1-12-2006

Semidefinite Programming and Stability of Dynamical System

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Under the Direction of Mihály Bakonyi

ABSTRACT

In the first part of the thesis we present several interior point algorithms for solving certain positive definite programming problems. One of the algorithms is adapted for finding out whether there exists or not a positive definite matrix which is a real linear combination of some given symmetric matrices $A_1, A_2, \ldots, A_m$.

In the second part of the thesis we discuss stability of nonlinear dynamical systems. We search using algorithms described in the first part, for Lyapunov functions of the form $\sum_{i=1}^{m} \alpha_i x_i + \frac{1}{2}\sum_{i=1}^{m} \lambda_i x_i^2$, $\lambda_i > 0$, $i = 1, \ldots, m$, and also of the form $x^T A x$, where A is a positive definite real matrix. A suitable Lyapunov function implies the existence of a hyperellipsoidal attraction region for the dynamical system, thus guaranteeing stability.

INDEX WORDS: Positive semidefinite programming, attraction region, Lyapunov function

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A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

Master's of Science

in the College of Arts and Science

Georgia State University

2005

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Electric Version Approved:

Office of Graduate Studies

College of Arts and Sciences

Georgia State University

December 2005

ACKNOWLEDGEMENTS

The results contained in this thesis are part of research project of Dr.Dr.Mihály Bakonyi.

I would like to thank the following people for their time, effort, and inspiration while I worked on the thesis: Dr.Mihály Bakonyi, Chair, Dr. Michael Stewart,and Dr.Andrey Shilnikov.

To list each of your contributions would be a considerable task, so please know I appreciate everything you did.

Contents

1 Introduction

By a positive semidefinite programming problem one understands a minimization problem of a variable $x \in \mathbf{R}^m$ of the form:

$$\begin{cases} \text{Minimize } f(x) \\ \text{subject to } F(x) \geq 0 \end{cases} \tag{1}$$

where $F(x) = F_0 + \sum_{i=1}^{m} x_i F_i$. Here $F_0, F_1, \ldots, F_m \in \mathbf{R}^{n \times n}$ are symmetric matrices and $F(x) \geq 0$ means $F(x)$ is positive semidefinite.

Karmarkar introduced in his landmark paper ([12]) a polynomial-time interior-point method for solving positive semidefinite programming problems. We refer to [16] for literature on this subject. We present in Section 2 solutions to several positive semidefinite programming problems which originally appeared in [5], [21], and [3]. We adapt an algorithm used to solve one of them for finding out whether there exists or not a positive definite matrix which is a real linear combination of given symmetric matrices $A_1, A_2, \ldots, A_m$. In particular, this algorithm can be used to find out in polynomial-time whether a subspace of $\mathbf{R}^n$ contains or not a vector which is entrywise positive. For the latter problem we also include another polynomial-time algorithm suggested to us by Florian Potra.

In Section 3, we apply the algorithms in Section 2 to study stability of nonlinear dynamical systems. We need first some well-known definitions in the area of dynamical systems (see [22] as a classical reference). A *dynamical system* is a manifold M called the phase (or state) space and a smooth evolution function f^t that for any

element of $t \in T$, the time, maps a point of the phase space back into the phase space. The notion of smoothness changes with applications and the type of manifold. There are several choices for the set T. We consider $T = [0, \infty)$, case in which the dynamical system is called a *semi-flow.* The evolution functions f^t are the solutions of a *differential equation of motion*

$$x' = v(x).$$

The equation gives the time derivative of a trajectory $x(t)$ on the phase space starting at some point x_0. The vector field $v(x)$ is a smooth function that at every point of the phase space M provides the velocity vector of the system at that point.

A compact region Ω in $\mathbf{R}^n$ is called an *attraction region* for a dynamical system if any trajectory for the system starting outside Ω enters Ω after a finite time interval T, determined by the initial distance to Ω, and no trajectory starting in Ω leaves Ω. The existence of an attraction region guarantees that all solutions of the system are stable. Research has shown that within an attraction region chaotic behavior and fractal attractors are common. The study of chaos and fractals are currently booming research areas, however, we do not want to enter into details here since it is not the aim of the present work. Our goal is to study based on our results in Section 2 the existence of a hyperellipsoidal attraction region, possibly of minimal diameter. That is equivalent to the existence of such a region Ω, such that for each solution $x(t)$ of the system there exists $T > 0$ which depends only on $\|x(0)\|$ with the condition of

$x(t) \in \Omega$ for $t > T$.

We say 0 is an *asymptotically stable trajectory* for a system if for each $\epsilon > 0$ there exists $T > 0$ such that for each solution $x(t)$, $\|x(t)\| < \epsilon$ for $t > T$. T must depend only on ϵ and $\|x(0)\|$.

A function $\Lambda : \mathbf{R}^n \to \mathbf{R}$ which is positive and has negative derivative over the trajectory of every solution of the sytem is called a Lyapnov function.

C. Jeffries ([10]) considered a Lyapunov function of the form

$$\Lambda(x) = \sum_{i=1}^{m} \alpha_i x_i + \frac{1}{2} \sum_{i=1}^{m} \lambda_i x_1^2 \tag{2}$$

$\lambda_i > 0$, $i = 1, \ldots, m$, and showed that if $\lambda = (\lambda_1, \ldots, \lambda_n)^T$ is a null-vector of a certain matrix determined by a dynamical system of a particular form, then there exists a hyperellipsoidal attraction region for the system. Conditions on the sign-pattern of this matrix which guarantee the existence of an entrywise positive null-vector were estabilished in [11]. From the sign-pattern point of view, the existence problem of an entrywise positive (respectivelly nonnegative) null-vector was completely settled later ([7], [1], and [2]). Based on an algorithm in Section 2, we can decide in polynomial-time whether there exists a choice of Lyapunov function of type (2) which implies the existence of an attraction region.

We also consider Lyapunov functions of the more general form $\Lambda(x) = x^T A x$, where A is a real $n \times n$ positive definite matrix. Thus, for a class of dynamical systems, a hyperellipsoidal attraction region for the system is guaranteed by the condition

$tr(AB_r) = 0$, where $B_r, r = 1, \ldots, m$, are some symmetric matrices determined by the system. The existence of such A is decided by an algorithm in Section 2. Interior-point methods were previously used (see [20], [6], and [4]) for finding Lyapunov functions for linear time-variant dynamical systems, but their application to nonlinear systems considered in the present thesis is new. We conclude Section 3 by an example of a dynamical system for which there exists a Lyapunov function of the form $\Lambda(x) = x^T Ax$ which guarantees the existence of an attraction region, but there is no such function of type (2).

2 Positive Definite Optimization Problems

Let $A_0, A_1, \ldots, A_m$ be $n \times n$ symmetric matrices. Assume $A_0 > 0$ and that there is no nonzero positive semidefinite matrix of the form $\sum_{i=1}^{m} x_i A_i$, where $x_1, \ldots, x_m \in \mathbf{R}$. Consider the set $\mathcal{S}= \{Q = A_0 + \sum_{i=1}^{m} x_i A_i : Q > 0, x = (x_1, \ldots, x_m)^T \in \mathbf{R}^m\}$, which is nonempty and bounded.

Let $\phi : \mathcal{S} \to \mathbf{R}$ be defined by $\phi(Q) = \log\det Q$. It is known ([8], Theorem 7.6.7) that ϕ is a concave function, and since near the boundary of $\mathcal{S}$, ϕ approaches $-\infty$, ϕ takes on a maximum value at a unique point $P_0 \in \mathcal{S}$. Next we present an algorithm to approximate P_0, mainly following the lines of [3].

Since ϕ is concave, P_0 is the only point in $\mathcal{S}$ such that $\frac{\partial\phi}{\partial x_i}(P_0) = 0, i = 1, \ldots, m$. Then,

$$\begin{aligned}\frac{\partial\phi}{\partial x_i}(P_0) &= \lim_{t\to 0}\frac{\phi(P_0 + tA_i) - \phi(P_0)}{t} = \lim_{t\to 0}\frac{\log\det(P_0 + tA_i) - \log\det(P_0)}{t} \\ &= \lim_{t\to 0}\frac{\log\det[(P_0 + tA_i)P_0^{-1}]}{t} = \lim_{t\to 0}\frac{\log\det(I + tA_iP_0^{-1})}{t}\end{aligned}$$

Define $F(t) = \log\det(I + tA_iP_0^{-1})$ for t in a sufficiently small neighborhood of 0. The last relation implies that $\frac{\partial\phi}{\partial x_i}(P_0) = F'(0)$, the latter being equal to the coefficient of t in the polynomial expansion of $\det(I + tA_iP_0^{-1})$, which equals $tr(A_iP_0^{-1})$. Thus P_0 maximizes ϕ over $\mathcal{S}$ if and only if $P = P_0^{-1}$ satisfies $tr(PA_i) = 0$, $i = 1, \ldots, m$.

The following algorithm is based on Newton's Method and can be found in §1.2 of [16]. We use here its description given in §3.4 of [5] (see also [3]).

Algorithm 1

Introduce $x = (0, 0, \ldots, 0)^T$,

$y = (trA_1, trA_2, \ldots, trA_m)^T$,

$X = A_0$,

error $= \max\{|tr(X^{-1}A_k)| : k = 1, \ldots, m\}$

For error $>$ tolerance, do $H = (H_{ij})_{i,j=1}^m$

$H_{ij} = tr(X^{-1}A_iX^{-1}A_j)$,

$v = H^{-1}y$,

$\delta = \sqrt{y^Tv}$,

$$\alpha = \begin{cases} 1 & \text{if } \delta < \frac{1}{4} \\ \frac{1}{1+\delta} & \text{if } \delta \geq \frac{1}{4} \end{cases} \tag{3}$$

$x = x + \alpha v$

$x = (x_1, x_2, \ldots, x_m)^T$

$X = A_0 + x_1A_1 + \ldots + x_mA_m$

$y_k = tr(X^{-1}A_k),\ k = 1, \ldots, m$

$y = (y_1, \ldots, y_m)^T$

$\|y\|_\infty = \max\{|y_k| : k = 1, \ldots, m\}$

The algorithm stops when $|tr(X^{-1}A_k)|$ is less than the tolerance for $k = 1, \ldots, m$ and then X is an approximation for P_0 (remember that P_0 is characterized by $tr(P_0^{-1}A_k) = 0$).

In [5], an algorithm to approximate the solution of the following optimization problem with variables $x \in \mathbf{R}^m$ and $\lambda \in \mathbf{R}$ is given.

Problem 1

$$\begin{cases} \text{Minimize } \lambda \\ \lambda B(x) - A(x) \geq 0 \\ B(x) > 0 \\ C(x) > 0 \end{cases}$$

The feasible set is the set of all $x \in \mathbf{R}^m$ for which all three inequalities are true. Here A, B, and C are symmetric matrix functions that depend affinely of $x \in \mathbf{R}^m$,

$$A(x) = A_0 + \sum_{i=1}^{m} x_i A_i,$$

$$B(x) = B_0 + \sum_{i=1}^{m} x_i B_i,$$

$$\text{and } C(x) = C_0 + \sum_{i=1}^{m} x_i C_i,$$

where $A_i = A_i^T$ and $B_i = B_i^T$ are all in $\mathbf{R}^{n \times n}$, and $C_i = C_i^T \in \mathbf{R}^{q \times q}$.

The following assumptions have to be made.

1. The problem is feasible and we are given an inital point, i.e., we know $\lambda^{(0)}$ and $x^{(0)}$ with $\lambda^{(0)} B(x^{(0)}) - A(x^{(0)}) > 0$, $B(x^{(0)}) > 0$, and $C(x^{(0)}) > 0$.

2. B is bounded away from singular on the feasible set, i.e., we know $b_{min} > 0$ such that $B(x) \geq b_{min} I$.

3. The feasible set is bounded, i.e., there is M such that $C(x) > 0 \Rightarrow \|x\| \leq M$.

If $X = X^T$ and $Y = Y^T > 0$ are both $n \times n$ real matrices, the maximum generalized eigenvalue of the pair X, Y, denoted $\lambda_{max}(X, Y)$, can be defined in several equivalent ways:

$$\lambda_{max}(X, Y) = \max\{\lambda \in \mathbf{R} : \det(\lambda Y - X) = 0\} = \lambda_{max}(Y^{-\frac{1}{2}} X Y^{-\frac{1}{2}})$$

$$= \inf\{\lambda \in \mathbf{R} : \lambda Y - X > 0\} = \sup\{v^T X v : v \in \mathbf{R}^n, v^T Y v \leq 1\}$$

$$= \sup\{\frac{tr(XU)}{tr(YU)} : U = U^T \geq 0, U \neq 0\}$$

Let λ^{opt} denote the solution to Problem 1. For $\lambda > \lambda^{opt}$, define

$$\mathcal{S}_\lambda = \{x \in \mathbf{R}^m : \lambda B(x) - A(x) > 0, B(x) > 0, C(x) > 0\}$$

Let $\phi_\lambda : \mathcal{S}_\lambda \to \mathbf{R}$, be defined by

$$\phi_\lambda(x) = \log\det[(\lambda B(x) - A(x)) \oplus C(x)] = \log\det(\lambda B(x) - A(x)) + \log\det C(x).$$

Since ϕ_λ is a concave function, and near the boundary of $\mathcal{S}_\lambda$, ϕ_λ approaches $-\infty$, there exists a unique $x^*(\lambda) \in \mathcal{S}_\lambda$ where ϕ_λ takes on a maximum value.

The following algorithm to solve Problem 1 described in [5] is based on the notion of analytic centers due to Lieu and Huard ([13] and [9]).

Algorithm 2

The algorithm is initialized with $\lambda^{(0)}$ and $x^{(0)}$, $\lambda^{(0)} B(x^{(0)}) - A(x^{(0)}) > 0$, $B(x^{(0)}) > 0$, and $C(x^{(0)}) > 0$ and proceeds as follows:

$$\lambda^{(k+1)} = \frac{1}{2}\lambda_{max}(A(x^{(k)}), B(x^{(k)})) + \frac{1}{2}\lambda^{(k)}$$

$$x^{(k+1)} = x^*(\lambda^{(k+1)}).$$

Since $C(x) > 0 \Rightarrow \|x\| \leq M$, there is no positive semidefinite matrix of the form $\sum_{i=1}^{m} x_i C_i$, thus $x^*(\lambda)$ can be approximated by using Algorithm 1.

Since the feasible set for Problem 1 is bounded, $B(x)$ is bounded on it, there exists b_{max} such that $B(x) \leq b_{max} I$, for all feasible x. Let $\alpha = \frac{nb_{max}}{b_{min}}$. As shown in [5], we have that

$$\lambda^{(k+1)} - \lambda^{opt} \leq (1 - \frac{1}{2\alpha})(\lambda^{(k)} - \lambda^{opt}),$$

implying

$$\lambda^{(k)} - \lambda^{opt} \leq (1 - \frac{1}{2\alpha})^k (\lambda^{(0)} - \lambda^{opt}).$$

Thus, $\lambda^{(k)}$ converges to λ^{opt} at least geometrically.

For our applications, we are interested in solving the following feasibility problem.

Problem 2

Given the matrices $A_i = A_i^T \in \mathbf{R}^{n \times n}$, $i = 1, \ldots, m$, determine whether there exists a positive definite matrix of the form $\sum_{i=1}^{m} x_i A_i$, where $x = (x_1, \ldots, x_m)^T \in \mathbf{R}^m$.

Problem 2 can be solved using Algorithm 2 to approximate the solution of a problem of a particular type.

Consider in Problem 1, $B(x) = I$ and $A(x) = I - \sum_{i=1}^{m} x_i A_i$. Define $C_0 = I_{2m}$ and $C_i = 0_{2(i-1)} \oplus \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \oplus 0_{2(m-i)}$, for $i = 1, \ldots, m$, where 0_d denotes the 0 matrix

of size $d \times d$. Then $C(x) = diag(1 - x_1, 1 + x_1, \ldots, 1 - x_m, 1 + x_m)$, so $C(x) > 0$ when $\mid x_i \mid < 1$, implying that the feasible set is bounded. The existence of a positive definite matrix of the form $\sum_{i=1}^{m} x_i A_i$ is equivalent to the existence of such a matrix with $\mid x_i \mid < 1$.

For our particular choice of $A(x)$, $B(x)$, and $C(x)$, Problem 1 becomes:

$$\begin{cases} \text{Minimize } \lambda \\ (\lambda - 1)I + \sum_{i=1}^{m} x_i A_i \geq 0 \\ |x_i| < 1 \end{cases} \tag{4}$$

If $\lambda^{opt} < 1$, then there exist $|x_i| < 1$ such that $\sum_{i=1}^{m} x_i A_i \geq (1 - \lambda^{opt})I$, thus there exists a positive definite matrix of the form $\sum_{i=1}^{m} x_i A_i$. It is easy to see that $\lambda^{opt} < 1$ is also a necessary condition for the exsitence of a positive definite matrix of the form $\sum_{i=1}^{m} x_i A_i$.

Algorithm 2 in this case works as follows. Initialize with $\lambda^{(0)} > 1$ and $x^{(0)} = 0 \in \mathbf{R}^m$. Let

$$\lambda^{(k+1)} = \frac{1}{2}\lambda_{\max}(A(x^{(k)})) + \frac{1}{2}\lambda^{(k)}$$

$$x^{(k+1)} = x^*(\lambda^{(k+1)})$$

Here $\lambda_{max}(A(x^{(k)}))$ is the largest eigenvalue of $I - \sum_{i=1}^{m} x_i^{(k)} A_i$, and $x^*(\lambda^{(k+1)})$ is the vector in $\mathbf{R}^m$ which maximizes $\log\det((\lambda^{(k+1)} - 1)I + \sum_{i=1}^{m} x_i A_i) + \sum_{i=1}^{m} \log(1 - x_i^2)$ subject to $(\lambda^{(k+1)} - 1)I + \sum_{i=1}^{m} x_i A_i > 0$ and $|x_i| < 1$.

We know that $\lambda^{(k)} \longrightarrow \lambda^{opt}$, and $\lambda^{opt} < 1$ or $\lambda^{opt} = 1$ decides whether there is a positive definite matrix of the form $\sum_{i=1}^{m} x_i A_i$ or not.

Problem 3

Given the matrices $A_i = A_i^T \in \mathbf{R}^{n \times n}$, $i = 1, \ldots, m$, determine whether there exists a positive semidefinite matrix of the form $\sum_{i=1}^{m} x_i A_i$, where $x = (x_1, \ldots, x_m)^T \in \mathbf{R}^m$.

We can reduce Problem 3 by duality to Problem 2. We first need to recall some notions. The set of all $n \times n$ real, symmetric matrices is an inner product space with $< A, B >= tr(AB)$. It is well-known that a matrix P is positive semidefinite if and only if $tr(PQ) \geq 0$ for every $Q \geq 0$. Moreover, P is positive definite if and only if $tr(PQ) > 0$ for every $Q \geq 0$, $Q \neq 0$.

The separation property ([18]) states in particular that the cone of positive semidefinite real matrices and the hyperplane $\sum_{i=1}^{m} x_i A_i$ have only the 0 matrix in common if and only if there exists a functional on $\mathbf{R}^{n \times n}$ which is strictly positive on the cone of positive semidefinite real matrices and vanishes on the hyperplane $\sum_{i=1}^{m} x_i A_i$. This is equivalent to the existence of $Q > 0$ such that $tr(QA_i) = 0$, $i = 1, \ldots, m$. The latter equations define a set of linear conditions for the entries of the symmetric matrix Q. We can thus find a set of symmetric matrices $B_1, \ldots, B_r$ which form a linear basis for the set $\{X : tr(XA_i) = 0, i = 1, \ldots, m\}$. Thus Problem 3 can be reduced to Problem 2 for the matrices $B_1, \ldots, B_r$, for which the existence of a positive definite matrix

Q of the form $\sum_{i=1}^{r} x_i B_i$ is equivalent to the fact that there is no positive semidefinite matrix of the form $\sum_{i=1}^{m} x_i A_i$.

Problem 4

Given the vectors $\mathbf{v_1}, \ldots, \mathbf{v_m} \in \mathbf{R}^n$, determine whether there exists a vector $\mathbf{v} = (v_1, \ldots, v_n)^T$, $v_i > 0$, for $i = 1, \ldots, n$, of the form $\sum_{i=1}^{m} x_i \mathbf{v_i}$, where $x = (x_1, \ldots, x_m)^T \in \mathbf{R}^m$.

Problem 4 can be solved as a particular case of Problem 2. Indeed, let $\mathbf{v_i} = (v_{i1}, \ldots, v_{in})^T$ and $A_i = \text{diag}(v_{i1}, \ldots, v_{in})$. Then, the existence of a solution for Problem 4 is equivalent to the existence of a solution for Problem 2 for the matrices A_i, $i = 1, \ldots, m$. The algorithm for solving Problem 2 simplifies to the following .

For $\lambda > 1$ let $\mathcal{S}_\lambda = \{x \in \mathbf{R}^m : \lambda - 1 - \sum_{i=1}^{m} x_i v_{ij} > 0 \text{ for } j = 1, \ldots, n\}$ and define $\phi_\lambda : \mathcal{S}_\lambda \to \mathbf{R}$ by $\phi_\lambda(x) = \sum_{j=1}^{n} \log(\lambda - 1 - \sum_{i=1}^{m} x_i v_{ij}) + \sum_{i=1}^{m} \log(1 - x_i^2)$.

Since near the boundary of $\mathcal{S}_\lambda$, ϕ_λ approaches $-\infty$, and ϕ_λ is a concave function, there exists a unique $x^*(\lambda) \in \mathcal{S}_\lambda$ where ϕ_λ takes on a maximum value.

We initialize the algorithm with $\lambda^{(0)} > 1$ and $x^{(0)} = 0 \in \mathbf{R}^m$ and proceed as follows.

Let $x^{(k+1)} = x^*(\lambda^{(k)})$ and let

$$\lambda_{max}(x^{(k)}) = \max\{1 + \sum_{i=1}^{m} x_i^{(k+1)} v_{ij} : j = 1, \ldots, n\}.$$

Define

$$\lambda^{(k+1)} = \frac{1}{2}(\lambda^{(k)} + \lambda_{max}(x^{(k)})).$$

Then, $\lambda^{(n)} \to \lambda^{opt}$ at least geometrically, and the existence of a vector $\mathbf{v} = (v_1, \ldots, v_n)^T \in \mathbf{R}^n$, $v_i > 0$ for $i = 1, \ldots, n$, of the form $\sum_{i=1}^{m} x_i \mathbf{v_i}$ for some $x = (x_1, \ldots x_m)^T \in \mathbf{R}^m$ is equivalent to $\lambda^{opt} < 1$.

By duality Problem 4 can also be treated as a null-vector problem. Let $B = [\mathbf{v_1}, \ldots, \mathbf{v_m}]$ (the $n \times m$ matrix with columns $\mathbf{v_1}, \ldots, \mathbf{v_m}$) and $C = B^T$. Then $N(C) = R(B)^{\perp}$. Let $\{\mathbf{w_1}, \ldots, \mathbf{w_r}\}$ be a linear basis for $N(C)$. Let $A = [\mathbf{w_1}, \ldots, \mathbf{w_r}]^T$. Then $N(A) = R(B) = \{\sum_{i=1}^{m} x_i v_i : x = (x_1, \ldots, x_m)^T \in \mathbf{R}^m\}$. The existence of a vector $\mathbf{v} = (v_1, \ldots, v_n)^T \in \mathbf{R}^n$, $v_i > 0$ for $i = 1, \ldots, n$, is equivalent to the existence of such a vector in the null-space of A. We develop an idea suggested to us by Florian Potra for finding out in polynomial-time the existence of an entrywise positive vector in the null-space of a matrix. Consider the following problem.

Problem 5

$$\begin{cases} \text{Minimize z} & \\ Ax - zAe = 0 & \\ \sum_{i=1}^{m} x_i = 1 & \\ x_i \geq 0, & i = 1, \ldots, m \\ z \geq 0 & \end{cases}$$

where e is the vector with all components equal to 1. Problem 5 is feasible since

$z = x_i = \frac{1}{n}$, $i = 1, \ldots, n$ is a solution. We can apply Algorithm 2.1 in [17] to Problem 5. Since the problem is feasible, the iterates will converge to (x^*, z^*), an optimal solution of Problem 5. We have then one of the following cases.

Case 1: $z^* > 0$, then there is no vector $x = (x_1, \ldots, x_n)^T$, $x_i > 0$ for $i = 1, \ldots, n$ in $N(A)$, and we stop.

Case 2: $z^* = 0$. If $x_i^* > 0$ for $i = 1, \ldots, n$, then we found a vector with all positive components in $N(A)$ and we stop.

If there exist $j \in \{1, \ldots, n\}$ such that $x_j^* = 0$, then reorder the columns of A and x^* such that $x_j^* > 0$ for $j \leq m_1 \leq n$. Continue then the algorithm for the matrix obtained by deleting the first m_1 columns of the reordered A. The procedure stops when:

I. At a certain step, $z^* > 0$. Then Problem 4 has no solution.

II. We obtain a sequence, $1 \leq m_1 < m_2 < \ldots < m_p = n$, which is increasing as a result of $\sum x_i = 1$. Then Problem 4 admits a solution. Each step determines more components of a solution vector x.

Remark

Given $A \in \mathbf{R}^{n\times m}$, in order to find out whether there exists $x = (x_1, \ldots, x_m)^T \in N(A)$, $x_i > 0$, $i = 1, \ldots, m$, one can apply the simplex algorithm for the problem

$$\begin{cases} \text{Minimize} \sum_{\text{i=1}}^{\text{m}} x_i \\ \quad Ax = 0 \\ \quad x_i \geq 1 \qquad i = 1, \ldots, m. \end{cases}$$

Even if this algorithm is easier to implement than the two ones described in this section , it is a well-known fact that the simplex algorithm does not run in polynomial-time. However, it works well for small values of m and n.

3 Stability of Nonlinear Dynamical Systems

Consider the dynamical system

$$x_i'(t) = -\epsilon_i x_i(t) + g_i(x) \tag{5}$$

where $\epsilon_i > 0$, and g_i is a smooth function, $i = 1, \ldots, n$. Such systems are most commonly studied. We assume here the existence of a linearly independent set of function $\{f_l(x)\}_{l=1}^m$ which span the set $\{g_i(x)\}_{i=1}^n$ as well as $\{x_k g_i(x) : i, k = 1, \ldots, n\}$. A typical situation for this is when each $g_i(x)$ is a polynomial in $x_1, x_2, \ldots, x_n$ and $\{f_l(x)\}_{l=1}^m$ is a large enough set of monomials. We can thus assume the system (5) is of the form

$$x_i'(t) = -\epsilon_i x_i(t) + \sum_{l=1}^{m} k_{li} f_l(x) \tag{6}$$

for $i = 1, \ldots, n$. We are searching in this case for a Lyapunov function (see [10]) of the form:

$$\Lambda(x) = \sum_{i=1}^{n} \alpha_i x_i + \frac{1}{2} \sum_{i=1}^{n} \lambda_i x_i^2 \tag{7}$$

where α_i and $\lambda_i > 0$ are unknown for $i = 1, \ldots, n$. The level sets of $\Lambda(x)$ are hyperellipsoids centered at $(-\frac{\alpha_1}{\lambda_1}, \ldots, -\frac{\alpha_n}{\lambda_n})$. Then (by denoting $\Lambda'(x) = \frac{d\Lambda(x(t))}{dt}$)

$$\begin{aligned}\Lambda'(x) = \sum_{i=1}^{n} \alpha_i x_i' + \sum_{i=1}^{n} \lambda_i x_i x_i' &= -\sum_{i=1}^{n} \epsilon_i \alpha_i x_i - \sum_{i=1}^{n} \epsilon_i \lambda_i x_i^2 \\ &+ \sum_{l=1}^{m} (\sum_{i=1}^{n} \alpha_i k_{li}) f_l(x) + \sum_{l=1}^{m} (\sum_{i=1}^{n} \lambda_i k_{li} x_i f_l(x))\end{aligned} \tag{8}$$

By our assumption we have that

$$x_i f_l(x) = \sum_{j=1}^{m} \alpha_j^{(il)} f_j(x)$$

Then

$$\sum_{l=1}^{m}\sum_{i=1}^{n} \lambda_i k_{li} x_i f_l(x) = \sum_{j=1}^{m}\sum_{i=1}^{n}(\sum_{l=1}^{m} k_{li}\alpha_j^{(il)})\lambda_i f_j(x) = \sum_{j=1}^{m}(\sum_{i=1}^{n} \beta_{ji}\lambda_i) f_j(x),$$

where $\beta_{ji} = \sum\limits_{l=1}^{m} k_{li}\alpha_j^{(il)}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, m$.

We try to find α_i and $\lambda_i > 0$ which reduce the last two terms of (8) to 0. This means that we want to make the coefficient of each $f_j(x)$, $j = 1, \ldots, m$, to vanish, namely that $\sum\limits_{i=1}^{n} k_{ji}\alpha_i + \sum\limits_{i=1}^{n} \beta_{ji}\lambda_i = 0$ for $j = 1, \ldots, m$.

Let $A = [\{k_{ji}\}_{i=1,j=1}^{n,m}, \{\beta_{ji}\}_{i=1,j=1}^{n,m}]$, which is an $m \times (2n)$ matrix and has $(\alpha_1, \ldots, \alpha_n, \lambda_1, \ldots, \lambda_n)^T$ as a null-vector. Let $\{v_1, \ldots, v_r\}$ be a linear basis for $\ker A$ and let for $t = 1, \ldots, r$, w_t be the vector in $\mathbf{R}^n$ which represents the last n entries of v_t. Since we want $\lambda_i > 0$, our problem reduces in finding a vector $\lambda = (\lambda_1, \ldots, \lambda_n)^T$ with $\lambda_i > 0$ of the form $\sum\limits_{t=1}^{r} x_t w_t$, for a certain $x = (x_1, \ldots, x_r)^T \in \mathbf{R}^r$. This is exactly Problem 4 discussed earlier, for which we have alternative ways to determine whether a solution exists. If the problem admits a solution, then we have

$$\Lambda'(x) = -\sum_{i=1}^{n} \epsilon_i \alpha_i x_i - \sum_{i=1}^{n} \epsilon_i \lambda_i x_i^2,$$

and $\Lambda'(x) < 0$ outside the hyperellipsoid of equation $\Lambda'(x) = 0$. At points where $\Lambda'(x) < 0$, $\Lambda(x)$ decreases, thus the trajectory gets closer to the point $(-\frac{\alpha_1}{\lambda_1}, \ldots, -\frac{\alpha_n}{\lambda_n})$.

Let $c \in \mathbf{R}$ be such that the region $\{x \in \mathbf{R}^n : \Lambda'(x) \geq 0\}$ is a proper subset of $\Omega = \{x \in \mathbf{R}^n : \Lambda(x) < c\}$. Then Ω is an attraction region for the dynamical system (6).

The best known example of a dynamical system of type (5) which admits an attraction region is the following one by Lorenz ([14]). This example triggered the research on attraction regions of the type considered in the present work. There are many other recent examples of such systems. We refer the reader to Appendix C in Part II of [19].

$$\begin{cases} x_1' = -10x_1 + 10x_2 \\ x_2' = -x_2 + 28x_1 - x_1x_3 \\ x_3' = -\frac{8}{3}x_3 + x_1x_2 \end{cases} \tag{9}$$

One is considering in the case a Lyapunov function of the form

$$\Lambda(x) = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 + \frac{1}{2}\lambda_1 x_1^2 + \frac{1}{2}\lambda_2 x_2^2 + \frac{1}{2}\lambda_3 x_3^2$$

which leads to

$$\Lambda'(x) = (-10\alpha_1 + 28\alpha_2)x_1 + (10\alpha_1 - \alpha_2)x_2 - \frac{8}{3}\alpha_3 x_3 -$$

$$10\lambda_1 x_1^2 - \lambda_2 x_2^2 - \frac{8}{3}\lambda_3 x_3^2 + (\alpha_3 + 10\lambda_1 + 28\lambda_2)x_1x_2 - \alpha_2 x_1 x_3 + (-\lambda_2 + \lambda_3)x_1x_2x_3$$

Let then $f_1(x) = x_1x_2$, $f_2(x) = x_1x_3$, and $f_3(x) = x_1x_2x_3$. We want their coefficients to vanish, which leads to $(\alpha_1, \alpha_2, \alpha_3, \lambda_1, \lambda_2, \lambda_3)^T$ being a null-vector of the matrix

$$A = \begin{bmatrix} 0 & 0 & 1 & 10 & 28 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \tag{10}$$

The simplest solution for which $\lambda_1, \lambda_2, \lambda_2 > 0$ is $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = -38$, $\lambda_1 = \lambda_2 = \lambda_3 = 1$. So we can consider

$$\Lambda(x) = -38x_3 + \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{2}x_3^2,$$

for which

$$\Lambda'(x) = -10x_1^2 - x_2^2 - \frac{8}{3}(x_3 - 19)^2 + \frac{2888}{3}.$$

Then $\Lambda'(x) < 0$ outside an ellipsoid centered at $(0, 0, 19)$ and let $c \in \mathbf{R}$ be such that $\{x \in \mathbf{R}^3 : \Lambda'(x) \geq 0\}$ is a proper subset of the ellipsoidal region $\Omega = \{x \in \mathbf{R}^3 : \Lambda'(x) < c\}$ (centered at $(0, 0, 38)$). Then Ω is an attraction region for the dynamical system (9).

We present next an example which is a slight generalization of the problem considered in [11]. Let Φ_l, $l = 1, \ldots, , M$, be monomials in the variables $x_1, \ldots, x_n$. Consider the dynamical system

$$x_i'(t) = -\epsilon_i x_i + \sum_{l=1}^{M} k_{li} \frac{\partial \Phi_l}{\partial x_i},$$

where $\epsilon_i > 0$ for $i = 1, \ldots, n$, and $\{k_{li}\}$ is an $M \times n$ real matrix. We try to find a Lyapunov function of type $\Lambda(x) = \frac{1}{2}\sum_{i=1}^{n} \lambda_i x_i^2$, $\lambda_i > 0$, $i = 1, \ldots, n$. Let $x_i \frac{\partial \Phi_l}{\partial x_i} = n_{li}\phi_l$

(n_{li} is the power of x_i in Φ_l), and then

$$\Lambda'(x) = \sum_{i=1}^{n} \lambda_i x_i x_i' = -\sum_{i=1}^{n} \epsilon_i \lambda_i x_i^2 + \sum_{i=1}^{n} \sum_{l=1}^{M} \lambda_i k_{li} n_{li} \Phi_l(x) =$$

$$-\sum_{i=1}^{n} \epsilon_i \lambda_i x_i^2 + \sum_{l=1}^{M} (\sum_{i=1}^{n} k_{li} n_{li} \lambda_i) \Phi_l(x).$$

If $A = \{k_{li} n_{li}\}_{l=1,i=1}^{M,n}$, we can search using one of the two algorithms in Section 2 for a null-vector $(\lambda_1, \ldots, \lambda_n)^T$, $\lambda_i > 0$, $i = 1, \ldots, n$, of A. For such a choice of λ_i, we have $\Lambda'(x) = -\sum_{i=1}^{n} \epsilon_i \lambda_i x_i^2 < 0$, implying 0 is an asymptotically stable trajectory for the system.

We consider now a Lyapunov function for a system of type (6) of the form,

$$\Lambda(x) = x^T A x = \sum_{i,j=1}^{n} a_{ij} x_i x_j,$$

where $A = \{a_{ij}\}_{i,j=1}^{n}$ is a positive definite matrix. Then

$$\Lambda'(x) = \sum_{i,j=1}^{n} a_{ij} x_i' x_j + \sum_{i,j=1}^{n} a_{ij} x_i x_j' =$$

$$\sum_{i,j=1}^{n} a_{ij} x_j (-\epsilon_i x_i + \sum_{l=1}^{m} k_{li} f_l(x)) + \sum_{i,j=1}^{n} a_{ij} x_i (-\epsilon_j x_j + \sum_{l=1}^{m} k_{lj} f_l(x))$$

By our assumption we have that $x_i f_l(x) = \sum_{j=1}^{m} \alpha_j^{(il)} f_j(x)$, thus

$$\Lambda'(x) = -\sum_{i,j=1}^{n} a_{ij} (\epsilon_i + \epsilon_j) x_i x_j + \sum_{i,j=1}^{n} a_{ij} \sum_{l=1}^{m} k_{li} \sum_{r=1}^{m} \alpha_r^{(jl)} f_r(x)$$
$$+ \sum_{i,j=1}^{n} a_{ij} \sum_{l=1}^{m} k_{lj} \sum_{r=1}^{m} \alpha_r^{(il)} f_r(x)$$

$$= -\sum_{i,j=1}^{n} a_{ij}(\epsilon_i + \epsilon_j)x_i x_j + \sum_{r=1}^{m}(\sum_{i,j=1}^{n} a_{ij} \sum_{l=1}^{m} k_{li}\alpha_r^{(jl)})f_r(x)$$
$$+ \sum_{r=1}^{m}(\sum_{i,j=1}^{n} a_{ij} \sum_{l=1}^{m} k_{lj}\alpha_r^{(il)})f_r(x)$$

Let us denote $\mu_{ij}^{(r)} = \sum_{l=1}^{m} k_{li}\alpha_r^{(jl)}$ and $E = \mathrm{diag}(\epsilon_1, \epsilon_2, \ldots, \epsilon_n)$. Then $AE = \{a_{ij}\epsilon_j\}_{i,j=1}^{n}$, $EA = \{\epsilon_i a_{ij}\}_{i,j=1}^{n}$. So $\Lambda'(x) = -x^T(AE + EA)x + \sum_{r=1}^{m}[\sum_{i,j=1}^{n} a_{ij}(\mu_{ij}^{(r)} + \mu_{ji}^{(r)})]f_r(x)$.

For $r = 1, \ldots, m$, let B_r denote the symmetric matrix $\{\mu_{ij}^{(r)} + \mu_{ji}^{(r)}\}_{i,j=1}^{n}$, so $\sum_{i,j=1}^{n} a_{ij}(\mu_{ij}^{(r)} + \mu_{ji}^{(r)}) = tr(AB_r)$.

We try to find a matrix $A > 0$ such that $AE + EA > 0$ and $tr(AB_r) = 0$ for $r = 1, \ldots, m$. This would then imply $\Lambda(x) > 0$ and $\Lambda'(x) = -x^T(AE + EA)x < 0$ for $x \neq 0$. As a consequence, 0 is an asymptotically stable trajectory for the system (6).

For finding A, we solve the dual problem. Consider the linear subspace $\mathcal{M} = \{X = X^T : tr(AX) = 0\}$ and find a basis $C_1, C_2, \ldots, C_s$ for $\mathcal{M}$. Let

$$F_i = \begin{bmatrix} C_i & 0 \\ 0 & EC_iE \end{bmatrix},$$

$i = 1, \ldots, s$. Then our problem is equivalent to the existence of a positive definite matrix

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$$

of the form $\sum_{i=1}^{s} x_i F_i$, for some $x = (x_1, \ldots, x_s)^T \in \mathbf{R}^s$. If such a matrix exists, it is of the form

$$\begin{bmatrix} A & 0 \\ 0 & EAE \end{bmatrix},$$

where $A > 0$ and $EAE > 0$, and $A = \sum_{i=1}^{s} x_i C_i$, so $tr(AB_r) = 0$ for $r = 1, \ldots, m$.

We can apply Algorithm 2 to solve the latter problem. This represents a new way for finding in polynomial-time a sufficient condition which guarantees 0 is an asymptotically stable trajectory for a system of type (6).

The next example describes a situation when a Lyapunov function of type $\Lambda(x) = x^T A x$ with $A > 0$ guarantees 0 is an asymptotically stable trajectory for the system, but any function of type $\Lambda(x) = \sum_{i=1}^{n} \alpha_i x_i + \frac{1}{2} \sum_{i=1}^{n} \lambda_i x_i^2$, with $\lambda_i > 0$, $i = 1, \ldots, n$, fails to provide this information. In this example $E = I$, so $A > 0$ automatically implies $AE + EA > 0$.

Example.

Consider the dynamical system:

$$\begin{cases} x_1' = -x_1 + x_1 x_2 + 6 x_2 x_3 + x_1 x_3 \\ x_2' = -x_2 + x_1 x_2 - 2 x_2 x_3 - 3 x_1 x_3 \\ x_3' = -x_3 - 3 x_1 x_2 - 2 x_2 x_3 + x_1 x_3 \end{cases} \tag{11}$$

By considering a Lyapunov function of the type

$$\Lambda(x) = \lambda_1 x_1^2 + \lambda_2 x_2^2 + \lambda_3 x_3^2 + \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$$

we cannot cancel in $\Lambda'(x)$ all the terms of the form $x_1^2 x_2$, $x_1 x_2^2$, $x_2^2 x_3$, $x_2 x_3^2$, $x_1^2 x_3$, $x_1 x_3^2$, and $x_1 x_2 x_3$ when $\lambda_1, \lambda_2, \lambda_3 > 0$.

Let us consider the case of a Lyapunov function of the form $\Lambda(x) = x^T A x$, where $A = (a_{ij})_{i,j=1}^{3}$ is a positive definite matrix such that $\Lambda'(x) = -2x^T A x$, which happens

in case all terms of the form $x_1^2x_2$, $x_1x_2^2$, $x_2^2x_3$, $x_2x_3^2$, $x_1^2x_3$, $x_1x_3^2$ and $x_1x_2x_3$ cancel in $\Lambda'(x)$. This leads to the equations:

$$\begin{cases} a_{11} + a_{12} - 3a_{13} = 0 \\ a_{12} + a_{13} - 3a_{23} = 0 \\ 3a_{12} - a_{22} - a_{23} = 0 \\ 3a_{13} - a_{23} - a_{33} = 0 \\ a_{11} - 3a_{12} + a_{13} = 0 \\ a_{13} - 3a_{23} + a_{33} = 0 \\ 6a_{11} - a_{12} - a_{13} - 3a_{22} + 2a_{23} - 3a_{33} = 0 \end{cases} \tag{12}$$

If we denote

$$B_1 = \begin{bmatrix} 2 & 1 & -3 \\ 1 & 0 & 0 \\ -3 & 0 & 0 \end{bmatrix},\ B_2 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 2 & -3 \\ 0 & -3 & 0 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 1 \\ 3 & -1 & -2 \end{bmatrix},\ B_4 = \begin{bmatrix} 0 & 3 & 0 \\ 3 & -2 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$B_5 = \begin{bmatrix} 2 & -2 & 1 \\ -3 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix},\ B_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -3 \\ 1 & -3 & 2 \end{bmatrix},$$

$$\text{and } \mathrm{B}_7 = \begin{bmatrix} 12 & -1 & -1 \\ -1 & -6 & 2 \\ -1 & 2 & -6 \end{bmatrix} \tag{13}$$

then the equations (12) are of the form $tr(AB_i) = 0$ for $i = 1, \dots, 7$.

A simple computation shows that the system (12) has a one-dimensional solution

set generated by $a_{11} = a_{22} = a_{33} = 2$ and $a_{12} = a_{13} = a_{23} = 1$. This leads to

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix},$$

for which $\Lambda'(x) = -2x^T Ax$, implying 0 is an asymptotically stable trajectory for the dynamical system.

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