7-28-2006

On Some Aspects of the Differential Operator

Panakkal Jesu Mathew

ON SOME ASPECTS OF THE DIFFERENTIAL OPERATOR

by

PANAKKAL JESU MATHEW

Under the direction of LIFENG DING

ABSTRACT

The Differential Operator D is a linear operator from $C_1[0,1]$ onto $C[0,1]$. Its domain $C_1[0,1]$ is thoroughly studied as a meager subspace of $C[0,1]$. This is analogous to the status of the set of all rational numbers Q in the set of the real numbers R.

On the polynomial vector space P_n the Differential Operator D is a nilpotent operator. Using the invariant subspace and reducing subspace technique an appropriate basis for the underlying vector space can be found so that the nilpotent operator admits its Jordan Canonical form. The study of D on P_n is completely carried out. Finally, the solution space V of the differential equation $\frac{d^n x}{dt^n} + a_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_1\frac{dx}{dt} + a_0 x = 0$ is studied. The behavior of D on V is explored using some notions from linear algebra and linear operators.

INDEX WORDS: Differential operator, linear operator, nilpotent operator, Jordan Canonical form.

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Panakkal J. Mathew

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

Master of Science

in the College of Arts and Sciences

Georgia State University

2006

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By

Panakkal J. Mathew

Major Professor: Dr. Lifeng Ding

Committee: Dr. Mihaly Bakonyi
Dr. Frank Hall
Dr. Marina Arav

Electronic Version Approved:

Office of Graduate Studies
College of Arts and Science
Georgia State University
May 2006

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INTRODUCTION

The Differential operator $D = \frac{d}{dt}$ is a well known linear operator. In a standard functional analysis course it is mentioned as an unbounded linear operator from the space $C_1[0,1]$ to C[0,1] under the sup or the uniform norm. When the Closed Graph Theorem is introduced, the differential operator serves as a counter example which asserts that although it is a closed operator, it is not bounded due to the fact that $C_1[0,1]$ is not a Banach Space. This reveals the significance of the question " the operator is defined from where to where?". It is regretful that many interesting aspects of the differential operator has not gained much attention.

In this thesis by using the well known Banach-Steinhaus Theorem we first prove that $C_1[0,1]$ is meager, or a subspace of first category in C[0,1]. So the status of $C_1[0,1]$ in C[0,1] is analogous to that of the rational number **Q** in the real number **R**.

It is not easy to describe the structure of D in $C_1[0,1]$ directly. However, if we restrict D to some nowhere dense subspaces we can get a clear "cross-sectional view" of D.

In chapter 3 we restrict D on P_n , polynomials of degree upto n-1. We see that D is nilpotent on P_n , and we go through the entire process to reach the Jordan

Canonical Form of D. The basis of P_n under which D admits its Jordan Canonical Form is obtained.

A linear operator T is algebraic if there is a polynomial p such that p(T) = 0. In chapter 4 we note that D is not an algebraic operator on $C_n[0,1]$. But we show that for any polynomial p the solution space V of p(D) = 0 is a finite dimensional subspace of $C_n[0,1]$. p is the minimal polynomial of D, so D is algebraic on V. As an algebraic operator on V, D has the advantage of Primary Decomposition. So its structure on V is fully obtained. In fact the polynomial p plays an important role. For some appropriate p, D is diagonalizable on V, and D is also semi-simple on V.

All the above cross-sectional views are obtained on finite subspaces of $C_1[0,1]$, which are nowhere dense subspaces of C[0,1].

We may ask the question " could we further reveal the structure of D on other "infinite dimensional subspaces of $C_1[0,1]$"? The answer is "yes". For example D on P, which is the collection of all polynomials. Under an appropriate basis of P, D is a unilateral shift operator. Many issues may arise. But that is some future research for the author.

CHAPTER 1

BANACH SPACES AND LINEAR OPERATORS

Throughout the thesis, the field F is either the field R of real numbers or the field C of complex numbers. The reader is expected to be familiar with the notion of vector spaces, normed vector spaces, norm on vector spaces and the linear operators. First we introduce some of the notations used in the thesis.

(a) The vector space C(X) is the set of all real valued continuous functions on the compact topological space X.

(b) P_n is the set of all the polynomials x(t) with complex coefficients of degree up to n-1, over the field F, in variable t.

(c) For any $x \in P_n$, let $x(t) = \sum_{j=0}^{n-1} \xi_j t^j$, then $D(x(t)) = \sum_{j=1}^{n-1} \xi_j t^{j-1}$, is the derivative of the polynomial x. Here, D is the differential operator and it is also a linear operator.

Example 1 In C(X), mentioned in (a) above, since a continuous real valued function on a compact topological space is bounded, we introduce the norm on C(X) as below:

$\| f \|_\infty = \sup \{ |f(x)| : x \in X\} < \infty \quad \forall \quad f \in C(X).$

$\| f \|_\infty$ is called the super norm.

For a normed vector space $(V, \| \,.\, \|)$ we can induce a metric by

$$d(x, y) = \|x - y\|, \quad x, y \in V.$$

It is called the metric induced by the norm.

So a normed vector space V is also equipped with a metric topology induced by its norm. We also call it a norm topology. Under this norm topology the notions of neighborhood, interior points, open sets, closed sets, compact sets and other concepts are defined in the conventional way.

In particular, a sequence $\{x_n\}$ in V converges to an element x in V if

$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} \|x_n - x\| = 0.$ Again, a sequence $\{x_n\}$ is a cauchy sequence if for each $\varepsilon > 0$ there is positive integer N such that $\|x_n - x_m\| < \varepsilon$ holds for $n, m > N$. It is noted that every convergent sequence is a Cauchy sequence but the converse is not true. If every Cauchy sequence in V converges in V, then V is called a Complete normed vector space. A complete normed vector space is called a Banach Space.

<u>**PROPOSITION 1**</u> The vector space C(X) in (a) is a Banach Space.

<u>Proof</u> We need to prove that C(X) is complete under the metric

$d(f, g) = \|f - g\|_\infty$. Let $\{f_n\}$ be a Cauchy sequence in C(X). Then, $\forall\ \varepsilon > 0, \exists\ N$ such that n, m > N implies $\|f_n - f_m\| < \varepsilon$.. So, for all $x \in X$,

$$|f_n(x) - f_m(x)| \le \|f_n - f_m\|_\infty < \varepsilon \quad \forall \quad n, m > N.$$

So, for each $x \in X, \{f_n(x)\}$ is a Cauchy sequence in R. Now, since R is complete, we know that there exists a real number f(x) such that $f_n(x) \to f(x)$. So we have a real valued function $f(x)$ defined on X and $f_n \to f$ pointwisely on X. We now fix n and let $m \to \infty$ so that $f_m(x) \to f(x)$. Hence, for each $x \in X$, $|f_n(x) - f(x)| < \varepsilon \quad \forall \quad n > N$. Then, $\sup\{|f_n(x) - f(x)| : x \in X\} \leq \varepsilon. \ \forall$ $n > N$. So, $\|f_n - f\|_\infty \leq \varepsilon \quad \forall \quad n > N$.

This means that f_n converges to f uniformly on X. Since the uniform limit of a sequence of continuous functions is a continuous function, f is continuous i.e, $f \in C(X)$. Thus, every Cauchy sequence $\{f_n\}$ in C(X) converges to a vector f in C(X). Hence, C(X) is complete. .

<u>THEOREM 1</u> [1] A linear operator acting between two normed vector spaces is continuous if and only if it is continuous at vector 0.

<u>PROOF</u> <u>Necessity</u> This is trivial.

<u>Sufficiency</u>

Suppose T is continuous at 0. So by definition, $\forall \quad \varepsilon > 0$, $\exists \quad \delta > 0$ such that for any $x \in X, \|x\| < \delta$ implies $\|Tx\| < \varepsilon$. Now let $x \in X$. Then for $y \in X$, with $\|y - x\| < \delta$, we have $\|T(y - x)\| < \varepsilon$ or $\|Ty - Tx\| < \varepsilon$. So T is continuous at x. Now, since x is arbitrary, T is continuous on X.

.

<u>DEFINITION 1</u> Let $T: X \to Y$ be a linear operator between two normed spaces. The *operator norm* of T is defined by

$$\|T\| = \sup\{\|Tx\| : \quad \|x\| = 1\}$$

If $\|T\|$ is finite, then T is called a bounded operator and if $\|T\| = \infty$, T is called an unbounded operator. Note that $\|Tx\| \le \|T\| \, \|x\|$ holds for all $x \in U$.

To see this, if $x = 0$, the inequality is trivial. For $x \ne 0$, the vector

$y = \frac{1}{\|x\|} x$ satisfies $\|y\| = 1$, and so $\frac{\|Tx\|}{\|x\|} = \left\| T\left(\frac{1}{\|x\|} x \right) \right\| = \|T(y)\| \le \|T\|$ holds.

Then, $\|Tx\| \le \|T\| \, \|x\|$. In particular, it follows from the previous inequality that $\|T\| = \sup\{\|Tx\| : \|x\| \le 1\}$. Indeed,

$$\|T\| = \sup\{\|Tx\| : \|x\| = 1\} \le \sup\{\|Tx\| : \|x\| \le 1\} \le \sup\{\|T\| \, \|x\| : \|x\| \le 1\}$$

$= \|T\| \sup\{\|x\| : \|x\| \le 1\} = \|T\|$. So, $\|T\| = \sup\{\|Tx\| : \|x\| \le 1\}$. For a linear operator the concepts of continuity and boundedness are equivalent. It is proved in the next theorem.

<u>THEOREM 2</u> [1] A linear operator acting between two normed spaces is continuous if and only if it is bounded.

<u>PROOF</u> <u>Necessity</u>

Suppose $T: X \to Y$ is continuous where X and Y are two normed vector spaces. As T is continuous at 0, so $\forall \quad \varepsilon > 0, \quad \exists \quad \delta > 0$ such that $\|x\| \le \delta \Rightarrow$

$\|Tx\| \le \varepsilon$. We pick $\varepsilon = 1$, so $\|x\| \le \delta \Rightarrow \|Tx\| < 1 \Rightarrow \frac{1}{\delta}\|x\| \le 1 \Rightarrow \|Tx\| < 1$.

Let, $y = \frac{1}{\delta}x$, and $x = \delta y$. So $\|y\| \le 1 \Rightarrow \|T(\delta y)\| < 1 \Rightarrow \delta\|Ty\| < 1$

$\Rightarrow \|Ty\| < \frac{1}{\delta} < \infty$ holds for all y with $\|y\| \le 1$.

So we have, $\|x\| \le 1 \Rightarrow \|Tx\| < \frac{1}{\delta} < \infty$. Hence, T is bounded.

b) Sufficiency

We assume T is bounded. So, $\sup\{\|Tx\| : \|x\| \le 1\} = M < \infty$. It suffices to prove that T is continuous at 0. Suppose, $\varepsilon > 0$ let $\delta = \frac{\varepsilon}{M}$. Then

$\|x\| < \delta \Rightarrow \|x\| < \frac{\varepsilon}{M} \Rightarrow \left\|\frac{Mx}{\varepsilon}\right\| < 1$. So, $\left\|T\left(\frac{Mx}{\varepsilon}\right)\right\| \le M \Rightarrow \|T(x)\| \le \varepsilon$.

So T is continuous at 0, hence T is continuous.

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DEFINITION 2 A scalar valued linear operator on a normed vector space is called a *Linear Functional.*

Example 2 [3] Let [a,b] be any finite interval on real t-axis, and let α be any complex valued integrable function defined on [a,b]. Define y by

$y(x) = \int_a^b \alpha(t)x(t)dt \quad \forall \quad x \in C[a,b]$, then y is a linear functional on C[a,b].

DEFINITION 3 Let $f : X \to R$ be a linear functional. Then the *Kernel* of f is defined as $\text{Ker } f = \{x \in X : f(x) = 0\}$.

THEOREM 3 Suppose $f : X \to R$ is a linear functional. Then f is continuous if and only if Ker f is a closed set.

PROOF Necessity

Observe that , $\text{Ker } f = f^{-1}\{0\}$. Now, $\{0\}$ is a closed set in R and since f is continuous $f^{-1}\{0\}$ is closed in X. Hence, Ker f is closed in X.

Sufficiency

Suppose Ker f is a closed set in X. If f were not continuous, then f is unbounded. Then, $\sup\{\|f(x)\| : \|x\| \le 1\} = \infty$. So for all N, there exists x_n with $\|x_n\| \le 1$ and $|f(x_n)| \ge n$. (Here the norm is the absolute value which means $\|f(x_n)\| = |f(x_n)|$). So there is a sequence $\{x_n\} \in X$ with $\|x_n\| \le 1$ and $f(x_n) \to \infty$. Since f is not bounded, $\text{Ker } f \neq X$. So, $\exists$ $x \notin \text{Ker } f$ and $x \in X$.

Let $y_n = x - \varepsilon_n x_n$, where $\varepsilon_n = \dfrac{f(x)}{f(x_n)}$, and $f(y_n) = f(x) - \varepsilon_n f(x_n) =$ f(x)-f(x) = 0. So $y_n \in$ Ker f $\forall$ n. Note that $\varepsilon_n \to 0$ as $n \to \infty$ So $y_n \to x$. So there exists a sequence $\{y_n\} \subseteq \text{Ker } f$, but $y_n \to x \notin \text{Ker } f$, which contradicts our assumption that Ker f is closed. .

Note that every linear subspace of a finite-dimensional normed linear space is a closed subspace. Since Ker f is a closed subspace in a finite dimensional normed linear space , we have

COROLLARY 1 Every linear functional on a finite-dimensional normed linear space is continuous.

COROLLARY 2 Every linear operator from a finite-dimensional normed space to another normed space is continuous.

THE SPACE L(X, Y)

Let $T : X \to Y$ be a linear operator. We denote the collection of all bounded linear operators from X to Y by L(X,Y). The addition and scalar multiplication are introduced to L(X,Y) in a conventional way so that L(X,Y) is a vector space.

We mention a theorem, the proof of which follows immediately from the definition.

THEOREM 4 [5] T is a bounded linear operator between two normed spaces X and Y if and only if there exists a real number $M \geq 0$ such that $\|T(x)\| \leq M\|x\|$ holds for all $x \in X$.

THEOREM 5 [1] Let X and Y be two normed spaces. Then $L(X, Y)$ is a normed vector space. Moreover, if Y is a Banach Space, then $L(X, Y)$ is likewise a Banach space.

PROOF The norm on T is given by $\|T\| = \sup\{\|Tx\| : \|x\| = 1\}$. Clearly, from its definition $\|T\| \geq 0$ holds for all $T \in L(X, Y)$. Also the inequality $\|Tx\| \leq \|T\|\,\|x\|$ shows that $\|T\| = 0$ if and only if $T = 0$. The proof of the identity $\|\alpha T\| = |\alpha| .\|T\|$ is straightforward. For the triangular inequality, let $S, T \in L(X, Y)$, and let $x \in X$ with $\|x\| = 1$. Then,

$$\|(S+T)(x)\| = \|S(x) + T(x)\| \leq \|S(x)\| + \|T(x)\| \leq \|S\| + \|T\|$$

holds, which shows that $\|S+T\| \leq \|S\| + \|T\|$. Thus, $L(X, Y)$ is a normed vector space.

Now assume that Y is a Banach space. To complete the proof we have to show that L(X,Y) is a Banach space. Let $\{T_n\}$ be a Cauchy sequence of L(X,Y). From the inequality $\|T_n(x) - T_m(x)\| \leq \|T_n - T_m\|\,\|x\|$, it follows that for each $x \in X$ the sequence $\{T(x_n)\}$ of Y is Cauchy and thus convergent in Y.

Let $T(x) = \lim T_n(x)$ for each $x \in X$, and note that T defines a linear operator from X to Y. Since $\{T_n\}$ is a Cauchy sequence, and Cauchy sequence is a bounded set in a normed vector space, there exists some $M > 0$ such that $\|T_n\| \leq M$ for every n. But the inequality $\|T_n(x)\| \leq \|T_n\|\,\|x\| \leq M\|x\|$, coupled with the continuity of the norm, shows that $\|T(x)\| \le M\|x\|$ holds for all $x \in X$. So from the previous theorem we have $T \in L(X, Y)$.

Finally, we show that $\lim T_n = T$ holds in L(X,Y). Let $\varepsilon > 0$. Choose K such that $\|T_n - T_m\| < \varepsilon$ for every $n, m \ge K$.

Now, the relation $\|T_m(x) - T_n(x)\| \le \|T_m - T_n\| . \|x\| \le \varepsilon\|x\|$ for all $n, m \ge K$ implies $\|T(x) - T_n(x)\| = \lim_{m\to\infty}\|T_m(x) - T_n(x)\| \le \varepsilon\|x\|$ for all $n \ge K$ and $x \in X$. That is, we have $\|T - T_n\| \le \varepsilon$ for all $n \ge K$, and therefore, $\lim T_n = T$ holds in L(X,Y).

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CHAPTER 2

THE DIFFERNTIAL OPERATOR $\mathbf{D = \frac{d}{dt}}$

The differential operator $\mathbf{D = \frac{d}{dt}}$ is a linear operator. Let us first discuss the question "D is defined from where to where", and then set the domain of D.

We start with the familiar case C[0,1], the Banach space of continuous real valued functions on the closed interval [0,1]. Consider the function

$$f(x)=\begin{cases} x^2 \sin\frac{1}{x}, \text{ if } & x \in (0,1] \\ 0 \qquad\quad , \text{ if } & x = 0. \end{cases}$$

It is easy to verify that f(x) is differentiable at each point of [0,1] and

$$(Df)(x)= f^{'}(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, \text{ if } & x \in (0,1] \\ \qquad 0 \qquad\qquad\quad , \text{ if } & x = 0. \end{cases}$$

We see that $f^{'}(x)$ is not continuous at 0. So the differential operator D could map a differentiable function (that is also continuous and hence a member of C[0,1]) to a function that is not a member of C[0,1].

Hence, if we choose the domain of D as the collection of differentiable function on [0,1] which is also a subspace of C[0,1], then the range of D would not be easy to determine. We impose a tougher condition : the domain of D consists of real valued functions with continuous derivatives denoted by $C_1[0,1]$, which is a subspace of C[0,1]. Then D maps every member of $C_1[0,1]$ to a member C[0,1]. In fact D is an onto map. Indeed, if the function f(x) is continuous at $x \in [0,1]$, then $F(x) = \int_0^x f(t)dt$ is differentiable at x and $F'(x) = f(x)$. Hence we observe that the mapping $D : C_1[0,1] \to C[0,1]$ is onto.

Next, we explore and get a better understanding of $C_1[0,1]$. Let us begin with a few notions.

DEFINITION 1 Suppose A is a subset of a metric space X. The set A is said to be *nowhere dense* set in X if the closure of A contains no interior points.

Note that if we denote the closure of A by $\overline{A}$, the interior of set E by $\overset{o}{E}$, and the complement of set E by $E^c = \{x \in X, x \notin E\}$, then the set A is nowhere dense if $\overset{o}{\overline{A}} = \phi$. Since, for any set E, $\overset{o}{E} = \left(\overline{E^c}\right)^c$, hence, $\overset{o}{\overline{A}} = \phi$ if and only if

$$\phi = \overset{o}{\overline{A}} = \left(\overline{\left(\overline{A}\right)^c}\right)^c.$$

Hence, $\overline{\left(\overline{A}\right)^c} = X$. In other words A is nowhere dense in X if $\left(\overline{A}\right)^c$ is dense in X i.e, the complement of the closure of A is dense in X. The typical examples of nowhere dense sets are finite sets of real numbers in R.

DEFINITION 2 A set is a *meager* set or of first category if it is the union of countably many nowhere dense sets. A set which is not meager is said to be of second category. For example the set Q is meager in R because Q is countable, so Q admits an enumeration : $Q = \{r_1, r_2, \ldots\}$ and hence $Q = \bigcup_{n=1}^{\infty} \{r_n\}$, where each $\{r_n\}$ is nowhere dense. We have a very well known result on complete metric space, called the Baire Category Theorem. It asserts that a complete metric space is not of first category.

PROPOSITION 1 In a complete metric space the complement of a meager set is dense and is of second category.

PROOF Let (X,d) be a complete metric space and A be a set of first category.

Now, since A is meager, $A = \bigcup_{n=1}^{\infty} A_n$, where each A_n is nowhere dense. We show that A^c is not meager. We prove this by contradiction. Suppose A^c is meager. Then, $A^c = \bigcup_{n=1}^{\infty} B_n$, where each B_n is nowhere dense. Now, $X = A \cup A^c$

$\Rightarrow X = \left(\bigcup_{n=1}^{\infty} A_n \right) \cup \left(\bigcup_{n=1}^{\infty} B_n \right)$. So X is the union of countably many nowhere dense sets. Since, (X,d) is complete, so it contradicts the Baire Category Theorem.

Now, we show A^c is dense in X.

$$A = \bigcup_{n=1}^{\infty} A_n \subseteq \bigcup_{n=1}^{\infty} \overline{A_n} \Rightarrow A^c \supseteq \bigcap_{n=1}^{\infty} \overline{(A_n)}^c$$

Since, each A_n is nowhere dense, $(\overline{A_n})^c$ is open and dense, so their countable intersection is also dense. This follows from the result that countable intersection of open dense sets is also dense in a complete metric space. Hence, $\bigcap_{n=1}^{\infty} \overline{(A_n)}^{c}$ is dense, so is A^c. .

Next we introduce a well know result on continuous linear operators on a Banach space : the Banach- Steinhaus Theorem.

BANACH –STEINHAUS THEOREM ([1])

Let $\{A_\alpha\}$ be a family of continuous linear operators defined on Banach space X and taking values in a normed vector space. In order that $\sup\{\|A_\alpha\| < \infty\}$, it is necessary and sufficient that the set $\{x \in X : \sup\|A_\alpha x\| < \infty\}$ be of second category in X.

PROOF Necessity

Assume $\sup_\alpha \|A_\alpha\| < \infty$. Then $\forall\, x, \|A_\alpha x\| \le \|A_\alpha\|\,\|x\| < \infty$. Hence we have $\{x \in X : \sup_\alpha \|A_\alpha x\| < \infty\} = X$. Now, since X is complete, by the Baire Category Theorem, X is of second category.

Sufficiency

Suppose $\{x \in X : \sup_\alpha \|A_\alpha x\| < \infty\}$ is of second category. Let us consider

$F_n = \{x \in X : \sup_{\alpha} \|A_\alpha x\| \leq n\}\ \forall\, n \in N.$

We note two things :

1) F_n is a closed subset of X since each F_n can be written as

$$F_n = \bigcap_{\alpha} A_\alpha^{-1}(\overline{B(0,n)}),$$

where each $\overline{B(0,n)}$ is the closed ball centred at 0 with radius n in the normed linear space. Since A_α is a continuous linear operator, each of the sets in the intersection is closed and arbitrary intersection of closed sets is closed, hence F_n is closed.

2) $\{x \in X : \sup_{\alpha} \|A_\alpha x\| < \infty\} = \bigcup_{n=1}^{\infty} F_n$.

By the assumption that $\{x \in X : \sup_{\alpha} \|A_\alpha x\| < \infty\}$ is of second category , so there is an F_m which is not nowhere dense. Hence F_m has a neighborhood, that is there is $r > 0$ and x_0 such that $S = \{x \in X : \|x - x_0\| \leq r\} \subseteq F_m$. For any $\|x\| \leq 1$, we have $x_0 + rx \in S$. Then, $A_\alpha x = \frac{1}{r} A_\alpha[(x_0 + rx) - x_0] = \frac{1}{r}[A_\alpha(x_0 + rx) - A_\alpha x_0]$.

Now, since x_0 and $x_0 + rx$ are in S,

$$\|A_\alpha x\| \leq \frac{1}{r}\|A_\alpha(x_0 + rx)\| + \frac{1}{r}\|A_\alpha x_0\| \leq \frac{1}{r} m + \frac{1}{r} m = \frac{2m}{r}.$$

This is true for every α and $\|x\| \leq 1$. So, we have $\sup\ \{\|A_\alpha x\| : \|x\| \leq 1\} \leq \frac{2m}{r} < \infty$

$$\Rightarrow \|A_\alpha\| \le \frac{2m}{r} < \infty \text{ for every } \alpha \Rightarrow \sup_\alpha \|A_\alpha\| \le \frac{2m}{r} < \infty.$$

NOTE We make a small note before we prove the next theorem.

For a real valued function x on [0,1] we show that for $t \in (0,1)$ if $x'(t)$ exists,

then $\lim_{h\to 0} \frac{x(t+h)-x(t-h)}{2h} = x'(t)$

Indeed, $\lim_{h\to 0} \frac{x(t+h)-x(t-h)}{2h} = \lim_{h\to 0} \frac{x(t+h)-x(t)+x(t)-x(t-h)}{2h}$

$$= \lim_{h\to 0}\left[\frac{x(t+h)-x(t)}{2h} - \frac{x(t-h)-x(t)}{2h}\right]$$

$$= \lim_{h\to 0} \frac{x(t+h)-x(t)}{2h} + \lim_{-h\to 0} \frac{x(t-h)-x(t)}{2(-h)}$$

$$= \frac{1}{2}x'(t) + \frac{1}{2}x'(t) = x'(t).$$

It is noted that the converse is not true i.e existence of $\lim_{h\to 0} \frac{x(t+h)-x(t-h)}{2h}$

does not imply that $x'(t)$ exists. For example, $x(t) = |t|, \quad t \in [0,1]$

$\lim_{h\to 0} \frac{x(t+h)-x(t)}{h}$ does not exist at t = 0 but $\lim_{h\to 0} \frac{x(t+h)-x(t-h)}{2h} = 0$ at

t = 0.

THEOREM 1 $C_1[0,1]$ is meager in $C[0,1]$.

<u>PROOF</u> Let $x \in C[0,1]$. Now, fix t in (0,1). So for large $n \in N$ we have $(t-\frac{1}{n}, t+\frac{1}{n}) \subseteq (0,1)$. Define a functional $f : C[0,1] \to R$ by

$$f_n(x) = \frac{x(t+\frac{1}{n}) - x(t-\frac{1}{n})}{\frac{2}{n}} = \frac{n}{2}\left[x(t+\frac{1}{n}) - x(t-\frac{1}{n})\right]$$

It is easy to verify that f_n is a linear functional. Now we show that $\|f_n\| = n$.

$$\|f_n\| = \sup\{|f_n(x)| : \|x\|_\infty = 1\} = \sup\left\{\frac{n}{2}\left|x(t+\frac{1}{n}) - x(t-\frac{1}{n})\right| : \|x\|_\infty = 1\right\}$$

$$\le \sup\left\{\frac{n}{2}\left|x(t+\frac{1}{n})\right| + \frac{n}{2}\left|x(t-\frac{1}{n})\right| : \|x\|_\infty = 1\right\}$$

$$\le \sup\left\{(\frac{n}{2}.1 + \frac{n}{2}.1) : \sup\{|x(t)| : t \in [0,1]\} = 1\right\} = n.$$

This shows $\|f_n\| \le n$.

We next show that there is a function "x" such that $f_n(x) = n$.

$$\text{Let } x(s) = \begin{cases} -1, & 0 \le s \le t-\frac{1}{n} \\ n(s-t), & t-\frac{1}{n} \le s \le t+\frac{1}{n} \\ 1, & t+\frac{1}{n} \le s \le 1. \end{cases}$$

Then, $x \in C[0,1]$ and $\|x\|_\infty = 1$. Then, $f_n(x) = \frac{n}{2}[1-(-1)] = n$. So we have $\|f_n\| = n$.

Hence, $\{f_n : n \in N\}$ is a subset of $L(C[0,1], R)$. But, $\{\|f_n\| : n \in N\}$ is not bounded. So, by the Banach-Steinhaus Theorem $B = \{x \in C[0,1] : \sup_n \|f_n(x)\| < \infty\}$ is meager.

Now, if x is differentiable at t, then $\lim_{n\to\infty} f_n(x(t)) = x'(t) \in R$, so if x is differentiable at t, then $\{f_n(x), n \in N\}$ is a bounded set of real numbers. So, the set T of all differentiable functions at t is a subset of B. Of course $C_1[0,1] \subseteq T \subseteq B$. Hence $C_1[0,1]$ is meager in $C[0,1]$. .

We note here that $C_1[0,1]$ is not a Banach space due to Baire Category Theorem.

Now, let P denote the set of all polynomials. By the well known Weierstrass Theorem, P is dense in C[0,1] under the uniform sup norm i.e, $\overline{P} = C[0,1]$.

Again, $P \subseteq C_1[0,1] \subseteq C[0,1]$. This shows that $\overline{C_1[0,1]} = C[0,1]$.

The above discussion tells us that most of the functions in C[0,1] are not differentiable, but the differentiable functions are dense in C[0,1].

Note that the set Q of rational numbers is meager in the space R of real numbers, and Q is dense in R under the Euclidean norm. So the status of $C_1[0,1]$ in C[0,1] is analogous to that of Q in R.

Next, we show that D is an unbounded operator. As we showed earlier $D : C_1[0,1] \to C[0,1]$. Let $\{x_n\} \in C_1[0,1]$ and $x_n(t) = t^n$ then, $\|x_n\|_\infty = 1$ and $\|D(x_n)\|_\infty = \sup\{nt^{n-1} : t \in [0,1]\} = n$ holds for each n, implying $\|D\| = \infty$. Hence, D is an unbounded operator. However, D is what we call a closed operator. We introduce a few notions below which will lead to a better understanding of closed operators.

CARTESIAN PRODUCT SPACE

If X and Y are normed vector spaces equipped with norms $\| \bullet \|_x$ and $\| \bullet \|_y$ respectively. The Cartesian product space is given by

$$X \times Y = \{(x, y), x \in X, y \in Y\}.$$

Then the norm defined on $X \times Y$ is given by $\|(x, y)\| = \|x\| + \|y\|$. This norm is called the product norm. There are other equivalent norms, such as $\|(x, y)\| = \left(\|x\|^2 + \|y\|^2\right)^{\frac{1}{2}}$ and $\|(x, y)\| = \max\{\|x\|, \|y\|\}$.

It should be noted that $\lim (x_n, y_n) = (x, y)$ holds in $X \times Y$ with respect to the product norm if and only if $\lim x_n = x$ and $\lim y_n = y$ both hold. Moreover, if both X and Y are Banach Space, then $X \times Y$ with the product norm is a Banach Space.

Now, let T be a linear operator from X and Y. The graph of T is the subset G of $X \times Y$ given by $G = \{(x, T(x)) : x \in X\}$ and $\|(x, T(x))\| = \|x\| + \|Tx\|$.

DEFINITION 3 A linear operator $T : X \to Y$ is a *closed operator* if the graph $G = \{(x, T(x)) : x \in X\}$ is a closed set in $X \times Y$.

Now, we show that differential operator D is a closed operator.

Let $D : C_1[0,1] \to C[0,1]$. The graph of D is given by

$$G = \{(x, Dx) : x \in C_1[0,1]\}$$

We show that $G \subseteq C_1[0,1] \times C[0,1]$ is a closed set in $C_1[0,1] \times C[0,1]$.

Let (x_n, Dx_n) converge to (x,y) in $C_1[0,1] \times C[0,1]$. Then, $x_n \to x$ and $Dx_n \to y$ under uniform sup norm. Now by the well known theorem :

"If f_n converges uniformly to f, and if all the f_n are differentiable, and if the derivatives f'_n converge uniformly to g, then f is differentiable and its derivative is g."

We have, x is differentiable and y = Dx. Then $(x, y) \in G$ is a closed set, and D is a closed operator.

Finally, we mention the Closed Graph Theorem that asserts that if X and Y are Banach spaces and $T : X \to Y$ is a linear operator, then the closed property of the graph $T = \{(x, T(x)) : x \in X\}$ in $X \times Y$ implies that T is a bounded operator.

In the above discussion D has a closed graph in $C_1[0,1] \times C[0,1]$. But the fact that D is not bounded gives a counterexample to the Closed Graph Theorem, where if X is not a Banach space then D is not necessarily bounded. We encounter this problem because $C_1[0,1]$ is not a Banach space, though C[0,1] is a Banach space.

CHAPTER 3

NILPOTENT OPERATORS AND JORDAN FORM

In this chapter we come across some properties of the differential operator D. We first review some notions below.

DEFINITION 1 Let T be a linear operator on a vector space V. If W is a subspace of V such that $T(W) \subseteq W$, we say W is *invariant under T or is T-invariant*. For example Ker T is an invariant subspace of V as $v \in \text{Ker T}$ then $T(v) = 0 \in \text{Ker T}$. So Ker T is an invariant subspace of T

If we have an invariant subspace in a finite dimensional vector space its matrix representation becomes much simpler as we see in the theorem below.

THEOREM 1 [6] Let W be an invariant subspace of a linear operator T on V. Then T has a matrix representation $\begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$, where A is the matrix representation of T_w that is the restriction of T on W.

DEFINITION 2 Let V be a vector space over a field F. Let M and N be two subspaces of V, such that

1) $M \cap N = \{0\}$

2) $\forall\, v \in V, \exists\, x \in M$ and $y \in N$ such that $v = x + y$.

Then V is called the *direct sum* of M and N. We write it as $V = M \oplus N$.

We state a theorem without proof, pertaining to the dimension of the direct sum.

THEOREM 2 [6] If V is a vector space and M and N are subspaces with dimensions m and n respectively, such that $V = M \oplus N$, then

$$\dim V = \dim M + \dim N \quad \text{i.e,} \quad \dim V = m + n.$$

DEFINITION 3 If M and N are two subspaces of V such that both are invariant under T and $V = M \oplus N$, then *T is reduced by the pair (M,N).*

The matrix representation of T is further simplified than the one mentioned in Theorem 1.

THEOREM 3 [6] If W and U are invariant subspaces of a linear operator T on a finite dimensional vector space V over F and $V = W \oplus U$, then there is a basis β of V such that the matrix of T with respect to β is $\begin{bmatrix} T_W & 0 \\ 0 & T_U \end{bmatrix}$ or $\operatorname{diag}(T_W, T_U)$, where T_W is the matrix of restriction of T on W and T_U is the matrix of restriction of T on U.

In general we note that the greater the number of invariant subspaces of a linear operator , the simpler will the matrix representation of the linear operator be.

Let us take the case of the differential operator D acting on P_n i.e, polynomials x in t of degree $\leq n-1$. We note here that each $P_1, P_2, \ldots, P_n$ is an invariant subspace under D. The basis for P_n is given by $x_i(t) = t^{i-1}, 1 \leq i \leq n$ and $\dim P_n = n$.

Now let us find the matrix representation of the differential operator D.

$$Dx_1 = 0 = 0x_1 + 0x_2 + \ldots + 0x_{n-1} + 0x_n.$$

$$Dx_2 = 1 = 1x_1 + 0x_2 + \ldots + 0x_{n-1} + 0x_n.$$

$$Dx_3 = 2t = 0x_1 + 2x_2 + \ldots + 0x_{n-1} + 0x_n.$$

.

.

$$Dx_n = (n-1)t^{n-2} = 0x_1 + 0x_2 + \ldots + (n-1)x_{n-1} + 0x_n.$$

The matrix representation of D is given by

$$D = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 2 & . & . & . & 0 \\ 0 & 0 & 0 & 3 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & n-1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We proceed to introduce and study a very special but useful class of linear operators called the nilpotent operators.

DEFINITION 4 A linear operator A is called nilpotent if there exists a positive integer p such that $A^p = 0$; the least such integer p is called the *index of nilpotence.*

We note that $D : P_n \to P_n$, is a nilpotent operator of index n.

THEOREM 4 [3] If T is a nilpotent linear operator of index p on a finite dimensional vector space V, and if ξ is a vector for which $T^{p-1}\xi \neq 0,$ then the vectors $\xi, T\xi, \ldots, T^{p-1}\xi$ are linearly independent. If H is a subspace spanned by these vectors, then there is a subspace K such that $V = H \oplus K$ and the pair (H,K)

(H,K) reduces T.

PROOF To prove the asserted linear independence, suppose that $\sum_{i=0}^{p-1} \alpha_i T^i \xi = 0$, and let j be the least index such that $\alpha_j \neq 0.$ (We do not exclude the possibility j=0). Dividing through by $-\alpha_j$ and changing the notation in a obvious way, we obtain a relation of the form $T^j\xi = \sum_{i=j+1}^{p-1} \beta_i T^i \xi = T^{j+1}(\sum_{i=j+1}^{p-1} \beta_i T^{i-j-1}\xi) = T^{j+1}y.$,where $y = \sum_{i=j+1}^{p-1} \beta_i T^{i-j-1}\xi$. It follows from the definition of index of p that

$$T^{p-1}\xi = T^{p-j-1}T^{j}\xi = T^{p-j-1}T^{j+1}y = T^{p}y = 0;$$

since this contradicts the choice of ξ ,we must have $\alpha_j = 0$ for each j.

It is clear that H is invariant under T; to construct K we go by induction on the index p of nilpotence. If p=1, then T = 0 and we have $V = \{0\} \oplus V$. we now assume the theorem is true for p-1.The range R of T is a subspace that is invariant under T ; restricted to R the linear operator T is nilpotent of index p-1. We write $H_0 = H \cap R$ and $\xi_0 = T\xi$; then H_0 is spanned by linearly independent vectors $\xi_0, T\xi_0, \ldots, T^{p-2}\xi_0$. The induction hypothesis may be applied, and we may conclude that R is the direct sum of H_0 and some other invariant subspace K_0.

We write K_1 is the set of all vectors x such that Tx is in K_0; it is clear that K_1 is a subspace. The temptation is great to set $K = K_1$ and to attempt to prove that K has the desired properties. But this need not be true ; H and K_1 need not be disjoint. (It is true, but we shall not use the fact, that the intersection of H and K_1 is contained in the null-space of T.) In spite of this, K_1 is useful because of the fact that $H + K_1 = V$. To prove this, observe that Tx is in R for every x, and, consequently, Tx = y + z with y in H_0 and z in K_0. The general element of H_0 is a linear combination of $T\xi, \ldots, T^{p-1}\xi$; hence we have

$$y = \sum_{i=1}^{p-1}\alpha_i T^i\xi = T(\sum_{i=0}^{p-2}\alpha_{i+1}T^i\xi) = Ty_1, \text{ where } y_1 \text{ is in H.}$$

It follows that $Tx = Ty_1 + z$, or $T(x - y_1)$ is in K_0. This means that $x - y_1$ is in K_1, so that x is the sum of an element (namely y_1) of H and an element (namely $x - y_1$) of K_1.

As far as disjointness is concerned, we can say at least that $H \cap K_0 = \{0\}$. To prove this, suppose that x is in $H \cap K_0$, and observe first that Tx is in H_0 (since x is in H). Since, K_0 is also invariant under T, the vector Tx belongs to K_0 along with x, so that Tx = 0. From this we infer that x is in H_0 .(Since x is in H, we have $x = \sum_{i=0}^{p-1} \alpha_i T^i \xi$; and therefore $0 = Tx = \sum_{i=1}^{p-1} \alpha_{i-1} T^i \xi$, from the linear independence of $T^j \xi$, it follows that $\alpha_0 = \ldots = \alpha_{p-2} = 0$, so that $x = \alpha_{p-1} T^{p-1} \xi$). We have proved that if x belongs $H \cap K_0$, then it also belongs to $H_0 \cap K_0$, and hence that x = 0.

The situation now is this : H and K_1 together span V, and K_1 contains the two disjoint subspaces K_0 and $H \cap K_1$. If we let $K_0{}^c$ be the complement of $K_0 \oplus (H \cap K_1)$ in K_1, that is if $K_0{}^c \oplus K_0 \oplus (H \cap K_1) = K_1$, then we may write $K = K_0{}^c \oplus K_0$; we assert that this K has the desired properties. In the first place, $K \subset K_1$ and K is disjoint from $H \cap K_1$; it follows that $H \cap K = \{0\}$. In the second place, $H \oplus K$ contains both H and K_1, so that $H \oplus K = V$. Finally, K is invariant under T, since the fact that $K \subset K_1$ implies that $AK \subset K_0 \subset K$. The proof of the theorem is complete. .

DEFINITION 5 Let V be a vector space over a field F, and T be a linear operator on V. For any vector ξ in V the subspace

$$Z_\xi = \{P(T)\xi : P \text{ is a polynomial in } F(x)\}$$

is called the *T-cyclic subspace* generated by ξ. If $Z_\xi = V$, then ξ is called a *cyclic vector* of T.

In particular, if T is nilpotent with index p and $T^{p-1}\xi \neq 0$, then

$Z_\xi = \langle \xi, T\xi, \ldots T^{p-1}\xi \rangle$ is the T-cyclic subspace generated by ξ.

Theorem 4 shows that every nilpotent operator T on a finite dimensional vector space has a T-cyclic subspace Z_ξ generated by vector ξ, and this cyclic subspace has a complementary T-invariant subspace V_0 such that the pair Z_ξ and V_0 reduce T.

Let us further analyze the result in Theorem 4. Suppose T is nilpotent on V with index p_1. Then there is a T-invariant subspace V_0 such that $V = Z_{\xi_1} \oplus V_0$, where $Z_{\xi_1} = \langle T^{p_1-1}(\xi_1), \ldots. T(\xi_1), \xi_1 \rangle$. From Theorem 3 we know that T is represented by a matrix of the form $\mathrm{diag}(A_1, B_1)$, relative to the basis for V consisting of a basis for Z_{ξ_1} and a basis for V_0. Relative to the ξ_1-basis for Z_{ξ_1}, $\{T^{p_1-1}(\xi_1), \ldots. T(\xi_1), \xi_1\}$, the restriction of T on Z_{ξ_1} is represented by

$$A_1 = \left[TZ_{\xi_1}\right]_{\xi_1} = \begin{bmatrix} 0 & 1 & 0 & . & . & 0 & 0 \\ 0 & 0 & 1 & . & . & 0 & 0 \\ . & . & . & . & . & 0 & 0 \\ . & . & . & . & . & 0 & 0 \\ . & . & . & . & . & 1 & 0 \\ 0 & 0 & 0 & . & . & 0 & 1 \\ 0 & 0 & 0 & . & . & 0 & 0 \end{bmatrix} = J_{p_1}(0).$$

We use the same notation for general $J_r(\lambda)$ which denotes the $r \times r$ matrix with λ on the diagonal, ones on the super diagonal, and zeroes elsewhere. $J_r(\lambda)$ is called a *simple Jordan block.*

Now, the restriction of T on V_0, T_{V_0} is nilpotent on V_0 of index $p_2 \le p_1$. From Theorem 4 we can find a T-invariant decomposition $V_0 = Z_{\xi_2} \oplus V_1$, where $T^{p_2-1}\xi_2 \neq 0$, $Z_{\xi_2} = \left\langle T^{p_2-1}(\xi_2), \ldots T(\xi_2), \xi_2 \right\rangle$. Then $V = Z_{\xi_1} \oplus Z_{\xi_2} \oplus V_1$.

As above, we have the matrix of T on Z_{ξ_2}, $\left[TZ_{\xi_2}\right]_{\xi_2} = J_{p_2}(0)$. This means that there exists a basis for V relative to which T is represented by $\mathrm{diag}(J_{p_1}(0), J_{p_2}(0), B_2)$.

Continuing in this way, we eventually find, since dim(V) is finite, a T-invariant direct-sum decomposition of the form $V = Z_{\xi_1} \oplus Z_{\xi_2} \oplus \ldots \oplus Z_{\xi_k}$, and a basis for V:

$$T^{p_1-1}(\xi_1), \ldots, T(\xi_1), \xi_1$$

$$T^{p_2-1}(\xi_2), \ldots, T(\xi_2), \xi_2$$

. . . . . .

$T^{p_k-1}(\xi_k), \ldots, T(\xi_k), \xi_k$

relative to which T is represented by $\mathrm{diag}(J_{p_1}(0), J_{p_2}(0), \ldots, J_{p_k}(0))$.

The above discussion can be summarized in the following theorem.

<u>THEOREM 5</u> [2] If T is a nilpotent operator of index p_1, then there exists an integer k, k distinct vectors $\xi_1, \ldots, \xi_k$, and integers $p_1 \geq p_2 \geq \ldots \geq p_k$ such that vectors

$T^{p_1-1}(\xi_1), \ldots, T(\xi_1), \xi_1$

$T^{p_2-1}(\xi_2), \ldots, T(\xi_2), \xi_2$

. . . .

$T^{p_k-1}(\xi_k), \ldots, T(\xi_k), \xi_k$

form a basis for V. Moreover V is the direct sum of the T-cyclic subspaces generated by ξ_i: $V = Z_{\xi_1} \oplus Z_{\xi_2} \oplus \ldots \oplus Z_{\xi_k}$. Relative to the above basis, T is represented by the matrix $\mathrm{diag}(J_{p_1}(0), J_{p_2}(0) \ldots J_{p_k}(0))$.

In fact, Theorem 5 concludes that for a nilpotent operator T on a finite dimensional vector space V we can find a suitable basis β of V such that T admits a Jordan canonical form, where the number k of distinct “simple Jordan blocks” is equal to the number of vectors $\xi_1, \ldots, \xi_k$. Note that the vectors

$T^{p_1-1}\xi_1, T^{p_2-1}\xi_2, \ldots, T^{p_k-1}\xi_k$ are linearly independent, and they are in the null space of T. So the nullity of T is greater than or equal to k. On the other hand

$$V = Z_{\xi_1} \oplus Z_{\xi_2} \oplus \ldots \oplus Z_{\xi_k}, \text{ and}$$

$$n = \dim V = \dim Z_{\xi_1} + \ldots + \dim Z_{\xi_k} = p_1 + \ldots + p_k.$$

Note that the rank of T is $(p_1 - 1) + \ldots + (p_k - 1) = (p_1 + \ldots + p_k) - k = n - k.$

Hence the nullity of T is equal to $n - (n - k) = k.$ So the geometric multiplicity of T is k and $\{T^{p_1-1}\xi_1, T^{p_2-1}\xi_2, \ldots, T^{p_k-1}\xi_k\}$ is a basis of null space of T. So we conclude that the number k of distinct simple Jordan blocks is exactly equal to the geometric multiplicity of T.

<u>Example 1</u> Let $T : R^5 \to R^5$.The matrix representation of T under the standard basis is $[T]_\varepsilon = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$, we apply Theorem 5 to T.

We note here that $T^3 = 0$ and $T^2 \neq 0.$ T is nilpotent on R^5.

Let $\xi_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$. So $T\xi_1 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$ and $T^2\xi_1 = \begin{bmatrix} 0 \\ 0 \\ 6 \\ 0 \\ 0 \end{bmatrix}$, where $\xi_1, T\xi_1$ and $T^2\xi_1$ are linearly independent.

Hence the cyclic subspace generated by ξ_1 is

$$Z_{\xi_1} = \langle \xi_1, T\xi_1, T^2\xi_1 \rangle = \left\langle \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 6 \\ 0 \\ 0 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \\ 0 \end{bmatrix} ; a_1, a_2, a_3 \in R \right\}$$

If the ordered basis is $\{T^2\xi_1, T\xi_1, \xi_1\}$, the simple Jordan block is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

If the ordered basis is $\{\xi_1, T\xi_1, T^2\xi_1\}$, the simple Jordan block is $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

Now, let $\xi_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ and $T\xi_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 0 \end{bmatrix}$.

So that $V_0 = \langle \xi_2, T\xi_2 \rangle = \left\langle \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 4 \\ 0 \end{bmatrix} \right\rangle = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ a_5 \end{bmatrix} ; a_4, a_5 \in R \right\}$.

If the ordered basis is $\{T\xi_2, \xi_2\}$, the matrix of representation of T on V_0 is $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If the ordered basis is $\{\xi_2, T\xi_2\}$, the matrix of representation of T on V_0 is $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. So the matrix representation of T on R^5 is

$$\text{either } \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

Using the terminology in Theorem 5, we have k = 2, $p_1 = 3$ and $p_2 = 2$.

Now, we recall $D : P_n \to P_n$, where P_n is the set of all polynomials with degree $\leq n-1$ in variable t, the cyclic vector is t^{n-1}. With respect to Theorem 4, the vector ξ is t^{n-1}. Note that the vectors $t^{n-1}, D(t^{n-1}), \ldots, D^{n-1}(t^{n-1})$ are linearly independent ,i.e, $t^{n-1}, (n-1)t^{n-2}, \ldots, (n-1)!t, (n-1)!$ are linearly independent. Hence, $H = \text{span}\{t^{n-1}, t^{n-2}, \ldots, t, 1\}$ and $K = \{0\}$, so that

$$P_n = H \oplus K.$$

The Jordan canonical form for a general linear operator is discussed below. We quote the next result without proof.

<u>THEOREM 6</u> [2] Let V be a finite-dimensional vector space over an algebraically closed filed F. For any linear operator T on V, there exist T-invariant subspaces M and N, such that $V = M \oplus N$ and T_M, the restriction of T on M, is nonsingular and T_N, the restriction of T on N, is nilpotent.

<u>THEOREM 7</u> [2] Let V be a finite-dimensional vector space over an algebraically closed field F. If T is a linear operator on V with characteristic polynomial $c(x) = \prod_{i=1}^{r}(x - \lambda_i)^{s_i}$, then there exist T-invariant subspaces $N_1, N_2, \ldots,$ N_r such that

a) $V = N_1 \oplus N_2 \oplus \ldots. \oplus N_r$,

b) $\dim(N_i) = s_i$,

c) $T_{N_i} = \lambda_i I + \eta_i$, where η_i is nilpotent.

<u>PROOF</u> Let $T_1 = \lambda_1 I - T$, be a linear transformation on V. Using Theorem 6, there are complementary T_1-invariant subspaces N_1 and M_1 $(V = N_1 \oplus M_1)$ such that $T_{1_{M_1}}$ is nonsingular and $T_{1_{N_1}}$ is nilpotent. Now, $T = \lambda_1 I - T_1$, so for $\alpha \in N_1$ we have $T(\alpha) = \lambda_1 \alpha - T_1(\alpha) \in N_1$. It follows that N_1 is also T-invariant and similarly M_1 is T-invariant. Now, $T_{N_1} = (\lambda_1 I - T_1)_{N_1} = \lambda_1 I - T_{1_{N_1}} = \lambda_1 I + \eta_1$, where $\eta_1 = -T_{1_{N_1}}$ is nilpotent by construction.

Suppose that $\dim(N_1) = n_1$. Relative to the basis for V, consisting of a basis for N_1 and a basis for M_1, T is represented by a matrix of the form $A = \text{diag}(A_1, A_2)$, where the $n_1 \times n_1$ matrix A_1 represents T_{N_1} and A_2 represents T_{M_1}. Now, $\det(xI - A) = c(x) = \det(xI - A_1)\det(xI - A_2)$ and $\det(\lambda_1 I - A_2) \neq 0$, since $\lambda_1 I - A_2$ represents $T_{1_{M_1}} = \lambda_1 I - T_{M_1}$ and is nonsingular by construction. Thus $x - \lambda_1$ is not a factor of $\det(xI - A_2)$ and hence $(x - \lambda_1)^{s_1}$ must be a factor of $\det(xI - A_1)$. Since the degree of $\det(xI - A_1)$ is n_1, we have $n_1 \geq s_1$.

Now, $(A_1 - \lambda_1 I)$ represents $(T - \lambda_1 I)_{N_1} = -T_{1_{N_1}} = \eta_1$, which is nilpotent by construction. By Theorem 5 we can choose a basis for N_1 such that $(A_1 - \lambda_1 I)$ is upper triangular matrix whose only nonzero elements are ones on the super diagonal. This A_1 has λ_1 on every diagonal position and zeroes and ones on the super diagonal so $\det(xI - A_1) = (x - \lambda_1)^{n_1}$. Thus $n_1 \leq s_1$ and , with the inequality established above, it follows that $n_1 = s_1$.

If $r = 1$, the proof is completed; if not, we repeat the above argument using the operator T_{M_1}. After r repetitions the proof is completed. .

In addition to the assertions in the above theorem, we have established that there is a basis for V relative to which T is represented by

$\text{diag}(A_1, A_2, \ldots A_r)$, where A_i has λ_i on the diagonal, zeroes and ones on the super diagonal, and zeroes elsewhere. We thus have the following theorem.

THEOREM 8 (Jordan) [2] Let A be an $n \times n$ matrix whose entries are from an algebraically closed field F. Suppose the characteristic polynomial of A is $c(x) = \det(xI - A) = \prod_{i=1}^{r}(x - \lambda_i)^{s_i}$, then A is similar to a matrix with the λ_i on the diagonal, zeroes and ones on the super diagonal, and zeroes elsewhere.

Note that if F is an algebraically closed field, then the hypothesis in Theorem 6 and 7 about the factorability of c(x) is always satisfied. From Theorem 5 we know that for each i there exists an integer k(i) and k(i) integers :

$$p_{i1} \geq p_{i2} \geq p_{i3} \geq \ldots \geq p_{ik(i)}$$

whose sum is s_i, such that the nilpotent matrix $(A_i - \lambda_i I)$ is similar to

$$\text{diag}(J_{p_{i1}}(0), J_{p_{i2}}(0), \ldots, J_{p_{ik(i)}}(0))$$

and hence that A_i is similar to

$$\text{diag}(J_{p_{i1}}(\lambda_i), J_{p_{i2}}(\lambda_i), \ldots, J_{p_{ik(i)}}(\lambda_i)).$$

We now see that a Jordan canonical matrix similar to $A \in F_n$ is completely determined by scalars : $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_r$,

$$s_1, s_2, s_3, \ldots, s_r,$$

$$p_{11}, p_{12}, p_{13}, \ldots, p_{1k(1)},$$

$$p_{21}, p_{22}, p_{23}, \ldots, p_{2k(2)},$$

$$\ldots.$$

$$p_{r1}, p_{r2}, p_{r3}, \ldots, p_{rk(r)}.$$

Note that the Jordan matrix is unique, except for the order in which simple Jordan forms appear on the diagonal. Note also that $\sum_{i=1}^{r} s_i = n, \sum_{j=1}^{k(j)} p_{ij} = s_i$ and that the minimal polynomial (its definition will be given in the next chapter) of A is given by $m_A(x) = \prod_{i=1}^{r} (x - \lambda_i)^{p_{i1}}$, p_{i1} being the size of the largest simple Jordan block associated with λ_i. We summarize with :

Theorem 9 [2] The Jordan canonical form similar to A in F_n is determined up to the order of the diagonal blocks, by constants.

$$\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_r,$$

$$s_1, s_2, s_3, \ldots, s_r,$$

$$p_{11}, p_{12}, p_{13}, \ldots, p_{1k(1)},$$

$$p_{21}, p_{22}, p_{23}, \ldots, p_{2k(2)},$$

$$\ldots.$$

$$p_{r1}, p_{r2}, p_{r3}, \ldots, p_{rk(r)}$$

For each $i = 1, 2, \ldots, r$, the integer k(i) is the number of simple Jordan blocks with eigenvalue λ_i. Equivalently, $k(i) = \dim(\ker(\lambda_i I - A))$, which is the number of linearly independent eigenvectors associated with λ_i.The largest simple Jordan block associated with λ_i is $J_{P_{i1}}(\lambda_i)$ and the minimal polynomial of A is

$m_A(x) = \prod_{i=1}^{r}(x-\lambda_i)^{p_{i1}}$.

MORE ABOUT JORDAN CANONICAL FORM [4]

As we have seen earlier a Jordan block is a k-by-k upper triangle matrix of the form

$$J_k(\lambda) = \begin{bmatrix} \lambda & 1 & 0 & . & . & . & 0 \\ 0 & \lambda & 1 & . & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 \\ 0 & 0 & 0 & . & . & . & \lambda \end{bmatrix}$$

There are $(k-1)$ terms "+1" in the super diagonal, the scalar λ appears k times on the main diagonal. All other entries are zero, and $J_1(\lambda) = [\lambda]$. A Jordan matrix $J \in M_n$, where M_n is the set of all $n \times n$ matrices, is the direct sum of the Jordan blocks.

$$J = \begin{bmatrix} J_{n_1}(\lambda_1) & 0 & . & . & . & 0 \\ 0 & J_{n_2}(\lambda_2) & . & . & . & 0 \\ . & . & . & . & . & . \\ . & . & . & . & . & . \\ 0 & 0 & . & . & . & J_{n_k}(\lambda_k) \end{bmatrix},$$

$n_1 + n_2 + \ldots + n_k = n,$ in which the orders n_i may not be distinct and the values λ_i need not be distinct. If $A \in M_n$ over an algebraically closed field, then there exists $P \in M_n$, which is non singular so that $A = PJP^{-1}$. Here, J is the Jordan canonical form of A.

We note the following points:

1) The number of Jordan blocks in J corresponding to eigenvalue λ_i of A = number of linearly independent eigenvectors corresponding to λ_i = null space of $(\lambda_i I - A)$ = geometric multiplicity of λ_i.
2) The sum of orders of all Jordan blocks corresponding to λ_i is the algebraic multiplicity of λ_i.
3) If λ is an eigenvalue of A, then the smallest integer k_1 such that $(A - \lambda I)^{k_1} = 0$ is the size of the largest block. The rank of $(A - \lambda I)^{k_1 - 1}$ is the number of blocks of size k_1, the rank of $(A - \lambda I)^{k_1 - 2}$ is twice the number of blocks of size $k_1 - 1$, and so forth. The sequence of ranks of $(A - \lambda I)^{k_1 - i}$, recursively determine the orders of blocks in J.
4) If k = n, then J is diagonalizable.

Again, let us consider the differential operator $D : P_n \to P_n$. As we have already seen that the matrix representation of D with respect to the basis $\{1, t, t^2, \ldots, t^{n-1}\}$ is given by

$$D = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 2 & . & . & . & 0 \\ 0 & 0 & 0 & 3 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & 0 & 0 & n-1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let us try to find the Jordan canonical form of D. We see that all eigenvalues of D are 0, so $\lambda = 0.$ The following results can also be easily verified

a) $D^n = 0.$

b) $D^{n-1} \neq 0.$

c) Nullity of D is 1.

So the largest Jordan block is of size n and obviously there is only one block of this size. So the Jordan canonical form of D is

$$J = \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ 0 & 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \\ 0 & 0 & 0 & . & . & . & 0 \end{bmatrix}$$

Let us find P for this differential operator D such that

$$D = PJP^{-1}.$$

$\Rightarrow$ DP = PJ.

$$\Rightarrow D\begin{bmatrix} P_1 & P_2 & . & . & P_n \end{bmatrix} = \begin{bmatrix} P_1 & P_2 & . & . & P_n \end{bmatrix} J$$

$$\begin{bmatrix} DP_1 & DP_2 & . & . & DP_n \end{bmatrix} = [P_1 \quad P_2 \quad . \quad . \quad P_n] \begin{bmatrix} 0 & 1 & 0 & . & . & . & 0 \\ 0 & 0 & 1 & . & . & . & 0 \\ 0 & 0 & 0 & 1 & . & . & 0 \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 \\ 0 & 0 & 0 & . & . & . & 0 \end{bmatrix}$$

$$\begin{bmatrix} DP_1 & DP_2 & . & . & DP_n \end{bmatrix} = \begin{bmatrix} 0 & P_1 & P_2 & . & . & P_{n-1} \end{bmatrix}$$

So, $DP_1 = 0, DP_2 = P_1, DP_3 = P_2, \ldots\ldots, DP_n = P_{n-1}$. We see that the solutions of

a) $DP_1 = 0$ is span $\begin{bmatrix} 1 \\ 0 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$ b) $DP_2 = P_1$ is span $\begin{bmatrix} 1 \\ 1 \\ 0 \\ . \\ . \\ 0 \end{bmatrix}$

c) $DP_3 = P_2$ is span $\begin{bmatrix} 1 \\ 1 \\ 1/2! \\ . \\ . \\ 0 \end{bmatrix}$ d) $DP_n = P_{n-1}$ is $\begin{bmatrix} 1 \\ 1 \\ 1/2! \\ . \\ . \\ 1/(n-1)! \end{bmatrix}$

Hence, the required P is of the form :

$$P = \begin{bmatrix} 1 & 1 & 1 & . & . & 1 & 1 \\ 0 & 1 & 1 & . & . & 1 & 1 \\ . & 0 & \frac{1}{2!} & . & . & \frac{1}{2!} & \frac{1}{2!} \\ . & . & 0 & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & \frac{1}{(n-2)!} & \frac{1}{(n-2)!} \\ . & . & . & . & . & 0 & \frac{1}{(n-1)!} \end{bmatrix}.$$

It may be noted that

$$\left\{(n-1)!, (n-1)!\,t, (n-1)\ldots 3t^2, \ldots (n-1)(n-2)t^{n-3}, (n-1)t^{n-2}, t^{n-1}\right\}$$

is a basis of P_n , under which D admits the Jordan canonical form.

CHAPTER 4

SOLUTION OF P(D) = 0 AND SOME RESULTS

We already learned that Jordan canonical form works on an algebraically closed field F, such as the complex C, but not on real R. The problem is the linear operator T on a finite dimensional vector space V over F may not have a single eigenvalue in R. Moreover, even if the characteristic polynomial factors complet- -ely over R into a product of linear polynomials, there may not be enough eigenvectors for T to span the space V. The primary decomposition takes care of these issues.

DEFINITION 1 Let T be a linear operator on a finite dimensional vector space V over a field F. The *minimal polynomial* for T is the unique monic generator of the ideal of polynomials over F which annihilate T.

We note here that if dimension of V is n, then dimension of L(V,V) is n^2.

Check that the first n^2+1 powers of T : $I, T, T^2, \ldots, T^{n^2}$ are linearly dependent.

Hence, there is a non-zero polynomial of degree n^2 or less which annihilates T. Note that the collection of polynomials p which annihilate T i.e, $P(T)=0$, is an ideal in the polynomial ring F[x]. Since the polynomial ring is a principal ideal ring, the generator of the above ideal exists. So the definition is reasonable.

<u>Theorem 1</u> <u>(Primary Decomposition Theorem)</u> [6] Let T be a linear operator on a finite dimensional vector space V over a field F. Let p be the minimal polynomial for T such that :

$$p(x) = p_1^{r_1} \dots p_k^{r_k} ,$$

where all the p_i are distinct irreducible monic polynomials over F and the r_i are positive integers. Let W_i be the null space of $P_i(T)^{r_i}$, i = 1, . . . k. Then

(A) $V = W_1 \oplus \dots \oplus W_k$.

(B) each W_i is invariant under T.

(C) if T_i is the operator restricted on W_i by T, then the minimal polynomial for T_i is $p_i^{r_i}$.

We notice that in the Primary Decomposition Theorem, it is not necessary that the vector space be finite dimensional, nor is it necessary for parts (A) and (B) that p be the minimal polynomial for T. In fact, if T is a linear operator on an arbitrary vector space and if there is a monic polynomial p such that p(T) = 0, then parts (A) and (B) in the theorem are valid for T. (Note that we will call T an algebraic operator which we discuss later). This is because the proof of the primary decomposition is based on the use of projections E_i (which are identity on W_i, and zero on the other W_j), and the fact that if $p_1, \dots, p_k$ are distinct prime polynomials , the polynomials $f_1, \dots, f_k$, where $f_i = \frac{p}{p_i^{r_i}} = \prod_{j \neq i} p_j^{r_j}$,

$i = 1, \ldots, k$, are relatively prime. Thus there are polynomials $g_1, \ldots, g_k$ such that $\sum_{i=1}^{k} f_i g_i = 1$.

Let us consider the differential equation

$$\frac{d^n x}{dt^n} + \alpha_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \ldots + \alpha_1 \frac{dx}{dt} + \alpha_0 x = 0, \ \ldots\ldots. \quad (*)$$

where $\alpha_0, \ldots, \alpha_{n-1}$ are some constants. Let $C_n[0,1]$ denote all n times continuo--usly differentiable functions on [0,1], which is a linear subspace of C[0,1]. The space V of solutions of this differential equation is a subspace of $C_n[0,1]$. Let p denote the polynomial

$$p(s) = s^n + \alpha_{n-1}s^{n-1} + \ldots + \alpha_1 s + \alpha_0.$$

Then the differentiable equation (*) can be denoted by $p(D)x = 0$. Hence the space V is the null space of the operator p(D). Therefore V is an invariant subspace of the differential operator D.

Let us regard D as a linear operator on V. Then $p(D) = 0$. It follows that p is the minimal polynomial of D on V. The polynomial p can be factored into the product of the powers of linear polynomials when we treat $C_n[0,1]$ and V as complex vector spaces, and $\alpha_0, \ldots, \alpha_{n-1}$ as complex numbers. So we have

$$p(s) = (s - \lambda_1)^{r_1}(s - \lambda_2)^{r_2} \ldots (s - \lambda_k)^{r_k}$$

where $\lambda_1, \lambda_2, \ldots \lambda_k$ are distinct complex numbers and $r_1, r_2, \ldots r_k$ are positive integers. In fact, $\lambda_1, \lambda_2, \ldots \lambda_k$ are eigenvalues of D. If W_j is the null space of the operator $(D - \lambda_j I)^{r_j}$, $j = 1, \ldots, k$ then by Primary Decomposition Theorem, we have the following direct sum for V :

$$V = W_1 \oplus W_2 \oplus \ldots \oplus W_k .$$

It is easy to see that $\left\{t^m e^{\lambda_j t} : m = 0,1, \ldots, r_j - 1\right\}$ is a basis for W_j, j = 1,. . . ,k.

Hence,

$$\left\{t^m e^{\lambda_j t} : m = 0,1, \ldots, r_j - 1; j = 1, \ldots, k\right\}$$

is a basis for V. Moreover, $r_1 + \ldots + r_k = n$, and the dimension of V is n.

Let N_j denote the restriction of the operator $D - \lambda_j I$ on W_j. Then N_j is nilpotent on W_j with index of nilpotence r_j. Since any function which satisfies the differential equation $(D - \lambda_j I)x = 0$ is the scalar multiple of $e^{\lambda_i t}$, the dimension of the null space of $D - \lambda_j I$ is 1, that is, the nullity of N_j is 1. Hence, corresponding to each eigenvalue λ_j there is only one elementary Jordan matrix block with size r_j. Thus, the Jordan canonical form for D on the space V is the direct sum of k elementary Jordan matrices, one for each eigenvalue λ_j with size r_j.

We next delve into some related issues.

DEFINITION 2 [5] A linear operator T on a vector space V is said to be *algebraic* if there is a polynomial p such that p(T) = 0 on V.

Hence D is algebraic on the space V of the solutions of the differential equation (*). However, D is not algebraic on $C_n[0,1]$. This is because for any polynomial g the space of the solutions of the differential equation g(D)x = 0 must be finite dimensional and clearly $C_n[0,1]$ is infinite dimensional. It follows that we cannot expect the Primary Decomposition would work on $C_n[0,1]$ for D.

We mention the fact that the n-dimensional V of the solutions of (*) is a closed subspace in $C_n[0,1]$. This is due to the fact that every finite dimensional subspace of a normed vector space is closed. Furthermore, V is nowhere dense in $C_n[0,1]$. Indeed, for every closed proper subspace X of a normed vector space Y, at each point $p \in X$ and for any $\varepsilon > 0$, the ε-open ball is $B(p,\varepsilon) = \{y \in Y : \|y - p\| < \varepsilon\}$. Let $q \in Y$ and $q \notin X$ and we a choose positive integer n satisfying $\frac{1}{n}\|q\| < \varepsilon$, then $p + \frac{1}{n}q \notin X$, however $\left\|(p + \frac{1}{n}q) - p\right\| = \frac{1}{n}\|q\| < \varepsilon,$ so $p + \frac{1}{n}q \in B(p,\varepsilon)$. This shows us that X contains no interior points, and X is nowhere dense in Y. So is V nowhere dense in $C_n[0,1]$.

The other question is whether the differential operator D is diagonalizable on the space V of the solutions of (*) . From linear algebra we know that a linear operator D is diagonalizable if and only if the minimal polynomial for D on V is the product of distinct linear polynomials , that is,

$$p(s) = (s - \lambda_1)(s - \lambda_2)\ldots(s - \lambda_k),$$

where $\lambda_1, \lambda_2, \ldots \lambda_k$ are distinct scalars. Therefore the diagonalizability of D on V depends on the polynomial p. Finally, we would like to ask the question: what is the advantage of diagonalizability of D on V ? We answer the question below.

DEFINITION 3 Let V be a finite dimensional vector space over the field F, and let T be the linear operator on V. We say that T is *semi-simple operator* if every T-invariant subspace has a complimentary T-invariant subspace.

It is known that if the minimal polynomial for T is irreducible over the scalar field F, then T is a semi-simple operator . Its converse is also true. Therefore , T is semi-simple if and only if the minimal polynomial p for T is of the form $p = p_1 \ldots p_k$, where $p_1, \ldots, p_k$ are distinct irreducible polynomials over the field F. It follows that if the scalar field F is algebraically closed, then T is a semi-simple if and only if T is diagonalizable.

Hence if the polynomial p is the product of distinct linear polynomials, then D is diagonalizable on the corresponding V, and hence D is also semi-simple on V. This may be the easiest way to handle D.

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