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Sara Malec

Under the Direction of Dr. Florian Enescu

ABSTRACT

This paper presents the theory of Noetherian filtrations, an important concept in commutative algebra. The paper describes many aspects of the theory of these objects, presenting basic results, examples and applications. In the study of Noetherian filtrations, a few other important concepts are introduced such as Rees algebras, essential powers filtrations, and filtrations on modules. Basic results on these are presented as well. This thesis discusses at length how Noetherian filtrations relate to important constructions in commutative algebra, such as graded rings and modules, dimension theory and associated primes. In addition, the paper presents an original proof of the finiteness of the intersection algebra of principal ideals in a UFD. It concludes by discussing possible applications of this result to other areas of commutative algebra.

INDEX WORDS: Graded Rings and Modules, Noetherian Filtrations, Rees Algebras, E.P.F. Filtrations, Intersection Algebras

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by

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Contents

Chapter 1

Introduction: Graded Rings and Modules

Noetherian filtrations are a class of mathematical objects which have certain nice properties. In this paper we will develop the theory of filtrations, and prove the Noetherianity of a certain class of filtrations.

All rings are assumed to be commutative with identity. We will begin the first chapter by introducing graded rings. Then we will review several notions from introductory commutative algebra, beginning with defining Noetherian rings and modules and presenting some related results. The rest of the first chapter contains additional definitions and results concerning graded Noetherian rings.

Once these fundamentals have been established, Chapter 2 defines the objects in which this thesis is primarily concerned: filtrations, Rees algebras, and associated graded rings. We again include a number of examples. Further, we compute the dimension of Rees algebras of an ideal in a Noetherian ring.

Chapter 3 summarizes many important results concerning a special class of filtrations, called Noetherian filtrations. These are studied in depth here. In the process,

we define essential powers filtrations and discuss their relationship to Noetherian filtrations. Examples are given. The chapter concludes with a number of important equivalent conditions that characterize Noetherian filtrations.

Finally, Chapter 4 presents the concept of finite intersection algebra of two ideals. We present an original proof of the finiteness of intersection algebra of two primary ideals in a UFD. The chapter concludes with some other related results.

We will start by giving a review of some basic facts from commutative algebra that will be needed later in this paper. In this thesis, all rings are assumed to be commutative with identity.

Definition 1.1. A *semigroup* **G** is a set together with a binary operation + which is closed under addition, associative, and has an identity. A semigroup is called cancellative if for any $a, b, c \in \mathbf{G}$, and $a + b = a + c$, then $b = c$.

Definition 1.2. A *graded ring* over a cancellative semigroup **G** is a ring R that can be written as a direct sum of abelian groups $R = \bigoplus_{i \in G} R_i$ with the additional constraint that $R_i R_j \subset R_{i+j}$. An element $r \in R$ is called *homogeneous* if there is some i such that $r \in R_i$. Then i is called the *degree* of r. A *homogeneous ideal* is an ideal generated by homogeneous elements.

It should be noted that while a ring can be graded over any cancellative semigroup, generally in this paper they are graded over $\mathbb{N}$ or $\mathbb{Z}$. Also, in this thesis, we will assume that $\mathbb{N}$ contains 0.

Definition 1.3. If I is an ideal of R, then the graded ideal I^* is defined to be the ideal generated by all of the homogeneous elements in I. An ideal is called *homogeneous* if $I = I^*$. Other equivalent definitions of a homogeneous ideal will be explored below.

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Example 1.4. The simplest example of a graded ring is the polynomial ring $R = k[x]$. Then $R = R_0 \oplus R_1 \oplus \cdots \oplus R_n \oplus \cdots$, where R_i is the collection of the terms of degree i. This is clearly a direct sum decomposition, since $R_i \cap \bigoplus_{j \neq i} R_j = \emptyset$ for all i, and $R_i R_j \subset R_{i+j}$ since $x^i x^j = x^{i+j}$.

Example 1.5. For another example, take $R = k[x, y]$ and use the $\mathbb{N}$-grading induced by the total order, i.e. for any monomial $x^i y^j \in R$, the degree of that monomial is $i + j$. Thus an example of a homogeneous ideal would be $(x^4 y^2, x^3 + y^3, xy^3 + x^2 y^2)$, where the first term is in R_6, the second in R_3, and the third in R_4.

Example 1.6. The same ring can have a different grading and produce different homogeneous elements. If we instead use the multidegree order, $R = k[x, y]$ is graded over $\mathbb{N} \times \mathbb{N}$. The degree of any element $x^i y^j$ is (i, j), and therefore $(x^3 y^4, xy, x)$ is a homogeneous ideal with the first element of degree $(3, 4)$, the second of degree $(1, 1)$ and the third of degree $(1, 0)$.

Proposition 1.7. Let R be a G-graded ring, where G is a cancellative semigroup. Then 1 is in R_0.

Proof. Since $1 \in R$, $1 = \sum_{i \in G} x_i$ with $x_i \in R_i$. We claim $x_0 = 1$, and thus $1 \in R_0$. Let y be homogeneous in R. Then $y = y \cdot 1 = \sum_{i \in G} y x_i$. We equate degrees on both sides. Note that all of the terms of the sum are of distinct degrees, for if $\deg(yx_i) = \deg(yx_j)$, then $\deg(y) + \deg(x_i) = \deg(y) + \deg(x_j)$, and since G is cancellative, $\deg(x_i) = \deg(x_j)$. So, as $\deg(x_0) = 0$, then $y \cdot x_0 = y$. $\square$

Example 1.8. (Yongwei Yao) An interesting example arises if G is not cancellative. Let $G = \{0, b\}$, where $b \neq 0$ is such that $b + b = b$. Note that this is a semigroup, as it is closed under associative addition. Let $R = 0 \oplus \mathbb{Z}$ with the natural multiplication on $\mathbb{Z}$, where 0 is R_0 and is of degree 0, and $\mathbb{Z}$ is R_b and with the elements of $\mathbb{Z}$ having

degree b.We claim this fits the requirements for a graded ring: $R_0 \cdot R_b \subseteq R_b$, since for any $z \in \mathbb{Z}, 0 \cdot z = 0 \in \mathbb{Z} = R_b$, and $R_b \cdot R_b \subseteq R_b$ as $\mathbb{Z}$ is closed under addition. But here, 1 is clearly in R_b and not in R_0.

Proposition 1.9. Let I be an ideal of a G-graded ring R. Let I^* be the ideal generated by the homogeneous elements of I, and I^{**} be the ideal generated by the homogeneous components of I. Then the following are equivalent:

1. If $f \in I$ and $f = f_1 + f_2 + \cdots + f_n$ with $f_i \in R_{g_i}$ and $g_i \neq g_j$, then $f_i \in I$;

2. $I = I^{**}$;

3. I is generated by homogeneous elements;

4. $I = I^*$;

Proof. $1 \Rightarrow 2$: First, note that $I^* \subseteq I \subseteq I^{**}$ always. So let $f \in I^{**}$ be a generator of I^{**}. So f is a homogeneous component of an element of I by definition of I^{**}, and so by hypothesis $f \in I$. Thus all of the generators of I^{**} are in I, and thus $I^{**} \subseteq I$, therefore they are equal.

$2 \Rightarrow 3$: Since I^{**} is generated by homogeneous elements and $I = I^{**}$, I is generated by homogeneous elements.

$3 \Rightarrow 4$: We know already that $I^* \subseteq I$, so now let $f \in I$ be a in the set of homogeneous generators for I. By hypothesis, f is homogeneous, and thus $f \in I^*$ by the definition of I^*. Since the generators of I are in I^*, $I \subset I^*$ and thus $I = I^*$.

$4 \Rightarrow 1$: Let $f \in I$, with $f = f_1 + f_2 + \cdots + f_n$, and f_i be homogeneous of degree g_i. Now $f \in I^*$, so $f = \sum r_j h_j$ where $r_j \in R$ and the h_j are homogeneous elements of l_j. Now each of the r_i is a sum of homogeneous elements, so multiply out the terms of f and identify the degrees. Then $f_k = \sum r'_l h_l$ where the r'_l are homogeneous, and thus $f_k \in I^*$, so $f_k \in I = I^*$.

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□

Before we can proceed, we need a general discussion of Noetherianity of rings and modules and a few other items from commutative algebra. The following summary is presented without proof, and a thorough treatment can be found in an introductory text such as [4], [9] or [5].

Definition 1.10. A ring R is said to be *Noetherian* if it satisfies the ascending chain condition (A.C.C.) on ideals, i.e. for any increasing chain $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ of ideals of R there exists an integer k such that $I_n = I_k$ for all $n \geq k$. A left R-module M is Noetherian if it satisfies the A.C.C. on submodules.

Definition 1.11. In a dual way, we can define an *Artinian* ring R as one that satisfies the descending chain condition, or D.C.C. That is for any decreasing chain of ideals $I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots$ there exists an integer k such that $I_n = I_k$ for all $n \geq k$. A left R-module M is Artinian if it satisfies the D.C.C. on submodules.

Proposition 1.12. The following are equivalent:

1. R is a Noetherian ring (module);

2. Every ideal (submodule) of R (M) is finitely generated;

3. Every nonempty family of ideals (submodules) of R (M) has a maximal element (under inclusion);

Proposition 1.13. Any homomorphic image of a Noetherian ring is Noetherian. In particular, if R is Noetherian with I an ideal of R, then R/I is Noetherian.

Theorem 1.14. *(Hilbert Basis Theorem) If R is a commutative Noetherian ring with identity, then so is $R[x_1, \ldots, x_n]$.*

Definition 1.15. Let R be a ring. The supremum of the lengths of chains of prime ideals of R is called the *dimension of R*, denoted $\dim R$.

Definition 1.16. Let P be a prime ideal of a ring R. Then the *height of P*, denoted $\mathrm{ht}(P)$, is the supremum of lengths of chains of prime ideals $P_0 \subset \cdots \subset P_n = P$.

Definition 1.17. Let L/K be a field extension. The *transcendence degree* of the extension is the largest cardinality of an algebraically independent subset of L over K.

Definition 1.18. Let R be a ring. If R has a unique maximal ideal $\mathfrak{m}$, then we say that R is a *local ring*, denoted $(R, \mathfrak{m})$.

Definition 1.19. Let R be a ring and S a subset of R with identity that is closed under multiplication. Then the *localization of R at S*, denoted $S^{-1}R$ or R_S, is defined to be $\{\frac{r}{s} | r \in R, s \in S\}$, with the additional requirement that $r/s = r'/s'$ if and only if there exists some $u \in S$ such that $u(s'r - sr') = 0$.

Definition 1.20. Let R and S be as above and let M be an R-module. Then the *localization of M at S*, denoted $S^{-1}M$, is defined to be $M \otimes_R R_S$.

In the above two definitions, if S is the complement of a prime ideal P in R, then the localization of the ring or module at S is called R_P or M_P respectively. In this case, S is automatically a multiplicative set.

Proposition 1.21. Let $(R, \mathfrak{m})$ be a Noetherian ring. Then $\dim(R)$ is finite.

Proposition 1.22. Let R be Noetherian. Then R is Artinian if and only if $\dim(R)=0$. Further, if R is local with maximal ideal $\mathfrak{m}$, then there exists an $n \in \mathbf{N}$ such that $\mathfrak{m}^n = 0$.

Definition 1.23. Let R be a ring. The collection of prime ideals of R is called the *spectrum* of R and denoted $Spec(R)$. The collection of *minimal primes* of R is denoted $Min(R)$.

Definition 1.24. Let M be an R-module with $P \in \mathrm{Spec}(R)$. We say that P is an *associated prime* if P is the annihilator of an element of M. The collection of associated primes is denoted $\mathrm{Ass}(M)$.

Definition 1.25. The support of a module M, denoted $Supp(M)$, is the set of prime ideals $P \in \mathrm{Spec}(R)$ such that $M_P \neq 0$.

Definition 1.26. Let I be an ideal in R. Then the *radical* of an ideal, denoted $\mathrm{Rad}(I)$ or $\sqrt{I}$, is defined to be $\mathrm{Rad}(I) = \{r \in R | r^n \in I \text{ for some } n \in \mathbb{N}\}$. Note that for any I, $I \subset \mathrm{Rad}(I)$.

Definition 1.27. Let R be a ring and P a prime ideal. Then the *nth symbolic power of P*, denoted $P^{(n)}$, is $P^n R_P \cap R$.

The following three results are presented by Bruns and Herzog in [2] on pages 29-30. We will follow their treatment closely.

Theorem 1.28. *Let R be an $\mathbb{N}$-graded R_0-algebra, and $x_1, \ldots, x_n$ homogeneous elements of positive degree. Then the following are equivalent:*

1. $x_1, \ldots, x_n$ generate the ideal $\mathfrak{m} = \bigoplus_{i=1}^{\infty} R_i$;

2. $x_1, \ldots, x_n$ generate R as an R_0-algebra.

In particular, R is Noetherian if and only if R_0 is Noetherian and R is a finitely generated $R_0 - algebra$.

Proof. $2 \Rightarrow 1$: By hypothesis, for any $r \in R$, there exists $f(T_1, \ldots, T_n) \in R_0[T_1, \ldots, T_n]$ such that $r = f(x_1, \ldots, x_n)$. Let $r \in \mathfrak{m}$ be a homogeneous element. Then we claim that $r = f(x_1, \ldots, x_n) = \sum_{I=(i_1, \ldots, i_n)} (r_I x_1^{i_1} \cdots x_n^{i_n}) \subseteq (x_1, \ldots, x_n)$. Since r is homogeneous, so f is homogeneous of the same degree. So we can match up the degrees. Since $r \in \mathfrak{m}$, $\deg r \geq 1$, and so each term of f has an x_i in it for some i. Hence $r = \sum r_i' \cdot x_i \in (x_1, \ldots, x_n)$. Clearly, $(x_1, \ldots, x_n) \subseteq \mathfrak{m}$, so $\mathfrak{m} = (x_1, \ldots, x_n)$.

$1 \Rightarrow 2$: Let $y \in R$ be homogeneous of degree d. We do induction on d. We want to show that $y = y_1 x_1 + \cdots + y_n x_n$ with $y_i \in R_0$. If $\deg y = 0$, we are done, as $y \in R_0$ already.

Now assume that the homogeneous elements of R of degree less than d are generated as an R_0-algebra by $x_1, \ldots, x_n$. By hypothesis, we know $y \in \bigoplus_{i \geq 1} R_i = \mathfrak{m} = (x_1, \ldots, x_n)$. So $y = y_1 x_1 + \cdots + y_n x_n$, with $y_i \in R_i$. So y is homogeneous, and the x_i are homogeneous, but the y_i may not be. Multiply out and combine like terms. Then we have $y = y_1' x_1 + \cdots + y_n' x_n$, where the y_i' are homogeneous of degree $\deg(y) - \deg(x_i)$, which is less than d. So by induction, there exists an $f_i \in R_0[T_1, \ldots, T_n]$, with $y_i' = f_i(x_1, \ldots, x_n)$. Now, non-homogeneous elements are sums of homogeneous elements, so the statement follows.

For the last statement, if R is Noetherian, then $R_0 \cong R / \bigoplus_{i \geq 1} R_i = R/\mathfrak{m}$, which implies that R_0 is Noetherian. Also, if R is Noetherian, $\mathfrak{m}$ is finitely generated by say $(x_1, \ldots, x_n)$, and by this Theorem, R is R_0 finitely generated by $(x_1, \ldots, x_n)$. For the other direction, if R_0 is Noetherian, then since $R = R_0[r_1, \ldots, r_m] = R_0[T_1, \ldots, T_h]/I$, which implies that R is Noetherian. $\qquad \square$

Theorem 1.29. *Let R be a $\mathbb{Z}$-graded ring. Then the following are equivalent:*

1. *Every graded ideal of R is finitely generated;*

2. R is a Noetherian ring;

3. R_0 is Noetherian, and R is a finitely generated R_0-algebra;

4. R_0 is Noetherian, and both $S_1 = \bigoplus_{i=0}^{\infty} R_i$ and $S_2 = \bigoplus_{i=0}^{\infty} R_{-i}$ are finitely generated R_0-algebras.

Proof. The above theorems make $4 \Rightarrow 3 \Rightarrow 2 \Rightarrow 1$ clear: assuming 4 shows that R is a finitely generated R_0-algebra, since it is a sum of S_1 and S_2. The previous theorem makes 2 clear, which clearly implies 1 since every ideal of R is finitely generated.

$1 \Rightarrow 4$: Note that R_0 is a direct summand of R as and R_0-module. So $IR \cap R_0 = I$ for any ideal I of R_0. We claim that R_0 is Noetherian.

Take an ascending chain of ideals $I_0 \subseteq I_1 \subseteq \cdots \subseteq I_n \subseteq I_{n+1} \subseteq \cdots$ in R_0. Extend these ideals to R. So $RI_0 \subseteq RI_1 \subseteq \cdots \subseteq RI_n \subseteq RI_{n+1} \subseteq \cdots$ is a chain of ideals in R. Since R is Noetherian, this chain stabilizes at say the nth position. Now contract this chain back to R_0 to get $RI_0 \cap R_0 \subseteq RI_1 \cap R_0 \subseteq \cdots \subseteq RI_n \cap R_0 = RI_{n+1} \cap R_0 = \cdots$. This chain obviously stabilizes, and since $IR \cap R_0 = I$, this chain is the same as the one we started with. A similar argument for chains of submodules shows that R_i is a finite R_0-module for every $i \in \mathbf{Z}$.

Now let $\mathfrak{m} = \bigoplus_{i=1}^{\infty} R_i$. We claim $\mathfrak{m}$ is a finitely generated ideal of S_1. By hypothesis, $\mathfrak{m}R$ has a finite system of generators $x_1, \ldots, x_m$, and assume each generator x_i is homogeneous of degree d_i. Let $d = \max\{d_1, \ldots, d_m\}$. Then $y \in \mathfrak{m}$ with $\deg y \geq d$ can be written as a linear combination of $x_1, \ldots, x_m$ with coefficients in S_1. Thus $x_1, \ldots, x_m$ together with the homogeneous generators spanning $R_1, \ldots, R_{d-1}$ over R_0 generate $\mathfrak{m}$ as an ideal of S_1. By (1.28), S_1 is a finitely generated R_0-algebra and S_2 follows by symmetry. $\square$

Theorem 1.30. *Let R be a $\mathbb{Z}$-graded ring.*

1. *For every prime ideal $\mathcal{P}$, the ideal $\mathcal{P}^*$ is a prime ideal.*

2. *Let M be a graded R-module*

   (a) *If $\mathcal{P} \in Supp(M)$, then $\mathcal{P}^* \in Supp(M)$.*

   (b) *If $\mathcal{P} \in Ass(M)$, then $\mathcal{P}$ is graded; furthermore $\mathcal{P}$ is the annihilator of a homogeneous element.*

Proof.   1. Let $a, b \in R$ with $ab \in P^*$. We can write $a = \sum_i a_i$, with $a_i \in R_i$, $b = \sum_j b_j$ with $b_j \in R_j$. We do a proof by contradiction.

Assume $a \notin P^*$ and $b \notin P^*$. Then there exists a $p, q \in \mathbf{Z}$ such that $a_p \notin P^*$ but $a_i \in P^*$ for $i < p$ and $b_q \notin P^*$, but $b_j \in P^*$ for $j < q$. Then the $(p+q)$th homogeneous component of $ab \in P^*$ is $\sum_{i+j=p+q} a_i b_j$. This sum is in P^*, since P^* is graded. All summands of this sum are also in P^* since P^* is a homogeneous ideal, so $a_p b_q \in P^*$. Since $P^* \subset P$ and P is prime, then $a_p \in P$ or $b_p \in P$. But a_p and b_q are homogeneous, so either a_p or $b_q \in P^*$

2. (a) Assume $P^* \notin Supp(M)$. So $M_{P^*} = 0$. Let $x \in M$ homogeneous. Then there exists an $a \in R \setminus P^*$ such that $ax = 0$. Since $x/1 \in M_{P^*} = 0$, there exists an $a \notin P^*$ with $ax = 0$. It follows that $a_i x = 0$ for any a_i a homogeneous component of a. Since $a \in R \setminus P^*$, there exists an i such that $a_i \notin P^*$. Since a_i is homogeneous, $a_i \notin P$. Thus $x/1 = 0$ in M_P, which is a contradiction.

   (b) Let $x \in M$ with $P = \text{Ann}(x)$. Let $x = x_m + \cdots + x_n$ with x_i homogeneous, and $a = a_p + \cdots + a_q \in P$. Since $ax = 0, \sum_{i+j=r} a_i x_j = 0$ for $r = m + p, \ldots, n + q$. Thus $a_p x_m = 0, a_{p+1} x_{m+1} = 0$, etc. We claim that $a_p^2 x_{m+1} = 0$.

Examine the $p+m+1^{\text{th}}$ degree terms: $a_p \cdot x_{m+1} + a_{p+1} \cdot x_m$. We know this must be 0. Thus $a_p(a_p \cdot x_{m+1} + a_{p+1}x_m = a_p^2 \cdot x_{m+1} + a_{p+1}(a_p \cdot x_m) = 0$, thus $a_p^2 \cdot x_{m+1} = 0$. By induction, $a_p^i x_{m+i-1} = 0$ for all $x \geq 1$. So a_p^{n-m+1} annihilates x. Since P is prime, $a_p \in P$, so each homogeneous component of a is in P, and thus P is graded.

For the second part, we need to show that $Ann_R(x) = Ann_R(x_m)$. Now $a \in P = P^*$, so $ax = 0$, and $ax_i = 0$ for all i, so $P \subseteq \mathrm{Ann}(x_i)$. Now $\mathrm{Ann}(x) = \bigcap_{i=m} \mathrm{Ann}(x_i)$. And $\bigcap_{i=m} \mathrm{Ann}(x_j) \subseteq P$. Since P is prime, there is an i such that $P \supseteq \mathrm{Ann}(x_i) \supseteq P$.

$\square$

Chapter 2

Filtrations and Rees Algebras

The fundamental objects that we will use in this paper are filtrations of ideals and the Rees algebras generated by them. This chapter begins with some examples, and then develops the basic ideas behind the dimension of Rees algebras of a power filtration. Basic facts on filtrations such as [2], [5], pages 147-150, [16], pages 93-95 and [9], pages 93-94, and a thorough treatment is given in a remarkable book by Rees. More specific aspects, not touched on upon here, can be found in [8].

Definition 2.1. Let R be a ring. We define a *filtration* of ideals of R to be a chain of ideals $\{I_n\}_n$ starting with $I_0 = R$, $I_n \subseteq I_{n-1}$ for all $n \geq 1$, with the additional requirement that the ideals satisfy $I_n \cdot I_m \subseteq I_{n+m}$. Let E be an R-module. Then we define a filtration on E, denoted $e = \{E_n\}_n$, to be a descending chain of R-submodules E_n of E such that $E_0 = E$.

Example 2.2. Let I be an ideal of R. A typical example of a filtration is the power filtration $\{I^n\}_n$. Then $I_0 = R$, $I_1 = I$, $I_2 = I^2$ and so on. Clearly this satisfies both properties of a filtration, since $I^n \supset I^{n+1}$ and $I^n I^m \subset I^{n+m}$.

Example 2.3. Note that if you "shift" a filtration up, it remains a filtration. Say,

using the I_i from above, that $J_0 = I_0 = R$, $J_1 = I_4$, $J_2 = I_5$, and so on. Then $\{J_n\}$ is a filtration as well, for the same reasons as above.

Example 2.4. Let P be a prime ideal in a ring R. Then for all n and m, $P^n \cdot P^m \subseteq P^{n+m}$, so $P^n R_P \cdot P^m R_P \subseteq P^{n+m} R_P$. Hence $(P^n R_P \cap R) \cdot (P^m R_P \cap R) \subseteq P^{n+m} R_P \cap R$. In other words, $P^{(n)} \cdot P^{(m)} \subseteq P^{(n+m)}$, and so $\{P^{(n)}\}_n$ forms a filtration.

Example 2.5. Another example is very close to a power filtration. Let $I_1 = (x^a y^b)$, $I_2 = (x^{2a} y^b)$, $I_n = (x^{na} y^b)$ in $R = k[x, y]$. Again, this clearly satisfies both properties of a filtration.

Example 2.6. A less obvious one is $I_n = (x^{\lceil \sqrt{n} \rceil})$ in $k[x]$ with k a field. While the inclusion is still trivial, the other requirement requires proof.

We need to show that $\lceil \sqrt{m+n} \rceil \leq \lceil \sqrt{m} \rceil + \lceil \sqrt{n} \rceil$. Obviously $\lceil \sqrt{m+n} \rceil \leq \lceil \sqrt{m} + \sqrt{n} \rceil$. Then, $\sqrt{m+n} \leq \sqrt{m} + \sqrt{n} \leq \lceil \sqrt{m} \rceil + \lceil \sqrt{n} \rceil$. Also, $\lceil \sqrt{m} \rceil + \lceil \sqrt{n} \rceil \in \mathbb{N}$. So by the definition of the ceiling, $\lceil \sqrt{m+n} \rceil \leq \lceil \sqrt{m} \rceil + \lceil \sqrt{n} \rceil$. So the second property of a filtration is fulfilled, and $\{I_n\}$ is a filtration of ideals on $k[x]$.

Definition 2.7. Let $f = \{I_n\}$ be a filtration of ideals of a ring R. Then we can define the *graded ring associated to the filtration* as

$$gr_f(R) = \bigoplus_{n \geq 0} \frac{I_n}{I_{n+1}}.$$

For $x \in I_n$ and $y \in I_m$, multiplication is defined to be $(x + I_{n+1})(y + I_{m+1}) = xy + I_{n+m+1}$. If the filtration is understood to be the power filtration, we can write the associated graded ring of an ideal I as $gr_I(R)$.

If A is a ring, with $I \leq A$ an ideal, and $f = \{I^n\}$ is the power filtration defined by an ideal I, then $gr_I(A)$ is generated over A/I by the elements of I/I^2. To see

this, notice that any element in I^n/I^{n+1} can be written as a linear combination of products of n elements of I/I^2.

Now, using filtrations, we introduce the notion of a Rees Algebra of a filtration.

Definition 2.8. Let $f = \{I_n\}$ be a filtration of a ring R. Then the *Rees Algebra* of f is

$$\mathcal{R} = \{F = \sum_{k=0}^{n} F_k t^k | F_k \in I_k\} \subseteq R[t].$$

By the properties of the filtration, this is a subring of $R[t]$. To check this, let $F = a_0 + a_1 t + \cdots + a_n t^n$ and $G = b_0 + b_1 t + \cdots + b_m t^m$ be in $\mathcal{R}$. Then because I_i is an ideal, $a_i + b_i$ stays in I_i, so $F + G$ is clearly still in $\mathcal{R}$. Also, $a_i t^i \cdot b_j t^j = a_i b_j t^{i+j}$, and since I is a filtration, $a_i b_j \in I_{i+j}$.

Definition 2.9. Let $u = t^{-1}$ and $f = \{I_n\}$ be a filtration. We define the *extended Rees Algebra*, $\mathcal{R}' = \cdots \oplus Ru^2 \oplus Ru \oplus R \oplus I_1 t \oplus I_2 t^2 \oplus \cdots \subseteq R[t, t^{-1}]$.

Proposition 2.10. Let R be a ring with a filtration f and $\mathcal{R}'$ the extended Rees algebra as defined above. Then $g_f(R) \cong \mathcal{R}'/u\mathcal{R}'$.

Proof. Let $r \in \mathcal{R}'$, with $r = \sum_n r_n t^n$, where $r_n \in I_n$ if $n \geq 0$ and $r_n \in R$ when $n < 0$. Construct a homomorphism $\varphi : \mathcal{R}' \to g_f(R)$, where $\varphi(r) = \sum_n \bar{r}_n$, where $r_n \in I_n/I_{n+1}$ for all $n \geq 0$ and $r_n \in R$ for $n < 0$. This is clearly a surjective homomorphism. Then, if $\varphi(r') = 0$, then $r' = \sum_n r'_{n+1} t^n$, where $r'_{n+1} \in I_{n+1}$, which is the same as $u\mathcal{R}'$. Thus, $g_f(R) \cong \mathcal{R}'/u\mathcal{R}'$. $\square$

Now we can compute the dimension of the Rees algebras of a power filtration. This result is shown as Theorem 5.1.4 in [16].

Theorem 2.11. *Let R be a Noetherian ring, and let I be a proper ideal of R. Then dim R is finite if and only if the dimension of either the Rees algebra or the extended Rees algebra is finite. Further, if dim R is finite, then:*

1.

$$\dim R[It] = \begin{cases} \dim R + 1, & \text{if } I \nsubseteq P \text{ for some prime ideal } P \text{ with } \dim(R/P) = \dim R; \\ \dim R & \text{otherwise.} \end{cases}$$

2. $\dim R[It, t^{-1}] = \dim R + 1$

3. If $\mathfrak{m}$ is the only maximal ideal in R, and if $I \subseteq \mathfrak{m}$, then $\mathfrak{m}R[It, t^{-1}] + ItR[It, t^{-1}] + t^{-1}R[It, t^{-1}]$ is a maximal ideal in $R[It, t^{-1}]$ of height $\dim R + 1$.

4. $\dim(gr_I(R)) = \dim R$.

Proof. First, let J be an ideal of R. Then,

$$J \subseteq JR[It] \cap R \subseteq JR[It, t^{-1}] \cap R \subseteq JR[t, t^{-1}] \cap R = J \tag{2.1}$$

so the above inclusions are all equalities. So, any ideal in R is a contraction of an ideal in $R[It]$ and $R[It, t^{-1}]$. In addition,

$$\frac{R}{J} \subseteq \frac{R[It]}{JR[t, t^{-1}] \cap R[It]} \subseteq \frac{R[It, t^{-1}]}{JR[t, t^{-1}] \cap R[It, t^{-1}]} \subseteq \frac{R[t, t^{-1}]}{JR[t, t^{-1}]}. \tag{2.2}$$

We claim that the two middle rings are isomorphic to the Rees algebra and the extended Rees algebra, respectively, of the image of I in R/J. To see this, let $\bar{I} = \frac{I+J}{J} \subseteq \frac{R}{J}$, and $\bar{R} = R/J$. Then let $r \in R[It] = r_0 + r_1 t + \cdots$, where $r_i \in I^i$. Define a homomorphism $\varphi : R[It] \to \bar{R}[\bar{I}t]$ by $\varphi(r) = \bar{r}_0 + \bar{r}_1 t + \cdots$, where $\bar{r}_i \in \bar{I}$. Then $\text{Ker}(\varphi) = \{r \in R[It] | \varphi(r) = 0\} = \{r \in R[It] | r_i \in J \text{ for all } i\}$, which is the same as saying that $r \in R[It]$ and $r \in JR[t] \subseteq JR[t, t^{-1}]$, which proves the isomorphism. The second is done in a similar way.

In particular, we claim that if P is a minimal prime of R, then $PR[t, t^{-1}] \cap R[It]$

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must be minimal in $R[It]$, and $PR[t,t^{-1}] \cap R[It,t^{-1}]$ must be minimal in $R[It,t^{-1}]$.
Call $PR[t,t^{-1}] \cap R[It] = \tilde{P}$. To show that $\tilde{P}$ is minimal, we need that $R[It]_{\tilde{P}}$ is Artinian, which is equivalent to having $\tilde{P}R[It]_{\tilde{P}}$ nilpotent. So we must show that every element in $\tilde{P}R[It]_{\tilde{P}}$ is nilpotent. We know that $\tilde{P} \cap R = P$, and that R_P is Artinian, so PR_P is nilpotent, and therefore $PR_P[t]$ is nilpotent. Let $S = R \setminus P \subseteq R[It] \setminus \tilde{P}$. But $S^{-1}\tilde{P} \subseteq P_P R_P[t]$, so it is nilpotent as well. Then $\tilde{P}R[It]_{\tilde{P}}$ is a further localization of the nilpotent ideal $S^{-1}\tilde{P}$, so it is nilpotent as well.

Then, any nilpotent element of $R[It]$ or $R[It,t^{-1}]$ is certainly nilpotent in $R[t,t^{-1}]$, so it has to lie in the intersection of the primes of $R[t,t^{-1}] = \bigcap_{P \in \text{Min}(R)} PR[t,t^{-1}]$. So all the minimal prime ideals of the Rees algebras are contractions of minimal primes of $R[t,t^{-1}]$ and are of the form $PR[t,t^{-1}]$. So,

$$\begin{aligned}
\dim R[It] =&\ \max_{Q \in \text{Min} R[It]}(\dim \tfrac{R[It]}{Q}) = \max_{P \in \text{Min} R}(\dim \tfrac{R[It]}{PR[t,t^{-1}] \cap R[It]}) \\
=&\ \max_{P \in \text{Min} R}(\dim \bar{R}[\bar{I}t]) = \max_{P \in \text{Min} R}(\dim \tfrac{R}{P}[\tfrac{I+P}{P}t])
\end{aligned}$$

Thus $\dim R[It] = \max\{\dim (\tfrac{R}{P}[\tfrac{I+P}{P}t]) | P \in \text{Min} R\}$, and similarly $\dim R[It,t^{-1}] = \max\{\dim (\tfrac{R}{P}[\tfrac{I+P}{P}t, t^{-1}]) | P \in \text{Min } R\}$. So, to calculate $\dim R[It]$, it is enough to show that for an integral domain R, $\dim R[It] = \dim R$ if I is the zero ideal and is $\dim R + 1$ otherwise. Thus we can assume that R is a domain.

Proposition 2.12. (Dimension Inequality) Let R be a Noetherian integral domain, with S a ring extension of R which is also a domain. Let Q be a prime ideal in S and $P = Q \cap R$. Then

$$\text{ht}Q + \text{tr. } \deg_{\kappa(P)}\kappa(Q) \le \text{ht}P + \text{tr.deg}_R S. \tag{2.3}$$

The above Proposition, proven as Theorem B.2.5 in [16], implies that, when $R = R$

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and $S = R[It]$, that for every prime ideal Q in $R[It]$,

$$\text{ht}Q + \text{tr.deg}_{\kappa(Q\cap R)}\kappa(Q) \le \text{ht}(Q\cap R) + \text{tr.deg}_R R[It]. \tag{2.4}$$

Clearly, $\text{tr.deg}_R R[It] = 1$, since the larger ring is simply R with one variable adjoined to it. Therefore, no matter what $\text{tr.deg}_{\kappa(Q\cap R)}\kappa(Q)$ is, ht $Q \le \text{ht}(Q\cap R) + 1 \le \dim R + 1$. So, the height of any prime in $R[It]$ is at most one larger than the height of any prime in R, which proves that $\dim R[It] \le \dim R + 1$. Clearly $\dim R[It] = \dim R$ if I is the zero ideal, since $R[(0)t] = R$. So assume that I is non-zero. Let $P_0 = ItR[It]$. Then $P_0 \cap R = (0), It \subseteq P_0$, $\text{ht}P_0 > 0$ (since $(0) \subsetneq P_0$), and $R[It]/P_0 \cong R$ which is an integral domain, proving that P_0 is prime. Since P_0 is another prime added to any chain of primes that can be made in R,

$$\dim R[It] \ge \dim R + 1.$$

This proves (1).

Similarly for (2), it is enough to show that when R is a domain,

$$\dim R[It, t^{-1}] = \dim R + 1.$$

Again by the dimension inequality, $\dim R[It, t^{-1}] \le \dim R+1$, and the other inequality follows from $\dim R[It, t^{-1}] \ge \dim R[It, t^{-1}]_{t^{-1}} = \dim R[t, t^{-1}] = \dim R + 1$.

Lastly, let $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_h = \mathfrak{m}$ be a saturated chain of prime ideals in R, with $h = \text{ht}\,\mathfrak{m}$. Set $Q_i = P_i R[t, t^{-1}] \cap R[It, t^{-1}]$. As $Q_i \cap R = P_i, Q_0 \subseteq Q_1 \subseteq \cdots \subseteq Q_h$ is a chain of distinct prime ideals in R. The biggest one is $Q_h = \mathfrak{m}R[t, t^{-1}] \cap R[It, t^{-1}] = \mathfrak{m}R[It, t^{-1} + ItR[It, t^{-1}]$, which is properly contained in the maximal ideal $Q_h + t^{-1}R[It, t^{-1}]$, which proves (3). $\qquad \square$

Chapter 3

Noetherian Filtrations

Filtrations of ideals represent an important concept in commutative algebra. They have a rich and long history and have been studied by many authors in various contexts. Noetherian filtrations are central among filtrations of ideals and their theory has been developed by authors such as W. Bishop, Okon, Petro, Rattliff, Rees, and Rush among others, see [1], [10], [11], [12], [13], and [14],.

In this chapter, we define and give examples of Noetherian filtrations, and show that they are an interesting class of filtrations with remarkable properties. Noetherian filtrations have finiteness conditions that are similar to power filtrations. This chapter will explain what those conditions are. Also, we will introduce and study the notion of an e.p.f. filtration. Our presentation follows closely [1], [12] and [13].

Proposition 3.1. Let R be a ring and I an ideal in R. Then if $\mathcal{R}$ is the Rees algebra generated by the power filtration of I, $\mathcal{R}$ is finitely generated over R whenever I is finitely generated. In this case, if $I = (a_1, \ldots, a_n)$, then $\mathcal{R}$ is generated by $a_1 t, \ldots, a_n t$.

Proof. Let $I = (a_1, \ldots, a_h)$. Then I^k is generated by products of k elements chosen from I. So any element of $\mathcal{R}$ looks like $F = i_0 + i_1 t + i_2 t^2 + \cdots + i_h t^h$, with $i_j \in I^j$. Let $i \in I^j$. Then i is an R-linear combination of products of the generators of I of the

form $a_1^{j_1} \cdots a_k^{j_k}$, where $j_1 + \cdots + j_h = j$. But $a_1^{j_1} \cdots a_k^{j_k} t^{j_1 + \cdots + j_k} = (a_1 t)^{j_1} \cdots (a_k t^{j_k})$. So every monomial in $\mathcal{R}$ can be written as a sum of powers of $a_i t$, where a_i is a generator of I. So $\mathcal{R} = R[a_1 t, \ldots, a_n t]$. $\qquad \square$

Example 3.2. Let $I = (x, y) \subset R[x, y]$, and construct a Rees algebra with the power filtration of I. So any element of $\mathcal{R}$ looks like $f = \sum_{k=0}^{n} a_k t^k$, with $a_k \in I^k$. But any element $a_k t^k$ can be written as products of powers of xt and yt, so any $f \in \mathcal{R}$ can be written as polynomial in xt and yt, so $\mathcal{R} = R[It]$.

Example 3.3. Now return to the example $I_n = (x^{\lceil \sqrt{n} \rceil}) \subseteq k[x]$. We claim $\mathcal{R}$ is not finitely generated. Assume the contrary. Then $\mathcal{R}$ is generated by some elements

$$\{x^{\lceil \sqrt{\alpha_1} \rceil} t^{\alpha_1}, x^{\lceil \sqrt{\alpha_2} \rceil} t^{\alpha_2}, \ldots, x^{\lceil \sqrt{\alpha_n} \rceil} t^{\alpha_n}\}.$$

We can write

$$x^{\lceil \sqrt{m} \rceil} t^m$$

as a polynomial over R in the above generators for all m.

So we need to find $a_1, a_2, \ldots, a_n$ such that:

$$a_1 \alpha_1 + a_2 \alpha_2 + \cdots + a_n \alpha_n \quad = \quad m \tag{3.1}$$

$$a_1 \lceil \sqrt{\alpha_1} \rceil + a_2 \lceil \sqrt{\alpha_2} \rceil + \cdots + a_n \lceil \sqrt{\alpha_n} \rceil \quad = \quad \lceil \sqrt{\alpha_1 + \alpha_2 + \cdots + \alpha_n} \rceil \tag{3.2}$$

Assume we have the a_i, $i = 1, \ldots, n$, such that equation (1) holds. Then, substituting (1) into (2) gives:

$$a_1 \lceil \sqrt{\alpha_1} \rceil + a_2 \lceil \sqrt{\alpha_2} \rceil + \cdots + a_n \lceil \sqrt{\alpha_n} \rceil = \lceil \sqrt{m} \rceil$$

We proved above that $\lceil \sqrt{a+b} \rceil \leq \lceil \sqrt{a} \rceil + \lceil \sqrt{b} \rceil$, so:

$$a_1 \lceil \sqrt{\alpha_1} \rceil + a_2 \lceil \sqrt{\alpha_2} \rceil + \cdots + a_n \lceil \sqrt{\alpha_n} \rceil \leq \lceil \sqrt{a_1 \alpha_1} \rceil + \lceil \sqrt{a_2 \alpha_2} \rceil + \cdots + \lceil \sqrt{a_n \alpha_n} \rceil$$

$$\leq \lceil \sqrt{a_1} \rceil \lceil \sqrt{\alpha_1} \rceil + \lceil \sqrt{a_2} \rceil \lceil \sqrt{\alpha_2} \rceil + \cdots + \lceil \sqrt{a_n} \rceil \lceil \sqrt{\alpha_n} \rceil$$

Therefore, $a_i \leq \lceil \sqrt{a_i} \rceil$ for all $i = 1, \ldots, n$. But, since $x \geq \lceil \sqrt{x} \rceil$ for any x, then $a_i = \lceil \sqrt{a_i} \rceil$ for all i. Thus, all of the inequalities above are in fact equality, and so:

$$\lceil \sqrt{a_1 \alpha_1 + a_2 \alpha_2 + \cdots + a_n \alpha_n} \rceil = \lceil \sqrt{a_1 \alpha_1} \rceil + \lceil \sqrt{a_2 \alpha_2} \rceil + \cdots + \lceil \sqrt{a_n \alpha_n} \rceil.$$

Since $x = \lceil \sqrt{x} \rceil$ only if $x = 0, 1$ or 2, this implies that for all i, a_i can be no larger than 2. Thus the largest α_i is certainly no larger than 2. So $m = \sum_i a_i \alpha_i \leq 2 \sum_i \alpha_i$. So m is bounded. But this is clearly impossible, so this Rees algebra is not finitely generated.

Definition 3.4. Let R be a ring with a filtration $f = \{I_n\}$. Recall that $gr_f(R) = \bigoplus_{n \geq 0} \frac{I_n}{I_{n+1}}$ is the graded ring associated to the filtration. We say that f is *Noetherian* if $gr_f(R)$ is Noetherian.

Theorem 3.5. *Let R be a Noetherian ring with $f = \{I^n\}$ the power filtration. Then f is Noetherian.*

Proof. Examine the following isomorphism:

$$\frac{R}{IR} = \frac{R[It]}{IR[It]} \simeq gr_f(R) = \bigoplus \frac{I_n}{I_{n+1}}$$

Thus, if R is Noetherian, then $\frac{R}{IR}$ is as well, and therefore $gr_f(R)$ is Noetherian, which is the definition of a Noetherian filtration. $\square$

Theorem 3.6. *(P. Roberts [15]) Let R be the polynomial ring $\mathbb{C}[x, y, z]$ localized at (x, y, z). Then there exists a prime ideal P in R such that $\bigoplus_{n \geq 0} P^{(n)}$ is not Noetherian.*

We've shown a few nice properties of power filtrations. But we can generalize the power filtration to a larger class of filtrations that behave nicely.

Definition 3.7. We say that a filtration $f = \{I_n\}$ of ideals of a ring R is an *essentially powers filtration* (or e.p.f.) if there exists an $m > 0$ such that $I_n = \sum_{i=1}^{m} I_{n-i} I_i$ for all $n \geq 1$. If $n - i < 0$, I_{n-1} is assumed to be R.

Let $f = \{I_n\}$ be a filtration on a ring R. Then we can prove a number of statements about e.p.f.'s.

Proposition 3.8. Let $f = \{I_n\}$ be a filtration on a ring R. Then the following are equivalent:

1. f is an e.p.f.;

2. $I_n = \sum(\prod^{k} I_j^{e_i})$, where m is as given in the definition of e.p.f.'s, and the sum is over all $e_i > 0$ such that $e_1 + 2e_2 + \cdots + ke_k = n$;

3. There exists an $m \in \mathbb{N}$ with the property that f is the least filtration on R whose first $m + 1$ terms are $R, I_1, I_2, \ldots, I_m$;

Proof. First, what does least mean here? And how do we know a smallest filtration exists? For the first question, for two filtrations $f = \{I_n\}$ and $g = \{J_n\}$, we say that $f \leq g$ if $I_i \subseteq J_i$ for all i. And we know that the smallest filtration with the given property exists, because we can simply take the intersection of all filtrations whose first $m + 1$ terms are $R, I_1, I_2, \ldots, I_m$.

$(1 \iff 2)$

Let $I_n = \sum_{i=1}^{k} I_{n-i} I_i$ for all $n \geq 1$, and let $I'_n = \sum(\prod^k I_j^{e_i})$. Then $a_i^{e_i} \in I'_n$ can be written as $a_i \cdots a_i$ (e_i times), which is inside I_{ie_i}. Since $I_{ie_i} = \sum_{j=1}^{k} I_{ie_i-j} I_j$, any monomial in I'_n can be written as a product of just two terms whose degrees sum to n, and thus it would be in I_n.

$I_n \subseteq I'_n$ by induction

$(2 \Longleftrightarrow 3)$ Let $g = \{J_n\}$ be any filtration on R such that $I_i = J_i$ for all $i = 0, \ldots, k$. So by the definition of a filtration,

$$\sum(\prod_{i=1}^{k} I_i^{e_i}) = \sum(\prod_{i=1}^{k} J_i^{e_i}) \subseteq J_n$$

Let $K_n = \sum(\prod_{i=1}^{k} I_i^{e_i})$ for all $n \geq k$, and $K_n = I_n$ for all $n < k$. Then let $h = \{K_n\}$, which is clearly a filtration. Since h is less than g, it is less than any filtration that agrees with f at first on R, so it's the smallest. So $h \leq f$, but $f = h$, by $(1 \Longleftrightarrow 2)$ □

Now assume further that R is Noetherian. Then there are a number of additional results that we can show. This is presented as Theorem (2.7) in [12].

Theorem 3.9. *Let R be a Noetherian ring with $f = \{I_n\}$ any filtration of R. Then the following are equivalent:*

1. *The extended Rees algebra $\mathcal{R}'$ of f, $\cdots \oplus Rt^{-2} \oplus Rt^{-1} \oplus R \oplus I_1 t \oplus I_2 t^2 \oplus \cdots$, is Noetherian;*

2. *$\mathcal{R}$ is Noetherian;*

3. *$\mathcal{R}$ is finitely generated over R;*

4. *f is an e.p.f.;*

Proof. Notice that, from the previous sections, 1 through 3 are equivalent, since $\mathcal{R}$ is graded and R is Noetherian. So we need only show that 4 is equivalent to the others.

$(4 \Rightarrow 3)$ Since f is an e.p.f., we know that $\mathcal{R} = R[tI_1, \dots, t^k I_k]$, since all terms can be gotten from the first k terms of the filtration. $(2 \Rightarrow 4)$ Let $f_1, \dots, f_m$ be a basis of $\mathcal{N}$. Since $\mathcal{N}$ is homogeneous, we can assume the f_i's are too, since if they are not, we can take the homogeneous components and add them to the list. So let $f_i = a_i t^{e_i}$ with $e_i > 0$. Let $k = max\{e_i | i = 1, \dots, m\}$, so $\mathcal{N} = (tI_1, t^2 I_2, \dots, t^k I_k)$. Let $n > k$ and $a \in I_n$, so $x = at^n \in \mathcal{N}$. But every element of $\mathcal{N}$ looks like $\sum g_i f_i$ for some $g_i \in \mathcal{R}$, hence $x = \sum g_i f_i$. Assume $g_i = b_i t^{n-e_i}$, and g_i is homogeneous. So:

$$x = \sum g_i f_i = \sum b_i t^{n-e_i} a_i t^{e_i} = \sum a_i b_i t^n = at^n$$

$$\Rightarrow a = \sum a_i b_i \in \sum_{i=1}^{n} I_{e_i} I_{n-e_i} \subseteq \sum_{i=1}^{k} I_j I_{n-j}$$

Thus, since $a \in I_n$, $I_n = \sum_{j=1}^{k} I_j I_{n-j}$ for $n > k$, which is the definition of an e.p.f. $\qquad\square$

Definition 3.10. Let $e = \{E_n\}$ be a filtration on an R-module E and $f = \{I_n\}$ a filtration on R. Then e is said to be *compatible* with f in case $I_m E_n \subseteq E_{m+n}$ for all m and n.

Definition 3.11. Let e be as above. Then e is said to be f-*good* in case e is compatible with f and there exists a positive integer m such that $E_n = \sum_{i=1}^{m} I_{n-i} E_i$ for all large n. In other words, f is f-good if and only if f is an e.p.f.

The following Proposition as well as the associated corollary are shown as (3.5) and (3.6) by Bishop in [1].

Proposition 3.12. Let R be a Noetherian ring with an e.p.f. $f = \{I_n\}$ and let E be a finitely generated R-module with an f-filtration $e = \{E_n\}$. Then e is f-good if and only if there exists a $k \geq 0$ such that $E_{k+i} = I_k E_i$ for all $i \geq k$.

Proof. Assume e is f-good. Then by definition, e is compatible with f and there exists an m such that $E_n = \sum_{i=1}^m I_{n-i} E_i$ for all large n, say $n > n_0$. Then we claim $\mathbb{E} = \sum E_i t^i$ is finitely generated over $S = R[tI_1, t^2 I_2, \ldots]$. Let $x_n \in E_n$, with $n > n_0$. Then $x_n \in \sum_{i=0}^m I_{n-i} E_i$, so $x_n t^n \in \sum_{i=1}^m I_{n-i} t^{n-i} E_i t^i$, which is in $SE_i t^i$. So if $x \in \mathbb{E}$, then $x = \sum_{i=0}^m x_n t^n$, which is in $\sum_{i=1}^m S(E_i t^i)$

We showed before that f is an e.p.f. if and only if $S = R[tI_1, t^2 I_2, \ldots]$ is finitely generated over R. So, there exists a $g > 0$ such that $S = R[tI_1, \ldots, t^g I_g]$ since f is an e.p.f.

Let j be the lcm of $2, 3, \ldots, g$. Then let m_i be the positive integer such that $im_i = j$ for all $i = 1, \ldots, g$. Then $(t^i I_i)^{m_i} \subseteq t^j I_j \subseteq A = R[t^j I_j]$. Thus any element of the form $t^i x$ with $x \in I_i$ is integral over A. Since S is finitely generated over A by integral elements, S is integral and finitely generated over $A = R[t^j I_j]$. Therefore $\mathbb{E}$ is finite A-module.

Let $\Theta_1, \ldots, \Theta_m$ be a homogeneous system of generators for $\mathbb{E}$ over A, with $\deg(\Theta_i) = d_i$ and $d = \max d_i$ for $i = 1, \ldots, m$. Let $n > \max\{d, j\}$ and let x be an element of E_n. So we can write $x = \sum_i h_i \Theta_i$ where h_i are homogeneous elements of A. These h_i are either 0 or of degree of degree $n - d_i$. By resubscripting if necessary, assume $h_i \neq 0$ for $i = 1, \ldots, m' \leq m$. Then $n - d_i \geq 1$, and since all of the elements of A have degree a multiple of j, thus for all $i = 1, \ldots, m'$ there exists a positive integer k_i such that $jk_i = n - d_i$. Thus:

$$x = \sum_{i=1}^{m'} h_i \Theta_i \subseteq \sum_{i=1}^{m'} I_j^{k_i} E_{d_i} \subseteq I_j (\sum^{m_i} I_j^{k_i-1} E_{d_i})$$

And since $I_j^{k_i-1}E_{d_i} \subseteq I_{j(k_i-1)}E_{d_i} \subseteq E_{j(k_i-1)+d_i} = E_{n-j}$, we have that $E_n \subseteq I_jE_{n-j}$. The opposite inclusion is obvious since e is compatible with f, so $E_n = I_jE_{n-j}$ for all $n > \max\{d, j\}$.

Last, let $k = jd$ and $i \geq k$. Then by the above equation, $E_{k+i} = E_{jd+i} = I_jE_{j(d-1)+i}$. Now $j(d-1)+i \geq \max(d, j) + 1$, so we can continue to pull out I_j until we are left with $I_j^dE_i$, which is in I_kE_i. Thus, $E_k \subseteq I_kE_i$.

For the converse, let there be a positive k such that $E_{k+i} = I_kE_i$ for all $i \geq k$. Then we claim $\mathbb{E} = \sum E_it^i$ is generated as a module over S by $E_1t, \ldots E_{2k-1}t$, since the smallest i which is not covered in the hypothesis is $i = k - 1$. Then according to (2.3) in [12], if $\mathbb{E}$ is finitely generated over S, then e is f. So it remains to show that $\mathbb{E}$ is finitely generated over S. Let $\mathbb{G}_i$ be the collection of generators for $E_i, i < 2k - 1$. This collection is finite, since by hypothesis, each E_i is finitely generated over R. So for every $e \in E_i$, $e = \sum_{finite} r_jx_j$, where $r_j \in R$ and $x_j \in E_j$. So to find the generators of $\mathbb{E}$, we need only to collect all the generators from each $\mathbb{G}_i$ and attach to them the appropriate power of t, i.e. the generators of $\mathbb{E}$ over S are all of the terms et^i, where $e \in \mathbb{G}_i$.

$\square$

Corollary 3.13. Let $f = \{I_n\}$ be a filtration on a Noetherian ring R. Then f is an e.p.f. if and only if there exists a $k > 0$ such that $I_{k+i} = I_iI_k$ for all $i \geq k$.

Proof. If f is an e.p.f. then f is e-good, so let $E = R$ and $e = f$ in (3.12) to see that there exists a k such that $I_{k+i} = I_iI_k$. If such a k exists, then if $n \geq 2k$, we can easily write

$$I_n = I_{n-k}I_k \subseteq \sum_{i=1}^{2k} I_{n-1}I_i \subseteq I_n,$$

so then f is an e.p.f. with $m = 2k$. If $n < 2k$, Then $I_n = \sum_{i=1}^{2k} I_{n-1}I_i$. Clearly, I_n is in this, if $i = n$, and I_n is also in both of them by the definition of a filtration. So f

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is an e.p.f. $\qquad\square$

We can now show another important equivalence, but first we need a few more results.

Proposition 3.14. Let R be a ring with a filtration $f = \{I_n\}_{n\geq 0}$ and let E be an R-module with an f-filtration $e = \{E_n\}_{n\geq 0}$ and let E be an R-module with an f-filtration $e = \{E_n\}_{n\geq 0}$ such that E_n is a finitely generated R-module for all $N \geq 1$. Then $G^+(E,e) = \sum_{n=1}^{\infty} E_n/E_{n+1}$ is a finitely generated $\mathrm{gr}_f(R)$-submodule of $G(E,e) = \sum_{n=0}^{\infty} E_n/E_{n+1}$ if and only if there exists a positive integer g such that, for all $j \geq g$, $E_{j+1} = I_j E_1 + \cdots, + I_{j-g+1} E_g + E_{j+2}$.

Proof. Assume that $G^+(E,e)$ is a finitely generated $\mathrm{gr}_f(R)$-submodule of $G(E_e)$. Construct the following submodules: let $A_{ij} = I_j E_1 + I_{j-1} E_2 + \cdots + I_{j-i+1} E_i + E_{j+2}$ and let $\bar{A}_i = \sum_{j=0}^{\infty} A_{ij}/E_{j+2}$. Then $\bar{A}_i$ is a $\mathrm{gr}_f(R)$-submodule of $G^+(E,e)$. Also $\bar{A}_i \subseteq \bar{A}_{i+1}$ and $\bigcup_{i=1}^{\infty} \bar{A}_i = G^+(E,e)$. Therefore the hypothesis implies that there exists a positive integer g such that $\bar{A}_g = \bar{A}_{g+t}$ for all $t \geq 0$ so it follows that $A_{gj}/A_{(g+t)j} = 0$ for all $j \geq 0$ and $t \geq 0$. In particular, if $j \geq g$ and $t \geq 1$, then $I_{j-g-t+1} E_{g+t} \subseteq I_j E_1 + I_{j-1} E_2 + \cdots + I_{j-g+1} E_g + E_{j+2}$ for all $j \geq g$, and since the opposite inclusion is obvious when $t = j - g + 1$, we obtain $E_{j+1} = I_j E_1 + I_{j-1} E_2 + \cdots + I_{j-g+1} E_g + E_{j+2}$ for all $j \geq g$.

Now let g be as given in the hypothesis. Then, for every $j \geq g$, $E_{j+1}/E_{j+2} = (I_j E_1 + \cdots + I_{j-g+1} E_g + E_{j+2})/E_{j+2} = (I_j/I_{j+1})(E_1/E_2) + \cdots + (I_{j-g+1}/I_{j-g+2})(E_g/E_{g+1})$. It follows that $G^+(E,e)$ is generated as a $\mathrm{gr}_f(R)$-submodule of $G(E,e)$ by E_n/E_{n+1} for $n = 1, \ldots, g$. Therefore, since each E_n is finitely generated, it follows that $G^+(E,e)$ is a finitely generated $\mathrm{gr}_f(R)$-submodule of $G(E,e)$. $\qquad\square$

Corollary 3.15. Let R be a Noetherian ring with a filtration $f = \{I_n\}_{n\geq 0}$ and let E be a finitely generated R-module with an f-filtration $e = \{E_n\}_{n\geq 0}$. If $G(E,e)$

is a finitely generated $\mathrm{gr}_f(R)$-module and for each positive integer n, there exists a positive integer $\rho(n)$ such that $E_{\rho(n)} \subseteq (\mathrm{Rad}(I_1))^n E_1$, then e is f-good.

Proof. Since $G(E, e)$ is a finitely generated $\mathrm{gr}_f(R)$-module, let g be as given in the previous proposition. So, by considering consecutive values of j, for all $j \geq g$, $E_{j+1} = I_j E_1 + \cdots + I_{j-g+1} E_g + E_{j+2}$. Since $E_{n+1} \subseteq E_n$, it follows that for all $j \geq g$,

$$E_{j+1} = I_j E_1 + \cdots + I_{j-g+1} E_g + E_t \tag{3.3}$$

for all $t \geq j + 2$ by induction on t..

Assume that the $\rho(n)$ described above exists. Since R is Noetherian, every ideal of R contains a power of its radical, so there exists a positive integer m such that $(\mathrm{Rad}(I_j))^m) \subseteq I_j$. Also, $(\mathrm{Rad}(I_1))^m = (\mathrm{Rad}(I_j))^m$, since $I_j \subseteq I_1$ and $I_1^j \subseteq I_j$. So $(\mathrm{Rad}(I_1))^m \subseteq I_j$. By assumption, or each positive integer n, there exists a positive integer $\rho(n)$ such that $E_{\rho(n)} \subseteq (\mathrm{Rad}(I_1))^n E_1$. Therefore, $E_{\rho(m)} \subseteq (\mathrm{Rad}(I_1))^m E_1 \subseteq I_j E_1$. Let $t = \rho(m)$ in equation (3.3). Then $E_{j+1} = I_j E_1 + \cdots + I_{j-g+1} E_g$ for all $j \geq g$. So for any m, n, we have $I_m E_n \subseteq E_{m+n}$, so e is f-good.

$\square$

Corollary 3.16. If $f = \{I_n\}_{n \geq 0}$ is a filtration on a Noetherian ring R, then the following are equivalent:

1. f is an e.p.f. ;

2. f is a Noetherian filtration and there exists a positive integer g such that $I_{gn} \subseteq (\mathrm{Rad}(I_1))^n$ for all large n;

3. f is a Noetherian filtration and for each positive integer n there exists a positive integer $\rho(n)$ such that $I_{\rho(n)} \subseteq (\mathrm{Rad}(I_1))^n$.

Proof. First, notice that since f is an e.p.f., then $\mathcal{R}$ is Noetherian. Since we showed in Chapter 2 that $\mathcal{R}/u\mathcal{R} \cong \mathrm{gr}_f(R)$, and quotients of Noetherian rings are Noetherian, then f is always a Noetherian filtration.

$(1 \Rightarrow 2)$ By the previous corollary we know that there exists a k such that $I_{k+i} = I_k I_i$ for all $i \geq k$. Therefore, $I_{gn} = I_g{}^{n-1} I_g$ for all $n \geq 1$. So $I_{gn} = I_g{}^{n-1} I_g \subseteq (I_1)^{n-1} I_1 \subseteq (\mathrm{Rad}(I_1))^n$ for all $n \geq 1$.

$(2 \Rightarrow 3)$ Clear.

$(3 \Rightarrow 1)$ By 3.15, if E is a finitely generated R-module with an f-filtration $e = \{E_n\}$, and if $G(E, e)$ is a finitely generated $\mathrm{gr}_f(R)$-module, and there exists a $\rho(n)$ such that $E_{\rho(n)} \subseteq (\mathrm{Rad}(I_1))^n E_1$, then e is f-good. Thus, if $E = R$ and $e = f$, by (2) f is f-good, i.e. f is an e.p.f. $\qquad\square$

Chapter 4

Finite Intersection Algebras

Now that we have established some properties of Noetherian filtrations, we can look at one example in depth that illustrates both the concepts of graded rings and Noetherian filtrations.

Definition 4.1. ([7], pages 126-127) Given a pair (I, J) of ideals of a ring R, call the algebra $\mathcal{B} = \bigoplus_{r,s} (I^r \cap J^s) u^r v^s$ the *intersection algebra* of I and J. If this algebra is finitely generated over R, we say that I and J have *finite intersection algebra*.

Let R be a Noetherian ring, and let I and J be ideals of R. Then denote $\mathcal{B}_{r,s} = (I^r \cap J^s) u^r v^s$. Note that, because $(I^{r'} \cap J^{s'}) \cdot (I^{r''} \cap J^{s''}) \subseteq I^{r'+r''} \cap J^{s+s''}$ we have that $\mathcal{B}_{r',s'} \cdot \mathcal{B}_{r'',s''} \subseteq \mathcal{B}_{r'+r'',s'+s''}$.

Denote $\mathcal{B}_n = \bigoplus_{r+s=n} \mathcal{B}_{r,s}$. With this notation, $\mathcal{B} = \bigoplus_{n \geq 0} \mathcal{B}_n$, which is $\mathbb{N}$-graded, because $\mathcal{B}_{n'} \cdot \mathcal{B}_{n''} \subseteq \mathcal{B}_{n'+n''}$. Then by (1.28), $\mathcal{B}$ is Noetherian if and only if $\mathcal{B}$ is a finitely generated R-algebra, as $\mathcal{B}_0 = R$ is Noetherian.

The purpose of this chapter is to prove the following theorem:

Theorem 4.2. *Let R be a UFD and I, J principal ideals in R. Then I, J have finite intersection algebra.*

Proof. By the above definition, I and J having finite intersection algebra is equivalent to $\mathcal{B}$ being finitely generated over R. With the notations from above, it is enough to show the following claim.

Claim: There exists an $N > 0$ such that for every $x \in \mathcal{B}_{r,s}$ there exists $y \in \mathcal{B}_{r',s'}$ and $z \in \mathcal{B}_{r'',s''}$, where $x = yz$, $r'' + r' = r$, $s'' + s' = s$ and $0 < r' + s' \leq N$.

First, we will show that the Claim implies that $\mathcal{B} = R[\mathcal{B}_{r,s}|r + s \leq N]$. In our case, $\mathcal{B}_{r,s}$ are R-free submodules of $\mathcal{B}$ of rank 1. So $\mathcal{B} = R[\mathcal{B}_{r,s}|r + s \leq N]$ implies that $\mathcal{B}$ is a finitely generated R-algebra, hence the Theorem. To show that the Claim implies $\mathcal{B} = R[\mathcal{B}_{r,s}|r + s \leq N]$, note first that $R[\mathcal{B}_{r,s}|r + s \leq N] \subseteq \mathcal{B}$, because $\mathcal{B}_{r',s'} \cdot \mathcal{B}_{r'',s''} \subseteq \mathcal{B}_{r'+r'',s'+s''}$. Denote $A = R[\mathcal{B}_{r,s}|r + s \leq N]$. For $\mathcal{B} \subseteq A$, it is enough to show that for every r, s, $\mathcal{B}_{r,s} \subseteq A$. We'll prove this by induction on $r + s$.

Let $x \in \mathcal{B}_{r,s}$. By the Claim, there exists $y \in \mathcal{B}_{r',s'}$ and $z \in \mathcal{B}_{r'',s''}$, where $x = yz$, $r'' + r' = r$, $s'' + s' = s$ and $0 < r' + s' \leq N$. Hence, $x = yz$, since $r'' + s'' < r + s$ by the induction assumption, it follows that $z \in A$. In conclusion, $x = yz \subseteq A$.

We will concentrate now on proving the Claim.

Let $a, b \in R$ such that $I = (a)$ and $J = (b)$. R is a UFD, so a and b can be uniquely decomposed into a product of prime elements. Thus, there exists $p_1, \ldots, p_n$ primes in R, $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_n \in \mathbb{N}$, not all zero, such that $a = p_1^{\alpha_1} \cdots p_n^{\alpha_n}$ and $b = p_1^{\beta_1} \cdots p_n^{\beta_n}$.

To illustrate our method of proving the Claim, we will treat first the cases $n = 1$ and $n = 2$, and then move on to the general case.

First, let us examine the case where I and J are generated by one prime element. So $I = (p^\alpha)$ and $J = (p^\beta)$, where where p is some prime and $\alpha, \beta \in \mathbb{N}$. Then $I^r \cap J^s = (p^\alpha)^r \cap (p^\beta)^s = (p^{\alpha r}) \cap (p^{\beta s}) = (p^{\max(\alpha r, \beta s)})$.

Examine a generic term from the algebra with its indexing dummy variables: $p^{\max(\alpha r, \beta s)} u^r v^s$. We need to find some N such that for every (r, s), there exists an

r', s' with $r' \leq r$, $s' \leq s$ and $r' + s' \leq N$ such that

$$\frac{p^{\max(\alpha r, \beta s)} u^r v^s}{p^{\max(\alpha r', \beta s')} u^{r'} v^{s'}} = p^{\max(\alpha(r-r'), \beta(s-s'))} u^{r-r'} v^{s-s'}.$$

This equation then simplifies to

$$\max(\alpha r, \beta s) - \max(\alpha r', \beta s') = \max(\alpha(r - r'), \beta(s - s')).$$

Let r^0, s^0 be such that $\alpha r^0 = \beta s^0 = [\alpha, \beta]$. Then, the Claim is satisfied for $r' = r^0$ and $s' = s^0$ as long as $r > r^0$ and $s > s^0$.

For the two prime case, let $I = (p^{\alpha_1} q^{\alpha_2})$, and $J = (p^{\beta_1} q^{\beta_2})$. We want to find N such that for every (r, s), there exists an r', s' with $r' \leq r$, $s' \leq s$ and $r' + s' \leq N$ such that

$$max((r - r')\alpha_i, (s - s')\beta_i) + max(r'\alpha_i, s'\beta_i) = max(r\alpha_i, s\beta_i)$$

for $i = 1, 2$. With an additional lemma, we can simplify these equations a bit more.

Lemma 4.3. For any $a, b, c, d \in \mathbb{N}$, $\max(a - b, c - d) + \max(b, d) = \max(a, c) \Leftrightarrow ((a - b) - (c - d))(b - d) \geq 0$.

Proof. First we show the forward implication. Let $b > d$. If $c - d > a - b$, then we have $c - d + b \leq \max(a, c)$. Our condition implies that $c - d + b > a$, so since $b > d$, $c > a$. But then the original equation can never hold. So if $b > d$, then $a - b > c - d$. A similar calculation shows the same results for $b < d$.

For the converse, assume that $((a-b)-(c-d))(b-d) \geq 0$. Then, either $a-b \geq c-d$ and $b \geq d$ or vice versa. Assume that this is the case. Then $a-b \geq c-d \Rightarrow a \geq c-d+b$, and since $d < b$, then $a \geq c$. Therefore, $\max(a - b, c - d) + \max(b, d) = \max(a, c)$

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□

So we can rewrite these equations as

$$((r - r')\alpha_i - (s - s')\beta_i)(r'\alpha_i - s'\beta_i) \geq 0 \text{ for all } i = 1, 2. \tag{4.1}$$

In this case, we will find two separate sets of (r', s') that will handle most of the r and s.

Let r_1^0 and s_1^0 such that $r_1^0\alpha_1 = s_1^0\beta_1 = [\alpha_1, \beta_1]$ and find r_2^0 and s_2^0 such that $r_2^0\alpha_2 = s_2^0\beta_2 = [\alpha_2, \beta_2]$. We will show the Claim for (r, s) as long as $r \geq r_i^0$ and $s \geq s_i^0$ up to a possible finite list of pairs.

Look at (4.1) with $r' = r_1^0$ and $s' = s_1^0$:

$$((r - r_1^0)\alpha_1 - (s - s_1^0)\beta_1)(r_1^0\alpha_1 - s_1^0\beta_1) \geq 0 \tag{4.2}$$

$$((r - r_1^0)\alpha_2 - (s - s_1^0)\beta_2)(r_1^0\alpha_2 - s_1^0\beta_2) \geq 0 \tag{4.3}$$

Note that the first equation will always hold, since $r_1^0\alpha_1 = s_1^0\beta_1$. So we look at the second one. If it holds as well, then this r', s' will work. If not, then $((r - r_1^0)\alpha_2 - (s - s_1^0)\beta_2)(r_1^0\alpha_2 - s_1^0\beta_2) < 0$. Then repeat this process with $r' = r_2^0$ and $s' = s_2^0$. This time, the second equation is automatically satisfied. If the first is as well, than this r' and s' will work, and if not, $((r - r_1^0)\alpha_2 - (s - s_1^0)\beta_2)(r_1^0\alpha_2 - s_1^0\beta_2) < 0$.

Since $r_i^0\alpha_i = s_i^0\alpha_i$, we can rearrange these two resulting equations to give

$$((r - r_2^0)\alpha_1 - (s - s_2^0)\beta_1)(\frac{\alpha_1}{\beta_1} - \frac{\alpha_2}{\beta_2}) < 0 \tag{4.4}$$

$$((r - r_1^0)\alpha_2 - (s - s_1^0)\beta_2)(\frac{\alpha_2}{\beta_2} - \frac{\alpha_1}{\beta_1}) < 0 \tag{4.5}$$

Order the $\frac{\alpha_i}{\beta_i}$, renumbering if necessary, so $\frac{\alpha_1}{\beta_1} \leq \frac{\alpha_2}{\beta_2}$. Then look at equation (4.4),

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the one with r_1^0 and s_1^0. Since this equation is strictly less than 0, $(\frac{\alpha_2}{\beta_2} - \frac{\alpha_1}{\beta_1})$ can't be

0, so it must be greater than 0. Thus $((r - r_1^0)\alpha_2 - (s - s_1^0)\beta_2) < 0$, which implies

that $r < \frac{1}{\alpha_2}((s - s_1^0)\beta_2) + r_1^0$.

Now repeat this with the other equation. Notice that now, $(\frac{\alpha_2}{\beta_2} - \frac{\alpha_1}{\beta_1})$ must be less

than 0. So now $r > \frac{1}{\alpha_1}((s - s_2^0)\beta_1) + r_2^0$.

Combining these two ranges, we get

$$\frac{1}{\alpha_1}((s - s_2^0)\beta_1 + r_2^0 \quad < \quad \frac{1}{\alpha_2}((s - s_1^0)\beta_2 + r_1^0 \tag{4.6}$$

$$\Rightarrow s(\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}) \quad < \quad r_1^0 - r_2^0 - \frac{\beta_2}{\alpha_2}s_1^0 + \frac{\beta_1}{\alpha_1}s_2^0. \tag{4.7}$$

The term on the left is positive by assumption. So the equations only fail when

$$s < \frac{r_1^0 - r_2^0 - \frac{\beta_2}{\alpha_2}s_1^0 + \frac{\beta_1}{\alpha_1}s_2^0}{\frac{\beta_1}{\alpha_1} - \frac{\beta_2}{\alpha_2}} = s_1^0 + s_2^0 \text{ and} \tag{4.8}$$

$$r < \frac{\beta_2}{\alpha_2}(s_1^0 + s_2^0 - s_1^0) + r_1^0 = r_1^0 + r_2^0. \tag{4.9}$$

This shows the Claim as long as $r \geq r_i^0$ and $s \geq s_i^0$, $i = 1, 2$.

Now to the n prime case. Let $a = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n}$ and $b = p_1^{\beta_1}p_2^{\beta_2}\cdots p_n^{\beta_n}$, where

p_i are prime in R and α_i, β_i are in $\mathbb{N}$. Then $I = (a)$ and $J = (b)$, and thus $I^r \cap$

$J^s = (p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_n^{\alpha_n})^r \cap (p_1^{\beta_1}p_2^{\beta_2}\cdots p_n^{\beta_n})^s = (p_1^{r\alpha_1}p_2^{r\alpha_2}\cdots p_n^{r\alpha_n}) \cap (p_1^{s\beta_1}p_2^{s\beta_2}\cdots p_n^{s\beta_n}) = $

$(p_1^{\max(\alpha_1 r, \beta_1 s)}\cdots p_n^{\max(\alpha_n r, \beta_n s)})$.

Let $x \in \mathcal{B}_{r,s}$. Then $x = c \cdot p_1^{\max(\alpha_1 r, \beta_1 s)}\cdots p_n^{\max(\alpha_n r, \beta_n s)}u^r v^s$, where $c \in R$. We

will find some N such that for all r, s, there exists an r', s' with $r' \leq r$, $s' \leq s$, and

$r' + s' \leq N$ such that

$$\frac{p_1^{\max(\alpha_1 r, \beta_1 s)} \cdots p_n^{\max(\alpha_n r, \beta_n s)} u^r v^s}{p_1^{\max(\alpha_1 r', \beta_1 s')} \cdots p_n^{\max(\alpha_n r', \beta_n s')} u^{r'} v^{s'}}$$
$$= p_1^{\max((r-r')\alpha_1, (s-s')\beta_1)} \cdots p_n^{\max((r-r')\alpha_n, (s-s')\beta_n)} u^{r-r'} v^{s-s'}.$$

If so, then by letting

$$y = p_1^{\max(\alpha_1 r', \beta_1 s')} \cdots p_n^{\max(\alpha_n r, \beta_n s))} u^{r'} v^{s'}$$

and

$$z = p_1^{\max((r-r')\alpha_1, (s-s')\beta_1)} \cdots p_n^{\max((r-r')\alpha_n, (s-s')\beta_n)} u^{r-r'} v^{s-s'},$$

the Claim is proven.

What is left to be proven simplifies to

$$\max(\alpha_i r, \beta_i s) - \max(\alpha_i r', \beta_i s') = \max(\alpha_i (r - r'), \beta_i (s - s')) \text{ for all } i = 1, \ldots, n$$

which, by Lemma (4.3), simplifies to the following:

$$((r - r')\alpha_i - (s - s')\beta_i)(r'\alpha_i - s'\beta_i) \geq 0 \text{ for all } i = 1, \ldots, n. \qquad (4.10)$$

We will produce $N > 0$ such that for all r, s there exists r', s' with $0 < r' + s' \leq N$ and $r \geq r'$, $s \geq s'$ such that (4.10) is satisfied. For clarity, we will label the ith equation of (4.10) by E_i.

Let r_i^0 and s_i^0 be such that $r_i^0 \alpha_i = s_i^0 \beta_i = [\alpha_i, \beta_i]$, and call $r_i^0 + s_i^0 = N_i$. We will show the Claim for all pairs (r, s) such that $r \geq r_i^0$, $s \geq s_i^0$ for all $i = 0, \ldots, n$. The other possibility is easier to deal with and be treated separately.

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For $r_i^0 = r'$ and $s_i^0 = s'$, the equation E_i is automatically satisfied. If, by some chance, all equations are satisfied for this choice of r' and s', then we are done, by simply letting $N = r_i^0 + s_i^0$.

If, however, one equation of 4.10 is not satisfied, then there exists some j_i such that $((r - r_i^0)\alpha_{j_i} - (s - s_i^0)\beta_{j_i})(r_i^0\alpha_{j_i} - s_i^0\beta_{j_i}) < 0$. Further, since $r_i^0\alpha_i = s_i^0\beta_i$, we can simplify the system once more to:

$$((r - r_i^0)\alpha_{j_i} - (s - s_i^0)\beta_{j_i})(\frac{\alpha_{j_i}}{\beta_{j_i}} - \frac{\alpha_i}{\beta_i}) < 0. \tag{4.11}$$

We will examine now this possibility:

For all i, there exists some j_i such that (4.11) happens.

Order the $\frac{\alpha_i}{\beta_i}$, renumbering if necessary, so that $\frac{\alpha_1}{\beta_1} \leq \frac{\alpha_2}{\beta_2} \leq \cdots \leq \frac{\alpha_n}{\beta_n}$. We can assume that all $\beta_i \neq 0$ in the system (4.10) because the equation E_i becomes $r \geq r'$ whenever $\beta_i = 0$, which is a constraint that we have to satisfy anyway.

Consider r_1^0, s_1^0. Hence, for some j_1, we have

$$((r - r_1^0)\alpha_{j_1} - (s - s_1^0\beta_{j_1}))(\frac{\alpha_{j_1}}{\beta_{j_1}} - \frac{\alpha_1}{\beta_1}) < 0. \tag{4.12}$$

Due to our renumbering, we know that $(\frac{\alpha_{j_1}}{\beta_{j_1}} - \frac{\alpha_1}{\beta_1})$ must be ≥ 0, and it cannot equal zero because of (4.12). Therefore

$$((r - r_1^0)\alpha_{j_1} - (s - s_1^0)\beta_{j_1}) < 0 \text{ or } r < \frac{1}{\alpha_{j_1}}((s - s_1^0)\beta_{j_1} + r_1^0. \tag{4.13}$$

Consider now $r_{j_1}^0, s_{j_1}^0$. There exists j_2 such that

$$((r - r_{j_1}^0)\alpha_{j_2} - (s - s_{j_1}^0)\beta_{j_2})(\frac{\alpha_{j_2}}{\beta_{j_2}} - \frac{\alpha_{j_1}}{\beta_{j_1}}) < 0. \tag{4.14}$$

If $(\frac{\alpha_{j_2}}{\beta_{j_2}} - \frac{\alpha_{j_1}}{\beta_{j_1}}) > 0$, then we get another upper bound on r:

$$r < \frac{1}{\alpha_{j_2}}((s - s_{j_1}^0)\beta_{j_2} + r_{j_1}^0.$$

If $(\frac{\alpha_{j_2}}{\beta_{j_2}} - \frac{\alpha_{j_1}}{\beta_{j_1}}) < 0$, then

$$r > \frac{1}{\alpha_{j_2}}((s - s_{j_1}^0)\beta_{j_2} + r_1^0.$$

If $(\frac{\alpha_{j_2}}{\beta_{j_2}} - \frac{\alpha_{j_1}}{\beta_{j_1}}) > 0$, we continue on with j_2 and j_3, until there is a k with $\frac{\alpha_{j_{k+1}}}{\beta_{j_{k+1}}} < \frac{\alpha_{j_k}}{\beta_{j_k}}$.

There will always be such a k, since our list of $\frac{\alpha_i}{\beta_i}$ is finite. Look at the equation relating these two terms, as well as the one previous to it, i.e. the one relating j_{k-1} to j_k.

$$\left(\frac{\alpha_{j_{k+1}}}{\beta_{j_{k+1}}} - \frac{\alpha_{j_k}}{\beta_{j_k}}\right)((r - r_k^0)\alpha_{j_{k+1}} - (s - s_k^0)\beta_{j_{k+1}}) < 0 \tag{4.15}$$

$$\left(\frac{\alpha_{j_k}}{\beta_{j_k}} - \frac{\alpha_{j_{k-1}}}{\beta_{j_{k-1}}}\right)((r - r_{k-1}^0)\alpha_{j_k} - (s - s_{k-1}^0)\beta_{j_k}) < 0 \tag{4.16}$$

Since $(\frac{\alpha_{j_{k+1}}}{\beta_{j_{k+1}}} - \frac{\alpha_{j_k}}{\beta_{j_k}}) < 0$, $((r - r_k^0)\alpha_{j_{k+1}} - (s - s_k^0)\beta_{j_{k+1}}) > 0$, and so

$$r < \frac{1}{\alpha_{j_{k+1}}}((s - s_{j_k}^0)\beta_{j_{k+1}}) + r_{j_k}^0.$$

Similarly, since $(\frac{\alpha_{j_k}}{\beta_{j_k}} - \frac{\alpha_{j_{k-1}}}{\beta_{j_{k-1}}}) < 0$, $((r - r_{k-1}^0)\alpha_{j_k} - (s - s_{k-1}^0)\beta_{j_k}) > 0$, and so

$$r > \frac{1}{\alpha_{j_k}}((s - s_{j_{k-1}}^0)\beta_{j_k}) + r_{j_{k-1}}^0.$$

Putting the two together as above, we get

$$s\left(\frac{\beta_{j_{k+1}}}{\alpha_{j_{k+1}}} - \frac{\beta_{j_k}}{\alpha_{j_k}}\right) < s_{j_k}^0\frac{\beta_{j_{k+1}}}{\alpha_{j_{k+1}}} - s_{j_{k-1}}^0\frac{\beta_{j_k}}{\alpha_{j_k}} + r_{j_{k-1}}^0 - r_{j_k}^0 \tag{4.17}$$

Since $(\frac{\beta_{j_{k+1}}}{\alpha_{j_{k+1}}} - \frac{\beta_{j_k}}{\alpha_{j_k}})$ is always positive, we always have an upper bound on s, which induces an upper bound on r as follows:

$$s < s_{j_k}^0 - s_{j_{k-1}}^0 \frac{\alpha_{j_{k+1}}(\alpha_{j_k}\beta_{j_{k-1}} - \alpha_{j_{k-1}}\beta_{j_k})}{\alpha_{j_{k-1}}(\alpha_{j_k}\beta_{j_{k+1}} - \alpha_{j_{k+1}}\beta_{j_k})} \tag{4.18}$$

$$r < r_{j_k}^0 - r_{j_{k-1}}^0 \frac{\beta_{j_{k+1}}(\alpha_{j_k}\beta_{j_{k-1}} - \alpha_{j_{k-1}}\beta_{j_k})}{\beta_{j_{k-1}}(\alpha_{j_k}\beta_{j_{k+1}} - \alpha_{j_{k+1}}\beta_{j_k})}. \tag{4.19}$$

Call $\mathcal{F}$ the set consisting of pairs r, s satisfying all possible equations of the form (4.18) and (4.19). Now let $N^0 = \max\{r + s | (r, s) \in \mathcal{F}\}$. If for all i, $r \geq r_i^0, s \geq s_i^0$, then our Claim follows for $N' = \max\{N^0, N_1^0, \ldots, N_n^0\}$. This is so because either there exists one pair (r_i^0, s_i^0) that works as (r', s') or $(r, s) \in \mathcal{F}$. It remains to deal with the case when there exists k such that for all i, $r < \max(r_i^0), s > \max(s_i^0)$ (the case $r > \max(r_i^0), s < \max(s_i^0)$ is similar).

Let $r' = 0, s' = 1$ in (4.10). Then there exists an i such that

$$(r\alpha_i - (s - 1)\beta_i)(-\beta_i) < 0 \text{ or } r > \frac{\beta_i}{\alpha_i}s - \frac{\beta_i}{\alpha_i}.$$

But $r < \max(r_i^0)$ implies that $s < \frac{\alpha_i}{\beta_i}\max(r_i^0) + 1$. Let $\mathcal{G}'$ be the set of all such pairs (r, s). $\mathcal{G}'$ is finite. Similarly the case where for all i $s < \max(s_i^0), r > \max(r_i^0)$ gives a finite set $\mathcal{G}''$ of possible pairs (r, s). Let $\mathcal{G} = \mathcal{G}' \cup \mathcal{G}''$, and $N'' = \max\{r + s | (r, s) \in \mathcal{G}\}$. If (r, s) are such that there exists a k with $r < r_k^0$ and $s > s_0^k$ or $r > r_k^0$ and $s < s_0^k$, then either $(0, 1)$ or $(1, 0)$ work as choices for (r', s') or $(r, s) \in \mathcal{G}$. In conclusion, we can let $N = \max\{N', N\}$, and the Claim follows.

$\square$

Corollary 4.4. Let R be a PID with I, J ideals in R. Then I and J have finite intersection algebra.

Proof. Since R is a PID, I and J are principal ideals, and R is a UFD. So, by the above theorem, I and J have finite intersection algebra. $\qquad\square$

Now let $f = \{I_n\} = \bigoplus_{m \geq n} \mathcal{B}_m$. This is a filtration on $\mathcal{B}$, first because clearly $I_{n+1} \subseteq I_n$. For the other part of the definition, let $x \in I_k$ and $y \in I_l$. Then $xy \subseteq I_{k+l}$ because $\mathcal{B}_{r',s'} \cdot \mathcal{B}_{r'',s''} \subseteq \mathcal{B}_{r'+r'',s'+s''}$.

Then compute $\mathrm{gr}_f(\mathcal{B})$.

$$\mathrm{gr}_f(\mathcal{B}) = \bigoplus_{n \geq 0} \frac{I_n}{I_{n+1}} = \bigoplus_{n \geq 0} \mathcal{B}_n = \mathcal{B}.$$

We proved above that $\mathcal{B}$ is Noetherian. Thus, $\mathrm{gr}_f(\mathcal{B})$ is Noetherian, and by definition f is Noetherian as well. Further, it can easily be shown that f is an e.p.f., since our Claim (see the proof of the above theorem) shows that there exists an $N > 0$ such that $I_n = \sum_{i=1}^{N} I_{n-i} I_i$ for every $n > 1$.

If I or J are not principal, I and J do not necessarily have finite intersection algebra. This was shown by Fields in [7] as follows.

Example 4.5. Let P and R as in Theorem (3.6) such that the algebra $R \oplus P^{(1)} \oplus P^{(2)} \oplus \cdots$ is not finitely generated. Fields has shown that there exists an $f \in R$ such that $(P^a : f^a) = P^{(a)}$ for all a. Then Fields shows in Lemma 5.6 in [7] that the algebra $\bigoplus_{n \geq 0} P^{(n)}$ is a homomorphic image of the intersection algebra between (f) and P. Since $\bigoplus_{n \geq 0} P^{(n)}$ is not Noetherian, (f) and P do not have finite intersection algebra.

Although this shows that in general the intersection algebra of I, J in R is not finite, we note that there are other known classes of ideals I and J that have finite intersection algebra. Fields has shown in [6] that if $R = k[[x_1, \ldots, x_n]]$ and I and J are monomial ideals in R, then I and J have finite intersection algebra. This result

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is a consequence of results from the theory of integer linear programming. Fields'
methods can be used to provide a proof of our theorem, 4.2. We have given a different
and original proof that also provides information on the degrees of the generators for
the finite intersection algebra. Fields' thesis explains how intersection algebras can
be applied to the asymptotic theory of ideals. The following result illustrates more
applications of intersection algebras.

Definition 4.6. Let R be a Noetherian ring, and I, J ideals in R with $J \subseteq \sqrt{I}$. Also
assume that I is not nilpotent and $\bigcap_k I^k = (0)$. Then for each positive integer m,
define $v_I(J, m)$ to be the largest n such that $J^m \subseteq I^n$. Also, we can examine the
sequence $\{v_I(J, m)\}_m$, which here we will abbreviate to $v(m)$.

The following Proposition appears as (3.2) in [3].

Proposition 4.7. Let I, J be ideals in a Noetherian local ring R such that $J \subseteq \sqrt{I}$,
the ideals I, J are not nilpotent, and $\bigcap_k I^k = (0)$. Assume that J is principal and
the ring $\mathcal{B} = \bigoplus_{m,n} J^m \cap I^n$ is Noetherian. Then there exists a positive integer t such
that $v(m + t) = v(m) + v(t)$ for all $m \geq t$.

This shows why it is of interest to establish that I and J have finite intersection
algebra.

Proposition 4.8. Let R be a UFD and I and J nonzero principal ideals in R such
that $J \subseteq \sqrt{I}$. Then there exists a positive integer t such that $v(m+t) = v(m)+v(t)$.

Proof. Theorem (4.2) and Proposition (4.7) combined imply the result. However, we
will give a direct proof without relying on these results. Let $J = (a) = (p_1^{\alpha_1} \cdots p_h^{\alpha_h})$
and $I = (b) = (p_1^{\beta_1} \cdots p_h^{\beta_h})$. Then $v(m)$ is the largest n such that $(a^m) \subseteq (b^n)$, which
is equivalent to being the largest n such that $n \cdot \beta_i \leq m \cdot \alpha_i$. Thus $n = \lfloor \min(\frac{m\alpha_i}{\beta_i} | i =$

$1, \ldots, h)\rfloor = v(m)$. Let t be the minimum number such that $\frac{t\alpha_i}{\beta_i} \in \mathbb{N}$ for all i. Then for all $m \geq t$,

$$
\begin{aligned}
&\lfloor \min(\frac{m\alpha_i}{\beta_i})|i = 1, \cdots, h)\rfloor \\
&\quad + \lfloor \min(\frac{t\alpha_i}{\beta_i})|i = 1, \ldots, n)\rfloor \\
&= \lfloor \min(\frac{(m+t)\alpha_i}{\beta_i})|i = 1, \ldots, h)\rfloor
\end{aligned}
$$

or $v(m) + v(t) = v(m + t)$. $\qquad\square$

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