Finding Connected-Dense-Connected Subgraphs and variants is NP-Hard

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finding Connected-Dense-Connected Subgraphs and variants is NP-Hard

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Abstract. Finding Connected-Dense-Connected (CDC) subgraphs from Triple Networks is NP-Hard. Finding One-Connected-Dense (OCD) subgraphs from Triple Networks is also NP-Hard. We present formal proofs of these theorems hereby.

Keywords: Triple Networks · Connected-Dense-Connected subgraphs · One-Connected-Dense subgraphs · NP-Hard

Theorem 1. Finding a CDC subgraph in a Triple Network is NP Hard.

Proof. We prove that finding a CDC subgraph is a reduction of set-cover problem. Let \( R = \{r_1, \ldots, r_p\} \) be a set and and \( C = \{C_1, \ldots, C_q\} \) be its cover with \( R = \bigcup^q_{i=1} C_i \). The aim of this set cover problem is to find minimum subset \( C_{opt} \subset C \), known as optimal set-cover, such that each \( r_j \in R \) belongs to at least one set of \( C_{opt} \). This problem is proved to be NP complete.

Let \( T = \{t_1, \ldots, t_p\} \) be a set of points, having the same cardinality as \( R \). Let \( D = \{D_1, \ldots, D_q\} \) be a set-cover of \( T \), analogous to \( C \), such that if \( r_i \in C_j \), then \( t_i \in D_j \). Hence, \( T, D \) can be considered as a copy of \( R, C \).

We construct the Triple Network as follows. Let \( V_a = \{h, r_1, \ldots, r_p, C_1, \ldots, C_q\} \), where node \( h \) is connected to every \( C_i \in C \) and node \( r_1 \) is connected to node \( C_j \) if \( r_i \in C_j \) in the set-cover problem. Similarly, let \( V_b = \{k, t_1, \ldots, t_p, D_1, \ldots, D_q\} \) be the analogous set to \( V_a \). We connect \( V_a \) and \( V_b \) by connecting all nodes \( \{r_1, \ldots, r_p, h\} \) to all nodes \( \{t_1, \ldots, t_p, k\} \).

Construction of this Triple Network is illustrated in figure 1 from an instance of set-cover problem \( C_1 = \{r_1, r_2\}, C_2 = \{r_1\}, C_3 = \{r_2, r_4\}, C_4 = \{r_2, r_3\}, C_5 = \{r_4\} \).

Let \( C_{opt} \subset C \) be an optimal solution to the set-cover problem of \( C \) and \( |C_{opt}| = q^* \leq q \). Similarly, let \( D_{opt} \) be the analogous optimal solution to \( D \) and \( |D_{opt}| = q^* \leq q \). Let \( H = \{h, r_1, \ldots, r_p\} \) and \( J = \{k, t_1, \ldots, t_p\} \). The subgraph induced by \( S_a = H \cup C_{opt} \) is connected in \( V_a \), and similarly, the subgraph induced by \( S_b = J \cup D_{opt} \) is connected in \( V_b \). Hence, the sub Triple Network \( G[S_a, S_b] \) has density \( \rho(S_a, S_b) = \frac{(p+1)^2}{(p+q^*+1)} \).

Let \( S_1 \) and \( S_2 \) be any nonempty node sets where \( G_a[S_1] \) and \( G_b[S_2] \) are connected. In general, \( S_1 = H' \cup C' \) where \( H' \subset H \) and \( C' \subset C \). Similarly, \( S_2 = J' \cup D' \) where \( J' \subset J \) and \( D' \subset D \). We show that \( \rho(S_1, S_2) \leq \rho(S_a, S_b) \),
making $G[S_a, S_b]$ a CDC subgraph. Let $|H'| = p_1$, $|C'| = q_1$, $|J'| = p_2$ and $|D'| = q_2$. Hence, $\rho(S_1, S_2) = \frac{(p_1+1)^2(p_2+q_2)}{(p_1+q_1)(p_2+q_2)}$.

First, we consider the case when $S_1$ contains all the nodes of $H$ and $S_2$ contains all the nodes of $J$. In this case, $p_1 = p_2 = p + 1$. Also, by definition of optimal set-cover, $q' \leq q_1$ and $q' \leq q_2$. Hence, $\rho(S_1, S_2) = \frac{(p+1)^2}{(p+q_1+1)(p+q_2+1)} \leq \frac{3}{2 \rho(S_a, S_b)}$.

Second, we consider the case when $S_1$ contains a subset of nodes $H' \subset H$. In this case, we first show that adding elements from $H \setminus H'$ to $S_1$ will only increase its density.

If $h \notin S_1$, then after adding $h$ to $S_1$, the resulting subgraph has density $\frac{(p_1+1)p_2}{(p_1+q_1)(p_2+q_2)} > \frac{p_1p_2}{(p_1+q_1)(p_2+q_2)} = \rho(S_1, S_2)$. This subgraph is also connected in $G_a$, since $h$ is connected to every $C_i \in C$. To add a node $r_j \in H \setminus H'$ and making it still connected, we need to add at most one node $C_i$ to $C'$ with $r_j \in C_i$. Hence, the density of this resulting subgraph is $\frac{(p_1+1)p_2}{(p_1+q_1+2)(p_2+q_2)} = \rho(S_1, S_2)$. We can repeat this process by adding remaining nodes of $H \setminus H'$ to $S_1$, while density of the resulting subgraphs keeps increasing.

Similarly, adding elements from $J \setminus J'$ to $S_2$ increases density of the resulting subgraphs. Since we proved in the first case that the density $\rho(S_1, S_2)$ when $H \subset S_1$ and $J \subset S_2$, we have hence completed the proof of the second case.

In summary, we proved that for any nonempty sets $S_1 \subset V_a$ and $S_2 \subset V_b$, $\rho(S_1, S_2) \leq \rho(S_a, S_b)$, making $G[S_a, S_b]$ a CDC subgraph. Also, $G[S_a, S_b]$ is the solution inducted by optimal set covers, an instance being $S_a = \{r_1, r_2, r_3, r_4, h, C_1, C_3, C_4\}$ and $S_b = \{s_1, s_2, s_3, s_4, k, D_1, D_3, D_4\}$ hence proving that finding a CDC subgraph is NP hard.

**Lemma 1.** Finding OCD subgraph in triple network is NP hard

**Proof.** We prove that finding OCD subgraph is also reduction of the set cover problem. We first construct the triple network same as in theorem 1. Let $S_a = H$ and $S_b = J \cup D_{opt}$. The subgraph $G[s_a, S_b]$ hence has density $\rho(S_a, S_b) = \sqrt{|D_{opt}|}$. This subgraph is also connected in $G_a$, since $h$ is connected to every $C_i \in C$. To add a node $r_j \in H \setminus H'$ and making it still connected, we need to add at most one node $C_i$ to $C'$ with $r_j \in C_i$. Hence, the density of this resulting subgraph is $\frac{(p_1+1)p_2}{(p_1+q_1+2)(p_2+q_2)} = \rho(S_1, S_2)$. We can repeat this process by adding remaining nodes of $H \setminus H'$ to $S_1$, while density of the resulting subgraphs keeps increasing.

Similarly, adding elements from $J \setminus J'$ to $S_2$ increases density of the resulting subgraphs. Since we proved in the first case that the density $\rho(S_1, S_2)$ when $H \subset S_1$ and $J \subset S_2$, we have hence completed the proof of the second case.
We claim that $G[S_a, S_b]$ is an OCD subgraph. We observe that $G[S_b]$ is connected.

Let $S_1$ and $S_2$ be any nonempty node sets where either $G[S_1]$ or $G[S_2]$ is connected. In general, $S_1 = H' \cup C'$ where $H' \subset H$. Similarly, $S_2 = J' \cup D'$ where $J' \cup J$. We show that $\rho(S_1, S_2) \leq \rho(S_a, S_b)$.

First, we consider the case when $S_1$ contains all the nodes of $H$ and $S_2$ contains all the nodes of $J$. In this case, $p_1 = p_2 = p + 1$. Also, by definition of optimal set-cover, $q^* \leq q_1$ and $q^* \leq q_2$. Hence, $\rho(S_1, S_2) = \frac{(p+1)^2}{\sqrt{(p+1)(p+q^*-1)(p+q^*+1)}} \leq \frac{(p+1)^2}{\sqrt{(p+q^*-1)(p+q^*+1)(p+q^*+1)}} = \rho(S_a, S_b)$.

Second, we consider the case when $S_1$ contains a subset of nodes $H' \subset H$. In this case, we first show that adding elements from $H \setminus H'$ to $S_1$ will only increase its density. Suppose, $G_a[S_1]$ is not connected and $G_b[S_2]$ is connected. Then, after adding element from $H \setminus H'$, the resulting subgraph has density $\rho(S_1, S_2) = \frac{p_1 p_2}{(p_1 + q_1)(p_2 + q_2)} = \rho(S_a, S_b)$. This includes adding $h$ to $S_1$ if $h \notin H'$, making resultant subgraph connected in $V_a$. Now suppose $G_a[S_1]$ is connected. Then, following the same case of theorem 1, we first add $h$ if it is not in $H'$ and then add element from $H \setminus H'$ and still show that the resultant subgraph is connected in $V_a$ and its density increases. Similarly, we conclude that when $S_2$ contains a subset of nodes in $J' \subset J$, adding elements from $J' \setminus J$ also increases the density of the resultant subgraph.

At last, we observe that if $G_a[S_2]$ is connected, then the resultant subgraph obtained by removing elements from $C'$ has density $\rho(S_1, S_2) > \frac{p_1 p_2}{\sqrt{(p_1 + q_1 - 1)(p_2 + q_2)}}$.

In summary, we have proved that for any nonempty sets $S_1 \subset V_a$ and $S_2 \subset V_b$ with either $G_a[S_1]$ or $G_b[S_2]$ connected has density $\rho(S_1, S_2) \leq \rho(S_a, S_b)$, making $G[S_a, S_b]$ an OCD subgraph. Also, $G[S_a, S_b]$ is the solution induced by optimal set-cover, an instance being $S_a = \{r_1, r_2, r_3, r_4, h\}$, $S_b = \{s_1, s_2, s_3, s_4, k, D_1, D_2, D_4\}$ hence proving that finding OCD subgraphs is NP hard.