3-18-2011

On the Lebesgue Integral

Jeremiah D. Kastine
Georgia State University

ON THE LEBESGUE INTEGRAL

by

JEREMIAH D. KASTINE

Under the Direction of Dr. Imre Patyi

ABSTRACT

We look from a new point of view at the definition and basic properties of the Lebesgue measure and integral on Euclidean spaces, on abstract spaces, and on locally compact Hausdorff spaces. We use mini sums to give all of them a unified treatment that is more efficient than the standard ones. We also give Fubini's theorem a proof that is nicer and uses much lighter technical baggage than the usual treatments.

INDEX WORDS: Lebesgue measure, Lebesgue integral, Fubini's theorem, Locally compact Hausdorff space, Riesz representation theorem

ON THE LEBESGUE INTEGRAL

by

JEREMIAH D. KASTINE

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

Master of Science

in the College of Arts and Sciences

Georgia State University

2011

ON THE LEBESGUE INTEGRAL

by

JEREMIAH D. KASTINE

Committee Chair: Dr. Imre Patyi

Committee: Dr. Alexandra Smirnova

Dr. Changyong Zhong

Electronic Version Approved:

Office of Graduate Studies

College of Arts and Sciences

Georgia State University

May 2011

ACKNOWLEDGMENTS

I would like to thank Dr. Imre Patyi for all of the mathematical wisdom he has bestowed upon me during my two years here at GSU, for continually challenging me, and for guiding me expertly in the completion of this thesis. I also thank all of the other professors that have inspired and encouraged me: Dr. Changyong Zhong, Dr. Robert Clewley, and Dr. Draga Vidakovic of Georgia State University; Dr. Gerardo Lafferriere, Dr. Sean Larsen, and Dr. John Erdman of Portland State University; and Dr. Philip Demarois and Dr. Marshall Tuttle of Mt. Hood Community College. Each one has profoundly influenced the way that I approach math, math education, and/or life in general.

I would also like to thank my wife, Val, for supporting me through my transition from a music major to a math major, for understanding the time commitment that I have had to make to mathematics, and for helping me through the process of writing this thesis. I also thank my family, for motivating my pursuit of education and especially my grandparents for providing me with the financial means to do so.

Finally, I would like to thank Dr. Changyong Zhong and Dr. Alexandra Smirnova for agreeing to be a part of my thesis committee.

TABLE OF CONTENTS

1. INTRODUCTION

In this thesis we look at four basic topics about the beginnings of the theory of the Lebesgue integral, namely, (i) the definition and fundamental convergence theorems of the Lebesgue integral on Euclidean spaces, (ii) Fubini's theorem, (iii) items (i) and (ii) for the Lebesgue integral on abstract measure spaces, and (iv) the Riesz representation theorem (see [R]) for positive linear functionals on the space of continuous functions with compact support on locally compact Hausdorff spaces.

In the traditional development (e.g., [MW], [S]) of the above topics we have a heavy technical baggage of notions that we are obliged to carry around but are not really useful for purposes other than just building up the above. The build up is so long winded and markedly unpleasant that the analysis textbook [L] drops it entirely, and prefers to spend the more than half a semester's worth development of (i-iv) on more interesting topics. We give here a simple and minimalistic treatment that is more economical than the usual ones. It has neither measure, measurable sets, measurable functions, Borel classes of sets, Baire or Young classes of functions, transfinite methods, nor gymnastics with algebras of sets such as the monotone class lemma, π-systems, and λ-systems. We only rely on the simplest properties of the real line such as the supremum axiom and notions of elementary point set topology.

2. THE LEBESGUE INTEGRAL ON EUCLIDEAN SPACES

Definition. Let $X = \mathbb{R}^{d(X)}$ be the usual Euclidean space of dimension $d(X) \in \mathbb{N}$, where we denote the variable by $x = (x_1, x_2, ..., x_{d(X)})$. Let $\mathbb{Z}^{d(X)}$ be the integer lattice in X, i.e., $\mathbb{Z}^{d(X)} = \{x = (x_1, x_2, ..., x_{d(X)}) \in X : x_i \in \mathbb{Z} \text{ for } 1 \leq i \leq d(X)\}$. A finite interval in X is a subset of the form $I = \prod_{i=1}^{d(X)} I_i$ where each I_i is an interval in $\mathbb{R}$ with endpoints $a_i \leq b_i$ in $\mathbb{R}$. The elementary volume of an interval $I = \prod_{i=1}^{d(X)} I_i$ is $|I|_X = \prod_{i=1}^{d(X)}(b_i - a_i)$. For any set $A \subseteq X$, the indicator function of A is defined and denoted by

$$1_A(x) = \begin{cases} 1 & \text{if } x \in A \\ \\ 0 & \text{if } x \in X \backslash A. \end{cases}$$

Let K be the family of all functions $f : X \to \mathbb{R}$ of the form $f = \sum_{i=1}^{n} y_i 1_{I_i}$ where $y_i \in \mathbb{R}$ and $I_i \subset X$ is a finite interval for all i. Let K^+ denote the set of $f \in K$ for which $f \geq 0$ on X. Any $f \in K$ can be represented in terms of pairwise disjoint I_i; furthermore, if $f \in K^+$, then f can be represented in terms of nonnegative y_i. Note that K is a function lattice, i.e., if $f, g \in K$ and $c, d \in \mathbb{R}$, then $cf + dg \in K$ and $|f| \in K$.

Proposition 1. *For a finite interval $I \subset \mathbb{R}$ with endpoints $a \leq b$ in $\mathbb{R}$,*

$$b - a - 1 \leq \text{card}(\mathbb{Z} \cap I) \leq b - a + 1$$

where $\text{card}(A)$ *is the cardinality of the set A.*

Proof. If $b - a < 1$, then $\text{card}(\mathbb{Z} \cap I) \in \{0, 1\}$. So

$$b - a - 1 < 0 \leq \text{card}(\mathbb{Z} \cap I) \leq 1 \leq b - a + 1.$$

If $b - a = 1$, then $\text{card}(\mathbb{Z} \cap I) \in \{0, 1, 2\}$. So

$$b - a - 1 = 0 \leq \text{card}(\mathbb{Z} \cap I) \leq 2 = b - a + 1.$$

Now suppose $b - a > 1$. Then $\mathbb{Z} \cap (a,b) \neq \emptyset$, so we can let $N_1 = \text{card}(\mathbb{Z} \cap (a,b))$ and $n_1 = \min(\mathbb{Z} \cap (a,b))$. Then $n_1 + N_1 - 1 = \max(\mathbb{Z} \cap (a,b))$ and

$$b \leq n_1 + N_1 \leq a + 1 + N_1$$

so that $b - a - 1 \leq N_1$. Now let $N_2 = \text{card}(\mathbb{Z} \cap [a,b])$ and $n_2 = \min(\mathbb{Z} \cap [a,b])$. Then $n_2 + N_2 - 1 = \max(\mathbb{Z} \cap [a,b])$ and

$$b \geq n_2 + N_2 - 1 \geq a + N_2 - 1$$

so that $b - a + 1 \geq N_2$. Note that $N_1 \leq \text{card}(\mathbb{Z} \cap I) \leq N_2$, since $(a,b) \subseteq I \subseteq [a,b]$. Therefore,

$$b - a - 1 \leq N_1 \leq \text{card}(\mathbb{Z} \cap I) \leq N_2 \leq b - a + 1. \qquad \square$$

Definition. Define a positive linear functional $\xi : K \to \mathbb{R}$ by

$$\xi(f) = \lim_{H \to \infty} H^{-d(X)} \sum_{n \in \mathbb{Z}^{d(X)}} f(n/H).$$

Proposition 2. *The following hold:*

(1) *If $I \subset X$ is a finite interval, then $\xi(1_I) = |I|_X$.*

(2) *If $f = \sum_{i=1}^{k} y_i I_i \in K$, then $\xi(f) = \sum_{i=1}^{k} y_i |I_i|_X$.*

(3) *$\xi : K \to \mathbb{R}$ is indeed a positive linear functional, i.e. $\xi(cf + dg) = c\xi(f) + d\xi(g)$ for $c, d \in \mathbb{R}$ and $|\xi(f)| \leq \xi(|f|)$.*

Proof. Parts (2) and (3) follow easily from the definition of ξ once we have established (1). To prove part (1), let $I = \prod_{i=1}^{d(X)} I_i$, where the $I_i \subset \mathbb{R}$ have endpoints $a_i \leq b_i$. Note that

$$\begin{aligned} H^{-d(X)} \sum_{n \in \mathbb{Z}^{d(X)}} 1_I(n/H) &= H^{-d(X)} \text{card}((H^{-1}\mathbb{Z}^{d(X)}) \cap I) \\ &= H^{-d(X)} \text{card}(\mathbb{Z}^{d(X)} \cap (HI)) \\ &= H^{-d(X)} \prod_{i=1}^{d(X)} \text{card}(\mathbb{Z} \cap (HI_i)). \end{aligned}$$

$$\le H^{-d(X)} \prod_{i=1}^{d(X)} (Hb_i - Ha_i + 1)$$

$$= \prod_{i=1}^{d(X)} (b_i - a_i + \frac{1}{H})$$

$$\searrow |I|_X$$

as $H \to \infty$. Similarly,

$$H^{-d(X)} \sum_{n \in \mathbb{Z}^{d(X)}} 1_I(n/H) = H^{-d(X)} \prod_{i=1}^{d(X)} \operatorname{card}(\mathbb{Z} \cap (HI_i))$$

$$\ge H^{-d(X)} \prod_{i=1}^{d(X)} (Hb_i - Ha_i - 1)$$

$$= \prod_{i=1}^{d(X)} (b_i - a_i - \frac{1}{H})$$

$$\nearrow |I|_X$$

as $H \to \infty$. So $\xi(1_I) = \lim_{H\to\infty} H^{-d(X)} \sum_{n\in\mathbb{Z}^{d(X)}} 1_I(\frac{n}{H}) = |I|_X$. □

Definition. For $f : X \to [0, \infty]$, let

$$\xi'(f) = \inf\Big\{ \sum_{n=1}^{\infty} \xi(f_n) : f_n \in K^+, f \le \sum_{n=1}^{\infty} f_n \text{ on } X \Big\}.$$

The sums in this definition converge (possibly to infinity) since all terms are positive. The set to which the infimum is applied is nonempty since we have $f \le \sum_{n=1}^{\infty} 1_{[-n,n]^{d(X)}}$ on X for all $f : X \to [0, \infty]$. So the value of $\xi'(f)$ is well-defined and nonnegative.

Proposition 3. *Let $f, f_n : X \to [0, \infty]$ and $g : X \to \mathbb{R}$. The following properties hold:*

(1) *If $c \in [0, \infty)$, then $\xi'(cf) = c\xi'(f)$. In particular, $\xi'(0) = 0$.*

(2) *If $f_1 \le f_2$ on X, then $\xi'(f_1) \le \xi'(f_2)$.*

(3) *(Markov's inequality) If $c \in (0, \infty)$, then $c\xi'(1_{\{|g|\ge c\}}) \le \xi'(|g|)$.*

(4) *If $f \le \sum_{n=1}^{\infty} f_n$ on X, then $\xi'(f) \le \sum_{n=1}^{\infty} \xi'(f_n)$.*

Proof. Part (1) is clear. Part (2) follows directly from the definition of ξ', since any series of functions in K^+ that dominates f_2 also dominates f_1. For part (3), we apply parts (1) and (2): $c\xi'(1_{\{|g|\geq c\}}) = \xi'(c1_{\{|g|\geq c\}}) \leq \xi'(|g|)$.

For part (4), if $\sum_{n=1}^{\infty} \xi'(f_n) = \infty$, then there is nothing to show. So suppose $\sum_{n=1}^{\infty} \xi'(f_n) < \infty$. Let $\varepsilon > 0$ and, for each n, choose $\{g_{nk}\}_{k=1}^{\infty} \subset K^+$ such that $f_n \leq \sum_{k=1}^{\infty} g_{nk}$ on X and $\sum_{k=1}^{\infty} \xi(g_{nk}) < \xi'(f_n) + 2^{-n}\varepsilon$. Then $f \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} g_{nk}$ and

$$\xi'(f) \leq \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \xi(g_{nk}) < \sum_{n=1}^{\infty} (\xi'(f_n) + 2^{-n}\varepsilon) = \sum_{n=1}^{\infty} \xi'(f_n) + \varepsilon.$$

Letting $\varepsilon \searrow 0$, we obtain $\xi'(f) \leq \sum_{n=1}^{\infty} \xi'(f_n)$. □

Definition. Let X be a topological space and $f : X \to \mathbb{R}$. We say that f is lower semicontinuous on X if $\{f > c\}$ is open in X for all $c \in \mathbb{R}$.

Proposition 4. *Let X be a topological space. The following hold:*

(1) *If $f : X \to \mathbb{R}$ is lower semicontinuous, then so is af for $a \in [0, \infty)$.*

(2) *If $f, g : X \to \mathbb{R}$ are lower semicontinuous, then so is $f + g$.*

(3) *If $G \subset X$ is open, then 1_G is lower semicontinuous.*

(4) *If $F \subset X$ is closed then -1_F is lower semicontinuous.*

(5) *If $f : X \to \mathbb{R}$ is lower semicontinuous and $f(x_0) > 0$ for an $x_0 \in X$, then there is an open neighborhood of U of x_0 such that $f > 0$ on U.*

Proposition 5. *If $f, f_n \in K^+$ for $n \geq 1$ and $f \leq \sum_{n=1}^{\infty} f_n$ on X, then for any $\varepsilon > 0$ there exists $N \geq 1$ such that $\xi(f) \leq \varepsilon + \sum_{n=1}^{N} \xi(f_n)$.*

The condition in the conclusion of Proposition 5 is, of course, equivalent to $\xi(f) \leq \sum_{n=1}^{\infty} \xi(f_n)$, which expresses a form of σ-subadditivity of ξ, and is the crucial point for all of our further development here.

Proof. Choose $M > 0$ such that $f \leq M$ on X and $f = 0$ on $X \backslash [-M, M]^{d(X)}$. Let $\varepsilon > 0$ and $\varepsilon' = \varepsilon / [(2M+2)^{d(X)} + M + 1])$. Write f and f_n as

$$f = \sum_{i=1}^{m} y_i 1_{I_i} \qquad f_n = \sum_{i=1}^{m_n} y_{ni} 1_{I_{ni}}$$

where $y_i, y_{ni} \geq 0$ and define

$$F = \sum_{i=1}^{m} y_i 1_{\overline{I_i}} \qquad F_n = \sum_{i=1}^{m_n} y_i 1_{J_{ni}}$$

where each J_{ni} is an open set that covers $\overline{I_{ni}}$ and $\xi(F_n) \leq \xi(f_n) + \varepsilon'/2^n$. This is possible since the elementary volume of an interval varies continuously with its edge lengths, as the explicit product formula shows. Since each ∂I_i is a null set, we can choose open intervals $\{G_{ik}\}_{k=1}^{K_i}$ such that $\bigcup_{k=1}^{K_i} G_{ik} \supseteq \partial I_i$ and $\sum_{i=1}^{m} \sum_{k=1}^{K_i} |G_{ik}|_X < \varepsilon'$. Let

$$G = \sum_{i=1}^{m} \sum_{k=1}^{K_i} 1_{G_{ik}}$$

$$Q = (-M-1, M+1)^{d(X)}$$

$$H_N = \varepsilon' 1_Q - F + MG + \sum_{n=1}^{N} F_n$$

$$H = \varepsilon' 1_Q - F + MG + \sum_{n=1}^{\infty} F_n.$$

Note that $H > 0$ on $C = [-M, M]^{d(X)}$, since $F \leq MG + \sum_{n=1}^{\infty} F_n$ on X and $C \subset Q$. For each $x \in C$, we can choose $N(x) \in \mathbb{N}$ such that

$$H_{N(x)}(x) = \varepsilon' 1_Q(x) - F(x) + MG(x) + \sum_{n=1}^{N(x)} F_n(x) > 0.$$

As $H_{N(x)}$ is lower semicontinuous, we can choose an open neighborhood $U(x)$ of x such that $H_{N(x)} = \varepsilon' 1_Q - F + MG + \sum_{n=1}^{N(x)} F_n > 0$ on $U(x)$. Since $\{U(x)\}_{x \in C}$ is an open cover of the compact set C, we can choose a finite subcover $\{U(x_i)\}_{i=1}^{s}$ of C. Fix $N = \max\{N(x_i)\}_{i=1}^{s}$. Then $H_N > 0$ on C and

$$0 \leq \xi(H_N) = \varepsilon' \xi(1_Q) - \xi(F) + M\xi(G) + \sum_{n=1}^{N} \xi(F_n).$$

So

$$
\begin{aligned}
\xi(f) &= \xi(F) \\
&\le \varepsilon'\xi(1_Q) + M\xi(G) + \sum_{n=1}^{N} \xi(F_n) \\
&\le \varepsilon'(2M+2)^{d(X)} + M\varepsilon' + \sum_{n=1}^{N} (\xi(f_n) + \varepsilon'/2^n) \\
&\le \varepsilon'[(2M+2)^{d(X)} + M + 1] + \sum_{n=1}^{N} \xi(f_n) \\
&= \varepsilon + \sum_{n=1}^{N} \xi(f_n). \qquad \square
\end{aligned}
$$

Proposition 6. *If $f \in K^+$, then $\xi'(f) = \xi(f)$.*

Proof. Letting $f_1 = f$ and $f_n = 0$ for $n \ge 2$, we see that $\xi'(f) \le \sum_{n=1}^{\infty} \xi(f_n) = \xi(f)$ by the definition of ξ'. To show the opposite inequality, let $\{f_n\}_{n=1}^{\infty} \subset K^+$ with $f \le \sum_{n=1}^{\infty} f_n$ on X. For $\varepsilon > 0$, we can choose $N \ge 1$ such that $\xi(f) \le \varepsilon + \sum_{n=1}^{N} \xi(f_n)$. Letting $N \to \infty$ and $\varepsilon \searrow 0$, we have $\xi(f) \le \sum_{n=1}^{\infty} \xi(f_n)$. Therefore, $\xi(f) \le \xi'(f)$. □

Definition. Let K_1 be the set of functions $f : X \to [-\infty, \infty]$ for which there is a sequence $\{f_n\}_{n=1}^{\infty} \subset K$ with $\xi'(|f - f_n|) \to 0$ as $n \to \infty$. Let $K_1^+ = \{f \in K_1 : f \ge 0 \text{ on } X\}$.

Definition. The positive and negative parts of a number $x \in \mathbb{R}$ are denoted and defined by

$$
x^+ = \frac{1}{2}(|x| + x),
$$

$$
x^- = \frac{1}{2}(|x| - x),
$$

and satisfy

$$
\begin{aligned}
(-x)^+ &= x^-, & |x^+ - y^+| &\le |x - y|, \\
(-x)^- &= x^+, & |x^- - y^-| &\le |x - y|, \\
|x| &= x^+ + x^-, & \big||x| - |y|\big| &\le |x - y|.
\end{aligned}
$$

Proposition 7. *The following hold:*

(1) *If* $f \in K_1^+$*, then there is a sequence* $\{g_n\}_{n=1}^{\infty} \subset K^+$ *with* $\xi'(|f-g_n|) \to 0$ *as* $n \to \infty$*.*

(2) *If* $f, g \in K_1$ *and* $c, d \in [0, \infty)$*, then* $cf + dg \in K_1$*.*

(3) *If* $f \in K_1$*, then* $|f| \in K_1^+$*.*

(4) *If* $f_1, f_2 \in K_1$*, then* $f_1 \wedge f_2 \in K_1$ *and* $f_1 \vee f_2 \in K_1$*.*

(5) *If* $f \in K_1^+$*, then* $\frac{f}{1+f} \in K_1^+$*.*

Proof. For part (1), choose $\{f_n\}_{n=1}^{\infty} \subset K$ with $\xi'(|f-f_n|) \to 0$ as $n \to \infty$. Let $g_n = f_n^+ \in K^+$. Then

$$\xi'(|f-g_n|) = \xi'(|f^+ - f_n^+|) \leq \xi'(|f-f_n|) \searrow 0$$

as $n \to \infty$.

For parts (2) and (3), choose $\{f_n\}_{n=1}^{\infty}, \{g_n\}_{n=1}^{\infty} \subset K$ such that $\xi'(|f-f_n|) \to 0$ as $n \to \infty$ and $\xi'(|g-g_n|) \to 0$ as $n \to \infty$. Then $cf_n + dg_n \in K$ and

$$\begin{aligned} \xi'(|cf+dg-(cf_n-dg_n)|) &\leq \xi'(c|f-f_n| + d|g-g_n|) \\ &\leq c\xi'(|f-f_n|) + d\xi'(|g-g_n|) \\ &\searrow 0 \end{aligned}$$

as $n \to \infty$. So $cf + dg \in K_1$. Also, $|f_n| \in K$ and

$$\xi'\big(\big||f| - |f_n|\big|\big) \leq \xi'(|f-f_n|) \searrow 0$$

as $n \to \infty$. So $|f| \in K_1^+$.

For part (4), note that

$$\begin{aligned} f_1 \wedge f_2 &= \frac{1}{2}(f_1 + f_2 - |f_1 - f_2|) \in K_1 \\ f_1 \vee f_2 &= \frac{1}{2}(f_1 + f_2 + |f_1 - f_2|) \in K_1. \end{aligned}$$

For part (5), choose $\{f_n\}_{n=1}^{\infty} \in K$ with $\xi'(|f - f_n|) \to 0$ as $n \to \infty$. Then $\frac{f_n}{1+f_n} \in K$ and

$$\xi'\Big(\Big|\frac{f}{1+f} - \frac{f_n}{1+f_n}\Big|\Big) = \xi'\Big(\Big|\frac{f - f_n}{(1+f)(1+f_n)}\Big|\Big) \leq \xi'(|f - f_n|) \searrow 0$$

as $n \to \infty$. □

Theorem 8. *(Monotone convergence theorem) The following hold:*

(1) If $f_n \in K_1^+$ for $n \geq 1$ and $\sum_{n=1}^{\infty} \xi'(f_n) < \infty$, then $\xi'(\sum_{n=1}^{\infty} f) = \sum_{n=1}^{\infty} \xi'(f_n)$ and $\sum_{n=1}^{\infty} f_n \in K_1^+$.

(2) If $f_n \in K_1^+$ for $n \geq 1$, $f_n \nearrow f$ pointwise on X as $n \to \infty$, and $\lim_{n\to\infty} \xi'(f_n) < \infty$, then $\xi'(f_n) \nearrow \xi'(f)$ as $n \to \infty$ and $f \in K_1^+$.

Proof. For part (1), note that $\xi'(\sum_{n=1}^{\infty} f_n) \leq \sum_{n=1}^{\infty} \xi'(f_n)$ by Proposition 3. To show the opposite inequality, let $\varepsilon > 0$ and, for each $n \geq 1$, choose $g_n \in K^+$ such that $\xi'(|f_n - g_n|) < \varepsilon/2^{n+1}$. Then

$$\begin{aligned}
\sum_{n=1}^{N} \xi'(f_n) &\leq \sum_{n=1}^{N} \xi'(g_n + |f_n - g_n|) \\
&\leq \sum_{n=1}^{N} \xi'(g_n) + \sum_{n=1}^{N} \xi'(|f_n - g_n|) \\
&\leq \sum_{n=1}^{N} \xi(g_n) + \sum_{n=1}^{\infty} \xi'(|f_n - g_n|) \\
&< \xi\Big(\sum_{n=1}^{N} g_n\Big) + \frac{\varepsilon}{2} \\
&= \xi'\Big(\sum_{n=1}^{N} g_n\Big) + \frac{\varepsilon}{2} \\
&\leq \xi'(\sum_{n=1}^{\infty} g_n) + \frac{\varepsilon}{2} \\
&\leq \xi'\Big(\sum_{n=1}^{\infty} (f_n + |f_n - g_n|)\Big) + \frac{\varepsilon}{2}
\end{aligned}$$

$$\leq \xi'\Big(\sum_{n=1}^{\infty} f_n\Big) + \sum_{n=1}^{\infty} \xi'(|f_n - g_n|) + \frac{\varepsilon}{2}$$

$$< \xi'\Big(\sum_{n=1}^{\infty} f_n\Big) + \varepsilon.$$

Letting $N \to \infty$ and $\varepsilon \searrow 0$, we have $\sum_{n=1}^{\infty} \xi'(f_n) \leq \xi'(\sum_{n=1}^{\infty} f_n)$.

Fix $N \geq 1$ such that $\sum_{n=N+1}^{\infty} \xi'(f_n) < \varepsilon/2$. Then $\sum_{n=1}^{N} g_n \in K^+$ and

$$\xi'\Big(\Big|\sum_{n=1}^{\infty} f_n - \sum_{n=1}^{N} g_n\Big|\Big) \leq \sum_{n=1}^{N} \xi'(|f_n - g_n|) + \sum_{n=N+1}^{\infty} \xi'(f_n) < \varepsilon.$$

Therefore $\sum_{n=1}^{\infty} f_n \in K_1^+$.

For part (2), we have $f_k - f_{k-1} \in K_1^+$ for each $k \geq 1$ (with $f_0 = 0$) by Proposition 7. So

$$\xi'(f_n) = \xi'\Big(\sum_{k=1}^{n} (f_k - f_{k-1})\Big)$$

$$= \sum_{k=1}^{n} \xi'(f_k - f_{k-1})$$

$$\nearrow \sum_{k=1}^{\infty} \xi'(f_k - f_{k-1})$$

$$= \xi'\Big(\sum_{k=1}^{\infty} (f_k - f_{k-1})\Big)$$

$$= \xi'(f)$$

and $f = \sum_{k=1}^{\infty} (f_k - f_{k-1}) \in K_1^+$ by part (1). □

Definition. If $f : X \to [-\infty, \infty]$ and $\xi'(f^+)$ or $\xi'(f^-)$ is finite, then we can define $\xi'(f) = \xi'(f^+) - \xi'(f^-)$. Clearly $\xi'(cf) = c\xi'(f)$ for $c \in \mathbb{R}$ whenever either side is defined.

We call a $f : X \to [-\infty, \infty]$ a null function if $\xi'(|f|) = 0$ and $A \subset X$ a null set if 1_A is a null function. We say that two functions f and g are equal almost everywhere if $\{x \in X : f(x) \neq g(x)\}$ is a null set. In this case, it is clear that $\xi'(f) = \xi'(g)$, $\xi'(f^+) = \xi'(g^+)$, $\xi'(f^-) = \xi'(g^-)$, and $\xi'(|f|) = \xi'(|g|)$.

If $A \subseteq B \subset X$ and B is a null set, then A is a null set, since $1_A \leq 1_B$ on X. Also, if $\xi'(|f|) < \infty$, then $A = \{|f| = \infty\}$ is a null set, since $1_A \leq 1_{\{|f| \geq n\}}$ for all n and $0 \leq \xi'(1_A) \leq \xi'(1_{\{|f| \geq n\}}) \leq \frac{1}{n}\xi'(|f|) \searrow 0$ as $n \to \infty$ by Proposition 3.

For $f \in K_1$, let

$$\hat{f}(x) = \begin{cases} f(x) & \text{if } |f(x)| < \infty \\ \\ 0 & \text{if } |f(x)| = \infty. \end{cases}$$

Note that $\hat{f} = f$ almost everywhere. For $f, g \in K_1$, we can define $f + g = \hat{f} + \hat{g}$.

Proposition 9. *If $f_n \in K_1^+$ for $n \geq 1$, then $\inf_{n \geq 1} f_n \in K_1^+$.*

Proof. Let $f_k = f_1 - \bigwedge_{n=1}^{k} f_n$. Then $f_k \nearrow f_1 - \inf_{n \geq 1} f_n$. By theorem 8, $f_1 - \inf_{n \geq 1} f_n \in K_1^+$. So $\inf_{n \geq 1} f_n = f_1 - (f_1 - \inf_{n \geq 1} f_n) \in K_1^+$. □

Theorem 10. *(Fatou's lemma) The following hold:*

(1) *If $f_n \in K_1^+$ for $n \geq 1$, then $\xi'(\liminf_{n \to \infty} f_n) \leq \liminf_{n \to \infty} \xi'(f_n)$. If, furthermore,* $\liminf_{n \to \infty} \xi'(f_n) < \infty$, then $\liminf_{n \to \infty} f_n \in K_1^+$.

(2) *If $f_n, g \in K_1^+$ and $g \geq f_n$ for $n \geq 1$, then $\limsup_{n \to \infty} \xi'(f_n) \leq \xi'(\limsup_{n \to \infty} f_n)$.*

Proof. For part (1), if $\liminf_{n \to \infty} \xi'(f_n) = \infty$, then the inequality obviously holds. So suppose that $\liminf_{n \to \infty} \xi'(f_n) < \infty$. Let $g_n = \inf_{k \leq n} f_k$. Note that $g_n \in K_1^+$ and $g_n \leq f_n$ for all n and $g_n \nearrow \liminf_{n \to \infty} f_n$ as $n \to \infty$. Also,

$$\lim_{n \to \infty} \xi'(g_n) = \liminf_{n \to \infty} \xi'(g_n) \leq \liminf_{n \to \infty} \xi'(f_n) < \infty.$$

So by Theorem 8,

$$\xi'(\liminf_{n \to \infty} f_n) = \xi'(\lim_{n \to \infty} g_n) = \lim_{n \to \infty} \xi'(g_n) \leq \liminf_{n \to \infty} \xi'(f_n)$$

and $\liminf_{n \to \infty} f_n = \lim_{n \to \infty} g_n \in K_1^+$.

For part (2), let $F_n = g - f_n$. Then

$$\xi'(g) = \xi'(\liminf_{n\to\infty} F_n + \limsup_{n\to\infty} f_n) \le \xi'(\liminf_{n\to\infty} F_n) + \xi'(\limsup_{n\to\infty} f_n)$$

and, by Theorem 8

$$\xi'(g) \quad = \quad \xi'(F_n + f_n) = \xi'(F_n) + \xi'(f_n)$$

Therefore

$$\begin{aligned}\limsup_{n\to\infty} \xi'(f_n) &= \limsup_{n\to\infty}(\xi'(g) - \xi'(F_n)) \\ &= \xi'(g) - \liminf_{n\to\infty} \xi'(F_n) \\ &\le \xi'(g) - \xi'(\liminf_{n\to\infty} F_n) \\ &\le \xi'(\limsup_{n\to\infty} f_n).\end{aligned}$$

□

Lemma. *If $f_1, f_2 \in K_1$, then $|\xi'(f_1) - \xi'(f_2)| \le \xi'(|f_1 - f_2|)$.*

Proof. First, we show that this result holds for $f_1, f_2 \in K_1^+$. Note that $\xi'(f_1) \le \xi'(|f_1 - f_2|) + \xi'(|f_2|)$. So $\xi'(f_1) - \xi'(f_2) \le \xi'(|f_1 - f_2|)$ and, similarly, $\xi'(f_2) - \xi'(f_1) \le \xi'(|f_1 - f_2|)$. Therefore, $|\xi'(f_1) - \xi'(f_2)| \le \xi'(|f_1 - f_2|)$ for $f_1, f_2 \in K_1^+$.

Now let $f_1, f_2 \in K_1$. Then $f_1^+ + f_2^- \in K_1^+$ and $f_1^- + f_2^- \in K_1^+$. So

$$\begin{aligned}|\xi'(f_1) - \xi'(f_2)| &= |\xi'(f_1^+) - \xi'(f_1^-) - \xi'(f_2^+) + \xi'(f_2^-)| \\ &= |\xi'(f_1^+ + f_2^-) - \xi'(f_1^- + f_2^-)| \\ &\le \xi'(|(f_1^+ + f_2^-) - (f_1^- + f_2^-)|) \\ &= \xi'(|f_1 - f_2|).\end{aligned}$$

□

Theorem 11. *(Dominated convergence theorem) Let $f_n, g \in K_1$ with $|f_n| \le g$ for all $n \ge 1$. If $f_n \to f$ pointwise on X, then $f \in K_1$, $\xi'(|f - f_n|) \to 0$ as $n \to \infty$, and $\xi'(f_n) \to \xi'(f)$ as $n \to \infty$.*

Proof. Note that $g \pm f_n \in K_1^+$. Also, $g \pm f = \liminf_{n\to\infty} g \pm f \in K_1^+$ by Theorem 10. Therefore, $f = \frac{1}{2}(g+f) - \frac{1}{2}(g-f) \in K_1$. Next, note that $|f - f_n| \in K_1^+$ and $|f - f_n| \leq 2g$ for each n. So

$$\lim_{n\to\infty} \xi'(|f - f_n|) = \limsup_{n\to\infty} \xi'(|f - f_n|) \leq \xi'(\limsup_{n\to\infty} |f - f_n|) = \xi'(0) = 0,$$

and by the previous lemma, $|\xi'(f) - \xi'(f_n)| \leq \xi'(|f - f_n|) \to 0$ as $n \to \infty$. □

Theorem 12. *(Bounded convergence theorem) If $A \subset X$, $1_A \in K_1^+$, $f_n \in K_1$, and $|f_n| \leq M1_A$ for some $M < \infty$ and all $n \geq 1$, and $f_n \to f$ pointwise on X, then $f \in K_1$ and $\xi'(|f - f_n|) \to 0$ as $n \to \infty$, and $\xi'(f_n) \to \xi'(f)$ as $n \to \infty$.*

Proof. It follows directly from the previous theorem with $g = M1_A$. □

Definition. Define $K_1(\text{loc})$ to be the set of all functions $f : X \to [-\infty, \infty]$ that such that $f1_{Q_n} \in K_1$ for all cubes $Q_n = [-n, n]^{d(X)}$ with $n \geq 1$. Let $K_1^+(\text{loc}) = \{f \in K_1^+(\text{loc}) : f \geq 0 \text{ on } X\}$. Let $\mathfrak{A}$ be the set of all $A \subset X$ with $1_A \in K_1^+(\text{loc})$.

Proposition 13. *If $f : X \to \mathbb{R}$ is continuous, then $f \in K_1(\text{loc})$.*

Proof. For $k \geq 1$, let $f_k = f(\frac{1}{k}\lfloor kx_1 \rfloor, \frac{1}{k}\lfloor kx_2 \rfloor, \ldots, \frac{1}{k}\lfloor kx_{d(X)} \rfloor)$ where $\lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\}$. Fix $n \in \mathbb{N}$. Note that $f_k 1_{Q_n} \in \mathcal{A}$ for all k and $f_k 1_{Q_n} \to f1_{Q_n}$. So by Theorem 12, $f1_{Q_n} \in K_1$. Therefore, $f \in K_1(\text{loc})$. □

Proposition 14. *The following hold:*

(1) *$\mathfrak{A}$ is a σ-algebra on X.*

(2) *If $f \in K_1(\text{loc})$, then $\{f > 0\} = \{x \in X : f(x) > 0\} \in \mathfrak{A}$. Also, for $c \in \mathbb{R}$,*

$$\{f > c\}, \{f \geq c\}, \{f < c\}, \{f \leq c\}, \{f = \infty\}, \{f = -\infty\} \in \mathfrak{A}$$

(3) $\mathfrak{A}$ *contains all open sets and all closed sets in* X *and, therefore, all Borel sets in* X *since it is a* σ*-algebra on* X.

Proof. For part (1), note that $\emptyset \in \mathfrak{A}$, since $1_\emptyset = 0 \in K_1^+(loc)$. If $A \in \mathfrak{A}$, then $X \backslash A \in \mathfrak{A}$ since

$$1_{X\backslash A}1_{Q_n} = 1_{Q_n} - 1_A 1_{Q_n} \in K_1^+$$

for all n. If $\{A_k\}_{k=1}^\infty \subset \mathfrak{A}$, then $A = \bigcup_{k=1}^\infty A_k \in \mathfrak{A}$, since

$$1_A 1_{Q_n} = 1_{Q_n} - 1_{X\backslash A}1_{Q_n} = 1_{Q_n} - \inf_{k\geq 1} 1_{X\backslash A_k}1_{Q_n} \in K_1^+.$$

For part (2), fix $n \geq 1$ and note that

$$1_{\{f>0\}}1_{Q_n} = \lim_{k\to\infty} \frac{kf^+}{1+kf^+}1_{Q_n} \in K_1^+$$

by Theorem 11, since $|\frac{kf^+}{1+kf^+}1_{Q_n}| \leq f^+ \in \mathcal{A}^+$. Therefore, $\{f > 0\} \in \mathfrak{A}$. The other sets are in $\mathfrak{A}$ since $\mathfrak{A}$ is a σ-algebra and $f \in K_1(\text{loc})$ if and only if $\pm f \pm c \in K_1(\text{loc})$.

For part (3), it is sufficient to show that any open set $\mathcal{O} \subseteq X$ is in $\mathfrak{A}$. Let $f(x) = d(x, X\backslash\mathcal{O}) = \inf_{y\in\mathcal{O}} |x-y|$. Then $f \in K_1^+(\text{loc})$, since f is continuous. Therefore, $\mathcal{O} = \{f > 0\} \in \mathfrak{A}$. □

Proposition 15. *The following hold:*

(1) *If* $f : X \to [0,\infty]$ *and* $\xi'(f) < \infty$*, then there exists* $F \in K_1^+$ *such that* $f \leq F$ *and* $\xi'(f) = \xi'(F)$*. We call* F *a hull of* f*.*

(2) *If* $f_n, f : X \to [0,\infty]$ *and* $f_n \nearrow f$ *pointwise on* X *an* $n \to \infty$*, then* $\xi'(f_n) \nearrow \xi'(f)$ *as* $n \to \infty$*.*

Proof. For part (1), choose $\{g_{nk}\}_{k=1}^\infty \subset K_1^+$ for each $n \geq 1$ such that $f \leq f_n = \sum_{k=1}^\infty g_{nk}$ on X and $\xi'(f) \leq \sum_{k=1}^\infty \xi(g_{nk}) \leq \xi'(f) + 1/n$. Note that $f_n \in K_1^+$ by Theorem 8 and $\xi'(f) \leq \xi'(f_n) \leq \sum_{k=1}^\infty \xi(g_{nk}) < \xi'(f) + 1/n$ for each $n \geq 1$. By Theorem 10, $F = \liminf_{n\to\infty} f_n \in K_1^+$

and

$$\xi'(F) \le \liminf_{n\to\infty} \xi'(f_n) \le \liminf_{n\to\infty} (\xi'(f) + \frac{1}{n}) = \xi'(f).$$

Also, $\xi'(f) \le \xi'(F)$ since $f \le F$. Therefore, $\xi'(f) = \xi'(F)$.

For part (2), clearly $\xi'(f_n) \nearrow L \le \xi'(f)$ as $n \to \infty$ for some $L \in [0, \infty]$. We must show that $\xi'(f) \le L$. Without loss of generality, suppose $L < \infty$. By part (1), we can choose $F_n \in K_1^+$ such that $f_n \le F_n$, and $\xi'(f_n) = \xi'(F_n)$. Then

$$\xi'(f) \le \xi'(\liminf_{n\to\infty} F_n) \le \liminf_{n\to\infty} \xi'(F_n) = \liminf_{n\to\infty} \xi'(f_n) = L. \qquad \square$$

Theorem 16. *(Riesz-Fischer theorem) If $f_n \in K_1$ for $n \ge 1$ and for each $\varepsilon > 0$ there exists N such that $\xi'(|f_n - f_m|) < \varepsilon$ for all $n, m \ge N$, then there exists $f \in K_1$ such that $\xi'(|f - f_N|) \to 0$ as $N \to \infty$.*

Proof. At first, let us suppose that $\sum_{n=1}^{\infty} \xi'(|f_n - f_{n-1}|) < \infty$ (with $f_0 = 0$). By Theorem 8,

$$f = \sum_{n=1}^{\infty} (f_n - f_{n-1})^+ - \sum_{n=1}^{\infty} (f_n - f_{n-1})^- \in K_1$$

Then

$$\begin{aligned}
\xi'(|f - f_N|) &= \xi'\Big(\Big|\sum_{n=1}^{\infty}(f_n - f_{n-1})^+ - \sum_{n=1}^{\infty}(f_n - f_{n-1})^- - \sum_{n=1}^{N}(f_n - f_{n-1})\Big|\Big) \\
&= \xi'\Big(\Big|\sum_{n=N+1}^{\infty}(f_n - f_{n-1})^+ - \sum_{n=N+1}^{\infty}(f_n - f_{n-1})^-\Big|\Big) \\
&\le \xi'\Big(\sum_{n=N+1}^{\infty}(f_n - f_{n-1})^+ + \sum_{n=N+1}^{\infty}(f_n - f_{n-1})^-\Big) \\
&\le \sum_{n=N+1}^{\infty} \xi'((f_n - f_{n-1})^+) + \sum_{n=N+1}^{\infty} \xi'((f_n - f_{n-1})^-) \\
&= \sum_{n=N+1}^{\infty} \xi'(|f_n - f_{n-1}|) \\
&\searrow 0
\end{aligned}$$

as $N \to \infty$.

Now, if $\sum_{n=1}^{\infty} \xi'(|f_n - f_{n-1}|) = \infty$, then we can choose a subsequence (using the Cauchy property) such that $\sum_{k=1}^{\infty} \xi'(|f_{n(k)} - f_{n(k-1)}|) < \infty$. Then

$$\xi'(|f - f_N|) \leq \xi'(|f - f_{n(N)}|) + \xi'(|f_{n(N)} - f_N|) \searrow 0$$

as $N \to \infty$. $\square$

Definition. Let X, Y, and $Z = X \times Y$ be Euclidean spaces with ξ, η, and ζ set up so that $\zeta(1_{A\times B}) = \xi(1_A)\eta(1_B)$ for finite intervals $A \subset X$, $B \subset Y$, and $A \times B \subset Z$. We use the notations $K(X)$, $K_1(X)$, $K_1^+(X)$,... and similarly for Y and Z. For $f : Z \to [-\infty, \infty]$ and $y \in Y$, define $[f]^y : X \to [-\infty, \infty]$ by $[f]^y(x) = f(x, y)$. Define $[f]_x$ similarly. If $f \in K(Z)$, we can interpret $\xi(f)$ as a function of $y \in Y$, i.e., $\xi(f) : Y \to [-\infty, \infty]$ defined by $\xi(f)(y) = \xi([f]^y)$. The same convention holds for $\eta(f)$, $\xi'(f)$ and $\eta'(f)$. With this convention, compositions such as $(\xi\eta)(f) = \xi(\eta(f))$ make sense.

Proposition 17. *The following hold:*

(1) *If $f \in K(Z)$, then $\eta(f) \in K(X)$, $\xi(f) \in K(Y)$, and $\zeta(f) = (\xi\eta)(f) = (\eta\xi)(f)$.*

(2) *If $f : Z \to [0, \infty]$, then $(\xi'\eta')(f) \leq \zeta'(f)$ and $(\eta'\xi')(f) \leq \zeta'(f)$.*

(3) *If $g : X \to [0, \infty]$, $h : Y \to [0, \infty]$, and $f : Z \to [0, \infty]$ with $f(x, y) = g(x)h(y)$ for $(x, y) \in Z$, then $\zeta'(f) \leq \xi'(g)\eta'(h)$ where $0 \cdot \infty = \infty \cdot 0 = 0$.*

(4) *In fact, given the conditions in part (3), $\zeta'(f) = \xi'(g)\eta'(h)$. In particular, $\zeta'(1_{A\times B}) = \xi'(1_A)\eta'(1_B)$ for any $A \subseteq X$ and $B \subseteq Y$.*

Proof. For part (1), Let $f = \sum_{i=1}^{n} u_i 1_{A_i \times B_i}$ where $A_i \subset X$ and $B_i \subset Y$ are finite interval and $u_i \in \mathbb{R}$. Then

$$\eta(f) = \eta\Big(\sum_{i=1}^{n} u_i 1_{A_i} 1_{B_i}\Big) = \sum_{i=1}^{n} u_i 1_{A_i} \eta(1_{B_i}) = \sum_{i=1}^{n} u_i |B_i|_Y 1_{A_i} \in K(X)$$

and

$$
\begin{aligned}
(\xi\eta)(f) &= \xi\Big(\sum_{i=1}^{n} u_i|B_i|_Y 1_{A_i}\Big) \\
&= \sum_{i=1}^{n} u_i|B_i|_Y \xi(1_{A_i}) \\
&= \sum_{i=1}^{n} u_i|A_i|_X|B_i|_Y \\
&= \sum_{i=1}^{n} u_i|A_i \times B_i|_Z \\
&= \zeta(f)
\end{aligned}
$$

The proof of the other claims in part (1) is directly analogous.

For part (2), if $\zeta'(f) = \infty$, then there is nothing to prove. So suppose $\zeta'(f) < \infty$, let $\varepsilon > 0$, and choose $\{f_n\}_{n=1}^{\infty} \subset K^+(Z)$ with $f \le \sum_{n=1}^{\infty} f_n$ on Z and $\sum_{n=1}^{\infty} \zeta(f_n) \le \zeta'(f) + \varepsilon$. Then

$$
\begin{aligned}
(\xi'\eta')(f) &\le (\xi'\eta')\Big(\sum_{n=1}^{\infty} f_n\Big) \\
&\le \sum_{n=1}^{\infty} (\xi'\eta')(f_n) \\
&= \sum_{n=1}^{\infty} (\xi\eta)(f_n) \\
&= \sum_{n=1}^{\infty} \zeta(f_n) \\
&\le \zeta'(f) + \varepsilon
\end{aligned}
$$

Letting $\varepsilon \searrow 0$, we have $(\eta'\xi')(f) \le \zeta'(f)$.

For part (3), if $\xi'(g)\eta'(h) = \infty$, then there is nothing to prove. We will first suppose that $\xi'(g) = 0$ and show that $\zeta'(f) = 0$. Choose $\{h_k\}_{k=1}^{\infty} \subset K^+(Y)$ with $h \le \sum_{k=1}^{\infty} h_k$ on Y. Let $\varepsilon > 0$ and $\varepsilon_k = \varepsilon \cdot 2^{-k}(1 + \eta(h_k))^{-1}$ for $k \ge 1$. Choose $\{g_{nk}\}_{n=1}^{\infty} \subset K^+(X)$ for each k such that $g \leq \sum_{n=1}^{\infty} g_{kn}$ and $\sum_{n=1}^{\infty} \xi(g_{kn}) \leq \varepsilon_k$. Let $f_{kn}(x,y) = g_{kn}(x)h_k(y)$ for $(x,y) \in Z$ and $k,n \geq 1$. Then $f_{kn} \in K^+(Z)$ and for $(x,y) \in Z$,

$$f(x,y) = g(x)h(y) \leq \sum_{k=1}^{\infty} g(x)h_k(y) \leq \sum_{k,n=1}^{\infty} g_{kn}(x)h_k(y) = \sum_{k,n=1}^{\infty} f_{kn}(x,y)$$

Therefore,

$$\begin{aligned}
\zeta'(f) &\leq \sum_{k,n=1}^{\infty} \zeta(f_{kn}) \\
&= \sum_{k,n=1}^{\infty} \xi(g_{nk})\eta(h_k) \\
&= \sum_{k=1}^{\infty} \eta(h_k) \sum_{n=1}^{\infty} \xi(g_{nk}) \\
&\leq \sum_{k=1}^{\infty} \eta(h_k)\varepsilon_k \\
&= \sum_{k=1}^{\infty} \frac{\varepsilon \eta(h_k)}{2^k(1+\eta(h_k))} \\
&< \varepsilon
\end{aligned}$$

Letting $\varepsilon \searrow 0$, we have $\zeta'(f) = 0$ as desired.

Now assume that both $\xi'(g)$ and $\eta'(g)$ are finite. Let $\varepsilon > 0$. Choose $\{g_n\}_{n=1}^{\infty} \subset K^+(X)$ such that $g \leq \sum_{n=1}^{\infty} g_n$ and $\sum_{n=1}^{\infty} \xi(g_n) \leq \xi'(g) + \varepsilon$ and choose $\{h_n\}_{n=1}^{\infty} \subset K^+(Y)$ such that $h \leq \sum_{n=1}^{\infty} h_n$ and $\sum_{n=1}^{\infty} \eta(h_n) \leq \eta'(h) + \varepsilon$. Let $f_{nk}(x,y) = g_n(x)h_k(y)$ for $(x,y) \in Z$. Then $f_{nk} \in K^+(Z)$, $f \leq \sum_{n,k=1}^{\infty} f_{nk}$, and

$$\begin{aligned}
\zeta'(f) &\leq \sum_{n,k=1}^{\infty} \zeta(f_{nk}) \\
&= \sum_{n,k=1}^{\infty} \xi(g_n)\eta(h_k) \\
&= \sum_{n=1}^{\infty} \xi(g_n) \sum_{k=1}^{\infty} \eta(h_k) \\
&\leq (\xi'(g)+\varepsilon)(\eta'(h)+\varepsilon)
\end{aligned}$$

Letting $\varepsilon \searrow 0$, we have $\zeta'(f) \leq \xi'(g)\eta'(h)$.

For part (4), note that

$$(\xi'\eta')(f) \leq \zeta'(f) \leq \xi'(g)\eta'(h)$$

from parts (2) and (3). But $(\xi'\eta')(f) = \xi'(g)\eta'(h)$, so

$$(\xi'\eta')(f) = \zeta'(f) = \xi'(g)\eta'(h).$$

□

Theorem 18. *(Lebesgue, Fubini, Tonelli) If $f \in K_1^+(Z, \mathrm{loc})$, then*

$$\zeta'(f) = (\xi'\eta')(f) = (\eta'\xi')(f).$$

Proof. We already have $(\xi'\eta')(f) \leq \zeta'(f)$. To show the opposite inequality, first suppose that $\zeta'(f) < \infty$. Then $f \in K_1^+(Z)$, so we can choose a sequence $\{f_n\}_{n=1}^{\infty} \subset K^+(Z)$ with $\zeta'(|f - f_n|) \to 0$ as $n \to \infty$. Then

$$\begin{aligned} \zeta'(f_n) &= \zeta(f_n) \\ &= (\xi\eta)(f_n) \\ &= (\xi'\eta')(f_n) \\ &\leq (\xi'\eta')(f) + (\xi'\eta')(|f - f_n|) \\ &\leq (\xi'\eta')(f) + \zeta'(|f - f_n|) \end{aligned}$$

Taking the limit of both sides as $n \to \infty$, we have $\zeta'(f) \leq (\xi'\eta')(f)$. So $\zeta'(f) = (\xi'\eta')(f)$.

Now suppose $\zeta'(f) = \infty$. Let $g_n = f 1_{[-n,n]^{d(X)}}$ and apply the previous result to the g_n to get $\zeta'(g_n) = (\xi'\eta')(g_n)$. Note that $g_n \nearrow f$ as $n \to \infty$, so by taking the limit as $n \to \infty$ we have, by Proposition 15, that $\zeta'(f) = (\xi'\eta')(f)$. □

Theorem 19. *The following hold:*

(1) *(Borel-Cantelli lemma) If* $A_n \subseteq X$*,* $\sum_{n=1}^{\infty} \xi'(1_{A_n}) < \infty$*, and* $A = \limsup_{n\to\infty} A_n = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} A_n$*, then* $\xi'(1_A) = 0$*, i.e.* A *is a null set.*

(2) *If* $\{a_n\}_{n=1}^{\infty} \subset [0,\infty)$ *and* $\sum_{n=1}^{\infty} a_n = A < \infty$*, then there exists* $\{b_n\}_{n=1}^{\infty} \subset \mathbb{N}$ *with* $b_n \nearrow \infty$ *such that* $\sum_{n=1}^{\infty} a_n b_n < \infty$*.*

(3) *If* $\phi_n : X \to [0,\infty]$ *for all* n *with* $\sum_{n=1}^{\infty} \xi'(\phi_n) < \infty$*, then* $\phi_n \to 0$ *for* ξ' *almost every* $x \in X$*.*

Proof. For part (1), note that $1_A \leq \sum_{n=N}^{\infty} 1_{A_n}$ for all N, so

$$\xi'(1_A) \leq \xi'\Big(\sum_{n=N}^{\infty} 1_{A_n}\Big) \leq \sum_{n=N}^{\infty} \xi'(1_{A_n}) \searrow 0$$

as $N \to \infty$.

For part (2), let $N(0) = 1$ and, for $k \geq 1$ a natural number choose $N(k) > N(k-1)$ such that $\sum_{n=N(k)}^{\infty} a_n \leq A/2^k$. Note that $\{N(k)\}_{k=1}^{\infty}$ is a strictly increasing sequence of natural numbers, so $N(k) \geq k \nearrow \infty$ as $k \to \infty$. For each $n \geq 1$, let b_n be the unique natural number such that $N(b_n - 1) \leq n < N(b_n)$. Then $N(b_n - 1) \leq n < n+1 < N(b_{n+1})$ for all n and, by the strict monotonicity of $\{N(k)\}_{k=1}^{\infty}$, we have $b_n - 1 < b_{n+1}$ for all n . Therefore, $\{b_n\}_{n=1}^{\infty}$ is monotonically increasing. Also, $N(b_{N(k)} - 1) \leq N(k)$ for all k and, by the strict monotonicity of $\{N(k)\}_{k=1}^{\infty}$, we have $b_{N(k)} \leq k+1$ for all k. So

$$\begin{aligned}\sum_{n=1}^{\infty} a_n b_n &= \sum_{k=1}^{\infty} \sum_{n=N(k-1)}^{N(k)-1} a_n b_n \\ &\leq \sum_{k=1}^{\infty} b_{N(k)} \sum_{n=N(k-1)}^{N(k)-1} a_n \\ &\leq \sum_{k=1}^{\infty} b_{N(k)} \frac{A}{2^{k-1}}\end{aligned}$$

$$\leq \sum_{k=1}^{\infty}(k+1)\frac{A}{2^{k-1}}$$
$$< \infty$$

by the ratio test.

For part (3), we can use part (2) to choose $\{b_n\}_{n=1}^{\infty}$ such that $\{b_n\}_{n=1}^{\infty} \subset \mathbb{N}$ with $b_n \nearrow \infty$ and $\sum_{n=1}^{\infty} \xi'(\phi_n) b_n < \infty$. Let $B_n = \{x \in X : \phi_n(x) \geq 1/b_n\}$ and $B = \limsup_{n\to\infty} B_n$. Note that $\sum_{n=1}^{\infty} \xi'(1_{B_n}) \leq \sum_{n=1}^{\infty} b_n \xi'(\phi_n) < \infty$ by Proposition 3. So by part (1), $\xi'(1_B) = 0$. Also, if $x \in X \backslash B$, then $0 \leq \phi_n(x) < 1/b_n$ for all but finitely many n, i.e. $\phi_n(x) \searrow 0$. So $X \backslash B \subseteq A$ and $\xi'(1_{X\backslash A}) \leq \xi'(1_B) = 0$. □

Theorem 20. *If $f \in K_1^+(Z)$, then*

(1) $\eta'(f) \in K_1^+(X)$.

(2) $[f]_x \in K_1^+(Y)$ *for* ξ' *almost every* $x \in X$.

Proof. For part (1), choose $\{f_n\}_{n=1}^{\infty} \subset K^+(Z)$ such that $\zeta'(|f - f_n|) \to 0$ as $n \to \infty$. Then $\eta'(f_n) = \eta(f_n) \in K^+(X)$ for each n and

$$\xi'(|\eta'(f) - \eta'(f_n)| \leq (\xi'\eta')(|f - f_n|) \leq \zeta'(|f - f_n|) \to 0$$

as $n \to \infty$.

For part (2), we can assume $\sum_{n=1}^{\infty} \zeta'(|f - f_n|) < \infty$ (otherwise, pass to a subsequence). Note that $[f_n]_x \in K(Y)$ for all $n \geq 1$ and $x \in X$ and let $h_n(x) = \eta'(|[f]_x - [f_n]_x|)$. Then

$$\sum_{n=1}^{\infty} \xi'(h_n) = \sum_{n=1}^{\infty} (\xi'\eta')|f - f_n| \leq \sum_{n=1}^{\infty} \zeta'|f - f_n| < \infty.$$

Therefore, $h_n(x) = \eta'(|[f]_x - [f_n]_x|) \to 0$ as $n \to \infty$ for ξ' almost every $x \in X$ by the previous theorem, i.e., $[f]_x \in K_1^+(Y)$ for ξ' almost every $x \in X$. □

Definition. For $A \subseteq X$, recall that the Lebesgue outer measure of A is

$$\lambda(A) = \inf\{\sum_{k=1}^{\infty} |I_k|_X : I_k \subset X \text{ finite intervals, and } \bigcup_{k=1}^{\infty} I_k \supseteq A\}.$$

Also, let $\mu(A) = \xi'(1_A)$.

Proposition 21. *The following hold:*

(1) *For $A \subseteq X$, $\mu(A) = \widetilde{\mu}(A) = \inf\{\mu(G) : G \text{ is open}, G \supseteq A\}$.*

(2) *For $A \subseteq X$, $\lambda(A) = \widetilde{\lambda}(A) = \inf\{\lambda(G) : G \text{ is open}, G \supseteq A\}$.*

(3) *For $A \subseteq X$, $\mu(A) \le \lambda(A)$.*

(4) *If $G \subseteq X$ is open, then $\mu(G) = \lambda(G)$.*

(5) *For $A \subseteq X$, $\mu(A) = \lambda(A)$.*

Proof. For part (1), it is clear that $\mu(A) \le \widetilde{\mu}(A)$, since for all open sets $G \supseteq A$, we have $\mu(A) = \xi'(1_A) \le \xi'(1_G) = \mu(G)$ by the monotonicity of ξ'. To show the opposite inequality, let $\varepsilon > 0$ and suppose $\mu(A) < \infty$. For $\varepsilon > 0$, choose $\{u_n\}_{n=1}^{\infty} \subset [0, \infty)$ and finite open intervals $\{I_n\}_{n=1}^{\infty}$ such that $1_A \le \sum_{n=1}^{\infty} u_n 1_{I_n}$ on X and $\sum_{n=1}^{\infty} u_n |I_n|_X \le \mu(A) + \varepsilon$. The function $g = \sum_{n=1}^{\infty}(1+\varepsilon) u_n 1_{I_n}$ is lower semicontinuous, so $G = \{x \in X : g(x) > 1)$ is open in X and $G \supseteq A$, so

$$\widetilde{\mu}(A) \le \mu(G) \le (1+\varepsilon) \sum_{n=1}^{\infty} u_n |1_{I_n}|_X \le (1+\varepsilon)(\mu(A)+\varepsilon).$$

Letting $\varepsilon \to 0$, we have $\widetilde{\mu}(A) \le \mu(A)$.

For part (2), it is clear that $\lambda(A) \le \widetilde{\lambda}(A)$, since for all open sets $G \supseteq A$, we have $\lambda(A) \le \lambda(G)$ by the monotonicity of λ. To show the opposite inequality, suppose that $\lambda(A) < \infty$ and let $\varepsilon > 0$. Choose finite open intervals $\{I_n\}_{n=1}^{\infty}$ in X such that $\bigcup_{n=1}^{\infty} I_n \supseteq A$ and $\sum_{n=1}^{\infty} |I_n|_X \le \lambda(A) + \varepsilon$. Then $G = \bigcup_{n=1}^{\infty} I_n$ is open in X with $G \supseteq A$ and

$$\widetilde{\lambda}(A) \le \lambda(G) \le \sum_{n=1}^{\infty} |I_n|_X \le \lambda(A) + \varepsilon.$$

Letting $\varepsilon \to 0$, we have $\widetilde{\lambda}(A) \le \lambda(A)$.

For part (3), suppose that $\lambda(A) < \infty$, let $\varepsilon > 0$, and choose finite intervals $\{I_n\}_{n=1}^{\infty}$ in X such that $\bigcup_{n=1}^{\infty} I_n \supseteq A$ and $\sum_{n=1}^{\infty} |I_n|_X \leq \lambda(A) + \varepsilon$. Then $\mu(A) \leq \sum_{n=1}^{\infty} \xi(1_{I_n}) = \sum_{n=1}^{\infty} |I_n|_X \leq \lambda(A) + \varepsilon$. Letting $\varepsilon \searrow 0$, we have $\mu(A) \leq \lambda(A)$.

For part (4), we only need to show that $\lambda(G) \leq \mu(G)$. Suppose $\mu(G) < \infty$. We can write $G = \bigcup_{n=1}^{\infty} I_n$ where the I_n are pairwise disjoint open intervals. Then

$$\lambda(G) \leq \sum_{n=1}^{\infty} |I_n|_X = \sum_{n=1}^{\infty} \xi'(1_{I_n}) = \xi'\Big(\sum_{n=1}^{\infty} 1_{I_n}\Big) = \xi'(1_G) = \mu(G)$$

by the monotone convergence theorem.

For part (5), we have

$$\mu(A) = \widetilde{\mu}(A) = \inf\{\mu(G) : G \text{ is open}, G \supseteq A\}$$

$$= \inf\{\lambda(G) : G \text{ is open}, G \supseteq A\} = \widetilde{\lambda}(A). \qquad \square$$

Definition. If $f : X \to [0, \infty]$, then we call $G(f) = \{(x, y) \in Z : 0 < y < f(x)\}$ the subgraph of the function f.

Proposition 22. *The following hold:*

(1) *If $f : X \to [0, \infty]$, then $\xi'(f) = (\xi'\eta')(1_{G(f)})$.*

(2) *If $f : X \to [0, \infty]$, then $\xi'(f) \leq \zeta'(1_{G(f)}) = \mu_Z(G(f))$.*

(3) *If $a \in [0, \infty)$ and $A \subset X$, then $G(a1_A) = A \times (0, a)$.*

(4) *If $f \in K^+(X)$, then $1_{G(f)} \in K^+(Z)$ and $\xi(f) = \zeta(1_{G(f)}) = \mu_Z(G(f))$.*

(5) *If $f : X \to [0, \infty]$, then $\xi'(f) = \zeta'(1_{G(f)}) = \mu_Z(G(f))$.*

Proof. For part (1), we have $1_{G(f)}(x, y) = 1_{(0, f(x))}(y)$ for all $(x, y) \in Z$. So $\eta'([1_{G(f)}]_x) = \eta'(1_{(0, f(x))}) = f(x)$ for all $x \in X$ and $(\xi'\eta')(1_{G(f)}) = \xi'(\eta'(1_{G(f)})) = \xi'(f)$.

For part (2), we have $\xi'(f) = (\xi'\eta')(1_{G(f)}) \leq \zeta'(1_{G(f)}) = \mu_Z(G(f))$.

Part (3) is clear from the definition.

For part (4), write $f = \sum_{n=1}^{N} u_n 1_{I_n}$ where $u_n \in [0,\infty)$ and the I_n are disjoint finite intervals in X. Then $1_{G(f)} \in K^+(Z)$, since $G(f) = \cup_{n=1}^{N} G(u_n 1_{I_n}) = \cup_{n=1}^{N}(I_n \times (0,u_n))$ is the disjoint union of finite intervals $I_n \times (0,u_n) \subset Z$. Also,

$$\xi(f) = \sum_{n=1}^{N} u_n |I_n|_X = \sum_{n=1}^{N} |I_n \times (0,u_n)|_Z = \zeta(1_{G(f)}).$$

For part (5), we already have $\xi'(f) \leq \mu_Z(G(f))$. To show the opposite inequality, suppose $\mu_Z(G(f)) < \infty$, let $\varepsilon > 0$, and choose $\{f_n\}_{n=1}^{\infty} \subset K^+(X)$ such that $F = \sum_{n=1}^{\infty} f_n \geq f$ on X and $\sum_{n=1}^{\infty} \xi(f_n) \leq \xi'(f) + \varepsilon$. Note that $F_N = \sum_{n=1}^{N} f_n \in K^+(X)$ for each N and $F_n \nearrow F$ as $N \to \infty$. So

$$\begin{aligned}
\mu_Z(G(f)) &\leq \mu_Z(G(F)) \\
&= \zeta'(1_{G(F)}) \\
&= \lim_{N\to\infty} \zeta'(1_{G(F_n)}) \\
&= \lim_{N\to\infty} \xi'(F_n) \\
&= \lim_{N\to\infty} \sum_{n=1}^{N} \xi(f_n) \\
&\leq \xi'(f) + \varepsilon.
\end{aligned}$$

Letting $\varepsilon \searrow 0$, we have $\mu_Z(G(f)) \leq \xi'(f)$. $\square$

In conclusion, let us say that $K(X)$ is the set of Riemann step functions f on X where $\xi(f)$ is the elementary Riemann integral of f. Then $\xi'(f)$ is the Lebesgue outer measure of the subgraph $G(f)$ of any $f : X \to [0,\infty]$, $K_1(X)$ is the usual Lebesgue class $L_1(X)$, $\xi'(f) = \int_X f(x)dx$ for Lebesgue summable functions f on X, $\mathfrak{A}$ is the σ-algebra of Lebesgue measurable sets on X (including all subsets of null sets), and μ_X is the usual Lebesgue outer measure on X.

Definition. Let $(X, \mathfrak{A}, \mu)$ be a measure space, i.e., $\mu \geq 0$ is a measure on a σ-algebra $\mathfrak{A}$ of subsets of a ground set X. Let K be the family of functions $f : X \to \mathbb{R}$ of the form $f = \sum_{i=1}^{n} y_i 1_{A_i}$, where $n \geq 1$, $y_i \in \mathbb{R}$, and $A_i \in \mathfrak{A}$ with $\mu(A_i) < \infty$. Let $K^+ = \{f \in K : f \geq 0 \text{ on } X\}$. For $f \in K$, define $\xi(f) = \sum_{i=1}^{n} y_i \mu(A_i) = \sum_{y \in \mathbb{R} \setminus \{0\}} y \mu(\{x \in X : f(x) = y\})$. Then ξ is a positive linear functional on the function lattice K. For any $f : X \to [0, \infty]$, define

$$\xi'(f) = \inf\Big\{ \sum_{n=1}^{\infty} \xi(f_n) : f_n \in K^+, f \leq \sum_{n=1}^{\infty} f_n \text{ on } X \Big\}$$

Let $K_1(\text{loc})$ be the family of functions $f : X \to [-\infty, \infty]$ such that $f 1_Q \in K_1$ for all $Q \in \mathfrak{A}$ with $\mu(Q) < \infty$. Also, let $K_1^+(\text{loc}) = \{f \in K_1(\text{loc}) : f \geq 0 \text{ on } X\}$.

In order to establish that Propositions/Theorems 3, 6-12, 15-20 hold in the setting of measure spaces (Theorems 18 and 20 for σ-finite measure spaces only), it is enough to show that Proposition 5 holds; all other arguments are generic.

Proposition 23. *If $f, f_n \in K^+$ for $n \geq 1$ and $f \leq \sum_{n=1}^{\infty} f_n$ on X, then for any $\varepsilon > 0$ there exists $N_0 \geq 1$ such that $\xi(f) \leq \varepsilon + \sum_{n=1}^{N_0} \xi(f_n)$.*

Proof. We can write $f = \sum_{i=1}^{m} y_i 1_{A_i}$ with $y_i \in [0, \infty)$ and $A_i \in \mathfrak{A}$. Let $A = \bigcup_{i=1}^{m} A_i$ and $Y = \sum_{i=1}^{m} y_i$. Note that $\mu(A) \leq \sum_{i=1}^{m} \mu(A_i) < \infty$. For $N \geq 1$, consider $g_N = -f + \sum_{n=1}^{N} f_n 1_A \in K$ and $B_N = \{x \in X : g_N(x) < 0\} \in \mathfrak{A}$. Since $B_N \searrow \emptyset$ as $N \to \infty$ and $\mu(B_1) \leq \mu(A) < \infty$, the decreasing continuity of the measure μ gives us $\mu(B_N) \searrow 0$ as $N \to \infty$. Let $\varepsilon > 0$ and choose $N_0 \geq 1$ such that $\mu(B_{N_0}) < \varepsilon / Y$. Then,

$$\begin{aligned} \xi(f) &= \xi\Big(- g_{N_0} + \sum_{n=1}^{N_0} f_n 1_A \Big) \\ &\leq \xi\Big(Y 1_{B_{N_0}} + \sum_{n=1}^{N_0} f_n \Big) \end{aligned}$$

$$= Y\mu(B_{N_0}) + \sum_{n=1}^{N_0} \xi(f_n)$$

$$< \varepsilon + \sum_{n=1}^{N_0} \xi(f_n).$$ □

Let us now assume a knowledge of the beginnings of the theory of the Lebesgue integral so that we may show that $\xi'(f) = \int_X f \, d\mu$.

Proposition 24. *If* $f : X \to [0, \infty]$ *and* $\int_X f \, d\mu$ *exists and is finite, then* $\xi'(f) = \int_X f \, d\mu$.

Proof. If $f \le \sum_{n=1}^{\infty} f_n$ on X for $f_n \in K^+$, then

$$\int_X f \, d\mu \le \int_X \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu = \sum_{n=1}^{\infty} \xi(f_n)$$

by the monotone convergence theorem for the Lebesgue integral with respect to μ. Thus, $\int_X f \, d\mu \le \xi'(f)$.

To show the opposite inequality, first suppose that f is a μ null function. Let $\varepsilon > 0$ and choose $\{A_n\}_{n=1}^{\infty} \subset \mathfrak{A}$ such that $f \le \sum_{n=1}^{\infty} 1_{A_n}$ on X and $\sum_{n=1}^{\infty} \mu(A_n) < \varepsilon$. Then

$$\xi'(f) \le \xi'(\sum_{n=1}^{\infty} 1_{A_n}) \le \sum_{n=1}^{\infty} \xi(1_{A_n}) = \sum_{n=1}^{\infty} \mu(A_n) < \varepsilon.$$

Letting $\varepsilon \searrow 0$, we have $\xi'(f) = 0 = \int_X f \, d\mu$.

Second, suppose that $f = \sum_{n=1}^{\infty} f_n$ on X for $f_n \in K^+$ and $\sum_{n=1}^{\infty} \int_X f_n \, d\mu < \infty$. Then

$$\xi'(f) \le \sum_{n=1}^{\infty} \xi(f_n) = \sum_{n=1}^{\infty} \int_X f_n \, d\mu = \int_X \sum_{n=1}^{\infty} f_n \, d\mu = \int_X f \, d\mu$$

by the monotone convergence theorem for the Lebesgue integral with respect to μ.

Third, consider the general case, in which f can be written as $f = g + h$ on X where g is a μ null function and $h = \sum_{n=1}^{\infty} h_n$ on X for $h_n \in K^+$. Then

$$\int_X f \, d\mu = \int_X g \, d\mu + \int_X h \, d\mu = \xi'(g) + \xi'(h) = \xi'(h) = \xi'(f)$$

since $h = f$ almost everywhere on X. □

Definition. Let X be a locally compact Hausdorff space and let $K = K(X)$ the set of all continuous functions $f : X \to \mathbb{R}$ with compact support. Endow K with the sup/max norm $\|f\| = \sup_{x \in X} |f(x)|$, which is finite for all $f \in K$. Let $K^+ = \{f \in K : f \geq 0\}$. Let $\xi : K \to \mathbb{R}$ be a positive linear functional. (Recall that a functional ξ is positive if $|\xi(f)| \leq \xi(|f|)$ for all $f \in K$.) For $f : X \to [0, \infty]$, let

$$\xi'(f) = \inf \Big\{ \sum_{n=1}^{\infty} \xi(f_n) : f_n \in K^+, f \leq \sum_{n=1}^{\infty} f_n \text{ on } X \Big\}.$$

In order to establish Propositions/Theorems 3 and 6-12 in the setting of locally compact Hausdorff spaces, it is enough to show that Proposition 5 holds; the rest of the arguments are generic.

Proposition 25. *If $f, f_n \in K^+$ for $n \geq 1$ and $f \leq \sum_{n=1}^{\infty} f_n$ on X, then for any $\varepsilon > 0$ there exists $N \geq 1$ such that $\xi(f) \leq \varepsilon + \sum_{n=1}^{N} \xi(f_n)$.*

Proof. Since f has compact support, there is a compact set $L \subset X$ with $f = 0$ on $X \backslash L$. Urysohn's lemma gives a continuous function $\chi : X \to [0, 1]$ with compact support and $\chi = 1$ on L.

Let $\varepsilon > 0$ and $\varepsilon' = \varepsilon/(1 + \xi(\chi))$. Since $\chi f = f \leq \sum_{n=1}^{\infty} f_n$ on X, the function $\phi = \varepsilon'\chi - f + \sum_{n=1}^{\infty} f_n$ is strictly positive on L. Let $\phi_k = \varepsilon'\chi - f + \sum_{n=1}^{k} f_n$ and note that $\phi_k \nearrow \phi > 0$ on L as $k \to \infty$. So for each point $x \in L$, we can choose $N(x) \in \mathbb{N}$ such that $\phi_{N(x)}(x) > 0$. Since $\phi_{N(x)}$ is continuous we can choose an open neighborhood $U(x)$ of x on which $\phi_{N(x)} > 0$. Then $\{U(x)\}_{x \in L}$ is an open cover of L which is compact. So there is a finite subcover $\{U(x_i)\}_{i=1}^{s}$ of L. Setting $N = \max\{N(x_i)\}_{i=1}^{s}$, we see that $\phi_N > 0$ on L.

Also, $\phi_N \geq 0$ on $X \backslash L$, since $f = 0$ on $X \backslash L$. So $\phi_N \geq 0$ on X and $\xi(\phi_N) \geq 0$. Therefore,

$$\xi(f) \leq \varepsilon' \xi(\chi) + \sum_{n=1}^{N} \xi(f_n) < \varepsilon + \sum_{n=1}^{N} \xi(f_n). \qquad \square$$

Definition. Define $\mu(A) = \xi'(1_A)$ for $A \subseteq X$ and $\mathfrak{A}$ as the σ-algebra on X generated by the G_δ compact subsets of X, i.e., $\mathfrak{A}$ is the Baire σ-algebra. Then μ is a measure on $\mathfrak{A}$ by the following proposition and the monotone convergence theorem. The σ-algebra $\mathfrak{A}$ contains all compact sets of the form $L = \{x \in X : f(x) \geq c\}$, where $f \in K^+$ and $c \in [0, \infty)$.

Proposition 26. *If $f \in K^+$ and $\varepsilon > 0$, then there is a continuous function $\lambda : X \to [0,1]$ with compact support such that $\lambda = 1$ on the compact set $L = \{x \in X : f(x) \geq 1\}$ and $\mu(L) \leq \xi(\lambda) \leq \mu(L) + \varepsilon$.*

Proof. For each $n \geq 1$, Urysohn's separation theorem gives us a continuous function $\Lambda_n : X \to [0,1]$ with

$$\Lambda_n = \begin{cases} 1 & \text{on } L_n = \{f \geq \frac{n+1}{n+2}\} \\ \text{continuous} & \text{in between} \\ 0 & \text{on } P_n = \{f \leq \frac{n}{n+1}\}. \end{cases}$$

Note that $\lambda_N = \bigwedge_{n=1}^{N} \Lambda_n \searrow 1_L$ as $N \to \infty$ and each λ_N is a continuous function with support in the compact set $\{f \geq \frac{1}{2}\}$. By the monotone convergence theorem, $1_L \in K_1^+$, $L \in \mathfrak{A}$, and $\xi(\lambda_N) = \xi'(\lambda_N) \searrow \xi'(1_L) = \mu(L)$ as $N \to \infty$. Thus, $\lambda = \lambda_N$ will work for sufficiently large N. $\square$

Theorem 27. *(Riesz representation theorem) If $f \in K$, then f is integrable in the measure space $(X, \mathfrak{A}, \mu)$ and $\int_X f \, \mathrm{d}\mu = \xi(f)$.*

Proof. By decomposing f as $f = f^+ - f^-$, it is enough to show that the theorem holds for $f \geq 0$. As f is bounded and both $\int_X$ and ξ are linear, we can assume f is normalized so that $\|f\| = \sup_{x\in X} |f(x)| \leq 1$. Since f is continuous, $\{f > c\}$ is open and, therefore, in $\mathfrak{A}$ for all $c > 0$ in $\mathbb{R}$. Thus, f is measurable with respect to the σ-algebra on $\mathfrak{A}$.

Choose a compact set $L \in \mathfrak{A}$ such that $f = 0$ on $X \backslash L$. Also, choose a continuous function $\chi : X \to [0,1]$ with compact support such that $\chi = 1$ on L. For $n \geq i \geq 1$, let $L_{ni} = \{f \geq \frac{i}{n}\} \in \mathfrak{A}$ and consider the step function $g_n = \frac{1}{n}\sum_{i=1}^{n} 1_{L_{ni}}$. Note that if $\frac{i}{n} \leq f(x) < \frac{i+1}{n}$, then $g_n(x) = \frac{i}{n}$. So $g_n \leq f < g_n + \frac{1}{n}1_L$ on X for all n.

For each of the compact sets L_{ni}, the previous proposition gives us a continuous function $\lambda_{ni} : X \to [0,1]$ in K with $1_{L_{ni}} \leq \lambda_{ni}$ and $\mu(L_{ni}) \leq \xi(\lambda_{ni}) \leq \mu(L_{ni}) + \frac{1}{n}$. Consider the function $h_n = \frac{1}{n}\chi + \frac{1}{n}\sum_{i=1}^{n} \lambda_{ni}$ which belongs to K^+ and satisfies $f \leq \frac{1}{n}1_L + g_n \leq h_n$ on X. Note that

$$\begin{aligned}\int_X f \,\mathrm{d}\mu &\leq \int_X (\frac{1}{n}1_L + g_n)\,\mathrm{d}\mu \\ &= \frac{1}{n}\mu(L) + \frac{1}{n}\sum_{i=1}^{n}\mu(L_{ni}) \\ &\leq \frac{1}{n}\xi(\chi) + \frac{1}{n}\sum_{i=1}^{n}\xi(\lambda_{ni}) \\ &= \xi(h_n)\end{aligned}$$

and

$$\begin{aligned}\xi(h_n) &= \frac{1}{n}\xi(\chi) + \frac{1}{n}\sum_{i=1}^{n}\xi(\lambda_{ni}) \\ &\leq \frac{1}{n}\xi(\chi) + \frac{1}{n}\sum_{i=1}^{n}(\mu(L_{ni}) + \frac{1}{n}) \\ &= \frac{1}{n}(1+\xi(\chi)) + \int_X g_n\,\mathrm{d}\mu \\ &\leq \frac{1}{n}(1+\xi(\chi)) + \int_X f\,\mathrm{d}\mu.\end{aligned}$$

So $\int_X f\,\mathrm{d}\mu \leq \xi(h_n) \leq \frac{1}{n}(1+\xi(\chi)) + \int_X f\,\mathrm{d}\mu$ for all n. Therefore, $\xi(h_n) \to \int_X f\,\mathrm{d}\mu$ as $n \to \infty$.

Next note that

$$\begin{aligned}
\xi(f) &= \xi'(f) \\
&\leq \xi'(\frac{1}{n}1_L + g_n) \\
&= \frac{1}{n}\xi'(1_L) + \xi'(g_n) \\
&= \frac{1}{n}\xi'(1_L) + \frac{1}{n}\sum_{i=1}^{n}\xi'(1_{L_{ni}}) \\
&\leq \frac{1}{n}\xi(\chi) + \frac{1}{n}\sum_{i=1}^{n}\xi'(\lambda_{ni}) \\
&= \xi(h_n)
\end{aligned}$$

and

$$\begin{aligned}
\xi(h_n) &= \frac{1}{n}\xi(\chi) + \frac{1}{n}\sum_{i=1}^{n}\xi'(\lambda_{ni}) \\
&\leq \frac{1}{n}\xi(\chi) + \frac{1}{n}\sum_{i=1}^{n}(\mu(L_{ni}) + \frac{1}{n}) \\
&= \frac{1}{n}(1 + \xi(\chi)) + \xi'(g_n) \\
&\leq \frac{1}{n}(1 + \xi(\chi)) + \xi'(f) \\
&= \frac{1}{n}(1 + \xi(\chi)) + \xi(f)
\end{aligned}$$

So $\xi(f) \leq \xi(h_n) \leq \frac{1}{n}(1 + \xi(\chi)) + \xi(f)$ for all n. Therefore, $\xi(h_n) \to \xi(f)$ as $n \to \infty$. Thus $\xi(f) = \lim_{n\to\infty} \xi(h_n) = \int_X f \, \mathrm{d}\mu$. □

REFERENCES

[L] Lieb, E., Loss, M., *Analysis*, Amer. Math. Soc., 2nd Ed., (2001),

[MW] McDonald, J. N., Weiss, N. A., *A course in real analysis*, (1999), Academic Press, San Diego.

[R] Riesz, F., *Sur les opérations fonctionelles linéaires*, C. R. Acad. Sci., Paris, 149, (1909), 974-977.

[S] Stroock, D. W., *A concise introduction to the theory of integration*, 3rd ed., (1999), Birkhäuser, Boston.