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A Group-theoretic Approach to Human Solving Strategies in Sudoku

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Introduction.

7	6		8	1			4	5
		5				6	8	
	1	2				3	9	
					4			
1				7				3
		3	5					
	3					4	1	
	9	7				5		
6				3				2

Figure 1: Sudoku Puzzle P

The goal of the popular game of Sudoku is to fill in the numbers 1 through 9 in such a way that there is exactly one of each digit in each row, column and 3×3 block. When a human, as opposed to a computer [1], solves a Sudoku puzzle with paper and pencil, strategies such as the following are used: "There is no 3 in the first row. The third, sixth and seventh cell are empty. But the third column already has a 3, and the top right block already contains a 3. Therefore, the 3 must go in the sixth cell." We call this a Human Solving Strategy (HSS for short). This particular strategy has a name, called the *hidden single strategy*. (For more information on solving strategies see [7].) The following strategy may be used for the cell in the second row, second column: "This cell cannot contain 5, 6 or 8 since these numbers are already in the second row. It cannot contain 3 or 9 since these are already in the second column. And it cannot contain 1, 2, or 7 since these numbers are already in the first block. Therefore, the only number this cell can contain is 4." This strategy is called the *naked single strategy*. If the Sudoku is "easy" enough, one can continue on in a strategic manner using these simple strategies and solve the whole puzzle.

[1] The way a human solves a Sudoku is strikingly different than the way a computer would solve the same puzzle. The most efficient computer algorithm for solving Sudoku is a brute force, depth first search. A popular such algorithm is Knuth's "Dancing Links" version of Algorithm X [5].

If the Sudoku is harder, one would need to use more complicated logical strategies. We discuss a few of these in Section 2.

1				5				2
	4	7				5		1
	2		7			6	9	
			9					3
7				2				5
					8			
	9	8				7	1	
		5				4	3	8
6				7				9

Figure 2: Sudoku Puzzle Q

Consider Sudoku Puzzle Q above. As an initial strategy, one may look at the last column and notice that there is no 7 in it. The third, fifth and seventh cell are available. But the third row already has a 7 and the bottom right block already has a 7. Therefore, the 7 must go in sixth cell. If one continues to solve this puzzle (and we strongly suggest that the reader attempt to solve both puzzles) one will most likely get the feeling of déjà vu. Sudoku Q is *essentially the same* puzzle as Sudoku Puzzle P. Sudoku Q is just Sudoku P rotated 90° clockwise with the entries relabeled by the permutation (159483726).

In this paper, we introduce a novel way to codify human solving strategies. In Section 1, we define precisely what we mean when we say two puzzles are "essentially the same" by discussing the Sudoku symmetry group found in the literature. At the end of the section, we present a theorem with a smaller generating set than that used in other papers. In Section 2 we describe some human solving strategies in terms of "non-clues" which we call *packets*. In section 3 we introduce a new algebraic group that can be associated to Sudoku, the group of solving symmetries. Finally in Section 4 we use solving symmetries to demonstrate the feeling of déjà vu described in this section.

1 Essentially the Same Sudoku Puzzles

What do we mean when we say two Sudoku puzzles are "essentially the same"? First we distinguish between a Sudoku "puzzle" and Sudoku "board". A **Sudoku board** is a 9×9 grid filled with the digits 1-9 according to the rules of the game. A **Sudoku puzzle** is any subset of a Sudoku board. We will say that two Sudoku boards, B and C are *essentially the same* if there is an action which maps B to C such that if that action is performed on *any* given Sudoku board, it would not create any inconsistencies in the rules of the game. As already mentioned in the introduction, rotating a puzzle 90 degrees will give another puzzle that is essentially the same as the original. Also, permuting the numbers used for clues in one puzzle leads to another essentially the same puzzle. In the following paragraphs, we determine other actions which lead to essentially the same puzzles.

First a bit of notation. The three three-row strips are called **bands** and the three three-column strips are called **stacks**. We can interchange any two rows in a band and not create any inconsistencies. We cannot, however, interchange two rows that are not in the same band. Similarly, we can interchange any two columns in a stack, but we cannot interchange two columns in different stacks. We can also interchange any two entire bands and any two entire stacks. Finally, we can reflect the entire puzzle across any of the lines of symmetry of the square: horizontal, vertical or diagonal.The set generated by all of the actions on Sudoku boards described above, form the full group of Sudoku symmetries [6]. Next we find a smaller generating set for this group.

Let r be the rotation 90 degrees clockwise and let t be the reflection across the diagonal from upper left to bottom right. The band and stack swaps can be defined as follows: Let $B12$ to be the swap of band 1 with band 2 and $B13$ to be the swap of band 1 with band 3. Notice that $B12 \circ B13 \circ B12$ swaps bands 2 and 3. $r \circ B12 \circ r^{-1}$ swaps stack 1 and 2. Similarly, we can swap the other stacks using a conjugation of a band swap and a rotation. Now, let $R12$ to be the swap of row 1 and row 2

and $R13$ to be the swap of row 1 and row 3. Together with the band swaps, we can now swap any two rows in the same band. Together with rotation, we can swap any two columns in the same stack. Now the group of all of the actions defined above can be generated by these six elements: $r, t, B12, B13, R12, R13$. We call this the *group of position symmetries*. This group permutes three sets of three rows, three bands, three sets of three columns, three stacks, and transpose. Therefore we see that this is a non-abelian group of size $[(3!)^3 \cdot 3!]^2 \cdot 2 = 2^9 3^8 = 3,359,232$. Using GAP [4], we confirm this calculation. This group is non-Abelian. In addition to the position symmetries, we can also relabel the entries in the cells. There are 9! ways to do this. The position symmetries commute with the relabeling symmetries. So the entire Sudoku symmetry group is $\mathcal{G} = \langle r, t, B12, B13, R12, R13 \rangle \times S_9$ [6, 2]. We give a much smaller generating set in the following theorem.

Theorem 1.1. *The Sudoku Symmetry Group, $\mathcal{G}$ can be generated by the three elements $B12, R12$ and r together with S_9. That is*

$$\mathcal{G} = \langle B12, R12, r \rangle \times S_9.$$

Proof. We just need to show that the generators above can be written as a combination of these three generators. $B23$ can be obtained by rotating twice, swapping band 1 and band 2 and then rotating back. That is

$$B23 = r^2 \circ B12 \circ r^2.$$

$R13$ can be obtained by rotating twice, swapping row 1 and row 2, rotating twice again, swapping bands 1 and 2, and again rotating twice and swapping bands 1 and 2. That is

$$R13 = B12 \circ r^2 \circ B12 \circ r^2 \circ R12 \circ r^2.$$

The reflection t is much more complicated, and we use GAP to find $t = r \circ B12 \circ r^2 \circ B12 \circ r^2 \circ R12 \circ B12 \circ r^2 \circ R12 \circ B12 \circ r^2 \circ R12 \circ B12 \circ r^2 \circ R12 \circ B12 \circ r^2 \circ R12 \circ B12 \circ r^2 \circ R12 \circ B12 \circ R12 \circ r^2 \circ R12 \circ B12 \circ R12$.

Alternatively, since one set of generators is contained in the other, we can use GAP to compute the size of both groups. We find that they have the same size, indicating that they generate the same group. □

Now we use $\mathcal{G}$ to define precisely what we mean by "essentially the same" Sudoku boards.

Definition 1.1. *Two Sudoku boards are* **essentially the same** *if they are in the same equivalence class induced by the the action of the group, $\mathcal{G}$ on the set of Sudoku boards. In other words, Sudoku boards B and C are essentially the same, if there exists a Sudoku symmetry, $g \in \mathcal{G}$ such that g applied to B yields C.*

In [6] an explicit calculation yields that there are 5,472,730,538 equivalence classes of Sudoku boards.

In the next section, we define another symmetry group relating to Sudoku: the group of solving symmetries. Before we do this, however, we introduce the idea of a "packet". A packet tells the solver what number does *not* go in a particular cell. Packets turn out to be a critical ingredient of solving symmetries.

2 Packets and Human Solving Symmetries

When humans solve a Sudoku puzzle with pencil and paper, they often look at a cell and decide which clues can NOT go in the cell. In the naked single strategy described in the introduction, one determines what numbers cannot go in a cell and based on this, one determines the only possible clue that can be placed in the cell. We give this type of clue the name *packet*.

Definition 2.1. *A* **packet** *is a representation that a number cannot be placed in a cell.*

For example, if we determine that 1, 3, 5 and 7 cannot go in the first cell, we represent the packets by putting an $\otimes$ in the first, third, fifth and

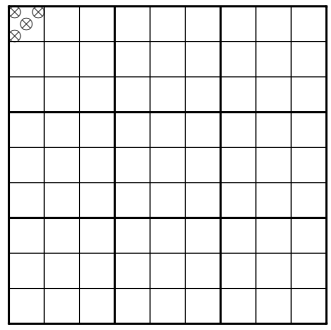

Figure 3: Packets in the first cell

seventh position, in the first cell. If we label the top left cell by the letter a, then we will refer to these packets as $a_1,\ a_3,\ a_5$ and a_7.

We can expand on the Human Solving Strategy (HSS), naked single, and discuss another HSS, naked triple. In Figure 4, we have packets in the 7-position in all but the last three cells in row 3. This implies that we must have packets in the 7-position in the top 6 cells of the top right block. In the diagram below, packets marked with $\otimes$ are given packets. Packets marked with $\times$ are immediately implied by the given packets.

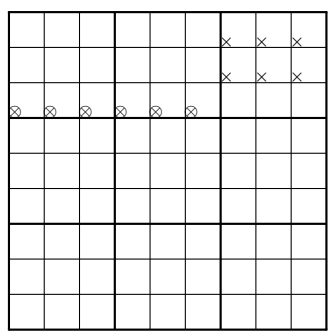

Figure 4: Packets representing a Naked Triple

Another HSS that takes advantage of packets is the "X-Wing" strategy. In this strategy, two rows have packets in all but two cells and these two cells are in the same column in the two rows. These given packets, represented by $\otimes$ in figure 5 below, imply packets in all but the two empty cells in the two columns. In the figure, the value 3 can only be in cell three or seven

in both rows one and eight. Therefore, 3 cannot be in any of the other cells in column three and column seven. If three belongs in the third cell in the first row, then it must also be in the seventh cell in the eighth row. If it belongs in the seventh cell in the first row, then it must also be in the third cell in the eighth row. The possibilities for the 3 form an X, hence the name.

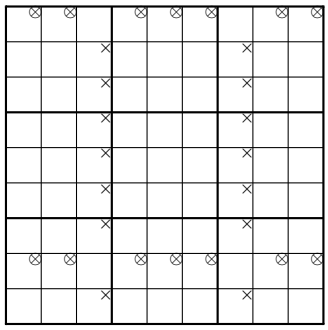

Figure 5: A Sudoku packet puzzle with an X-Wing

In the next section we use the idea of given packets and implied packets to algebraically define these human strategies for solving Sudoku puzzles.

3 Solving Symmetries

Before defining solving symmetry, we first give a general context that will relate the group of solving symmetries to the Sudoku symmetry group, $\mathcal{G}$, described in Section 1. There are 729 possible packets in a 9×9 Sudoku grid (9 possible packets for each of the 81 cells). We denote this set of 729 packets by $\mathcal{K}$. We can now identify a Sudoku board as a subset of $\mathcal{K}$, with eight packets in each cell, corresponding to the solution of the puzzle. The packet that is missing from each cell represents the entry in the cell. A Sudoku puzzle is also a subset of $\mathcal{K}$. We call a subset of $\mathcal{K}$ a **packet set**. Note that not all packet sets are Sudoku puzzles. There may be inconsistencies such as nine packets in a cell, or a packet in the 7-position of every cell in a given row. We call packet sets that are consistent and lead

to a valid Sudoku board, **packet puzzles**. We summarize these definitions below.

Definition 3.1. *The set of 729 packets in a Sudoku grid is denoted by $\mathcal{K}$. A subset of $\mathcal{K}$ is called a* **packet set**. *The set of all packet sets is the power set of $\mathcal{K}$ denoted $\mathcal{P}(\mathcal{K})$. A* **packet puzzle** *is an element of $\mathcal{P}(\mathcal{K})$ that is consistent with the rules of Sudoku. A* **Sudoku board**, *can be considered as an element of $\mathcal{P}(\mathcal{K})$ with all but one packet filled in each cell, consistent with the rules of Sudoku.*

Let $\mathcal{B}$ denote the set of all Sudoku boards. If a packet puzzle, P, is a subset of a board, $B \in \mathcal{B}$, then we say that P **completes to** B. Note that a packet puzzle can complete to more than one board. This would not be a fun game for humans to play, but it is necessary to consider for our mathematical context.

Now we come to an important concept: the variety of a packet puzzle.

Definition 3.2. *The* **variety** *of a packet puzzle, P, denoted $V(P)$, is the set of all boards to which P completes. $V(P) \subseteq \mathcal{B}$ is the set of all completions of P.*

So why do we call this a variety when there are no ideals in sight? It turns out that we can represent Sudoku constraints as polynomials. If we use Boolean polynomials in 729 variables, one for each packet, then we can create a packet puzzle ideal by adding the packet variables to the ideal generated by the Sudoku constraint polynomials. If we call the packet puzzle ideal P, then $V(P)$ is in fact the variety of the ideal P which represents all possible Sudoku boards to which the puzzle P completes. This is all we will mention in this article about polynomials and ideals. See [1, 3] for more information about polynomial representation of Sudoku.

Note that if $B \in \mathcal{B}$ is a Sudoku board, and $P \in \mathcal{P}(\mathcal{K})$ is a packet puzzle such that $P \subseteq B$. Then $B \in V(P)$. We also have the following statement, based on the fact that $V(P)$ is an actual variety: Let P and Q be packet puzzles. If $P \cup Q$ is a packet puzzle, then $V(P \cup Q) = V(P) \cap V(Q)$.

This small, simple statement helps to show how a Sudoku solver can take a given set of packets and add new packets. If a good Sudoku puzzle solver is adding a set of packets S to a starting set P, then we would expect $V(P \cup S) = V(P)$. A Sudoku solver should only add new packets to the puzzle which do not add any new constraints, thus preserving the variety of the original puzzle. This brings us to the concept of *solving symmetries*.

In order to give context to both the group of solving symmetries and the group of Sudoku symmetries defined in Section 1, we define a universal set that encompasses all invertible functions on packet sets.

$$\mathcal{F} = \{f | f : \mathcal{P}(\mathcal{K}) \to \mathcal{P}(\mathcal{K}), f \text{ invertible}\}$$

It is easy to see that $\mathcal{F}$ is a group under composition. Since $\mathcal{P}(\mathcal{K})$ is finite with order 2^{729} the set of invertible functions on $\mathcal{P}(\mathcal{K})$ is isomorphic to the symmetric group on 2^{729} elements, $S_{2^{729}}$. The group of Sudoku symmetries, $\mathcal{G}$, described in Section 1 consists of invertible functions acting on packet sets. Therefore $\mathcal{G}$ is a subgroup of $\mathcal{F}$.

Definition 3.3. *A* **solving symmetry** *is an invertible function $\sigma \in \mathcal{F}$ such that for all packet puzzles $P \in \mathcal{P}(\mathcal{K})$, $V(\sigma P) = V(P)$. In other words σ maps a packet puzzle to another packet puzzle with the same set of solutions. Let $\mathcal{S}$ denote the set of all solving symmetries.*

It is not hard to see that $\mathcal{S}$ is also a subgroup of $\mathcal{F}$ under composition. The following is an example of a human solving strategy that we would like to represent as a group element of $\mathcal{S}$. We can call the first cell in the Sudoku grid a and the remaining cells in the row b, c, d, e, f, g, h and i. In this example, we have a set of 8 packets in the first cell in all positions except the 1-position. Call this set $T = \{a_2, a_3, a_4, a_5, a_6, a_7, a_8, a_9\}$. From this information, the naked single strategy implies that the first cell must contain a 1. The human solver then concludes that a 1 cannot be present in any other cell in the first row. So we know that the packet set $S = \{b_1, c_1, d_1, e_1, f_1, g_1, h_1, i_1\}$ is implied by the packet set T. Now we can define

$$\rho : T \mapsto T \cup S$$

shown in Figure 6. Here the elements of the packet set T are represented by the symbol $\otimes$ and the elements of the packet set S are represented by $\times$.

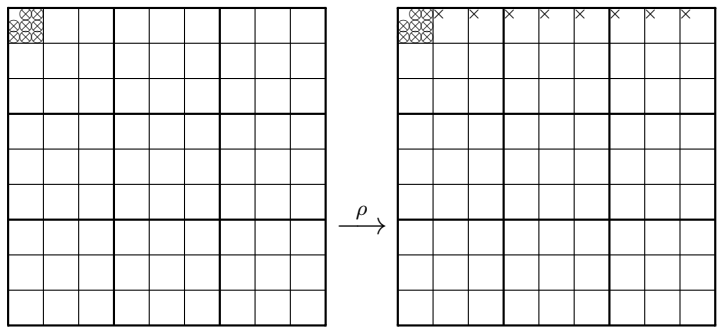

Figure 6: ρ acting on its triggering set

The solving symmetry ρ will act on a packet puzzle P if and only if P contains the set T. Hence, we call T the *triggering set* for ρ. The problem is that ρ is not an invertible function on the set of all packet sets, $\mathcal{P}(\mathcal{K})$. Say, for example, a packet puzzle, P, already has a packet in the b_1 position. If $T \subseteq P$ and we apply ρ to P, then we get a new packet puzzle Q such that $Q = P \cup S$. If we define $\rho^{-1}(Q)$ to be $Q \setminus S$, then we see that $\rho^{-1}(Q) \neq P$, since it is missing the packet in the b_1 position. This issue, however, can be corrected by considering one packet at a time and taking the symmetric difference of P and S, denoted $P \Delta S$, rather than the union. Now every primitive solving symmetry is invertible with order two.

Definition 3.4. *Suppose a packet set, T, implies a packet set S. By this we mean that $V(T) = V(T \cup S)$. Then we call T a* **triggering set** *for S and S the* **implied packet set** *for T.*

Next, we define the building block for Solving Symmetries.

Definition 3.5. *Given a triggering set, T, and a packet $k \notin T$, we define a* **primitive solving symmetry** *to be a function*
$F_{T,k} : \mathcal{P}(\mathcal{K}) \to \mathcal{P}(\mathcal{K}) \in \mathcal{F}$ *as follows:*

$$F_{T,k}(P) = \begin{cases} P \Delta \{k\} & \text{if } T \subseteq P \\ P & \text{if } T \not\subseteq P \end{cases}$$

where T is a triggering set for the single packet k. If the triggering set T is in the packet set P, then we form the symmetric difference, $P\Delta\{k\}$ and add it to T. In other words, we add the packet k to P if it is not already there, and take it away if it is there. If the triggering set T is **not** *a subset of the puzzle P, then we do nothing.*

Next we show that a primitive solving symmetry is indeed a solving symmetry.

Theorem 3.1. *Given a triggering set T and a single packet k implied by T, the primitive solving symmetry $\sigma = F_{T,k}$, is a solving symmetry.*

Proof. Let $\sigma = F_{T,k}$ be a primitive solving symmetry. σ is an invertible function from $\mathcal{P}(\mathcal{K})$ to $\mathcal{P}(\mathcal{K})$. Note that $\sigma^{-1} = \sigma$. Now consider an arbitrary packet set $P \in \mathcal{P}(\mathcal{K})$. To show that σ is a solving symmetry, we need to show that $V(\sigma P) = V(P)$. Suppose that $T \not\subseteq P$. Then $\sigma P = P$, so $V(\sigma P) = V(P)$.

Suppose now that $T \subseteq P$. We have two cases. Either the packet k is in P or it is not in P. Suppose $k \in P$. Then $\sigma(P) = P \setminus \{k\}$. We want to show that $V(P \setminus \{k\}) = V(P)$. Since T is a triggering set for $\{k\}$, $V(T \cup \{k\}) = V(T)$. Since $T \subseteq P$, we have that
$V(P) = V(P \cup T) = V(P \setminus \{k\} \cup \{k\} \cup T) = V(P \setminus \{k\}) \cap V(\{k\} \cup T) = V(P \setminus \{k\}) \cap V(T) = V(T \cup P \setminus \{k\}) = V(P \setminus \{k\})$ which is what we wanted to show.

Suppose now that $k \notin P$. Then $\sigma(P) = P \cup \{k\}$. We want to show that $V(P \cup \{k\}) = V(P)$. Now we have that $V(P \cup \{k\}) = V(P \cup T \cup \{k\}) = V(P) \cap V(T \cup \{k\}) = V(P) \cap V(T) = V(P \cup T) = V(P)$.

So in both cases, $V(\sigma P) = V(P)$ □

Does the set of primitive solving symmetries form a group? The answer is no. The identity is, in fact, not a primitive solving symmetry. And the composition of two primitive solving symmetries is not a primitive solving symmetry. The composition of two primitive solving symmetries is, however, a solving symmetry, which prompts our next definition.

Definition 3.6. *A* **simple solving symmetry** *is a primitive solving symmetry or the composition of two or more primitive solving symmetries with the same triggering set. If τ is a simple solving symmetry with triggering set T which implies a packet set, S, then*

$$\tau(P) = \begin{cases} P \Delta S & \text{if } T \subseteq P \\ P & \text{if } T \not\subseteq P \end{cases}$$

A primitive solving symmetry has a triggering set that implies a single packet at a time. Simple solving symmetries, which are just compositions of primitive solving symmetries with the same triggering set, imply more than one packet. For example, the naked triple HSS in Figure 4, is a simple solving symmetry. The triggering set consists of the six packets represented by the $\otimes$ symbols and the implied packet set consists of six packets represented by the $\times$ symbols. This simple symmetry can be broken down into the composition of six primitive symmetries, each with one of the $\times$ packets as its implied set.

Note that two simple solving symmetries with the same triggering set commute. Simple solving symmetries with different triggering sets may not commute.

The next theorem shows that the compositions of simple solving symmetries form a group.

Theorem 3.2. *Denote by $\mathcal{S}^3$ the set of compositions of simple solving symmetries together with the identity solving symmetry. Then $\mathcal{S}^3$ is a group under composition.*

Proof. The identity symmetry in $\mathcal{S}^3$ can be described in several ways. A simple way to describe the identity of this group is, an element of $\mathcal{S}^3$ with its triggering set being the empty set and its implied packet set also being the empty set. If we call this element e, then clearly, for any $a \in \mathcal{S}^3$ we have $a \circ e = e \circ a$.

If $a, b \in \mathcal{S}^3$ then a and b are each compositions of simple solving symmetries. Hence the composition, $a \circ b$, is also a composition of simple solving symmetries.

Each simple solving symmetry is invertible since it is the composition of invertible symmetries of order two. Therefore $\mathcal{S}^3$ is a group under composition. $\square$

Now any human solving symmetry can be represented by a simple solving symmetry in $\mathcal{S}^3$. In fact, given a puzzle that completes to a unique board, the set of all packets in the puzzle can be considered a triggering set, T, and the remaining packets needed to complete the board can be considered the set of implied packets, S. So any Sudoku puzzle can be solved in one fell swoop with a single solving symmetry! However, most humans are not so adept at recognizing such a large triggering set and implied packet set, so one solves the puzzle one small step at a time and uses a composition of solving symmetries.

Now we have the group of simple solving symmetries, $\mathcal{S}^3$, sitting inside the group of all solving symmetries, $\mathcal{S}$. An obvious question to ask is, "Are there any solving symmetries that are not compositions of simple solving symmetries?"

Conjecture 3.1 *There are no solving symmetries other than the composition of simple solving symmetries and the identity. In other words,* $\mathcal{S} = \mathcal{S}^3$.

4 Déjà Vu

Now we return to the question of déjà vu that we get when solving the two puzzles from the introduction. Why do we get a feeling of déjà vu when we solve two puzzles that are essentially the same? The answer should be clear by now: Because we use the same solving symmetry to solve both.

Sudoku P and Sudoku Q are the puzzles from the introduction. Suppose we have a Sudoku puzzle, P (here represented with clues rather than packets for readability) that completes to a Sudoku board, B. Now suppose

P

7	6		8	1			4	5
		5				6	8	
	1	2				3	9	
					4			
1				7				3
		3	5					
	3					4	1	
	9	7				5		
6				3				2

$\pi P = Q$

1				5				2
	4	7				5		1
	2		7			6	9	
			9					3
7				2				5
					8			
	9	8				7	1	
		5				4	3	8
6				7				9

$\xrightarrow{\pi}$ $\xleftarrow{\pi^{-1}}$

$\downarrow \sigma$ $\qquad$ $\downarrow \pi\sigma\pi^{-1}$

7	6	9	8	1	3	2	4	5
3	4	5	2	9	7	6	8	1
8	1	2	6	4	5	3	9	7
5	7	6	3	8	4	1	2	9
1	2	4	9	7	6	8	5	3
9	8	3	5	2	1	7	6	4
2	3	8	7	5	9	4	1	6
4	9	7	1	6	2	5	3	8
6	5	1	4	3	8	9	7	2

B

1	8	6	4	5	9	3	7	2
9	4	7	3	6	2	5	8	1
5	2	3	7	8	1	6	9	4
8	5	2	9	4	7	1	6	3
7	1	9	6	2	3	8	4	5
3	6	4	5	1	8	9	2	7
4	9	8	2	3	5	7	1	6
2	7	5	1	9	6	4	3	8
6	3	1	8	7	4	2	5	9

$\pi B = C$

there is a Sudoku symmetry, $\pi \in \mathcal{G}$, the Sudoku Symmetry Group, that maps P to another Sudoku puzzle Q. In our example, $\pi = r \circ (159483726)$, the permutation of the entries followed by a rotation of 90 degrees. So $Q = \pi(P)$. The packet set Q completes to the board $\pi(B) = C$. Now, suppose the solving symmetry needed to take P to B is σ. Here is the explanation for the déjà vu: The solving symmetry needed to take Q to C is $\pi^{-1}\sigma\pi$. It is essentially the same solving symmetry, σ.

5 Conclusion

There is a wealth of mathematics to be discovered and explored concerning Sudoku puzzles. The idea of a packet enables us to define several algebraic groups that act on Sudoku boards and puzzles. This concept also allows us to mathematically codify how humans solve Sudoku puzzles. We will

continue to study and hope to understand better this very large, but finite group of Solving Symmetries.

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