5-15-2014

Fan-Linear Maps and Fan Algebras

John J. Hull
College of Arts and Sciences

FAN-LINEAR MAPS AND FAN ALGEBRAS

JOHN HULL

Abstract. Fan algebras arise from fan-linear maps, a special class of functions defined on partitions of the nonnegative integer lattice in the plane. These algebras are natural objects of study in commutative algebra as they include many classical examples of commutative rings. Additionally, the ubiquity of this structure has only recently been identified, therefore little is known regarding the properties of these algebras. We begin our study by classifying all fan-linear maps via the conditions imposed on them by their domains. This classification includes a general result regarding the universal group of cones in lattices of arbitrary dimension. We then go on to show that the set of all fan-linear maps on any fixed partition is necessarily a finitely-generated affine semigroup. Finally, this leads to the conclusion that the set of fan algebras corresponding to a fixed partition and a fixed set of ideals forms a finitely generated semigroup. This is accomplished through the identification of generating maps in the semigroup of all fan-linear maps with generating algebras and the description of a natural additive operation.

Contents

1. Introduction

Let $I_1, \dots, I_n$ be a collection of ideals in a Noetherian domain R, let $f = \{f_1, \dots, f_n\}$ be a collection of special functions $f_i \colon \mathbb{N}^2 \to \mathbb{N}$ called *fan-linear maps*, and let u and v be indexing variables. Fan algebras are bi-graded rings of the following form:

$$\mathcal{B}(\Sigma_{\mathbf{a},\mathbf{b}}, f) = \bigoplus_{r,s} I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} u^r v^s \text{ as } (r,s) \text{ ranges over } \mathbb{N}^2$$

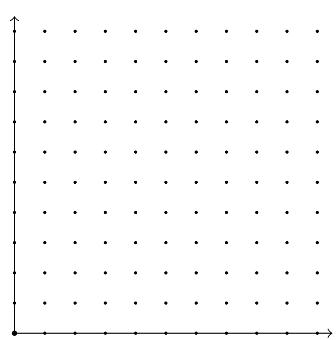
Fan algebras first appeared in the work of Sara Malec (see [2],[3]) and we cite here an important property of these objects.

Theorem 1.1. *([2], Theorem 2.3.3) Fan algebras are finitely generated as R-algebras.*

In what follows, we extend Malec's work by presenting a classification of all fan-linear maps. The study begins by examining the nonnegative integer lattice in the plane in order to deduce the general properties that it imposes on fan-linear maps (which have this lattice as their domain). Subsequently, we use these properties to describe all such maps and derive a correspondence between the set of all such maps and other semigroups similar to $\mathbb{N}^2$. Finally, we present a brief examination of fan algebras in this new context.

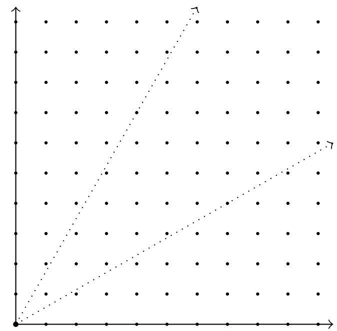
2. Preliminaries

Note that throughout this paper we take the convention that $0 \in \mathbb{N}$ and $I^0 = R$ for any nonzero ideal I of an integral domain R. To begin, we will develop an intuitive grasp of what a fan-linear map is and then discuss some general elements of semigroup theory necessary for further study of the objects before us. The first treatment of fan-linear maps will be concrete - we will avoid, for the time being, general and abstract conclusions. An *affine semigroup* is most simply described as a semigroup that can be identified as subsemigroup of $\mathbb{Z}^d$. The affine semigroup $\mathbb{N}^2$, where addition is defined componentwise, is best visualized as the integer coordinate pairs in the northeastern quadrant of the real plane adjoined with the integer coordinates on the positive axes:



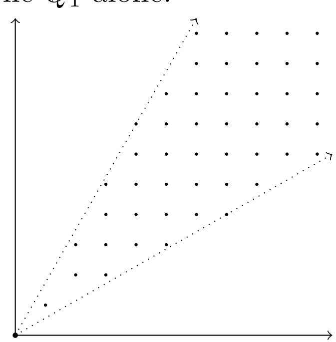
$\mathbb{N}^2$

A *fan of cones in* $\mathbb{N}^2$ is somewhat of a partition of $\mathbb{N}^2$ (although certainly not in the strictest sense); to construct a fan, we first n-sect the entire quadrant with positively oriented rays of rational slope meeting at the origin:



$\Sigma_{\mathbf{a},\mathbf{b}}$

Above, we see that the entirety of $\mathbb{N}^2$ is a union of these slices, and this segmenting is exactly what we mean when we say a *fan*. We use the notation $\Sigma_{\mathbf{a},\mathbf{b}}$ to denote one of these fans, the details of which we will explain later. By convention, we order and index each of the slices in a clockwise manner and the collection of the coordinates that reside within or on the defining rays of any one of these slices is what we call a *cone* in $\mathbb{N}^2$. In the example above, we have a *fan of* 3 *cones* which we may label Q_0, Q_1 and Q_2. The integer coordinates that fall on either ray of the two rays (including the axes when necessary) that define any of these cones Q_i constitute a *face* of that cone. Note that this implies that a non-axis face can be seen as the intersection of two cones (hence why a fan is not a true partition of $\mathbb{N}^2$). For example, below we show the cone Q_1 alone:



Q_1

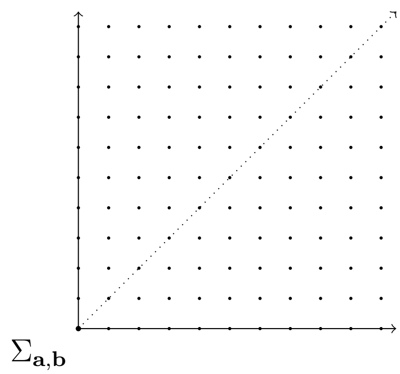
With this picture in mind, we may construct a more formal criterion for cone membership. In the case above, we see that the integer coordinates in $\mathbb{N}^2$ which lie in Q_1 can be described entirely by the integer coordinates between (inclusive) the lines of rational slope $y = \frac{7}{4}x$ and $y = \frac{4}{7}x$. For this to occur at the point (r, s), we must have $\frac{4}{7} \leq \frac{s}{r} \leq \frac{7}{4}$. Consequently, we may say that in this case, the faces of Q_1 are *defined by* $(4, 7)$ *and* $(7, 4)$. Note that in the same manner, the faces of Q_0 are defined by $(0, 1)$ and $(4, 7)$, while the faces of Q_2 are defined by $(1, 0)$ and $(7, 4)$. This leads to the conclusion that we may define cone membership as follows:

Definition 2.1. (Cone Membership Criterion) For any cone Q_i in an arbitrary fan $\Sigma_{\mathbf{a},\mathbf{b}}$ where the faces of Q_i are defined by (p_i, q_i) and (p_{i+1}, q_{i+1}), the element $(r, s) \in \mathbb{N}^2$ is a member of Q_i if and only if $\frac{q_{i+1}}{p_{i+1}} \leq \frac{s}{r} \leq \frac{q_i}{p_i}$ where we take the convention that $\frac{a}{b} < \frac{1}{0}$ for all $(a, b) \in \mathbb{N}^2$.

2.1. **Fan-Linear Maps.** A fan-linear map is a function $f\colon \mathbb{N}^2 \to \mathbb{N}$ defined with two specific requirements corresponding to a given fan. The first requirement is that for any

$(r_1, s_1), (r_2, s_2)$ in a given cone Q_i, we must have that $f[(r_1, s_1)+(r_2, s_2)] = f(r_1, s_1)+f(r_2, s_2)$. In this case, we may say that f is *additive on* Q_i, but since Q_i is a subsemigroup of $\mathbb{N}^2$ (closure under addition is easily verified) and $\mathbb{N}$ is a semigroup as well, this is the same as saying that f is a semigroup homomorphism when restricted to Q_i. The second requirement is that for $(r_1, s_1) \in Q_i$ and $(r_2, s_2) \in Q_j$ (where i may or may not be equal to j), $f[(r_1, s_1) + (r_2, s_2)] \leq f(r_1, s_1) + f(r_2, s_2)$. We call this property subadditivity.

Example 2.2. Consider the function $\max(r, s)\colon \mathbb{N}^2 \to \mathbb{N}$. We may view this function as fan-linear map on the fan described as $\mathbb{N}^2$ segmented by the ray $y = x$:



As the convention dictates, we take Q_0 to be the coordinates on or above the line $y = x$ and Q_1 to be the coordinates on or below it. When we restrict the $\max(r, s)$ function to say Q_1 and agree that $\max(r, s) = r$ when $r \geq s$, we have that $\max(r + r', s + s') = r + r' = \max(r, s) + \max(r', s')$, so $\max(r, s)$ is a semigroup homomorphism when restricted to Q_1. It follows similarly that this is the case when $\max(r, s)$ is restricted to Q_0. To verify subadditivity, assume that $(r, s) \in Q_0$ and $(r', s') \in Q_1$ and $(r + r', s + s') \in Q_0$. Then $\max(r + r', s + s') = s + s' \leq s + r' = \max(r, s) + \max(r', s')$. If $(r + r', s + s') \in Q_1$, then $\max(r + r', s + s') = r + r' \leq s + r' = \max(r, s) + \max(r', s')$. Collecting these observations, $\max(r, s)$ is a semigroup homomorphism on each cone and is subadditive on all of $\mathbb{N}^2$, hence $\max(r, s)$ is a fan-linear function.

Upon closer examination, we see that we may regard the function $\max(r, s)$ as a piecewise function where each piece is a semigroup homomorphism:

$$\max(r, s) = \begin{cases} s & \text{if } (r, s) \in Q_0 \\ r & \text{if } (r, s) \in Q_1 \end{cases}$$

Indeed, for any fan-linear map $f\colon \mathbb{N}^2 \to \mathbb{N}$ defined on a fan of any number of cones n, we may write f in such a way. Where $g_i\colon Q_i \to \mathbb{N}$ is a semigroup homomorphism for $i = 1, \ldots, n$, f can be considered as the following piecewise function:

$$f(r,s) = \begin{cases} g_1(r,s) & \text{if } (r,s) \in Q_0 \\ g_2(r,s) & \text{if } (r,s) \in Q_1 \\ \vdots & \\ g_n(r,s) & \text{if } (r,s) \in Q_n \end{cases}$$

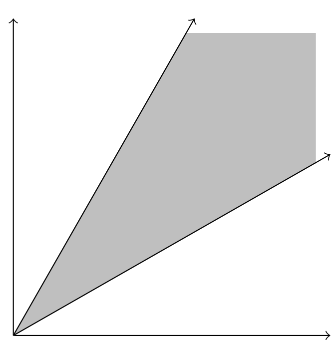
These semigroup homomorphisms must meet other requirements, but because they are the essential building blocks of fan-linear maps and our task is to classify every such map, the first step is to deduce what form each of the $g_i \colon Q_i \to \mathbb{N}$ can take. This is handled in section 3, but before we move on to that development, it is essential that we gather some tools to examine the concerned semigroups more rigorously.

2.2. **Cone Semigroup Properties.** Much of the theory of affine semigroups that applies to cones as we know them from our previous discussion is stated in terms of the properties in systems of linear inequalities in $\mathbb{R}^2$. For this reason, we will reconcile our understanding of a cone in $\mathbb{N}^2$ with this other view. We illustrate this concept with an example.

Example 2.3. Consider the following set:

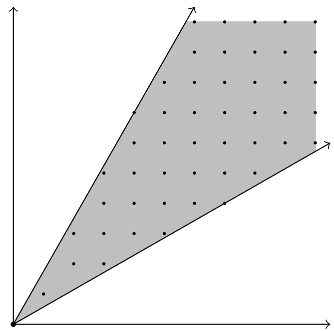
$$A = \left\{(x,y) \in \mathbb{R}^2 \mid y - \frac{4}{7}x \geq 0\right\} \cap \left\{(x,y) \in \mathbb{R}^2 \mid y - \frac{7}{4}x \leq 0\right\}$$

This is exactly the set of real coordinates above and below (inclusive) the lines $y = \frac{4}{7}x$ and $y = \frac{7}{4}x$ respectively in the first quadrant of $\mathbb{R}^2$.



A

Now consider the integer coordinates that lie within and on the boundaries of A.



A

It is clear then that the cone Q_1 that we previously examined can be viewed as the set $A \cap \mathbb{N}^2$. Considering this construction of Q_1 has some advantages that we will exploit in the classification of all fan-linear maps. In order to facilitate our handling of affine semigroups that are not necessarily contained within $\mathbb{N}^2$, we will present all of the following semigroup theory in full generality. First, we give some important definitions.

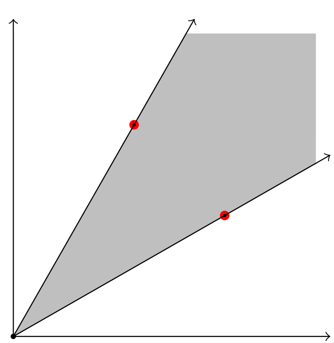
Definition 2.4. For fixed $x_1, \ldots, x_k, b \in \mathbb{R}$, a *hyperplane* in $\mathbb{R}^k$ is a set $A = \left\{(a_1, \ldots, a_k) \in \mathbb{R}^k \mid a_1x_1 + \cdots + a_kx_k = b\right\}$. A *half-space determined by* A is the set $\left\{(a_1, \ldots, a_k) \in \mathbb{R}^k \mid a_1x_1 + \cdots + a_kx_k \geq b\right\}$ or the set $\left\{(a_1, \ldots, a_k) \in \mathbb{R}^k \mid a_1x_1 + \cdots + a_kx_k \leq b\right\}$.

Definition 2.5. A *polyhedral cone* C in $\mathbb{R}^d$ is the intersection of finitely many closed half-spaces in $\mathbb{R}^d$ each determined by a hyperplane containing the origin. A hyperplane $H \subset \mathbb{R}^d$ containing the origin is called a *supporting hyperplane* if $H \cap C \neq \{\mathbf{0}\}$ and C is a subset of a closed half-space determined by H. In this case, $H \cap C$ is called a *face* of C.

Note that the sets $\left\{(x, y) \in \mathbb{R}^2 \mid y - \frac{4}{7}x \geq 0\right\}$ and $\left\{(x, y) \in \mathbb{R}^2 \mid y - \frac{7}{4}x \leq 0\right\}$ form half spaces, and the supporting hyperplane of each is $y - \frac{4}{7}x = 0$ and $y - \frac{7}{4}x = 0$ respectively. By the above definitions, the set A in example 2.3 is a polyhedral cone in $\mathbb{R}^2$.

Theorem 2.6. *([4], Corollary 7.1a) (Farkas-Minkowski-Weyl Theorem) A cone C is polyhedral if and only if it is finitely generated, i.e. $C = \{\lambda_1 c_1 + \cdots + \lambda_r c_r | \lambda_i \in \mathbb{R}_{\geq 0}\}$ for $c_1, \ldots, c_r \in \mathbb{R}^d$.*

For the set A in example 2.3, we may choose the integer pairs $(4, 7)$ and $(7, 4)$ as generators so that $A = \{\lambda_1(4, 7) + \lambda_2(7, 4) \mid \lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}\}$. The polyhedral cone A can now be further classified.



A

Definition 2.7. A polyhedral cone $C = \{\lambda_1 c_1 + \cdots + \lambda_r c_r | \lambda_i \in \mathbb{R}_{\geq 0}\}$ is said to be *rational* if we may choose all $c_1, \ldots, c_r \in \mathbb{Q}^d \subset \mathbb{R}^d$. We say that C is *trivial* when all $\mathbf{c}_i = \mathbf{0}$ and that C is *nontrivial* otherwise. We say that C is a *ray* if $C = \{\lambda \mathbf{c} \mid \lambda \in \mathbb{R}_{\geq 0}\}$ for some $\mathbf{c} \in \mathbb{R}^d$.

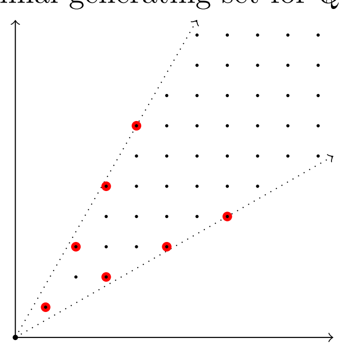
We may then conclude that A is a rational cone since both $(4, 7)$ and $(7, 4)$ are elements of $\mathbb{Z}^2 \subset \mathbb{Q}^2$. It is already clear that Q_1 is an affine semigroup, but the following theorem gives us the ability to readily identify affine semigroups embeddable in $\mathbb{Z}^d$ where $d > 2$.

Theorem 2.8. *([5], Proposition 7.16) (Gordan's Lemma) If $C \subset \mathbb{R}^d$ is a rational cone and A is any subgroup of $\mathbb{Z}^d$, then $C \cap A$ is an affine semigroup.*

We say that a commutative semigroup with identity S is *finitely generated* if S is trivial or there exists $s_1, \ldots, s_k \in S$ such that $S = \{n_1 s_1 + \cdots + n_k s_k \mid n_i \in \mathbb{N}\}$. Finding such generators greatly simplifies the study of such semigroups, especially if there exists a unique minimal set of such generators. The next theorem implies that most of the semigroups encountered in this study have this property.

Theorem 2.9. *([5], Proposition 7.15) A semigroup C is said to be* pointed *if it contains the identity and the identity is the only invertible element in C, i.e. $C \cap (-C) = \{\mathbf{0}\}$. Any pointed affine semigroup C has a unique finite minimal generating set.*

Consider $Q_1 = A \cap \mathbb{N}^2$ from before. We see that as $(0, 0) \in Q_1$ and that $Q_1 \cap (-Q_1) = \{(0, 0)\}$ (where $-Q_1$ denotes the reflection of Q_1 about the origin). This shows that Q_1 is a pointed affine semigroup and in fact, this is true for any cone in $\mathbb{N}^2$ [as $\mathbb{N}^2 \cap (-\mathbb{N}^2) = \{(0, 0)\}$]. We may then extract a unique minimal generating set for Q_1 which we highlight below:



Q_1

For any finitely generated subsemigroup of $\mathbb{N}^2$, we call its minimal generating set a *Hilbert basis.* There is a symmetry in the Hilbert basis for $Q_1 = A \cap \mathbb{N}^2$, and it is of a manageable size, however, in general this is not to be expected. Depending on the slopes of the bounding hyperplanes defining a cone in $\mathbb{N}^2$, finding a unique minimal generating set can be very cumbersome. Once it is found, it can be even more difficult to verify that it is indeed the unique minimal generating set. We have chosen to rely on the **Normaliz** package executable in **Macaulay2** to extract minimal generating sets of pointed affine semigroups when necessary; this tool saves a great deal of time.

In familiar territory, finite generation can be a simplifying property of extreme value. As an example, if B is a linearly independent generating set for a free module F, every R-module homomorphism $\phi \colon F \to M$ is completely determined the images of the basis elements $\phi(b)$. The converse of this is also true; that is, every map $b \mapsto m \in M$ defined for all $b \in B$ determines a *unique* R-module homomorphism $\lambda \colon R^n \to M$. This property of free

R-modules not only makes their study more accessible; since every R-module is the quotient of a free R-module, this property can be leveraged in the study of R-modules that are not necessarily free.

In the unfamiliar territory of cones $Q \subset \mathbb{N}^2$, it is true that every semigroup homomorphism defined on Q can be completely determined by the images of the basis elements; however, the converse property that every set map on the unique minimal set of generators induces a semigroup homomorphism is not true in general (and specifically not true in the case of homomorphisms $g\colon Q \to \mathbb{N}$ for nontrivial cones Q in $\mathbb{N}^2$). In the following section, we give a theorem that supports the classification of all fan-linear maps with relative ease.

3. First Orthant Cone Homomorphisms

Consider a semigroup homomorphism $g : Q \to \mathbb{N}$ where Q is a cone in $\mathbb{N}^2$ and assume that $\{h_1, \dots, h_k\}$ is the Hilbert basis for Q. We know that the image of g is determined by the images of the generators for Q; that is, if $q \in Q$, $q = n_1 h_1 + \cdots + n_k h_k$, then $g(q) = n_1 g(h_1) + \cdots + n_k g(h_k) \in \mathbb{N}$. The issue arises in that the minimal generating set for Q will rarely give a means of uniquely representing the elements of Q. There may be several ways to write $q \in Q$ in terms of the h_i, and even if there were a dependable rule for choosing such a representation, it would do little to simplify the piecewise form of fan-linear maps:

$$f(r,s) = \begin{cases} g_1(r,s) & \text{if } (r,s) \in Q_0 \\ g_2(r,s) & \text{if } (r,s) \in Q_1 \\ \vdots & \\ g_n(r,s) & \text{if } (r,s) \in Q_n \end{cases}$$

If we chose to study a fan-linear map by its mapping of the generators for the cones on which it is defined, we would immediately encounter problems. First, it is unlikely that the cones of a given fan would even have a comparable number of generators between them, so studying the interactive properties of the map between each cone could be exceedingly difficult. Secondly, we do not have a universal mapping property that allows us to define arbitrary maps in terms of the generators and the ability define such maps is essential to the study and utility of fan algebras. It is best then to escape the classification of cone homomorphisms in terms of Hilbert bases and somehow state $g(r, s)$ in terms of r and s. This problem is what motivates the main result of this section. In what follows, we classify the *universal group* for cones in $\mathbb{N}^d$ for all positive d and obtain the desired description of all maps from these cones into $\mathbb{N}$.

Definition 3.1. ([1], page 32-33) For a semigroup S, the universal group of S is a group $G(S)$ together with a homomorphism $\gamma\colon S \to G(S)$ such that for every homomorphism $\varphi\colon S \to G$ where G is a group, there exists a unique group homomorphism $\xi\colon G(S) \to G$ such that $\varphi = \xi \circ \gamma$. The group $G(S)$ exists for all semigroups S and is unique up to isomorphism.

Definition 3.2. Let S be a commutative, additive semigroup S with identity and let $a, b, c \in S$. S is called *reduced* if $a + b = 0$ implies that $a = b = 0$, *cancellative* if $a + c = b + c$ implies that $a = b$, and *power cancellative* if $n \cdot a = n \cdot b$ implies that $a = b$.

Even when a semigroup S is a subsemigroup of a group G, it is not generally true that the universal group $G(S)$ is the group generated by the elements of S (i.e. by closing S under sums of inverses). The following proposition is therefore essential to our arguments.

Proposition 3.3. *([1], pg. 36) Let S be a nonempty cancellative commutative semigroup. An abelian group G is isomorphic to the universal group $G(S)$ if and only if S is isomorphic to a subsemigroup T of G such that every $g \in G$ can be written as $a - b$ for some $a, b \in T$.*

The following theorem shows that the universal group $G(Q)$ of a cone is free of rank corresponding to the rank of the "smallest" real vector space containing A when $Q = A \cap \mathbb{N}^d$.

Theorem 3.4. *Let $A \subset \mathbb{R}^d$ be a pointed rational cone in the first orthant such that $d = \min\{n \mid A \hookrightarrow \mathbb{R}^n\}$ and let $Q = A \cap \mathbb{N}^d$ so that Q is a cone in $\mathbb{N}^d$. The following must be true:*

(1) *The universal group $G(Q)$ is a free $\mathbb{Z}$-module of rank d.*
(2) *A map $\phi\colon Q \to \mathbb{Z}$ is a homomorphism of semigroups if and only if $\phi(r_1, \ldots, r_d) = a_1 r_1 + \cdots + a_d r_d$ for some $a_1, \ldots, a_d \in \mathbb{Z}$.*
(3) *If $A = \{\lambda_1 x_1 + \cdots + \lambda_q x_q \mid \lambda_i \in \mathbb{R}_{\geq 0}\}$, then $Im(\phi) \subset \mathbb{N}$ if and only if $\phi(\beta_i) \in \mathbb{N}$ for at least one $\beta_i = \lambda_i x_i \in Q$ for each $i = 1, \ldots, q$.*

Proof. To prove (1), note that since Q is a subsemigroup of the group $\mathbb{Z}^d$, Q is cancellative and commutative. Since Q is pointed, we may extract a Hilbert basis $\mathcal{H}_Q$ for Q. Consider the group $H = \langle h \mid h \in \mathcal{H}_Q \rangle$ as a subgroup of $\mathbb{Z}^d$. Then Q is isomorphic to a subsemigroup T of H where T is the semigroup of all finite sums of elements $h \in \mathcal{H}_Q$. We also have that for all $g \in H$, $g = n_1 h_1 + \cdots + n_k h_k$ for some $n_i \in \mathbb{Z}$, so after grouping the nonnegative and negative terms we may write $g = a - b$ for $a, b \in T$. By Proposition 3.3, we may conclude that $G(Q) \cong H$ so that $G(Q)$ is identifiable as a subgroup of $\mathbb{Z}^d$.

Since $G(Q)$ is a subgroup and therefore a $\mathbb{Z}$-submodule of $\mathbb{Z}^d$ and $\mathbb{Z}$ is a principal ideal domain, $G(Q)$ is free of rank $d' \leq d$. We will proceed to show that $d' = d$ so that $G(Q) \cong \mathbb{Z}^d$. First, we claim that for any two rational cones $C_1, C_2 \subset \mathbb{R}^d$ in the first orthant, if $C_1 \cap \mathbb{N}^d = C_2 \cap \mathbb{N}^d$, then $C_1 = C_2$. By contrapositive, if $C_1 \neq C_2$, then without loss of generality there must exist some generator x for C_1 such that $x \notin C_2$ (otherwise $C_1 = C_2$). Since $x \in \mathbb{Q}^d$, we may choose some nonzero $N \in \mathbb{N}$ such that $N \cdot x \in \mathbb{N}^d$. Also, observe that $N \cdot x \in C_1 \backslash C_2$ (otherwise $\frac{1}{N} N \cdot x \in C_2$) and therefore $N \cdot x \notin C_2 \cap \mathbb{N}^d$. Consequently, $C_1 \cap \mathbb{N}^d \neq C_2 \cap \mathbb{N}^d$, proving our claim. Now consider the following rational cone in $\mathbb{R}^d$:

$$A' = \{\lambda_1 h_1 + \cdots + \lambda_k h_k \mid \lambda_i \in \mathbb{R}_{\geq 0}, h_i \in \mathcal{H}_Q\}$$

This is the rational cone generated by the Hilbert basis of Q, and since $Q = \{m_1 h_1 + \cdots + m_k h_k \mid m_i \in \mathbb{N}, h_i \in \mathcal{H}_Q\}$, we see immediately that $Q = A' \cap \mathbb{N}^d$. Consequently, $A \cap \mathbb{N}^d = A' \cap \mathbb{N}^d$, so $A = A'$. If $d' < d$ where again d' is the free rank of $G(Q)$, then $\mathcal{H}_Q$ does not contain d elements that are linearly independent over $\mathbb{Z}$. It follows then that A does not contain d elements that are linearly independent over $\mathbb{R}$; but this implies $A \hookrightarrow \mathbb{R}^s$ for some $s < d$, contradicting the minimality of d. Conclude that $d' = d$ so that $G(Q) \cong \mathbb{Z}^d$.

The proof of (2) follows naturally; since $G(Q) \cong \mathbb{Z}^d$, every semigroup homomorphism $\phi\colon Q \to G$ factors through a unique $\mathbb{Z}$-module homomorphism $\Phi\colon \mathbb{Z}^d \to G$ that we may define on the standard basis of $\mathbb{Z}^d$. In the case where $G = \mathbb{Z}$, any such homomorphism $\phi\colon Q \to \mathbb{Z}$ must be given by $(r_1, \ldots, r_d) \mapsto r_1\Phi(e_1) + \cdots + r_d\Phi(e_d) = a_1r_1 + \cdots + a_dr_d$ for some $a_i = \Phi(e_i) \in \mathbb{Z}$. For the converse, the linearity of $\phi\colon Q \to \mathbb{Z}$ defined by $(r_1, \ldots, r_d) \mapsto a_1r_1 + \cdots + a_dr_d$ is easily verified.

For (3), the forward implication is trivial. For the converse, note that for any $\omega \in Q$, $\omega = \lambda_1x_1 + \cdots + \lambda_qx_q$ where $\lambda_i \in \mathbb{R}_{\geq 0}$ and $x_i \in \mathbb{Q}^d$ by the assumption that the cone A is rational. As $\mathbb{R}$ is a field, we may factor out the denominators that appear in each x_i and replace each with some $\alpha_i \in \mathbb{Z}^d$ so that $\omega = p_1\alpha_1 + \cdots + p_q\alpha_q$. Since $\omega, \alpha_1, \ldots, \alpha_q \in \mathbb{Z}^d$, it is easily verified that each $p_i \in \mathbb{Q}_{\geq 0}$; to see this, simply let the α_i generate $\mathbb{Q}^d$ so that ω is the solution to a solvable overdetermined system in $\mathbb{Q}$. Choose some positive integer N that clears the denominator of all the p_i. By the implicit hypothesis that ϕ is well-defined and since each $Np_i \in \mathbb{N}$, we have the following:

$$\begin{aligned} N\omega &= Np_1\alpha_1 + \cdots + Np_q\alpha_q \\ \phi(N\omega) &= \phi(Np_1\alpha_1) + \cdots + \phi(Np_q\alpha_q) \\ N\phi(\omega) &= Np_1\phi(\alpha_1) + \cdots + Np_q\phi(\alpha_q) \end{aligned}$$

Since N is a positive integer, we may divide through to obtain the identity:

$$\phi(\omega) = p_1\phi(\alpha_1) + \cdots + p_q\phi(\alpha_q)$$

By hypothesis, at least one $\phi(\beta_i) \in \mathbb{N}$ where $\beta_i = \lambda_ix_i = p_i\alpha_i$, so by the above identity, $p_i^{-1}\phi(\beta_i) = \phi(\alpha_i)$. Consequently, $\phi(\alpha_i) \in \mathbb{N}$ since $p_i \geq 0$ and it follows then that $\mathrm{Im}(\phi) \subset \mathbb{N}$. This completes the proof of the theorem. ☐

We immediately obtain the following corrollary which can be directly applied to the problem of classifying fan-linear maps on $\mathbb{N}^2$.

Corollary 3.5. *Let $A = \{\lambda_1x_1 + \lambda_2x_2 \mid \lambda_i \in \mathbb{R}_{\geq 0}\}$ be a rational cone in the first quadrant of $\mathbb{R}^2$ so that the x_i are linearly independent in $\mathbb{R}^2$. If $Q = A \cap \mathbb{N}^2$, then $\phi\colon Q \to \mathbb{Z}$ is a semigroup homomorphism if and only if $\phi(r, s) = ar + bs$ for some $a, b \in \mathbb{Z}$. Furthermore, $Im(\phi) \subset \mathbb{N}$ if and only if $\phi(\beta_i) \in \mathbb{N}$ for some $\beta_i = \lambda_ix_i \in Q$ for $i = 1, 2$.*

4. A Complete Classification of Fan-Linear Maps

We now move on to the main task of this study, the classification of all functions from which fan-algebras arise. First, we give formal definitions to some of the objects with which we have already dealt intuitively.

Definition 4.1. For a finite indexing set I, a *fan* is a collection $\Sigma \subset \mathcal{P}(\mathbb{R}^d)$ of cones $\{C_i\}_{i\in I}$ so that the following conditions are satisfied:

(1) No C_i is a ray.
(2) The faces of each C_i are also in Σ.
(3) The intersection of any pair of cones, when the intersection is not $\{\mathbf{0}\}$, is a shared face of the two cones.

Definition 4.2. Let $\Sigma = \{C_i\}_{i=0}^n \subset \mathcal{P}(\mathbb{R}^2)$ be a fan such that $\bigcup_{i=0}^n C_i$ gives exactly the first quadrant of $\mathbb{R}^2$ and each C_i is rational. Let $\mathbf{a} = \{p_0, \ldots, p_{n+1}\}$ and $\mathbf{b} = \{q_0, \ldots, q_{n+1}\}$ so that each (p_i, q_i) and (p_{i+1}, q_{i+1}) define the faces of C_i, each p_i and q_i are relatively prime, and $\frac{q_{i+1}}{p_{i+1}} < \frac{q_i}{p_i}$ for $i = 0, \ldots, n$. Define a *fan in* $\mathbb{N}^2$ *of* $n+1$ *cones* to be the collection $\Sigma_{\mathbf{a},\mathbf{b}} = \{Q_i\}_{i=0}^n$ where each $Q_i = C_i \cap \mathbb{N}^2$ is a cone in $\mathbb{N}^2$, noting that $\bigcup_{i=0}^n Q_i = \mathbb{N}^2$. For $Q_i \in \Sigma_{\mathbf{a},\mathbf{b}}$, $Q_i \cap Q_{i+1}$ is a *face* of $\Sigma_{\mathbf{a},\mathbf{b}}$ and we say that each (p_i, q_i) and (p_{i+1}, q_{i+1}) *define a face* of Q_i.

Herein, we will be discussing only fans in $\mathbb{N}^2$, therefore when we say a fan of $n+1$ cones, we mean fan in $\mathbb{N}^2$ of $n+1$ cones. Then next few definitions and propositions give a formal definition of fan-linear maps and show that for any fixed fan $\Sigma_{\mathbf{a},\mathbf{b}}$, the set of all fan-linear maps on $\Sigma_{\mathbf{a},\mathbf{b}}$ is itself a semigroup.

Definition 4.3. Let f be a function so that addition is an associative binary operation defined on both the domain and range of f. Further assume that Range(f) is totally ordered. We say that f is *subadditive* if $f(x+y) \le f(x) + f(y)$ for all $x, y \in$ Domain(f).

Definition 4.4. Let $\Sigma_{\mathbf{a},\mathbf{b}}$ be a fan of $n+1$ cones. A *fan-linear map on* $\Sigma_{\mathbf{a},\mathbf{b}}$ is a map $f : \mathbb{N}^2 \to \mathbb{N}$ such that for cones $Q_0, Q_1, \ldots, Q_n \in \Sigma_{\mathbf{a},\mathbf{b}}$, f is a semigroup homomorphism when restricted to each Q_i and f is subadditive on $\mathbb{N}^2 = \bigcup_{i=1}^n Q_i$.

As a result of the discussion in the previous section, we may write any fan-linear maps in a much simplified form.

Proposition 4.5. *Let $f : \mathbb{N}^2 \to \mathbb{N}$ be a fan-linear map on $\Sigma_{\mathbf{a},\mathbf{b}} = \{Q_0, Q_1, \ldots, Q_n\}$. Then f can be written as a piecewise function:*

$$f(r,s) = \begin{cases} a_0 r + b_0 s & \text{if } (r,s) \in Q_0 \\ a_1 r + b_1 s & \text{if } (r,s) \in Q_1 \\ \vdots & \\ a_n r + b_n s & \text{if } (r,s) \in Q_n \end{cases}$$

where each $a_i, b_i \in \mathbb{Z}$.

Proof. This follows immediately from Corollary 3.5 and the definition requiring that f is a semigroup homomorphism when restricted to each cone Q_i. □

Proposition 4.6. *If f and g are subadditive on $\mathbb{N}^2$, then $f+g$ is subadditive on $\mathbb{N}^2$.*

Proof. If f and g are subadditive on $\mathbb{N}^2$ then for all $x, y \in \mathbb{N}^2$, $(f+g)(x+y) = f(x+y) + g(x+y) \le f(x) + f(y) + g(x) + g(y) = (f+g)(x) + (f+g)(y)$. □

Definition 4.7. Let $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ denote the collection of all fan-linear maps on a fixed fan $\Sigma_{\mathbf{a},\mathbf{b}}$.

Proposition 4.8. *The collection $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ is a commutative, additive semigroup with identity under the operation of function addition.*

Proof. Let $\Sigma_{\mathbf{a},\mathbf{b}} = \{Q_0, Q_1, \ldots, Q_n\}$. If $f, g \in Fan(\Sigma_{\mathbf{a},\mathbf{b}})$, then for each $x, y \in Q_i$, $(f+g)(x+y) = f(x+y) + g(x+y) = f(x) + f(y) + g(x) + g(y) = (f+g)(x) + (f+g)(y)$. By Proposition 4.6, $f+g$ is subadditive on $\mathbb{N}^2$ and as function addition is associative, $(\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}}), +)$ is a semigroup. Since the zero function $\mathbf{0} : \mathbb{N}^2 \to \mathbb{N}$ is linear on each Q_i and subadditive on $\mathbb{N}^2$, $\mathbf{0} \in \mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$, and as addition commutes in the codomain $(\mathbb{N}, +)$, $(\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}}), +)$ is commutative. □

The following map is essential to the classification of all fan-linear maps and leads to the conclusion that the set $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ of all fan-linear maps defined on a given fan $\Sigma_{\mathbf{a},\mathbf{b}}$ is an affine semigroup.

Definition 4.9. Let p and q be relatively prime positive integers. Define $f_{p,q} : \mathbb{N}^2 \to \mathbb{N}$ so that:

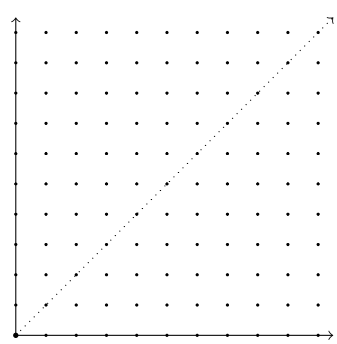
$$f_{p,q}(r,s) = \begin{cases} 0 & \text{if } \frac{s}{r} > \frac{q}{p} \\ qr - ps & \text{if } \frac{s}{r} \leq \frac{q}{p} \end{cases}$$

We call $f_{p,q}$ the *determining map of* (p,q) and note here that as $qr \geq ps$ for all $(r,s) \in \mathbb{N}^2$ such that $\frac{s}{r} \leq \frac{q}{p}$, $f_{p,q}$ is nonnegative on $\mathbb{N}^2$.

Proposition 4.10. *For any relatively prime pair $(p,q) \in \mathbb{N}^2$, the determining map $f_{p,q}$ is subadditive on $\mathbb{N}^2$.*

Proof. Choose (r,s) and (r',s') in $\mathbb{N}^2$ so that $\frac{s}{r} > \frac{q}{p}$ and $\frac{s'}{r'} \leq \frac{q}{p}$. If $\frac{s+s'}{r+r'} > \frac{q}{p}$, then $f_{p,q}(r+r', s+s') = 0 \leq f_{p,q}(r',s') = 0 + f_{p,q}(r',s') = f_{p,q}(r,s) + f_{p,q}(r',s')$. If $\frac{s+s'}{r+r'} \leq \frac{q}{p}$, then $f_{p,q}(r+r', s+s') = q(r+r') - p(s+s') = qr + qr' - ps - ps'$. Since $\frac{s}{r} > \frac{q}{p}$, it follows that $ps > qr$ and $qr + qr' - ps - ps' - (qr' - ps') = qr - ps < 0$ which implies that $f_{p,q}(r+r', s+s') \leq 0 + f_{p,q}(r',s') = f_{p,q}(r,s) + f_{p,q}(r',s')$. It follows that $f_{p,q}$ is subadditive on $\mathbb{N}^2$. □

Example 4.11. Recall the fan given in Example 2.2 which has faces determined by the coordinates $(0,1), (1,1), (1,0)$.



$\Sigma_{\mathbf{a},\mathbf{b}}$

Consider the $\max(r,s)$ function from same example which we wrote in piecewise form.

$$\max(r,s) = \begin{cases} s & \text{if } (r,s) \in Q_0 \\ r & \text{if } (r,s) \in Q_1 \end{cases}$$

The determining map gives us the ability to rewrite this function as the sum of a semigroup homomorphism mapping $Q_0 \to \mathbb{Z}$ and $f_{1,1}$. The determining map $f_{1,1}$ is given by:

$$f_{1,1} = \begin{cases} 0 & \text{if} \frac{s}{r} > 1 \\ r - s & \text{if} \frac{s}{r} \leq 1 \end{cases}$$

Let $f(r,s) = s + f_{1,1}(r,s)$. If $(r,s) \in Q_1$, then $f(r,s) = s + r - s = r$. If $(r,s) \in Q_0$, then $f(r,s) = s + 0 = s$. This shows that $f(r,s) = \max(r,s)$ for all $(r,s) \in \mathbb{N}^2$. In this manner, we may use the determining maps of coordinates defining the faces $\Sigma_{\mathbf{a},\mathbf{b}}$ of to give necessary and sufficient conditions on all fan-linear maps defined according to any fan $\Sigma_{\mathbf{a},\mathbf{b}}$.

Theorem 4.12. *Fix a fan $\Sigma_{\mathbf{a},\mathbf{b}}$ of $n+1$ cones. Assume that for $\Sigma_{\mathbf{a},\mathbf{b}} = \{Q_0, Q_1, \ldots, Q_n\}$, the faces of $\Sigma_{\mathbf{a},\mathbf{b}}$ are defined by $\{(p_i, q_i) \in \mathbb{N}^2\}_{i=0}^{n+1}$. A function f on $\Sigma_{\mathbf{a},\mathbf{b}}$ is fan-linear if and only if:*

$$f(r,s) = a_0 r + b_0 s + k_1 \cdot f_{p_1,q_1}(r,s) + \cdots + k_n \cdot f_{p_n,q_n}(r,s)$$

With the following conditions:

(1) $k_i \geq 0$ *for* $i = 1, \ldots, n$
(2) $f(\alpha) \geq 0$ *for all* $\alpha \in \{(p_i, q_i) \in \mathbb{N}^2\}_{i=0}^{n+1}$

Proof. Let f be a map that is fan-linear on $\Sigma_{\mathbf{a},\mathbf{b}}$. By Proposition 4.5, we may write f as follows:

$$f(r,s) = \begin{cases} a_0 r + b_0 s & if\ (r,s) \in Q_0 \\ a_1 r + b_1 s & if\ (r,s) \in Q_1 \\ \vdots & \\ a_n r + b_n s & if\ (r,s) \in Q_n \end{cases}$$

For some $i \in \{1, \ldots, n\}$, choose two cones Q_{i-1} and Q_i so that these cones share the face (p_i, q_i). By the assumption that f is well defined, we must have $a_{i-1}p_i + b_{i-1}q_i = a_i p_i + b_i q_i$

which implies that $(a_{i-1} - a_i)p_i = (b_i - b_{i-1})q_i$. As p_i and q_i are relatively prime, we must have that $a_{i-1} - a_i = dq$ and $b_i - b_{i-1} = dp$ so that $a_i = a_{i-1} - dq_i$ and $b_i = b_{i-1} + dp_i$ for some $d \in \mathbb{Z}$. Then we may rewrite f:

$$f(r,s) = \begin{cases} a_0 r + b_0 s & if\ (r,s) \in Q_0 \\ \vdots & \\ a_{i-1} r + b_{i-1} s & if\ (r,s) \in Q_{i-1} \\ (a_{i-1} - dq_i)r + (b_{i-1} + dp_i)s & if\ (r,s) \in Q_i \\ \vdots & \\ a_n r + b_n s & if\ (r,s) \in Q_n \end{cases}$$

Define a connected subfan of $\Sigma_{\mathbf{a},\mathbf{b}}$ to be the sequential union $Q_j \cup Q_{j+1} \cup \cdots \cup Q_k$ for $j \leq k$, $j,k \in \{0,1,\ldots,n\}$. Let f_{Q_{i-1},Q_i} denote the restriction of f to the connected subfan $Q_{i-1} \cup Q_i$. It is clear that $f_{Q_{i-1},Q_i}(r,s) = a_{i-1}r + b_{i-1}s + (-d)f_{p_i,q_i}(r,s)$ where f_{p_i,q_i} is the determining map of (p_i,q_i).

Now choose some $(r,s) \in Q_{i-1}$ and $(r',s') \in Q_i$ so that the sum $(r+r', s+s')$ lies in Q_i. The map f is subadditive, which implies that:

$$\begin{aligned} f(r+r', s+s') &\leq f(r,s) + f(r'+s') \\ a_{i-1}(r+r') + b_{i-1}(s+s') + (-d)f_{p_i,q_i}(r+r', s+s') &\leq a_{i-1}(r+r') + b_{i-1}(s+s') + (-d)f_{p_i,q_i} + (-j)f_{p,q}(r',s') \\ -d \cdot f_{p_i,q_i}(r+r', s+s') &\leq -d \cdot f_{p_i,q_i}(r',s') \\ -d[q_i(r+r') - p_i(s+s')] &\leq -d(q_i r' - p_i s') \\ -d(q_i r - p_i s) &\leq 0 \\ -dq_i r &\leq -dp_i s \end{aligned}$$

Since $(r,s) \in Q_{i-1} \subset \mathbb{N}^2$, r and s must satisfy the inequality $q_i r \leq p_i s$. It follows then that for $-dq_i r \leq -dp_i s$ to hold, we must have $-d \geq 0$ which implies that $d \leq 0$. Let $k_i = -d \geq 0$ so that we may now write $f_{Q_{i-1},Q_i}(r,s) = a_{i-1}r + b_{i-1}s + k_i \cdot f_{p_i,q_i}(r,s)$ for $a_{i-1}, b_{i-1} \in \mathbb{Z}$, $k_i \in \mathbb{N}$.

It follows then that if $a_{i-1} = a_0 + k_1 q_1 + \cdots + k_{i-1}q_{i-1}$ and $b_{i-1} = b_0 - k_1 p_1 - \cdots - k_{i-1}p_{i-1}$, the restriction of f to the connected subfan $\bigcup_{j=0}^{i} Q_j$ is given by $f_{Q_0,\ldots,Q_i} = a_0 r + b_0 s + k_1 \cdot f_{p_1,q_1}(r,s) + \cdots + k_{i-1} \cdot f_{p_{i-1},q_{i-1}}(r,s) + k_i \cdot f_{p_i,q_i}(r,s)$. Consequently, we see that for all $(r,s) \in \mathbb{N}^2$, $f(r,s) = a_0 r + b_0 s + k_1 \cdot f_{p_1,q_1}(r,s) + \cdots + k_n \cdot f_{p_n,q_n}(r,s)$ for $k_1, \ldots, k_n \in \mathbb{N}$.

Note that this implies condition (1). Since f is fan-linear and this implies that $f(Q_i) \subseteq \mathbb{N}$ for each $i = 0, 1, \ldots, n$, condition (2) follows.

For the converse, assume that f is a map satisfying the given condition that $f(r, s) = a_0 r + b_0 s + k_1 \cdot f_{p_1,q_1}(r, s) + \cdots + k_n \cdot f_{p_n,q_n}(r, s)$ for all $(r, s) \in \mathbb{N}^2$. Assume further that $k_i \geq 0$ for all $i = 1, \ldots, n$ and $f(\alpha) \geq 0$ for all $\alpha \in \{(0, 1) = (p_0, q_0), (p_1, q_1), \ldots, (p_n, q_n), (1, 0) = (p_{n+1}, q_{n+1})\}$. In order to show that f is fan-linear, we must show that f is a semigroup homomorphism that maps into $\mathbb{N}$ when restricted to each cone Q_i and also that f is subadditive on all of $\mathbb{N}^2$.

Let $(r, s) \in Q_i$ for any $i \in \{0, 1, \ldots, n\}$. As f_{p_j,q_j} vanishes when $\frac{s}{r} > \frac{p_j}{q_j}$, we see that $f(r, s) = a_0 r + b_0 s + k_1 \cdot f_{p_1,q_1}(r, s) + \cdots + k_i \cdot f_{p_i,q_i}(r, s) = (a_0 + k_1 q_1 + \cdots + k_i q_i)r + (b_0 - k_1 p_1 - \cdots - k_i p_i)s$ for all $(r, s) \in Q_i$. Let $(r, s), (r', s') \in Q_i$ and let $a' = a_0 + k_1 q_1 + \cdots + k_i q_i$ and $b' = b_0 - k_1 p_1 - \cdots - k_i p_i$. Then $f(r + r', s + s') = a'(r + r') + b'(s + s') = a'r + b's + a'r' + b's' = f(r, s) + f(r', s')$ which shows that f is a semigroup homomorphism when restricted to Q_i. Since $f_{|Q_i}$ is a semigroup homomorphism into $\mathbb{Z}$, by Corollary 3.5 and the conditions of the hypothesis, $\mathrm{Im}(f_{Q_i}) \subset \mathbb{N}$ for each i. Note that the map $(r, s) \mapsto a_0 r + b_0 s$ is linear on $\mathbb{N}^2$, which implies that it is trivially subadditive on $\mathbb{N}^2$. By Proposition 4.10, f_{p_i,q_i} is subadditive for each $i = 1, \ldots, n$. Since the sum of subadditive functions is subadditive by Proposition 4.6, $k_i \cdot f_{p_i,q_i}$ is subadditive for each $i = 1, \ldots, n$. It follows then that since f is the sum of subadditive functions, $f(r, s) = a_0 r + b_0 s + k_1 \cdot f_{p_1,q_1}(r, s) + \cdots + k_n \cdot f_{p_n,q_n}(r, s)$ is subadditive on $\mathbb{N}^2$. This completes the proof. □

4.1. **Representative Semigroup and Correspondence with $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$.** We now show that $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ is naturally isomorphic to an affine semigroup that embeds in $\mathbb{Z}^{n+2}$ when $\Sigma_{\mathbf{a},\mathbf{b}}$ is composed of $n + 1$ cones.

Definition 4.13. Fix a fan $\Sigma_{\mathbf{a},\mathbf{b}} = \{Q_0, Q_1, \ldots, Q_n\}$ of $n + 1$ cones with faces defined by $\{(p_i, q_i) \in \mathbb{N}^2\}_{i=0}^{n+1}$. Let $C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})} = \{(a, b, k_1, \ldots, k_n) \in \mathbb{Z}^{n+2}\}$ so that:

(1) $k_i \geq 0$ for $i = 1, \ldots, n$

(2) $ap_i + bp_i + k_1(q_1 p_i - q_i p_1) + \cdots + k_{i-1}(q_{i-1} p_i - q_i p_{i-1}) \geq 0$ for all $i = 0, 1, \ldots, n$.

We call $C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}$ the *correspondence semigroup of* $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$.

Proposition 4.14. *$C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}$ is an affine semigroup and has a finite and unique minimal set of generators.*

Proof. Let B be the set of coordinates $(a, b, k_1, \ldots, k_n) \in \mathbb{R}^{n+2}$ that satisfy the following inequalities:

$$a \cdot 1 + b \cdot 0 + k_1 \cdot q_1 + \cdots + k_n \cdot q_n \geq 0$$
$$a \cdot p_n + b \cdot q_n + k_1 \cdot (q_1 p_n - q_n p_1) + \cdots + k_{n-1} \cdot (q_{n-1} p_n - q_n p_{n-1}) + k_n \cdot 0 \geq 0$$
$$a \cdot p_{n-1} + b \cdot q_{n-1} + k_1 \cdot (q_1 p_{n-1} - q_{n-1} p_1) + \cdots + k_{n-2} \cdot (q_{n-1} p_n - q_n p_{n-1}) + +k_{n-1} \cdot 0 + k_n \cdot 0 \geq 0$$
$$\vdots$$
$$a \cdot p_2 + b \cdot q_2 + k_1 \cdot (q_1 p_2 - q_2 p_1) + k_2 \cdot 0 + \cdots + k_n \cdot 0 \geq 0$$
$$a \cdot p_1 + b \cdot q_1 + k_1 \cdot 0 + \cdots + k_n \cdot 0 \geq 0$$
$$a \cdot 0 + b \cdot 1 + k_1 \cdot 0 + \cdots + k_n \cdot 0 \geq 0$$

Then B is the intersection of finitely many closed linear half-spaces in $\mathbb{R}^{n+2}$. Considering the resulting equations by setting the inequalties to equations, it follows that each bounding hyperplane of these half-spaces contains the origin. B is therefore a polyhedral cone in $\mathbb{R}^{n+2}$, and by Theorem 2.6, $B = \{\lambda_1 b_1 + \cdots + \lambda_r b_r | \lambda_i \in \mathbb{R}_{\geq 0}\}$.

Since the bounding hyperplanes of B are all solutions to linear equations with integer coefficients, it is clear that each has a solution set in $\mathbb{Q}^{n+2}$, so we may take $b_1, \ldots, b_r \in \mathbb{Q}^{n+2}$ which implies that B is a rational cone. Since $C_{\mathcal{F}(A)} \subset B$ consists of all coordinates in B with integer entries, it follows that $C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})} = B \cap \mathbb{Z}^{n+2}$ and by Theorem 2.8, $C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}$ is an affine semigroup.

We now claim that $C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}$ is pointed. Consider $(a, b, k_1, \ldots, k_n) \in C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}$. With the possibility that $a < 0$, two cases arise. First, if not all $b, k_1, \ldots, k_n$ are zero, then as $b \geq 0$ and $k_i \geq 0$ for all $i = 1, \ldots, n$, $(-a, -b, -k_1, \ldots, -k_n) \notin C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}$. Second, if all $b = k_1 = \cdots = k_n = 0$ and $a \neq 0$, then we must have that $a > 0$ by condition (3) of definition 4.13. This implies that $(-a, 0, 0, \ldots, 0) \notin C_{\mathcal{F}(A)}$. It follows then that $C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})} \cap (-C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}) = \{\mathbf{0}\}$ and by Theorem 2.9, $C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}$ has a unique finite minimal generating set. □

Now we establish the natural isomorphism.

Theorem 4.15. *There is an isomorphism of semigroups between the semigroup of fan-linear maps $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ and the affine semigroup $C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}$. Note that in particular, this shows that $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ is an affine semigroup with a finite and unique minimal set of generators.*

Proof. Define the set map Φ from $C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}$ to $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ by $(a, b, k_1, \ldots, k_n) \mapsto f(r, s) = ar + bs + k_1 \cdot f_{p_1,q_1}(r, s) + \cdots + k_n \cdot f_{p_n,q_n}(r, s)$. This map is clearly well defined and the conditions of definition 4.13 imply that $k_i \geq 0$ for $i = 1, \ldots, n$ and $f(\alpha) \geq 0$ for all $\alpha \in \{(0, 1) = (p_0, q_0), (p_1, q_1), \ldots, (p_n, q_n), (1, 0) = (p_{n+1}, q_{n+1})\}$. Then by Theorem 4.12, Φ is surjective.

Suppose that $ar + bs + k_1 \cdot f_{p_1,q_1}(r,s) + \cdots + k_n \cdot f_{p_n,q_n}(r,s) = a'r + b's + k'_1 \cdot f_{p_1,q_1}(r,s) + \cdots + k'_n \cdot f_{p_n,q_n}(r,s)$ for all $(r,s) \in \mathbb{N}^2$. Then evaluating the given fan-linear maps at $(0,1)$, we see that $b = b'$. Next, evaluating at (p_1, q_1), we have that $ap_1 + bq_1 = a'p_1 + b'q_1$. As $b = b'$, it follows that $a = a'$. Similarly, by successively evaluating at (p_i, q_i) for each $i = 2, \ldots, n+1$, we have that $k_i = k'_i$ for each $i = 1, \ldots, n$. This shows that Φ is injective.

Since $\Phi[(a, b, k_1, \ldots, k_n) + (a', b', k'_1, \ldots, k'_n)] = \Phi[(a + a', b + b', k_1 + k'_1, \ldots, k_n + k'_n)] = (a + a')r + (b + b')s + (k_1 + k'_1) \cdot f_{p_1,q_1}(r,s) + \cdots + (k_n + k'_n) \cdot f_{p_n,q_n}(r,s) = ar + bs + k_1 \cdot f_{p_1,q_1}(r,s) + \cdots + k_n \cdot f_{p_n,q_n}(r,s) + a'r + b's + k'_1 \cdot f_{p_1,q_1}(r,s) + \cdots + k'_n \cdot f_{p_n,q_n}(r,s) = \Phi[(a, b, k_1, \ldots, k_n)] + \Phi[(a', b', k'_1, \ldots, k'_n)]$, we see that Φ is a semigroup homomorphism and therefore an isomorphism. □

Because $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ is a pointed affine semigroup, we now can assume that for any fixed fan $\Sigma_{\mathbf{a},\mathbf{b}}$, there exists a unique, minimal set of fan-linear maps $g_1, \ldots, g_k \in \mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ such that *every* fan-linear map can be written as $f(r,s) = n_1 g_1(r,s) + \cdots + n_k g_k(r,s)$. In fact, the above correspondence gives us a method to find such generating functions; we simply need to find the generators for $C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}$ which are uniquely determined by $\Sigma_{\mathbf{a},\mathbf{b}}$.

5. Fan Algebras

For this section, assume that we are given a fan $\Sigma_{\mathbf{a},\mathbf{b}}$. First, we recall the definition of a fan algebra.

Definition 5.1. Given ideals $I_1, \ldots, I_n$ in a Noetherian domain R, a fan $\Sigma_{\mathbf{a},\mathbf{b}}$ of cones in $\mathbb{N}^2$, and fan-linear maps $f_1, \ldots, f_n$, define a fan-algebra of $f = (f_1, \ldots, f_n)$ on $\Sigma_{\mathbf{a},\mathbf{b}}$ as:

$$\mathcal{B}(\Sigma_{\mathbf{a},\mathbf{b}}, f) = \oplus_{r,s} I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} u^r v^s$$

In order to study fan-algebras in the context of semigroups, we need an associative binary operation that allows to classify certain sets of fan algebras as such.

Proposition 5.2. *Let $\mathcal{B}_{\mathbf{a},\mathbf{b}}$ be the set of all fan algebras on a fixed fan $\Sigma_{\mathbf{a},\mathbf{b}}$. For $A, B \in \mathcal{B}_{\mathbf{a},\mathbf{b}}$, $A = \bigoplus_{r,s} I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} u^r v^s$ and $B = \bigoplus_{r,s} J_1^{g_1(r,s)} \cdots J_m^{g_m(r,s)} u^r v^s$, define $G(A,B) = C = \bigoplus_{r,s} I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} J_1^{g_1(r,s)} \cdots J_m^{g_m(r,s)} u^r v^s$. Then $G \colon \mathcal{B}_{\mathbf{a},\mathbf{b}} \times \mathcal{B}_{\mathbf{a},\mathbf{b}} \to \mathcal{B}_{\mathbf{a},\mathbf{b}}$ is an associative, commutative binary operation with identity so that $\mathcal{B}_{\mathbf{a},\mathbf{b}}$ forms a semigroup.*

Proof. First, we must show that the operation above is well defined. Assume that $\bigoplus_{r,s} I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} u^r v^s = \bigoplus_{r,s} I_1'^{f'_1(r,s)} \cdots I_n'^{f'_n(r,s)} u^r v^s$ and $\bigoplus_{r,s} J_1^{g_1(r,s)} \cdots J_m^{g_m(r,s)} u^r v^s = \bigoplus_{r,s} J_1'^{g'_1(r,s)} \cdots J_m'^{g'_m(r,s)} u^r v^s$. Because u and v are indexing variables, it follows that $\bigoplus_{r,s} I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} u^r v^s = \bigoplus_{r,s} I_1'^{f'_1(r,s)} \cdots I_n'^{f'_n(r,s)} u^r v^s$ if and only if $I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} = I_1'^{f'_1(r,s)} \cdots I_n'^{f'_n(r,s)}$ for all $(r,s) \in \mathbb{N}^2$. To show well definition, it is enough to show that under the assumptions above $I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} J_1^{g_1(r,s)} \cdots J_m^{g_m(r,s)} = I_1'^{f'_1(r,s)} \cdots I_n'^{f'_n(r,s)} J_1'^{g'_1(r,s)} \cdots J_m'^{g'_m(r,s)}$ for any fixed $(r,s) \in \mathbb{N}^2$, but since the left and the right side of this equation are ideals in R, it is clear then that the equation holds. Associativity follows similarly since $A(BC) = (AB)C$ for any ideals $A, B, C \subseteq R$, and commutativity follows from the assumption that R is commutative. We may also take the polynomial ring in u and v, that is $R[u,v] = \oplus_{r,s} R u^r v^s$, to be the identity element of this operation so that $\mathcal{B}_{\mathbf{a},\mathbf{b}}$ forms a semigroup with the claimed properties. □

To highlight the fact that the above operation is a commutative semigroup operation, we write $G(A, B)$ as $A + B$ and for $n \in \mathbb{N}$, $n \cdot A = \underbrace{A + \cdots + A}_{n \text{ times}}$. For convenience herein, we will use the notation $\alpha \in \mathbb{N}^2$ meaning that $\alpha = (r, s)$, and we write $u^r v^s = \mathbf{u}^\alpha$. Next, we use the semigroup properties of the affine semigroup $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ to show that fixing a set of ideals and considering all of the fan algebras constructed on those ideals establishes a finitely generated subsemigroup of $\mathcal{B}_{\mathbf{a},\mathbf{b}}$.

Proposition 5.3. *Let $\mathcal{J} = \{I_1, \ldots, I_n\}$ be a finite family of nonzero ideals in R and let $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ denote the set of all fan algebras of the form $A = \oplus_\alpha I_1^{f_1(\alpha)} \cdots I_n^{f_n(\alpha)} \mathbf{u}^\alpha$. Then $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ is a finitely generated subsemigroup of $\mathcal{B}_{\mathbf{a},\mathbf{b}}$.*

Proof. Let $A = \oplus_\alpha I_1^{f_1(\alpha)} \cdots I_n^{f_n(\alpha)} u^\alpha$ and $B = \oplus_\alpha I_1^{g_1(\alpha)} \cdots I_n^{g_n(\alpha)} u^\alpha$. Then $A + B = \oplus_\alpha I_1^{f_1(\alpha)} \cdots I_n^{f_n(\alpha)} I_1^{g_1(\alpha)} \cdots I_n^{g_n(\alpha)} u^\alpha = I_1^{(f_1+g_1)(\alpha)} \cdots I_n^{(f_n+g_n)(\alpha)} u^\alpha$, and as each $f_i + g_i$ is fan-linear, $A + B \in \mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$.

Now consider $A \in \mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$, $A = \oplus_\alpha I_1^{f_1(\alpha)} \cdots I_n^{f_n(\alpha)} u^\alpha$. As $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ is a pointed affine semigroup by Theorem 4.15, there exists a unique minimal set of fan-linear maps $\{b_1, \ldots, b_m\}$ such that $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}}) = \{c_1 b_1 + \cdots + c_m b_m \mid c_i \in \mathbb{N}\}$. Then we may rewrite $f_1, \ldots, f_n$:

$$\begin{aligned} f_1(\alpha) &= c_{1,1} b_1(\alpha) + \cdots + c_{1,m} b_m(\alpha) \\ &\vdots \\ f_n(\alpha) &= c_{n,1} b_1(\alpha) + \cdots + c_{n,m} b_m(\alpha) \end{aligned}$$

Let $\mathcal{B}_{i,j} = \oplus_\alpha I_i^{b_j(\alpha)} u^\alpha$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$, noting that since 0 is a fan-linear map, $\mathcal{B}_{i,j} \in \mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$. We may write A:

$$\begin{aligned} A &= \oplus_\alpha I_1^{f_1(\alpha)} \cdots I_n^{f_n(\alpha)} u^\alpha \\ &= \oplus_\alpha I_1^{c_{1,1}b_1(\alpha)+\cdots+c_{1,m}b_m(\alpha)} \cdots I_n^{c_{n,1}b_1(\alpha)+\cdots+c_{n,m}b_m(\alpha)} u^\alpha \\ &= \oplus_\alpha I_1^{c_{1,1}b_1(\alpha)} \cdots I_1^{c_{1,m}b_m(\alpha)} \cdots I_n^{c_{n,1}b_1(\alpha)} \cdots I_n^{c_{n,m}b_m(\alpha)} u^\alpha \\ &= \oplus_\alpha I_1^{c_{1,1}b_1(\alpha)} \cdots I_n^{c_{n,1}b_1(\alpha)} I_1^{c_{1,2}b_{1,2}(\alpha)} \cdots I_1^{c_{1,m}b_m(\alpha)} \cdots I_n^{c_{n,m}b_m(\alpha)} u^\alpha \\ &= \oplus_\alpha I_1^{c_{1,1}b_1(\alpha)} u^\alpha + \cdots + \oplus_\alpha I_n^{c_{n,1}b_1(\alpha)} u^\alpha + \cdots + \oplus_\alpha I_1^{c_{1,m}b_m(\alpha)} u^\alpha + \cdots + \oplus_\alpha I_n^{c_{n,m}b_m(\alpha)} u^\alpha \\ &= \oplus_\alpha (I_1^{b_1(\alpha)})^{c_{1,1}} u^\alpha + \cdots + \oplus_\alpha (I_n^{b_1(\alpha)})^{c_{n,1}} u^\alpha + \cdots + \oplus_\alpha (I_1^{b_m(\alpha)})^{c_{1,m}} u^\alpha + \cdots + \oplus_\alpha (I_n^{b_m(\alpha)})^{c_{n,m}} u^\alpha \\ &= c_{1,1} \cdot \mathcal{B}_{1,1} + \cdots + c_{n,1} \cdot \mathcal{B}_{n,1} + c_{1,2} \cdot \mathcal{B}_{1,2} + \cdots + c_{1,m} \cdot \mathcal{B}_{1,m} + \cdots + c_{n,m} \cdot \mathcal{B}_{n,m} \end{aligned}$$

It follows that the set of $\mathcal{B}_{i,j} = \oplus_\alpha I_i^{b_j(\alpha)} u^\alpha$ generate $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$. □

Now we know that for any finite family $\mathcal{J}$ of ideals in R, the semigroup $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ is finitely generated, so it is of immediate interest to know when $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ is an affine semigroup.

5.1. **Affine Semigroup Criteria.**

Definition 5.4. A family $\mathcal{I}$ of ideals in R is called *cancellative* if $IJ_1 = IJ_2$ implies that $J_1 = J_2$ for all $I, J_1, J_2 \in \mathcal{I}$, and *power cancellative* if $J_1^n = J_2^n$ implies that $J_1 = J_2$.

Theorem 5.5. *([1], page 56) A finitely generated commutative semigroup S is affine, i.e. S embeds into $\mathbb{Z}^d$ for some $d \in \mathbb{N}$ if and only if S is reduced, cancellative, and power cancellative.*

Definition 5.6. Let $\mathcal{I}$ be a family of nonzero ideals in R with $R \in \mathcal{I}$. The *multiplicative semigroup of* $\mathcal{I}$, denoted $S(\mathcal{I})$, is the collection of all finite products of elements of $\mathcal{I}$:

$$S(\mathcal{I}) = \left\{ \prod_{\text{finite}} I \mid I \in \mathcal{I} \right\}$$

Proposition 5.7. *The multiplicative semigroup $S(\mathcal{I})$ is a commutative semigroup with identity under the operation of ideal multiplication. Furthermore, $S(\mathcal{I})$ is reduced and if $|\mathcal{I}| < \infty$, then $S(\mathcal{I})$ is finitely generated.*

Proof. For any two ideals I and J in R, $IJ = \{\sum_{\text{finite}} ij \mid i \in I, j \in J\}$ is an associative and commutative binary operation, hence $S(\mathcal{I})$ is a commutative semigroup. Since $R \in \mathcal{I}$ and $RI = IR = I$ for any ideal I of R, R serves as the identity element of $S(\mathcal{I})$. Since $IJ = R$ implies that $R \subseteq I$, it follows that $S(\mathcal{I})$ is reduced. Finite generation of $S(\mathcal{I})$ under the hypothesis that $\mathcal{I}$ has finite cardinality follows directly from the definition of $\mathcal{I}$. □

Note that by construction, the semigroup of fan algebras $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ can be viewed as a subset of the set of all sequences on elements of $S(\mathcal{I})$ indexed by $\mathbb{N}^2$. Addition in $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ can be seen as component-wise multiplication of sequences $\{a_\alpha\}_{\alpha\in\mathbb{N}^2}$ where each $a_\alpha \in S(\mathcal{I})$, and for this reason, important semigroup properties of $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ are completely determined by the semigroup properties of $S(\mathcal{I})$. For instance, we observe here without proof that $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ is clearly reduced for any nonempty family $\mathcal{I}$ of ideals in R, as is the case with $S(\mathcal{I})$. Next, we show show that other important properties of $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ are dependent on the properties of $S(\mathcal{I})$.

Proposition 5.8. *Let $\mathcal{I}$ be a family of nonzero ideals in R so that $R \in \mathcal{I}$. The semigroup $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ is cancellative and respectively power cancellative if and only if $S(\mathcal{I})$ is is cancellative and respectively power cancellative.*

Proof. Assume that $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ is cancellative and respectively power cancellative. Note that the map $\pi_1\colon (r,s) \mapsto r$ is fan-linear on any given fan $\Sigma_{\mathbf{a},\mathbf{b}}$ (as it is linear), hence $n\cdot\pi_1\colon (r,s) \mapsto n\cdot r$ is fan-linear for all $n \in \mathbb{N}$. To show that $S(\mathcal{I})$ is cancellative, let $I_1, \ldots, I_k \in \mathcal{I}$ and assume that $I_1^{j_1+k_1} \cdots I_n^{j_n+k_n} = I_1^{j_1+h_1} \cdots I_n^{j_n+h_n}$. It follows then that:

$$\bigoplus_{\alpha} \left(I_1^{(j_1+k_1)} \cdots I_n^{(j_n+k_n)} \right)^{\pi_1(\alpha)} \mathbf{u}^\alpha = \bigoplus_{\alpha} \left(I_1^{(j_1+h_1)} \cdots I_n^{(j_n+h_n)} \right)^{\pi_1(\alpha)} \mathbf{u}^\alpha$$

$$\bigoplus_{\alpha} I_1^{(j_1+k_1)\pi_1(\alpha)} \cdots I_n^{(j_n+k_n)\pi_1(\alpha)} \mathbf{u}^\alpha = \bigoplus_{\alpha} I_1^{(j_1+h_1)\pi_1(\alpha)} \cdots I_n^{(j_n+h_n)\pi_1(\alpha)} \mathbf{u}^\alpha$$

Since $\mathcal{B}_{\mathbf{a},\mathbf{b}}$ is cancellative, we may factor the term $\bigoplus_\alpha I_1^{j_1\pi_1(\alpha)} \cdots I_n^{j_n\pi_1(\alpha)} \mathbf{u}^\alpha$ from both sides and achieve:

$$\bigoplus_{\alpha} I_1^{k_1\pi_1(\alpha)} \cdots I_n^{k_n\pi_1(\alpha)} \mathbf{u}^\alpha = \bigoplus_{\alpha} I_1^{h_1\pi_1(\alpha)} \cdots I_n^{h_n\pi_1(\alpha)} \mathbf{u}^\alpha$$

Considering $\alpha = (1,0)$, we have that $I_1^{k_1} \cdots I_n^{k_n} u = I_1^{h_1} \cdots I_n^{h_n} u$ which implies that $I_1^{k_1} \cdots I_n^{k_n} = I_1^{h_1} \cdots I_n^{h_n}$. This shows that $S(\mathcal{I})$ is cancellative. To show that $S(\mathcal{I})$ is power cancellative, assume that $I_1^{m\cdot k_1} \cdots I_n^{m\cdot k_n} = I_1^{m\cdot h_1} \cdots I_n^{m\cdot h_n}$. Then as before:

$$\bigoplus_{\alpha} I_1^{m\cdot k_1\pi_1(\alpha)} \cdots I_n^{m\cdot k_n\pi_1(\alpha)} \mathbf{u}^\alpha = \bigoplus_{\alpha} I_1^{m\cdot h_1\pi_1(\alpha)} \cdots I_n^{m\cdot h_n\pi_1(\alpha)} \mathbf{u}^\alpha$$

$$m\cdot\bigoplus_{\alpha} I_1^{k_1\pi_1(\alpha)}\cdots I_n^{k_n\pi_1(\alpha)}\mathbf{u}^\alpha = m\cdot\bigoplus_{\alpha} I_1^{h_1\pi_1(\alpha)}\cdots I_n^{h_n\pi_1(\alpha)}\mathbf{u}^\alpha$$

Since $\mathcal{B}_{\mathbf{a},\mathbf{b}}$ is power cancellative, we see in the same manner as before that $I_1^{k_1}\cdots I_n^{k_n} = I_1^{h_1}\cdots I_n^{h_n}$, hence $S(\mathcal{I})$ is power cancellative. Now assume that $S(\mathcal{I})$ is a cancellative and respecitively power cancellative semigroup. If $\bigoplus_\alpha I_1^{f_1(\alpha)+g_1(\alpha)}\cdots I_n^{f_n(\alpha)+g_n(\alpha)}\mathbf{u}^\alpha = \bigoplus_\alpha I_1^{f_1(\alpha)+h_1(\alpha)}\cdots I_n^{f_n(\alpha)+h_n(\alpha)}\mathbf{u}^\alpha$, then $I_1^{f_1(\alpha)+g_1(\alpha)}\cdots I_n^{f_n(\alpha)+g_n(\alpha)} = I_1^{f_1(\alpha)+h_1(\alpha)}\cdots I_n^{f_n(\alpha)+h_n(\alpha)}$ for all $\alpha\in\mathbb{N}^2$. Since $S(\mathcal{I})$ is cancellative, this implies that $I_1^{g_1(\alpha)}\cdots I_n^{g_n(\alpha)} = I_1^{h_1(\alpha)}\cdots I_n^{h_n(\alpha)}$ for all $\alpha\in\mathbb{N}^2$, so $\bigoplus_\alpha I_1^{g_1(\alpha)}\cdots I_n^{g_n(\alpha)}\mathbf{u}^\alpha = \bigoplus_\alpha I_1^{h_1(\alpha)}\cdots I_n^{h_n(\alpha)}\mathbf{u}^\alpha$ and therefore $\mathcal{B}_{\mathbf{a},\mathbf{b}}$ is cancellative.

Similarly, if $n\cdot\bigoplus_\alpha I_1^{g_1(\alpha)}\cdots I_n^{g_n(\alpha)}\mathbf{u}^\alpha = n\cdot\bigoplus_\alpha I_1^{h_1(\alpha)}\cdots I_n^{h_n(\alpha)}\mathbf{u}^\alpha$ for some $n>0$, then $I_1^{n\cdot g_1(\alpha)}\cdots I_n^{n\cdot g_n(\alpha)} = I_1^{n\cdot h_1(\alpha)}\cdots I_n^{n\cdot h_n(\alpha)}$ for all $\alpha\in\mathbb{N}^2$. Since $S(\mathcal{I})$ is power cancellative, we have that $\bigoplus_\alpha I_1^{g_1(\alpha)}\cdots I_n^{g_n(\alpha)}\mathbf{u}^\alpha = \bigoplus_\alpha I_1^{h_1(\alpha)}\cdots I_n^{h_n(\alpha)}\mathbf{u}^\alpha$. □

Corollary 5.9. *Let $\mathcal{J}$ be a finite family of nonzero ideals in R. The semigroup of fan-algebras $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ is affine if and only if $S(\mathcal{J})$ is affine.*

Proof. Here, we simply take the additional hypothesis that $\mathcal{J}$ is finite from which it follows that both $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ and $S(\mathcal{J})$ are finitely generated and reduced as semigroups. The claim then follows immediately from Proposition 5.8. □

Lemma 5.10. *Let $\mathcal{J}$ be a finite family of ideals with $\mathcal{J}\subset\mathcal{I}$, where $\mathcal{I}$ is a family of nonzero ideals of R and $R\in\mathcal{I}$. If $\mathcal{I}=S(\mathcal{I})$ (i.e. $\mathcal{I}$ is closed under finite products) and $\mathcal{I}$ is cancellative, then $S(\mathcal{J})$ is cancellative; if $\mathcal{I}$ is power cancellative, then $S(\mathcal{J})$ is power cancellative. In particular, if $\mathcal{I}$ is cancellative and power cancellative, then $S(\mathcal{J})$ is an affine semigroup.*

Proof. Note that $\mathcal{J}\subset\mathcal{I}$ implies that $S(\mathcal{J})\subset S(\mathcal{I})=\mathcal{I}$. If for any $I,J_1,J_2\in\mathcal{I}$, $IJ_1 = IJ_2$ implies that $J_1=J_2$, then for any $I_1,\ldots,I_n\in\mathcal{J}$, the equation $I_1^{j_1+k_1}\cdots I_n^{j_n+k_n} = I_1^{j_1+h_1}\cdots I_n^{j_n+h_n}$ implies that $I_1^{k_1}\cdots I_n^{k_n} = I_1^{h_1}\cdots I_n^{h_n}$ since $I_1^{j_1}\cdots I_n^{j_n}$ is an element of $\mathcal{I}$. The power cancellative property follows similarly and the third assertion is therefore clear. □

The above lemma is useful in simplifying arguments regarding when a finite family $\mathcal{J}$ generates an affine semigroup $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$.

Proposition 5.11. *Let $\mathcal{J}$ be a finite family of nonzero principal ideals in a unique factorization domain R. The semigroup of fan algebras $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ is affine.*

Proof. Note that the product of two principal ideals is principle, hence the family $\mathcal{I}$ of all nonzero principal ideals in any domain is closed under finite products. By Lemma 5.10, it is enough to show that $\mathcal{I}$ is cancellative and power cancellative. Assume that $IJ_1 = IJ_2$ for $I,J_1,J_2\in\mathcal{I}$. Then where $I=\langle i\rangle$, $J_1=\langle j_1\rangle$ and $J_2=\langle j_2\rangle$, $ij_1 = uij_2$ for some unit $u\in R$, so $j_1 = uj_2$ and hence $J_1=J_2$. If $J_1^n = J_2^n$, then $j_1^n = uj_2^n$ for some unit u in R. Because R is a unique factorization domain, it is easy to see that $j_1 = u'j_2$ for some unit $u'\in R$ and hence $J_1=J_2$. □

We note that the above proposition implies that the same is true for any finite family of ideals in a principal ideal domain. Also, the proof shows that the cancellative property holds for principal ideals in *any* domain R.

Proposition 5.12. *Let K be a field and $R = K[x_1, \dots, x_n] = K[\mathbf{x}]$ and let $\mathcal{J}$ be a finite family of monomial ideals in R. The semigroup of fan algebras $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ is affine.*

Proof. First, note that the family $\mathcal{I} \cup R$, where $\mathcal{I}$ is the family of all monomial ideals of R, is closed under multiplication; for $I = \langle \mathbf{x}^\alpha \rangle_{\alpha \in A}$ and $J = \langle \mathbf{x}^\beta \rangle_{\beta \in B}$ where A and B are finite sets of multidegrees, $IJ = \langle \mathbf{x}^{\alpha+\beta} \rangle_{(\alpha,\beta) \in A \times B}$. To show that $\mathcal{I}$ cancellative and power cancellative, we proceed by contrapositive. Assume that $J_1 = \langle \mathbf{x}^{\beta_1} \rangle_{\beta_1 \in B_1}$ and $J_2 = \langle \mathbf{x}^{\beta_2} \rangle_{\beta_2 \in B_1}$. If $J_1 \neq J_2$, without loss of generality we may assume that there is some $\beta_1 \in B_1$ such that $\mathbf{x}^{\beta_1} \notin J_2$ (otherwise $J_1 \subseteq J_2$). Observe that for this to occur, the multidegree $\beta_1 < \beta_2$ for all $\beta_2 \in B_2$ under some appropriately chosen monomial order. Then for any ideal $I \in \mathcal{I} \cup R$, I is nonzero and hence $I\mathbf{x}^{\beta_1} \not\subset IJ_2 = \sum_{\beta_2 \in B_2} I\mathbf{x}^{\beta_2}$, so $IJ_1 \neq IJ_2$. In the same manner, assuming again that $\mathbf{x}^{\beta_1} \notin J_2$, it is clear that since $\beta_1 < \beta_2$ for all $\beta_2 \in B_2$, $n \cdot \beta_1 < \sum_{i=1}^n \beta_{2_i}$ for any n (possibly repeated) $\beta_{2_i} \in B_2$. It follows that $\mathbf{x}^{n \cdot \beta_1} \notin J_2^n = \sum_{\beta_2 \in B_2} J_2^{n-1} \mathbf{x}^{\beta_2}$ so that $J_1^n \neq J_2^n$. Then $\mathcal{I} \cup R$ is cancellative and power cancellative which implies that $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ is affine. $\square$

We also wish to know when a family $\mathcal{J}$ is such that $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ is necessarily *not* affine. We give some conditions below.

Example 5.13. If $\mathcal{J}$ is a finite family of nonzero ideals of R such that $\mathcal{J}$ contains a nilpotent or a proper, nonzero idempotent ideal, then $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ is not affine. If R is an Artinian ring, then for any finite family $\mathcal{J}$ containing I such that $\langle 0 \rangle \subsetneq I \subsetneq R$, $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ is not affine.

Proof. If $I \in \mathcal{J}$ and $I^k = \langle 0 \rangle$ for some k, then $I^k = \langle 0 \rangle^k$ but $I \neq \langle 0 \rangle$ (since $\mathcal{J}$ is a family of nonzero ideals). It follows that $S(\mathcal{J})$ is not power cancellative and hence $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ is not affine. If $R \neq I \in \mathcal{J}$ and $I = I^2$, then $RI = I^2$ and $I \neq R$, hence $S(\mathcal{J})$ and subsequently $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ is not cancellative and therefore not affine. Similar to the case where I is idempotent, assume that R is Artinian and choose some $k \in \mathbb{N}$ such that $I^k = I^{k+i}$ for all $i \in \mathbb{N}$. Then $RI^k = I^{k+1}$ but $I \neq R$, hence $S(\mathcal{J})$ is not affine. $\square$

Recall the definition of the determining map:

$$f_{p,q}(r,s) = \begin{cases} 0 & \text{if } \frac{s}{r} > \frac{q}{p} \\ qr - ps & \text{if } \frac{s}{r} \leq \frac{q}{p} \end{cases}$$

Due to the conditions defining membership in $C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}$, it is easy to see that if $(p,q) \neq (1,0)$ or $(0,1)$ is an element on a face of $\Sigma_{\mathbf{a},\mathbf{b}}$, then the map $f_{p,q}$ will be included in the unique minimal set of generators for $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$. Then as implied in Proposition 5.3, we may take $\bigoplus_\alpha I^{f_{p,q}(\alpha)} \mathbf{u}^\alpha$ to be a generator for $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ whenever $I \in \mathcal{J}$. It is useful then to describe the fan algebra $\bigoplus_\alpha I^{f_{p,q}(\alpha)} \mathbf{u}^\alpha$ when possible.

Proposition 5.14. *Let $\mathcal{J} \subset \mathcal{I}$ so that $\mathcal{I} = S(\mathcal{I})$ is cancellative. If $I \in \mathcal{J}$ and $(I^{qr} \colon I^{ps}) \in \mathcal{I}$ for all $(r,s) \in \mathbb{N}^2$, then $\bigoplus_\alpha I^{f_{p,q}(\alpha)} \mathbf{u}^\alpha = \bigoplus_{(r,s)} (I^{qr} \colon I^{ps})\, \mathbf{u}^{(r,s)}$.*

Proof. Observe that if $\frac{s}{r} > \frac{q}{p}$, then $qr < ps$ and $I^{qr} \subset I^{ps}$, so $I^{f_{p,q}(r,s)} = I^0 = R = (I^{qr} \colon I^{ps})$. If $\frac{s}{r} \leq \frac{q}{p}$, then $qr \geq ps$ so that $qr - ps \geq 0$. Now $(I^{qr} \colon I^{ps})\, I^{ps} \subseteq I^{qr} = I^{qr-ps} I^{ps}$ by definition, and since $I^{qr-ps} \subseteq (I^{qr} \colon I^{ps})$, we have that $I^{qr-ps} I^{ps} \subseteq (I^{qr} \colon I^{ps})\, I^{ps}$. Then

$(I^{qr} \colon I^{ps})\, I^{ps} = I^{qr-ps} I^{ps}$ and by the cancellative property of $\mathcal{I}$, we have that $(I^{qr} \colon I^{ps}) = I^{qr-ps} = I^{f_{p,q}(r,s)}$. □

Note that the above situation always occurs when R is a principle ideal domain and also for monomial ideals (it is known that the ideal quotient of monomial ideals is again a monomial ideal). As a final effort for this study, we return to the exploration of intersection algebras and provide a simple proof that the intersection algebra of co-maximal ideals in a domain R is finitely generated.

Proposition 5.15. *Let I and J be co-maximal ideals in a Noetherian domain R. Then the intersection algebra of I and J is finitely generated.*

Proof. First, we recall that for any co-maximal ideals I and J in R, $I \cap J = IJ$. This follows easily; if $a \in I \cap J$, since $1 = i + j$ for some $i \in I$, $j \in J$, $a = ai + aj$ and it follows that since $ai, aj \in IJ$, $a \in IJ$. As $IJ \subseteq I \cap J$ trivially, $IJ = I \cap J$. Note also that $I^r + J^s = R$ for any such ideals and any $(r,s) \in \mathbb{N}^2$; otherwise, I^r is proper and contained in some maximal ideal $\mathfrak{m}$, and so is J^s. Since $\mathfrak{m}$ is prime, I, J and hence $I + J$ are contained in $\mathfrak{m}$, false by the assumption that I and J are co-maximal. As $I^r \cap J^s = I^r J^s$, we may write $\bigoplus_{(r,s)} I^r \cap J^s u^r v^s = \bigoplus_{(r,s)} I^r J^s u^r v^s = \bigoplus_{(r,s)} I^{\pi_1(r,s)} J^{\pi_2(r,s)} u^r v^s$ where $\pi_1 \colon (r,s) \mapsto r$ and $\pi_2 \colon (r,s) \mapsto s$. Since each π_i is additive on $\mathbb{N}^2$ and therefore subadditive, both π_i are fan-linear on any fan $\Sigma_{\mathbf{a},\mathbf{b}}$, so it follows that the the intersection algebra of I and J is in fact a fan algebra and therefore finitely generated. □

Acknowledgments: The author would like to thank Florian Enescu for his guidance and collaboration throughout this project. This project was undertaken in concert with the Research Initiations in Mathematics, Mathematics Education, and Statistics (RIMMES) program at Georgia State University.

References

[1] P.A. Grillet, *Commutative Semigroups*, Kluwer Academic Publishers, 2001.

[2] S. Malec, *Intersection Algebras and Pointed Rational Cones*, Ph.D. Thesis in Mathematics, 2013.

[3] S. Malec, *On the Intersection Algebras of Principal Ideals*, aXiv:1302.0312v1, 2013.

[4] A. Schrijver *Theory of Linear and Integer Programming*, Wiley and Sons, 1998.

[5] B. Sturmfels, E. Miller *Combinatorial Commutative Algebra*, Springer, 2005.

Department of Mathematics and Statistics, Georgia State University, Atlanta GA 30303
E-mail address: hull.john@gmail.com