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Spectrally Arbitrary Tree Sign Pattern Matrices

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by

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ABSTRACT

A sign pattern (matrix) is a matrix whose entries are from the set $\{+, -, 0\}$. A sign pattern matrix A is a spectrally arbitrary pattern if for every monic real polynomial $p(x)$ of degree n there exists a real matrix B whose entries agree in sign with A such that the characteristic polynomial of B is $p(x)$. All 3×3 SAP's, as well as tree sign patterns with star graphs that are SAP's, have already been characterized. We investigate tridiagonal sign patterns of order 4. All irreducible tridiagonal SAP's are identified. Necessary and sufficient conditions for an irreducible tridiagonal pattern to be an SAP are found. Some new techniques, such as innovative applications of Gröbner bases for demonstrating that a sign pattern is not potentially nilpotent, are introduced. Some properties of sign patterns that allow every possible inertia are established.

Keywords: Sign pattern matrix, Spectrally arbitrary pattern (SAP), Inertially arbitrary pattern (IAP), Tree sign pattern (tsp), Potentially nilpotent pattern, Gröbner basis, Potentially stable pattern, Sign nonsingular, Sign singular

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1. Introduction and Preliminaries

In qualitative and combinatorial matrix theory, we study properties of a matrix based on combinatorial information, such as the signs of entries in the matrix. After P. Samuelson, the Nobel Prize winner of economics, pointed to the need to solve certain problems in economics [16] and other areas based only on the sign of entries of the matrices, the study of sign pattern matrices has become somewhat synonymous with qualitative matrix theory. The dissertation of C. Eschenbach [6], directed by C.R. Johnson, studied sign pattern matrices that require or allow certain properties and summarized the work on sign pattern up to that point. In 1995, Richard Brualdi and Bryan Shader produced a through treatment [1] on sign pattern matrices from the sign solvability vantage point. Since 1995, there has been a considerable number of papers on sign patterns and some generalized notions such as ray patterns. For a current survey with extensive bibliography, see Hall and Li [9]. A matrix whose entries are from the set $\{+, -, 0\}$ is called a *sign pattern matrix* (or sign pattern, pattern). We denote the set of all $n \times n$ sign pattern matrices by Q_n. For a real matrix B, $\text{sgn}(B)$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by $+$ (respectively, $-$, 0). For a sign pattern matrix A, the *sign pattern class of A* is defined by

$$Q(A) = \{ B : \text{sgn}(B) = A \}.$$

A *subpattern* of a sign pattern A is a sign pattern matrix obtained by replacing some (possibly none) of the $+$ or $-$ entries in A with 0. The *identity sign pattern* $I_n \in Q_n$ is the diagonal pattern of order n with $+$ diagonal entries.

A square sign pattern matrix P is called a *permutation pattern* if all the entries except exactly one $+$ entry in each row and column are 0. It is clear that an $n \times n$ permutation sign pattern P satisfies $P^T P = PP^T = I_n$ and thus P^T can be viewed as the "inverse sign pattern" of P. Two sign pattern matrices A_1 and A_2 are said to be *permutationally equivalent* if there exists permutation patterns P_1 and P_2 such that $A_2 = P_1 A_1 P_2$. Two sign patterns A_1 and A_2 are said to be *permutationally similar* if there exists a permutation pattern P such that $A_1 = P^T A_2 P$.

A diagonal sign pattern is said to be a *signature (sign) pattern* if all of its diagonal entries are nonzero. A signature sign pattern $S$of order n satisfies $S^2 = I_n$ and hence, S can be viewed as the "inverse sign pattern" of S. Two sign pattern matrices A_1 and A_2 are said to be *signature equivalent* if there exists signature patterns S_1 and S_2 such that $A_2 = S_1 A_1 S_2$, and more specifically, *signature similar* if there is a signature pattern S such that $A_2 = SA_1S$.

A sign pattern $A \in Q_n$ is said to be *sign nonsingular* (SNS) if every matrix $B \in Q(A)$ is nonsingular. Since the determinant is a continuous function of the entries of a matrix, this means that $\det(B)$ is positive (or negative) for all $B \in Q(A)$. It is well known that A is sign nonsingular if and only if $\det A = +$ or $\det A = -$, that is, in the standard expansion of $\det A$ into $n!$ terms, there is at least one nonzero term, and all the nonzero terms have the same sign. A is said to be *sign singular* if every matrix $B \in Q(A)$ is singular, or equivalently, if $\det A = 0$.

A square sign pattern is *combinatorially symmetric* if $a_{ij} \neq 0$ iff $a_{ji} \neq 0$. The graph of *combinatorially symmetric* $n \times n$ sign pattern matrix $A = [a_{ij}]$ is the graph with vertex set $\{1, 2, 3, \ldots, n\}$ where $\{i, j\}$ is an edge iff $a_{ij} \neq 0$.

A *tree sign pattern (tsp)* matrix is a combinatorially symmetric sign pattern matrix whose graph is a tree (possibly with loops).

If $A = [a_{ij}]$ is an $n \times n$ sign pattern matrix, then a simple k-cycle (*cycle of length* k) is a formal product of the form $\gamma = a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1}$, where each of the elements is nonzero and the index set $\{i_1, i_2, \ldots, i_k\}$ consists of distinct indices. A simple cycle is said to be positive or negative depending on whether the actual product of the entries is positive or negative.

If $\gamma = \gamma_1\gamma_2 \ldots \gamma_m$, where the index sets of the $\gamma_i's$ are mutually disjoint, then γ is said to be *composite cycle*. We also note that if A is an $n \times n$ sign pattern then each nonzero term in $\det(A)$ is a cycle of length n properly signed.

For a matrix B the set $\sigma(B)$ of all the eigenvalues of is known as spectrum of B.

The *inertia* of a square matrix B is the ordered triplet $i(B) = (i_+, i_-, i_0)$, where i_+, i_-, i_0 are the number of eigenvalues with positive, negative, and zero real parts, respectively. For a square sign pattern A, the inertia set is $\mathrm{i}(A) = \{\mathrm{i}(B) : B \in \mathrm{Q}(A)\}$. A matrix is called *stable* if all of its eigenvalues have negative real parts. A sign pattern A is called *sign stable* if every matrix in $Q(A)$ is stable, and it is said to be *potentially stable* if there is some stable matrix in $Q(A)$.

A $n \times n$ square pattern A is an *inertially arbitrary pattern* (IAP) if for any nonnegative integers $n_1,\ n_2,\ n_3$ such that $n_1 + n_2 + n_3 = n$ there exits a matrix $B \in Q(A)$ such that $i(B) = (n_1, n_2, n_3)$. An $n \times n$ square pattern A is a *spectrally arbitrary pattern* (SAP) if for every given real monic polynomial $g(x)$ of degree n, there exists a real matrix in $\mathrm{Q}(A)$ that has characteristic polynomial $g(x)$. Equivalently, A is an SAP iff, given any self conjugate set of n complex numbers, there exists a matrix $B \in Q(A)$ with that set as its spectrum.

Clearly, if A is an SAP, then A is an IAP. We give an example later to show that the converse is not true. A sign pattern A is said to be *potentially nilpotent* if it allows nilpotence, namely, if there is a matrix $B \in Q(A)$ such that $B^n = 0$. It is obvious that an SAP must be potentially nilpotent. However, it can be shown that not every IAP is potentially nilpotent. Potentially nilpotent matrices have been studied in several papers, see for example [7], [8] and [14].

An SAP A is said to be *minimal spectrally arbitrary pattern* (MSAP) if replacement of any nonzero entry by zero yields a pattern that is no longer an SAP. An IAP A is said to be *minimal inertially arbitrary pattern* (MIAP) if replacement of any nonzero entry by zero yields a pattern that is no longer an IAP .

Of course, not every potentially nilpotent pattern is even an IAP. $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is an example of a potentially nilpotent pattern that is not an IAP. Also, $\begin{bmatrix} - & 0 \\ + & - \end{bmatrix}$ is an example of a potentially stable pattern that is not an IAP.

In [2], it is shown that every spectrally arbitrary sign pattern of order n must have at least $2n-1$ nonzero entries and it is conjectured that every spectrally arbitrary sign pattern of order n must have $2n$ nonzero entries. This conjecture is known as the $2n$-conjecture.

Tree Sign Patterns (tsp), especially those whose graphs (excluding loops) are paths, are examined in [10]. Implicit function method to check whether a pattern is SAP or not is developed there. A class of spectrally arbritrary pattern is constructed in [12] with the help of Soules matrix. All potentially stable star sign patterns are discussed in [15]. The inertia of matrices having a symmetric star sign pattern are examined in [17]. Potentially nilpotent star sign patterns are considered in [14], in which explicite characterization for orders 2 and 3 is done, and a recursive characterization for patterns of general order n is proved.

2. Basic Properties

In this section we examine some basic properties of SAP's and IAP's.

Lemma 2.1. The classes of all $n \times n$ SAP's and IAP's are closed under negation, transposition, signature similarity, and permutation similarity.

Recall that for any $n \times n$ matrix B, the characterization polynomial of B is

$$p_B(t) = t^n - E_1(B)t^{n-1} + E_2(B)t^{n-2} - \cdots + (-1)^n E_n(B),$$

where $E_k(B)$ is the sum of all the $k \times k$ principal minors of B. Note that, $E_k(B)$ is also equal to the sum of all k-fold products of eigenvalues of B.

Theorem 2.2. Every SAP of order n allows a positive and a negative principal minor of order k for $k = 1, 2, \ldots, n$

Proof. Suppose A is an SAP. Then there exists $B \in Q(A)$ such that $p_B(t) = t^n + t^{n-k}$, so that $E_k(B) = 1$ or -1 depending on k is even or odd respectively. If k is even, $E_k(B) = 1$, there must be at least one positive principal minor of order k. For the same k we can consider matrix $C \in Q(A)$ such that $P_C(t) = t^n - t^{n-k}$, and achieve a negative principal minor of order k. ■

The converse of theorem 2.2 does not hold. In Chapter 4, we provide an example of a sign pattern that satisfies the minor conditions of the above theorem but it is not an IAP.

Proposition 2.3. If A is an $n \times n$ $(n \geq 2)$ IAP, then A has at least one positive diagonal entry and at least one negative diagonal entry.

Proof. Let A be an IAP. Then there exists $B \in Q(A)$ such that $i(B) = (0, 1, n-1)$. The one eigenvalue with negative real part is negative and sum of eigenvalues with zero real part is zero. So, $\mathrm{tr}(B)$ is negative. Hence, B has at least one negative diagonal entry so A must have one negative diagonal entry. Using $iB) = (1, 0, n-1)$ we get the result for positive diagonal entry. ■

Proposition 2.4. If A is an IAP, then A is not SNS, and also A has two oppositely signed cycles of length n.

Proof. Suppose A is an $n \times n$ IAP. Consider $B \in Q(A)$ with $i(B) = (n-1, 1, 0)$. Thus, $\det(B)$, which is the product of eigenvalues, is negative (because product of even number of nonzero complex numbers with their conjugates is positive). Also, if we take $C \in Q(A)$ with $i(C) = (n, 0, 0)$ then $\det(C)$ is positive. This shows that $Q(A)$ allows a positive and a negative determinant, and hence, A is not SNS. ■

Note that the above propositions confirms that every $n \times n$ IAP has a positive and negative principal minor of order 1 and of order n.

Proposition 2.5. If A is an IAP of order 3, then $Q(A)$ allows a positive and a negative principal minor of order k for all $k = 1, 2, 3$.

Proof. We know that in general for an IAP of order n the result is true for $k = 1$ and $k = n$. suppose $k = 2$. Now for any matrix B, $E_2(B) = \sum \lambda_i \lambda_j$. Take $B_1 \in Q(A)$ such that

$$i(B_1) = (3, 0, 0).$$

Then $\sigma(B_1) = \{a + ib, a - ib, c\}$ where $a, c > 0$ and

$$E_2(B_1) = \sum_{1 \leq i < j \leq 3} \lambda_i \lambda_j = a^2 + b^2 + 2ac > 0.$$

Since $E_2(B_1) > 0$, some 2×2 principal minor of B_1 is positive. Using $B_2 \in Q(A)$ with $i(B_2) = (1, 1, 1)$, we get $E_2(B_2) = \lambda_1 \lambda_3 < 0$ (with $\lambda_2 = 0$), and hence, some principal minor of A_2 is negative. ■

Proposition 2.6. If A is an IAP of order 4 then $Q(A)$ allows a positive and a negative principal minor of order k for all $k = 1, 2, 3, 4$.

Proof. We only need to prove the result for $k = 2$ and $k = 3$. Consider the inertia triple (4,0,0). Then either all the eigenvalues are positive, or the eigenvalues are in the form $a + ib, a - ib, c, d$ with $a, c, d > 0$, or $a + ib, a - ib, c + id, c - id$ with $a, c > 0$. In all the above cases, $E_2 = \sum_{1 \leq i < j \leq 4} \lambda_i \lambda_j > 0$ (by direct computation using the eigenvalues). Also, an IAP must have a positive diagonal entry a_{ii} and a negative diagonal entry a_{jj}. By choosing suitable values of those entries we can find a matrix $B \in Q(A)$ such that $E_2(B)$ is negative. Similarly the inertias $(3, 0, 1)$ and $(0, 3, 1)$ give $E_3(B)$ to be positive and negative, respectively. $\blacksquare$

Note that the proof for the negative principal minor of order 2 is independent of the order of the matrix.

Proposition 2.7. Up to equivalence,

$$T_2 = \begin{bmatrix} - & + \\ - & + \end{bmatrix}$$

is the only SAP, IAP, MSAP, MIAP.

Proof. For $B = \begin{bmatrix} -a & b \\ -c & d \end{bmatrix}$, $P_B(t) = t^2 - (d - a) + (bc - ad)$, which yields every monic polynomial of degree 2 with a, b, c, d varying over all positive numbers. So, T_2 is an SAP, and hence is an IAP. Now, it is easily seen that any 2×2 IAP can not have any zero entry. So, then T_2 is a MSAP as well as a MIAP. The uniqueness is straightforward and can be seen from the following discussions.

Out of all the 2×2 sign pattern having all 4 nonzero entries the following four are found to be equivalent to T_2

$$\begin{bmatrix} + & + \\ - & - \end{bmatrix}, \begin{bmatrix} - & - \\ + & + \end{bmatrix}, \begin{bmatrix} - & + \\ - & + \end{bmatrix}, \begin{bmatrix} + & - \\ + & - \end{bmatrix}.$$

Note that these pattern are MSAP's (MIAP's) because a pattern of order 2 requires 4 nonzero entries to be SAP (IAP). and the following are found not be IAP

$$\begin{bmatrix} - & * \\ * & - \end{bmatrix}, \begin{bmatrix} + & * \\ * & + \end{bmatrix}, \begin{bmatrix} + & + \\ + & - \end{bmatrix}, \begin{bmatrix} + & - \\ - & - \end{bmatrix}, \begin{bmatrix} - & + \\ + & + \end{bmatrix}, \begin{bmatrix} - & - \\ + & + \end{bmatrix}$$

■

Theorem 2.8. ([2]) Up to equivalence, every 3×3 SAP is equivalent to a superpattern of one of the following patterns

$$T_3 = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix}, U_3 = \begin{bmatrix} + & - & + \\ + & - & 0 \\ + & 0 & - \end{bmatrix}, V_3 = \begin{bmatrix} + & - & 0 \\ + & 0 & - \\ + & 0 & - \end{bmatrix}, W_3 = \begin{bmatrix} + & + & - \\ + & 0 & - \\ + & 0 & - \end{bmatrix}.$$

Note that the above mentioned patterns are MSAP's.

The complete characterization of 3×3 SAP's is done in [2] and [4]. The main results on 3×3 sign patterns in [2] and [4] can be summarized as the following theorem.

Theorem 2.9. For a 3×3 sign pattern A the following are equivalent.

(1). A is an SAP.

(2). A is an IAP.

(3). A is an irreducible potentially nilpotent pattern with a positive and a negative diagonal entry.

3. Tree Sign Patterns

In this section, we discuss tree sign patterns. From Section 2, it is clear that T_2 is the only 2×2 SAP which is a tsp, and it has 4 nonzero entries.

For a tsp A, since G(A) is a tree, G(A) has $n-1$ edges. So, A has $2(n-1)$ nonzero off-diagonal entries. In addition, if A is an IAP, we then have a positive and a negative diagonal entry, and hence, A has at least $2n$ nonzero entries.

Proposition 3.1. If A is tree sign pattern of order 3, then A is permutationally similar to a tridiagonal pattern.

Proof. Let A be a tree 3. Then $G(A)$ is a tree. Since the only tree on three vertices is a path, we may assume that $G(A)$ is the path from 1 to 3 of length 2. Hence it follows that:

$$A = \begin{bmatrix} * & a_{12} & 0 \\ a_{21} & * & a_{23} \\ 0 & a_{32} & * \end{bmatrix}.$$

■

By analyzing 3×3 tsp with 6 nonzero entries we can see that

$$\begin{bmatrix} - & + & 0 \\ + & 0 & + \\ 0 & - & + \end{bmatrix}, \quad \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & + & + \end{bmatrix}$$

are SNS, so not IAP's. Also the following pattern do not contain a positive 2-cycle, so are not IAP's

$$\begin{bmatrix} - & + & 0 \\ + & 0 & + \\ 0 & + & + \end{bmatrix}, \begin{bmatrix} - & - & 0 \\ - & 0 & - \\ 0 & - & + \end{bmatrix}.$$

In fact, as mentioned in [10], up to equivalence,

$$T_3 = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix}$$

is the only 3×3 tsp MSAP (MIAP). From [10], we also know that the only 3×3 tsp SAP's (IAP's) up to equivalence are

$$T_3 = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix}, \quad U = \begin{bmatrix} - & + & 0 \\ - & + & + \\ 0 & + & - \end{bmatrix}, \quad \text{and } \tilde{T_3} = \begin{bmatrix} - & + & 0 \\ - & + & + \\ 0 & - & + \end{bmatrix}.$$

Generalizing T_3, we have the $n \times n$ antipodal pattern.

Theorem 3.2. For every n, $2 \leq n \leq 16$,

$$T_n = \begin{bmatrix} - & + & 0 & \dots & \dots & 0 \\ - & 0 & + & \ddots & & \vdots \\ 0 & - & 0 & + & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & - & 0 & + \\ 0 & \dots & \dots & 0 & - & + \end{bmatrix}$$

is a SAP.

This theorem follows from [10] and [5].

Up to equivalence, a 4×4 tsp is either a star or a tri-diagonal pattern. From [10] it can be seen that the only 4×4 tsp SAP's (IAP's) with 8 nonzero entries are

$$T_4 = \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & + \end{bmatrix}, \text{ and } \; H = \begin{bmatrix} - & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & + \end{bmatrix}.$$

Note that these patterns are MIAP's.

It is also easy to see that $\begin{bmatrix} - & + & 0 & 0 \\ - & + & 0 & 0 \\ 0 & 0 & - & + \\ 0 & 0 & - & + \end{bmatrix}$ is an SAP. Obviously, it is not a tsp, since it is a reducible pattern

A more general question to consider is what about the 4×4 tsp's with more than 8 nonzero entries. Also what are other 4×4 SAP's which are not trees.

The following star patterns are proved to be SAP's [13].

$$Z_{41} = \begin{bmatrix} 0 & + & + & + \\ - & - & 0 & 0 \\ + & 0 & + & 0 \\ - & 0 & 0 & + \end{bmatrix}, Z_{42} = \begin{bmatrix} 0 & + & + & + \\ - & - & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & + \end{bmatrix}, Y_4 = \begin{bmatrix} + & + & + & + \\ - & - & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & - \end{bmatrix}.$$

Also being superpattern of Z_{41} and Z_{42} the following star patterns are SAP's

$$Z_{41}^{+} = \begin{bmatrix} + & + & + & + \\ - & - & 0 & 0 \\ + & 0 & + & 0 \\ - & 0 & 0 & + \end{bmatrix}, Z_{42}^{+} = \begin{bmatrix} + & + & + & + \\ - & - & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & + \end{bmatrix}.$$

Using same arguments the following patterns are also SAP's

$$Z_{41}^{-} = \begin{bmatrix} - & + & + & + \\ - & - & 0 & 0 \\ + & 0 & + & 0 \\ - & 0 & 0 & + \end{bmatrix}, Z_{42}^{-} = \begin{bmatrix} - & + & + & + \\ - & - & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & + \end{bmatrix}.$$

By analyzing all the potentially stable 4×4 tridiagonal tsp's given in [11], we get the following results.

(a) The following patterns are equivalent to T_4 or a superpattern of T_4, so they are SAP's (IAP's):

$$\begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}.$$

(b) The following patterns are equivalent to H or $\tilde{H}$ (a superpattern of H), so they are SAP's (IAP's):

$$\begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & - \end{bmatrix},$$

$$\begin{bmatrix} + & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}.$$

(c) The following patterns are not SAP's (IAP's) because they are SNS:

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & 0 \end{bmatrix}.$$

(d) The following patterns are not SAP's because they are not potentially nilpotent (more specifically, A^4 is not compatible with zero):

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix}.$$

(e) The following patterns are not SAP's (IAP's) because they do not have a positive and a negative diagonal entry

$$\begin{bmatrix} 0 & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & - \end{bmatrix},$$

$$\begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & - \end{bmatrix},$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} - & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ + & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & 0 \end{bmatrix}.$$

A very common and very well known method to show that a pattern is an SAP is the *Nilpotent-Jacobian method* ((N-J) method) which first appeared in [10].

The (N-J) method: To show that an irreducible pattern A is an SAP, let $B \in Q(A)$ with the absolute values of the nonzero entries denoted by positive parameters $b_1, b_2, \ldots, b_k$. If B has a nilpotent realization $\tilde{B}$ with $(b_1, b_2, \ldots, b_k) = (\tilde{b_1}, \tilde{b_2}, \ldots, \tilde{b_k})$ and there are n b_i's such that the Jacobian of the coefficients of $t^{n-1}, t^{n-2}, \ldots, t^0$ in the characteristic polynomial $\det(tI - B)$ with respect to those b_i's is nonzero at $(\tilde{b_1}, \tilde{b_2}, \ldots, \tilde{b_k})$

The (N-J) method was restated in [2]. The new version is as follows.

Theorem 3.3. Let A be an $n \times n$ sign pattern, and suppose that there exists some nilpotent $B \in Q(A)$ with at least n nonzero entries $(b_1, \ldots, b_n)$. Let X be a matrix obtained by replacing these entries in B by variables $(x_1, \ldots, x_n)$. let

$$P_B(t) = t^n + c_1 t^{n-1} + c_2 t^{n-2} + \ldots + c_{n-1} t + c_n.$$

If the Jacobian matrix J of $c_1, c_2, \ldots, c_n$ with respect to $x_1, \ldots, x_n$ is nonsingular at $(x_1, \ldots, x_n) = (b_1, \ldots, b_n)$, then A and every superpattern of A is an SAP.

Theorem 3.4. The following tridiagonal tsp's are SAP's.

$$\begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix},$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix},$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & + & - \end{bmatrix},$$

Proof. The proof follows the (N-J) method. We give below the sign pattern, its nilpotent realization, and the matrix used for computing the Jacobian (calculated using the above mentioned method) which turns out to be nonsingular.

$$\begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} 15 & 625/3 & 0 & 0 \\ -1 & -14 & 1/3 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \begin{bmatrix} x_1 & x_2 & 0 & 0 \\ -1 & -14 & x_3 & 0 \\ 0 & 1 & 0 & x_4 \\ 0 & 0 & -1 & -1 \end{bmatrix};$$

$$\begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & -2 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} x_1 & x_2 & 0 & 0 \\ -1 & 0 & x_3 & 0 \\ 0 & -1 & -2 & x_4 \\ 0 & 0 & -1 & 1 \end{bmatrix};$$

$$\begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} 3 & 1 & 0 & 0 \\ -1 & 0 & 55 & 0 \\ 0 & 1 & \frac{-186}{55} & \frac{194481}{3025} \\ 0 & 0 & -1 & \frac{21}{55} \end{bmatrix}, \begin{bmatrix} 3 & x_1 & 0 & 0 \\ -1 & 0 & x_2 & 0 \\ 0 & 1 & \frac{-186}{55} & x_3 \\ 0 & 0 & -1 & x_4 \end{bmatrix};$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} 0 & 1/2 & 0 & 0 \\ -1 & 1 & 1/2 & 0 \\ 0 & -1 & -2 & 2 \\ 0 & 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & x_1 & 0 & 0 \\ -1 & 1 & x_2 & 0 \\ 0 & -1 & -x_3 & x_4 \\ 0 & 0 & -1 & 1 \end{bmatrix};$$

$$\begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & 0 \end{bmatrix}, \begin{bmatrix} -6 & 48 & 0 & 0 \\ -1 & 8 & 1 & 0 \\ 0 & -1 & -2 & 3 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} -6 & x_1 & 0 & 0 \\ -1 & 8 & x_2 & 0 \\ 0 & -1 & -x_3 & x_4 \\ 0 & 0 & -1 & 0 \end{bmatrix};$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} 0 & 2 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & -1 & 2 \end{bmatrix}, \begin{bmatrix} 0 & x_1 & 0 & 0 \\ -1 & -1 & x_2 & 0 \\ 0 & 1 & -x_3 & x_4 \\ 0 & 0 & -1 & 2 \end{bmatrix};$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & 2 & 1 & 0 \\ 0 & -8 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & x_1 & 0 & 0 \\ 4 & 2 & x_2 & 0 \\ 0 & -8 & -x_3 & 1 \\ 0 & 0 & 1 & x_4 \end{bmatrix};$$

$$\begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 2 & 1 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \begin{bmatrix} 0 & x_1 & 0 & 0 \\ -1 & 0 & x_2 & 0 \\ 0 & -8 & -x_3 & x_4 \\ 0 & 0 & -1 & -1 \end{bmatrix};$$

$$\begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} -1 & 1/3 & 0 & 0 \\ -1 & 0 & 1/3 & 0 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & -1 & -1 \end{bmatrix}, \begin{bmatrix} -1 & x_1 & 0 & 0 \\ -1 & 0 & x_2 & 0 \\ 0 & 1 & -x_3 & x_4 \\ 0 & 0 & -1 & -1 \end{bmatrix};$$

$$\begin{bmatrix} - & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} -1 & 1 & 0 & 0 \\ 1/5 & -2 & 1 & 0 \\ 0 & -81/5 & 4 & 1 \\ 0 & 0 & 5 & -1 \end{bmatrix}, \begin{bmatrix} -x_1 & 1 & 0 & 0 \\ 1/5 & -2 & x_2 & 0 \\ 0 & -81/5 & 4 & x_3 \\ 0 & 0 & 5 & -x_4 \end{bmatrix}. \quad \blacksquare$$

Corollary 3.5. The following tridiagonal tsp's are SAP's.

$$\begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix},$$

$$\begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & + & + & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix},$$

$$\begin{bmatrix} - & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix}.$$

Proof. They are superpatterns of the above mentioned patterns. ■

Gröbner basis: The following discussion about Gröbner basis is based on [3]. Let R be a commutative ring. Consider a subset S of the multivariable polynomial ring $R[x_1, x_2, \ldots, x_n]$. A *zero* or a *solution* of S in R (or some super-ring of R) is an n-tuple $(r_1, r_2, \ldots, r_n) \in R^n$ with $P(r_1, r_2, \ldots, r_n) = 0$ for every polynomial $p \in S$. It can be seen that an n-tuple $(r_1, r_2, \ldots, r_n) \in R^n$ is a solution of S iff it is a solution of the ideal generated by S. The **Hilbert's Basis Theorem** states that every ideal of a polynomial ring over a field is finitely generated. From this point on R is the field of real numbers. Let M be a set of monomials in $R[x_1, x_2, \ldots, x_n]$. Suppose certain ordering of all the monomials is prescribed. Let $init(P)$, the initial monomial of a polynomial P, be the largest monomial appearing in P. If S be a subset of $R[x_1, x_2, \ldots, x_n]$ and $init(S)$ is the ideal generated by $\{init(s) : s \in S\}$. Let I be an ideal of $R[x_1, x_2, \ldots, x_n]$, then a finite subset $G = \{g_1, g_2, \ldots, g_k\}$ of I is called a *Gröbner basis* of I if $\{init(g_1), init(g_2), \ldots, init(g_k)\}$ generates $init(I)$. For any ideal I of $R[x_1, x_2, \ldots, x_n]$ the following are true.

(1) I has a Gröbner basis relative to any monomial ordering.

(2) Every Gröbner basis G of I generates I.

It can be seen that for every subset S of $R[x_1, x_2, \ldots, x_n]$ and a Gröbner basis G of the ideal generated by S, the solution set of S is the same as the solution set of G.

Lemma3.6. An $n \times n$ real matrix B is nilpotent iff $\text{tr}(B) = 0$, $\text{tr}(B^2) = 0$, $\text{tr}(B^3) = 0, \ldots, \text{tr}(B^n) = 0$.

Proof 3.6. Let $\sigma(B) = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$. It is well known that the trace of a matrix is equal to the sum of its eigenvalues. Thus, the necessity is obvious.

We now prove sufficiency. If all the eigenvalues are zero, then B is unitarily similar to a strictly upper triangular matrix, and hence B is nilpotent. Now assume that B has some nonzero eigenvalues. Let $\lambda_1, \lambda_2, \ldots, \lambda_k$ denote the distinct nonzero eigenvalues of B, with multiplicities $m_1, m_2, \ldots, m_k$. The system of equations $\mathrm{tr}(B^s) = 0, 1 \le s \le n$, can be written as:

$$\begin{aligned} m_1\lambda_1 + m_2\lambda_2 + \ldots + m_k\lambda_k &= 0 \\ m_1\lambda_1^2 + m_2\lambda_2^2 + \ldots + m_k\lambda_k^2 &= 0 \\ &\vdots \\ m_1\lambda_1^n + m_2\lambda_2^n + \ldots + m_k\lambda_k^n &= 0 \end{aligned}$$

Regarding $m_1, m_2, \ldots, m_k$ as the variables, the coefficient matrix F of the first k equations in the above system is

$$F = \begin{bmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_k^2 \\ \vdots & & & \vdots \\ \lambda_1^k & \lambda_2^k & \ldots & \lambda_k^k \end{bmatrix}.$$

Then,

$$F = \begin{bmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \ldots & \lambda_k^2 \\ \vdots & & & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \ldots & \lambda_k^{k-1} \end{bmatrix} \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k)$$

where the first factor is a Vandermonde matrix. Thus F is nonsingular since the k parameters $\lambda_1, \lambda_2, \ldots, \lambda_k$ are nonzero and distinct. Thus, the above system has only the trivial solution $m_1 = m_2 = \cdots = m_k = 0$. But by assumption, $m_1, m_2, \ldots, m_k$ are the multiplicities of distinct eigenvalues, and so can not be zero's. Hence each eigenvalue of B must be zero. That is, B is nilpotent. ■

Remark: The above result remains valid when the last condition $\mathrm{tr}(B^n) = 0$ is replaced by $\det(B) = 0$.

An SAP must be potentially nilpotent. Using Maple to compute a Gröbner basis of polynomials obtained using the necessary and sufficient conditions mentioned in the above remark for a matrix to be nilpotent, we can get the following results.

Theorem 3.7. The following patterns with 9 nonzero entries are not potentially nilpotent so they are not SAP's

$$\begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & 0 \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & 0 \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & + \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & - & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & + & - & + \\ 0 & 0 & - & + \end{bmatrix},$$

$$\begin{bmatrix} 0 & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ + & 0 & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix},$$

$$\begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & + & + \\ 0 & 0 & + & - \end{bmatrix}.$$

Proof. Let $B \in Q(A)$ where the absolute values of the nonzero entries of B are independent variables and A is one of the above mentioned patterns. A Gröbner basis of the system of equations consisting of $\mathrm{tr}(B) = 0$, $\mathrm{tr}(B^2) = 0$, $\mathrm{tr}(B^3) = 0$, $\mathrm{tr}(B^4) = 0$ contains an equation that does not have a positive solution. The following are the details of the above mentioned process. Let

$$A = \begin{bmatrix} - & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & 0 \end{bmatrix}.$$

By performing a suitable diagonal similarity if necessary, we may assume that a matrix $B \in Q(A)$ has the following form

$$B = \begin{bmatrix} -a & b & 0 & 0 \\ -1 & c & d & 0 \\ 0 & -1 & -e & f \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

where the variables can take on any positive values. Using $\mathrm{tr}(B) = 0$, $\mathrm{tr}(B^2) = 0$, $\mathrm{tr}(B^3) = 0$, $\mathrm{tr}(B^4) = 0$, we get the following system of polynomial equations.

$a + c - e = 0$ .... (1)

$a^2 - 2b + c^2 - 2d + e^2 + 2f = 0$ .... (2)

$-(a^2-b)a+ab-bc+(a-c)b+(-b+c^2-d)c-cd+de+(-c+e)d-(-d+e^2+f)e-2ef =$
0 .... (3)

$(a^2 - b)^2 + 2(-ab + bc)(a - c) + 2bd + (-b + c^2 - d)^2 + 2(cd - de)(-c + e) - 2df +$
$(-d + e^2 + f)^2 + 2e^2f + f^2 = 0$ .... (4)

The Gröbner basis with total degree ordering (a, b, c, d, e, f) for the system of above equations consists of the following polynomials, $a - c + e, c^2 - ce + e^2 - b - d + f, -df + e^2f + f^2, fcd - dfe + ef^2, e^3 + cd - 2de + 2ef,$
$bdf + cef^2, bdef + ef^2d - cf^3 - ef^3, bd^2f + d^2f^2 - 2df^3 + f^4.$

Note that the solutions of the original system of equations are the same as the zeros of the system of polynomials in a Gröbner basis. The equation $bdf + cef^2 = 0$ can not have a positive solution for a, b, c, d, e, f. Thus the original system can not have a solution where all the variables are positive. ■

Remark: The following are the matrix realization of the remaining above mentioned patterns and the monomial ordering of used for Gröbner basis computation with the help of Maple.

$$\begin{bmatrix} -a & b & 0 & 0 \\ -1 & c & d & 0 \\ 0 & 1 & -e & f \\ 0 & 0 & -1 & 0 \end{bmatrix}, \mathrm{tdeg}(a, b, c, d, e, f).$$

$$\begin{bmatrix} -a & b & 0 & 0 \\ 1 & c & d & 0 \\ 0 & -1 & -e & f \\ 0 & 0 & 1 & 0 \end{bmatrix}, \mathrm{tdeg}(f, b, c, d, e, a).$$

$$\begin{bmatrix} 0 & a & 0 & 0 \\ -1 & -b & c & 0 \\ 0 & -1 & -d & e \\ 0 & 0 & -1 & f \end{bmatrix}, \mathrm{tdeg}(f, b, c, d, e, a).$$

$$\begin{bmatrix} a & b & 0 & 0 \\ -1 & c & d & 0 \\ 0 & -1 & -e & f \\ 0 & 0 & -1 & 0 \end{bmatrix}, \mathrm{tdeg}(f, b, c, d, e, a).$$

$$\begin{bmatrix} 0 & a & 0 & 0 \\ -1 & b & c & 0 \\ 0 & 1 & -d & e \\ 0 & 0 & -1 & f \end{bmatrix}, \mathrm{tdge}(f, e, c, d, b, a).$$

$$\begin{bmatrix} 0 & a & 0 & 0 \\ -1 & b & c & 0 \\ 0 & -1 & -d & e \\ 0 & 0 & 1 & -f \end{bmatrix}, \mathrm{tdge}(f, b, c, d, e, a).$$

$$\begin{bmatrix} a & b & 0 & 0 \\ -1 & -c & d & 0 \\ 0 & 1 & 0 & e \\ 0 & 0 & 1 & -f \end{bmatrix}, \mathrm{tdge}(a, b, c, d, e, f).$$

$$\begin{bmatrix} -a & b & 0 & 0 \\ 1 & 0 & c & 0 \\ 0 & -1 & d & e \\ 0 & 0 & -1 & -f \end{bmatrix}, \mathrm{tdge}(f, b, c, d, e, a).$$

$$\begin{bmatrix} a & b & 0 & 0 \\ -b & 0 & c & 0 \\ 0 & -c & -d & 1 \\ 0 & 0 & 1 & -f \end{bmatrix}, \mathrm{lexdeg}([d], [a, b, c, f]).$$

$$\begin{bmatrix} -a & b & 0 & 0 \\ -b & 0 & c & 0 \\ 0 & -c & d & 1 \\ 0 & 0 & 1 & -f \end{bmatrix}, \mathrm{lexdeg}([d, a, c], [b, f]).$$

Theorem 3.8. The following patterns with 10 nonzero entries are not potentially nilpotent so they are not SAP's

$$\begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & - & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & + & + & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix},$$

$$\begin{bmatrix} - & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & + & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} + & + & 0 & 0 \\ - & - & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & - \end{bmatrix}$$

$$\begin{bmatrix} - & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix} \begin{bmatrix} + & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & - & - \end{bmatrix}, \begin{bmatrix} - & + & 0 & 0 \\ + & - & + & 0 \\ 0 & + & + & + \\ 0 & 0 & - & - \end{bmatrix}.$$

Proof. The following are the matrix realizations and the term orderings for the Gröbner basis for the above mentioned patterns.

$$\begin{bmatrix} a & b & 0 & 0 \\ -b & -c & d & 0 \\ 0 & d & -e & 1 \\ 0 & 0 & 1 & -f \end{bmatrix}, \mathrm{tdeg}(a, b, c, d, e, f).$$

$$\begin{bmatrix} a & b & 0 & 0 \\ -b & -c & d & 0 \\ 0 & d & e & 1 \\ 0 & 0 & 1 & -f \end{bmatrix}, \mathrm{lexdeg}([a], [b, c, d, e, f]).$$

$$\begin{bmatrix} a & b & 0 & 0 \\ -b & c & d & 0 \\ 0 & -d & -e & 1 \\ 0 & 0 & 1 & -f \end{bmatrix}, \mathrm{lexdeg}([e], [a, b, c, d, f]).$$

$$\begin{bmatrix} -a & b & 0 & 0 \\ -b & -c & d & 0 \\ 0 & -d & e & 1 \\ 0 & 0 & 1 & -f \end{bmatrix}, \mathrm{lexdeg}([d, e, c], [a, b, f]).$$

$$\begin{bmatrix} -a & b & 0 & 0 \\ b & -c & d & 0 \\ 0 & -d & e & 1 \\ 0 & 0 & -1 & -f \end{bmatrix}, \mathrm{lexdeg}([d, e, f], [a, b, c]).$$

$$\begin{bmatrix} a & b & 0 & 0 \\ -b & -c & d & 0 \\ 0 & -d & -e & 1 \\ 0 & 0 & 1 & -f \end{bmatrix}, \mathrm{lexdeg}([d,c,a],[b,e,f]).$$

$$\begin{bmatrix} -a & b & 0 & 0 \\ b & c & d & 0 \\ 0 & -d & e & 1 \\ 0 & 0 & -1 & -f \end{bmatrix}, \mathrm{lexdeg}([c,d,e],[a,b,f]).$$

$$\begin{bmatrix} a & b & 0 & 0 \\ b & -c & d & 0 \\ 0 & -d & e & 1 \\ 0 & 0 & -1 & -f \end{bmatrix}, \mathrm{lexdeg}([c,d,e],[a,b,f]).$$

$$\begin{bmatrix} -a & b & 0 & 0 \\ b & -c & d & 0 \\ 0 & d & e & 1 \\ 0 & 0 & -1 & -f \end{bmatrix}, \mathrm{lexdeg}([c,d,e],[a,b,f]).$$

Note: For some of the above mentioned patterns, applying suitable diagonal similarity and multiplying by suitable scalar matrix, the absolute values of symmetric entries is made same and the absolute values of (3, 4) and (4,3) entries are made 1.

With the help of above discussions and theorems on tridiagonal patterns we conclude with the following theorem which provides necessary and sufficient conditions for a 4×4 irreducible tridiagonal pattern to be an SAP.

Theorem 3.8. A 4×4 irreducible tridiagonal pattern is an SAP iff it is potentially stable and potentially nilpotent with at least one nonzero diagonal entry.

4. Some Observations

For 3×3 patterns it has been proved that every IAP is an SAP, But it is not true in general. It has been shown [4] that the sign pattern

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & - & - \\ + & + & 0 & 0 \\ 0 & 0 & - & - \end{bmatrix}$$

is an IAP but it is not an SAP. It has been proved [2] that, a 3×3 irreducible sign pattern (with at least one nonzero diagonal entry) is SAP if and only if it is potentially nilpotent, but this is not true in general. The pattern

$$\begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & - & 0 & + \\ - & 0 & 0 & - \end{bmatrix}$$

is potentially nilpotent but not an SAP. However, the above mentioned patterns are not trees. In case of 4×4 tsp's, if it is a star, it has been proved that a potentially nilpotent star pattern (with at least one nonzero diagonal entry) which is a potentially stable is an SAP. We have demonstrated that this conclusion is true for tridiagonal patterns too.

As stated earlier, the minor conditions are not sufficient conditions for SAP. Consider the pattern

$$A = \begin{bmatrix} + & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & + & 0 & + \\ 0 & 0 & + & - \end{bmatrix}.$$

We may assume that $B \in Q(A)$ has the following form

$$B = \begin{bmatrix} a & b & 0 & 0 \\ -1 & 0 & d & 0 \\ 0 & 1 & 0 & e \\ 0 & 0 & 1 & -f \end{bmatrix}$$

for some positive real numbers a, b, d, e, f. The characteristic polynomial of B is

$$x^4 + (f - a)x^3 + (-e - af + b - d)x^2 + (ae - df + bf + ad)x - be + adf.$$

If $(0, 0, 4) \in \mathrm{i}(B)$, then the coefficient of x^3 and the coefficient of x must be simultaneously zero. Which is not possible, because $f = a$ implies $ae - df + bf + ad = bf + ae > 0$. Thus A is not an IAP. But, A satisfies the minor conditions.

As proved in the theorem 3.4 the pattern

$$\begin{bmatrix} - & + & 0 & 0 \\ + & - & + & 0 \\ 0 & - & + & + \\ 0 & 0 & + & - \end{bmatrix}$$

is an SAP so that its negative must be potentially stable. But, the pattern

$$\begin{bmatrix} + & + & 0 & 0 \\ + & + & + & 0 \\ 0 & - & - & + \\ 0 & 0 & + & + \end{bmatrix}$$

is not mentioned to be potentially stable in [11].

For general n the following patterns are proved to be SAP ([2] and [4])

$$V_n = \begin{bmatrix} + & - & 0 & \dots & 0 & 0 \\ + & 0 & - & 0 & \vdots & 0 \\ + & 0 & 0 & \ddots & 0 & \vdots \\ \vdots & \vdots & \vdots & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & 0 & 0 & - \\ + & 0 & \dots & 0 & 0 & - \end{bmatrix} (n \geq 3)$$

and

$$D_{n,r} = \begin{bmatrix} - & + & 0 & 0 & \dots & \dots & \dots & 0 \\ - & 0 & + & 0 & & & & \vdots \\ - & 0 & 0 & + & \ddots & & & \vdots \\ \vdots & \vdots & & \ddots & \ddots & \ddots & & 0 \\ - & 0 & & & \ddots & \ddots & 0 & 0 \\ 0 & - & 0 & & & & + & 0 \\ \vdots & \ddots & \ddots & \ddots & & & 0 & + \\ 0 & \dots & 0 & - & 0 & \dots & 0 & + \end{bmatrix} (n \leq 2r),$$

where r negative entries in the first column.

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