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Primary Decomposition in Non Finitely Generated Modules

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PRIMARY DECOMPOSITION

IN NON-FINITELY GENERATED MODULES

by

SOMAYA MUINY

Under the Direction of Dr.Yongwei Yao

ABSTRACT

In this paper, we study primary decomposition of any proper submodule N of a module M over a Noetherian ring R. We start by briefly discussing basic facts about the well known case where M is a finitely generated module over a Noetherian ring R, then we proceed to discuss the general case where M is any module over a Noetherian ring R. We put a lot of emphasis on the associated primes that occur with the primary decomposition, essentially studying their uniqueness and their relation to the associated primes of M/N.

INDEX WORDS: Noetherian modules, Primary submodules, Associated primes, Annihilator, General primary decomposition

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This thesis is dedicated to my parents,

and to my husband.

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Chapter 1

INTRODUCTION

1.1 Elementary Facts

In this chapter we give some basic definitions and some well known elementary facts from commutative algebra. In this paper all rings are assumed to be commutative with unity.

Definition 1.1.1. [2] A commutative ring R is called *Noetherian* if for every ascending sequence of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots$ there exists a positive integer k such that $I_k = I_{k+1} = I_{k+2} = \cdots$ or $I_k = I_{k+i}$ for every $i \geq 0$. In other words, R is *Noetherian* if and only if every ascending chain of ideals of R is eventually stationary.

Definition 1.1.2. An R-module M is called *Noetherian* if for every ascending sequence of R-submodules $M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$ there exists a positive integer k such that $M_k = M_{k+1} = M_{k+2} = \ldots$ or $M_k = M_{k+i}$ for every $i \geq 0$. In other words, M is *Noetherian* if and only if every ascending chain of submodules is eventually stationary.

Theorem 1.1.3. *For an R-module M, the following are equivalent:*

1. *M is* Noetherian*.*

2. *Every submodule of M is finitely generated.*

3. *Every non-empty set of submodules of M has a maximal element.*

Definition 1.1.4. [2] Let R be a commutative ring, and let N be a nonzero submodule of M over$R$. Then the ideal quotient $(0 : N) = \{r \in R \,|\, rN = 0\} = \{r \in R \,|\, rn = 0 \text{ for all } n \in N\}$ is called the *annihilator* of N and is denoted by $Ann(N)$. Also for $m \in M$, the *annihiliator of* m is the quotient ideal

$$Ann(m) = (0 : m) = \{r \in R \,|\, rm = 0\}\,.$$

Definition 1.1.5. [4] Let R be a ring and M an R-module. A prime ideal P of R is called an *associated prime* ideal of M if $P = Ann(m)$ for some $m \in M$. The set of associated primes of M is denoted by $Ass(M)$.

The above definition can be paraphrased as: $P \in Ass(M)$, iff there is an injection $R/P \hookrightarrow M$. This is because the submodule of M generated by m is isomorphic with R/P iff the annihilator of m is P. Also note that the element m with a prime annihilator can never be 0, since the annihilator of 0 is the unit ideal.

Definition 1.1.6. If $Ass(M/N) = \{P\}$ for any submodule N of an R-module M, we say that N is a *P-primary* submodule.

Theorem 1.1.7 ([4])**.** *A submodule N of an R-module M is P-primary if and only if $P = \sqrt{Ann(M/N)}$ is a prime ideal and elements of $R - P$ are not zero divisors on M/N.*

In other words, the above theorem states N is P-primary if and only if for any $a \in R$ and $m \in M$, with $am \in N \Rightarrow a \in \sqrt{Ann(M/N)} = P$ or $m \in N$.

In the special case where $M = R$ and $N = I$ i.e I is an ideal of R, I is P-primary $\Leftrightarrow \forall\, a,\ b \in R$ with $ab \in I$ then $a \in \sqrt{I} = P$ or $b \in I$.

Remark 1.1.8. For any submodule N of an R-module M, where R is Noetherian, the union of the associated primes is the set of elements of R that are zero divisors of M/N.

Theorem 1.1.9 ([4])**.** *Let R be a Noetherian ring, and let M be an R-module, then $M \neq 0 \Leftrightarrow Ass(M) \neq \emptyset$.*

Proof. $\Leftarrow$ Assume $Ass(M) \neq \emptyset$, say $P \in Ass(M)$. This means that $\exists\ u \in M$ such that $P = Ann(u)$, forcing u to be a nonzero element of M, for if $u = 0$, then $Ann(0) = R$. Hence $M \neq 0$.

$\Rightarrow$ Assume $M \neq 0$, let $\Omega = \{Ann(x) \,|\, 0 \neq x \in M\}$. Each element of Ω is an ideal of R and therefore Ω has a maximal element as R Noetherian, say $P = Ann(u)$, where $0 \neq u \in M$. To show P is prime, for a contradiction, assume it is not. That is $\exists\, a, b \in R$ where $ab \in P$, $a \notin P$, and $b \notin P$ implying that $au \neq 0$. Consider $Ann(au)$, we know that $b \in Ann(au)$ and therfore $P = Ann(u) \subsetneq Ann(au)$ contradicting the fact that P is a maximal in Ω. And hence P is prime and the $Ass(M) \neq \emptyset$. $\square$

Lemma 1.1.10 ([4])**.** *For any proper submodule $N \subset M$ we have the exact sequence*

$$0 \longrightarrow N \longrightarrow M \longrightarrow M/N \longrightarrow 0$$

and it follows that

$$Ass(N) \subseteq Ass(M) \subseteq Ass(N) \bigcup Ass(M/N).$$

Proof. It is clear that for any $P \in Ass(N)$ we have $P \in Ass(M)$, and therefore $Ass(N) \subseteq Ass(M)$.

To prove that $Ass(M) \subseteq Ass(N) \cup Ass(M/N)$, let P be one of the associated prime of M, then P is the annihilator of some element in M, say u, i.e. $P = Ann(u)$. Consider the homomorphism $f : R \longrightarrow Ru$, this homomorphism is onto with kernel P, therfore, by the 1^{st} Homomorphism Theorem we have $Ru \cong R/P$, and consequently we have $Ass(Ru) = Ass(R/P) = \{P\}$. Now, observing the submodule $Ru \cap N$, we have 2 cases to consider

Case 1: If $Ru \cap N = \{0\}$, then by the 2^{nd} homomorphism we have

$$M/N \supseteq (Ru + N)/N \cong Ru/(Ru \cap N) \cong Ru$$

taking the associates of everything in the above expression we get

$$Ass(M/N) \supseteq Ass\left((Ru + N)/N\right) = Ass\left(Ru\right) = \left\{P\right\},$$

hence $P \in Ass(M/N)$.

Case 2: If $Ru \cap N \neq \{0\}$, then $Ass(Ru \cap N) \subseteq Ass(Ru) = \{P\}$. The fact that $Ru \cap P \neq 0$ means that $Ass(Ru \cap P) \neq \emptyset$, by Theorem 1.1.9, hence $Ass(Ru \cap P) = \{P\}$. As $Ass(Ru \cap N) \subseteq Ass(N)$, we conclude that $P \in Ass(N)$.

Putting both cases together, we get

$$Ass(M) \subseteq Ass(N) \bigcup Ass(M/N). \qquad \square$$

Lemma 1.1.11. *If $M = \bigoplus_{i \in \Lambda} M_i$, then $Ass(M) = Ass(\oplus_{i \in \Lambda} M_i) = \bigcup_{i \in \Lambda} Ass(M_i)$.*

Proof. Let $P \in \bigcup_{i \in \Lambda} Ass(M_i)$, then $P \in Ass(M_k)$ for a specific $k \in \Lambda$. As $M_k \subseteq \bigoplus_{i \in \Lambda} M_i$ up to isomorphism, we have $P \in Ass(M_k) \subseteq Ass(\bigoplus_{i \in \Lambda} M_i)$. Hence $\bigcup_{i \in \Lambda} Ass(M_i) \subseteq$

$Ass(\bigoplus_{i\in\Lambda} M_i)$.

For the other direction, we first prove the case when $\Lambda = \{1, 2, \ldots, n\}$ is finite. There exists a short exact sequence

$$0 \longrightarrow M_1 \longrightarrow M_1 \bigoplus M_2 \longrightarrow M_2 \longrightarrow 0.$$

By Theorem 1.1.10, we have

$$Ass(M_1 \bigoplus M_2) \subseteq Ass(M_1) \bigcup Ass(M_2).$$

Proceding by induction, we get

$$Ass(\bigoplus_{i\in\Lambda} M_i) \subseteq \bigcup_{i\in\Lambda} Ass(M_i), \text{ where } \Lambda \text{ is finite.}$$

Now, for the general case when Λ is infinite. Let $P \in Ass(\bigoplus_{i\in\Lambda} M_i)$, this means that $P = Ann(u)$ for some $u \in \bigoplus_{i\in\Lambda} M_i$ and hence $u \in \bigoplus_{j=1}^{n} M_{i_j}$ for some $\{i_1, i_2, \ldots, i_n\} \subseteq \Lambda$ implying that $P \in Ass(\bigoplus_{j=1}^{n} M_{i_j}) = \bigcup_{j=1}^{n} Ass(M_{i_j}) \subseteq \bigcup_{i\in\Lambda} Ass(M_i)$.

Therefore, $Ass(\bigoplus_{i\in\Lambda} M_i) \subseteq \bigcup_{i\in\Lambda} Ass(M_i)$. Hence $Ass(\bigoplus_{i\in\Lambda} M_i) = \bigcup_{i\in\Lambda} Ass(M_i)$. $\square$

Theorem 1.1.12 ([3])**.** *Let R be a Noetherian ring and let M be a non-zero finitely generated module over R. Then there exists a chain of submodules*

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \cdots \subseteq M_{n-1} \subseteq M_n = M$$

such that $Mi/M_{i-1} \cong R/P_i$ for some $P_i \in Spec(R)$, $i = 1, \ldots, n$.

Lemma 1.1.13. *If M is a finitely generated module over a Noetherian ring R, then $|Ass(M)|$ is finite.*

Proof. If $M = 0$, then $|Ass(M)| = 0$. Now assume $M \neq 0$. By the above thorem there exists a chain

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \subseteq M_n = M,$$

where $Mi/M_{i-1} \cong R/P_i$ for some $P_i \in Spec(R)$, $i = 1, \ldots, n$. Note that $Ass(M_i/M_{i-1}) = Ass(R/P_i) = \{P_i\}$, for all $i = 1, \ldots, n$. And by Lemma 1.1.10, we have $Ass(M_2) \subseteq Ass(M_1) \bigcup Ass(M_2/M_1) = \{P_1, P_2\}$, Similarly, $Ass(M_3) \subseteq Ass(M_2) \bigcup Ass(M_3/M_2) = Ass(M_2) \bigcup \{P_3\} = \{P_1, P_2, P_3\}$. Working inductively we can show that $Ass(M) \subseteq \{P_1, P_2, \ldots, P_n\}$. Hence M has a finite number of associated primes. □

Remark 1.1.14. If M is not finitely generated, then $|Ass(M)|$ could be infinite as in Example 1.1.15.

Example 1.1.15. Consider $M = \bigoplus_{p\,\text{prime}} \mathbb{Z}_p$, i.e, $M = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \cdots$ over the ring $R = \mathbb{Z}$. Every non-zero prime ideal of $\mathbb{Z}$ is actually in $Ass(M)$, since $Ass(M) = Ass(\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \ldots) = Ass(\mathbb{Z}_2) \cup Ass(\mathbb{Z}_3) \cup Ass(\mathbb{Z}_5) \cup \cdots = \{(2), (3), (5), \ldots\} = \{(p) \,|\, p \text{ is prime in } \mathbb{Z}\}$ and therefore $|Ass(M)| = \infty$.

1.2 Primary Decomposition of Finitely Generated Modules

Definition 1.2.1. Let M be a finitely generated R-module, and let $N \subseteq M$ be submodule of M. If there exists primary submodules $Q_1, Q_2, Q_3, \ldots Q_r$ of M such that

$$N = Q_1 \cap Q_2 \cap Q_3 \cdots \cap Q_r,$$

then we say that N admits a *primary decomposition* in M.

Definition 1.2.2. Given a primary decomposition of a proper submodule $N \subset M$ of an R-module M, we say that this decomposition is *irredundant* if the following hold

1. The collection $\{P_i \,|\, i = 1, \dots n\}$ of associated primes to which $\{Q_i \,|\, i = 1, \dots n\}$ are primary to, are mutually distinct.

2. No Q_i can be removed from the intersection without changing the intersection.

Theorem 1.2.3 ([2])**.** *Let M be a finitely generated module over a Noetherian ring R, and N a proper submodule of M. Then N admits a primary decomposition in M, and this decomposition contains a finite number of primary components.*

Proof. This theorem has many classical proofs, see [2] and [3]. However in this paper we shall benefit from Remark 2.2.5 which is based on Theorem 2.2.1. □

Theorem 1.2.4 ([2])**.** *Let N be a submodule of a finitely generated R-module M over a Noetherian ring R, and let*

$$N = Q_1 \cap Q_2 \cap \cdots \cap Q_r \text{ where } Q_i \text{ is } P_i\text{-primary for } i = 1, \dots, r$$

and

$$N = Q'_1 \cap Q'2 \cap \cdots \cap Q_{r'} \text{ where } Q'_i \text{ is } P'_i\text{-primary for } i = 1, \dots, r'$$

be two irredundant primary decomposition of N in M. Then $r = r'$ and we have

$$\{P_1, P_2, \dots, P_r\} = Ass(M/N) = \{P'_1, P'_2, \dots, P'_r\}$$

In other words, the number of terms and the collection of primes appearing in an irredundant primary decomposition of N is independent of the choice of that decomposition.

Proof. By Remark 3.1.9 we have $\{P_i\}_{i=1}^{r} = Ass(M/N)$ is valid for the first primary decoposition and $\{P_i'\}_{i=1}^{r'} = Ass(M/N)$ is also valid for the second primary decomposition. Hence $r = r'$ and $\{P_1, P_2, \ldots, P_r\} = Ass(M/N) = \{P_1', P_2, \ldots, P_r'\}$ as claimed. □

Theorem 1.2.5 ([2])**.** *Let N be a submodule of a finitely generated R-module M over a Noetherian ring R, and let*

$$N = Q_1 \cap Q_2 \cap \cdots \cap Q_r \text{ where } Q_i \text{ is } P_i\text{-primary for } i = 1, \ldots, r$$

and

$$N = Q_1' \cap Q_2' \cap \cdots \cap Q_r' \text{ where } Q_i' \text{ is } P_i\text{-primary for } i = 1, \ldots, r$$

be two irredundant primary decomposition of N in M. Suppose that P_j is a minimal member of $\{P_1, P_2, \ldots, P_r\}$ with respect to inclusion. Then $Q_j = Q_j'$.

The above theorem means that if P is one of the minimal primes, then the primary submodule to P that occurs in the primary decomposition is uniquely determined by N and is independent of the choice of irredundant primary decomposition.

Proof. The following proof is based on Sharp's treatment of the case of ideals.

If $r = 1$, then there is nothing to prove, and we therefore suppose that $r > 1$. Let P_j be a minimal prime in the collection $\{P_1, P_2, P_3, \ldots, P_r\}$, and let $a_i \in P_i \setminus P_j$ for all $i = 1, \ldots, r$

and $i \neq j$. Now, let $a = \prod_{\substack{1 \leq i \leq r \\ i \neq j}} a_i$, this a has the property that

$$a \in \prod_{\substack{1 \leq i \leq r \\ i \neq j}} P_i \setminus P_j \subseteq \bigcap_{\substack{1 \leq i \leq r \\ i \neq j}} P_i \setminus P_j.$$

Hence $a \notin P_j$, but $a \in P_i = \sqrt{Ann(M/Q_i)}$ for all $i = 1, \ldots, r$ and $i \neq j$, and it follows that there exist $h_i \in \mathbb{N}$ such that $a^{h_i} \in Ann(M/Q_i)$. Let

$$t \geq max \left\{h_1, h_2, \ldots, h_{j-1}, h_{j+1}, \ldots, h_n\right\}.$$

Then $a^t \in Ann(M/Q_i)$ for all $i = 1, \ldots, r$ and $i \neq j$, and it follows that

$$(N : a^t) = \left(\bigcap_{i=1}^{r} Q_i : a^t\right) = \bigcap_{i=1}^{r} (Q_i : a^t)$$

Claim: The equality $(Q_i : a^t) = M$ holds for all $i = 1, \ldots, r$ and $i \neq j$. *Proof of claim.* We have shown above that $a^t \in Ann(M/Q_i)$ which means that $a^t M \subseteq Q_i$. As $(Q_i : a^t) = \{m \in M | a^t m \in Q_i\}$, we conclude that $(Q_i : a^t) = M$.

Claim: For j, $(Q_j : a^t) = Q_j$. *Proof of claim.* Since $a^t \notin P_j = Ass(M/Q_j)$ means that a^t is not a zerodivisor of M/Q_j, we know that there exists no non zero element $\bar{m} \in M/Q_j$ with $a^t \bar{m} = \bar{Q_j} = 0_{M/Q_j}$. Now, let $m \in (Q_j : a^t)$, this implies that $a^t m \in Q_j$ and hence $a^t \bar{m} = 0_{M/Q_j}$. As a^t is not a zero divisor of M/Q_j, it is clear that $\bar{m} = 0_{M/Q_j}$ and that $m \in Q_j$, therefore $(Q_j : a^t) \subseteq Q_j$. The other direction $Q_j \subseteq (a^t : Q_j)$ is trivial.

Therefore, by the above argument, we have

$$(N : a^t) = \bigcap_{i=1}^{r} (Q_i : a^t) = M \cap M \cap \ldots \cap Q_j \cap \ldots M = Q_j$$

After we have shown that $(N : a^t) = Q_j$ whenever t is sufficiently large, in the same manner, we can prove $Q'_j = (N : a^t)$ when t is sufficiently large. Hence $Q_j = Q'_j$ as claimed. $\square$

Chapter 2

PRIMARY DECOMPOSITION IN NON-FINITELY GENERATED MODULES

2.1 A Very Important Lemma

Lemma 2.1.1. *Let R be Noetherian, $M \neq 0$ an R-module and let $P \in Ass(M)$. Then there exists a submodule $M(P)$ such that*

1. $Ass(M/M(P)) = \{P\}$, *and*

2. $Ass(M(P)) = Ass(M) - \{P\}$.

Proof. Consider $\Omega = \{N \subseteq M \mid Ass(N) \subseteq Ass(M) - \{P\}\}$, which is Ω non-empty since $\{0\} \in \Omega$. Let $N_1 \subseteq N_2 \subseteq \cdots$ be a chain of submodules in Ω and consider $\bigcup_i N_i$. We claim that $Ass(\bigcup_i N_i) = \bigcup_i Ass(N_i)$. Indeed, as $N_i \subseteq \bigcup_i N_i$, we have $Ass(N_i) \subseteq Ass(\bigcup_i N_i)$ and therefore $\bigcup_i Ass(N_i) \subseteq Ass(\bigcup_i N_i)$. To show the other direction, let $P \in Ass(\bigcup_i N_i)$, this implies that P is the annihilator of some element $u \in \bigcup_i N_i$, i.e $u \in N_k$ for a specific k implying that $P \in Ass(N_k)$ and hence, $P \in \bigcup_i Ass(N_i)$. Therefore

$$Ass\left(\bigcup_i N_i\right) = \bigcup_i Ass(N_i) \subseteq Ass(M) - \{P\}$$

Now, as Ω contains all submodules N with the property that $Ass(N) \subseteq Ass(M) - \{P\}$

means that

$$\bigcup_i N_i \in \Omega.$$

Therefore, every chain in Ω has an upper bound in it and hence we can apply Zorn's lemma and say that there exists a maximal element in Ω, call it $M(P)$. This means that $Ass(M(P)) \subseteq Ass(M) - \{P\}$ and there is no proper submodule A in Ω such that $M(P) \subsetneq A$.

Now, to prove that $Ass(M/M(P)) = \{P\}$, we are going to use Theorem 1.1.9. Note that $Ass(M/M(P)) \neq \emptyset$, because if $Ass(M/M(P)) = \emptyset$, then it follows that the quotient submodule itself is zero, i.e, $M/M(P) = 0$, which means that $M(P) = M$ and $Ass(M(P)) = Ass(M) \ni P$ which contradics that $M(P)$ is an element of Ω. So $M(P) \subsetneq M$, and hence we can apply lemma 1.1.10 to get $Ass(M) \subseteq Ass(M(P)) \cup Ass(M/M(P))$. Since $P \in Ass(M)$, we have $P \in Ass(M(P)) \cup Ass(M/M(P))$, the fact that $M(P) \in \Omega$ means that $P \notin Ass(M(P))$ which forces $P \in Ass(M/M(P))$, and hence $\{P\} \subseteq Ass(M/M(P))$. Now, to show that $\{P\} = Ass(M/M(P))$ assume for a contradiction that there exists another prime ideal $P' \neq P$ such that $P' \in Ass(M/M(P))$. This implies that there exists $x + M(P) \in M/M(P)$ such that $P' = Ann(x + M(P))$ where $x \notin M(P)$. Now consider the submodule $Rx + M(P)$ of M, realizing that $M(P) \subsetneq Rx + M(P) \subseteq M$ we can define a homomorphism f as follows:

$$f : R \longrightarrow (Rx + M(P))/M(P)$$

by:

$$f(r) = rx + M(P).$$

This homomorphism f is onto with kernel P'. Therefore by the 1^{st} homomorphism theorem we have

$$(Rx + M(P))/M(P) \cong R/P'$$

and it follows that

$$Ass(Rx + M(P)/M(P)) = Ass(R/P') = \{P'\}\,.$$

Now, according to Lemma 1.1.10, we have

$$Ass(Rx + M(P)) \subseteq Ass(M(P)) \bigcup Ass(Rx + M(P)/M(P)) = Ass(M(P)) \bigcup \{P'\}\,,$$

where $P \notin Ass(M(P)) \cup \{P'\}$. Hence, $(Rx+M(P)) \in \Omega$ and it is strictly larger than $M(P)$, which contradicts that $M(P)$ is maximal in Ω. And therefore $Ass(M/M(P)) \subseteq \{P\}$. This fact along with the result shown above $\{P\} \subseteq Ass(M/M(P))$ forces that

$$Ass(M/M(P)) = \{P\}\,,$$

proving first part of the theorem.

Now, knowing that $Ass(M) \subseteq Ass(M(P)) \bigcup Ass(M/M(P)) = Ass(M(P)) \bigcup \{P\}$, we can use set theory to conclude that $Ass(M(P)) \supseteq Ass(M) - \{P\}$ and hence

$$Ass(M(P)) = Ass(M) - \{P\}$$

which proves the second part of the theorem. □

2.2 Existence of Primary Decomposition

Theorem 2.2.1. *Let R be a Noetherian ring and $M \neq 0$ a general module over R, then every proper submodule N of M can be expressed as the intersection of primary submodules, i.e $N = \bigcap_{\lambda\in\Lambda} Q_\lambda$ where each Q_λ is P_λ-primary in M.*

Proof. As a first step, we are going to study the primary decomposition of $0_M \subset M$. By the previous lemma for each $P \in Ass(M)$, there exists $M(P)$ such that

1. $Ass(M/M(P)) = \{P\}$, i.e., $M(P)$ is P-primary in M, and
2. $Ass(M(P)) = Ass(M) - \{P\}$.

We claim that $0 = \bigcap_{P\in Ass(M)} M(P)$. Indeed, we notice that

$$Ass\left(\cap_{P\in Ass(M)} M(P)\right) \subseteq Ass(M(P)) = Ass(M) - \{P\}, \forall P \in Ass(M),$$

which implies that

$$Ass\left(\cap_{P\in Ass(M)} M(P)\right) \subseteq \cap_{P\in Ass(M)} Ass(M(P)) = \cap_{P\in Ass(M)} \{Ass(M) - \{P\}\} = \emptyset.$$

Hence $Ass(\cap_{P\in Ass(M)} M(P)) = \emptyset$, and therefore by Theorem 1.1.9 we conclude that

$$\bigcap_{P\in Ass(M)} M(P) = 0,$$

i.e, the zero submodule admits a primary decomposition in M.

Now, we are going to prove the theorem in general for any proper submodule $N \subseteq M$

over R. By the obove, we know that

$$0_{M/N} = \bigcap_{P \in Ass(M/N)} \frac{M}{N}(P)$$

where $\frac{M}{N}(P)$ is a primary submodule of M/N. Rewriting this submodule$\frac{M}{N}(P)$ as $B(P)/N$ for some submodule $B(P)$ of M where $N \subseteq B(P) \subsetneq M$, we can write the following using the 3^{rd} Homomorphism Theorem

$$M/B(P) \cong \frac{M/N}{B(P)/N} \cong \frac{M/N}{\frac{M}{N}(P)}$$

Taking the associates of every thing in the above expression, we get

$$Ass(M/B(P)) = Ass\left(\frac{M/N}{\frac{M}{N}(P)}\right) = \{P\}$$

as $\frac{M}{N}(P)$ is P-primary in M/N. Hence $B(P)$ is also P-primary in M, and by the Correspondence Theorem, we can express N as

$$N = \bigcap_{P \in Ass(M/N)} B(P).$$

In other words, N can be expressed as the intersection of primary submodules. $\square$

Remark 2.2.2. In this paper, we will call any primary decomposition with (possibly infinitely many components) a *General Primary Decomposition*.

Remark 2.2.3. If R is not Noetherian, then the primary decomposition of any proper submodule $N \subsetneq M$ over R is not guaranteed as in the following example.

Example 2.2.4. [2] Let $R = C[0,1]$ the ring of all real-valued continous functions on the inteval [0,1]. This ring is not Noetherian and the zero ideal in this ring does not have a primary decomposition.

Remark 2.2.5. The proof of the above theorem shows that every proper submodule $N \subset M$ over a Noetherian ring R admits a primary decomposition, and the primary submodules that appear in the intersection are in one-to-one correspondence with $Ass(M/N)$. Therefore, if $|Ass(M/N)| < \infty$ (which happens when M is finitely generated or more generally when M/N is finitely generated), then N can be expressed as the intersection of a finite number of primary submodules. This fact proves the existence of a finite primary decomposition of any proper submodule $N \subsetneq M$ over a Noetherian ring R.

2.3 Examples of General Primary Decomposition

In this section, we demonstrate two examples of the general irredundant primary decomposition.

Example 2.3.1. Let M be as described in example 1.1.15 where $M = \bigoplus_{p\,\text{prime}} \mathbb{Z}_p$ and consider the primary submodules of M:

$$Q_2 = 0_{\mathbb{Z}_2} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7 \oplus \cdots$$

$$Q_3 = \mathbb{Z}_2 \oplus 0_{\mathbb{Z}_3} \oplus \mathbb{Z}_5 \oplus \mathbb{Z}_7 \oplus \cdots$$

$$Q_5 = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus 0_{\mathbb{Z}_5} \oplus \mathbb{Z}_7 \oplus \cdots$$

Generally, for any prime $p \in \mathbb{Z}$, we can define $Q_p = \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \ldots \oplus 0_{\mathbb{Z}_p} \oplus \ldots$. Note that

$$M/Q_2 = (\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \ldots)/(0_{\mathbb{Z}_2} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_5 \oplus \ldots) \cong \mathbb{Z}_2$$

and this means that $Ass(M/Q_2) = Ass(\mathbb{Z}_2) = \{(2)\}$, in other words, Q_2 is (2)-primary. In general, $Ass(M/Q_P) = Ass(\mathbb{Z}_p) = \{(p)\}$ and therefore Q_p is (p)-primary.

It is clear that if we take the intersection of all these primary submodules, we will obtain:

$$0_M = \bigcap_{p \,\text{prime in}\, \mathbb{Z}} Q_p,$$

which is a primary decomposition of 0_M in M. Moreover, to show that this decomposition is irredundant we will remove one primary component from the intersection, say Q_2, by removing it we get $\cap_{p\neq 2} Q_p = \mathbb{Z}_2 \oplus 0_{\mathbb{Z}_3} \oplus 0_{\mathbb{Z}_5} \oplus \ldots \neq 0_M$. In general, if we remove any Q_i, where i is prime, then we get

$$\bigcap_{p \neq i} Q_P = 0_{\mathbb{Z}_2} \oplus 0_{\mathbb{Z}_3} \oplus \ldots \oplus \mathbb{Z}_i \oplus \ldots \neq 0_M.$$

Hence, the submodule 0_M admits an irredundant primary decomposition of in M.

Example 2.3.2. Let $M = \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \cdots$ or simply $M = \prod_{p \text{ prime}} \mathbb{Z}_p$ and consider the following primary submodules of M:

$$Q_2 = 0_{\mathbb{Z}_2} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_7 \times \cdots$$

$$Q_3 = \mathbb{Z}_2 \times 0_{\mathbb{Z}_3} \times \mathbb{Z}_5 \times \mathbb{Z}_7 \times \cdots$$

$$Q_5 = \mathbb{Z}_2 \times \mathbb{Z}_3 \times 0_{\mathbb{Z}_5} \times \mathbb{Z}_7 \times \cdots$$

In general, we can define Q_p for any prime $p \in \mathbb{Z}$ in a similar fashion. Note that $M/Q_p \cong \mathbb{Z}_p$ meaning that $Ass(M/Q_p) = \{(p)\}$ and hence Q_p is (p)-primary. By taking the intersection of these primary submodules, we get

$$0_M = \bigcap_{p \text{ prime}} Q_p.$$

This is a primary decomposition of 0_M in M, and by an argument similar to the one used in the previous example, we can show that this is an irredundant primary decomposition of 0_M in M.

Chapter 3

PROPERTIES OF THE GENERAL PRIMARY DECOMPOSITION

In the previous chapter we proved that any proper submodule N of a non empty module M over a Noetherian ring R admits a primary decomposition of the form $N = \bigcap \{Q_\lambda \mid \lambda \in \Lambda\}$ where Q_λ is P_λ-primary. In this chapter we investigate the collection of associated primes $\{P_\lambda \mid \lambda \in \Lambda\}$, but before we do that we need to study the following example.

Example 3.0.3. Consider the module mentioned in Example 2.3.2, i.e, $M = \prod_{p \text{ prime}} \mathbb{Z}_p$, we note that $\mathbb{Z}_2 \cong \mathbb{Z}_2 \times 0 \times 0 \ldots \subseteq M$ which means that $Ass(\mathbb{Z}_2) = \{(2)\} \subseteq Ass(M)$. In general, for every $p \in \mathbb{Z}$, the ideal (p) is in $Ass(M)$, and the question here is about the zero ideal: Does $0_\mathbb{Z} \in Ass(M)$? To show that $0_\mathbb{Z} \in Ass(M)$, we need to show that $\{0_\mathbb{Z}\} = Ann(m)$ for some $m \in M$. Consider $m = (\bar{1}, \bar{1}, \bar{1}, \dots) \in M$ and Let $r \in R$ be such that $r \in Ann(m)$, i.e, $rm = 0$. To find an integer $r \in \mathbb{Z}$ that kills all the entries in m, we must have $p|r$ for every prime p which is impossible except for $r = 0$, and therefore $Ann(m) \subseteq \{0_\mathbb{Z}\}$. The other direction, $\{0_\mathbb{Z}\} \subseteq \{r \in R \mid rm = 0\} = Ann(m)$ is true for all $m \in M$. Hence

$$Ass(M) = Ass\left(\prod_{p \text{ prime}} \mathbb{Z}_p\right) = \{0_\mathbb{Z}, (2), (3), (5), \dots\}$$

.

3.1 Relation Between the Collection of Associated Primes P_λ and $Ass(M/N)$

We begin this section by studying the well known case where $|Ass(M/N)| < \infty$ and M/N may not be finitely generated.

Lemma 3.1.1. *Let N be a proper submodule of an R-module M where N admits a finite primary decomposition, i.e., $N = Q_1 \cap Q_2 \cdots \cap Q_n$, where each Q_i is P_i primary then $Ass(M/N) \subseteq \{P_1, P_2, \ldots, P_n\}$.*

Proof. Define a homomorphism $f : M \longrightarrow M/Q_1 \oplus M/Q_2 \cdots \oplus M/Q_n$ by

$$f(m) = (m + Q_1, m + Q_2, \ldots, m + Q_n), \text{ for m } \in M.$$

Here $Ker(f) = N = Q_1 \cap Q_2 \cap \cdots \cap Q_n$ and hence $M/N \subseteq \bigoplus_{i=1}^{n} M/Q_i$ up to isomorphism which implies that

$$Ass(M/N) \subseteq Ass\left(\bigoplus_{i=1}^{n} M/Q_i\right) = \{P_1, P_2, \ldots, P_n\} \qquad \square$$

The above argument raises the question whether the same result will still hold if the submodule N is not finitely generated and it admits a primary decomposition of infinitely many components. To answer the question we introduce the following lemma.

Lemma 3.1.2. *Let N be a proper submodule of a module M over a ring R, and let N admits a primary decomposition of the form $N = \bigcap_{\lambda \in \Lambda} Q_\lambda$ where each Q_λ is P_λ primary for all $\lambda \in \Lambda$. Then*

$$Ass(M/N) \subseteq Ass\left(\prod_{\lambda\in\Lambda} M/Q_\lambda\right).$$

Proof. As in the previous lemma, we define a homomorphism $f : M \longrightarrow \prod_{\lambda\in\Lambda} M/Q_\lambda$, where for all $m \in M$ the λ^{th} component of $f(m)$ is $(m + Q_\lambda) \in M/Q_\lambda$. It is clear that the kernel of f is $Ker(f) = N$ and therefore $M/N \hookrightarrow \prod_{\lambda\in\Lambda} M/Q_\lambda$ hence

$$Ass(M/N) \subseteq Ass\left(\prod_{\lambda\in\Lambda} M/Q_\lambda\right) \qquad \square$$

Question 3.1.3. In the case of irredundant primary decomposition where the number of associated primes is finite , i.e Λ is finite we know that

$$Ass(M/N) = Ass(\prod_{\lambda\in\Lambda} M/Q_\lambda)$$

and we ask: Does this equality hold in general for Λ is infinite .

To answer our question, we study the following example.

Example 3.1.4. Let $M = \bigoplus \mathbb{Z}_p$ as in example 2.3.1, we know that 0_M admits an irredundant primary decomposition, and the collection of associated primes that corresponds to this decomposition is $\{P_\lambda \,|\, \lambda \in \Lambda\} = \{(2), (3), (5), \dots\}$, and $Ass(M/N) = Ass(M) = \{(2), (3), (5), \dots\}$, on the other hand, $Ass(\prod_{p \text{ prime}} M/Q_p) = Ass(\prod_{p \text{ prime}} \mathbb{Z}_p) = \{0_\mathbb{Z}, (2), (3), (5), \dots\}$.

In other words, this example agrees with the lemma 3.1.2 stated above, i.e, $Ass(M/N) \subsetneq Ass(\prod_{p \text{ prime}} M/Q_p)$, and shows that the equality mentioned in Question 3.1.3 in not guaranteed for Λ infinite.

Remark 3.1.5. Recall that for all $\lambda \in \Lambda$, we have $M/Q_\lambda \subseteq \prod_{\lambda\in\Lambda} M/Q_\lambda$ up to isomorphism, we easily see that $Ass(M/Q_\lambda) \subseteq Ass(\prod_{\lambda\in\Lambda} M/Q_\lambda)$ or simply

$$\{P_\lambda \,|\, \lambda \in \Lambda\} \subseteq Ass(\prod_{\lambda\in\Lambda} M/Q_\lambda).$$

Note that the collection $\{P_\lambda \,|\, \lambda \in \Lambda\}$ may be properly contained in the $Ass(\prod_{p \text{ prime}} M/Q_\lambda)$ as in the following example.

Example 3.1.6. Let M be as mentioned in Example 2.3.2 i.e $M = \prod_{p \text{ prime}} \mathbb{Z}_p$, for all p prime in $\mathbb{Z}$, we know that $N = 0_M = \bigcap Q_p$ is an irredundant general primary decomposition, where Q_p is as defined in the example and Q_p is (p)-primary. Here the collection of associated primes that corresponds to this decomposition is $\{P_\lambda \,|\, \lambda \in \Lambda\} = \{(p) \,|\, p \text{ prime in } \mathbb{Z}\}$, We also know form Example 3.0.3 that $Ass(\prod_{p \text{ prime}} M/Q_p) = Ass(\prod_{p \text{ prime}} \mathbb{Z}_p) = \{0_\mathbb{Z}, (2), (3), (5), \ldots\}$. Clearly: $\bigcup_{p \text{ prime}} Ass(\mathbb{Z}_p) \subsetneq Ass(\prod_{p \text{ prime}} \mathbb{Z}_p)$, in other words $\{p_\lambda \,|\, \lambda \in \Lambda\} \subsetneq Ass(\prod_{p \text{ prime}} M/Q_p)$ as the above remark stated.

By the above example, we can say that there are examples of general primary decomposition with the property that

$$\bigcup_{\lambda\in\Lambda} Ass(M_\lambda) \subsetneq Ass\left(\prod_{\lambda\in\Lambda} M_\lambda\right).$$

Theorem 3.1.7. *Let N be a proper submodule of a module M over a ring R. If N admits an irredundant primary decomposition of the form $N = \bigcap_{\lambda\in\Lambda} Q_\lambda$, where each Q_λ is P_λ primary, then $\{P_\lambda \,|\, \lambda \in \Lambda\} \subseteq Ass(M/N)$.*

Proof. Let $\lambda_0 \in \Lambda$ be arbitrary, then $Ass(M/Q_{\lambda_0}) = \{P_{\lambda_0}\}$ as each Q_λ is P_λ-primary. The assumption that the primary decomposition of N is irredundant guarantees that $N \subsetneq$

$\bigcap_{\lambda\neq\lambda_0} Q_\lambda$, implying that the quotient submodule $(\bigcap_{\lambda\neq\lambda_0} Q_\lambda)/N = (\bigcap_{\lambda\neq\lambda_0} Q_\lambda)/(\bigcap_{\lambda\in\Lambda} Q_\lambda)$ is a non-zero submodule of M/N. Applying the 2^{nd} Homomorphism Theorem, we get

$$(\bigcap_{\lambda\neq\lambda_0} Q_\lambda)/(\bigcap_{\lambda\in\Lambda} Q_\lambda) \cong (Q_{\lambda_0} + \bigcap_{\lambda\neq\lambda_0} Q_\lambda)/Q_{\lambda_0},$$

where the new quotient module $(Q_{\lambda_0} + \bigcap_{\lambda\neq\lambda_0} Q_\lambda)/Q_{\lambda_0}$ is a nonzero submodule of M/Q_{λ_0}. Taking the associates of everything in the above statement, we get

$$Ass(M/N) \supseteq Ass((\bigcap_{\lambda\neq\lambda_0} Q_\lambda)/(\bigcap_{\lambda\in\Lambda} Q_\lambda)) = Ass((Q_{\lambda_0}+\bigcap_{\lambda\neq\lambda_0} Q_\lambda)/Q_{\lambda_0}) \subseteq Ass(M/Q_{\lambda_0}) = \{P_{\lambda_0}\}.$$

As mentioned above, the submodule $(\bigcap_{\lambda\neq\lambda_0} Q_\lambda)/N$ is non zero, forcing it to have non empty associates, which means that the submodule isomorphic to it has also non-empty associates as well, in other words $\emptyset \neq Ass((Q_{\lambda_0} + \bigcap_{\lambda\neq\lambda_0} Q_\lambda)/Q_{\lambda_0}) \subseteq \{P_{\lambda_0}\}$ implying that

$$Ass\left((\bigcap_{\lambda\neq\lambda_0} Q_\lambda)/(\bigcap_{\lambda\in\Lambda} Q_\lambda)\right) = Ass\left(Q_{\lambda_0} + \bigcap_{\lambda\neq\lambda_0} Q_\lambda)/Q_{\lambda_0})\right) = \{P_{\lambda_0}\}.$$

The submodule $(\bigcap_{\lambda\neq\lambda_0} Q_\lambda)/(\bigcap_{\lambda\in\Lambda} Q_\lambda) = (\bigcap_{\lambda\neq\lambda_0} Q_\lambda)/N$ is contained in the quotient module M/N and by the above argument we have

$$\{P_{\lambda_0}\} = Ass\left((\bigcap_{\lambda\neq\lambda_0} Q_\lambda)/(\bigcap_{\lambda\in\Lambda} Q_\lambda)\right) \subseteq Ass(M/N).$$

Since P_{λ_0} was arbitrarily chosen, we can generalize the following fact

$$\{P_\lambda \,|\, \lambda \in \Lambda\} \subseteq Ass(M/N). \qquad \square$$

It is well known that in the case of an irredundant primary decomposition with a finite number of components, we have $\{P_i \mid i = 1, \ldots, n\} = Ass(M/N)$. And in the above theorem, we proved that in the general primary decomposition (with possibly infinitely many components), the collection of associated primes has the property that: $\{P_\lambda \mid \lambda \in \Lambda\} \subseteq Ass(M/N)$ and this makes us question whether the equality $\{P_\lambda \mid \lambda \in \Lambda\} = Ass(M/N)$ still holds in the general case. This question motivates us to go back to our previous example and investigate its associated primes in a different way.

Example 3.1.8. Let $M = \prod_{p\,\text{prime}} \mathbb{Z}_p$ and consider the irredundant primary decomposition of $N = 0_M = \bigcap Q_p$, where Q_p is as defined in example 2.3.2 and Q_p is (p)-primary for every prime $p \in \mathbb{Z}$. Here $\{P_\lambda \mid \lambda \in \Lambda\} = \{(2), (3), (5), \ldots, \}$, and we showed that $Ass(M/N) = Ass(M) = \{0_\mathbb{Z}, (2), (3), (5), \ldots\}$. Clearly $\{(2), (3), \ldots\} \subsetneq \{0_\mathbb{Z}, (2), (3), \ldots\}$, therefore there are cases of general primary decomposition where

$$\{P_\lambda \mid \lambda \in \Lambda\} \subsetneq Ass(M/N).$$

In other words, in the general primary decomposition the equal relationship between the collection of $\{P_\lambda\}$ and $Ass(M/N)$ is not guaranteed.

Remark 3.1.9. Considering the case when N admits a primary decomposition of a finite number of primary components i.e $N = \bigcap_{i=1}^{r} Q_i$ where Q_i is P_i-primary, Lemma 3.1.1 states that the $Ass(M/N) \subseteq \{P_1, P_2, P_3, \ldots, P_r\}$. Also by Theorem 3.1.7, we know that for any Λ the collection of associated primes $\{P_\lambda \mid \lambda \in \Lambda\} \subseteq Ass(M/N)$, this means that when Λ is finite, i.e, when $\Lambda = r$ for some $r \in \mathbb{N}$ we have $\{P_1, P_2, P_3, \ldots, P_r\} \subseteq Ass(M/N)$. Hence the well known result for finite primary decomposition

$$Ass(M/N) = \{P_1, P_2, \ldots, P_r\}$$

3.2 About the Uniqueness of the Collection of Associated Primes P_λ

In the first chapter, Theorem 1.2.4 stated that the collection of associated primes $\{P_i \,|\, i = 1, \ldots, n\} = Ass(M/N)$ is uniquely determined by $N \subsetneq M$ regardless of the particular irredundant primary decomposition of N in M. And in this section we investigate this uniqueness property in the general primary decomposition, i.e, if $N \subsetneq M$ and N admits two irredundant general primary decomposition $N = \bigcap_{\lambda\in\Lambda} Q_\lambda = \bigcap_{i\in I} Q'_i$ where Q_λ is P_λ-primary, Q'_i is P'_i primary, Λ and I are both infinite sets. The question here is: Does $\{P_\lambda \,|\, \lambda \in \Lambda\} = \{P'_i \,|\, i \in I\}$? To answer the question we introduce the following interesting example.

Example 3.2.1. Let $M = \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \cdots$. Consider the submodule 0_M in M, this submodule admits the following irredundant primary decomposition $O_M = Q_0 \cap Q_2 \cap Q_3 \cap Q_5 \cap Q_7 \ldots$, where

$$Q_0 = 0_\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \cdots$$

$$Q_2 = \mathbb{Z} \times 0_{\mathbb{Z}_2} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \cdots$$

$$Q_3 = \mathbb{Z} \times \mathbb{Z}_2 \times 0_{\mathbb{Z}_3} \times \mathbb{Z}_5 \times \cdots$$

Note that $Ass(M/Q_0) = Ass\left((\mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \cdots)/(0_\mathbb{Z} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \cdots)\right) = Ass(\mathbb{Z}) = \{0_\mathbb{Z}\}$, similarly $Ass(M/Q_2) = \{(2)\}$, $Ass(M/Q_3) = \{(3)\}$ and so on, and hence the collection of associated primes that occurs along with this irredundant primary decomposition is exactly $\{0_{\mathbb{Z}}, (2), (3), (5), (7), \dots\}$.

Also 0_M admits another irredundant primary decomposition $0_M = Q'_2 \cap Q'_3 \cap Q'_5 \cap \cdots$ where

$$Q'_2 = (2) \times 0_{\mathbb{Z}_2} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \cdots$$

$$Q'_3 = (3) \times \mathbb{Z}_2 \times 0_{\mathbb{Z}_3} \times \mathbb{Z}_5 \times \cdots$$

$$Q'_5 = (5) \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times 0_{\mathbb{Z}_5} \times \cdots$$

Here $Ass(M/Q'_2) = Ass(\mathbb{Z}_2 \times \mathbb{Z}_2) = Ass(\mathbb{Z}_2) \cup Ass(\mathbb{Z}_2) = \{(2)\}$, and similarly,$Ass(M/Q'_p) = \{(p)\}$ for every prime in $\mathbb{Z}$. Therefore, the collection of associated primes that corresponds to this primary decomposition is $\{(2), (3), (5), \dots\}$, which is different from the above collection, i.e, $\{P_\lambda \,|\, \lambda \in \Lambda\} \neq \{P'_i \,|\, i \in I\}$.

Hence, we can say that in a general primary decomposition the collection of associated primes is not necessarily unique and is dependent on the choice of the primary components of the primary decomposition.

3.3 About the Minimal Primes in the Collection of Associated Primes P_λ

In the first chapter , Theorem 1.2.5 stated that if a proper submomodule $N \subsetneq M$ admits a primary submodule of finitely many primary components, and if P_j is one of the minimal primes, then the primary submodule to P_j that occurs in the primary decomposition is uniquely determined by N and is independent of the choice of irredundant primary decomposition.

Well, in the general case the situation is different, i.e if $N = \bigcap_{\lambda\in\Lambda} Q_\lambda = \bigcap_{i\in I} Q'_i$ where Λ an I are not finite and P_j is a minimal associated prime that appears in both decomposition, then it may happen that $Q_j \neq Q'_j$ as in the following example.

Example 3.3.1. Let $M = \mathbb{Z}\times\mathbb{Z}\times\mathbb{Z}_2\times\mathbb{Z}_3\times\mathbb{Z}_5\times\ldots$, and consider the following submodules

$$Q_0 = 0_{\mathbb{Z}} \times 0_{\mathbb{Z}} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \ldots$$

$$Q_2 = \mathbb{Z} \times \mathbb{Z} \times 0_{\mathbb{Z}_2} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \ldots$$

$$Q_3 = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2 \times 0_{\mathbb{Z}_3} \times \mathbb{Z}_5 \times \ldots$$

Similarly, for any prime $p \in \mathbb{Z}$, we can define $Q_p = \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}_2 \times \ldots \times 0_{\mathbb{Z}_p} \times \ldots$, note that

$$Ass(M/Q_0) = Ass(\mathbb{Z} \times \mathbb{Z}) = Ass(\mathbb{Z}) \cup Ass(\mathbb{Z}) = \{0_{\mathbb{Z}}\}$$

$$Ass(M/Q_p) = Ass(\mathbb{Z}_p) = \{(p)\}$$

Hence Q_p is p-primary, and if we take the intersection of these submodules we get

$$0_M = Q_0 \bigcap (\cap_{\,p \text{ prime}}\, Q_p)$$

This is a general primary decomposition of 0_M submodule in M. Furthermore, if we remove Q_0 from the intersection we get $\bigcap_{\lambda\neq 0} Q_\lambda = \mathbb{Z} \times \mathbb{Z} \times 0_{\mathbb{Z}_2} \times 0_{\mathbb{Z}_3} \times \cdots \neq 0_M$ where $\Lambda =$

$\{0, 2, 3, \dots\}$, and in general, if we remove any Q_p, we get

$$\bigcap_{\lambda \neq p} Q_\lambda = 0_{\mathbb{Z}} \times 0_{\mathbb{Z}} \times 0_{\mathbb{Z}_2} \times \dots \times \mathbb{Z}_p \times \dots \neq 0_M.$$

This means that this decomposition is an irredundant one, and the collection of associated primes that corresponds to it is $\{0_{\mathbb{Z}}, (2), (3), (5), \dots\}$.

Now, for the same module M consider the following submodules

$$Q'_0 = 0_{\mathbb{Z}} \times \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \dots$$

$$Q'_2 = \mathbb{Z} \times (2) \times 0_{\mathbb{Z}_2} \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \dots$$

$$Q'_3 = \mathbb{Z} \times (3) \times \mathbb{Z}_2 \times 0_{\mathbb{Z}_3} \times \mathbb{Z}_5 \times \dots$$

In a similar manner, we can define Q_p for any prime $p \in \mathbb{Z}$, where $M/Q_0 \cong \mathbb{Z}$ implying that $Ass(M/Q_0) = \{0_{\mathbb{Z}}\}$ and $M/Q_2 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ implying that $Ass(M/Q_2) = \{(2)\}$. In general $Ass(M/Q_p) = \{(p)\}$, and if we take the intersection of these submodules we obtain

$$0_M = Q_0 \bigcap \left(\cap_{\,p \text{ prime}} Q_p\right).$$

This is another primary decomposition of 0_M in M, and by an argument similar to the one used in the previous decomposition we can show that it is an irredundant one, and the corresponding collection of associated primes is $\{0_{\mathbb{Z}}, (2), (3), (5), \dots\}$.

The associated prime $0_{\mathbb{Z}}$ is a minimal prime with respect to inclusion in both decomposition, but the corresponding primary submodules Q_0 and Q'_0 that appear in the decompositions are different: $Q_0 = 0_{\mathbb{Z}} \times 0_{\mathbb{Z}} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \ldots$, and $Q'_0 = 0_{\mathbb{Z}} \times \mathbb{Z} \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \ldots$ i.e., $Q_0 \neq Q'_0$.

Remark 3.3.2. Although all examples in this paper are over the ring of integers $\mathbb{Z}$, we can generate other examples using other rings. So for the module $M = \prod_{\text{p prime}} \mathbb{Z}/(p)$ that we studied in depth, we can replace it with $M = \prod K[x]/(p(x))$ where K is a field and $p(x)$ runs all over monic prime polynomials. Similarly, we can replace the module $M = \bigoplus_{p \text{ prime}} \mathbb{Z}/(p)$ by $M = \bigoplus K[x]/(p(x))$.

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