-analyzing. Both Tom and Kate used building-with evaluative-analyzing when they concluded they had already done what Mike suggested. Although the level of cognitive demand was maintained, the level of cognitive demand was in danger of decline due to the frustration level exhibited by Kate.

In the sixth segment, the discussion moved back to understanding the task and the level of mathematical thinking exhibited was lowered to recognizing comprehending. Tom suggested they ask me for help. Mike very reluctantly agreed. Kate explained what they know. Tom argued they did not know the information Kate suggested. Although Kate had repeatedly told the others how the rope was connected, she asked me how the rope was connected. I told her the rope was tied in the corners. At this point the students needed a precise answer. Tom somehow inferred the ropes crossed. Mike referred to his previous question regarding the inaccuracy of his drawing. I looked over his sketch and remarked the problem could be because he drew a rectangle. Mike briefly discussed his
inability to accurately draw squares. Watching as he drew I replied, “Maybe you should concentrate on drawing the final product. You just drew another rectangle.” I assumed my hint would move them forward. I was wrong.

The sixth segment consisted of 11 verbal exchanges involving what they know, agreement, argument, explanation, and challenge. The level of mathematical thinking exhibited after the beginning discussion was building-with synthetic-analyzing as students were attempting to make connections between the drawing of the situation and what they needed to do next.

In the seventh segment the students worked again to understand the task. Mike originally evaluated his drawing shortly after beginning the problem and suggested the goats were tied in the corner and could only graze in an area with a radius of 50 m. Kate reiterated the goats were tied at the corners in line Intro G1 102 and I told the group the goats were tied at the corner. Also, the problem clearly stated the goats were tied at each corner of a square field. Mark completed the square, drew the circles accurately, and then explained what his drawing represented. Tom asked what that did for them and Kate inquired the location of the rope. The students remained in a quandary. They continued to make the simple complex (Lovecky, 1994).

The seventh segment consisted of 13 verbal exchanges related to understanding the task. The verbal exchanges involved inquiry and clarification. The level of mathematical thinking was building-with synthetic-analyzing. The students tried to make connections and draw conclusions from Mike’s pictorial representation of the situation (Williams, 2000).
Although the focus of the discussion in the eighth segment was also on understanding the task, I separated it from the previous segment because it signaled a decline in the level of cognitive demand. Again there were 13 verbal exchanges and most were inquiry and clarification. Several of the statements did not contribute to the mathematical thinking at all. Tom told Mike to draw the goats and to include the horns. The crux of their problem was illuminated in Mike’s next statement, “What I don’t get is when it says 3 of the goats are moved, when moved where? The rope of the 4th goat is lengthened.” Kate persisted at trying to make sense of what the students already knew, “He can graze over an area equal to a combined area. So the area that the four goats can go is 7853.98.” Mike divided that by $\pi$ and stated the radius would be 2500 m. Kate incredulously inquired of what? Kate was very patient doing the math suggested by the others. But she had enough. Frustrated, Kate called me over again. Mike and Kate exhibited building-with analyzing by continuing to familiarize themselves with the problem using Mike’s drawing (Williams, 2000). The following discussion occurred when I joined the group discussion.

Intro G1 203 Mike: We can’t figure out where the ropes go.
Intro G1 204 Me: The three goats are taken away and the rope is tied at the corner.
Intro G1 205 Mike: That’s if 3 goats are moved?
Intro G1 206 Me: The three goats are moved away.
Intro G1 207 Mike: Oh, so the goats are moved to a new field and they combine the ropes.
Intro G1 208 Me: Yes, but they don’t necessarily remain the same length.
Intro G1 209 Mike: So would it be like that? (Pointed to picture)
Intro G1 210 Me: Like that
Intro G1 211 Kate: So we need to find the length of the rope right there.
Intro G1 212 Me: What is the key to being able to solve the problem?

Both Kate and Tom responded to my question in Intro G1 212. Tom assured his group he had it, and launched into doing the math. Unfortunately, he led the group in
doing the same thing they had already done. He divided 7853.98 by 4, had Kate divide that value by $\pi$ and take the square root of the answer. Lastly, he told the group the answer plus 50 $\pi$ represented the length of the rope. An argument ensued. Kate got 24.99 and Tom had 74.9 and both wanted verification from me if he or she was correct. Twenty-three verbal exchanges took place while doing the math. These included inquiry, verbalizing doing the math, explanation, giving answers, and challenge. The level of mathematical thinking was building-with analyzing as the students applied known mathematical procedures (Dreyfus et al., 2001). The level of cognitive demand declined slightly at one point during this segment due to circularity involved in doing the math. Kate and Mike helped maintain the high-level of cognitive.

The last segment involved my interaction with the students during the last few minutes of class. I asked Mike about the portion of the circle drawn in his picture. Again, I asked about the key to the problem. Kate told me the area that the goat could graze was equal to the combined area. The group had struggled far too long. Hershkowitz et al. (2001) suggested that providing a hint can help students think at the build-with level. Rather than risk the decline in the level of cognitive demand, I told the students the area of this part (pointing to Mike’s drawing) must equal the four circles they had drawn. I did not lower the level of cognitive demand because the students still had to make the connections themselves. Mike replied they had that already. I told him I realized they had that earlier but they put it equal to $\pi r^2$. Mike countered that represented the area of a circle. I thought Tom understood the hint I had just provided when he said it was a fourth of the circle, but then he said to divide by $\pi$ first. Mike disagreed and verbalized what he did to get 9998. I asked to see what he did and suggested he leave his values in the
calculator instead of rounding. Tom got the answer of 10,000. I asked what this represented. Mike said it was the area. My look of surprise caused him to amend his answer. Tom said take the square root, it is 100. Class was over, but Mike needed verification that his drawing accurately depicted the situation. He stayed behind to make sure his drawing was correct and to explain again how they did the math.

This last segment was doing the math and involved 22 verbal exchanges that included inquiry, clarification, explanation, argument and giving answers. The levels of mathematical thinking were building-with analyzing and building-with evaluative-analyzing. The students applied known mathematical procedures to solve the problem (Dreyfus et al., 2001) and evaluated the reasonableness of their method and solution (Williams, 2000).

The level of cognitive demand of the mathematical task was doing mathematics due to the complex thinking and reasoning required to achieve a solution (Stein et al., 2000). Factors associated with maintenance of the level of cognitive demand for doing mathematics include building on previous knowledge, scaffolding, and sustained press for explanation and meaning (Stein et al., 2000). The levels of mathematical thinking exhibited during the interaction of the students were recognizing-comprehending and building-with synthetic-analyzing. The students adapted previous knowledge to the task (Dreyfus et al., 2001) and constructed an accurate pictorial representation of the grazing areas within the square. Students used building-with evaluative-analyzing to evaluate whether their method was reasonable (Williams, 2000). I provided scaffolding to the students by using Mike’s representation and by stating the areas were equal. The students still had to make the connection. Kate showed insight at the beginning of the problem by
integrating what she knew to the new context, representing constructing-synthesizing. The levels of mathematical thinking exhibited represented the levels of complex thinking and reasoning characteristic of doing mathematics. The level of cognitive demand of doing math was maintained.

*Group 2.* Sally reminded the others they had to work together and there should be no going off on one’s own. She read the problem, said be yourself and then started to laugh. Sally recognized and stated the dimensions. April drew a little goat and Chad made a comment. Sally responded, “Cute guys” and then refocused the group to the task finding the length of the rope. Chad reread the last lines of the problem. When I heard Chad rereading the last lines, I knew what was going on and I closely observed this group. Chad had the ability to see things others did not immediately see (Sfard, 1991). His shirt during this task read, “Scientific Theory Proven.”

April inquired if they should find the area first. Chad picked up this thinking (Pirie & Schwarzenberger, 1988) and directed the others to find the area of a circle with a radius of 50 m. The short period of understanding the task was over and they moved to doing the math. The level of mathematical thinking involved in the 10 verbal exchanges was recognizing comprehending, but ended with building-with analyzing as Chad led the application of known mathematical procedures. April verbalized doing the math as a collaborative check. Terry entered the discussion by giving the answer. April wanted him to just say the numbers. Chad repeated the answer.

The next segment focused on strategy. Sally wanted to know what the numbers meant and April pointed to a picture and said the area of that. Terry argued the rope could not go all the way around. Chad showed them how it should go. Sally challenged him.
Terry clarified what Chad said. Chad suggested they say 75. Sally inquired what and Terry inquired why. Chad admitted he just guessed. He used the problem-solving strategy guess and test. Sally responded, “What are you talking about? No, we’re not guessing. We’re using formulas.” This demonstrated how differently the two students think. I noted Chad continued to work with his calculator and his fingers flew across the calculator keys. April drew a new picture with only one goat. April then inquired about the length of the radius for the whole circle. Sally said to plug it into the “little thingie.” April pointed to her drawing and inquired if “that” equals $\pi r^2$. Terry argued “it” needs to go to 79% of “that.” Chad started to pursue this when April interrupted to inquire what 79% of 7854 was. She corrected herself and asked if it was 79% of 10,000. April inquired if her drawing was 79% of the whole thing. Terry clarified all four of them was 79% of the whole thing. April then inquired why they had to use percents. Sally restated what they needed to do when Chad asserted, “It’s 100 exactly.” Sally asked if it was 50 m. Terry pressed Chad for justification.

The focus of this segment was on strategy with 30 verbal exchanges involving clarification, explanation, inquiry, agreement, collaborative checking and insight. Most of the mathematical thinking was either building-with synthetic-analyzing or building-with evaluative-analyzing. Students discussed different ways to solve the problem (Krutetskii, 1976; Williams, 2000) and evaluated the reasonableness of using each method discussed (Williams). Terry integrated concepts to create new insight when he saw that the area of all four together represented 79% of the whole thing which is constructing-synthesizing. Chad also used constructing-synthesizing level to arrive at his answer (Krutetskii; Williams).
Even though Chad’s earlier guess of 75 was dismissed by Sally and he was
warned they were to use formulas, Chad continued with his line of thinking. His replied
he just plugged in numbers until he got it right. Sally asked if it was the Pythagorean
Theorem. Chad told them to watch. He explained 100 squared times $\pi$. Terry started to
argue, but Chad continued. The following discussion indicated Chad was constructing-
synthesizing as he formulated a mathematical argument to explain his reasoning
(Williams, 2000).

Intro G2 63 Chad: Ready, this is the number. Then you will have to divide it by 4.
Just watch. You get the entire circle and divide that by 4. What number is that?

Intro G2 64 Sally: How did you get it? Explain.

Intro G2 65 Chad: I kind of did the work in my head so I kind of lost the work in my
head. So you do this over 4. I’ll prove it works.

Intro G2 66 Sally: How do you know that it is going to be a fourth of a circle?

Intro G2 67 Chad: Because since this is 90 right there, then you have to have another
90, another 90, another 90 and that’s 360 for a circle.

Intro G2 68 Sally: OK, I get it now.

Intro G2 69 Chad: So how long is the rope? 100 m

Chad knew I was observing him. He commented to the others that I was watching
while he explained. I joined the group and pressed Chad for justification. As a
consequence of Chad’s explanation and Terry’s integrating his knowledge with Chad’s
thinking, Terry came up with an algorithm for all cases in a similar situation. The
following discussion relates Chad’s explanation and Terry’s subsequent rule.

Intro G2 83 Me: Can you draw a picture for me?

Intro G2 84 Chad: If you can read Chad (referring to my understanding of his
intuitiveness)

Intro G2 85 Me: I can read Chad.

Intro G2 86 Terry: How?

Intro G2 87 Me: I can see exactly what he has on there except his circle is wacky,
but that’s a Chad thing.

Intro G2 88 Chad: I drew the arc though.

Intro G2 89 Me: You did. How did you come up with this? I heard you and
automatically knew you had it. I just want to know what you did.
Intro G2 90 Chad: First I really guessed. First I did 75, but that didn’t work.
Intro G2 91 Me: I heard Sally tell you your group wasn’t going to guess. So you guessed another number and then worked backwards?
Intro G2 92 Chad: Yeah
Intro G2 93 Terry: That would be the radius, then 2 times the radius, all that divide by 4. Would that be the formula? Always?

Chad cheered that his group won because they were the first group to get the problem. Chad also checked the time. The mathematical thinking involved in the last minute or so reached the highest levels of cognitive activity. Chad’s explanation was constructing synthesizing (Williams, 2000). Terry also used constructing synthesizing to develop new insights given Chad’s explanation (Krutetskii, 1976) adding to existing knowledge (Cobb & Yackel, 1996). Terry then moved to constructing evaluating when he reflected on Chad’s solution, and developed a formula that could always be applied to similar problems (Dreyfus et al., 2001). Terry extended his thinking based on Chad’s discovery.

Sheffield (2000) stated that communicating results through talk with peers often helps students make connections not previously seen. The social interaction of the classroom, specifically within the group, influenced Terry’s construction of knowledge (Hershkowitz et al., 2001; Wood et al., 2006). Terry made a generalization based on a pattern he recognized. The resulting equation was elegant and an extension of mathematical thinking (Hekimoglu, 2004; Sheffield, 2000; Sriraman, 2003). Although completed in a short time frame, the mathematical discussion and thinking involved in the collaborative effort moved students beyond just finding a solution, but demonstrated how students think about and do mathematics creatively in ways that are elegant. The discussion above describing the levels of mathematical thinking involved in the task
provide evidence that the factors associated with the maintenance of the high-level
cognitive demand of doing mathematics were met (Stein et al., 2000).

*Group 3.* Amy recognized the given radius meant they needed to draw circles.
Bruce pointed out that the grazing area was bounded by the square and from that
statement Karol recognized that there are four circles with their centers at the corners of
the field. Bruce recognized the four pieces of the circles inside the square equaled one
circle and that they needed to find the total area of the four quarter pieces. Amy inquired
if finding the area was all they had to do and then did the math simultaneously with the
others.

Understanding the task involved 15 verbal exchanges of stating what they know,
explanation, clarification, and agreement. The levels of mathematical thinking exhibited
by the students were recognizing applying nested within building-with analyzing as they
are applied previous knowledge in a new context (Dreyfus et al., 2001).

The students did the math by applying what they knew from the task and
collaboratively checked their results. Joey and Bruce verbalized doing the math. Amy
gave the answer, Kelly agreed, and Joey verbalized his result to verify with the others.
This short segment included eight brief exchanges of verification. The level of
mathematical thinking was recognizing applying. Students applied the formula for the
area of the circle with a radius of 50 m to determine the area the four goats can graze.
Then they returned to the problem to understand the task.

In the next six segments the students moved back and forth from understanding
the task to discussing or implementing strategy. The students returned to understanding
the task when one student challenged another’s thinking while discussing strategy.
Therefore, I discuss these six segments in the context that understanding the task was nested within the discussion of strategy or implementation. Once the students found the grazing area of the four goats, they moved back to the problem. Bruce stated the ropes were the same length and when three goats leave, the fourth goat got them. Amy inquired if they should use circumference. Bruce responded they had to find the area. Joey inquired if the rope represented the diameter. Karol said the diameter would be 100 m. Karol and Amy argued whether they needed to find the area, or the length of the rope. Karol acquiesced. She then suggested that adding 50 + 50 + 50 + 50. Frustrated, Joey suggested they should guess and check. Joey, Amy, and Karol continued to argue strategy in terms of what they thought the problem meant. Bruce’s silence indicated he was processing (Hoyles, 1985). Finally, Karol asked Bruce what he had done. Bruce spoke slowly, and hesitatingly. He told them to make one circle. He barely had time to explain before Joey got frustrated. The others said they were lost too. The following discussion indicates how one student’s explanation helped others build-within the construction of knowledge (Hershkowitz et al., 2001; Wood et al., 2006).

Intro G3 89 Bruce: Oh, its divide by four because you are trying to find the four like this (points to the circles) but the new area is divided by four because you aren’t finding the area of the whole circle, just this.
Intro G3 90 Karol: Oh, I get it.
Intro G3 91 Bruce: See what I mean?
Intro G3 92 Joey: So each one of these is like one of the small ones.
Intro G3 93 Bruce: Each one of what?
Intro G3 94 Joey: So each fourth is going to be equal to this area (pointing).
Intro G3 95 Bruce: Yeah
Intro G3 96 Joey: But what I don’t get is when is says (Amy interrupts)
Intro G3 97 Amy: You’re finding the little area. Sorry, I’m just trying to understand. OK, proceed.

Even though Amy followed Bruce’s explanation, she suggested setting up a proportion. Bruce questioned her use of a proportion to find the length, but listened to her
justification. While Amy was explaining how she set up the proportion, she inquired if the group ever found the area of the four circles. Bruce gave the answer for the combined areas as $2500\pi$. He then inquired what Amy was attempting to do. The group picked up Amy’s idea (Pirie & Schwarzenberger, 1988) and set up a proportion transitioning to doing the math.

The six segments occurred in three repetitions of understanding the task, then strategy. There were a total of 143 verbal exchanges of inquiry, explanation, agreement, correction, challenge, giving answers, clarification, justification, and argument. The levels of mathematical thinking progressed from recognizing comprehending to recognizing applying, to building-with analyzing to building-with synthetic-analyzing in the first four of the six segments. The students recognized and understood how the ropes were tied and differentiated why they needed to use area as opposed to circumference. During the sixth segment the students’ mathematical thinking was building-with evaluative-analyzing as they listened to Bruce’s explanation and evaluated his reasoning (Williams, 2000). Bruce’s level of thinking was constructing synthesizing because he formulated a mathematical argument to explain his discovery (Williams).

Amy attempted to set up a proportion using the radius of a circle with the same area of the large one-fourth of a circle to the radius of a circle with the same area of the small one-fourth of a circle equal to the area of the large fourth of a circle to the area of the small fourth of a circle. Collaboratively they did the math. Equation 1 represents the proportion the students used:

$$\frac{7853.98}{1963.50} = \frac{x}{50}$$ (1)
The group came up with an answer of approximately 200 m. Amy’s method would have worked had the students used the radius of the small fourth of a circle to its area equal to \( x \) over the area of the larger fourth of a circle. Bruce wondered if they are on the right track. Rather than take the time to verify their conclusion, Amy suggested they ask me for verification. She then inquired, “Y’all, do realize that 200 is all of the radius combined?” As they wait for me to come over, the discussion turned to voice mail and was not social chat (Pirie & Schwarzenberger, 1988) as a way of processing. The level of cognitive demand of the task had declined. Amy, Joey, and Karol were content to seek verification from the teacher (Stein et al., 2000).

I joined their group during the last segment. Karol asked if I could tell them if their answer was correct. Bruce somewhat embarrassed told me he tried to explain. Joey promptly told Bruce he was not a team player. Rather than answer Amy’s question, I asked her to explain what she did. Amy told me she set up a proportion. I looked over each of their sketches and Bruce started to explain what he concluded at the very beginning. I asked him if he could sketch what he explained. Once Bruce drew a pictorial representation, Amy and Joey said they understood. Amy said she did not get why what she did would not have worked. She was pleased to know that I had heard her pursuing this line of thinking earlier and knew she was on the right track. Had the class not ended, I would have suggested she revisit her proportion to verify the additional pathway to a solution and extend her thinking (Sheffield, 2000).

There were 23 verbal exchanges involving seeking verification, inquiry, teacher questioning, explanation, understanding, and agreement. The levels of mathematical thinking included building-with evaluative-analyzing. As Bruce sketched what he
explained, the students were able to evaluate the reasonableness of his explanation (Williams, 2000). Seeing what Bruce described helped the others build—with contributing to their mathematical understanding of the problem (Hershkowitz et al., 2001; Sheffield, 2000). The level of cognitive demand in this segment was raised to doing mathematics as the students had to draw conclusions from Bruce’s pictorial representation.

Bruce recognized in the opening segment that each of the four quarters of circles inside the square equaled a quarter of the new circle. But, the other students did not pursue his idea. After finding the area of the four quarters, Amy, Joey, and Karol argued strategy. Finally, they consulted Bruce who explained his reasoning. Amy suggested using a proportion and the group picked up her idea (Pirie & Schwarzenberger, 1988) and did the math. Once they arrived at a solution, they waited for me to verify its accuracy. I did not lower the level of cognitive demand by telling. Rather I pressed Bruce to continue with his reasoning. His drawing and explanation enabled the others to understand the underlying mathematical structure (Hershkowitz et al., 2001). Although the level of cognitive demand declined once the students arrived at a conclusion, Bruce’s explanation of his thinking was able to raise the level of cognitive demand to doing mathematics. The students built on prior knowledge, I provided scaffolding by encouraging Bruce to draw what he described, and pressed for justification from the students.

*Group 4.* The composition of this group remained the same as in the Pilot Study II. A change was made after this task to balance another group and to separate two close friends. For the introductory task, Bob, Daren, Ethan and Rita worked together. Erickson (1977) argued that making sense of the sometimes outrageous behavior is the “tour de force” (p. 61) of the ethnographer. The males in this group of gifted students exhibited
outrageous behavior at times during problem solving, but after data analysis, I realized they processed their thinking during these episodes and they fluidly moved in and out of the discussion (Saul, 1999). Rita often was the target of their tirades, but she was always amiable and kept them moving along.

After a brief interlude of Bob and Daren testing their secret service routine with the tape recorder, Bob read the first part of the problem and began the segment of understanding the task. For this group understanding the task involved only eight short verbal exchanges primarily reading the problem and verbalizing what they know. Most of the mathematical thinking was recognizing comprehending. Bob cautioned the others not to draw anything until they understood the task. Rita had already started drawing so Bob said she needed to be kicked out of the group. He told her they had to finish reading the directions before anyone could do anything. She replied by giving the dimensions of the square. She was undaunted. Bob continued to read the problem and before he finished he told the group they should put the problem into a proportion which showed insight and represented building-with analyzing as he applied previous knowledge in a new context (Dreyfus et al., 2001). High-level gifted students often grasp the essence of a problem immediately (Lovecky, 1994). This group’s immediate understanding of the task contrasted with the struggle of Group 1 to understand the task and illustrated the diversity in gifted student’s thinking. Bob read until he got to the portion about the ropes and Rita took over.

Discussing strategy was the focus of the next segment as Bob suggested using a proportion to obtain dimensions to scale for their drawings and to set up a proportion for solving. There were 10 short verbal exchanges of inquiry, explanation and collaborative
checking (Pirie, 1998). While Rita questioned her drawing, Bob said he needed to solve the proportion. Daren inquired why. Bob replied, “10 cm for this 100 thing and then you can say a 50 m radius.” Bob verbalized as he wrote. Bob decided 10 cm was too small and Rita told him it would work, just to use the notebook. Bob said, “The notebook.” They briefly discussed the movie with the same title and the significance of the notebook as a compilation of the experiences of an individual, written through reflections. The notebook the students used for work and to respond to my thoughts and observations throughout data collection took on new significance to the students and was always referred to as the notebook throughout the remainder of the study. The level of mathematical thinking exhibited in this segment was both recognizing analyzing and building-with analyzing. Each of the students used the notebook to accurately draw the square represented in the problem according to scale which set up the next phase which was exclusively collaborative checking.

Ethan began the sequence of collaborative checking by showing Rita his picture and inquiring about the size of the circle. Ethan drew four complete circles with the centers at the corners of the square. Rita argued he could not do that because the goats are only on the inside. He said he knew that. Looking at his drawing again, he inquired if anyone’s grazed area was touching. The students drew the squares and the circles at the corners to scale indicating they know when to use a mathematical idea (Williams, 2000), recognizing applying; created an accurate pictorial representation for the situation (Williams), building-with analyzing; and evaluated the reasonableness of their drawings (Williams), building-with evaluative-analyzing. This represents the nesting of mathematical thinking described by Hershkowitz et al., 2001 and Wood et al., (2006).
There were 19 verbal exchanges in reaching a consensus about an accurate pictorial representation. The accuracy of their drawing helped them progress toward achieving a solution as indicated in the next discussion. Intro represents Introductory Task, G4 represents Group 4, and the number represents the line number in the transcription.

Intro G4 60 Bob: Mine almost touch.
Intro G4 61 Rita: They touch.
Intro G4 62 Bob: They should touch.
Intro G4 63 Rita: No they shouldn’t.
Intro G4 64 Ethan: What are you talking about? They definitely shouldn’t touch.
Intro G4 65 Bob: Where at?
Intro G4 66 Ethan: Actually, I think they should touch. Because it’s 5 cm.
Intro G4 67 Bob: Yeah, its 5 cm from each side.
Intro G4 68 Daren: Yeah, it should, I did mine too small
Intro G4 69 Ethan: Ok, we’ve got this. Let’s do it!

Collectively they returned to the problem. This segment moved work related to understanding the task to discussing strategy. There were 26 verbal exchanges of clarification, inquiry, justification, explanation agreement, and extending an idea. Again building-with analyzing was nested in building-with synthetic-analyzing and building-with evaluative-analyzing. The students applied known mathematical procedures, (Williams, 2000), made an independent generalization (Williams), and evaluated and reflected on the process (Hershkowitz et al., 2001; Williams). Rita asked what was meant by three goats are moved. Bob replied it meant they just move around. Rita reread the problem. Bob inquired if they could just find the area of the circles and divide by four to get one-fourth. Rita said that the goats can move into all the areas. Bob clarified they can move into the combined areas. Rita reread the problem again and agreed with Bob. But, through this discussion Bob had an opportunity to evaluate his own thinking and realized they needed to multiply by 4 because there were four circles. Ethan made a small generalization (Williams) when he added that represented how far the goat can graze.
Bob said right, so find the area of the one circle. Ethan offered an answer and Daren inquired if the ropes are connected for the whole area. Rita clarified, but Daren realized she did not understand what he was asking. Daren used the drawing to inquire. Rita understood his question and explained the new area the goat can graze is only inside the square. Ethan argued. Rita justified what she said and ended, “which is $\pi r^2$.”

In the next segment of doing the math, Bob picked up on Rita’s idea (Pirie & Schwarzenberger, 1988). He stated, “Fifty squared times pi is the area of the circle and that’s the area he can travel.” Daren admitted he did not think about that. Rita gave the answer of 7853.98. Daren praised Bob causing Bob to move into character. As quickly as Bob moved into character he returned to the discussion moving it to the next phase of doing the math by focusing on the area the fourth goat can graze.

Building on Bob’s statement, Ethan added the goat can go anywhere. Rita added it can go the area of the combined goats. Daren looked over the drawing then verbalized his thinking. Bob gave the answer. Daren asserted they had to find the radius. Drawing the arc, Rita proposed, “What if you do this? What if it’s saying you can go in this area (pointed to the area the arc bounds)? Do the calculation of $\pi r^2$ with a radius of 100.” No one picked up her idea though. Ethan disagreed. Daren thought the problem meant the whole thing. Ethan gave an incorrect answer of 140 m. Rita questioned his reasoning. Ethan explained his answer was the diameter. Rita challenged Ethan. Daren suggested multiplying 50 times 4. Again, Rita insisted she had the answer, but Bob and Daren continued to ignore her. Bob discussed Daren’s idea and Ethan gave the answer of 200 m if the ropes are 50 m each. Again Rita argued, “No, it’s 100 m, the rope is 100 m.” Ethan continued to argue. Rita returned to the drawing and showed them the picture of the rope
she drew. She explained that the four ropes together equal this pointing to her picture as in figure 4 rope. Daren wanted to know where Rita was getting the magic number of 100. The following discussion indicates how Rita explained, Bob restated her explanation, and then Rita used her drawing shown in Figure 4 explain again.

Intro G4 144 Rita: Are you going to let me explain? Listen, let me explain. You have four. You take part of the circle and the radius is 100 and find the area of the whole circle and divide it by 4 you get the same exact thing.

Intro G4 147 Bob: She took a circle of 100 radius and dragged it across the field and that’s the fourth of the circle and then she dragged this one out and it goes 100 m when it touches that which is the radius of the circle.

Intro G4 150 Rita: Ok, all of this right here, that 4 like circles equals all four of these. It can’t go right here. That is as far as it can go is 100 m. I figured it out.

Figure 4. Rita’s drawing used in mathematical argument.

There were 29 verbal exchanges in this segment of problem solving involving challenge, inquiry, explanation, argument, clarification, and giving answers. The
mathematical thinking in most instances of the discussion was building-with evaluation as the students evaluated each other’s thinking related to what constituted the length of rope. Rita’s thinking was constructing synthesizing (Williams, 2000). Her “what if” proposal demonstrated building-with when she drew an arc from one corner to another corner and recognized that this quarter of a circle must represent the same area as the four smaller circles. Rita’s thinking demonstrated simultaneous use of recognizing and building-with (Hershkowitz et al., 2001). While the others continued to discuss various possibilities for the rope, Rita calculated the area of a circle, divided by four to get the same answer as the combined areas. She formulated a mathematical argument, constructing synthesizing, to explain what she had discovered (Williams). Then Bob formulated another way to explain her solution to the others (Williams). Bob’s thinking also represented constructing synthesizing with nested recognizing and building-with (Hershkowitz et al.; Wood et al., 2006). The level of cognitive demand of doing mathematics was maintained.

The students used what their previous knowledge about circles and scale to draw a pictorial representation of the situation and then used collaborative checking to evaluate the accuracy of their drawings. Through collaborative discussion they determined what was meant for the new area to be equal to the combined areas. Rita made a conjecture, acted on her idea, and arrived at a solution. Through her explanation of her discovery, Bob was able to restate what she explained to clarify the understanding of Ethan and Daren. These actions represent factors associated with maintenance in the level of cognitive demand of doing mathematics (Stein et al., 2000).
Comparison and Contrast of Group Experiences on Introductory Task

The students in group 1 encountered difficulty understanding the task, specifically how the ropes were connected when the fourth goat’s rope was extended. Mike thought the problem lacked critical information. Kate’s early insight was not picked up (Pirie & Schwarzenberger, 1988) and the group moved back and forth from strategy, doing the math, and understanding the task through eight segments. Students verbalized what they did, used collaborative checking, and appeared to process their thinking through silence. The students used recognizing comprehending, recognizing applying, building-with analyzing, and building-with synthetic-analyzing to connect previous knowledge to construct an accurate pictorial representation of the grazing areas, and do the math (Dreyfus et al., 2001; Krutetskii, 1976; Williams, 2000). Students used building-with evaluative-analyzing to evaluate whether a method was reasonable (Williams). Kate’s continued insight represented constructing synthesizing. Unfortunately, she was unable, or not given the opportunity to formulate a mathematical argument to explain her insight. I provided scaffolding so the students could make the connections to obtain a solution.

The levels of mathematical thinking exhibited represented complex thinking and reasoning characteristic of doing mathematics (Stein et al., 2000). The level of cognitive demand was maintained throughout the eight segments of the task although was at risk during one segment as indicated by the group interaction. The social capital operating within the group was evident in the group interactions. The students were cognizant of other students listening, observing, and possibly reacting to his or her input (Gillies & Ashman, 2003). This social capital (Bourdieu, 1991) can influence the individual’s contribution to achievement of the goal. Examples include Tom’s concern with sounding
intelligent, his frustration that his thinking was challenged, and Mike’s insistence the problem lacked critical information, and his reluctance to seek my assistance.

The experience of group 2 represented the converse of the experience of group 1. Sally focused on ensuring the effort was collaborative and that the students used mathematical procedures. Understanding the task for this group was very brief and they transitioned immediately into doing the math. The students like the first group, verbalized doing the math and used collaborative checking. Processing for Chad appeared to occur when the problem was read. Processing for the others was through the collaborative communication (Sheffield, 1999). Chad worked with the others through understanding the task, but displayed an intuitiveness Sfard (1991) referred to as an understanding that precedes explanation by guessing a radius, evaluating the reasonableness of the solution, and then working backwards to formulate an argument. His thinking represented constructing synthesizing (Krutetskii, 1976; Williams, 2000). Terry showed flexibility using percents (Sheffield, 2000). Terry demonstrated constructing synthesizing and constructing evaluating through his insight to an equation that could be employed in the future for similar situations (Krutetskii; Williams). Group 1 worked through the entire period before finding a solution and then Mike remained after class to verify his understanding. In contrast, group 2 completed the task in less than 5 min. This contrast in the time, indicated the diverse ways gifted students think about and do mathematics to achieve the same goal (Clark, 1997). The mathematical thinking of group 2 reached the level of constructing synthesizing and Terry reached the level of constructing evaluating whereas group 1 demonstrated limited constructing synthesizing.
The students in group 1 like the students in group 3, also worked for the entire period. Also like group 1, the students moved back and forth from understanding the task to discussing or implementing strategy, and doing the math. The students used collaborative checking in all nine segments. The thinking of group 2 reached higher levels of thinking as a result of Chad’s mathematical argument. Like Chad in group 2, Bruce’s argument could have moved the group’s thinking to a high-level had they pursued his insight. Joey later admitted in his reflection that he sometimes shut out other’s thinking. The group pursued Amy’s idea of using proportions instead. Her alternative approach, represented fluency (Sheffield, 2000). The students collaboratively did the math, but did not move to verify the accuracy of their solution through collaborative checking. Rather they sought verification from me, the teacher. I also provided scaffolding to the group, like group 1, by asking Bruce to sketch what he explained. His pictorial representation and mathematical argument helped the other students build-with contributing to their mathematical understanding (Hershkowitz et al., 2001). The level of cognitive demand declined when the students did not seek to verify their solution and relied on the teacher for verification (Stein et al., 2000). The level of cognitive demand of doing mathematics was raised with a press for explanation and justification through teacher questioning.

The students in group 4 used their previous knowledge about circles and scale to draw a pictorial representation of the situation and then used collaborative checking to evaluate the accuracy of their drawings. Unlike group 1 and group 3, the students in group 4 returned to understanding the task only once after the initial segment. Like the other groups, they verbalized doing the math and used collaborative checking. The males
in the group appeared to process their thinking by moving into character (Saul, 1999). Rita obviously was able to think on multiple levels while engaged in conversation with the males as they moved in and out of character. Through collaborative discussion they determined what was meant for the new area to be equal to the combined areas. Rita demonstrated a relational understanding (Skemp, 1987) of the underlying mathematical structure through her insight. The group acted on Rita’s idea and arrived at a solution. Bob restated Rita’s mathematical argument to clarify the understanding of Ethan and Daren. Unlike group 3, the group collaboratively verified their findings. The students, working collaboratively, demonstrated building-with previous knowledge, scaffolding, and sustained press for explanation and meaning (Stein et al., 2000), factors associated with maintenance of the level of cognitive demand for doing mathematics.

The levels of mathematical thinking exhibited and maintained by the students in all four groups represented the levels of complex thinking and reasoning characteristic of doing mathematics (Stein et al., 2000). The exception was a slight decline of group 3 due to their failure to verify their solution prior to a press for justification by the researcher. The levels of mathematical thinking exhibited across the groups included recognizing-comprehending and recognizing applying nested in building-with analyzing and building-with synthetic-analyzing, constructing synthesizing, constructing evaluating. Students applied previous knowledge of circles and area (Dreyfus et al., 2001), constructed pictorial representations of the grazing areas, calculated areas the goats could graze, reasoned how to find the radius of the quarter circle that represented the combined areas the goats could graze, and evaluated the reasonableness of both methods and solutions.
(Williams, 2000). One group extended their thinking by developing a formula applicable for similar situations.

Task 1: Basketball Court Renovation

Maple Street Middle School is undergoing considerable renovation. The first phase of the renovation is refinishing the basketball court. The shaded area will be painted. The remaining area will receive a hardwood finish. Give a plan for finding the area of the court to be painted. Give a plan for finding the area to receive the hardwood finish. Find the area to be painted. Show your work. Find the area to be given a hardwood finish. Find the circumference of the center circle. If eight players stand around the center circle for a jump ball, what is the arc length available to each player?

The task involved the hypothetical renovation of the middle school gym. The first phase of the renovation was refinishing the basketball court. The shaded area will be painted and the remaining area will receive a hardwood finish. The first two parts of the task required students to devise a plan to determine the area to be painted and the area to receive hardwood finish. After making a plan, the students had to implement the plan to find the solution. The last question asked the students to find the arc length available for each student around the center circle for a jump ball. Although the students recognized the court dimensions were in feet, they rarely used the measure when expressing an answer. I reported the students’ answers as numbers in the same context without the measures to more accurately indicate what transpired.

Group 1. Tom, Mike, and Kate are joined by Ethan for the remainder of the problem-solving tasks. In this section, understanding the task and devising a plan, or strategy, overlap as the task required the students to think about what they should do
before actually implementing the plan. The students were unaccustomed to writing a plan before doing the math and this was disconcerting to them. I commented on this observation in my notes to the students and received a variety of responses. Mike read the problem aloud and Tom immediately suggested finding the area of the large rectangle and then subtracting the area of the smaller rectangle the area of all the things inside the rectangle. A period of silence occurred while students processed their thinking (Hoyles, 1985). Mike added they needed to add the two small rectangles as well. Tom recapped what was said, but erroneously said subtract the shaded areas inside the smaller rectangle. Mike thought they should add those sections but used inquiry for clarification.

Tom recapped again and did the same thing. Mike tried to refocus and inquired about finding the painted area as the goal. Tom interrupted. Mike verbalized what he wrote, “I have to find the area of the big rectangle and then subtract the area from the small rectangle.” Tom argued they did not know that area yet. Ethan also inquired about subtracting the smaller rectangle from the larger rectangle. Kate correctly and succinctly stated, “Subtract everything that is shaded from the big one.” Again Tom argued his plan. Kate then said just find the areas of the shaded shapes and then you do not have to find the area of the big rectangle. Mike verbalized his plan in full. Tom got aggravated and told them to write whatever they wanted because there were many ways to do the problem. Tom recognized there were several pathways to a solution (Stein et al., 2000). Kate and Ethan began discussing the fact Ethan has freckles and the fact Ethan did not know he had freckles. Mike was writing, Ethan and Kate were engaged in the freckle discussion while Tom told the group what he did. It appeared no one was listening.

Kate’s attentiveness to the problem while carrying on another conversation represents a
method of processing (Hoyles, 1985). Once Tom verbalized what he did Kate very simply said, “Find the area of all the shaded shapes, add together, and then you’re done.” Her strategy represented a simple yet succinct plan to achieve the goal.

The first segment involved 31 verbal exchanges involving suggesting strategy, inquiry in the form of questioning steps, explanation, agreement, argument, and verbalizing what I did. The level of mathematical thinking was recognizing comprehending and recognizing applying. The cognitive activity represented understanding the concepts behind known strategy (Dreyfus et al., 2001; Williams, 2000). The students recognized they would need to use area formulas to find the areas of the shaded regions and subtract those areas from the area of the large rectangle. Tom’s plan and work on the basketball task is shown in figure 5.

The next segment also involved understand the task with an overlap of strategy as the students had to devise a plan for finding the area to receive the hardwood finish. Mike refocused the group by inquiring if the group was ready to move on. Next, he read the second part of the task. Kate attempted to speak but was interrupted by Tom who said add 94 plus eight and incorrectly gave the sum as 112. Tom was already implementing the plan before the group devised a plan for finding the area of the court to receive the hardwood finish. Mike attempted to speak and Kate took over and again succinctly stated her plan, “Find the area of the overall shape and subtract the shaded area.” Tom rejoined the group after getting a calculator, corrected his error, and stated the dimensions of the larger rectangle, “That’s 102, not 112 times 56. That’s 5-7-1-2.” Ethan asked if they just did that, then realized the group had moved to implementing the plan. This brief segment included ten verbal exchanges involving refocus statement, verbalizing doing the math,
explanation, and inquiry. The levels of mathematical thinking exhibited were recognizing comprehending, recognizing applying as students recognized the use of area and when to use it, and building-with analyzing when Tom began to implement the plan.

Maple Street Middle School is undergoing considerable renovation. The first phase of the renovation is refinishing the basketball court. The shaded area will be painted. The remaining area will receive a hardwood finish.

1. Give a plan for finding the area to be painted.
   1) Find area of large rectangle
   2) Find area of smaller rectangle
   3) Find one of shapes in a small rectangle
   4) Subtract the total area of the shapes from the area of the big rectangle

2. Give a plan for finding the area to receive the hardwood finish.
   1) Find area of the small rectangle
   2) Find one of shapes in a basketball
   3) Subtract the total area from the area of the rectangle

3. Find the area to be painted. Show your work.
   \[ A = \frac{6.85 \times 12}{2 \times 12} \]
   \[ A = 1644.19 \]

4. Find the area to be given a hardwood finish.
   \[ A \approx 4171.19 \]

5. Find the circumference of the center circle. If eight players stand around the center circle for a jump ball, what is the arc length available to each player?
   \[ C = 3.1417 \]

Figure 5. Tom’s work on Basketball Court Renovation
Mike refocused the group again signaling the transition into doing the math. He inquired if they were supposed to implement the plan next. Tom had been working ahead. He said after subtracting the smaller rectangle from the larger rectangles you get 1012 and then find the area of the things inside and just add it up. Kate inquired what they got for the shaded part of the rectangle. Tom could not answer even though he just told Mike the area was 1012. Kate verbalized finding the area of the smaller rectangle. Tom moved ahead and verbalized finding the area of the rectangle. Ethan got aggravated and told Tom to do it his way. Kate continued to verbalize doing the math. Ethan, Mike, and Kate continued to do the math. To get the shaded area outside the smaller rectangle, they first multiplied $50 \times 4 \times 2$ and then multiplied $94 \times 3 \times 2$. Tom gave the answer, “I think y'all should get 1694.19.” Kate inquired what Tom did for the circles. Tom replied just do $36\pi \times 2$. Ethan argued you do not multiply by two because the top of the free throw line is just half a circle. Tom explained there are two circles because the two half circles make one full circle and the circle at half court also has the same radius. Kate and Tom in unison said “That plus $456 + 1012 = 1694.19.” Ethan questioned what Kate and Tom did to clarify his understanding. Kate further explained they had to find the area of the whole thing and subtract 1694.19. Mike needed confirmation that his answer was correct so he inquired, “What did you guys get? 1964.19?” Kate continued to do the math and told the others they have to show 5712 minus 1694.19. Kate gave the answer as 4017.81. Mike clarified that represented the area for the hardwoods.

Doing the math segment consisted of 45 verbal exchanges involving verbalizing doing the math, explanation, inquiry, clarification, agreement, argument, and giving answers. The level of mathematical thinking involved was building-with analyzing,
building-with synthetic-analyzing, and building-with evaluative analyzing. The students used building-with analyzing when they applied known area formulas to find the area of the shaded regions. The students used more than one pathway (Krutetskii, 1976; Williams, 2000) to find the area of the shaded regions representing building-with synthetic-analyzing. Throughout doing the math, the students evaluated the reasonableness of the method (Williams). The students also evaluated their result. Each of the students did the math and verified the results with the others. Satisfied with the collaborative check the students moved to the final question of the task.

The question asked students to determine the arc length available for each player if eight players stand around the circle for the jump ball. During this segment understanding the task and doing the math overlapped. Collaborative checking was ongoing. Both Kate and Mike questioned why they have to find the circumference of a semicircle. I could not ascertain why the students kept referring to a semicircle. My only thought was the half court line in the drawing divides the circle into two semicircles. Obviously, they corrected that error. Kate inquired about the formula for finding circumference. Mike told Kate circumference is $2\pi r$. Tom interjected circumference is $\pi d$ so it was $12\pi$. Kate gave the answer of 27.70 and Tom agreed. Kate reread the question. Tom asserted, “so 360 divided by 8, that’s 45.” Mike verbalized doing the math while dividing 27.70 by eight and got 4.71. Tom argued with Mike that you do not divide the circumference by 8 rather 360. Mike inquired why 360. Ethan inquired what 360 divided by 8 represented. Tom replied it is the angle for one player. Tom confused arc angle with arc length. Mike attempted to speak, but Tom interrupted him and said there is a formula. Not willing to listen to what the group had to say Tom called for me. Tom
asked me if the formula for arc length is the secant of a circle time $\pi d$. The following
discussion relates my interaction with the group in response to Tom’s question. Task 1
represents the Basketball Court Renovation, G1 represents Group 1, and the number
represents the line number of the transcription.

Task 1 G1 149  Tom:  [Teacher], the formula for an arc length is the secant of a circle
times $\pi d$.
Task 1 G1 150  Me:  Do you mean sector?
Task 1 G1 151  Tom:  Sector, that’s what I mean.
Task 1 G1 152  Me:  Is there another way to do that?
Task 1 G1 153  Mike:  I got 12.5
Task 1 G1 154  Ethan:  There’s a formula?
Task 1 G1 155  Me:  There is a formula, but is there some other way to do it?
Task 1 G1 156  Mike:  I know, but I did it a different way.
Task 1 G1 157  Tom:  Oh, it’s exactly, no that’s an arc.
Task 1 G1 158  Mike:  I got 12.5%. I did like, I did like 27.7 divided by 8 and then I
did that (pointing) over the area that they had total.
Task 1 G1 159  Me:  Why did you divide by 8?
Task 1 G1 160  Mike:  Because they had 8 players
Task 1 G1 161  Tom:  Alright, ok
Task 1 G1 162  Me:  What does that represent in terms of the circle?
Task 1 G1 163  Mike:  It represents one eighth of the circle on the arc.
Task 1 G1 164  Me:  What do you mean?
Task 1 G1 165  Mike:  The circumference.
Task 1 G1 166  Me:  Good.
Task 1 G1 167  Mike:  So that’s the answer.
Task 1 G1 168  Me:  What are you asked to do?
Task 1 G1 169  Mike:  Find the arc length.
Task 1 G1 170  Me:  So how did you write it?
Task 1 G1 171  Mike:  As a percentage. I didn’t mean to do that.

The last segment of doing the math and collaborative checking involved 46 verbal
exchanges of inquiry, explanation, argument, giving answers, verbalizing what I did, and
justification. The levels of mathematical thinking were building-with analyzing and
building-with evaluative-analyzing. The students applied what they knew about circles
and arc length to find the arc length available for each player. Mike and Ethan evaluated
the reasonableness of Tom’s method for finding the arc length through collaborative
checking. Mike approached the question differently and found the percentage of the circumference the arc represented which indicated Mike’s thinking involved building-with synthetic-analyzing. The level of cognitive demand of procedures with connections was maintained.

Looking back, Ethan responded to my observational notes by saying they talked through the math so anyone who did not understand could figure it out. He did not have anything additional to add. Tom was absent the day I provided my observation notes and students wrote in their notebooks. Kate stated she thought making a plan was weird. She elaborated, “When I solve problems I don’t think of what I’m going to do next, I just do it.” She collaborated what Ethan said about talking through the math to help each other understand. Mike responded he thought making the plan was awkward because “we were just warming up our minds to think mathematically.” He elaborated further that the group wanted to talk to each other in order to agree on a plan to solve the problem. Mike concluded, “I like doing these problems because they challenge me to think in different ways that I normally would not.” The students have lacked the opportunities to realize their full potential by engaging in challenging complex investigations, collaborative problem-solving experiences, and higher level mathematical thinking (Sheffield, 2000). Mike’s concluding statement indicates gifted students’ desire engagement with challenging investigations.

**Group 2.** Chad’s shirt read, “Try not to let you mind wander.” Sally read the problem. Terry immediately inquired what give a plan meant and the group discussed what a plan means. The students are so accustomed to just doing rather than thinking about what they are doing that Terry felt the group needed my clarification on what give
a plan meant. I commented on his quandary in my notes to the students. In response to his question, I answered with a question about their approach. Sally suggested they find the whole thing and then subtract the part. Terry argued that would make it harder. April suggested simply adding the shaded areas. Chad agreed. April wrote, “Give formulas of areas and plug in measurements.” Everyone wrote some version of April’s suggestion. Sally questioned this, but wrote it anyway. April read the next part of the task, gave a plan, and inquired if that was right. No one argued and they all wrote the same thing. April interjected she was told in third grade not to say minus anymore because Chad wrote “minnis” on his paper.

The first segment, understanding the task, consisted of 30 verbal exchanges, 15 of which were related to understanding what give a plan meant. Lovecky (1994) reported many tasks that are simple for most children seem complex for highly gifted students due to the multiple possibilities of an answer. The verbal exchanges involved inquiry, clarification, and verbalizing what they will do. The level of mathematical thinking was recognizing comprehending. The level of cognitive demand was maintained.

The students implemented the plans they devised in the next segment. Chad immediately recognized that four must be added to each end of the length of the court and verbally justified why the length was 102. April picked up on Chad’s insight (Pirie & Schwarzenberger, 1988) and did the math. Chad interrupted her verbalizing what he is going to do. Sally reminded the group they are supposed to be working together. Terry argued she was not doing anything. Chad continued to talk and calculate. He will not be deterred. Chad actually got a little perturbed that Sally wanted the group to slow down and move together. Terry moved on to the rectangle of the free-throw lane. Terry and
Sally recognized the width of the free-throw lane was the diameter of the top of the key, or the semicircle and that the two semicircles make one whole circle. April interjected the frame, the shaded region between the inner court and the outside of the court, was 1012. Terry continued doing the math with the circles and April moved on to the rectangles of the free-throw lanes. Chad inquired if Sally and Terry had found the area of the frame. Sally responded they found the area of the circles and inside rectangles. April stated she had the area of all the shaded areas and gave the answers. As she was giving the answers, the others were collaboratively checking. Chad tried to move on to the next phase, but the others were still verifying their answers. He did not wait. Sally was not willing to accept just an answer. She wanted to understand the process of how they arrived at their answers. Sally continued to inquire for clarification as she worked through the math. Terry tried to clarify Sally’s understanding while simultaneously doing the math related to the second phase. April implemented the plan and subtracted the painted regions from the total area to get the area to receive the hardwood finish. Rather than give the answer she inquired if the others got 4017.8.

Doing the math involved 65 verbal exchanges. The verbal exchanges included making sure the effort remained collaborative, verbalizing doing the math, inquiry, clarification, giving answers, and collaborative checking. The level of mathematical thinking exhibited included building-with analyzing and building-with evaluative-analyzing. The group applied known mathematical procedures to the situation (Dreyfus et al., 2001) when they implemented the plan they devised. Building-with evaluative-analyzing was evident through their collaborative check of the results (Williams, 2000). The level of cognitive demand was maintained.
The final segment of doing the math was related to finding the arc length for each of eight players when standing around the center circle for a jump ball. April read the problem and then Chad read the problem. Chad told them to find the circumference of the center circle. April stated the formula and gave the answer. Verifying his understanding of what the question asked, Chad told them to divide by 8. April questioned if that represented arc length. Terry verified what Chad said. Chad then inquired if they should leave the answer in terms of pi. Everyone but Sally did the math and agreed that was the answer. There were 22 verbal exchanges in this segment involving inquiry, verbalizing doing the math, verification, and checking. The level of mathematical thinking exhibited was recognizing applying and limited building-with evaluative-analyzing. The pathway for using circumference was given. The students simply applied the formula and then checked their results. The high-level cognitive demand of procedures with connections was maintained.

The purpose of looking back was an opportunity for me to share with my students my observations and to allow them to provide their feedback about what I thought occurred and their interpretation. In my notes, I commented to the students, that problem solving was not a race. I also pointed out because I was observing four groups at one time that I could not possibly see or hear everything they meant when they said “this” or “that”. The student responses were varied. Chad responded, “Chad will try to think out loud. Chad will not go off in his own world to find the solution.” Terry admitted he knew what devise a plan meant, but he just wanted to make sure there was no other meaning of the word that would make him look stupid. I reminded Terry mathematics is not about tricks rather logical thinking. Otherwise, mathematics would be reduced to rote memory.
He concluded by reiterating black is amazing (The group had pursued a discussion on wearing black colored clothing earlier and this became the theme for the group). Sally was concerned that whenever she suggested an idea, the others would tell her to do something else. She said, “I think the rest of my group just jumps right into a problem before stepping back and evaluating what needs to be done.” For the record, she also stated black was depressing. April responded to my comment about racing. She said, “If we tell you every step we take in our math (thinking), I think beating our own time isn’t bad.”

Students had an opportunity to discuss my notes within their group before responding individually. The group discussion allowed students to talk about aspects of the process that I commented on and to voice their feelings in the group and to me. Giving students this voice not only strengthened the research, but also added to the cohesiveness of the groups. In addition, sharing problems I encountered while observing helped the students become more cognizant of the need for their explanations.

Group 3. Understanding the task included 19 verbal exchanges involving inquiry, clarification, and giving a plan. The level of mathematical thinking exhibited was recognizing comprehending. Before the group ever got around to reading the problem there was a discussion about where the recorder would be placed. Amy made some derogatory comment concerning Joey’s obsession with the recorder and he implied the comment hurt. Amy told Joey because they were Asian they could be honest with one another. Finally, Joey read the problem. Amy stated a plan which would yield the area to receive the hardwood finish rather than the area to be painted. Silence followed as they processed. No one commented on the plan Amy suggested. Amy inquired what
represented the non-shaded area and Joey clarified the non-shaded region represented the area to receive the hardwood finish. Karol restated what Amy stated earlier, “So find the area of the whole thing. Subtract the shaded area.” As Karol verbalized what she wrote, Amy caught the mistake, and through inquiry caused the group, at least Karol and Joey, to reflect on whether to subtract the shaded or non-shaded regions. Everyone except Bruce reversed the plan they devised for finding the area to receive the hardwood finish.

The second segment involved doing the math. Karol read the task. This group also automatically understood the addition to the length and width to obtain the dimensions of the larger rectangle. They verbalized their agreement by repeating the dimensions. Karol gave the answer of 5712. Joey repeated the answer as inquiry and Karol clarified that 5712 was the whole thing. Amy inquired what the whole thing meant. Karol and Joey explained why they needed the area of the whole thing. Bruce disagreed with their answer. Joey explained how they got the dimensions of the larger rectangle. Bruce simply replied, “Got ya.” Joey verbalized doing the math for finding the area of the smaller rectangle. Everyone agreed and then subtracted.

Next, they focused on the rectangles that represented the free-throw lanes. Joey used inquiry to check the accuracy of his dimensions. This caused the others to use what they know about circles to draw a conclusion. Bruce questioned why Joey used six. Joey’s justification led Bruce to correct his own thinking and concluded the width was 12. Karol disagreed and Joey justified how he determined the dimensions. Karol inquired if they should add the area of the two rectangles to the shaded area they found earlier. Bruce said yes. Amy, Karol, and Joey first determined how many circles were represented in the drawing and then did the math. Bruce wanted to know why they
multiplied by 2 and Joey explained there were two circles. Amy gave the answer as 1694.2 ft then added squared. This was the first group to mention a measure. Amy suggested they subtract the answer they just found from the whole thing. Both Joey and Karol praised Amy. Each of them did the subtraction and concluded the answer was 4117.8. They forget to include the measure this time.

The two phases of the doing the math included 96 verbal exchanges. The verbal exchanges included verbalizing doing the math, inquiry, explanation, clarification, agreement, self correction, argument, justification, and giving answers. The levels of mathematical thinking exhibited included recognizing applying nested in building-with analyzing and building-with evaluative analyzing. When interpreting the dimensions from the drawing the students’ thinking was recognizing applying (Hershkowitz et al., 2001). Recognizing applying was also nested within building-with analyzing as the group knew to use a formula and then applied the formula. The students continuously reflected on the process of doing the math and took time to explain what they are doing to one another which represents building-with (Hershkowitz et al.).

The last segment involved understanding the task, strategy, doing the math, and collaborative checking. After reading the problem Joey stated they divide by 8 and then he did the math for finding the circumference of the circle. He told the others to divide by 8. Amy, a basketball player, asked why they are dividing by 8. Joey started to explain, but stumbled on the word arc. He confused arc length with arc measure. This was followed by silence as they appeared to process (Hoyles, 1985). I noted at this point they are trying to listen to Terry’s group. Joey told them to divide by 360. Karol disagreed. Joey returned to the circumference, suggested they divide by 8, and gave the answer as
4.71 then he told the others to divide 360 by 4.71. Karol argued they did not need to divide again. Amy did the math and inquired if that was the answer. Bruce confirmed her answer. Once again, Amy wanted verification from me that the group’s answer was correct. Once Amy realized I was not going to tell them, she suggested they check their calculations. Before they turned off the recorder, Bruce told the group he got a different answer for the area of the painted region. Bruce verbalized each step he did to find the area. Amy was doing the math with him through every step. At the conclusion they checked their answers. Bruce recognized when adding each of the shaded regions that he had written 4750 for the area of the shaded region outside the smaller rectangle rather than 4700. Amy’s collaborative check allowed Bruce to correct a mistake and recognize the mistake was an error in recording rather than understanding.

Sixty-five verbal exchanges occurred in the last segment. The verbal exchanges included verbalizing doing the math, agreement, inquiry, explanation, argument, giving answers, clarification, student correction, and justification. The levels of mathematical thinking were recognizing applying and building-with analyzing. The students applied previous knowledge of area formulas in a new context (Dreyfus et al., 2001) and reflected on each process explaining and clarifying as needed to further mathematical understanding (Hershkowitz et al., 2001). One person would put forth an idea, the students pursued the idea together, clarifying and justifying as needed based on another’s inquiry, and collaborative checking, as they did the math and verified their solutions. The collaborative checking among peers represented a sustained press for justification and explanation, a factor associated with maintenance of high-level cognitive demand (Stein et al., 2000).
Looking back, Amy stated giving plans was a waste of time. She admitted when she sees a problem she immediately begins thinking about how to get the answer. Amy preferred to explain after she got an answer. Amy concluded the time she spent writing the plan could have been better spent doing the math. Amy’s statement that planning the math was a waste of time could be justified by the issue of showing work discussed earlier. Showing work, the menial task of recording each step, for gifted students often counters the goal of thinking deeply about the mathematics involved in problem solving and can stifle the creativeness involved in problem solving (Sheffield, 1999). Counter to this justification is the underlying problem that many students are conditioned through previous experience in mathematics (Bishop, 1988) to obtain an answer quickly without thinking deeply about the mathematics involved. Based on the results, I would argue Amy’s statement was justified from the gifted aspect. Bruce wrote he did not like writing a plan because he usually knows what he is doing. Bruce also stated that listening to the others was difficult for him. Joey said the plan Amy stated in the beginning was so simple that they group rejected it. I had commented to the group that sometimes simplicity was elegance. Joey explained how Bruce would follow his idea silently while the remainder of the group worked until he got to a “checkpoint” and then Bruce would share his ideas with the group and seek verification. Karol said the project was fun and intellectual at the same time. She stated the group worked well together and she thought everyone in the group was very smart.

**Group 4.** Like the first two groups, the first segment involved understanding the task overlapped with strategy. Daren read the problem to the group and promptly questioned what kind of plan. Rita said first you find the area of everything. Daren agreed
and moving into character sent Bob off to get calculators as if he was Batman. Bob returned and whispered, “We’re on to Rita. We’re under attack. Since we are underground waiting for the zombies to attack we might as well work on this problem. Give a plan for finding the area to be painted. Find the area of the entire court.” Ignoring the fact that Bob was in character, Rita picked up his idea (Pirie & Schwarzenberger, 1988) and added minus the hardwood. Daren corrected Rita’s use of the word minus. Bob’s statement appeared to be outrageous, but he was processing his thinking about the task (Saul, 1999). Rita verbalized the plan as she wrote it down.

There were 12 verbal exchanges in the first segment excluding Bob and Daren’s lines in character. The verbal exchanges included inquiry, student correction, and verbalizing their plan. The level of mathematical thinking was recognizing comprehending and recognizing applying. The cognitive activity represented by these levels of thinking included understanding the previous knowledge that correlated to the task (Dreyfus et al., 2001; Williams, 2000). The students recognized they would need to use area formulas to find the areas of the shaded regions and subtract those areas from the area of the large rectangle.

The next segment was implementing the plan for finding the area to be painted, or doing the math. Rita suggested they find the area of the outside part first. Daren suggested a different way. He said they could just add up all the shaded areas. Rita reminded Daren they had to find the area of the hardwoods too. Daren incorrectly thought the dimensions for the hardwoods were not provided. The dimensions were provided only for the shaded region. The students had to extend their thinking to get the dimensions of the larger rectangle. Bob told them just to find the area of the whole thing. Bob pointed
out the dimensions were 50 times 94. Daren made the connection these are the dimensions for the hardwoods after all and argued they are not the dimensions for the larger rectangle and Rita agreed. Daren stated the length of the court was 102. Bob asked why 102. But before Daren responded he incorrectly said it is times 53. Daren then answered Bob’s inquiry, “94+4+4.” Rita agreed. Bob corrected Daren by telling him it would be 56 times 102. Rita argued those were not the dimensions. Bob used the drawing to show Rita there were 3 ft on both sides of the 50 ft width and 4 ft on both sides of the 94 ft length. Bob’s explanation with the use of the picture helped Rita understand. While Bob explained the math to Rita, Daren did the math. He gave the answer of 5712 ft. Rita took the lead and told the group to subtract 94 times 50. Bob questioned why. Rita explained subtracting 94 times 50 would give the area of the outer shaded region. Daren gave that answer as 4700 ft. Bob asked Rita why they just did that. Obviously, Daren did not understand either. Rita explained it gave the area of the outside part. At first Daren still did not understand what Rita meant by the outside part. Daren got very frustrated with himself when he did not see something instantly. Before Rita could explain further, he made the connection. Then Bob did not understand. He made the connection earlier that 3 had to be added to both sides of the width and 4 had to be added to both sides of the length to get the dimensions, 56 times 102 of the outside rectangle. Daren explained you subtract the area of the smaller rectangle from the larger rectangle. These exchanges repeated four times until Bob finally understood.

The first segment of doing the math involved 79 verbal exchanges. These verbal exchanges consisted of what to do, inquiry, agreement, challenge, argument, explanation, correcting, giving answers, verbalizing the math, and understanding. The levels of
mathematical thinking exhibited were building-with analyzing, building-with synthetic analyzing, building with evaluative-analyzing, and constructing synthesizing. The students applied known area formulas (Dreyfus et al., 2001) to find the area of the larger rectangle and the smaller rectangle which was building-with analyzing. While arguing the reasonableness of the result as well as the pathways used to obtain a solution, students were building-with evaluative analyzing (Williams, 2000). Daren was building-with synthetic analyzing when he suggested a different method for finding the total area of the shaded regions (Krutetskii, 1976; Williams). Bob was constructing synthesizing when explaining to Rita the discovered pattern of adding the indicated dimensions to both the length and width of the interior of the court to obtain the dimensions of the outside of the court (Williams).

Daren transitioned into the next phase of doing the math by telling the group they had to add all of the shaded regions inside. Rita agreed and started to find the area of the free-throw lane. Daren admitted he did not know what to multiply. The rectangle representing the free throw lane had a length of 19. The width of the rectangle which was the free-throw line was not given. That dimension had to be deduced from the radius given at the top of the key, the semi-circle. Daren wanted some clarification on what the 6 ft mark in the drawing represented and started to call me over. Bob sang, “There are arrows, there are arrows.” Daren replied, “Got it.” Rita explained the width was the diameter of the semi-circle, was 12. Daren did the math but used the diameter in the area formula for a circle rather than the radius. Rita did the math and gave the answer 26.5. Daren disagreed and said it was 28.26. Daren was correct, checked his math, and agreed with Rita. Daren did not use the correct numbers in his check, and did not catch the error.
Bob moved into character for a minute or so and involved Rita in the exchange. While this is going on Daren processed his thinking and obviously corrected his error. Rita inquired if their figure was correct and Daren correctly explained, “Times that (referring to the lane) by 2 and you get 569.04 plus 1012.” He continued, “Equal to 1581.04 as our answer plus” but Rita interrupted and asked if the circle was times two. Frustrated, Bob wanted to know what they had done. Bob needed to understand. This transitioned the group to an extended phase of collaborative checking.

The second segment of doing the math involved 70 verbal exchanges. The verbal exchanges used were mostly inquiry and explanation, but also clarification, doing the math, agreement, correction, argument and verbalizing doing the math. The levels of mathematical thinking exhibited were recognizing applying, building-with analyzing, building-with synthetic-analyzing, building-with evaluative-synthesizing, and an instance of constructing synthesizing. Most of the cognitive activity involved applying known mathematical procedures (Dreyfus et al., 2001) which was nested in building-with synthetic-analyzing as the students interconnected their assumptions (Williams, 2000) to determine how to find the area of the outside shaded region. Bob recognized the meaning of the arrows earlier and continued to carry this small discovery through (Williams) when he explained the meaning of the arrows in the top of the key, or semi-circle. This thinking was building-with synthetic analyzing. Rita constructed synthesized when she formulated a mathematical argument (Williams) to explain how to find the width of the free-throw lane. As the students worked, they constantly evaluated each other’s arguments for flaws or strengths (Williams).
Bob’s need to understand from the previous segment set up an extended phase of collaborative checking as Rita and Daren’s explanations were punctuated by Bob’s inquiry for clarification. As they talked, they were all doing the math again which served as a collaborative check (Pirie & Schwarzenberger, 1988). Rita stated she got 1694.2. Daren asked for what and Rita told him it was the whole thing. Bob gave a different answer and then Rita and Daren questioned their answer. Verbalizing the math and explanation helped their understanding. The following excerpt indicates the extent of their collaborative checking and that just getting an answer was unacceptable for them.

Task 1 represents the Basketball Court Renovation problem, G4 represents Group 4, and the number represents the line number of the transcription.

Task 1 G4 239    Bob:    What’s that for?
Task 1 G4 240  Rita:  This (pointing)
Task 1 G4 241 Daren: No that can’t be right.
Task 1 G4 242 Rita:  But you got the same thing I got so how did that work?
Task 1 G4 243 Daren: I don’t know. It’s 56.52 times 2.
Task 1 G4 244 Rita:  Yeah, well I have 55 because I rounded up once
Task 1 G4 245 Bob:  What’s 56.52? How did you get that?
Task 1 G4 246 Daren: Dude, 6 squared times pi
Task 1 G4 247 Rita:  56.55 times 4 equals 226.2
Task 1 G4 248 Bob:  Why did you times it by 2?
Task 1 G4 249 Rita:  You times it by 4
Task 1 G4 250 Daren: You don’t multiply 113.04 times 2
Task 1 G4 251 Rita:  Yeah you do
Task 1 G4 252 Daren: No you don’t
Task 1 G4 253 Rita:  Listen, are you ready? That right there is 56.55 so is that, so is that, so 56.55 times 4
Task 1 G4 254 Bob:  How do you know that is congruent to all that?
Task 1 G4 255 Daren: You’re talking about all these. Oh, I thought you were talking about just the middle.
Task 1 G4 256 Rita:  No
Task 1 G4 257 Daren: Ok then you’re right.
Task 1 G4 258 Bob:  How do you know? Oh I see why.
Task 1 G4 259 Rita:  Ok, so that’s the area of the painted region and you just subtract to get the area of the non-painted region.
Task 1 G4 260 Bob:  So wait, I don’t know what’s going on.
Task 1 G4 261 Rita:  We found the area of the painted region correct?
Bob: Right, ok, so let me see what I’ve got.

The segment of collaborative checking included 78 verbal exchanges involving explanation, inquiry, clarification, verbalizing doing the math, agreement, argument, justification and understanding. The levels of mathematical thinking exhibited included building-with analyzing nested within building-with evaluative analyzing, and constructing synthesizing. Bob inquired about how the group arrived at their answer and either Rita or Daren would explain. Bob listened to their explanations and asked for Bob was not content to just get an answer. He wanted to understand each step in the process of obtaining a solution. Doing the math again with Rita and Daren helped Bob understand and served as a collaborative check (Pirie & Schwarzenberger, 1988). The mathematical thinking involved in these interactive exchanges was nested within each other. Daren and Rita used mathematical argument to explain their discoveries (Williams, 2000) while Bob interconnected what he knows (Williams) with the arguments Rita and Daren made, evaluated the reasonableness of their methods allowing him to integrate and construct new knowledge. Hershkowitz et al., (2001) suggested that constructing is often the simultaneous use of recognizing and building-with, and constructions can be nested over several activities. In this case Bob’s constructions were nested in each repetitive cycle of the interactive exchanges.

The effort of the Daren and Rita to verify their solution to Bob left very little time for them to work on the last question. Rita read the question aloud. Daren started doing the math as she read. Rita must have been cognizant of what Daren was doing because she added divide by 8. Daren asked what she got. She repeated divided by 8 and gave the answer 4.71. Daren agreed. There was no further explanation given. Obviously, both Rita
and Daren recognized they just needed to divide the circumference of the circle by 8 and did the math. They demonstrated a keen awareness of what the other was doing although this was not verbally communicated. Bob, Daren, and Rita often operated with a keen awareness of the mathematical ideas they shared regardless of whether Bob and Daren were being outrageous. It was as if Rita also recognized the characters Bob and Daren portrayed simply was a cover for them while they processed their thinking. The high-level cognitive demand was maintained.

Looking back, Bob stated he had a good understanding of how to do the parts of the problem because he was applying concepts he had learned. Daren said he really enjoyed solving the problem. He said the questions were challenging and fun to do. He also informed me he chose drama as an elective in high school. Rita confirmed Bob and Daren always acted in the group. She stated, “After we recognized what we had to do to accomplish the problem, we all worked together to figure it out.”

Comparison and Contrast of Group Experiences for Task 1: Basketball Court Renovation

The activity of group 1 progressed through five segments involving several segments involving overlap of understanding the task and strategy, and several segments of doing the math. Making a plan was disconcerting for the students. The students used more than one pathway to find the area of the shaded regions applying known mathematical formulas (Krutetskii, 1976; William, 2000). Collaborative checking was ongoing through inquiry, clarification, explanation, and verification. The levels of mathematical thinking exhibited by group 1 included recognizing comprehending, recognizing applying, building-with analyzing, building-with synthetic analyzing, and
Group 2 experienced difficulty understanding what write a plan meant similar to the difficulty group 1 experienced understanding the introductory task. I attributed this to gifted students making the simple complex (Lovecky, 1994). Once the students got past writing a plan, they moved to doing the math. The work of group 2 progressed through three segments involving understanding the task, doing the math to find the areas to receive paint or hardwood finish, and doing the math to find the arc length available to each player. Like the first group, the students verbalized doing the math and used collaborative checking. Unlike the first group, the students intuitively understood how to find the arc length available for each player. Also, like the previous group, the levels of mathematical thinking exhibited by the students involved recognizing comprehending, recognizing applying, building-with analyzing, building-with synthetic analyzing, and building-with evaluative-analyzing. The level of cognitive demand of procedures with connections was maintained.

There was little discussion in group 3 about writing a plan, unlike the previous two groups. Like group 2, the students in group 3 intuitively understood the addition to the length and width to obtain the dimensions of the larger rectangle. The students worked through four segment of understanding the task, two extended segments of doing the math, and a combination of understanding the task, strategy, doing the math, and collaborative checking. The students in group 3 worked well collaboratively by one person putting forth an idea, pursuing the idea together, clarifying and justifying as needed based on another’s inquiry, and collaborative checking as they did the math and
verified their solutions. Like the two previous groups, the levels of mathematical thinking exhibited by the students in group 3 included recognizing comprehending, recognizing applying, building-with analyzing, building-with synthetic analyzing, and building-with evaluative-analyzing. Also like the other groups, the collaborative checking among peers represented a sustained press for justification and explanation, a factor associated with maintenance of high-level cognitive demand (Stein et al., 2000).

The students in group 4, like the students in group 3, gave little thought to writing a plan, but like group 1 had to think through the addition to the length and width to obtain the dimensions of the larger rectangle. The students in group 4 worked through four segments involving understanding the task, two phases of doing the math, and an extended segment of collaborative checking. The students in group 4, like the students in group 3, applied known area formulas to find the area of the larger rectangle and the smaller rectangle. Through argument, students evaluated the reasonableness of pathways used and the results (Williams, 2000). Students in group 4 also used collaborative checking among peers as a sustained press for justification and explanation, a factor associated with maintenance of the level of cognitive demand (Stein et al., 2000). Like students in group 2, students in group 4 intuitively knew how to find the arc length available to each student. The levels of mathematical thinking exhibited by the students in group 4 like the students in the other groups included recognizing comprehending, recognizing applying, building-with analyzing, building-with synthetic analyzing, and building-with evaluative-analyzing. The students in group 4 worked through the entire class period like the students in groups 1 and 3. As in the introductory task, the males appeared to process by moving into character. Bob, Daren, and Rita operated with a keen
awareness of the mathematical ideas they shared regardless of whether Bob and Daren were being outrageous. I did not see this level of shared understanding or level of camaraderie among the other groups.

The level of cognitive demand for the basketball court renovation task was procedures with connections (Stein et al., 2000) due to explicit pathways. The level of cognitive demand was maintained throughout the task by all groups. The levels of mathematical thinking exhibited across the groups included recognizing-comprehending and recognizing applying nested in building-with analyzing and building-with synthetic-analyzing, and constructing synthesizing. Finding the areas of the basketball court to be painted the school colors and those that would receive a hardwood finish built on students’ prior knowledge of area. Prior knowledge of circumference was used to find the arc length for available to each player during a jump ball. Building on the students’ prior knowledge of area allowed the students to draw conceptual connections. Students discussed what they had to do by devising a plan and then implemented the plan to obtain a solution through several pathways (Sheffield, 2000; Stein et al.). The students’ constant use of collaborative checking (Pirie, 1998) represented their sustained press for explanation, justification, and meaning (Stein et al.). The higher-level thinking indicated students’ work on the basketball court task translated into a deeper understanding of the mathematical processes, concepts, and the relationships involved (Hiebert, 2003).

Task 2: Smiley

Phase 2 of the Renovation Process: A dream has come true. The [teacher’s] lobbying efforts have paid off and finally there will be a math lab complete with Geometer’s Sketchpad. A picture made from Sketchpad has been enlarged to form a
mural on the wall of the math lab. In keeping with Maple Street colors, the circle will be black and the shaded regions will be painted blue. Determine the paint needed for each feature of “Smiley”. Show you work in an orderly manner.

The second mathematical task once again involved a fictional remodeling of the school. This time the task represented a mural for the wall in a newly designated math lab. Students previously used a geometry computer application program extensively in the study of polygons and circles and we often discussed the need for our own math lab due to difficulty with coordinating our class time with computer lab availability. The face pictured in this task was composed of shaded regions inside a circle. In keeping with the color scheme of the school, the students had to determine how much blue paint was needed for the shaded regions and how much black paint was needed for the unshaded regions. The face consisted of an equilateral triangle, right triangles, circles, portions of concentric circles, and portions of sectors. The task represented an open-ended task with no explicit or implicit pathways for solving. The level of cognitive demand was doing mathematics.

*Group 1.* The recorder used by this group did not work properly. I was unaware that there was a mechanical failure until the last task because the recorder was new. Consequently, the discussion within this group as they worked on this task was not captured on audio recording. I was left with only my field notes, student artifacts, observation notes to students, and their reflections for analysis.

Using my field notes, Tom suggested they find the area of the eyes first. Students’ work indicated the group followed Tom’s suggestion. They got $2\pi$ as the answer for the eyes. Once they found the area of the eyes they moved to the mouth. Tom again stated he
assumed the mouth was an isosceles triangle. It was an equilateral triangle. Obviously, Tom corrected his mistake. The students used special properties of 30-60-90 triangles to find the length of the base of the small triangles embedded in the equilateral triangle. Mike recognized the longer leg of each 30-60-90 triangle was 4. Using previous knowledge he put \( 4 = x \sqrt{3} \). Mike solved for the variable \( x \) and got 2.309 rounded to the thousandth place. The thinking represented was recognizing applying and building-with analyzing nested within building-with synthetic-analyzing. The students deduced the smaller triangles embedded in the equilateral triangle were 30-60-90 triangles from their previous knowledge of properties of equilateral triangles. They applied special properties of 30-60-90 triangles to find the value of \( x \). The students also used algebraic principles to solve the equation involving a radical. The high-level cognitive demand was maintained.

Next, the students divided the triangles in the shaded regions representing the nose and mouth into four congruent triangles. But rather than finding the area of one of the smaller triangles and multiplying by four, Tom returned to the large equilateral triangle and subtracted the area of the two congruent white triangles. From my field notes, I noted Mike rather ingeniously rotated and flipped one of the two congruent white triangles to form a rectangle with the other white triangle. His method represented originality. The students used special properties of 30-60-90 triangles to find the height of the large triangle. The height was the leg opposite the 60 degree angle so they solved for \( x \) and correctly got \( \sqrt{48} \) as the height. Using \( \sqrt{48} \) as the height, they found the area of the equilateral triangle and then subtracted the area of the rectangle composed of the two congruent white triangles. They got 18.477 as their answer. The levels of mathematical thinking represented involved recognizing applying and building-with analyzing nested
within building-with synthetic-analyzing. The students’ thinking involved building-with evaluative-analyzing when collaboratively checking their results. They used collaborative checking for the height of the equilateral triangle and they checked their results for the total area of the shaded regions as Tom and Mike used two different ways to find the area. The level of cognitive demand was maintained.

My notes indicated Tom and Mike explained how they could find the area of the hair by using an inscribed hexagon. Their discussion drew my attention. This group was the only group to use an inscribed hexagon and showed originality (Hekimoglu, 2004; Sheffield, 2000). The students found the area of the hexagon by multiplying the area of the equilateral triangle by 6 because there were six congruent equilateral triangles in the hexagon. These were clearly sketched on the task as shown in Figure 6. In addition, Mike used the formula for finding the area of a regular polygon using the apothem. His use of two methods represented flexibility. Next, they subtracted the area of the hexagon from the area of the circle to get the region outside the hexagon, but inside the circle. Then they divided by 6 because there were six sectors in the circle. This left the remaining area in the sector that represents the hair. I did not have any notes regarding how this group found the area of the eyebrows. Because there was limited work on their papers, I could only deduce they subtracted $9\pi$ from $16\pi$. I could see from Mike’s work he added each of the areas representing parts of the face together for the total area to receive blue paint. Each student correctly obtained the answer. The levels of mathematical thinking represented by the work of this group were recognizing applying and building-with analyzing nested within building-with synthetic-analyzing, and the highest level of constructing synthesizing. Although I only noted occasional instances of collaborative
Phase 2 of the Renovation Process: A dream has come true. Mrs. lobbying efforts have paid off and finally there will be a math lab complete with geometer's sketchpad. A picture made from sketchpad has been enlarged to form a mural on the wall of the math lab. In keeping with Maple Street colors, the circle will be black and the shaded regions will be painted blue. Determine the paint needed for each feature of "Smiley".

Figure 6. Mike’s work on Smiley.

cHECKING, from the level of their work, I assume they used building-with evaluative-analyzing. These students demonstrated deep thinking through their work on the task and maintained the high-level cognitive demand of doing mathematics. Unlike their work on
the first task, the students did not indicate they had any difficulty understanding the task, nor did they ask me any questions, or seek verification from me in any way.

Looking back, Mike was concerned that the recorder did not operate properly. He commented, “I really like these group problems because they challenge me and when I finally get it, it gives me satisfaction.” Kate indicated once the group figured out the hexagon was inscribed in the circle, the problem was easier to complete. Ethan simply stated he did not have anything to add to my observations. Tom was serving as a guide for the 5th-grade student orientation during this phase of data collection. Although asked to come in to provide his feedback, he did not which was unusual for Tom.

Group 2. I noted Chad was not wearing his usual tee shirt. The first segment involved understanding the task and strategy. A short discussion on why black was still amazing preceded the reading of the problem. Using a British accent, Terry read the problem. When I heard him reading, I knew I had made an error in the wording of the problem. I commented in my notes to the students this indicated Terry was actually decoding as he read. April also pointed out I misspelled a word. She also commented the mural did look like a smiley face. Sally disagreed and said it was more of a frowny face. Chad used inquiry for clarification, “So we determine the paint in the shade areas?” He also commented that he was supposed to think out loud from now on and then inquired if they could tell if they beat their own time for problem solving. First, Sally stated she had no idea how they were going to do this, but then suggested a plan and reasoned the radius would be 8. Chad started to give the area of the circle when Sally told him to shut-up. Chad reminded Sally the recorder was on. She ignored him and suggested they find the area of the circle and subtract the white areas to find the shaded areas. Chad argued that
would be more difficult than just finding the areas of the shaded regions. Terry suggested they find “this one” and Sally reprimanded him for not being clear about what he was talking about. He clarified he meant the white circle between the shaded circles. Terry argued the radius was 7 by counting the pieces. April pointed out he missed one piece and clarified the radius was 8. Sally summarized, “We all agree that the radius is equal to 8.” This recap was the transition to the next segment.

The first segment consisted of 50 verbal exchanges involving understanding the task and suggesting strategy. The verbal exchanges inquiry, general comments about the task, explanation, suggesting strategy, argument, clarification, and summarizing what they know. The levels of mathematical thinking involved recognizing comprehending as the students interpreted the task and recognized the inherent structure and recognizing applying when they adapted existing knowledge to the task (Hershkowitz et al., 2001).

The second segment involved doing the math for the area of the whole circle and discussing and implementing strategy for finding the area of the shaded regions in the equilateral triangle. April picked up on Sally’s summation and did the math to find the area of the large circle. Terry responded, “Wait, what?” Terry used this phrase frequently and appeared to be his way of processing. Terry concluded the big triangle was equilateral and reasoned because the radius was 8, both the unmarked sides of the triangles were 8 as well. Chad added the triangles inside the equilateral triangle were congruent. Sally told him he could not do that (make an assumption) and Terry accused Chad of relying on the art to draw a conclusion. Chad did not justify his statement, but he was correct in his thinking. Terry suggested they split the equilateral triangle and use the Path-a-go-rean Theorem. I commented on his mispronunciation in my notes to the
students. Through Terry’s response I learned this was an intentional mispronunciation and was an inside joke from the previous year and the source of the mispronunciation was his teacher. Sally used the Pythagorean Theorem and gave the answer of 6.9 as the height of the triangle. Then she directed someone to find the area of the triangle. April verbalized doing the math and gave the answer for the area of the equilateral triangle as 27.6. Chad told them to divide next and counted the small triangles inside the equilateral triangle to determine how many were inside. April inquired if they were all equal. Chad said they were, but Sally wanted justification. Chad explained, “You draw a line straight down the center you get all the right triangles that are given are equal. The white triangles given are all the same.” He tried very hard to articulate his thinking and admitted thinking out loud was hard for him. Terry summarized Chad’s explanation and Sally said she understood. Next, they collaboratively did the math to find the area of the shaded triangles that composed the mouth and nose. They verbalized both the process and the checks. April summarized the shaded areas inside the equilateral triangles equaled 18.4.

The second segment consisted of 72 verbal exchanges involving verbalizing doing the math, processing statements, inquiry, clarification, agreement, general statements, explanation, affirmation, giving answers, extending an idea, understanding, and summarizing. The levels of mathematical thinking included recognizing-applying and building-with analyzing nested within building-with synthetic-analyzing, and constructing synthesizing. The students applied previous knowledge of a procedure (Williams, 2000) to find the area of the whole circle. Building on the knowledge of the radius, Terry determined the large triangle was equilateral. He also recognized the equilateral triangle could be split and the Pythagorean Theorem applied to find the height
of the triangle. Chad recognized there were six congruent triangles inside the equilateral triangle and both he and Terry formulated mathematical arguments to explain discovered patterns (Williams).

The third segment involved doing the math for the eyes, eyebrow, hair, and getting a total for the shaded regions. April suggested they find the area of the two eyes. No one in the group discussed how they would do this. Terry gave an answer of 6.3, and inquired if he was right as a collaborative check. April and Chad agreed with his answer. Building on the knowledge each eye had a radius of 1, Terry concluded they could find the eyebrow. April thought they needed to divide by 2, but Terry explained because there are two eyebrows you did not need to divide. They collaboratively did the math and then checked their results. April summarized what they did, “4 squared times $\pi$ equals that minus 3 squared times $\pi$ equals” and Sally finished her statement, “22.” Next, they moved to finding the area of the hair.

Terry exclaimed he knew how to do it over and over. His explanation of how to find the hair was simple, yet elegant (Hekimoglu, 2004; Sheffield, 2000). He explained, “Alright, so the area of this entire circle is 201.6. Each of these sections are the same. Divide that by 6, then minus the area of that equilateral triangle from before.” Each member of the group affirmed him in some way. From here they implemented the strategy Terry laid out for them and compared their answers. April inquired if they just added up all of their values. Chad thought they needed to determine the number of gallons of paint. Terry assured him they only had to find the area that would receive the paint. He gave the answer for the total area as 52.61 units squared. I was amazed that someone actually included the measure. April summarized again as a final collaborative
check, “So we have nose, mouth, eyes, eyebrows, and hair. Ok.” Sally who was quite the pessimist at the beginning of the task remarked they worked together better this time. April called me over. I asked them if I could ask a few questions. I remarked that I had heard Terry’s explanation about the six congruent triangles inside. Terry did not take the credit though and told me it was Chad who recognized the triangles were congruent. Terry explained how they found the area of the hair. I asked for the correct name of the sector and only April could remember the word sector. Terry wanted to know if the group could have another problem. Terry desired more challenging opportunities to think deeply about mathematics (Hiebert, 2003; Sheffield).

The third segment consisted of 110 verbal exchanges involving inquiry, general statements, suggesting strategy, agreement, verbalizing doing the math, explanation, clarification, giving answers, affirmation, teacher questioning, and summarizing. The levels of mathematical thinking included recognizing-applying and building-with analyzing nested within building-with evaluative-analyzing, and constructing synthesizing. The students applied known mathematical procedures (Williams, 2000), the area of circles to find the area of the eyes and built on that information to determine the area of the eyebrows. Their use of collaborative checking was building-with evaluative-analyzing. Terry’s simple, but elegant explanation (Sheffield, 2000) of how to find the hair was constructing synthesizing.

Looking back, Chad said I covered most everything in my observation notes except that Tom needed to learn to say “Pothagerion” and then added he, meaning Chad, needed to learn to spell. Terry began by stating black was still amazing. He pointed out that in order to find out how much paint would be needed, the group needed to know how
many square units a gallon of paint covered. Terry responded he did not realize he used the expression, “Wait, what?” Terry stated the group went slower and explained more, but still finished first. Terry concluded, “Like black, we’re amazing.” April agreed that the group worked better on this task. She said that sharing her thinking out loud was still hard. April also admitted that her group was a little obsessed with beating their own time, getting the solution before other groups, and the color black. She also provided the reason Terry intentionally mispronounced Pythagorean Theorem. April was the only student to mention an interest in finishing ahead of the other groups. Although Sally insisted the face, smiley, looked more like a frowny face, she thought the problem was “cool”. She responded, “I think it’s cool how everything that we’ve learned this year is being applied in one problem.” I commented that she seemed to enjoy the challenge and addressed that as a group, in the past, they had not been challenged to think.

Group 3. The first segment involved understanding the task and strategy. Amy read the problem and commented on the ugliness of the smiley face. Joey inquired about the colors used and then suggested they find the diameter and radius of the circle indicating he knew where to start. Amy said they had to determine each feature and Karol said they had to determine each section. Bruce suggested they find the area of the white stuff and then subtract to get the area of the shaded regions. Amy, Karol, and Joey returned to the question of which sections represented blue paint and which represented black paint. Joey suggested they ask me which they did. Once again this is an example of making the simple complicated (Lovecky, 1994). Without any other hints or lowering the level of cognitive demand, I simply responded the shaded areas are the mouth, nose, and eyes when Karol interrupted and clarified through a statement the shaded areas were each
feature of the face. Amy made a connection to the castle problem they had earlier where they had to find the volume of all the features of the castle. This acknowledgement signaled they had the information needed to proceed to doing the math.

The first segment included 31 verbal exchanges involving general statements about the task, inquiry, clarification, agreement, questioning the teacher, and suggesting strategy. The levels of mathematical thinking exhibited were recognizing comprehending and recognizing applying. Amy, Karol, and Joey’s thinking remained recognizing comprehending while Bruce’s thinking indicated he understood what the task required when he suggested a strategy for finding the area of the features.

The second segment was the first phase in a repetitive cycle of collaboratively doing the math for each feature followed by collaborative checking. One student would begin with a statement and another student would pickup the idea (Pirie & Schwarzenberger, 1988) followed by another. This group started with the eye. There was no discussion about what constituted the radius of the eye as in some groups. Joey said let’s start and before he could finish Bruce interjected the eyes. Amy added the formula, Joey verbalized doing the math, Bruce doubled it because there are two eyes and Karol summarized, “Eye is $\pi$ so eyes is $2\pi$.” Bruce agreed and Amy shared what she did on her paper with the others. Next they focused on the eyebrows.

The second segment consisted of 10 brief verbal exchanges involving stating a formula, explanation, inquiry, and agreement. The level of mathematical thinking was recognizing-applying embedded in building-with analyzing. The students applied a known mathematical procedure in a new context (Dreyfus et al., 2001) when they recognized the radius and used the area formula for a circle to find the area of one eye.
Then the students recognized they must multiply by 2 to get the area of two eyes. The level of cognitive demand was maintained.

The third segment involved finding the area of the eyebrows. Joey asserted the radius of the big circle was 3 and to divide the area of the big circle by two. Karol argued the radius would be 4. Joey corrected the radius and said to find the area of the big circle and subtract the smaller circle. Bruce asked for clarification on which circle Joey was referring to. Amy clarified and Joey again verbalized doing the math. Joey squared 4 to get 16 and squared 3 to get 9. Operating in terms of $\pi$, he then subtracted. Bruce gave the answer as $7\pi$. There was a brief collaborative check. Joey thought they needed to add in another pi. Bruce disagreed. Joey justified why he thought another pi should be added and in the process corrected his own thinking. He realized he was adding in the eyes again. Bruce added $7\pi$ represented the whole thing. Joey picked up on Bruce’s lead and suggested dividing by 2. Amy concluded each eye was 11, Karol confirmed her answer, and Joey agreed both were equal to 22. Joey transitioned the group into finding the area of the nose.

The third segment consisted of 55 verbal exchanges involving explanation, verification, inquiry, clarification, verbalizing doing the math, argument, justification, and collaborative checking. The levels of mathematical thinking exhibited included recognizing-applying, building-with analyzing, and building-with evaluative-analyzing. The students recognized the radii of the semi-circles that composed the eyebrow. They applied what they knew about concentric circles to find the area of the shaded region between the two semi-circles. The students collaboratively evaluated both their methods.
for finding the area and the reasonableness of their solution (Williams, 2000). The level of cognitive demand was maintained.

During the fourth segment, the group vacillated between doing the math for finding the area of the nose, mouth, and hair. Joey suggested splitting the nose and mouth area down the middle. Amy offered a different way of splitting the triangle. Karol looked at the combined nose and mouth as two distinct triangles. Joey finally suggested splitting it down the middle to get two right triangles and use the Pythagorean Theorem assuming they found enough measures. Next, students silently processed their thinking (Hoyles, 1985). Next, they tried to apply previous knowledge to find the measures. Joey started by stating the radius was the same as the hypotenuse of the triangle so it must be 4 because it was the radius of the circle as well. Bruce and Amy argued the radius was not 4. Bruce explained the radius was 8. Joey realized the triangle was equilateral, but did not make the connection from what Bruce had just stated. Amy did not make the connection either as she thought Joey was just making assumptions. At this point the task began to decline because Joey suggested they measure the dashed lines. Amy and Karol refocused the discussion on finding another way to proceed and maintained the level of cognitive demand. Amy suggested another way, but repeatedly used the word “this.” She was in the process of clarifying when she dropped the recorder. At this point I checked out the group’s progress.

I asked what they were doing. Amy started to explain when Joey interrupted to ask me if they could assume the side of the triangle was a radius. Bruce clarified he meant half the radius. I told them they would have to tell me why they would do that. I gave no hint. Joey said it would be nice if the dashes were equal. I responded it would be
nice. I asked if they were asking me if a point was the midpoint. Joey continued to question me about making an assumption. Amy cautioned him again about assumptions so he inquired what the group wanted to do. They agreed to make the assumption that the point was the midpoint. Amy wanted to know how they could figure out the radius. I returned her question with a question about what represented the radius. She determined the radius was four. Bruce corrected her again and said it was 8. Then she returned to the eyebrows and reasoned that since the radius of the eyebrows was 4, then the radius of the big circle must be 8. Joey argued it was 10. Karol and Amy said it was definitely 8. I asked for clarification and then asked about the other side. My question led them to recognize that the triangle was equilateral. Amy recognized the two right triangles were congruent and said she knew how to find the unknown measures. She did not have an opportunity to pursue this line of thinking because Joey switched the focus of the group to the hair. More group indecision about how to proceed followed. Joey began to discuss arc measure. Amy informed him that would not help. Karol and Joey discussed finding the area of the whole circle and dividing, but they did not pursue this line of thinking either. They returned to the mouth and nose. Again silence followed this decision. The level of cognitive demand was in danger of decline. Although they did not pursue their ideas, Amy recognized how to find the area of the right triangles and Joey and Karol suggested a strategy for finding the area of the hair.

Karol proposed a conditional, “If this is 4, then this would be a 30-60-90.” Joey picked up on the idea (Pirie & Schwarzenberger, 1988) and summarized what they knew. Amy argued that the triangle could be 45-45-90. Joey countered the triangle would have to be isosceles if it was 45-45-90. Joey indicated the three pair of triangles inside the
equilateral triangle were congruent. Bruce wanted justification. Joey answered it was because of their angle measures. Bruce was not convinced and inquired if they could just be similar. Joey then tried to force Bruce’s consensus. He told him to say he understood. Bruce refused to say he understood when he did not understand. Karol wanted to know why he did not understand. Amy intimated that if she understood, then he should. Karol, Amy, and Joey verbalized doing the math. Unfortunately, Karol took half of the half radius. Amy actually drew an accurate sketch of the situation, but listened to Karol and Joey’s justification of why the side was 2. This was the second time Amy had the correct information, but did not pursue what she figured out or use her drawing to find the area of the nose and mouth. Karol explained what she was doing, “Because this is 4, and because it is 30-60-90, you know this little fat side would be half of 4 which is 2.” Karol used the wrong side to draw this conclusion. She then calculated the area using the base of 8 and the height of 2 and got 16 for the mouth. Joey argued if the two pieces of the nose were rotated, they would be congruent to the mouth as well so the area of the nose was 16 as well. Bruce who has been silent since Joey tried to force him to say he understood followed them through doing the math and said, “I think your calculations are off.” Rather than give Joey a chance to justify what the others did, he began to share his thinking about finding the area of the hair. It appeared that Bruce did not transition back to the mouth and nose when the others did.

The fourth segment consisted of 164 verbal exchanges related to discussing strategy and doing the math for the nose, mouth, or hair. The verbal exchanges involved suggesting strategy, inquiry, clarification, explanation, verbalizing doing the math, assertion, questioning the teacher, student correction, questioning the students, argument,
agreement, trying to force consensus, justification, and collaborative checking. The levels of mathematical thinking exhibited involved recognizing-applying and building-with analyzing nested with building-with synthetic-analyzing, building-with evaluative analyzing, and constructing synthesizing. Joey recognized the triangle could be split down the middle and the Pythagorean Theorem could be applied. Amy, Karol, and Bruce suggested different pathways for solving the problem (Krutetskii, 1976; Williams, 2000). His thinking represented flexibility (Hekimoglu, 2004; Sheffield, 2000). Hershkowitz et al., (2001) suggested that building-with can be used incorrectly as with Karol when she used applied 30-60-90 triangle measures using the wrong leg. Bruce constantly evaluated the reasonableness of the methods and the results of what the group was doing (Williams) and refused to be forced into a consensus without understanding. Collectively, the students exhibited building-with while engaged in the problem-solving process of explaining, understanding, and reflecting on the process (Hershkowitz et al.). They also exhibited constructing by reorganizing information to refine their methods for solving the task (Hershkowitz et al.).

The fifth segment involved doing the math for the hair following Bruce’s explanation and the collaborative checking of the results of the area for all the features. Bruce explained half the circle was $8\pi$. He suggested finding the area of one triangle and multiplying by 3 and then subtracting that figure from $8\pi$. Joey tried to recap what Bruce said, but did so incorrectly. Bruce corrected him and then told the group the triangle at the top was congruent to the triangle on the bottom which was equilateral. Rather than give Bruce time to continue, Joey interrupted and verbalized doing the math. He arrived at the answer of 8.9 as the height of the triangle. Bruce disagreed saying the height was
the $\sqrt{48}$ . Amy used Bruce’s height to find the area of the triangle. Rounding she gave the answer as 28. Then she multiplied by 3 to get 84. Joey inquired how she got the answer. Bruce responded Amy found the area of the three triangles. Joey checked again by verbalizing doing the math using the smaller triangle with measures that Karol found by incorrectly applying properties of 30-60-90 triangles. Amy did the math again too, and then told Joey that both she and Bruce got the same answer. Joey inquired what they did. Amy explained she used the Pythagorean Theorem. Joey insisted he too had used the Pythagorean Theorem. Amy explained how she and Bruce arrived at 6.9 as the height of the triangle. Joey and Karol worked as she explained. Karol got the same answer and Joey recognized that he had added 4 squared and 8 squared rather than subtracting 4 squared from 8 squared. This collaborative check allowed Joey and Karol to correct their thinking. Joey then subtracted the answer from $8\pi$ and got a negative causing him to realize the group should have subtracted from $64\pi$ rather than $8\pi$. Karol, Amy, and Bruce all agreed. Next, they divided by 2 because of the semi-circle. They continued to do the math collaboratively and constantly verified answers with one another. The group subtracted the area of the three triangles inscribed inside the semi-circle and divided by three to get the area of the hair. Then, each member of the group added the area for each feature together and verified the answer of 50.08 with the other members. Even though Bruce suggested earlier their calculations were off, he did not address how they should find the area of the shaded triangles within the equilateral triangle. The fact that he also got 50.08 led me to believe that his earlier remark was specific to the height of the triangle only. Karol wondered if they had completed the task. Bruce reminded them they
needed to find the black region too. Karol suggested they just subtract the area to be painted blue from the area of the whole thing.

The fifth segment consisted of 102 verbal exchanges involving explanation, inquiry, agreement, justification, clarification, argument, collaborative checking, verbalizing doing the math, student correction, and giving answers. The levels of mathematical thinking exhibited involved recognizing-applying and building-with analyzing nested with building-with synthetic-analyzing, building-with evaluative analyzing, and constructing synthesizing. Bruce exhibited constructing synthesizing through his mathematical argument to explain his insight (Williams, 2000). Following Bruce and Amy’s explanation, Karol and Joey reorganized and integrated what they learned, representing constructing (Hershkowitz et al., 2001). Both recognizing and building-with, such as comparing the two methods, are nested within constructing. Joey’s thinking exhibited building-with evaluative-analyzing when he got a negative answer, realized there was a flaw in their method (Williams), and led the group to correct their thinking.

The last segment involved my interaction with the students. I noticed the group was concluding their work on the task and approached them to ask what they did. I had noted earlier Karol used the wrong leg when she applied special properties of 30-60-90 triangles. I assumed the collaborative checking would reveal the error. Karol simply showed me the answers for the area of the blue and black regions. I asked the group if I could ask a few questions and then inquired how they found the area of the mouth and nose. The following discussion relates my press for justification of the students’ solution.
Task 2 G3  420  Joey:  Ok, So we have, we know that its an equilateral triangle 8-8-8. OK, then we know this is a 90 triangle and because this is a central angle (pointing), this is 30 and this is 60. Add these two together its 120 and because it is isosceles these are 30 and 30. And then these two triangles are congruent, you know that that height is the same as this height which would be four. We found the area of this triangle by doing 2 times 8 divided by 2.

Task 2 G3  421  Me:  Why?
Task 2 G3  422  Joey:  You got all the angles right?
Task 2 G3  423  Me:  Yeah
Task 2 G3  424  Joey:  Ok, then so you know these two triangles are congruent.
Task 2 G3  425  Me:  Ok.
Task 2 G3  426  Joey:  Ok, if you flip it like this, Ok, we know that, Ok we flip it Ok, Ok. I’ve got it. This is the base which is 8 and since it splits it in two then we know that would be four. This would be 4 (pointing at each corresponding part), this would be 4 and this would be 4.

Task 2 G3  427  Me:  Yes, but what do you know about the altitude?
Task 2 G3  428  Joey:  I didn’t mean the altitude. I meant that (referring to the smallest leg of the white triangle).
Task 2 G3  429  Me:  How did you find the height?
Task 2 G3  430  Amy & Karol:  We used the Pythagorean Theorem.
Task 2 G3  431  Bruce:  This leg would be half of this, not this.
Task 2 G3  432  Me:  Pardon me?
Task 2 G3  433  Bruce:  This leg is half of this, not this.
Task 2 G3  434  Amy:  (Gasps)
Task 2 G3  435  Me:  I don’t understand.
Task 2 G3  436  Amy:  Bruce, what did you say?
Task 2 G3  437  Karol:  Oh my gosh. He’s right.
Task 2 G3  438  Amy:  Oh my gosh. What’s going on?
Task 2 G3  439  Joey:  What did you say?
Task 2 G3  440  Bruce:  What we did was we said this is 2 because we thought it is half of this (referring to the larger leg of the white right triangle). But it is not. It’s half of that (referring to the hypotenuse).
Task 2 G3  441  Karol:  We used 30-60-90 and we did that wrong.
Task 2 G3  442  Joey:  No…………
Task 2 G3  443  Bruce:  We mixed up our legs.

Questioning the students about the method they used to find the area of the nose and mouth caused them to reexamine what they did. Bruce recognized without specific
probing questions from me that the smaller leg was not half of the larger leg as they had assumed, rather was half of the hypotenuse. Bruce’s statement caused the others to also recognize their mistake. I was surprised that they had not corrected this earlier because Bruce and Amy had already told the group the height of the triangle was 6.9 when they found the area of the hair. I summarized everything they told me. Once they got the area of the triangle I asked them what they should do next. Karol took the lead and said divide by 6 because there were six congruent triangles. My last question was a leading question, but did not cause a decline in the level of cognitive demand. Hershkowitz et al. (2001) suggested students thinking can exhibit building-with when the teacher provides a hint. Each student did the math again and took turns giving his or her answer. Bruce asked one final question, “Are we done?”

The last segment consisted of 55 verbal exchanges involving teacher questioning to elicit student reflection on the process they used to find the area of the mouth and nose. The verbal exchanges included teacher comments and questions, explanation, clarification, student correction, verbalizing doing the math, and collaborative checking. The levels of mathematical thinking included recognizing applying nested within building-with analyzing, building-with synthetic-analyzing, and building-with evaluative analyzing. As a result of my questioning, the students evaluated the reasonableness of their method (Williams, 2000). They reflected on the process they used (Hershkowitz et al., 2001), and Bruce recognized the error the group made. His discovery caused the other members of the group to also recognize their error and correct their thinking.

Looking back, Bruce stated he could not think of anything to add. Joey said the problem was a huge success and that his group did a much better job of working together
on this problem. He admitted that sometimes he likes to jump to a conclusion and shut people out and apologized saying, “My bad.” Amy liked the fact that Joey and Karol always explained their thinking to her and admitted her frustration with Bruce because at times he did not explain more. I asked her how she could encourage him to share more. In my observations, I think Bruce attempted to share more, but his slow methodical explanations were “shut out” as Joey admitted. Karol specifically addressed my notes. She said they took a long time getting started with the social chat because they wanted to make sure I heard their “beautiful voices.” Karol agreed that Joey’s idea of counting the dashes was “goofy”, but added he realized he was incorrect. She also commented on how smart Bruce was and that they should listen to him more. Overall, she thought the group worked well together.

**Group 4.** Bob read the problem and processed while he read. There was a brief discussion about which part of the circle would get the black paint and which section would get the blue paint. Bob then stated he knew what to do. Rita and Daren were skeptical and challenged how he knew. The group launched into doing the math. This segment involved ten verbal exchanges involving inquiry and what they know. The level of mathematical thinking exhibited was recognizing comprehending. Bob’s thinking ahead represented recognizing applying because he understood what mathematical idea to use. The level of cognitive demand was maintained.

The second segment was the first phase of doing the math. Bob started with the eyes and stated they have a radius of one. Rita and Daren thought the one was just part of the picture. Bob argued the dotted line between the pupil and the eyebrow was two. Daren said he understood. Rita acknowledged Bob was correct and concluded the half
circle of the eyebrow had a radius of four. Rita acknowledged Bob was correct, picked up the idea, and concluded the eyebrow had a radius of four. Daren gave the answer 3.14. Bob clarified each eye was 3.14. Bob took on the character of Jerry Lewis and said, “Each eye is approximately 3.14 or $1\pi$. Each eye equals a $\pi$.” Daren inquired if they should leave it in terms of $\pi$. Bob said to leave it in terms of $\pi$ because it was more accurate. Rita called to me to ask if they should use pi. While they waited for me to come over to address their question, Rita decided the area of the pupils could not be $2\pi$ because you have to multiply $\pi$ times $\pi$. Bob tried to explain, but she argued. Daren told her it was not 3.14 times 3.14 rather 3.14 plus 3.14. Rita said, “If its $6\pi$ and $3\pi$ then it would be $9\pi$.” Obviously, she did not realize what she said. She got agitated at herself. After Daren repeated what she said, she finally understood she was just adding like terms.

This caused Bob and Daren to move into their characters. I included the following section of the dialogue to indicate how the temporary move into characters actually could serve as a method of processing for these students (Saul, 1999).

| Task 2 G4 68 | Daren: | We voted you off the island |
| Task 2 G4 69 | Rita: | Oh, the island again. I thought I was off already. |
| Task 2 G4 70 | Daren: | Let’s sacrifice her to the aliens |
| Task 2 G4 71 | Bob: | We’re going to broil, boil |
| Task 2 G4 72 | Daren: | Bob, these are our choices: Fried, baked, broiled, boiled, grilled, nuked, reheated, deep thawed, steamed, sautéed, pickled, smoked, salted. That’s about it. |
| Task 2 G4 73 | Bob: | To find the eyebrows (Rita interrupts) |
| Task 2 G4 74 | Rita: | That’s $2\pi$. Two $\pi$ for the eyes. |
| Task 2 G4 75 | Bob: | Now to find the eyebrows, you have to find first the big circle which has a radius of four |

After going through the outrageous routine, in Task 2 G4 73 Bob returns to the mathematical discussion and in Task 2 G4 75 completes his explanation about how the group should proceed.
The second segment involved doing the math related to the eyes and included 32 verbal exchanges. These verbal exchanges involved explanation, inquiry, argument, justification, agreement, verbal checking, and verbalizing doing the math. The levels of mathematical thinking exhibited included recognizing applying nested in building-with analyzing. The students knew when to use a mathematical idea (Williams, 2000) and applied known mathematical procedures in a new context (Dreyfus et al., 2001). The level of cognitive demand was maintained.

During the third segment, the students worked collaboratively to find the area of the eyebrows. Daren stated he had $16\pi$. Bob said they had to subtract. Rita exclaimed, “I get it! I get it! The radius is 3, yes, the radius is 3.” Bob incorrectly assumed you could subtract $3\pi$ from $4\pi$ and get $1\pi$. After Daren explained Bob had to square 4 and then divide by $1/8$, all three started to argue, and verbalize their calculations. Rita explained if you subtract $9\pi$ from $16\pi$ you get $7\pi$ for the eyebrows. Bob agreed with Rita. Daren admitted he was wrong. Rita inquired if he understood. Bob verbalized what they did. Daren started to add the eyebrows, but Rita told him the eyebrows are combined.

The third segment of doing the math included 28 verbal exchanges involving argument, inquiry, explanation, agreement, checking understanding, and verbalizing the math. The levels of mathematical thinking exhibited in this section were recognizing applying and building-with analyzing nested in building-with evaluative-analyzing. The students applied known mathematical procedures in a new context (Dreyfus et al., 2001) and evaluated the reasonableness of their methods (Williams, 2000). Daren recognized his error and corrected his thinking from Rita’s explanation of the process she used. The level of cognitive demand was maintained.
The fourth segment involved strategy as the group explored possible methods for finding the area of the hair. Daren pointed to the right angle symbols in the big triangle (they have not recognized the triangle is equilateral) and said that was all they have. Bob pointed out the measure of the base of the triangle was 8. Daren picked up the idea (Pirie & Schwarzenberger, 1988) and inquired if it was vertical because it intersects in the center and that segment would also be 8. Rita made a connection, “Wait I know this. This is. OK ready? This side length is 8 because it is the radius of the bigger circle. Because the radius of half of the radius of the big circle is 4. So, the radius of the bigger circle is 8.” Daren concluded all the triangle sides were 8. Rita correctly stated they were equilateral triangles. Once the students had this major insight, Bob returned to social chat. He appeared to be processing again. Bob returned to the mathematical discussion. Rita had another insight. Her explanation and Bob’s inquiry lead him to understand how to find the area of the hair. Again, Bob showed his need to understand rather than just except an answer. The following discussion shows how Rita’s mathematical argument aided Bob’s understanding.

Task 2 G4 157 Rita: Wait, I know how to find this. If you find the area of the whole circle and divide it by 6, that will get the area of each pie piece. You take the area of the whole circle and divide it by 6, that will give you each one of these pieces (meaning the sector).

Task 2 G4 158 Daren: Take 1/6th of the remaining stuff.

Task 2 G4 159 Rita: And if we take the area of this triangle and subtract from 1/6 of the big circle you’re going to get the area that piece of the arc area right there.

Task 2 G4 160 Bob: How are you going to get all of these by dividing by 6?

Task 2 G4 161 Rita: Ok, if you take the area of the bigger circle and divide by 6 to get each pizza piece.

Task 2 G4 162 Bob: by 6 to get each pizza piece.

Task 2 G4 163 Rita: Let’s say we have that pizza there.

Task 2 G4 164 Bob: 8 by 8 by 8 (meaning the equilateral triangle)

Task 2 G4 165 Rita: If you take the area of this triangle subtracted from this little pizza piece, then you’ll get that. The pizza piece. [Teacher],
what is that called? $1/6^{th}$ of the triangle. Symmetrical slayer? Slant height, slice

**Task 2 G4 166** Bob: I don’t understand how you got that. What did you subtract?

**Task 2 G4 167** Rita: The whole piece of that pie and subtract that triangle your going to get that little piece left. Get it?

**Task 2 G4 168** Bob: Ah hah! To find that little piece you find $1/6^{th}$ of the pizza cone and inside that cone there is a triangle that is 8 by 8 by 8, and subtract that you get the area of the shaded region.

The fourth segment included 38 verbal exchanges involving inquiry, explanation, insight, what they know, and verification. The levels of mathematical thinking are recognizing applying, building-with synthetic-analyzing, building-with evaluative-analyzing, constructive synthesizing. Hershkowitz et al. (2001) suggested students construct when they use what they know to build more complex structures and they are simultaneously using recognizing and building. Rita integrated concepts to create new insight (Krutetskii, 1976; Williams, 2000) regarding the length of the sides of the triangles and how to determine the area represented by the hair. In addition, Rita formulated mathematical arguments to explain her insight (Williams). Bob constantly evaluated the information Rita provided to build with it to achieve an understanding (Williams). The level of cognitive demand was maintained.

During the fifth segment, the students collaboratively did the math for the strategy they devised in the previous segment. Rita stated to find the area of the triangle they had to find the height. Daren recognized he could use 30-60-90 triangles to find the height of the triangle and Bob told him to drop an altitude. They did the math and got two different answers. Rita thought they had to use the apothem. Daren explained you could use the Pythagorean Theorem, but he used trigonometry. Daren demonstrated fluency (Hekimoglu, 2004; Sheffield, 2000) by recognizing there were several approaches he could use to find the area of the triangle. Rita verbalized doing the math. Satisfied their
answers agreed, they found the area of the equilateral triangle and gave the answer as 27.7. Daren told them to multiply the area of the triangle by 6. Rita began carrying on two different conversations with both males. She followed Daren’s lead while she explained to Bob what she had done. Bob gave the area of the circle as $64\pi$ and one slice as $10.6\pi$. Daren explained he subtracted all the areas of the triangles from the circle so he had 6 crusts left over and then divided by 6 to get the area of the individual crust, also known as the hair. Rita verbalized doing the math. Even though Bob gave the answers for both the circle and each slice, he had Rita explain the process to him again. Once again, Bob did the math as he followed her explanation. She would explain and Bob would give the answer. Bob summarized what they did to get the area for the hair and then inquired if they wanted to get it out of terms of $\pi$. Rita gave the answer as 5.8. They continued to check their results collaboratively for an extended period prior to moving to the nose.

The fifth segment of doing the math related to finding the area of the hair included 133 verbal exchanges. The verbal exchanges included inquiry, explanation, argument, clarification, correction, verbalizing doing the math, and verbalizing what I did. The levels of mathematical thinking exhibited were building-with analyzing, building-with synthetic-analyzing, and building-with evaluative-analyzing. Hershkowitz et al. (2001) suggested that using structures from previous activity for further action represents building-with. The students used what they built-with and constructed in the previous segment to find the area of the hair. In the process, they applied previous knowledge of mathematical procedures to interconnect numerous operations to achieve a goal (Dreyfus et al., 2001). The level of cognitive demand was maintained.
The last segment was strategy and doing the math. Daren recognized the midpoint of the side of the equilateral triangle and dropped an altitude. Bob summarized what they know. Rita and Daren did the math and compared their answers, but were unhappy with the results. Rita explained, “If the one leg of the right triangle is 4 because it is half the radius of the big circle and the little angle is 30, then to find the hypotenuse you can do adjacent over the hypotenuse.” Bob inquired why they want to find the area of the two white right triangles when all they have to do is find the area of the shaded region. Rita clarified they needed the area of the two white triangles to subtract from the equilateral triangle to get the area of the shaded region. Daren gave the answer as 4.5. Rita challenged him. When Bob did not get what the others got, he started from the beginning to check his calculations. I came over to check on the group and asked how they got their solution for the nose and mouth. Daren explained what they did. He told me they used the midpoint to divide the triangle into two parts, used trigonometry to get the area of the white triangles, and then subtracted from the larger triangle.

The last segment included 124 verbal exchanges related to strategy and doing the math. The segment also included 43 verbal exchanges coded as social chat that served as a way of processing. The verbal exchanges involved clarification, summarizing, inquiry, argument, explanation, agreement, and collaborative checking. The levels of mathematical thinking exhibited were building-with analyzing, building-with synthetic-analyzing, and building-with evaluative analyzing. The students applied known mathematical procedures in a new context (Dreyfus et al., 2001) often using more than one pathway (Krutetskii, 1976; Williams, 2000), and evaluated the reasonableness of the pathway and the solution (Williams). The level of cognitive demand was maintained.
Looking back, Bob explained he wanted to leave his answers in terms of $\pi$. Regarding his sometimes outrageous behavior, he said “Rita holds me and Daren in line.” Daren said I should continue to provide collaborative problem-solving opportunities for my students in coming years. Rita expressed her pleasure that she was able to make connections to the things she learned previously to new concepts and to explain to others how to do the math. I had remarked that if she could put up with Bob and Daren she could do anything. She also expressed her enjoyment of the problem-solving experience even through all the outrageous behavior.

**Comparison and Contrast of Group Experiences on Task 2: Smiley**

The students in group 1 showed improvement in their collaborative effort on this task, perhaps as Mike put it, because their minds were warmed up to thinking mathematically. The students also appeared to be enjoying the challenge. Once they understood they had to find the area of the shaded features of the face, they launched into doing the math. The students easily found the area represented by the eyes and eyebrows and moved to the mouth. They used special properties of 30-60-90 triangles to find the length of the base of the smaller triangles embedded in the equilateral triangle. This group ingeniously used rotation and reflection to create a rectangle from two congruent triangles representing the white area, found the area of the equilateral triangle, and then subtracted the area of the rectangle. Group 1 was the only group to use an inscribed hexagon to find the area of the region that represented the hair. The students also used several pathways to find the total area of the hexagon. The group concluded their work within one class period. The levels of thinking exhibited by the students in group 1 was
recognizing comprehending, recognizing applying, building-with analyzing, building-
with synthetic analyzing, and building-with evaluative-analyzing.

The students in group 2 approached the problem by first finding the area of the whole circle, then the equilateral triangles. The work of group 2 progressed through three segments involving an overlap of understanding the task and strategy, and two phases of doing with math. The students used the Pythagorean Theorem to find the area of equilateral triangle rather than special properties of 30-60-90 triangles used by the students in group 1. Like the first group, the students verbalized doing the math and collaborative checking. The students in this group, like those in the first group, had no difficulty finding the area of the eyes or eyebrows. Terry provided a simple but elegant strategy for finding the area of the hair by dividing the area of the whole circle by 6 then subtracting the area of the equilateral triangle. The students in group 2 exhibited the same levels of mathematical thinking as the students in group 1. The levels of mathematical thinking included recognizing comprehending, recognizing applying, building-with analyzing, building-with synthetic analyzing, and building-with evaluative-analyzing.

Like group 1, when the collaborative effort was improved, the students appeared to enjoy the challenge, and the work was completed in one class period.

The students in group 3 worked through six segments before arriving at a solution. The segments involved an overlap of understanding the task and strategy, three phases of doing the math, an overlap of strategy and understanding the task, doing the math, and teacher questioning to elicit student reflection on processes. Unlike the other groups, the students in group 3 encountered difficulty understanding which regions were to be painted blue and which regions were to be painted black. This group was the only
group to make the connection to a previous castle problem in which students had to find the volume and surface area of various geometrically shaped towers to find the total surface area and volume of the castle. Like the other groups, the students easily found the area of the eyes. Finding the area of the eyebrows required some clarification. Unlike the other groups, this group vacillated between doing the math for finding the area of the nose, mouth, and hair. The level of cognitive demand declined when Joey pursued measuring the dashed lines. The others students were able to refocus the discussion and maintain the level of cognitive demand. I provided scaffolding to move their thinking regarding the equilateral triangle. The other groups did not require scaffolding. The students used special properties of 30-69-90 triangles, like group 1, to find the area of the equilateral triangles, but used the wrong leg in their calculations. The students recognized their mistake only when pressed for justification of their answer. The students in group 3 exhibited the same levels of mathematical thinking as the students in groups 1 and 2. The levels of mathematical thinking included recognizing comprehending, recognizing applying, building-with analyzing, building-with synthetic analyzing, and building-with evaluative-analyzing. The collaborative effort of group 3 on this task lacked the cohesiveness represented in the interactions of the other three groups.

The students in group 4 understood the task like the students in groups 1 and 2. Group 4, like group 3, carried out the work through six segments. The segments involved a brief understanding the task, two phases of doing the math, strategy, doing the math, and an overlap of strategy and doing the math. The students in group 4, like the other groups, easily found the area of the eyes and the eyebrows. Unlike the students in the other groups, the students recognized, discussed, and implemented multiple pathways for
finding the area of the equilateral triangle. Collectively, group 4 was the only group to relate the circle, equilateral triangles, and the hair to a pizza, slice, and crust respectively. The males of this group processed their thinking by moving into character. The work of the group remained collaborative in all respects.

The level of cognitive demand for the task was doing mathematics (Stein et al., 2000) due to multiple pathways for solving. The high-level cognitive demand was maintained throughout the task by all groups, except for one instance in group 3. The levels of mathematical thinking exhibited across the groups included recognizing-comprehending and recognizing applying nested in building-with analyzing and building-with synthetic-analyzing, and constructing synthesizing. The task built on students’ previous knowledge of equilateral triangles, right triangles, circles, portions of concentric circles, and portions of sectors. Multiple pathways (Sheffield, 2000; Stein et al.) could be used to solve the problem. Except for several segments of work by group 3, the students’ constant use of collaborative checking (Pirie, 1998) continued to represent sustained press for explanation, justification, and meaning (Stein et al.). Sheffield (1999) stated students recognize the high-level cognitive demand required of peers engaged in problem solving, challenging them to extend their own thinking. I recognized this affect across all groups. The students in each group expressed their enjoyment involved in the challenge of the smiley task.

Task 3: Julie’s Wheel

Julie has three wheels from bikes and things that she stacked against the shed. Each wheel fit so neatly together that she couldn’t resist taking thing photograph. The
radius of the largest wheel is 16 cm and the radius of the middle-sized wheel is 9 cm.

What is the radius of Julie’s smallest wheel?

The third mathematical task used was called Julie’s Wheel (nzmaths, n.d.). Julie had three bicycle wheels she stacked against a shed. Julie noticed each wheel neatly fit together. The larger wheel was tangent to the middle-sized wheel. The smaller wheel, positioned between the larger wheel and the middle-sized wheel, was tangent to both wheels. Students were told the radius of the larger wheel and the radius of the middle-sized wheel only. Students had to find the radius of the small wheel. Julie’s Wheel represented an open-ended task with no explicit or implicit pathways for solving. The level of cognitive demand was doing mathematics (Stein et al., 2000).

Several problems occurred during data collection for Julie’s Wheel. The 8th-grade students traditionally go to Washington, D.C. for three days in May leaving only one week prior to semester exams with three of those days designated for review. I collected data up to the very day students left for the Washington, D.C. trip. I planned accordingly leaving two days for exploration with Julie’s Wheel. Julie’s Wheel was an extremely high-level problem for the students requiring knowledge of algebraic and geometric principles. The task built on the students’ knowledge of the Pythagorean Theorem, perfect square trinomials, and simplifying radicals. Students had to first deduce these concepts were needed. I realized from the results of the Pilot Study the students would encounter difficulty with this task and their frustration could lead to a decline in the level of cognitive demand. Consequently, I was prepared to provide a hint (see Appendix G).

The first day I briefed students that the problem was not obvious. I told them if they felt like they had exhausted all of the resources they knew to apply, they could ask
for the hint. The students were determined to solve the problem without the hint. Most groups spent the entire first day working without resolution although several groups came close. I had a choice to make: Stop the task and provide the students with notes on the work I observed thus far to get their feedback, or continue with the task and forego the opportunity to get the students’ feedback. The focus of the study was mathematical thinking as evidence of understanding rather than student reflection. I decided it would be best to continue to allow students to develop a deeper understanding of the mathematics related to the problem rather than ending the work with no resolution. I also realized after the first day, some students needed a more structured hint (see Appendix H).

The structured hint was used on the second day. I provided three sketches of Julie’s Wheel on one sheet, one for each embedded right triangle and asked them to find the missing leg. Then, I asked the students to look closely at segments HC and CD and find the sum. Several groups had made progress with the simple hint and I felt certain others could have completed the task with the simple hint had we had more time to work. Providing the structure lowered the level of cognitive demand to procedures with connections, but it was worth the trade-off in terms of my students progressing toward mathematical understanding. The hint still afforded the students the opportunity to make connections to the relationships involved in the task.

The second problem I encountered with Julie’s Wheel was a nonfunctioning audio recorder. One of the newer audio recorders was assigned to Group 1 and worked fine for the introductory task and the basketball court task. The recording for the third task was blank. I assumed the group forgot to turn the recorder on. For the final task, the students made sure the indicator light was on while discussing the task. Shortly into their
discussion, the participants realized the light went off. They reset the recorder and the same problem occurred again. At this point we realized there was a problem with the recorder. I quickly changed the batteries and they tried again only to encounter the same problem. Not anticipating such a problem, I sent the students to the library for a tape recorder. Unfortunately, the library recording equipment was quite antiquated and although the rather large machine appeared to be recording, that tape was blank also. Therefore, there was no recording of the mathematical discussion for the last task for Group 1 other than the first several minutes.

*Group 1.* The results are from my interpretation of the students’ work on the task, my field notes, and one page of transcription that was salvaged. Kate drew what appeared to be a right triangle. Then she drew the hypotenuse of what she thought was a right triangle. Kate did not include any notation that she added the two radii together. The larger leg of what she thought was a right triangle emanated from the center of the larger circle, but was not tangent to the auxiliary segment drawn along the bottom of the wheels. Kate dropped perpendicular segments from the center of the large and middle-sized circles to the auxiliary segment. Then, she drew parallel lines, one from the center of the middle-sized circle and the other passing through the center of the smaller circle. Kate marked two 45 degree angles in the sketch. It appeared Kate rotated the radius of the larger triangle to mark a segment 16 cm on the auxiliary segment and did the same with the middle-sized circle getting the two 45-45-90 triangles. She added the two rotated segments to get the sum 25 cm. From there she used properties of 45-45-90 triangles to get the two legs. From the short transcription, I learned Tom suggested the triangles were 45-45-90. It was also Tom who gave the sum of 25 for the distance between the two
perpendicular segments. Had Kate used a compass she would have realized the two segments overlapped and therefore her method was not reasonable. The vertices of all the segments she drew are labeled. No other work for day one is evident on Kate’s paper. Kate failed to see the triangle she drew was not a right triangle. She also failed to recognize the two radii formed a hypotenuse of a different right triangle. Hershkowitz et al., (2001) suggested that students are building-with even when doing so with “inappropriate structure” (p. 217). With the simple hint, Kate correctly labeled the hypotenuse for each embedded triangle. She did not label the legs of the embedded right triangles. There is no indication she recognized each hypotenuse was the composition of two radii. On the second day, Kate successfully found the radius of the smaller circle using the more structured hint. She applied the Pythagorean Theorem to find leg IB = 24, correctly labeled the second triangle in terms of x, and substituted the squared binomials into the Pythagorean Theorem correctly. Kate simplified the radicals, added like terms, squared both sides of the equation, and correctly gave the answer for the radius of the smaller circle as 144/49 for the radius of the smaller wheel. The level of mathematical thinking was building-with analyzing due to lowering the level of cognitive demand to procedures with connections. Kate incorporated numerous procedures to achieve the goal of finding the radius of the smaller circle.

The work on Tom’s paper supported the collaborative effort of using rotation to create 45-45-90 triangles. Tom did not mark the right triangle Kate incorrectly included in her sketch. Included on Tom’s paper is a 45-45-90 triangle embedded in the smaller circle with the radius marked \(\sqrt{2}\). Also, inside the smaller circle Tom has the radius marked x in two other places. Like Kate, he got 16\(\sqrt{2}\) for the hypotenuse of the right
triangle he created rotating the radius of the larger circle, even though this was done incorrectly. He reasoned he could subtract the diameter of the smaller circle, $2x\sqrt{2}$ according to his math, from $16\sqrt{2}$ to find solve for $x$. Tom failed to recognize he had two different measures for the radius of the same circle. Like Kate, Tom was building-with based on incorrect structure (Hershkowitz et al., 2001). Equation 2 represents the proportion Tom used:

$$\frac{16}{9} = \frac{16\sqrt{2} - 2x\sqrt{2}}{9\sqrt{2}}$$

(2)

With the simple hint, Tom also correctly labeled the embedded triangles. Even though he labeled the hypotenuse $AB$ as 25, he did not pick up this figure when he applied the Pythagorean Theorem to find leg $HB$. He used only part of the hypotenuse of 16 instead which caused him to get the wrong value for $HB$. Tom tried to set up another equation, but used incorrect values. I could not figure out how he got the binomials he used to set up the second equation. Tom attempted to use the FOIL method, multiplication of first terms, outside terms, inside terms, and last terms to find the product of the square of a binomial. On the second day with the help of the structured hint, Tom correctly labeled the drawings, applied the Pythagorean Theorem, squared the binomials, simplified radicals, and combined like terms. Tom did not isolate $\sqrt{x}$ before squaring both sides of the equation, but arrived at the same answer. Tom’s work indicated he used the FOIL method to square the binomials. On day one, Tom built-with using incorrect structure. On day two, Tom’s thinking was building-with-analyzing and building-with evaluative-analyzing because he applied numerous procedures to find the radius of the smaller wheel and evaluated the result (Krutetskii, 1976; Williams, 2000).
Mike’s paper included markings not seen on Kate, Tom, or Ethan’s papers. Mike drew numerous right triangles in his sketch. He also appeared to rotate the radius of the large circle and the radius of the middle-sized circle to get the length of the segment along the auxiliary segment. He did not indicate these legs formed 45-45-90 triangles like the others. Mike correctly labeled the hypotenuse of one embedded right triangle $16 + x$, the hypotenuse of the smaller embedded right triangle $9 + x$, and the hypotenuse of the third embedded triangle 25. His paper indicated he attempted to set up a proportion, but erased what he had. Mike had the information necessary to solve for $x$. Given more time, he may have solved the problem without the hint. One of the factors associated with maintaining high-level cognitive demand is appropriate time (Stein et al., 2000).

Although Mike did not need the simple hint because he had recognized the embedded triangles previously, on day two, Mike added segments to the structured hint, and like the others applied the Pythagorean Theorem, squared binomials, simplified radicals, combined like terms, and solved for $x$. Mike extended his thinking beyond building-with analyzing to constructing synthesizing and constructing evaluating (Krutetskii, 1976; Williams, 2000) by establishing a rule for finding $x$, the radius of the smaller wheel given the radius of the larger wheel, and the radius of the middle-sized wheel. Mike gave the following rule where $r_1$ is the radius of the larger wheel and $r_2$ is the radius of the middle-sized wheel and $x$ is the radius of the smaller wheel. His rule is shown as equation 3.

$$\left(\sqrt{4r_1x} + \sqrt{4r_2x}\right)^2 = (r_1 + r_2)^2 - (r_1 - r_2)^2$$

Even though the level of cognitive demand of the task was lowered to procedures with connections, Mike demonstrated doing mathematics, the highest level of cognitive demand (Stein et al., 2000) by relating, evaluating, investigating, and creating a
mathematical rule (Hekimoglu, 2004; (Krutetskii, 1976; Sheffield, 2000). Mike demonstrated a deep understanding of the mathematical concepts involved and showed originality and fluency by establishing a rule.

Ethan’s paper had the least amount of work. He correctly labeled the circle with the radii of both circles, wrote 16 along the piece of the hypotenuse for the large circle, and 9 along the piece of the hypotenuse for the middle-sized circle. He did not add these two together to find the length of the hypotenuse. Although Ethan’s sketch was simple, the three embedded triangles are clearly visible. On day two Ethan correctly labeled each hypotenuse of the three embedded right triangles. There is no indication from his work that he moved beyond the labeling.

**Group 2.** Chad’s tee shirt for the day read, “Jenius.” Sally read the problem and first suggested they use guess and test, then suggested they used trigonometry. Terry inquired if they could just draw lines from the center of each circle to the points of tangency. Both Chad and Sally commented Terry was on the right track. Terry verbalized his thinking which caught my attention and I joined the group. He had constructed perpendicular segments. I inquired how they could create additional right triangles. Terry connected the centers of the circles to one another and then drew lines parallel to the auxiliary line through the centers of the circles. Terry marked all of the right angles in his sketch. I asked the group about the composition of the hypotenuse in one of the triangles Terry recognized was a right triangle. He responded one hypotenuse was 16 plus whatever x was and the other hypotenuse was 9 plus whatever x was. Terry said he saw how to find the other leg, but commented they would not know what they had to take away. I asked, “Why wouldn’t you?” From this he realized that it was a part and said 16-
x was one side. Sally asked for an explanation. Terry explained, “Now we are drawing a line across here to make that a right triangle and when we make this go across, we are taking this x length from 16.” He then told them to use the Pythagorean Theorem.

This first segment consisted of 57 verbal exchanges related to understanding the task and strategy. The verbal exchanges involved general statements, inquiry, suggesting strategy, focus statements, verbalizing doing the math, affirmation, inquiry, insight, summarizing, clarification, and explanation. The levels of mathematical thinking exhibited were recognizing comprehending and recognizing applying nested in building-with analyzing, and building-with synthetic-analyzing. Terry created right triangles and reasoned the hypotenuse lengths were the composition of the radius of the larger circle plus x and the other the radius of middle-sized circle plus x. These insights represented constructive synthesizing (Krutetskii, 1976; Williams, 2000).

The second segment involved doing the math. April summarized what they know about the leg of one right triangle. Terry added the length of the hypotenuse of the right triangle and then inquired what the other leg was. Sally and Chad guessed 16 and Terry disagreed. Rather than use the Pythagorean Theorem with the information they had, they worked to find another way to find the length of the leg. Sally, April, and Chad experimented with scale and proportions while Terry continued to work. April added the radius of the larger circle and the radius of the middle-sized circle to get the length of 25 for the hypotenuse. Terry let the unknown leg equal y and tried to solve. He multiplied the binomials incorrectly as indicated in Figure 7. The others picked up the idea (Pirie & Schwarzenberger, 1988) and tried to solve for x. Terry realized he could not take the square root of negative 2 and called me to over.
Figure 7. Terry’s work on Julie’s Wheel on Day 1.

Joining the group, I looked over his work and asked, “If this is 16 and this is x like you said and this is 16 and this is x and you are subtracting to get this leg, which is the hypotenuse?” I then asked what happens when you square a binomial. Sally responded they could use FOIL, which represents the multiplication of the first terms, multiplication of the outside terms, multiplication of the inside terms, and multiplication of the last terms, followed by combining like terms. Terry was persistent and asked if his answer was correct. I responded he still had another right triangle to find. He inquired
why the other two triangles were needed when he could subtract $x^2$ from $(16 + x)^2$ to solve for $x$. I clarified $x$ represented the length of the missing leg, not just the little piece.

The second segment of doing the math consisted of 91 verbal exchanges. The verbal exchanges involved verbalizing doing the math, clarification, agreement, inquiry, argument, general statement, focus statements, suggesting strategy, explanation, justification, and teacher questioning. The levels of mathematical thinking included recognizing applying and building-with analyzing nested within building-with synthetic-analyzing and building-with evaluative-analyzing. Terry integrated what he knew and correctly set up an equation to solve for the unknown leg. Even though he multiplied the binomials incorrectly, he was building-with synthetic-analyzing. He evaluated the reasonableness (Williams, 2000) of his answer, building-with evaluative-analyzing and realized there was a problem. Terry also demonstrated building-with synthetic-analyzing when he questioned me as to why the other right triangles were needed. The high-level cognitive demand of doing mathematics was maintained.

The third segment was a continuation of doing the math. April stated they should just find the length of the hypotenuse for each of the right triangles. Terry argued that one right triangle did not have an $x$ embedded in a length. April clarified which triangle she meant. Terry and Chad picked up her idea (Pirie & Schwarzenberger, 1988) and indicated they knew how to proceed. Terry verbalized doing the math. April compared her expansion of the binomial squared with Terry and suggested they cancel each other out. Terry reminded her she had to distribute the negative. Chad gave the time remaining in class.
Joining the group again, Terry insisted the leg was 64 \( x \). I responded, “If this is 7 and this 25, can you find the length of this?” April then said to put that equal to what they just found. Sally suggested \( 24 = 64 \times + 12.73 \) and April argued. April realized that they needed to put 24 equal to the values for each piece of the leg. April and Terry had failed to simplify the radicals \( \sqrt{64x} \) and \( \sqrt{36x} \) to get the value of the leg. Chad gave the time remaining in class.

The third segment consisted of 109 verbal exchanges related to doing the math and social chat. The verbal exchanges included inquiry, clarification, agreement, argument, general statements, focus statements, verbalizing doing the math, explanation, and teacher questioning. The levels of mathematical thinking were recognizing, building-with analyzing, and limited building-with evaluative-analyzing. The students continued building-with analyzing until they could go no further due to algebraic errors. April’s thinking exhibited building-with evaluative-analyzing when she argued the equation Sally came up with was not reasonable. April realized the legs of the two smaller right triangles equaled 24, but due to the algebra errors in solving for \( x \), they could not make any further progress until this mistake was corrected. Chad’s preoccupation with time contributed to the frustration level. This group experienced insight while working on the previous two tasks and not arriving at a solution was distressing for them. Span & Overtoom-Corsmit (1986) suggested the failure of gifted students to reach a solution allows them to examine the reasons why in order to stimulate thinking to employ new strategies. The level of cognitive demand of doing mathematics declined in this segment simply because they could not progress due to the algebra error.
On day 2, the students corrected their algebra error and obtained a solution based on their progress from day 1. Each student completed the parts of the structured hint to ensure their mathematical understanding of what they did on day 1. The level of cognitive demand was lowered to procedures with connections due to the structured hint. The level of mathematical thinking was building-with analyzing since students applied the Pythagorean Theorem, combined like terms, simplified radicals, and solved for x. The collaborative checking of answers was building-with evaluative-analyzing as the students checked the reasonableness of their answers (Williams, 2000). Closure for this group preserved their joint efficacy (Gillies & Ashman, 2003).

Group 3. The Asian sensation and two white kids signed on, read the problem, and worked to understand the task and discuss strategy. Amy said the task was easy because all they had to do was add the two diameters. Bruce admitted he did not remember a lot about tangents. Joey took the lead. First, he subtracted 9 from 16 and said x must be seven. Next, he asserted the diameter of the smaller circle was half the radius of the middle-sized circle. Finally, he measured. Amy admonished him for making assumptions and Bruce reminded him the drawing would have to be to scale. Bruce suggested drawing segments from the center of the circles to the points of tangency on the auxiliary segment. I noted Joey displayed behavior that could not be attributed to processing. Karol asked about a formula. The group resorted to the textbook.

The first segment consisted of 75 verbal exchanges related to understanding the task and discussing strategy. The verbal exchanges included inquiry, suggesting strategy, assertion, argument, agreement, explanation, and insight. The level of mathematical thinking was recognizing comprehending as the students tried to understand the task.
Bruce interjected a few ideas, but no one picked up the idea and progressed toward achieving the goal.

I recognized there was no activity in the group and offered the hint. Stein et al., (2000) suggested that without support, high-level cognitive demand can decline into unsystematic exploration. Bruce took the hint although Amy and Joey pressured Bruce not to look at it. Karol found the length of the hypotenuse of the large right triangle by adding 16 and 9. Sally determined the segment Bruce dropped earlier formed a right angle and that the hypotenuse of the corresponding right triangle was equal to $16 + x$. Amy asked me if the problem involved the Pythagorean Theorem. I said yes and looked at her paper. Her sketch was more thorough than the sketch in the hint I provided. I told Amy because she understood the composition of the hypotenuse of each right triangle she should now be able to make some progress. Karol subtracted 9 from 16 to get 7 and then applied the Pythagorean Theorem using the hypotenuse of 25 and the leg 7. Bruce gave the answer as 24. Next they worked to find the two legs whose lengths equaled 24. Karol used FOIL to multiply the binomials and thought they canceled each other out. She forgot to distribute the negative through the quantity. Amy used $(16 - x)^2$ twice in her equation rather than subtracting from $(16 + x)^2$. Joey stated he got $2x$. The period ended and Amy’s departing comment was, “So frustrated”.

The second and last segment involved doing the math. There were 54 verbal exchanges in this segment excluding Joey’s nonmathematical monologue. The verbal exchanges involved general statements verbalizing doing the math, explanation, inquiry, argument, and collaborative checking. The levels of mathematical thinking involved recognizing applying, building-with analyzing, building-with synthetic-analyzing, and
building-with evaluative-analyzing. Karol and Amy recognized components of the hypotenuse, made connections about lengths of legs of the right triangles, and applied the Pythagorean Theorem using the information they had. The collaborative checking of the process and solutions was building-with evaluative analyzing (Williams, 2000). The level of cognitive demand was doing mathematics for Karol and Amy because they progressed without using the hint. The level of cognitive demand was procedures with connections for Bruce and Joey because Bruce used the hint and Joey was operating on information provided by the others rather than from insight.

During the second day the students were able to easily label each sketch and do the math because they had progressed to this point already. They took the square root of both sides of the equation, simplified the radicals, added like terms, set up the third equation, and correctly solved for x. Karol commented the structured hint helped her visualize the embedded triangles. The level of cognitive activity declined to procedures with connections, but still involved higher cognitive activity of building-with analyzing since the students applied numerous mathematical procedures in a new context (Dreyfus et al., 2001) and building-with evaluative-analyzing through constant collaborative checking.

**Group 4.** The first segment involved understanding the task and strategy. Rita read the problem and stated this was easy. Bob instructed the group to draw a line on the base circle and label it 16 cm. Daren inquired if anyone remembered how to find the lines that pass through a tangent. Rita asked if they could just measure the radius of the smaller circle. Daren reminded Rita the sketch was not to scale. Rita guessed a radius of three and said she would work backwards. Several formulas were suggested. As a result of Bob’s explanations, I joined the group. Bob had the others draw a segment from the center of
the large circle through the center of the smaller circle that intersected the auxiliary line. Rita asked if they could do areas. I asked her why. Bob tried to apply the theorem for two secants of a circle drawn from an exterior point. Daren argued there was no secant to use.

The first segment consisted of 77 verbal exchanges involving understanding the task and discussing strategy. The verbal exchanges involved inquiry, suggesting strategy, verbalizing what they already know, agreement, argument, teacher questioning, self correction, and justification. The levels of mathematical thinking exhibited were recognizing comprehending when students are interpreted the structure of the task. The level of cognitive demand was maintained.

Rita began drawing and Bob summarized what Rita did. He said, “She drew a horizontal and vertical radius in the large circle and connected them to make a 45-45-90 triangle.” He continued, “Then she did the Pythagorean thing and got 22.6 equal to the hypotenuse of the triangle in the big circle. And now we don’t know what to do.” Rita laughed and said she was experimenting. Rita said she needed to draw a new picture because she messed up and needed to see it. I gave her another copy. Rita showed insight when she concluded the circles all had to connect in some way. Rita took a different approach and then there would be silence again. After several cycles of this, I asked how they were doing. Rita wanted the hint, but Daren and Bob did not. I let them work a while longer. The discussion turned to non-processing social chat and the level of cognitive demand began to decline.

There were 53 verbal exchanges in the second segment excluding the social chat. The verbal exchanges included inquiry, verbalizing what they did, insight, argument, suggesting strategy, agreement, explanation, and clarification. The level of mathematical
thinking was building-with analyzing because the students were testing structural knowledge to build with to achieve a solution (Hershkowitz et al., 2001). The level of cognitive demand was maintained at the beginning of the segment. The extended social chat that differed from possible processing signaled a decline in the level of cognitive demand.

Bob started the last segment by asking for the hint. I joined the group again and told them some of the things they had sketched looked like my hint. I did not give them the hint. I used probing questions to guide them to see they could connect segments. Rita exclaimed, “You can connect those hypotenuses!” Daren commented he had already voted Rita off the island once. Bob commented how Rita stomped them in the ground every single day in mathematics. She asked them if they realized how close they were meaning close to a solution. Rita continued to lead the group in exploring different possibilities. Her drawing indicated she connected the segments to create the hypotenuses of the right triangles as I questioned her. Rita then drew the right triangle with the hypotenuse composed of the radius of the smaller circle and the middle-sized circle and concluded it was a 30-60-90 triangle. Rita used the sine function to find the length of the smaller leg. Daren questioned Rita about the larger leg with the hypotenuse composed of the radius of the larger circle and the radius of the middle-sized circle. Rita explained how she arrived at the answer of 24 for the length of the larger leg. Daren repeated her answers with each explanation responding each time either with agreement, further questioning, or argument.

I returned to the group again to see how they had progressed. I looked over their sketches and asked what they did. Daren first said he was stuck and then looked down at
his sketch again and remarked, “It’s the Pythagorean Theorem.” Bob asked Daren to show it to him. Rita concluded they needed to find the equations for the triangles.

The last segment consisted of 123 verbal exchanges as the students continued to test strategies and do the math. The verbal exchanges involved teacher questioning, what they know, insight, explanation, inquiry, agreement, argument, general statements, self checking, clarification, collaborative checking, and justification. The levels of mathematical thinking included building-with analyzing, building-with synthetic-analyzing, and building-with evaluative-analyzing. The students used previous knowledge as they explored different approaches to achieve the goal (Krutetskii, 1976; Williams, 2000). Rita used constructing synthesizing through her explanations of what she did. Daren and Bob were able to build-with as a result of evaluating Rita’s justifications (Hershkowitz et al., 2001; Williams). Although the students did not arrive at a solution, the level of cognitive demand was raised to doing mathematics and was maintained.

At the close of class on day 1 the students recognized they could use the Pythagorean Theorem. On day 2, students built on the knowledge gained on day 1 and with the help of the structured hint, applied the Pythagorean Theorem using the binomials, combined like terms, solved for x, and simplified the radicals arriving at the solution.

Comparison and Contrast of Group Experiences on Task 3: Julie’s Wheel

Unlike the other groups, the students of group 1 experienced frustration with inoperative equipment. Dealing with the unexpected event put them behind the others in time spent on the task. The students rallied once they got started. The students applied
properties of 45-45-90 triangles and tried to set up a proportion except for Mike. All students completed the task on the second day with the structured hint. Mike extended his thinking to constructing synthesizing and constructing evaluating by establishing a rule for finding the radius of the smaller wheel. The level of mathematical thinking included recognizing comprehending, recognizing applying, building-with analyzing, building-with synthetic analyzing, building-with evaluative-analyzing, constructing synthesizing, and constructing evaluating.

The students in group 2 made progress through three segments on the first day due to Terry’s insight. The segments included an overlap of understanding the task and strategy, and two phases of doing the math. The students worked to determine the composition of the leg and hypotenuse lengths. The students encountered some difficulty multiplying binomials. The simple hint did not help these students because their work had already progressed to that level. They achieved a solution once they corrected the algebra error. The students completed parts of the structured hint to ensure their own understanding of the complex parts involved in the task. The level of mathematical thinking included recognizing comprehending, recognizing applying, building-with analyzing, building-with synthetic analyzing, building-with evaluative-analyzing, and constructing synthesizing.

The work of group 3 progressed through only two segments. The first segment involved an overlap of understanding the task and strategy. The second segment involved doing the math. I intervened in group 3 when I recognized the level of cognitive demand was declining. Bruce was the only willing student to take the hint. I pressed the students for justification of what they had done. Amy had progressed further than she realized and
had the lengths expressed as binomials. On the second day, the group picked up where they left off solving the equations. The structured hint was much easier for them because they had already progressed to this point on day 1. Like groups 1 and 2, the structure hint helped them understand the underlying mathematical structure (Hershkowitz et al., 2001). The levels of mathematical thinking exhibited included recognizing comprehending, recognizing applying, building-with analyzing, building-with synthetic analyzing, building-with evaluative-analyzing, and constructing synthesizing.

The students in group 4 first approached the problem similarly to the students in group 1. The work of group 4 progressed through three segments. The segments included an overlap of understanding the task and strategy, strategy, and teacher questioning. I also intervened in group 4 when I recognized the level of cognitive demand was declining. When pressed them for justification, the students made connections like group 3. Similar to students in the other groups, Rita made some progress, but needed some guidance to make connections to what she had to move forward. The progress Rita made on day one carried over on day two as she explained to Bob and Daren what she had done. The students were able to complete the task using the structured hint like the other groups. The levels of mathematical thinking included recognizing comprehending, recognizing applying, building-with analyzing, building-with synthetic analyzing, building-with evaluative-analyzing, and constructing synthesizing. The high-level cognitive demand of procedures with connections was maintained.

Julie’s Wheel was an extremely high-level cognitive demand problem without explicit or implicit pathways for solving. Solving the problem required application of algebraic and geometric principles once the students grasped the mathematical structure.
embedded in the task. The task built on the students’ knowledge of the Pythagorean Theorem, perfect square trinomials, and simplifying radicals. The students had to first recognize these concepts were needed. Several groups accurately depicted the embedded right triangles, and correctly represented the composition of the leg and hypotenuse lengths. Several groups made progress the first day. Each of the groups completed the task with the scaffolding provided through the structured hint on the second day. Even the students who had achieved some success on the task without the hint completed the structured hint. Julie’s wheel challenged students’ thinking and provided an opportunity for them to develop a deeper understanding of mathematical concepts involved.

Summary

I began the chapter with a restatement of the purpose of the study and the research questions that guided the study. The purpose of this study was to examine the ways mathematically gifted student think about and do mathematics creatively as indicators of deep while working collaboratively on open-ended tasks with high-level cognitive demand. The questions that guided the study were: How is mathematical understanding of 8th-grade gifted geometry students elicited through exploration using open-ended problems? What levels of mathematical thinking do 8th-grade gifted geometry students demonstrate when engaged in collaborative problem-solving tasks with high-level cognitive demand?

Next, I presented an overview of issues pertinent to understanding the data reported. I remind the reader, although students worked collaboratively, individual gifted students approach problems in diverse ways. In addition, I described the students’ interactive talk in the context of mathematical discussion (Pirie & Schwarzenberger,
A review of the Mathematical Tasks Framework (Stein et al., 2000) and the work of Dreyfus et al., (2001), Hershkowitz et al. (2001), Williams (2000, 2002), and Wood et al. (2006) on mathematical thinking followed due to the centrality of these frameworks in reporting the data.

Then, I described the method used in reporting the results. Goetz & LeCompte (1984) described the essence of ethnographic research as the “holistic depiction of unconstrained group interaction” (p. 51). Reporting followed the segments of discussion derived from the data analysis of each group’s transcribed audio tapes in order to preserve the functional relevance, the relationship of the parts to the whole, of what was occurring in the problem-solving discussions (Erickson, 1977). I weaved the analysis and interpretation of my field notes, student work, and my reflections, or notes to the students into the reporting of the transcribed segments to better depict the whole. After a review of the context of the mathematical task, I presented each group’s results nested within that task. I concluded the presentation of the results for each group related to the basketball court task and the smiley task with a look back, using the students’ reflections. The lack of student reflection for the introductory task and Julie’s Wheel was explained in the presentation of the results. I then compared and contrasted the groups’ experiences within each task. The tasks included the introductory task, the basketball court renovation, smiley, and Julie’s wheels.

Following the discussion on the method of reporting, I introduced the group participants. I provided a brief description of each student participant within each group. Next, I reviewed the problem-solving task and then described the group problem-solving experiences within the task. I reported the data for each of the four groups by
The introductory task was intended to acquaint students with the audiotaping. The levels of mathematical thinking evident through the transcribed audio recording, student artifacts, and my field notes related to this task revealed creative thinking that was “original, fluent, flexible, and elegant” (Sheffield, 1999) and warranted inclusion in the discussion. The level of cognitive demand according to the MTF (Stein et al., 2000) was doing mathematics and was maintained.

The first task referred to as basketball court renovation, a modified version of a similar task, required students to apply area formulas previously studied. The level of cognitive demand of the task according to the mathematical task analysis was procedures with connections (Stein et al., 2000) and was maintained. The second mathematical task also involved a fictional remodeling of the school. The task represented a mural for the wall in a newly designated math lab. The mural was a circle containing shaded geometric figures representing features of a face. The task represented an open-ended task with no explicit pathways for solving. The level of cognitive demand was doing mathematics (Stein et al., 2000) and was maintained. The third mathematical task used was called Julie’s Wheel (nzmaths, n.d.). Students had to find the radius of the small wheel between and tangent to larger-sized wheels. Julie’s Wheel represented an open-ended task with no explicit or implicit pathways for solving. The level of cognitive demand was doing mathematics (Stein et al., 2000). The level of cognitive demand of doing mathematics
was maintained the first day. On the second day, the high-level cognitive demand of doing mathematics was lowered to procedures with connections due to instructional decisions to provide a hint. Providing the hint still afforded my students the opportunity to make connections to the relationships involved and achieve mathematical understanding.

The levels of cognitive demand of doing mathematics and procedures with connections required higher-level thinking. Students within groups consistently exhibited mathematical thinking at the levels of building-with analyzing, building-with synthetic-analyzing, building-with evaluative-analyzing, constructing-synthesizing, and constructing evaluating (Dreyfus et al., 2001; Williams, 2000; Wood et al., 2006).
CHAPTER 5
DATA ANALYSIS

This chapter begins with the findings of the study positioned in the context of the literature and the research questions. The discussion includes the results of this study in the context of the studies that supported my research, collaborative grouping, the mathematical thinking framework utilized by Wood et al. (2006) and the MTF (Stein et al., 2000). Next, I summarize the findings of the study situated in the social constructivist theoretical views that framed the study. I conclude the chapter with a brief summary.

Findings in the Context of the Literature

Sriraman (2003) examined the experiences of nine students in an accelerated ninth-grade algebra class working five increasingly complex combinatorial problems over the course of 3 months based on the conjecture that the more mathematically talented students would be able to abstract and generalize. Sriraman narrowed his focus on the experiences of four gifted students to show how these experiences support Krutetskii’s (1976) conclusions that generalization occurs through abstraction involving specific content and recognition of similar structure. Sriraman findings suggested a relationship between mathematical giftedness and the ability to abstract and generalize. The results of this study support Sriraman’s (2003) findings and
support the research question: What levels of mathematical thinking do 8th-grade gifted geometry students demonstrate when engaged in collaborative problem solving on task with a high-level cognitive demand? Two specific instances of generalization occurred during problem solving, one on the introductory task and the other on Julie’s Wheel.

The introductory task involved the area a goat could graze when its rope was lengthened. Chad worked with the others through understanding the task, but displayed an intuitiveness Sfard (1991) referred to as an understanding that precedes explanation by guessing a radius. Sfard referred to this as reason without rules. In Skemp’s (1987) framework on kinds of understanding, Chad’s thinking would fall within the relational understanding and the intuition mode of thinking. Chad evaluated his solution by doing the calculations as discussed in the data reporting chapter. Once he verified his solution made sense, he worked backwards to formulate an argument to justify his reasoning to the other group members. Chad used building-with evaluative analyzing to justify his solution and constructing synthesizing to formulate his explanation (Williams, 2000). Given Chad’s argument, Terry developed new insights, constructing synthesizing (Krutetskii, 1976; Williams) adding to existing knowledge (Cobb & Yackel, 1996; Hershkowitz et al., 2001). Based on his thinking about the insight, constructing synthesizing, Terry developed a formula that could always be applied to similar problems (Dreyfus et al., 2001), constructing evaluating. Terry extended his thinking based on Chad’s discovery and facilitated the construction of knowledge of the other group members.

The second instance of generalization occurred in group 1 while working on Julie’s Wheel. On the first day, students did not reach a solution due to algebraic errors.
On the second day, Mike lead his group not only in finding the solution to the problem, but extended his thinking beyond the problem by establishing a formula for finding the radius of the smaller wheel given similar situations involved in Julie’s Wheel. Mike’s generalization demonstrated his thinking was at the level of constructing synthesizing and constructing evaluating.

How do these levels of thinking relate to Sriraman’s findings of a relationship between mathematical giftedness and the ability to abstract and generalize? Hershkowitz, Schwarz, and Dreyfus (2001) defined abstraction as “an activity of vertically reorganizing previously constructed mathematics into a new mathematical structure” (p. 202). They also noted that new structure is in terms of new structure for the student and not the field of mathematics. Context, meaning everything students bring to the problem solving activity and the social interaction is itself, is implied in the definition. They argued that the “process of abstraction is occurring and constituted by” (p. 218) observing characteristics of nesting of the epistemic actions and that “constructing is mediated by human interaction and by a material tool” (p. 220). Chad, Terry, and Mike, gifted students, were part of small group interaction. The students were thinking, at the levels of recognizing and building-with nested within constructing synthesizing and constructing evaluating. Based on the operational definition of Hershkowitz et al., the generalizations of Terry and Mike constitute the process of abstraction. These two instances support Sriraman’s claim of a relationship between mathematical giftedness and the ability to abstract and generalize and provided evidence that these students were thinking at the levels of abstraction and generalization.
Hekimoglu (2004) used teaching experiment methodology based on the work of Steffe and Thompson (2000) to investigate differences in abstract reasoning of a gifted student and an average student. Results indicated that self-efficacy can impact mathematics performance and that the gifted student exhibited mathematical creativity, inventiveness, flexibility, and originality in solving problems.

First, findings of this study support Hekimoglu’s (2004) findings that self-efficacy can impact mathematics performance. This was evident during the first task in group 1. Tom was frustrated that his thinking was challenged. The group was frustrated because doing the math using Tom’s strategy put them in a loop. Four groups were operating within the classroom. The students were cognizant of other students listening, observing, and possibly reacting to his or her input (Gillies & Ashman, 2003). This social capital (Bourdieu, 1991) can influence the individual’s contribution to achievement of the goal. Tom expressed some frustration early during the basketball court renovation task that his way of thinking was not pursued exclusively. Soon he recognized the focus was on the mathematical ideas of all participants and no longer perceived performance differences as ability differences (Ames, 1981).

Second, findings of this study support Hekimoglu’s (2004) findings that gifted students exhibit mathematical creativity, inventiveness, flexibility, and originality in solving problems. During the introductory task involving the goats, although warned by a participant he had to use formulas, Chad showed inventiveness and originality using guess and test to determine the radius of the quarter circle the goat can graze with the lengthened rope. He then worked backwards to formulate a mathematical argument. Based on his explanation, Terry developed an equation for similar situations. Group 3 set
up a proportion to find the radius of the quarter circle with the combined area the four goats could graze. Group 4 used a scale drawing for the introductory task which aided the students’ understanding of the situation.

During work on smiley, the students in group 1 showed flexibility by using an inscribed hexagon to find the area of the shape representing the hair. Several students multiplied the area of the equilateral triangle by 6 to find the area. Mike found the apothem and used the formula for regular polygons. Group 1 also ingeniously rotated and flipped one of two white congruent triangles to form a rectangle and then subtracted the area of the rectangle from the equilateral triangle to find the area of the shaded region representing the nose and mouth. Other groups used trigonometry, while others used the Pythagorean Theorem to find the area of the shaded regions representing the nose and mouth. Some groups found the area of whole regions, and other group found the area of the parts. Daren recognized and verbalized several pathways to find the area of the large equilateral triangle. Mike in group 1 demonstrated a deep understanding of the mathematical concepts involved in Julie’s wheel and showed originality, and fluency in by establishing a rule for finding the radius of the smaller wheel in Julie’s Wheel.

Hekimoglu’s (2004) findings that gifted students exhibit mathematical creativity, inventiveness, flexibility, and originality in solving problems overlapped with Sriraman’s findings. Repeatedly in the data reporting section, instances of mathematical creativity, inventiveness, flexibility, and originality in solving problems are provided. These characteristics are subsumed in the higher-level categories of mathematical thinking evidenced in the students’ work related to the tasks with high cognitive demand. The
connections of these characteristics as indicators of deep mathematical understanding will be addressed later.

Dosemagen (2004) conducted a two part action research study on how her Advance Placement (AP) Calculus students viewed their mathematical understanding and how the students thought that understanding developed. The results of Dosemagen’s study indicated students deemed visualization, connections among concepts, and the application of concepts important to their understanding of mathematics. The four tasks used in this study provided students opportunities to apply previous knowledge, to visualize, and connect previous knowledge to new concepts representing relational understanding (Skemp, 1976). Students drew visual representations of the areas the goats could graze before and after the lengthening of the ropes, and applied what they knew about circles to reach a conclusion to the introductory problem. One group drew the representation to scale and used collaborative checking to verify the accuracy enabling them to progress toward a solution. Both the basketball court renovation and the smiley tasks involved area and were based on fictitious, but realistic, renovation projects in the school. Amy connected the smiley problem to the castle problem the students worked on earlier involving volume. Sally stated, “I think it’s cool how everything that we’ve learned this year is being applied in one problem.” Sally’s statement indicated the tasks used in this study enabled students to visualize, connect, and apply previous knowledge to develop a deeper understanding of the relationships between mathematical concepts.

The second finding by Dosemagen (2004) that impacted this study was her students’ perception of the importance of the impact of the classroom environment on
individual mathematical understanding. This finding supported my theoretical perspective of social constructivism and the instructional strategy of small group problem solving. Social interaction within the learning community is an essential component in social constructivism learning theory (Ernest, 1998a). Social interaction was also considered central to dynamically nested epistemic actions (Hershkowitz et al. 2001; Wood et al. 2006). Based on this key tenet and the indication that high ability mathematics students can achieve significantly more in small-group instruction (Peterson, Janiack, & Swing, 1981), I used the research on collaborative groups to guide the use of groups in my study.

While the focus of my study was not on the individual student’s perceptions, the voice represented by my students speaks loudly about ways the environment, in this case small groups, met the learner’s needs, influenced student understanding and self-efficacy (Hekimoglu, 2004). The students spoke through their reflections. The students discussed how they talked through the math for others to understand, liked the challenge of the problems because they had to think in different ways, and felt satisfaction when they got it. Others thought the problems were fun and intellectual, liked the group work, and were impressed with how smart the participants were. After completing work on smiley with time left over in the period, Terry asked, “Can we have another problem?” Daren specifically stated I should continue to provide collaborative problem-solving opportunities for my students in the future.

The research of Wood et al. (2006) of public mathematical thinking versus individual thinking indicated the importance of examining students’ mathematical thinking within the context of the classroom amidst all the interaction of the students.
They argued that the social interaction of the classroom influences students’ construction of knowledge. Wood et al. concluded, “it is the social cognitive processes of joint attention and understanding of others’ communicative intentionality that is the medium by which mathematical thought develops through meaning making with others” (p. 250).

Although my research differed from the inquiry/argument classroom of Wood et al. (2006), I followed their model utilizing the mathematical thinking framework to justify examining the possibilities of higher-level mathematical thinking shared among gifted students within small group learning communities. The results enabled me to address the research question: What levels of mathematical thinking do 8th-grade gifted geometry students demonstrate when engaged in collaborative problem-solving tasks with high-level cognitive demand?

Wood et al. (2006) indicated students make connections between mathematical ideas when using synthetic-analyzing and evaluative-analyzing. The mathematical thinking exhibited by the students within the groups in this study was consistently at the levels of building-with analyzing, building-with synthetic analyzing, building-with evaluative analyzing, constructing synthesizing, and occasionally constructing evaluating. The general pattern was one person would put forth an idea, the group pursued the idea together, verbalizing the math as they worked explaining, clarifying, and justifying as needed based on another’s inquiry, and collaboratively checked as they verbalized the math and verified their solutions. This represented the “social cognitive process of joint attention” (Wood et al., p. 250). The students’ constant use of collaborative checking (Pirie, 1998) represented building-with evaluative analyzing, and also represented their sustained press for explanation, justification, and meaning. Students were not content
with answers only. Their need to know, to understand, was evident repeatedly. When students needed clarification, another student would explain while the group reworked the math. The inquiry, argument, agreement, clarification, verbalizing doing the math, justification, challenge, explanation, and collaborative checking contributed to the students’ understanding of the mathematical concepts and relationships involved. The social interaction of the groups while working collaboratively on the open ended tasks exemplified “communicative intentionality that is the medium by which mathematical thought develops through meaning making with others” (Wood et al., p. 250).

The results of this study related to the Mathematical Tasks Framework (Stein et al., 2000) also address the research question: What levels of mathematical thinking do 8th-grade gifted geometry students demonstrate when engaged in collaborative problem solving tasks with high-level cognitive demand? Cognitive demand referred to the “kind of thinking and level of thinking required of students” (Stein et al., p. 11) when working on a mathematical task. I used the Mathematical Tasks Framework to evaluate the cognitive level required for each of the four tasks selected for this study. Then the tasks were set-up and implemented at the following levels. The level of cognitive demand required for the introductory task, smiley, and Julie’s wheel represented the highest level of cognitive demand of doing mathematics. The level of cognitive demand required for the basketball court renovation task was procedures with connections. The maintenance or decline of the level of cognitive demand for each segment of the group problem-solving activity was evaluated using the factors associated with maintenance or decline (Stein et al.).
The level of cognitive demand of the introductory task was doing mathematics due to the complex thinking and reasoning required to achieve a solution and was maintained by all four groups. The exception was a slight decline of group 3 due to their failure to verify their solution prior to a press for justification by the researcher. The level of cognitive demand for the basketball court renovation task was procedures with connections (Stein et al., 2000) due to explicit pathways. The high-level cognitive demand of the tasks was maintained by all groups. The level of cognitive demand for the smiley task was doing mathematics (Stein et al., 2000) due to multiple pathways for solving. The level of cognitive demand was maintained throughout the task by all groups except for one instance of a slight decline in one segment involving group 3. Julie’s Wheel was an extremely high-level problem without explicit or implicit pathways for solving. The high-level cognitive demand was maintained on the first day. The level of cognitive demand was reduced to procedures with connections on day 2 due to a decision by the researcher to provide a structured hint to allow students to develop a deeper understanding of the mathematics related to the problem rather than ending the work on the task with no resolution. The students’ levels of mathematical thinking involved on the four problem-solving tasks represented the highest levels of cognitive demand.

Doing mathematics was described as the active process of exploring situations, problems or tasks, searching for patterns, conjecturing, defending, debating, justifying, generalizing or abstracting mathematics (Teppo, 1998, Stein et al.). My students consistently explored, searched for patterns, conjectured, defended, debated, argued, justified, explained, and sometimes generalized and abstracted. If students are doing mathematics, then students use the processes described above, and if students use the
processes described above, then their thinking is higher-level thinking. Syllogistically, if my students were doing mathematics, (as indicated by the continued maintenance of the level of cognitive demand) then my students also used higher-level thinking. Procedures with connections, also considered to require a higher level of thinking represented student thinking that drew on previous knowledge of procedures to construct a deeper understanding of mathematical concepts related to a task (Stein et al.).

The use of small groups has been widely researched (Barkley, Cross, & Major, 2005). Johnson, Maruyame, Johnson, Nelson, and Skon (1981) conducted a meta-analysis of 122 studies and found cooperative learning promotes higher achievement and motivation to learn over competitive and individual learning. A follow up meta-analysis (Johnson, Johnson, & Maruyame, 1983) of studies of homogeneously and heterogeneously grouped students indicate there is greater interpersonal attraction within homogenously grouped students. According to Bruffee (1993), collaborative learning is grounded in social constructivism, involves higher order knowledge, and the teacher becomes part of the community of learners in the construction of knowledge. Common elements in collaborative learning include positive interdependence, promotive interaction, individual and group accountability, social skills, and group processing (Johnson & Johnson, 2004).

Although specific elements of collaborative learning were not the focus of the research, the success of the environment, the small groups, addressed earlier through students’ voices, influenced the mathematical thinking involved in the interaction and ultimately mathematical understanding. The collaboration among the students empowered the individual student to achieve difficult tasks, provided a sense of
accomplishment, fostered interdependence, advanced commitment, added significance to the tasks, and encouraged active engagement with the task (Gillies & Ashman, 2003). This positive interdependence was represented when Chad used guess and test to find the radius of the new grazing area and because of his justification Sally repeatedly praised him, the group constructed an understanding and extended their thinking as a result of Terry’s equation for similar situations. Chad cheered, April said, “Wow!”, and the class knew they had a record time. Chad’s self-image as well as the group perception of him improved and the group worked like well-oiled machinery on the remaining tasks.

Although group 1 encountered some difficulty during the first task, the general pattern observed among the groups was one student would put forth an idea, the group would pursue the idea together, clarifying and justifying as needed based on another’s inquiry, and collaboratively checked while doing the math and verifying solutions. The results of this study indicate positive interdependence of the groups increased as the experience with the tasks increased (Gillies & Ashman).

Summary of the Findings

Goetz and LeCompte (1984) described the essence of ethnographic research as the “holistic depiction of uncontrived group interaction” (p. 51). I strived to provide as realistic a picture as possible of the “uncontrived group interaction” of my students, and the mathematical discussions that provided evidence of their higher-level thinking and their mathematical understandings. As I prepared to summarize the findings of this study, I obviously returned to the research questions: How is the mathematical understanding of 8th-grade gifted geometry students elicited through exploration using open-ended problems? What levels of mathematical thinking do 8th-grade gifted geometry students
demonstrate when engaged in collaborative problem solving on tasks with high-level cognitive demand? Once again, I had to step back and consider the functional relevance, the relationship of the parts to the whole. In other words, how did higher levels of mathematical thinking (Erickson, 1977) of my students influence their understanding of the mathematics concepts involved in their work on the problem-solving tasks?

I returned to the work of Hershkowitz, Schwarz, and Dreyfus (2001) related to the discussion about their model of abstraction. Constructing was considered the first and most important of the three epistemic actions of constructing, recognizing and building-with because recognizing and building-with were nested in constructing, or were necessary parts of constructing. Likewise, I considered mathematical understanding the most important goal for my students and the levels of mathematical thinking exhibited by my students as necessary parts for mathematical understanding to occur, much like Skemp’s relational understanding. Then, like Sfard’s analogy of the two legs making forward movement possible, I realized sometimes understanding was a necessary part for high-level mathematical thinking to occur. This would have to be true for constructing as well because to recognize structure in a task indicates understanding. Regardless of the circularity involved, in essence, the whole represented the interconnections of higher-level thinking and understanding.

To summarize, I begin with the content from a statement by the National Research Council (NRC) in 1989 as the statement described elements that comprised my study. The National Research Council (1989) stated research provides evidence that students develop deep mathematical understanding when actively involved in their own construction of knowledge through group problem solving experiences that allow
investigation and communication. The NRC statement indicated given situations as described, students can construct their own mathematical understandings. This study involved my gifted geometry students while engaged in collaborative problem solving. Collaborative learning as described earlier by Bruffee (1993) means students had to latitude to be in charge of their own learning, free to discuss, argue, justify, and reason by way of examining, representing, transforming, solving, applying, proving, and communicating (National Research Council, 1989). The evidence lay in my students’ actions and explanations. The questions remained, how can that understanding be measured? Hiebert and Carpenter (1992) suggested evidence of students’ understanding occurs through their explanations and their understanding is inferred by the measurer from their explanations. I used the explanations of my students and their actions to conclude that mathematical understanding occurred and was interconnected with students’ higher-level thinking while working on the tasks. Next, I summarize the findings in the context of the social constructivist theoretical views that framed the study.

The social constructivist theoretical views that framed this study included the following (I extended the overarching views applicable to my study by adding the term gifted):

1. The gifted learner is both acquisitional and participationist in the process of coming to know (Cobb & Yackel, 1996; Ernest, 1998b; Jaworski, 1996; Palinscar, 1998).

My gifted students’ need to know, to understand, during work on the open-ended tasks was evident. Their constant use of collaborative checking represented their own sustained press for explanation, justification, and meaning (Stein et al, 2000). The
students used argument, agreement, clarification, verbalizing doing the math, justification, challenge, and explanation through collaborative checking to understand the mathematical concepts and relationships involved. My students, the individuals came to know (acquisition) by using newly constructed knowledge gained through the social interaction (participation) to amend, refine, or add to their existing knowledge (Cobb & Yackel, 1996; Ernest, 1998a; Hershkowitz et al., 2001).

2. Learning occurs when new knowledge is integrated with previous knowledge (Ernest, 1998b; Jaworski, 1996).

Students recognized and used what they had previously constructed to construct new knowledge as they explored various pathways to solutions to the tasks. Students used what they knew to build more complex meaning and in the process exhibited mathematical creativity, inventiveness, flexibility, and originality through the problem-solving activity. Students visualized, connected, and applied previous knowledge to develop a deeper understanding of the relationships between mathematics concepts on all four tasks. The mathematically gifted students demonstrated many of Krutetskii’s structure of abilities of the mathematically gifted: My students, formally grasped mathematical information, processed the information logically, demonstrated flexibility in their thinking, often changed directions in processing, curtailed, or shortened mathematical thinking, and generalized.

3. Social interaction within the gifted learning community is essential (Cobb & Yackel, 1996; Vygotsky, 1978).

The use of the instructional strategy of collaborative small groups met my students’ learning needs and influenced their understanding. The social interaction and
collaboration among the students within the small groups enabled students to talk through the mathematics for others to understand, encouraged students to think in different ways, empowered the individual students to achieve difficult tasks, provided a sense of accomplishment, fostered interdependence, advanced commitment to the task, added significance to the task and, encouraged active engagement with the task. The collaborative checking that Pirie and Schwarzenberger (1988) noted in their study was the most powerful tool students employed in this study and it permeated all phases of interaction during work on all four tasks.

As indicated earlier, I believe in Hiebert’s (2003) conditional that given the opportunity, students can construct deep mathematical understanding. In this study, I provided the opportunity to my gifted geometry students to move beyond repetitive textbook problems, to engage in challenging, complex investigations involving higher-level thinking. The results of this study certainly supported a true conditional, but the mathematical thinking of my students and the ways that thinking was expressed surpassed my expectations.

Summary

In this chapter, I provided findings of the study positioned in the context of the context of the studies that supported my research, collaborative grouping, the mathematical thinking framework used by Wood et al., 2006, and the MTF (Stein et al., 2000). Then, I connected the findings of the study to the social constructivist theoretical views that framed the study.

Sriraman discussed the experiences of four gifted students to show how these experiences support Krutetskii’s (1976) conclusions that generalization occurs through
abstraction involving specific content and recognition of similar structure. Sriraman findings suggested a relationship between mathematical giftedness and the ability to abstract and generalize. Chad, Terry, and Mike, gifted students, were part of small group interaction. The students were thinking, at the levels of recognizing and building-with nested within constructing synthesizing and constructing evaluating. Based on the operational definition of Hershkowitz et al., the generalizations of Terry and Mike constitute the process of abstraction. These two instances support Sriraman’s claim of a relationship between mathematical giftedness and the ability to abstract and generalize and provided evidence that these students were thinking at the levels of abstraction and generalization.

Hekimoglu (2004) used teaching experiment methodology based on the work of Steffe and Thompson (2000) to investigate differences in abstract reasoning of a gifted student and an average student. Results indicated that self-efficacy can impact mathematics performance and that the gifted student exhibited mathematical creativity, inventiveness, flexibility, and originality in solving problems.

First, findings of this study support Hekimoglu’s (2004) findings that self-efficacy can impact mathematics performance. Four groups were operating within the classroom. The students were cognizant of other students listening, observing, and possibly reacting to his or her input (Gillies & Ashman, 2003). Second, repeatedly in the data reporting section, instances of mathematical creativity, inventiveness, flexibility, and originality in solving problems are provided. These characteristics are subsumed in the higher-level categories of mathematical thinking evidenced in the students’ work related to the tasks with high cognitive demand.
Dosemagen (2004) conducted a two part action research study on how her Advance Placement (AP) Calculus students viewed their mathematical understanding and how the students thought that understanding developed. The results of Dosemagen’s study indicated students deemed visualization, connections among concepts, and the application of concepts important to their understanding of mathematics. The four tasks used in this study provided students opportunities to apply previous knowledge, to visualize, and connect previous knowledge to new concepts that allowed students to experience relational understanding (Skemp, 1976).

The second finding by Dosemagen (2004) that impacted this study was her students’ perception of the importance of the impact of the classroom environment on individual mathematical understanding. While the focus of my study was not on the individual student’s perceptions, the voice represented by my students speaks loudly about ways the environment, in this case small groups, met the learner’s needs, influenced student understanding, and self-efficacy (Hekimoglu, 2004). The students spoke through their reflections. The students discussed how they talked through the math for others to understand, liked the challenge of the problems because they had to think in different ways, and felt satisfaction when they got it.

The research of Wood et al. (2006) of public mathematical thinking versus individual thinking indicated the importance of examining students’ mathematical thinking within the context of the classroom amidst all the interaction of the students. Wood et al. (2006) indicated students make connections between mathematical ideas when using synthetic-analyzing and evaluvative-analyzing. The mathematical thinking exhibited by the students within the groups in this study was consistently at the levels of
building-with analyzing, building-with synthetic analyzing, building-with evaluative analyzing, and constructing synthesizing, and occasionally constructing evaluating. The inquiry, argument, agreement, clarification, verbalizing doing the math, justification, challenge, explanation, and collaborative checking contributed to the students’ understanding of the mathematical concepts and relationships involved.

The results of this study related to the Mathematical Tasks Framework (Stein et al., 2000) also address the research question: What levels of mathematical thinking do 8th-grade gifted geometry students demonstrate when engaged in collaborative problem solving tasks with high-level cognitive demand? The level of cognitive demand required for the introductory task, smiley, and Julie’s wheel represented the highest level of cognitive demand of doing mathematics. Doing mathematics was described as the active process of exploring situations, problems or tasks, searching for patterns, conjecturing, defending, debating, justifying, generalizing or abstracting mathematics (Teppo, 1998, Stein et al.). My students consistently explored, searched for patterns, conjectured, defended, debated, argued, justified, explained, and generalized and abstracted. The level of cognitive demand required for the basketball court renovation task was procedures with connections. Procedures with connections, also considered to require a higher level of thinking represented student thinking that drew on previous knowledge of procedures to construct a deeper understanding of mathematical concepts related to a task (Stein et al.).

According to Bruffee (1993), collaborative learning is grounded in social constructivism, involves higher order knowledge, and the teacher becomes part of the community of learners in the construction of knowledge. Common elements in
collaborative learning include positive interdependence, promotive interaction, individual and group accountability, social skills, and group processing (Johnson & Johnson, 2004). The collaboration among the students empowered the individual student to achieve difficult tasks, provided a sense of accomplishment, fostered interdependence, advanced commitment, added significance to the tasks, and encouraged active engagement with the task (Gillies & Ashman, 2003). One student would put forth an idea, the group would pursue the idea together, clarifying and justifying as needed based on another’s inquiry, and collaboratively checked while doing the math and verifying solutions. The results of this study indicate positive interdependence of the groups increased as the experience with the tasks increased (Gillies & Ashman).

In the summary of the findings, I first discussed how the whole of my study represented the interconnections of higher-level thinking and understanding. I began with the statement of the NRC about understanding and then connected the elements of my study. Then, I summarized the finding in the context of the social constructivist theoretical views that framed the study. My students, the individuals came to know (acquisition) by using newly constructed knowledge gained through social interaction (participation) to amend, refine or add to existing knowledge (Cobb & Yackel, 1996; Ernest, 1998a; Hershkowitz et al., 2001). Students visualized, connected and applied previous knowledge to develop a deeper understanding of the relationships between mathematics concepts. The social interaction and collaboration among the students within the small groups enabled students to talk through the mathematics for others to understand, encouraged students to think in different ways, empowered the individual students to achieve difficult tasks, provided a sense of accomplishment, fostered
interdependence, advanced commitment, added significance to the task and, encouraged active engagement with the task.
CHAPTER 6
SUMMARY AND DISCUSSION

This chapter begins with a summary of the study. Next, the significance of the study related to needed scholarly empirically assessed practice is discussed. The implications for further research are discussed in terms of mathematical perspectives and paradigm shifts in gifted education. The limitations of the study follow. The chapter closes with concluding remarks.

Summary of the Study

Implementation of standards to improve the mathematical abilities for all students has fallen short and there has been little focus on gifted students of mathematics (House, 1999). The reluctance to teach for understanding impedes the development of gifted students of mathematics (Graffam, 2003; Sheffield, 1999; Usiskin, 1999). Gifted students of mathematics often just do more problems at a faster rate without opportunities to construct mathematical understandings (Sheffield). Each year I, the researcher and teacher, receive a new class of 8th-grade gifted geometry students who are good “technicians who can follow rules and apply those rules to routine exercises” (Sheffield, p. 45). Many are conditioned through previous experience in mathematics (Bishop, 1988) to determine what it takes to get the good grade (Wheatley, 1999) rather than seeking to understand. In the past, my students have lacked the opportunity to become mathematical thinkers. Hiebert (2003) proposed a conditional: “If students have more opportunity to construct mathematical understandings, they will construct them more often and more
deeply” (p.16). Although my students were accelerated, the problem was they have lacked the opportunities to realize their full potential by engaging in challenging complex investigations, collaborative problem-solving experiences, and higher-level mathematical thinking. In this study, my students were given an opportunity to engage in challenging complex investigations while working collaboratively.

The purpose of this study was to examine the ways mathematically gifted students think about and do mathematics creatively as indicators of deep understanding while working collaboratively on open-ended tasks with high-level cognitive demand. The questions that guided this study were: How is mathematical understanding of 8th-grade gifted geometry students elicited through exploration using open-ended problems? What levels of mathematical thinking do 8th-grade gifted geometry students demonstrate when engaged in collaborative problem-solving tasks with high-level cognitive demand?

A learning community with a focus on inquiry mathematics as the intersection of a sociocultural context, social constructivist theory, and gifted education, was relevant to this study. Social constructivism, a theory that suggests the individual comes to know (acquisition) by using newly constructed knowledge gained through social interaction (participation) to amend, refine or add to existing knowledge (Cobb & Yackel, 1996; Ernest, 1998a) framed the study.

The social constructivist theoretical views that framed this study included the following (I extended the overarching views applicable to my study by adding the term gifted):
1. The *gifted* learner is both acquisistionist and participationist in the process of coming to know (Cobb & Yackel, 1996; Ernest, 1998b; Jaworski, 1996; Palincsar, 1998).

2. Learning occurs when new knowledge is integrated with previous knowledge (Ernest, 1998b; Jaworski, 1996).

3. Social interaction within the *gifted* learning community is essential (Cobb & Yackel, 1996; Vygotsky, 1978).

Participants were 15 of the 16 gifted students enrolled in my gifted geometry class. The class was comprised of six females and nine males. Two of the students were Asian and the remainder White. I examined how student understanding was affected by working within small groups on four open-ended problems and described the levels of mathematical thinking involved through ethnographic case study methodology. Case study was viewed as a subsystem of ethnography and the collaborative small groups as smaller cases embedded within the larger case. Means of data collection included participant observation, field notes, student artifacts (work on tasks), audio recording of each group’s problem-solving activity, my reflections typed for students’ feedback, and students’ reflections.

Inductive data analysis was used to gain insight as soon as data collection began (Goetz & LeCompte, 1984; Merriam, 1998). Understanding was constantly verified by the sources of data, in my case, the students (Lincoln & Guba, 1985; Merriam, 1998). Drawing on the work of Pirie (1998), and Pirie and Schwarzenberger (1988), data analysis of the transcribed audio tapes included discourse analysis (Sinclair & Coulthard, 1975), “repeated reanalysis of existing data” (Pirie, p. 90) to refine the categories based
on constant comparative method of analysis (Glaser & Strauss, 1967) for category development. Then the conceptual framework utilized by Wood et al. (2006) was used to categorize cognitive activity (Dreyfus et al., 2001; Williams, 2000) representative of levels of mathematical thinking. The MTF (Stein et al., 2000) and factors associated with decline or maintenance of the level of cognitive demand was used throughout data collection and analysis, but especially as a cross check of levels of mathematical thinking. The results were “illustrative and generative” (Ernest, 1998b, p. 34) and related through thick description (Geertz, 1973) characteristic of case study. The experiences of each group were reported within each task, the groups compared by task, and findings summarized by task.

The level of cognitive demand of the introductory task was doing mathematics due to the complex thinking and reasoning required to achieve a solution and was maintained by all four groups. The exception was a slight decline of group 3 due to their failure to verify their solution prior to a press for justification by the researcher. The levels of mathematical thinking exhibited across the groups included recognizing-comprehending and recognizing applying nested in building-with analyzing and building-with synthetic-analyzing, constructing synthesizing, constructing evaluating. Students used previous knowledge of circles and area (Dreyfus et al., 2001), constructed pictorial representations of the grazing areas, calculated areas the goats could graze, reasoned how to find the radius of the quarter circle that represented the combined areas the goats could graze, and evaluated the reasonableness of both methods and solutions (Williams, 2000). One group extended their thinking by developing a formula applicable for similar
situations. The levels of mathematical thinking exhibited represented the levels of complex thinking and reasoning characteristic of doing mathematics (Stein et al., 2000).

The level of cognitive demand for the basketball court renovation task was procedures with connections (Stein et al., 2000) due to explicit pathways. The level of cognitive demand was maintained throughout the task by all groups. The levels of mathematical thinking exhibited across the groups included recognizing-comprehending and recognizing applying nested in building-with analyzing and building-with synthetic-analyzing, and constructing synthesizing. Finding the areas of the basketball court to be painted the school colors and those that would receive a hardwood finish built on students’ prior knowledge of area. Prior knowledge of circumference was used to find the arc length for available to each player during a jump ball. Building on the students’ prior knowledge of area allowed the students to draw conceptual connections. Students discussed what they had to do by devising a plan and then implemented the plan to obtain a solution through several pathways (Sheffield, 2000; Stein et al.). The students’ constant use of collaborative checking (Pirie, 1998) represented their sustained press for explanation, justification, and meaning (Stein et al.). The higher-level thinking indicated students’ work on the basketball court task translated into a deeper understanding of the mathematical processes, concepts, and the relationships involved (Hiebert, 2003).

The level of cognitive demand for the task was doing mathematics (Stein et al., 2000) due to multiple pathways for solving. The high-level cognitive demand was maintained throughout the task by all groups, except for one instance in group 3. The levels of mathematical thinking exhibited across the groups included recognizing-comprehending and recognizing applying nested in building-with analyzing and building-
with synthetic-analyzing, and constructing synthesizing. The task built on students’ previous knowledge of equilateral triangles, right triangles, circles, portions of concentric circles, and portions of sectors. Multiple pathways (Sheffield, 2000; Stein et al.) could be used to solve the problem. Except for several segments of work by group 3, the students’ constant use of collaborative checking (Pirie, 1998) continued to represent sustained press for explanation, justification, and meaning (Stein et al.). Sheffield (1999) stated students recognize the high-level cognitive demand required of peers engaged in problem solving, challenging them to extend their own thinking. I recognized this affect across all groups. The students in each group expressed their enjoyment involved in the challenge of the smiley task.

Julie’s Wheel was an extremely high-level cognitive demand problem without explicit or implicit pathways for solving. Solving the problem required application of algebraic and geometric principles once the students grasped the mathematical structure embedded in the task. The task built on the students’ knowledge of the Pythagorean Theorem, perfect square trinomials, and simplifying radicals. The students had to first recognize these concepts were needed. Several groups accurately depicted the embedded right triangles, and correctly represented the composition of the leg and hypotenuse lengths. Several groups made progress the first day. Each of the groups completed the task with the scaffolding provided through the structured hint on the second day. Even the students who had achieved some success on the task without the hint completed the structured hint. Julie’s wheel challenged students’ thinking and provided an opportunity for them to develop a deeper understanding of mathematical concepts involved. The levels of mathematical thinking included recognizing comprehending, recognizing
Significance in Terms of Research

This study, supported by scholarly research, adds to the body of knowledge needed in terms of gifted education and empirically assessed classroom practice (Friedman-Nima et al., 2005). Albert (1969) surveyed the professional literature dealing with all forms of genius, creativity, and giftedness from 1927 to 1969 to determine conceptual shifts in gifted terminology. Freidman-Nima et al. extended the work of Albert by investigating the conceptual changes in the professional literature from 1965 to 2000. Over 28,000 articles from three data bases were identified based on a search of the terms gifted, creative, talent, gifted and disabled, gifted and disadvantaged, as well as lexicographic relatives. Friedman-Nima et al. found only 160 of the 723, or less than 25% of articles related to gifted education included supporting scholarly research and very few of the studies in gifted education add to the body of knowledge through empirically assessed practice.

Similarly to the problem with gifted literature, Lester and Kehle (2003) indicated research on problem solving is lacking with few connections for classroom practice. Searching scholarly literature for mathematical thinking and problem solving involving groups, I found only one study, the work of Wood et al. (2006) involving mathematical thinking within a group of students while engaged in problem solving. This finding supports Good, Mulryan, and McCaslin (1992) earlier suggestion that research has ignored students’ thinking and learning through the small group interaction on collaborative tasks. Clearly, there is a lack of research specific to the mathematical
thinking of gifted students working in groups engaged in problem solving. Building on the work of Wood et al. (2006), this study offers a glimpse into the mathematical thinking involved in group problem solving situated in a gifted geometry class and how high levels of mathematical thinking can contribute to understanding.

Paradigm Shifts: Implications for Future Research

Ernest (1991) described five educational ideologies and the view of ability, society, education, and mathematics from each perspective. The Progressive Educator ideology focused on the individual; the Mathematician’s ideology focused on pure mathematics; both the Industrial Trainer and the Technological Pragmists ideologies focused on the needs of society; and the Public Educator ideology focused on access to knowledge for all learners. These ideologies are not time periods, but perspectives based on beliefs and goals. They provided a framework to compare the views and the shifts in thinking about gifted education in the areas of concept of intelligence, identification of gifted students, curriculum differentiation, the equity versus excellence dilemma, and the impact of technology on the roles of teacher and student (Gallagher, 2003).

Through a Progressive Educator perspective, teachers needed a way to identify and serve students. The challenge of this perspective is to shift means of identification to match the shift in thinking about who is gifted. The results of this study indicate using the instructional strategy of group problem solving involving high-level cognitive demand could provide a means of recognizing mathematical promise of students who have been marginalized through psychometric means of measure (Matthews & Foster, 2006) by “measuring the processes of mathematical thought” (Krutetskii, 1976, p. xvi) based on observable cognitive actions of students at work on problem-solving tasks (Dreyfus,
Hershkowitz, & Schwarz, 2001; Williams, 2000; Wood et al., 2006). Krutetskii argued that identification of mathematically gifted students should occur through observation of students at work on problem-solving tasks based on the mathematical abilities he identified, rather than test scores alone.

The participants in this study were gifted geometry students identified early in elementary school through psychometric means typical of a Progressive Educator’s perspective. As a result of their identification, they have been served through acceleration aligned with a Mathematician’s perspective. My view of giftedness from a Technological Pragmatist view could have indirectly impacted this study in the affective domain as students’ were aware of my belief that they should willingly give back to society and not because it is demanded.

The Public Educator ideology focuses on the philosophy of social constructivism. The Public Educator perspective views ability as fluid and influenced by experience (Krutetskii, 1976) and focuses on teaching methods that include student to student discussion, student to teacher discussion, group work and problem solving for engagement, mastery, creativity, critical thinking, conflict, and social relevance for empowerment (Ernest, 1991). This study was grounded in social constructivism. The students engaged in group work and problem solving involving mostly student to student discussion with some student to teacher discussion. This study also represents my insistence of opportunities for students of mathematical promise to use their gifts and talents through collaborative problem solving rather than the status quo of individual seat work doing repetitive problems; Critical thinking versus mathematical indoctrination.
The results of this study revealed two additional concepts worthy of further research. The students in this study continually used collaborative checking (Pirie, 1998) as inquiry, explanation, clarification, verification, and evaluation of processes and solutions that greatly influenced the mathematical thinking and understanding. The term collaborative checking evolved as a result of constant comparison of data generated from student discussion from the work of Pirie and Schwarzenberger (1988) and goes beyond the idea of one individual checking the work of the group. Further research is needed on the specific use of collaborative checking in the manner used by my students and recognized earlier by Pirie and Schwarzenberger as mathematical communication during small group problem solving. The results of the study also indicated the importance of processing of thinking and the different ways processing of thinking can occur. Some students processed their thinking through silence (Hoyles, 1985), while others processed through social chat, and others processed through what could be viewed to an outsider as outrageous behavior (Saul, 1999). Regardless of which mode of processing students used, the students always returned to the discussion by adding, amending, or refining what had previously been said. Research focused on the ways students process their thinking while working in small groups could add greatly to what we know about students working collaboratively in problem solving.

Limitations

Lather (1986) suggested that value free research is impossible and the researcher should openly admit the researcher’s values influence the research. Lather (2001) advocated “work that attests to the possibilities of its time yet, in the very telling, registers the limits of itself as a vehicle for claiming truth” (p. 486). The interpretations
and conclusions drawn from my study will offer the reader an opportunity to step inside my classroom, sit among my students, listen to their discussions, and draw their own conclusions regarding my research. The “illustrative and generative” (Ernest, 1998b, p. 34) interpretations are told through my view, the teacher, the researcher, and a research instrument. As a research instrument, one has the ability to “explore the atypical or idiosyncratic responses to achieve a higher level of understanding” (Lincoln & Guba, 1985, p. 195) and in the process other opportunities are missed. Due to the rapport with my students, I could have missed something that the relationship assumed. The rapport established with my students and the homogenously grouping of gifted students for mathematics instruction impact transferability.

First, the rapport established with my students was a result of extended time spent with these students over the course of the school year. Through the year, I come to know my students and the educational, social, and cultural influences they bring to the learning environment. I know the students’ parents, where the students live, previously places lived, previous schools, previous teachers, the students’ past interests, their current interests, their future hopes and dreams, their successes and perceived failures, and a host of areas of knowledge that privilege me as the researcher. Stinson (2004) suggested this rapport with students motivates learning and in this case appeared to be true which in turn could also impact transferability.

Second, gifted students grouped homogenously for geometry instruction rarely occurs because gifted students usually are served through acceleration. Accelerated classes also serve mathematically talented students and not necessarily identified as
gifted students. Also, limiting transferability, the study of geometry as a course of study, rarely is offered at the eighth grade level.

Concluding Remarks

As a teacher of 8th-grade gifted geometry students, I recognized that my students, although accelerated, had lacked the opportunities to realize their full potential by engaging in challenging complex investigations, collaborative problem-solving experiences, and higher-level mathematical thinking. I believe Hiebert’s (2003) conditional: “If students have more opportunity to construct mathematical understandings, they will construct them more often and more deeply” (p. 16). Based on my belief in Hiebert’s conditional, the purpose of this study was to examine the ways mathematically gifted students think about and do mathematics creatively as indicators of deep understanding while working collaboratively on open-ended tasks with high-level cognitive demand. Although means for measuring mathematical understanding remains as elusive (Hiebert & Carpenter, 1992) as a definition of mathematical understanding, measurement of students’ understanding can be described, based on interpretations of the measurer, in this case the teacher, from some observable evidence (Hiebert & Carpenter) based on students’ explanations.

Through a social constructivist theoretical lens and the instructional strategy of collaborative small groups, I used my students’ explanations for observable evidence, and based on that evidence, I described their understandings. My students worked in collaborative groups on four tasks and were in charge of their own learning, free to discuss, argue, justify, examine, represent, transform, solve, construct, and prove. My gifted students’ need to know, to understand, during work on the open-ended tasks was
evident. Their constant use of collaborative checking represented their own sustained press for explanation, justification, and meaning (Stein et al, 2000).

I used two conceptual frameworks to provide observable evidence from my students’ explanations. The first conceptual framework was used by Wood et al. (2006) to categorize observable cognitive activity (Dreyfus et al., 2001; Williams, 2000) representative of mathematical thinking. The second conceptual framework was the MTF framework (Stein et al., 2000) for factors associated with maintenance or decline of the level of cognitive demand. In other words, I categorized what my students said, the actions I saw, and the work they produced based on indicators or descriptions from previous research as evidence that my students’ used higher-level thinking.

According to the categories of mathematical thinking used by Wood et al. (2006), the mathematical thinking exhibited by my students within the groups in this study was consistently at the levels of building-with analyzing, building-with synthetic analyzing, building-with evaluative analyzing, constructing synthesizing, and sometimes constructing evaluating (Dreyfus et al., 2001; Williams, 2000; Wood et al.). My students’ levels of mathematical thinking involved on the four problem-solving tasks represented the highest levels of cognitive demand (Stein et al.) of procedures with connections and doing mathematics and served as indicators of a deep understanding, the mathematical connections of ideas and concepts to existing knowledge.

The social interaction and collaboration among my students enabled them to talk through the mathematics for others to understand, to use what they knew to build more complex meaning, and in the process, my students exhibited mathematical creativity, inventiveness, flexibility, and originality through the problem-solving activity. Like my
gifted students, given the opportunity to engage in challenging complex investigations and collaborative problem-solving experiences, students can construct deep mathematical understanding.
References


APPENDIXES

APPENDIX A

Georgia State University

Department of Middle-Secondary and Instructional Technology

Child Assent Form

**Title:** Promoting Mathematical Understanding Through Open-ended Tasks: Explorations and Experiences of an Eighth Grade Gifted Geometry Class

**Principal Investigator:** Christine Thomas, PhD.

**Student Investigator:** Carol A. Taylor

You have been invited to volunteer to participate in a research study. In order to participate, your parents or legal guardian must give their parental permission. However, you can refuse to be in this study, and your parents or legal guardian cannot force you. If your parents provide permission and you agree, you may drop out at any time.

The purpose of this study is to describe the mathematical understanding and the levels of mathematical thinking your child demonstrates while solving problems. The research will occur during class several times a week for four weeks. The tasks are open-ended problems. The tasks allow you to use higher order thinking skills and to transfer knowledge to new situations. The tasks are applications of concepts in four chapters of your geometry textbook. The chapters include Chapter 8: Applying Right Triangles and Trigonometry; Chapter 9: Analyzing Circles; Chapter 10: Exploring Polygons and Area; and Chapter 11: Investigating Surface Area and Volume.

First, you will work together in small groups of four on a sample problem. Then you will work together on three separate open-ended tasks. You will use higher order thinking skills to solve the problems. The teacher will observe students working together. Each group will be audio recorded. Students’ discussions can help us understand how students think about mathematics. The teacher may ask you about how you solved a problem. The teacher may also use questions to get students back on track without telling the students how to solve the problem. The work students do will be analyzed for mathematical thinking. Your group discussions will also be analyzed. You may also be asked to write about your group’s work on the task and tell what you learned. Communication will be informal as normal practice. Performance is anonymous.
In this study, you will not have any more risks than in a normal day of life. You will not be subject to risk or discomfort physically, psychologically, socially, or academically because of participation. Benefits include opportunities to reason and solve problems and to demonstrate understanding of key geometry concepts through an alternative method rather than traditional assessment.

If you are willing to volunteer for this research, please sign below.

__________________________________________________   ____________________
Child Assent                                  Date
APPENDIX B

Georgia State University

Department of Middle-Secondary and Instructional Technology

Parental Permission Form

Title: Promoting Mathematical Understanding Through Open-ended Tasks: Explorations and Experiences of an Eighth Grade Gifted Geometry Class

Principal Investigator: Christine Thomas, PhD.
Student Investigator: Carol A. Taylor

I. Purpose:

Your child is invited to participate in a research study. The purpose of this study is to describe the mathematical understanding and the levels of mathematical thinking your child demonstrates while solving problems. Sixteen students will be in this study. The research will occur during class several times a week for four weeks. The tasks are open-ended problems. The tasks allow your child to use higher order thinking skills and to transfer knowledge to new situations. The tasks are applications of concepts in four chapters of your child’s geometry textbook. The chapters include Chapter 8: Applying Right Triangles and Trigonometry; Chapter 9: Analyzing Circles; Chapter 10: Exploring Polygons and Area; and Chapter 11: Investigating Surface Area and Volume.

II. Procedures:

First, students will work together in small groups of four on a sample problem. Then students will work together on three separate open-ended tasks. The students will use higher order thinking skills to solve the problems. The teacher will observe students working together. Each group will be audio recorded. Students' discussions can help us understand how students think about mathematics. Your child will only interact with other students in the class and the teacher. The teacher may ask students about how they solved a problem. The teacher may also use questions to get students back on track without telling the students how to solve the problem. The work students do will be analyzed for mathematical thinking. The students’ discussions will also be analyzed. Students may also write about their group’s work on the task and tell what they learned. Communication between the students and the teacher will be informal as normal practice. Performance is anonymous.
III. **Risks:**

In this study, your child will not have any more risks than in a normal day of life. Your child will not be subject to risk or discomfort physically, psychologically, socially, or academically.

IV. **Benefits:**

Participation in this study may benefit your child by providing opportunities to reason and solve problems. Students can show their understanding of key geometry concepts. Your child may also benefit from learning through peer collaboration. Overall, we hope to gain information about how students come to know mathematics while working together in the classroom. This can help us understand what constitutes mathematical learning. The results could provide another means for measuring mathematical thought.

V. **Voluntary Participation and Withdrawal:**

Participation in this research is voluntary. Your child does not have to be in this study. Your child can also drop out at any time. Students will not be penalized in any way.

VI. **Confidentiality:**

We will keep your child's records private to the extent allowed by law. Your child’s name and other facts that might point to your child will not appear when we present this study or publish its results. The findings will be summarized and reported in group form. Your child will not be personally identified. The results of this study will be published as a dissertation. Information regarding the study will be kept no more than five years and then will be shredded. The audio recordings will be destroyed after they are transcribed. The transcriptions and other research data will be stored in a locked file cabinet in the researcher’s classroom in a locked and alarmed school building and a computer that is password and firewall protected.

VII. **Contact Persons:**

Call Dr. Christine Thomas at 404 651-2515 or cthomas11@gsu.edu, or Carol Taylor at 770 460-8904 or taylor.carol@fcboe.org if you have questions about this study. If you have questions or concerns about your child’s rights as a participant in this research study, you may contact Susan Vogtner in the Office of Research Integrity at 404 463-0674 or svogtner1@gsu.edu.

VIII. **Copy of Parental Permission Form:**

We will give you a copy of this permission form. If you are willing to allow your child to volunteer for this research and to be audio recorded, please sign below.

_________________________________________
Print Your Child’s Name

__________________________________________________   _________________
Parent/Guardian                             Date

__________________________________________  _______________
Researcher Obtaining Permission     Date
APPENDIX C

Introductory Task

Four goats were tied, one at each corner of a square field that measures 100 m x 100 m. The ropes allow each goat to graze over an area with a 50 m radius. This leaves an ungrazed portion in the center of the field. When three of the goats are moved, the roped tying the fourth goat is lengthened. This allows him to graze over an area equal to the combined area of the four goats. How long is the rope?
APPENDIX D

Task One: Basketball Court Renovation

Maple Street Middle School is undergoing considerable renovation. The first phase of the renovation is refinishing the basketball court. The shaded area will be painted. The remaining area will receive a hardwood finish.

1. Give a plan for finding the area of the court to be painted.

2. Give a plan for finding the area to receive the hardwood finish.

3. Find the area to be painted. Show your work.

4. Find the area to be given a hardwood finish.

5. Find the circumference of the center circle. If eight players stand around the center circle for a jump ball, what is the arc length available to each player?
APPENDIX E

Task Two: Smiley

Phase 2 of the Renovation Process: A dream has come true. The [teacher’s] lobbying efforts have paid off and finally there will be a math lab complete with sketchpad. A picture made from sketchpad has been enlarged to form a mural on the wall of the math lab. In keeping with Maple Street colors, the circle will be black and the shaded regions will be painted blue. Determine the paint needed for each feature of “Smiley”.
APPENDIX F

Task Three: Julie’s Wheel

Julie has three wheels from bikes and things that she stacked against the shed. Each wheel fit so neatly together that she couldn’t resist taking thing photograph. The radius of the largest wheel is 16 cm and the radius of the middle-sized wheel is 9 cm. What is the radius of Julie’s smallest wheel?
Simple Hint for Julie’s Wheel

Julie’s Wheel

Julie has three wheels from bikes and things that she stacked against the shed. Each wheel fit so neatly together that she couldn’t resist taking this photograph. The radius of the largest wheel is 16cm and the radius of the middle-sized wheel is 9cm. What is the radius of Julie’s smallest wheel?

Hint: Look at the 3 embedded rt. Δ’s. What do you notice about the hypotenuse of each rt. Δ?

Δ AHB
Δ AGC
Δ BDC
APPENDIX H

Structured Hint for Julie’s Wheel

Find IB

Find HC

Find CD

Also look closely at HC + CD. What is the turn?