The Teaching and Learning of Parametric Functions: A Baseline Study

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THE TEACHING AND LEARNING OF PARAMETRIC FUNCTIONS: 
A BASELINE STUDY

by

HARRISON E. STALVEY

Under the Direction of Draga Vidakovic

ABSTRACT
This dissertation reports on an investigation of fifteen second-semester calculus students’ understanding of the concept of parametric function, as a special relation from a subset of \( \mathbb{R} \) to a subset of \( \mathbb{R}^2 \). A substantial amount of research has revealed that the concept of function, in general, is very difficult for students to understand. Furthermore, several studies have investigated students’ understanding of various types of functions. However, very little is known about how students reason about parametric functions. This study aims to fill
this gap in the literature. Employing Action–Process–Object–Schema (APOS) theory as the
guiding theoretical perspective, this study proposes a preliminary genetic decomposition for
how a student might construct the concept of parametric function. To determine whether
the students in this study made the constructions called for by the preliminary genetic de-
composition or other constructions not considered in the preliminary genetic decomposition,
data is analyzed regarding students’ reasoning about parametric functions. In particular,
this study explores (1) students’ personal definitions of parametric function; (2) students’
reasoning about parametric functions given in the form \( p(t) = (f(t), g(t)) \); (3) students’
reasoning about parametric functions on a variety of tasks, such as converting from para-
metric to standard form, sketching a plane curve defined parametrically, and constructing
a parametric function to describe a real-world situation; and (4) students’ reasoning about
the invariant relationship between two quantities varying simultaneously when described in
both a graph and a real-world problem. Then the genetic decomposition is revised based
on the results of the data analysis. This study concludes with implications for teaching the
concept of parametric function and suggestions for further research on this topic.

INDEX WORDS: Parametric function, APOS, Student understanding, Student reasoning,
Sketching parametric curve, Conversion from parametric to standard,
Invariance of ratio
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In memory of

Mrs. Jean Tarpley
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Chapter 1

INTRODUCTION

The purpose of this study is to investigate students’ understanding of parametric functions.\(^1\) Parametric functions are an indispensable topic in mathematics and science. They are directly related to vector-valued functions, useful in modeling trajectories of moving objects, necessary for optimizing multivariable functions, and used explicitly or implicitly in numerous computer applications. Parametric functions are particularly important for describing dynamical systems through the analysis of the “complicated interplay and changes that occur among many different variables as time passes” (Allen, 2006, p. 1). In general, parametric functions enable us to take a problem in several dimensions and reduce it to several problems in one dimension (Dubinsky, Schwingendorf, & Mathews, 1995, p. 567), making it easier to apply analytical techniques to study that problem. Despite the importance of parametric functions in mathematics and science, few studies have investigated students’ understanding of the concept.

The main issue regarding students’ understanding of parametric functions (and of the function concept in general) that is of interest to this study is conceptual growth. Schwingendorf, Hawks, and Beineke (1992) distinguish between two types of conceptual growth of the function concept—horizontal (growth in breadth) and vertical (growth in depth of formal understanding). Dreyfus and Eisenberg (1982) connect the ideas of horizontal and vertical growth of the function concept to the issue of transfer of learning. They describe horizontal transfer as applying function concepts across different representations, while vertical transfer is generalizing function concepts to apply to different types of functions. Similarly, Harel and Kaput (1991) state that the construction of conceptual entities lies in the realm of vertical growth, while the translation of mathematical ideas to different representational systems.

\(^1\)In this study, **parametric function** refers to a function from a subset of \(\mathbb{R}\) to a subset of \(\mathbb{R}^2\), based on a characterization by Dubinsky, Schwingendorf, and Mathews (1995).
and the application of mathematics to model real-world situations are examples of horizontal growth.

Horizontal growth of the function concept has been investigated in the context of equations and Cartesian graphs of real-valued functions of one variable (Schwingendorf et al., 1992) and polar graphs of real-valued functions (Montiel, Vidakovic, & Kabaël, 2008; Montiel, Wilhelmi, Vidakovic, & Elstak, 2009; Moore, Paoletti, & Musgrave, 2013). Vertical growth has been investigated in the context of sequences (Mamona, 1990; McDonald, Mathews, & Strobel, 2000) and multivariable functions (Martínez-Planell & Gaisman, 2012; Trigueros & Martínez-Planell, 2010; Yerushalmy, 1997). Parametric functions are a particularly interesting topic regarding conceptual growth due to their connections to curves in $\mathbb{R}^2$ (horizontal growth) and their status as functions in their own right with a range made up of ordered pairs (vertical growth). Therefore, this study was designed with conceptual growth as its underlying theme with the intention to contribute to the vast amount of literature on students’ understanding of the function concept.

In the remaining pages of this chapter, I illustrate why a study is needed regarding students’ conceptions of parametric functions; refine the goals of the present study; and establish the theoretical perspective guiding the study.

1.1 Statement of the problem

The concept of function is one of the most fundamental concepts in mathematics. Despite its emphasis in secondary mathematics curriculum, researchers have reported that undergraduate students continually demonstrate an impoverished understanding of the function concept (Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Carlson, 1998; Oehrtman, Carlson, & Thompson, 2008; Thompson, 1994; Vinner & Dreyfus, 1989). As a result, there have been calls for “instructional shifts that promote rich conceptions and powerful reasoning abilities” (Oehrtman et al., 2008, p. 27). To serve as a backdrop for developing such rich conceptions, Oehrtman et al. (2008) recommend that students should experience diverse
function types\textsuperscript{2} with an emphasis on multiple representations, including different coordinate systems (p. 29). Moreover, in the *Principles and Standards for School Mathematics*, the National Council of Teachers of Mathematics has called for curriculum to incorporate parametric equations as representations of functions and relations (NCTM, 2000, p. 296). In order to effectively respond to these instructional calls, it is crucial to consider students’ conceptions (and misconceptions) of parametric functions. A review of existing literature reveals that minimal research has been done in this area.

Although literature that addresses students’ conceptual development of parametric functions is sparse, a substantial amount of research has contributed to our knowledge of how students reason about functions in general. Breidenbach et al. (1992) assert that it is important for students to develop a conception of function as a process that transforms in some imaginable way an input into an output. A process conception has been shown to foster a dynamic perception of covariation in which one quantity changes in response to changes in another quantity (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Johnson, 2012b; Oehrtman et al., 2008; Thompson, 1994). On the other hand, as acknowledged by Johnson (2012a), a student may also reason about covariation without requiring that one quantity change in response to the other. This perspective is aligned with that of Confrey and Smith (1991, 1994, 1995), who describe covariation as the coordination of two sequences that are generated independently (1995, p. 67). Johnson (2013) calls this *simultaneous-independent reasoning* and describes it as “comparisons as if each quantity were changing independently of the other in relation to time” (p. 707).

Oehrtman et al. (2008) describe something similar to simultaneous-independent reasoning and assert that it can be applied to the study of parametric functions. This assertion is supported by the results of a study by Bishop and John (2008) who found that covariation can support high school students’ ability to reason about parametric functions. With the exception of the study by Bishop and John, much of existing research concentrates on stu-

\textsuperscript{2}I interpret *type* to encompass more than the word *class*. That is, instead of referring to just polynomials, exponentials, etc., I interpret *type* to also refer to, for example, sequences, parametric functions, multivariable functions.
dents using simultaneous-independent reasoning (which amounts to parameterization as a cognitive tool) to graph the relationship between two explicit variables (e.g., Johnson, 2012a; Saldanha & Thompson, 1998). That is, the parameter (time or otherwise) was a variable that was implicitly involved in the situation and was absent from the final representation of the situation. However, a formal understanding of parametric functions not only requires viewing the parameter, say \( t \), as an explicit independent variable on which multiple dependent variables \( x \) and \( y \) depend, it also requires viewing those dependent variables as a single entity \((x, y)\). This assertion is aligned with that of Thompson (2011), who says, “I intend the pair \((x, y)\) to represent conceiving of a multiplicative object—an object that is produced by uniting in mind two or more quantities simultaneously” (p. 47).

Conceiving of two quantities as a single object is a significant conceptual shift. Moreover, making sense of the possibility that two quantities can together form a single output of a function is an example of vertical growth (generalization) of the function concept. The task of generalizing the function concept has been acknowledged to be nontrivial for most students (Martínez-Planell & Gaisman, 2012; Trigueros & Martínez-Planell, 2010; Yerushalmy, 1997). Referring to students’ failure to view sequences as functions, Mamona (1990) says, “…the subtlety and scope of the notion of function is often underestimated […] , mostly because the ‘well-behaved’ and concrete nature of all the functions the students are likely to have met so far” (p. 337). With this observation in mind, it is likely that the difficulty of vertical growth is a potential factor inhibiting students’ development of formal conceptions of parametric functions.

In addition to issues related to the learning of parametric functions, there are matters of curriculum that must be addressed. Researchers have expressed their concern that there is a lack of coherence in how parametric functions are emphasized in mathematics courses. For example, in comparing the uses of parametric equations in calculus and differential equations, Czocher, Tague, and Baker (2013) state that “in [calculus] the main emphasis is on translation between coordinate systems, while in [differential equations] there is a need to be able to represent and work with functions and relationships within a chosen coordinate
system” (p. 682). In other words, in calculus, the focus is more on teaching parametric equations as analytic representations of graphs of curves than on the parametric function concept itself. That is not to say, however, that no calculus textbooks approach parametric topics from a functional perspective. Dubinsky et al. (1995) actively engage the student in graphing to explore the process of $x$ and $y$ varying simultaneously with respect to $t$ before ever introducing the idea of switching an analytic representation from parametric to standard rectangular form. Calculus textbooks intended for scientists and engineers, such as Briggs, Cochran, and Gillett (2013), emphasize a modeling approach to parametric functions. I hypothesize that a combination of these approaches with an emphasis on the function concept would be more effective than only guiding students through the mechanics of switching between representations. As asserted by Thompson (1994), we should focus on graphs, expressions, and tables as “representations of something that, from the students’ perspective, is representable, such as aspects of a specific situation” (p. 39). Then we should guide students to make connections between representations to produce in their minds a subjective sense of invariance.

Ultimately, the goal of this study is to investigate and document students’ conceptual development of the notion of parametric function. The results of this investigation are intended to broaden our knowledge of students’ conceptual growth in the context of function and to inform instructional strategies for teaching parametric functions. As a consequence, instructors and curriculum developers will hopefully be in a better position to aid future students in developing rich parametric function conceptions that can be flexibly applied in and beyond the calculus sequence.

1.2 Research questions

The following research questions and subquestions guide this study:

1. What are students’ personal definitions of parametric function?
2. How do students reason about parametric functions given in the form $p(t) = (f(t), g(t))$?

   (i) What are students’ misconceptions when reasoning about parametric functions given in the form $p(t) = (f(t), g(t))$?

   (ii) Is their reasoning related to their reasoning about real-valued functions given in the form $y = f(x)$?

3. How do students reason about parametric functions when:

   (i) Sketching a graph of a curve given parametrically;

   (ii) Converting from parametric form to standard form of a curve;

   (iii) Constructing a parametric function to describe a real-world situation.

4. How do students reason about the invariant relationship between two variables?

   (i) Are students able to perceive the invariant relationship between two quantities varying simultaneously with respect to a third quantity when described in a graphical representation?

   (ii) Are students able to perceive the invariant relationship between two quantities varying simultaneously with respect to a third quantity when described in a real-world problem?

1.3 Theoretical perspective

Theories of learning play a vital role in education research. According to Dubinsky and McDonald (2001),

---

3Here $f(t)$ and $g(t)$ refer to $x$ and $y$, respectively. Despite the common convention of using $x(t)$ and $y(t)$ to refer to a parametric function, these notations are not well-defined. On one hand, the notation $x(t)$, for example, denotes the value of the function $x$ at $t$, while on the other hand, $x$ denotes the horizontal coordinate. Using one symbol for two different things causes difficulty. Therefore, I reserve $x$ and $y$ to refer to the coordinates and use $f$ and $g$ as the functions which define those coordinates. I thank Ed Dubinsky for his valuable comments regarding the ambiguity of the notations $x(t)$ and $y(t)$. 

---
a theory of learning mathematics can help us understand the learning process by providing explanations of phenomena that we can observe in students who are trying to construct their understandings of mathematical concepts and by suggesting directions for pedagogy that can help in this learning process. (Dubinsky & McDonald, 2001, p. 275)

Before a researcher utilizes a theory of learning, he or she must establish an epistemological viewpoint—a philosophical belief about the nature of knowledge. The epistemological stance to which I subscribe is constructivism. There is a variety of trends within the constructivist epistemology, and an attempt to contrast them all can be quite disorienting, but one tenet that they all seem to share is that knowledge is in the mind of an individual. The radical constructivism of Ernst von Glasersfeld assumes that “knowledge, no matter how it be defined, is in the heads of persons, and that the thinking subject has no alternative but to construct what he or she knows on the basis of his or her own experience” (Glasersfeld, 1995, p. 1). Annie and John Selden describe the constructivism of mathematics education research by explaining, “what is constructed is some kind of (personal) knowledge, i.e., a structure in an individual mind which may not even be fully describable in words and which might, or might not, arise from discovery of mathematics” (Selden & Selden, 1998, s. 2).

The views of constructivism do not deny that society plays a role in our construction of knowledge. According to Selden and Selden (1998), social constructivism is a philosophy of science in which scientists are viewed as socially constructing knowledge of the external world. In other words, the mathematical community—often over the course of many decades or centuries—constructs concepts and establishes practices that collectively constitute institutional knowledge, as opposed to personal knowledge. I subscribe to the view that it should be a goal of mathematics education research to provide models of how an individual might construct his or her personal knowledge.

One such model, which serves as the specific theoretical perspective guiding this study, is APOS theory. APOS is a theory of cognition based on Jean Piaget’s theory of reflective abstraction and is used to describe the development of mathematical concepts in the minds

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4 Institutional and personal knowledge in the sense of the onto-semiotic approach (Godino, Batanero, & Font, 2007).
of students, particularly at the high school and collegiate level. The philosophy of APOS theory is based on the following viewpoint of mathematical knowledge:

An individual’s mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with the situations. (Asiala et al., 1996, p. 7)

The words actions, processes, objects, and schema refer to mental constructions involved in mathematical understanding, and these constructions are the motivation behind the acronym, APOS.

APOS theory was developed by Ed Dubinsky and colleagues over the course of many refinements (e.g., Breidenbach et al., 1992; Dubinsky, 1986, 1991; Cottrill et al., 1996). Asiala et al. (1996) provide a complete description of each of the APOS constructions, and Arnon et al. (2014) present a comprehensive review of APOS theory as it exists at the present time. The following paragraph outlines the mental constructions of APOS theory, descriptions of which can be found in any of the aforementioned literature.

An action is a transformation of objects by reacting to external cues that give precise details on what steps to take. When an action is repeated, and the individual reflects on it, the action can be interiorized into a process. An individual who has a process conception can reflect on or describe the steps of the transformation without actually performing those steps. Additionally, new processes can be constructed by the means of reversal of a process or the coordination of two or more processes. When an individual becomes aware of the process as a totality and can perform additional actions or processes on it, then the process has been encapsulated into an object. Objects can be de-encapsulated to obtain the processes from which they came, which is often important in mathematics. The individual’s collection of actions, processes, and objects organized in a structured manner is his or her schema. When an individual learns to apply an existing schema to a wider collection of phenomena, then we can say that the schema has been generalized.
constructions of actions, processes, objects, and schemas, as well as the mental mechanisms\textsuperscript{5} by which those constructions are possibly achieved, is called a \textit{genetic decomposition}.

As acknowledged by Arnon et al. (2014), the mental mechanism of generalization is closely related to Piaget’s notions of assimilation and accommodation. Since generalization is important for vertical conceptual growth, the notions of assimilation and accommodation also guide this study. According to Arnon et al. (2014), assimilation occurs when an individual successfully uses his or her existing schema to make sense of new phenomena. This is not always possible—often there are conflicts or obstacles between an individual’s knowledge and the situation he or she is attempting to make sense of. These conflicts put an individual’s schema in disequilibrium. In order for a schema’s equilibrium to be restored, accommodation must occur. That is, the individual must modify the existing schema before assimilating the new phenomena. Once a phenomenon has been assimilated, the schema reaches a new equilibrium. Accommodation is very difficult because, according to Skemp (1987), schemas are self-perpetuating, especially if they have been of great value to the individual in the past. Therefore, what does not fit into an individual’s schema is frequently not learned or is learned and quickly forgotten, potentially having a negative effect on future learning.

1.4 Outline of the study

In the remaining chapters, this study will attempt to answer the research questions (as stated in Section 1.2) regarding the learning of the parametric function concept. Chapter 2 extends the discussion of the research explicated in this chapter by presenting a more comprehensive review of the broad body of literature informing the present study. This literature includes prior research conducted on students’ reasoning about functions, parametric functions, and variables. Chapter 3 presents a conceptual analysis (Thompson, 2008) of the notion of parametric function. This analysis considers the presentation of the concept in textbooks, the historical development of the concept, and my own understanding. Based on this conceptual analysis, as well as previous literature, I propose a preliminary genetic decomposition for

\textsuperscript{5}E.g., interiorization, reversal, coordination, encapsulation, de-encapsulation, and generalization.
how an individual might construct the concept of parametric function. Moreover, Chapter 3 presents a description of the methodology and methods for data collection and data analysis. Chapter 4 reports on the results of the data analysis in order to determine if the students in the study made the constructions called for by the preliminary genetic decomposition. Finally, Chapter 5 provides a synthesis of the entire study in light of the research questions and the findings of other researchers. Then Chapter 5 presents revision of the genetic decomposition and offers recommendations for teaching parametric functions. The study is concluded with suggestions for future research on the topic of teaching and learning parametric functions.
Chapter 2

LITERATURE REVIEW

In the previous chapter, I provided a brief synthesis of relevant literature to serve as an introduction to and support for this study. The purpose of this chapter is to provide a comprehensive literature review that situates the present study in a larger body of existing research. In the following sections, I review mathematics education literature related to the following topics: (1) the general concept of function; (2) covariational reasoning; (3) vertical and horizontal growth of the function concept; and (4) parametric functions.

2.1 The general concept of function

The notion of function is a fundamental concept in all areas of mathematics. As traced by Kleiner (1989), the term function was first introduced by Leibniz in 1692 to refer to a geometric object; Bernoulli and Euler in the 1700s gave a function definition in terms of variables and constants; and one of the first modern definitions of the function can be attributed to Dirichlet, who in the early 1800s gave the following definition:

\[ y \text{ is a function of a variable } x, \text{ defined on the interval } a < x < b, \text{ if to every value of the variable } x \text{ in this interval there corresponds a definite value of the variable } y. \text{ Also, it is irrelevant in what way this correspondence is established.} \] (Kleiner, 1989, p. 291)

Dirichlet’s definition of function as an arbitrary correspondence between real numbers emerged in the 19th century in the midst of efforts to make calculus more rigorous. In the latter part of the century, perhaps as a result of developments in set theory and topology, functions as arbitrary correspondences were extended to abstract sets, giving rise to the notions of domain and range (Malik, 1980, p. 491). In 1939, Bourbaki gave the following definition:
Let $E$ and $F$ be two sets, which may or may not be distinct. A relation between a variable element $x$ of $E$ and a variable element $y$ of $F$ is called a **functional relation** in $y$ if, for all $x \in E$, there exists a unique $y \in F$ which is in the given relation with $x$.

We give the name of **function** to the operation which in this way associates with every element $x \in E$ the element $y \in F$ which is in the given relation with $x$; $y$ is said to be the **value** of the function at the element $x$, and the function is said to be **determined** by the given functional relation. Two equivalent functional relations determine the **same** function. (Kleiner, 1989, p. 299)

In a later chapter of the same text, Bourbaki observed that a function is a subset of the Cartesian product $E \times F$, i.e., a function is a set of ordered pairs (Malik, 1980, p. 491).

Cooney and Wilson (1993) provide a historical account of the implementation of the function concept in mathematics curriculum. At the turn of the 20th century there were recommendations that functions receive an increased emphasis in school mathematics. Response to these recommendations was mostly uneven until the 1960s when mathematics education underwent a reform\footnote{The *New Math* movement.} that called for the introduction of more formal mathematical ideas, such as set, in secondary school mathematics. Textbooks written for this movement defined functions as sets of ordered pairs and expressed functions using set notation. This more abstract and static definition was in contrast to earlier textbooks’ definitions which emphasized functions as correspondences or dependence relations. Many advocated the ordered-pair approach as mathematically precise, while others criticized it as pedagogically unsound. In 1967, Willoughby (as cited in Cooney & Wilson, 1993, p. 139) stated:

> The ordered-pair definition of function is correct and convenient to use; however, it has serious defects from a pedagogical point of view. The ordered-pair idea gives a static impression to the pupil, where a dynamic impression is far more appropriate. Even though it may not be as elegant, or as formally simple, a dynamic impression of a function will be far more appealing to children, and will put them in a much better position to use their knowledge about functions.

Following this period, the approach to teaching functions from a multiple representation perspective arose in hopes that it would facilitate the transfer of learning (Eisenberg, 1991, p. 141). Based on my own brief review of a few college mathematics textbooks, it
appears that today the ordered-pair definition of function (using set notation) is mostly re-
served for courses taken after calculus, while the correspondence and multiple representation
approaches receive priority before and during the calculus sequence.

The textbook *Calculus One and Several Variables* (Salas, Hille, & Etgen, 2007) defines
function as follows:

> Let’s suppose $D$ is some set of real numbers and that $f$ is a function defined on
> $D$. Then $f$ assigns a unique number $f(x)$ to each number $x$ in $D$. (p. 25)

Similarly, *Calculus Single and Multivariable* (Hughes-Hallett et al., 2009) gives the following
definition:

> A function is a rule that takes certain numbers as inputs and assigns to each a
definite output number. (p. 2)

Moreover, the *Common Core State Standards for Mathematics* (Governors Association Cen-
ter for Best Practices & Council of Chief State School Officers, 2010) state that by the end
of the 8th grade students are expected to:

> Understand that a function is a rule that assigns to each input exactly one output.
> (p. 55)

The textbook *A Transition to Advanced Mathematics* (Smith, Eggen, & St. Andre, 2010)
provides two definitions of function—one in the preface and another in a later chapter on
functions. The first definition defines function as a correspondence:

> A function (or a mapping) is a rule of correspondence that associates to each
element in a set $A$ a unique element in a second set $B$ [...] The elements of $A$ are
sometimes called the arguments or inputs of the function. (Smith et al. 2010,
pp. xvi–xvii)

Meanwhile, the second definition by Smith et al. (2010, p. 186) emphasizes that a function
is a set of ordered pairs:

> A function (or mapping) from $A$ to $B$ is a relation $f$ from $A$ to $B$ such that
> (i) the domain of $f$ is $A$, and
(ii) if \((x, y) \in f\) and \((x, z) \in f\), then \(y = z\).

The concept of function—whether it is defined as a correspondence or as a set of ordered pairs—is a prominent topic in college mathematics courses. Despite its strong emphasis in high school, function remains to be a difficult concept for even our best college students (Breidenbach et al., 1992; Carlson, 1998). Over three decades of research have contributed to our knowledge of students’ understanding of the general concept of function. Remainder of this section reports on this research.

For several decades, researchers have theorized about how advanced mathematical concepts might develop in the mind of an individual. David Tall and Shlomo Vinner (Tall & Vinner, 1981; Vinner, 1983) proposed the duality of concept image and concept definition. A concept definition is made up of the words that are used to designate a mathematical object. The definition may be formal, meaning it is accepted by the mathematical community, or it may be a personal construction or reconstruction by an individual in response to his or her interpretation of the formal definition or of examples intended to illustrate the formal definition. In contrast to a concept definition, which falls in the realm of linguistics, an individual’s concept image is a cognitive structure that encompasses all of the ways in which he or she thinks about the concept, such as representations, examples, and relations to other concepts. The elements of an individual’s concept image may be limited or may even lack mathematical correctness.

A pair of studies sought to identify the concept definitions and images of the function commonly held by high school students (Vinner, 1983), as well as college students and junior high school teachers (Vinner & Dreyfus, 1989). These studies found that students commonly think of functions as formulas, correspondences defined by a rule, or graphs. Moreover, the studies showed that students’ images were limited to functions that behaved nicely. That is, functions that were discontinuous or piecewise, for example, were not thought to be functions. Perhaps the most interesting conclusion of these studies is the fact that many of the students who gave the formal Dirichlet-Bourbaki definition of function did not use this definition in their mathematical discourse, which often led to inconsistent behavior and errors. This is an
indication that the ability to state a concept’s definition (especially an abstract definition) is not a sufficient condition for developing rich conceptual understanding.

Breidenbach et al. (1992), Dubinsky (1991), and Dubinsky and Harel (1992) present a theory in terms of the APOS constructions—actions, processes, and objects—to describe how an individual might cognitively develop the concept of function (see also Arnon et al., 2014, pp. 29–30). According to this theory, an action is a transformation of objects in response to external cues. An individual with an action conception of function is limited to evaluating functions that are defined by algebraic expressions. Unlike an action conception, which is static, a process conception of function involves a dynamic transformation of objects by some repeatable means. An individual with a process conception can imagine many inputs being transformed, one after the other, and can describe the transformation in general terms without actually performing it. The notions of domain, one-to-one, and inverse function become more accessible to an individual with a process conception of function. Finally, a function can be conceived of as an object if actions or processes can be performed on it, such as transforming, combining, collecting, or comparing functions. An object conception, however, is not a permanent cognitive state. That is, an individual should be able to switch back and forth between a process and an object conception.

Breidenbach et al. (1992) report on the development of the process conception of function by students during an instructional treatment involving computers. Activities during the instructional treatment involved students using the programming language ISETL to construct different types of functions. The inputs of the ISETL functions included both numbers and sets, while the outputs could be numbers or boolean values (e.g., true/false). In comparison to students’ function conceptions prior to the instructional treatment, the authors noticed that students made significant progress in developing a process conception of function. These results suggest that engaging in constructive activities is crucial for the development of the function concept.

Several researchers (Breidenbach et al., 1992; Cuoco, 1994; Dubinsky & Harel, 1992; Sfard, 1992; Thompson, 1994) have asserted that a process conception is not only necessary
for a rich understanding of function, it also *precedes* an object conception. Sfard (1992), who offers the operational-structural duality similar to the process-object duality, claims that imposing\(^2\) a structural (object) conception of function on students too soon can cause them to construct what she calls a *pseudostructural conception*, characterized by treating expressions and graphs as objects themselves rather than as representations of an object. Sfard further argues that such a “potentially harmful” conception is almost unavoidable with the usual structural way of teaching functions.

Although there have been concerns that introducing mathematical concepts as processes will inhibit the development of a formal (object) conception, researchers have acknowledged that the difficulty in moving to a formal conception is a result of insufficient development of a dynamic process conception (Cottrill et al., 1996). This is most certainly the case with the concept of function, as reported by Cuoco (1994). Cuoco describes what he calls the “\(\mathbb{R}\) to \(\mathbb{R}\) curriculum” of mathematics leading up to calculus. According to him, this curriculum emphasizes treating functions as objects, ignoring the development of a process conception. In a series of anecdotes, Cuoco provides evidence suggesting that constructing processes to model functions and finding alternate processes to model functions contributes to a process conception. Furthermore, evidence in Cuoco’s paper suggests that asking students to provide arguments for the equivalence of two different function processes can bring about the development of an object conception of function. Therefore, guiding students to develop a rich and flexible process conception and providing them with situations that may foster development of the object conception of function, by means of the mechanisms of encapsulation and de-encapsulation, might possibly lead students to develop a better conceptual understanding of function.

It has also been reported that students have difficulties with variables described by a function (Trigueros & Jacobs, 2008; Trigueros & Ursini, 2003). Trigueros and Ursini (2003) proposed a framework for understanding variable. The framework includes three perspec-

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\(^2\)I use this word ironically (and with the traditional negative connotation), since we cannot “impose” conceptions on students—students construct them.
tives: variable as an unknown, variable as a general number, and variables in a functional relationship. The authors conducted a study on university students’ understanding of variable, analyzing data according to the proposed framework. Their results found that, despite many years in the educational system, most students could not distinguish between the different uses of variable, except for in elementary examples.

Drijvers (2003) described the notion of parameter as a higher-order variable. While variation in a first-order variable leads to variations in the value of an expression, variation in a parameter leads to variation in the whole expression (see also Ursini & Trigueros, 2004). Variation in a parameter is related to the idea of transformations of functions and families of functions. Drijvers distinguished between four roles of parameter: placeholder, changing quantity, generalizer, and unknown. As a placeholder, a parameter acts as a spot in a formula that can be replaced with a number that is then treated as fixed. A parameter is treated as a changing quantity when it is no longer viewed as a fixed number and instead viewed as a dynamic variable. A parameter is treated as generalizer when it is used to create a family a functions. Conversely, finding a specific function in a family that satisfies certain conditions involves treating a parameter as an unknown.

This section provided a summary of different issues involved in learning the general concept of function. The following section reports on literature related to the notions of covariational reasoning and quantitative reasoning, which has been shown favorable for developing a robust conception of function.

### 2.2 Covariational reasoning

A process conception of function has become widely recognized as crucial for understanding topics in collegiate mathematics, and several researchers have sought to characterize ways of thinking about functions that support a strong process conception (e.g., Carlson, 1998; Carlson et al., 2002; Oehrtman, 2008; Thompson, 1994). One way of thinking about functions that has been shown to be favorable for students developing an image of function as a dynamic process is imagining the covariation (simultaneous change) of two quantities.
The notion of covariation first appeared in mathematics education literature by Confrey and Smith (1991, 1994, 1995). They described covariation by stating, “as one quantity changes in a predictable or recognizable pattern, the other also changes, typically in a differing pattern” (Confrey & Smith, 1991, p. 57). In a later paper, they described covariation as “the juxtaposition of two sequences, each of which is generated independently through a pattern of data values” (Confrey & Smith, 1995, p. 67). In addition to being able to describe the pattern of each sequence individually, students with a strong understanding of covariation can coordinate changes in values of one sequence with changes in values of the other. Covariation is a more general description of the function concept and can be used to describe one-to-many correspondences, which are typically rejected in a traditional correspondence approach to functions. For this reason, Confrey and Smith advocate a more balanced approach to the study of functions that incorporates both covariation and the traditional correspondence approach.

A natural representation of covariation is a table that enables an individual to coordinate changes in values from one column with changes in values from the other. However, such a representation is of discrete covariation. Saldanha and Thompson (1998) hypothesized that students, through graphing, can construct an image of continuous covariation. The authors described continuous covariation as follows:

In this regard, our notion of covariation is of someone holding in mind a sustained image of two quantities (magnitudes) simultaneously. It entails coupling the two quantities, so that, in one’s understanding, a multiplicative object is formed of the two. As a multiplicative object, one tracks either quantity’s value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value. (Saldanha & Thompson, 1998, p. 298)

Saldanha and Thompson (1998) documented the problem-solving of an 8th-grade student, Shawn, during a teaching experiment designed to support thinking about continuous covariation. The tasks in the teaching experiment were graphing activities that used Geometer’s Sketchpad and involved tracking and describing the behavior of the distances between a car and two cities as it moves along a road. A sliding point \( C \), representing the car, was placed on a segment, which represented the road, and two fixed points \( A \) and \( B \), representing
the two cities, were placed off of the segment. Then the varying distances of \( C \) from \( A \) and \( B \), respectively, were tracked on the vertical and horizontal axes. The tasks of this teaching experiment were administered in the three following phases:

- **Engagement**—which allowed Shawn to explore each distance individually, each represented by line segments—one on a vertical axis and one on a horizontal axis—of varying length, and then both distances simultaneously, which were represented by a moving point \( P \) and its locus in the plane. Shawn was said to have internalized these actions when he could *imagine* moving the car and describe the anticipated effects on the point \( P \).

- **Move to representation**—which supported Shawn’s internalization of covariation. He was presented with various road-city configurations and was asked to imagine moving the car and describe the effect it would have on the point \( P \).

- **Move to reflection**—which was intended to have Shawn come to imagine completed covariation. He was presented with various graphs of \( P \)’s locus and was asked to predict the locations of the two cities relative to the road.

The results of the study by Saldanha and Thompson (1998) indicate that it is non-trivial for students to understand graphs as representing a continuum of states of covarying quantities.

Based on the notions of covariation of Confrey and Smith (1994, 1995), Saldanha and Thompson (1998), and others, Carlson et al. (2002) proposed a framework for investigating covariational reasoning employed by students when describing dynamic events. The authors defined covariational reasoning to be “the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other” (Carlson et al., 2002, p. 354). Their framework consists of five mental actions (MA) and associated behaviors, which are described in Table 2.1. The mental actions are not hierarchical—it is possible, for example, for a student to exhibit MA5 reasoning without exhibiting each of MA1–MA4 reasoning. However, the covariation framework is hierarchical
by associating a level of reasoning with a student who is able to perform a certain mental action and the mental actions with a lower number. Most students in the study by Carlson et al. (2002) exhibited all of the mental actions MA1–MA3, while students who demonstrated MA5 reasoning were unable to “unpack” the mental action in terms of MA1–MA4.

Carlson et al. (2002) concluded their study by recommending that students have the opportunity to think about covariation in real-world problems. They further suggested that, although their study focused on graphing tasks and visual imagery, covariational reasoning should be extended to working with other representations of functions. Additionally,
they suggested that future research consider how students use the implicit time variable in covariational reasoning.

Oehrtman et al. (2008) asserted that a process conception of function (in the sense of APOS) is foundational for covariational reasoning. According to them, a student with a strong process conception can view a formula as a way of transforming an input value into an output value, and, by transiting between representations, the student can begin to explore how increases in the input value effect the output value. As stated by Carlson et al. (2002, p. 376), “[...] the covariation framework may be used to infer information not just about the developmental level of student images of covariation but also about the internal structure of these images.” In other words, investigating students’ covariational reasoning abilities can provide a more refined description of their process conception of function.

As previously stated, Saldanha and Thompson (1998) described continuous covariation as the mentally pairing of two covarying quantities to create a multiplicative object. Then “one tracks either quantity’s value with the immediate, explicit, and persistent realization that, at every moment, the other quantity also has a value” (Saldanha & Thompson, 1998, p. 298). In a much later paper, Thompson (2011) reformulated this description mathematically to capture the dynamic aspects of covariation. He accomplished this by utilizing the notion of covering and by making time \( t \) an explicit variable. First, the set of all values of \( t \) is covered by intervals of size \( \epsilon \): \([t, t+\epsilon)\). Then, as \( t \) is varied, the interval coverings represent variation in chunks, and \( \epsilon \) suggests that these chunks are infinitesimally small. Therefore, if \( t_\epsilon = [t, t+\epsilon) \), then \((x(t_\epsilon), y(t_\epsilon))\) represents two quantities \( x \) and \( y \) continuously covarying as a single entity \((x, y)\) as time passes. This is comparable to Thompson’s (1994) earlier distinction between the notions of ratio and rate. He defined a ratio to be “the result of comparing two quantities multiplicatively” (p. 190). Thompson further explained that when a ratio is reconceived as applying “outside of the phenomenal bounds in which it was originally conceived, then one has generalized that ratio to a rate” (1994, p. 192). In other words, two quantities are paired by some quantitative operation, and then the relationship between the quantities is perceived as invariant as they simultaneously change with respect to
Castillo-Garsow (2010, 2011) distinguished between two types of reasoning about continuous variation—chunky reasoning and smooth reasoning. Chunky reasoning is thinking about change in completed chunks of time, whereas smooth reasoning is thinking about change in progress. As asserted by Castillo-Garsow (2011), chunky reasoning is inherently discrete and, therefore, counter to continuous variation, since it inhibits the ability to imagine change in real time. In order to employ chunky reasoning to consider variation within a chunk of time, the chunk must be re-conceptualized into smaller chunks. Smooth reasoning, on the other hand, is inherently continuous. It entails imagining passing through every moment within a larger unit of time. Castillo-Garsow (2010, p. 230) interpreted Confrey and Smith’s (1994, 1995) perspective of covariation as chunky reasoning and Thompson’s (2011) perspective as a mixture of chunky and smooth reasoning.

The results of Castillo-Garsow’s dissertation (2010) showed that one student, Derek, demonstrated proficiency in both chunky and smooth reasoning when thinking about situations involving exponential relationships, and, when appropriate, Derek adopted one way of reasoning over the other. Another student, Tiffany, was limited mostly to chunky reasoning, and it put her at a disadvantage when attempting to describe the overall behavior of functions. For this reason, Castillo-Garsow claimed that chunky reasoning can suffice for numerical calculations, but students whose thinking about covariation is limited to chunky reasoning might not be able to accomplish other tasks. Therefore, it is necessary for students to develop both chunky and smooth reasoning abilities.

Johnson (2013) distinguished between two ways of reasoning about quantities varying together—change-dependent reasoning and simultaneous-independent reasoning. Change-dependent reasoning is characterized by treating one quantity as the independent variable and the other as the dependent variable, so that a change in the second quantity is in response to a change in the first quantity, whereas simultaneous-independent reasoning is characterized by viewing each quantity as changing independently of the other in relation to time. In one report, Johnson (2012b) documented her observations of a student—Hannah—
using change-dependent reasoning, while in another report, Johnson (2012a) documented her observations of three students—Austin, Mason, and Jacob—using simultaneous-independent reasoning. Hannah was able to use change-dependent reasoning to describe variation in the rate-of-change of two quantities, while Austin, Mason, and Jacob’s simultaneous-independent reasoning only enabled them to compare amounts of change in two quantities. She concluded, “such comparisons merely related quantities through division rather than representing a newly constructed quantity” (Johnson, 2012a, p. 52).

2.3 Horizontal and vertical growth of the function concept

In the early 1980s, Dreyfus and Eisenberg (1982, 1984) proposed a framework for a systematic approach to curriculum development pertaining to the concept of function. The framework includes three dimensions: settings or representations (e.g., tables, equations, graphs, etc.), concepts (e.g., image, zeros, extrema, domain and range, injectiveness, surjectiveness, etc.), and generalization and abstraction (extending function concepts to multivariable functions, complex-valued functions, etc.). The authors summarized this framework by what they called the *function block* with three mutually perpendicular axes. Along the $x$-axis are the representations of the function concept; along the $y$-axis are the formal concepts related the notion of function; and along the $z$-axis are the different levels of abstraction of the function concept. According to the framework, at any level of abstraction (on the $z$-axis), movement can be along the $x$-axis, to consider different representations of that function, and/or along the $y$-axis, to consider the formal concepts related to that function. Dreyfus and Eisenberg (1984) referred to movement along the $x$-axis as *horizontal transfer* and movement along the $z$-axis as *vertical transfer*. The authors asserted, “ultimately, ways have to be found to extend the discussion of selected concepts to functions of several variables, complex valued functions, implicitly defined functions, etc.” (Dreyfus & Eisenberg, 1984, p. 78).

A considerable amount of research over the past three decades has investigated students’ understanding of many of the elements described in Dreyfus and Eisenberg’s (1982, 1984) function block. This section provides an overview of such research.
A significant amount of literature from the 1980s and 1990s focused on students’ understanding of different representations of functions (Even, 1998; Hitt, 1998; Janvier, 1987; Moschkovich, Schoenfeld, & Arcavi, 1993; Schwingendorf et al., 1992). However, Thompson (1994) reflected on the multiple representation perspective, reconsidering that the “core concept of function” is not represented by the common representations of function. He stated:

Tables, graphs, and expressions might be multiple representations of functions to us, but I have seen no evidence that they are multiple representations of anything to students. In fact, I am now unconvinced that they are multiple representations even to us, but instead may be areas of representational activity among which, as Moschkovich, Schoenfeld, and Arcavi have said, we have built rich and varied connections. (Thompson, 1994, p. 39)

Thompson went on to assert, “we should focus on them as representations of something that, from the students’ perspective is representable, such as aspects of a specific situation” (p. 39).

David Tall and colleagues investigated the “core concept of function” and the representable “something” referred to by Thompson (Akkoç & Tall, 2002, 2003, 2005; McGowen, DeMarois, & Tall, 2000; Tall, McGowen, & DeMarois, 2000). Tall, McGowen, and DeMarois (2000) suggested that the “something” referred to by Thompson could be an embodied image of the function concept, such as the function machine. Meanwhile, Akkoç and Tall (2003) describe the “core concept of function” as a mathematically simplistic idea that generates ideas that are often too complicated for students who do not make connections among the different representations of function. Arnon et al. (2014) claimed that the reason students have difficulty transitioning from one representation of function to another is that instruction focuses on the direct translation without passing through the cognitive meaning of the function concept. From the perspective of APOS, the authors suggested that the student should determine the process of the function represented and use that process conception to transfer to a new representation.

Oehrtman et al. (2008) recommended that students be given more opportunities to experience multiple representations functions in different coordinate systems. While there
is a substantial amount of literature on students’ understanding of Cartesian graphs of functions, only recently have researchers begun to investigate students’ transferring their knowledge of the function concept other coordinate systems (Montiel et al., 2008, 2009, 2011; Moore et al. 2013).

Montiel et al. (2008) investigated the relationship between students’ understanding of functions in the Cartesian and polar coordinate systems in a single-variable context. The authors stated that when standard textbooks introduce the polar coordinate system, “very little emphasis is placed on the actual function concept, as it is supposed that by this state the student has developed the concept and that it will transfer” (p. 58). However, the authors found that the majority of students in the study were not able to transfer the notion of function to the polar coordinate system. Many of them took the vertical line test as the definition of function instead of a definition in terms of input and output. As the authors stated, “the generic definition of function […] seems to often be lost amongst the different representations students are exposed to, without recognizing any implicit hierarchy” (p. 64).

Moore et al. (2013) investigated two pre-service secondary mathematics teachers’ ways of thinking when graphing in the polar coordinate system. They found that by engaging in covariational reasoning, the students were able to conceive graphs in different coordinate systems as representative of the same relationship. As a result, the authors asserted that providing students with opportunities to graph relationships in multiple coordinate systems can establish a need for covariational reasoning, which can promote less problematic conceptions of functions and their graphs.

Several studies have shown that students have difficulty generalizing their conception of function. For example, Mamona (1990) observed that students had misconceptions about sequences. She found that many students treated sequences as series, as they felt the need to sum the list of numbers generated by a sequence in order to find its limit. Mamona conjectured that this misconception might be due to students’ lack of acclimation to abstract concepts. That is, she argued that students feel the need to “do” something to the numbers, such as addition. Moreover, Mamona (1990) found that students resisted to con-
sider sequences as functions, due to the discrete nature of the domain. She conjectured that this is a consequence of students’ previous exposure to only “well-behaved” functions that correspond to a continuous graph.

Similar to Mamona’s (1990) findings, McDonald et al. (2000) reported that students tend to construct two notions of sequence, one as a list—SEQLIST—and the other as a function—SEQFUNC—with domain $\mathbb{N}$. While the notion of SEQLIST was found to be easy for students to construct, the notion of SEQLIST was much more difficult, especially for students who had a weak function schema. For these students, it was inconceivable for a sequence to have a domain.

Martínez-Planell, Gonzalez, DiCristina, and Acevedo (2012) built on the study by McDonald et al. (2000) by investigating students’ conceptions of infinite series. The authors argued that students construct two notions for infinite series, one as a never-ending process of addition—SERLIST—and the other as sequence of partial sums—SERFUNC—each of which corresponds to a positive integer. They observed that weak schemas for function and limit caused students difficulty in constructing a strong SERFUNC conception.

A pair of studies by Trigueros and Martínez-Planell (Martínez-Planell & Trigueros Gaisman, 2012; Trigueros & Martínez-Planell, 2010) investigated students’ reasoning about two-variable functions. In these studies, the authors proposed a preliminary genetic decomposition (in the sense of APOS theory) for how students might generalize their understanding of one-variable function to two-variable functions. The genetic decomposition involved the essential constructions of a schema for $\mathbb{R}^3$, set, and function, that should be coordinated through the actions and processes of assigning a unique height to each element of a given subset of $\mathbb{R}^2$. The first study (Trigueros & Martínez-Planell, 2010) focused on students’ development of $\mathbb{R}^3$ and their understanding of graphs of two-variable functions, while the second study (Martínez-Planell & Trigueros Gaisman, 2012) investigated students’ general understanding of two-variable function, specifically the notions of domain, range, uniqueness of function value, and the arbitrary nature of function. These studies revealed that students have difficulty generalizing their notion of function to include relations between $\mathbb{R}^2$ and $\mathbb{R}$. 
They reported that students have difficulty recognizing that the domain of a two-variable function is a set of points in the plane. Furthermore, they reported that students have difficulty finding the range of two-variable functions when the domain is restricted, which is linked to their inability to imagine a graph of a function of two-variable functions given algebraically.

Weber and Thompson (2014) described a way of thinking about graphs of two-variable functions defined by $z = f(x, y)$. They said that by thinking of $x$ or $y$ as a fixed parameter, an individual can imagine “sweeping out” a point to create a graph in the plane. Then by varying the parameter, the individual can imagine sweeping the graph through space to create a surface. The authors found that covariational reasoning can support the visualization of graphs of two-variable functions, rather than relying on the memorization of prototypes.

2.4 Parametric function

While a substantial amount of research has investigated students’ understanding of the general function concept, representations of function, and different types of functions, very little is known about how students reason about parametric functions. Clement (2001) reported on asking five precalculus the following question:

A caterpillar is crawling around on a piece of graph paper, as shown below. If we wished to determine the creature’s location on the paper with respect to time, would this location be a function of time? Why or why not? (Clement, 2001, p. 745)

The authors reported that only two of the five students answered correctly, while the other three students answered incorrectly based on using the vertical line test. Although this example does not explicitly mention the parametric function concept, it is essentially asking students about a parametric function. Students’ incorrect responses suggest that they were unaware of what the function is in this situation. They interpreted the question to be about the points on the caterpillar’s path, instead of being about those points in relation to time. This example illustrates the potentially complex nature of the parametric function concept.
In the context of a differential equations course, Keene (2007) investigated the mathematical activity of students when solving problems that involve the parameter time as an implicit changing quantity. She distinguished between parameter as a changing quantity, as described by Drijvers (2003), and time as a dynamic parameter by explaining that in former case the parameter is always explicitly defined, while in the latter case, the parameter can be implicit. Although the goal of Keene’s study did not directly investigate students’ construction of the concept of parametric function, it explored various ways of reasoning in situations that involve relationships described parametrically. Dynamic reasoning is defined as “developing and using conceptualizations about time as a dynamic parameter that implicitly or explicitly coordinates with other quantities to understand and solve problems” (p. 231). Keene observed the following types of mathematical activity by students while engaging in dynamic reasoning:

- Making time an explicit quantity
- Using the metaphor of time as “unidimensional space”
- Using time to reason both quantitatively and qualitatively
- Using three-dimensional visualization of time related functions
- Fusing context and representation of time related functions
- Using the fictive motion metaphor for function

Making time an explicit quantity refers to the student introducing time as an additional variable and reasoning how each explicit variable changes with respect to time. Using the metaphor of time as “unidimensional space” refers to the student describing time as a value that “keeps on going.” Using time to reason both quantitatively and qualitatively refers to the student using both particular numerical values and general (behavioral) terminology to describe the change in dynamic variables. Using three-dimensional visualization of time related functions refers to the student introducing a third axis for \( t \) and visualizing a three-dimensional version of a plane curve. Fusing context and representation of time related
functions refers to the student discussing both mathematical information (e.g., a graph) and contextual information at the same time. And using the fictive motion metaphor for function refers to the student viewing solutions of a function as moving points that create a trace.

Keen concluded the study by asserting that little is known about how students reason about parametric functions in the calculus sequence and earlier. As a result, she suggested that further research explore this domain.

Oehrtman et al. (2008) asserted that covariational reasoning can be used to reason about functions defined parametrically. They claimed that by using covariational reasoning students can conceptualize a curve \((x, y) = (f(t), g(t))\) by generating the graphs of \(f(t)\) and \(g(t)\) separately and then tracking the values of \(x\) and \(y\) as \(t\) varies. Moreover, they described an application of covariation similar to using three-dimensional visualization as described by Keene (2007). Oehrtman et al. (2008) said that by perceiving \(t\) as an axis “coming straight at your eyes,” you can imagine that points on plane curve are actually some distance toward you (p. 38).

In their master’s project under the direction of Patrick Thompson, Bishop and John (2008) investigated the application of covariational reasoning to the teaching and learning of parametric functions in a high school algebra course. Their study resulted in three findings. One finding was that, although high school curriculum places little emphasis on covariational reasoning, students in the study were able to learn to reason covariationally. Another finding was that students strengthened their notion of variable. And the last finding was that students have difficulty making connections between the rate of change of the components of a parametric function to the rate of change in \(y\) with respect to \(x\).

2.5 Overview

This chapter provided an overview of existing literature regarding students’ understanding of the concept of function. Studies have shown that students have difficulty with many aspects related to the function concept, such as stating or applying the general definition of function in various problems (Breidenbach et al., 1992, Vinner & Drefyus, 1989), reasoning
about the different representations of functions (Moschkovich et al., 1993; Akkoç & Tall, 2003), transferring the function concept to unfamiliar coordinate systems (Montiel et al., 2008; Moore et al., 2013), and generalizing the concept of function to account for unfamiliar functions (Mamona, 1990; Martínez-Planell & Trigueros Gaisman, 2012; McDonald, 2000). However, literature addressing the concept of parametric function is sparse, with only two studies that explicitly address students’ reasoning about the concept. One of these studies addresses high school students’ understanding of parametric functions (Bishop & John, 2008), while the other study investigated how students’ reason about parametric functions in a differential equations course (Keene, 2007). However, no study has investigated specifically calculus students’ understanding of parametric functions.

As stated by Czocher et al. (2013), there are “epistemological mismatches” between how parametric functions (or parametric equations) are taught in the calculus sequence and how they are used in courses after the calculus sequence. For this reason, a comprehensive research agenda is needed regarding calculus students’ reasoning about parametric functions, in order to develop curriculum that addresses the epistemological mismatches. The current study is aimed to address this need.
Chapter 3

METHODOLOGY

This is a qualitative research study, which is “an approach to social science research that emphasizes descriptive data in natural settings, uses inductive thinking, and emphasizes understanding the [subject’s] point of view” (Bogdan & Biklen, 2007, p. 274). According to Crotty (1998), there are four main elements of qualitative research: the methods of data collection, the methodology that informs the methods, the researcher’s theoretical perspective, and the researcher’s epistemology. At the end of Chapter 1, I established my epistemology and theoretical perspective (constructivism and APOS theory). In this chapter, I describe the methodology and methods of this study, focusing on how they are informed by the theoretical perspective, APOS theory.

3.1 Role of APOS theory

According Asiala et al. (1996), research that employs APOS theory to investigate students’ understanding of mathematical concepts begins with the development of a genetic decomposition\(^1\) for how a learner might construct an understanding of a concept. At this stage, the genetic decomposition is preliminary and is a mere hypothesis for how a concept might be constructed by an individual. When developing the preliminary genetic decomposition, the researcher considers his or her own understanding of the concept, the results of existing research on the concept, and the historical development of the concept. The researcher may also consider how a topic is presented in textbooks. After the preliminary genetic decomposition is developed, instruction is designed to hopefully guide students to make the constructions called for by the preliminary genetic decomposition. After the instructional

\(^1\)Recall from Section 1.3 that a genetic decomposition is a description of students’ possible constructions of actions, processes, objects, and schemas, as well as the mental mechanisms by which those constructions are possibly achieved.
treatment, qualitative data is collected and analyzed to determine whether students made
the constructions described in the genetic decomposition and whether students’ responses
indicated any constructions that were not included in the genetic decomposition. The ge-
netic decomposition is then revised based on the results of the data analysis. This research
cycle may be repeated for as long as it is necessary.

3.2 Conceptual analysis

Glasersfeld (1995) described conceptual analysis as a way of determining “what mental op-
erations must be carried out to see the presented situation in the particular way one is seeing
it” (p. 78). Thompson (2008) asserted that conceptual analysis can be used, for example,
in building models of students’ knowledge and to describe ways of knowing that might be
favorable for students’ learning. Thompson’s description of conceptual analysis is highly
complementary with the methodology of the APOS framework in that a conceptual analysis
can serve as a foundation for developing a genetic decomposition. In this section, I present
a conceptual analysis of the concept of parametric function. It begins with a brief survey
of three calculus textbooks, followed by a discussion of the historical development of the
concept, as well as my own synthesis of what I believe are crucial elements of a coherent un-
derstanding of the parametric function concept. Based on this section, a preliminary genetic
decomposition will be proposed in Section 3.3.

3.2.1 Survey of textbooks

In the text Calculus One and Several Variables by Salas, Hille, and Etgen (2007), the authors
introduce parametric topics by stating,

So far, we have specified curves by equations in rectangular coordinates or by
equations in polar coordinates. Here we introduce a more general method. We
begin with a pair of functions $x = x(t)$, $y = y(t)$ differentiable on the interior of
an interval $I$. At the endpoints of $I$ (if any) we require only one-sided continuity.
For each number $t$ in $I$ we can interpret $(x(t), y(t))$ as the point with $x$-coordinate
$x(t)$ and $y$-coordinate $y(t)$. Then, as $t$ ranges over $I$, the point $(x(t), y(t))$ traces
out a path in the $xy$-plane. We call such a path a *parametrized curve* and refer to $t$ as the *parameter*. (Salas et al., 2007, p. 496)

Based on the first few examples and exercises, it seems that in order to develop a foundational understanding of parametric functions, the main objective of this text is for students to learn how to convert from parametric form of a curve to standard form.\(^2\) Converting from parametric form to standard form is accomplished either by isolating a term in one equation and making a substitution into the other equation or by using a trigonometric identity, resulting in a familiar equation (e.g., of a line, parabola, circle, or ellipse) that can be graphed either from memory or by using primitive techniques. Twenty-two exercises ask students to sketch a curve without the use of a graphing utility. Of these 22 exercises, 18 contain curves given parametrically that can be expressed in standard form, which can then be used to sketch the curve. The remaining 4 of these 22 exercises contain curves given parametrically that cannot be expressed in standard form. This is troubling because the text does not explicitly address how to sketch such curves—there is a subtle suggestion that it is useful to first find the locations of the vertical and horizontal tangent lines, but no other strategies are offered. Applications of parametric equations in this text involve particle movement, but minimal exercises ask students to parametrize a particle's path. Instead, most exercises give the particle's path in parametric form.

*Calculus for Scientists and Engineers* by Briggs et al. (2013) introduces parametric equations by stating,

So far, we have used functions of the form $y = f(x)$ to describe curves in the $xy$-plane. In this section we look at another way to define curves, known as parametric equations. As you will see, parametric curves enable us to describe both common and exotic curves; they are also indispensable for modeling the trajectories of moving objects. (Briggs et al., 2013, p. 724)

The authors go on to say,

In general, parametric equations have the form $x = g(t), y = h(t)$, where $g$ and $h$ are given functions and the parameter $t$ typically varies over a specified interval,\(^2\)

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\(^2\)An equation in $x$ and $y$. 
such as $a \leq t \leq b$. The **parametric curve** described by these equations consists of the points in the plane that satisfy $(x, y) = (g(t), h(t))$, for $a \leq t \leq b$. (Briggs et al., 2013, p. 725)

In a side note, the authors emphasize that $t$ is the independent variable and $x$ and $y$ are two dependent variables. The first few examples and exercises suggest that developing an understanding of functions begins with using parametric equations to construct a table of values in order to sketch a corresponding curve. In addition to having columns for values of $t$, $x$, and $y$, the authors include a column for values of $(x, y)$, which seems to place a careful emphasis on the pairing of $x$ and $y$ to construct a new variable quantity $(x, y)$. Overall, Briggs et al. (2013) present a well-balanced approach to topics involving parametric functions by offering exercises containing pure mathematical problems, a variety of modeling problems, and problems involving exotic curves.

*Calculus, Concepts & Computers* was written by Dubinsky et al. (1995) for the calculus reform movement of the 1990s. In this text, the authors introduce what is referred to as the **parametric form** of a curve in $\mathbb{R}^2$, which is represented by two equations

$$x = f(t), \ y = g(t)$$

or vector notation $v(t) = (f(t), g(t))$. The first objective is for students to coordinate equations of $f$ and $g$ to sketch a corresponding curve, first by hand and then by using a computer. The authors use function concepts to explain that converting to standard form can be accomplished by the relation $g \circ f^{-1}$, but the authors also demonstrate the standard ways of using algebra and trigonometric identities to make the conversion.

There are many conceptual ideas in the text by Dubinksy et al. (1995) that foster a more general view of parametric functions. Two of these are stated as follows:

- “The concept of *parameter* refers to an intermediate variable which relates two or more other variables” (p. 549).

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3The $C^4L$ project.
“A curve in \( \mathbb{R}^2 \) is a function whose domain is a set in \( \mathbb{R}^1 \) and whose range is \([\text{in}]\ \mathbb{R}^2\)” (p. 567).

The above statements by Dubinsky et al. (1995) form the basis of a pivotal characterization because they emphasize that a parametric function, regardless of how it is represented, accepts a real number as an input and returns one output in the form of an ordered pair of real numbers.

### 3.2.2 Historical considerations

Perhaps one the earliest presentations of parametric functions in history can be contributed to Euler in his 1748 *Introductio in Analysin Infinitorum* (Dubinsky et al., 1995). Book II of this text is where we can find Euler’s extensive presentation on the study of curves. Many of Euler’s descriptions of curves are comparable to ideas discussed in the literature review, such as the reports by Keene (2007) and Oehrtman et al. (2008).

Book II, Chapter 1 of Euler’s text was devoted to the discussion of curves in general. In Section 7, he described how a curve is defined by a function:

> In this way the curve which results from the function \( y \) is completely known, since each of its points is determined by the function \( y \). At each point \( P \), the perpendicular \( PM \) is determined, and the point \( M \) lies on the curve. Indeed, each point on the curve is found in this way. In whatever way the curve may be viewed, for each point on the curve there is one point on the straight line \( RS \), which is on the perpendicular to the line through the point on the curve. In this way we obtain the interval \( AP \), which gives the value of \( x \), and the length of the perpendicular \( PM \), which represents the value of the function \( y \). It follows that there is no point of the curve which is not defined by the function \( y \). (Euler, 1990, p. 5)

Immediately following, in Section 8, Euler acknowledged the dynamic aspect of curves:

> Although many different curves can be described mechanically as a continuously moving point, and when this is done the whole curve can be seen by the eye, still

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4What is interesting is that none of the textbooks that I reviewed explicitly described parametric functions in this way. In fact, none of these texts, including *Calculus, Concepts & Computers* by Dubinsky et al. (1995), mention the phrase *parametric function*. Instead, they use phrases like *parametric equation* or *parametric form*. 


we will consider these curves as having their origin in functions, since then they will more apt for analytic treatment and more adapted to calculus. (Euler, 1990, p. 5)

In Section 14 of Chapter 1, Euler stated the following:

Since \( y \) is a function of \( X \), either \( y \) is equal to an explicit function of \( x \), or we have an equation in \( x \) and \( y \) where \( y \) is defined by \( x \). In either case we have an equation which is said to repress the nature of the curve. For this reason every curve is expressed by an equation in two variables \( x \) and \( y \). (Euler, 1990, p. 8)

In the previous excerpt Euler expressed that every curve (presumably plane curve) can be expressed by an equation between between two variables. Although Euler initially referred to these variables as \( x \) and \( y \), in later chapters he described using other coordinate systems to represent curves. It was not until his discussion of three-dimensional objects that Euler used three variables in one representation. For example, in Chapter VI, Section 133 of the appendix, Euler gave a presentation on the intersection of two surfaces by introducing the notion of a non-planar curve. This is where it appears that Euler was describing what is considered today to be the concept of parametric function:

The nature of any non-planar curve is most conveniently expressed by two equations in three variables, for example, \( x \), \( y \), and \( z \) which represent mutually perpendicular coordinates. By means of the two equations, two of the variables can be determined by the third. For instance, \( y \) and \( z \) are equal to some functions of \( x \). We can even choose one of the variables arbitrarily for elimination, so that we can find three equations in only two variables: one in \( x \) and \( y \), another in \( x \) and \( z \), and a third in \( y \) and \( z \). Of these three equations, any one is automatically determined by the other two; given equations in \( x \) and \( y \) and in \( x \) and \( z \), the third can be found by eliminating \( x \) from these two. (Euler, 1990, p. 453)

In the last line of the previous excerpt, Euler described two parametric equations with independent variable (parameter) \( x \) and the process of eliminating the parameter to obtain an equation in \( y \) and \( z \). This is precisely the idea of converting from parametric to standard form as it appears in modern calculus textbooks today.

In Section 134, Euler described constructing a plane curve from a non-planar curve by dropping perpendiculars from each point on the curve to a plane:
Suppose that some non-planar curve is given. In figure 148 we let $M$ represent any point on the curve. We arbitrarily choose three mutually perpendicular axes, $AB$, $AC$, $AD$, by means of which the three mutually perpendicular planes $BAC$, $BAD$, and $CAD$ are determined. From the point $M$ on the curve we drop the perpendicular $MQ$ to the plane $BAC$ from the point $Q$ we draw $QP$ perpendicular to the axis $AD$. Then $AP$, $PQ$, and $QM$ are the three coordinates in which the two equations will determine the nature of the curve. We let $AP = x$, $PQ = y$, and $QM = z$. From the two given equations in $x$, $y$, and $z$ we eliminate $z$ to obtain an equation in only the two variables $x$ and $y$. This will determine the position of the points $Q$ in the plane $BAC$. All of the points $Q$ corresponding to points $M$ produce a curve $EQF$, whose nature is given by the equation in $x$ and $y$. (Euler, 1990, p. 453)

In the next section, Euler went on to explain that this method can be used to “discover the nature of the curve $EQF$” (p. 453). Furthermore, he called the plane curve $EQF$ the projection of the non-planar curve onto the plane $BAC$.

It is worth mentioning that Euler’s description of curves being generated by a “continuously moving point” is comparable with Keene’s (2007) observation of students using the fictive motion metaphor, while Euler’s description of non-planar curves is exactly how Oehrtman et al. (2008) described imagining a $t$-axis perpendicular to the $x$- and $y$-axes.

Based on the characterization by Dubinsky et al. (1995) of curves in $\mathbb{R}^2$ as functions from $\mathbb{R}$ to $\mathbb{R}^2$, what follows is my own presentation of what I call a function approach to the study of curves defined parametrically.

### 3.2.3 A function approach to curves defined parametrically

This section describes the concept of parametric function, as I have come to understand it, which is based on the characterization by Dubinsky et al. (1995) of curves in $\mathbb{R}^2$ as functions from $\mathbb{R}$ to $\mathbb{R}^2$.

A **parametric function** is a function $p : D \rightarrow \mathbb{R}^2$ where $D \subseteq \mathbb{R}$. That is, $p$ is a correspondence that associates to each $t \in D \subseteq \mathbb{R}$ a unique element $(x, y) \in \mathbb{R}^2$. The set $D$ is called the **domain** of the parametric function and the set of ordered pairs defined by $p(D)$ is called the **range**.

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5This definition can be easily generalized to higher dimensions.
A parametric function $p$ is defined by an ordered pair $(f, g)$ of functions, each defined from $D$ to $\mathbb{R}$. The largest possible domain of $p$ is the intersection of the largest possible domains of $f$ and $g$, i.e., $D = \text{Dom}(p) = \text{Dom}(f) \cap \text{Dom}(g)$. The range $p(D)$ corresponds to a curve in $\mathbb{R}^2$. If two parametric functions have the same range, then they trace the same curve in $\mathbb{R}^2$.

From this characterization, a parametric function (the abstract concept) can then be represented by tables, graphs, and equations. A tabular representation would involve a two-column table with one column corresponding to values of $t$ and the other column corresponding to values of $(x, y)$. Parametric functions can be expressed analytically in parametric form by a pair of equations, $x = f(t)$ and $y = g(t)$, or in vector form, $p(t) = (f(t), g(t))$. Because a parametric function defined by $(x, y) = (f(t), g(t))$ involves three variables, $t$, $x$, and $y$, it is natural to assign each variable to an axis in the Cartesian coordinate system, thereby constructing $\mathbb{R}^3$. Oehrtman et al. (2008) describe $t$ as an axis coming toward you perpendicular to the $x$- and $y$-axes. From this perspective, a plane curve in $\mathbb{R}^2$ is actually the $xy$-projection of a space curve moving toward you (as $t$ increases) and away from you (as $t$ decreases). This is very similar to Euler’s presentation of non-planar curves (see Section 3.2.2).

This approach to the parametric function concept is a generalization of the notion of real-valued function of one variable. The aspect of generalization on which the present study focuses is generalizing the generic function definition (in terms of input and output) to apply to parametric functions as a functions that transform a real number into an ordered pair.

### 3.3 Preliminary genetic decomposition

This section proposes a preliminary genetic decomposition\(^6\) for the initial learning of parametric functions in the context of a second-semester course in single-variable calculus. Prior to the study of parametric functions, it is hypothesized that the student should have constructed the following schemas: a schema for $\mathbb{R}^2$ that includes points as objects and curves.

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\(^6\)I would like to thank Ed Dubinsky for his valuable comments on the preliminary genetic decomposition.
as made up of points; and a schema for function that includes single-variable, real-valued functions as objects, and includes the concepts of function inverse and composition of functions. Note that an individual with an object conception of function should be able to de-encapsulate the object, when necessary, to obtain the process from which it came.

The following steps hypothesize the constructions that an individual should pass through in order to construct the concept of parametric function:

1. Coordinate the processes of evaluating two functions \( f \) and \( g \) at values of \( t \) to imagine \( f(t) \) and \( g(t) \) changing simultaneously as \( t \) increases through values shared by \( \text{Dom}(f) \) and \( \text{Dom}(g) \). At this step, the individual can compare changes in \( f(t) \) with changes in \( g(t) \) over small intervals of \( t \).

2. Coordinate the process in Step 1 with the schema for \( \mathbb{R}^2 \) to construct elements of \( \mathbb{R}^2 \) defined by \( (x, y) = (f(t), g(t)) \). At this step, an individual with an object conception of point in \( \mathbb{R}^2 \) can describe the process of \( (f(t), g(t)) \) tracing out a curve in \( \mathbb{R}^2 \).

3. Encapsulate the process in Step 2 to compare and contrast congruent curves in \( \mathbb{R}^2 \) in terms of their points and the rate at which they are generated. At this step, the individual can perceive the relationship between \( x \) and \( y \) as being independent of \( t \).

4. Reverse the process of one function, say \( f \), in Step 1 and compose it with the other function, in our case \( g \), to represent the relationship between \( x \) and \( y \) in Step 3 by a single function.

5. Reverse the process in Step 4 to describe the relationship between \( x \) and \( y \) parametrically.

6. Generalize the function schema to apply to parametric functions by encapsulating the processes in Step 1 to pair \( f \) with \( g \). Then coordinate the function schema with the \( \mathbb{R}^2 \) schema to conceptualize \( (f, g) \) as one function that transforms \( t \) into a unique element \( (f(t), g(t)) \) of \( \mathbb{R}^2 \).
3.4 Research setting

This study was conducted at a large public university in the southeastern United States during the Fall 2013 semester. The subjects for this study were recruited from one section of the course Calculus of One Variable II, for which I was the instructor. The class met each week from 8:00 a.m. to 9:45 a.m. on Tuesday and Thursday.

3.4.1 Course description

The course Calculus of One Variable II includes the following topics in its official description: applications and techniques of integration; transcendental and trigonometric functions; polar coordinates; infinite sequences and series; indeterminate forms; and improper integrals. The students who typically enroll in Calculus of One Variable II are undergraduates majoring in mathematics, physics, computer science, geology, or actuarial science. Occasionally, students with other majors, such as biology or chemistry, enroll in this course, although this university offers a calculus sequence for life science majors. The prerequisite for this course is a grade of C or better in Calculus of One Variable I, which covers the concepts of limits and continuity; differentiation; mean value theorem for derivatives; applications of differentiation; definition of the integral; fundamental theorem of calculus; and applications of integration to area.

The assessments for the particular section of Calculus of One Variable II in which this investigation took place consisted of weekly homework assignments, a semester project, three tests, and a comprehensive final exam. In compliance with a departmental policy, graphing calculators were not allowed to be used on tests or the final exam. Non-graphing scientific calculators were allowed, although test and exam problems did not necessarily require them.

The textbook for the course was Calculus One and Several Variables by Salas et al. (2007). The topic of curves given parametrically appears in Chapter 10 of this text. Although this chapter includes eight sections, the curriculum for Calculus of One Variable II covers only the following six:

10.2 Polar Coordinates.
10.3 Sketching Curves in Polar Coordinates.

10.4 Area in Polar Coordinates.

10.5 Curves Given Parametrically.

10.6 Tangents to Curves Given Parametrically.

10.7 Arc Length and Speed.

This investigation focused primarily on Section 10.5, which demonstrates how to convert from parametric form to standard form; presents formulas for parametrizations of a line, circle, and ellipse; and distinguishes collision points from intersection points.

3.4.2 Description of instruction

The text *Calculus One and Several Variables* by Salas et al. (2007) has little to no emphasis on the abstract parametric function concept that was described in Section 3.2.3. The instructor for the course addressed this issue supplementarily by defining a parametric function as a function that accepts a real number as an input and returns one ordered pair of real numbers as an output. This definition was distinguished from familiar functions that accept a real number as an input and return one real number as an output. An instructional task was assigned in class in order for students to explore in groups how to model two quantities changing simultaneously in situations involving time as either an implicit or explicit variable (see Appendix A). Overall, there were no major deviations from the objectives of the textbook. Therefore, this investigation qualifies more as an observational study than as a teaching experiment.

3.5 Subjects

In the section of *Calculus of One Variable II* described above, there were forty-seven students initially enrolled and three withdrew. The remaining forty-four students were asked for permission to use their written work on relevant assignments as data. Twenty-one students consented, and from these, fifteen students volunteered to participate in a semi-structured
interview. Only data was analyzed from the fifteen students who participated in both aspects of the data collection. These fifteen students varied in gender, ethnicity, and levels of mathematical knowledge. A pseudonym was assigned to each student.

### 3.6 Data collection

The data collected for this study included:

- Students’ written responses to five relevant problems on the final exam. These problems involved converting from parametric form to standard form, sketching graphs of curves, and discriminating between curves given their graphs. Moreover, students were asked to give their definition of parametric function. (See Appendix B for the instrument.)

- Students’ written and verbal responses to nine questions in a semi-structured interview. The purpose of these problems was to investigate students’ reasoning about functions in the form $y = f(x)$ and $p(t) = (f(t), g(t))$, determine how students reason about the invariant relationship between two variables changing simultaneously with respect to time, and investigate students’ reasoning when constructing a parametric function to describe a situation. (See Appendix C for the instrument.)

The interview sessions were conducted in a private meeting room on the university campus and were video-recorded. There were a total of nine interview sessions, each of which had one to four participants. The interview was preceded by a 45-minute problem-solving period during which each student completed the questionnaire without any guidance from the interviewer or influence from other participants. During the 45 minutes following the problem-solving period, the interviewer asked each participant to explain his or her answers. Furthermore, the interviewer asked probing questions in order to gain additional insight into the subject’s conceptual understanding. Because the interview sessions were distributed among three researchers, a common protocol was used to insure consistency in

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7Since there are both pros and cons to group and individual interviews, I did not choose one design over the other.
the data collection. (See Appendix D for the interview protocol.) As I was the instructor for the course, I was not present during any of the interviews so that the students would not feel any pressure in terms of their grade.

3.7 Data analysis

The data analysis for this investigation took place in multiple phases. While transcribing the interview data, I kept a journal in which I noted my immediate observations of each student. These notes helped insure that I did not overlook any details during my data analysis. After multiple passes through the interview transcriptions, each utterance by a student was color-coded according to the level of understanding, in terms of APOS theory, that the utterance indicated. Then the responses (not the students) were grouped according to the indicated level of understanding. Once the data was organized in this manner, the analysis was further refined to determine whether the preliminary genetic decomposition proposed in Section 3.3 is a reasonable model for how an individual might construct the concept of parametric function. Arnon et al. (2014) describe a comparable procedure for data analysis, using a two-column format instead of color-coding.
Chapter 4

DATA ANALYSIS AND RESULTS

This chapter reports the results of the investigation into second-semester calculus students’ conceptions of parametric functions. The presentation of the results is organized around four main sections, each addressing one of the research questions posed in Section 1.2. Section 4.1 reports on students’ personal definitions of the parametric function concept (research question 1). Section 4.2 reports on how students reason about real-valued functions given in the form \( y = f(x) \) and parametric functions given in the form \( p(t) = (x(t), y(t)) \) (research question 2). Section 4.3 reports on how students reason about parametric functions on certain tasks involving the concept (research question 3). And Section 4.4 reports on how students reason about the invariant relationship between two quantities varying simultaneously with respect to a third quantity (research question 4).

To address the research questions, students’ responses to relevant questions from the final exam and the interview were collected as data and analyzed according to APOS theory, as described in Section 1.3. (See Appendices B and C, respectively, for the final exam questions and the interview questions.) Section 4.1 reports on students’ written responses to question 3 from the final exam. Section 4.2 reports on students’ written responses and verbal explanations to question 1 from the interview. Section 4.3 reports on students’ written responses to questions 1 and 2 from the final exam, as well as their written responses and verbal explanations to question 3 from the interview. Section 4.4 reports on students’ written responses and verbal explanations to question 2 from the interview.

It should be mentioned that, in terms of APOS, a student can exhibit different levels of understanding a concept depending on the situation. This was certainly the case in this study. However, the goal was not to classify students, but to determine whether the preliminary genetic decomposition is a reasonable model for how an individual might construct
the concept of parametric function. Therefore, throughout this chapter, when a student is considered to have a particular APOS conception, it should be understood to be in reference to that particular task or situation.

4.1 Students’ personal definitions of parametric function

On question 3 of the final exam, students were asked to explain in their own words what is a parametric equation or parametric function (see Appendix B for details). In this report, it is considered *a priori* that a student with a strong understanding of parametric function should have a personal definition that relates, in some way, to the following *operational definition*\(^1\) of parametric function: At a value of the independent variable (or parameter) \(t\), there is a value of the dependent variable \(x\) and a value of the dependent variable \(y\), which together form a unique quantity \((x,y)\). Based on this operational definition, each student’s definition of parametric function was analyzed according to whether it satisfied the following four criteria:

(i) Identify an independent variable (or parameter) \(t\).

(ii) Identify a dependent variable \(x\).

(iii) Identify a dependent variable \(y\).

(iv) Coordinate \(x\) and \(y\) to form an ordered pair \((x,y)\).

Despite the fact that the parametric function concept was defined in class as a function that accepts a value of \(t\) as an input and returns an ordered pair \((x,y)\) as an output, students rarely formulated their definition explicitly in such terms. Nevertheless, it was possible to relate aspects of students’ definitions to at least some of the four criteria of the above operational definition. Based on the analysis, students’ definitions were grouped into three categories: pre-action, action, and process.

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\(^1\)Since the focus of this study was on single-variable calculus students’ understanding of parametric functions, characterizations are limited to two dimensions.
If a student possesses a personal definition of parametric function that satisfies criteria (i)–(iii) of the above operational definition, then he or she is considered to have at least a foundational understanding of the parametric function concept, even if his or her definition does not meet criterion (iv). On the other hand, if a student’s definition does not satisfy one or more of criteria (i)–(iii), then the student has little understanding of the parametric function concept. Such definitions are considered to be at the pre-action level of understanding of parametric function. Out of fifteen, two students did not give a response, and four students gave a response that was categorized as pre-action.

The following definition by Griffin is an example of a pre-action level definition:

A function that takes into account time as a variable.

Griffin’s definition could possibly refer to a dependent variable as a function of time, but it is too vague. At most, his definition satisfies criteria (i) and (ii) of the operational definition of parametric function. Since there is no indication that he was thinking of two variables \( x \) and \( y \) that are depending on a parameter, say \( t \), that should be further coordinated to form an ordered pair, his definition was classified to be at the pre-action level.

Peggy also gave a definition that satisfied at most criteria (i) and (ii) and, thus, was classified as pre-action. She said wrote:

A 2-dimensional function with respect to \( t \) and \( x \).

It is not clear what Peggy meant by her definition. She could be saying that \( y \) is a function of both \( t \) and \( x \), but this is not necessarily true, since there is no requirement that \( y \) be a function of \( x \) on a curve defined parametrically. However, she does indicate that \textit{something} is a function with respect to \( t \). Therefore, her definition was considered to satisfy criteria (i) and (ii).

Some definitions appeared to have some merit on the surface, but a closer analysis found them to be pre-action, such as the following definition by Bailey:

A parametric equation is a representation of position vs time express[ed] on a radial graph.
The issue with Bailey’s definition is that it is not clear whether he meant “position” to be a one- or two-dimensional value. The textbook used for the calculus sequence at the university where this study was conducted initially introduces particle movement in the context of straight-line motion:

On the line of motion we choose a point of reference, a positive direction, a negative direction, and a unit distance. [...] There is no loss of generality in taking the line of motion as the \( x \)-axis. (Salas et al., 2007, p. 209)

With this in mind, Bailey’s definition is no more general than the one-dimensional case described above, which satisfies only criteria (i) and (ii) of the operational definition. Moreover, it is not clear what he meant by “radial graph.”

Unlike Griffin and Bailey’s definitions, which were categorized as pre-action because they did not satisfy criterion (iii) of the operational definition, Cindi’s definition was categorized as pre-action because it did not satisfy criterion (i):

A parametric equation is a representation of the path of a particle which forms a curve and can be graphed.

Cindi’s statement, “path of a particle which forms a curve,” seems to refer to two-dimensional motion, possibly satisfying criteria (ii) and (iii) of the operational definition. However, Cindi’s definition does not explain how the path of the particle relates to a parameter \( t \).

If a student’s definition, in some way, satisfies criteria (i)–(iii) or (i)–(iv) of the operational definition of parametric function, then the definition was further analyzed to determine whether it qualifies as an action- or process-level definition. A definition was considered to be action-level if it expressed a reliance on superficial properties of parametric equations without expressing how parametric equations contribute to a more general process of transforming a value of \( t \) into a value of \( x \) and \( y \). Note that a definition would also be considered action-level if it relied on a particular example. However, none of the students in this study gave such a definition.
Only one student, Kevin, gave a parametric function or equation definition that appeared to be action-level:

When \(x \& y\) are with respect to \(t\).

Kevin’s definition satisfies criteria (i)–(iii) of the operational definition by identifying variables \(x, y,\) and \(t\). However, it does not clearly utilize the function concept. By stating, “with respect to \(t\),” it is possible that Kevin’s definition is in reference to a superficial property of parametric equations—the variables \(x\) and \(y\) are equal to expressions involving \(t\). Therefore, with this in mind, Kevin’s definition was interpreted as indicating an action conception of parametric function.

A student’s definition was categorized as process-level if, in addition to satisfying criteria (i)–(iii), it described some sort of general process of \(x\) and \(y\) being determined by \(t\). Seven students gave definitions that were categorized as process-level, two of which satisfied only criteria (i)–(iii). For example, Alex defined a parametric function as follows:

Parametric function is a function in which \(x\) and \(y\) [are] dependent on the parameter \(t\), which is time.

Alex’s definition states that \(x\) and \(y\) are dependent on \(t\), satisfying criteria (i)–(iii) of the operational definition. Unlike Kevin’s “with respect to \(t\)” definition above, which suggests \(x\) and \(y\) are represented in terms of \(t\), Alex’s “dependency” definition suggests functional relationships between \(t\) and \(x\) and between \(t\) and \(y\) that are external to their representation. It is not surprising that Alex’s interpretation of the parameter \(t\) is limited to time, as that was the application of \(t\) that was frequently emphasized in class. Viewing \(t\) as time does not make his definition any less of a process-level definition, since it still describes a process of an application of parametric functions.

Out of the seven students whose parametric function definitions were categorized as process-level, five gave a definition that satisfied all of criteria (i)–(iv) of the operational definition. Ron, for example, gave the following definition of parametric equation:
A parametric equation is one in which one measures not only the \((x, y)\) coordinate points, but more importantly the specific time that the given graph passes through those points. It displays the time as each equation passes through its natural destination.

Ron’s definition refers in general to values of \(x\), \(y\), and \(t\), satisfying criteria (i)–(iii) of the operational definition. Furthermore, it also coordinates the values of \(x\) and \(y\) to make an ordered pair \((x, y)\), thereby satisfying criterion (iv). A process conception is indicated by the use of the phrase, “passes through,” which suggests that Ron has a dynamic perception of how an \(xy\)-curve is generated by a parametric equation.

Of the five students who gave definitions that satisfied all of criteria (i)–(iv), only one student, Lee, gave a close to formal definition of parametric function:

It [is] a new type of function where you have one input = \(t\) and your one output is an ordered pair or a set of values, i.e. \((x(t), y(t))\) or \((v(t), h(t))\). It is determining the position, height, or volume of something based on the value of an implicit variable [sic].

Lee’s definition is an explicit generalization of the definition of function. Her definition emphasizes that the output is an ordered pair instead of a single number. Furthermore, her definition is indicative of a process conception because the input, transformation, and output were general (Breidenbach et al., 1992).

One student, Whitney, gave a potentially correct response that did not fit the criteria used to discern students’ understanding. She wrote:

A parametric equation is a function in respect to time. It’s more 3-dimensional, while a regular \(y\) and \(x\) function will only be a 2-dimensional graph.

Whitney’s definition appears to satisfy criteria (i)–(iii) of the operational definition. When she states, “it’s more 3-dimensional,” she might be referring to a space curve. This is consistent with Euler’s use of parametric equations in Book II of his 1748 Introductio
Table 4.1 Students’ personal parametric function definition.

<table>
<thead>
<tr>
<th>Subject</th>
<th>Definition criteria</th>
<th>APOS level</th>
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<tbody>
<tr>
<td>Alex</td>
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<td>Process</td>
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<tr>
<td>Bailey</td>
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<td>Pre-action</td>
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<td>Cindi</td>
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<td>Griffin</td>
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<td>Hannah</td>
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<td>Action</td>
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<td>Lee</td>
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<td>Process</td>
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<td>Mary</td>
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<td>Nicole</td>
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<td>Peggy</td>
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<td>Ron</td>
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<tr>
<td>Sam</td>
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<td>Process</td>
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<tr>
<td>Whitney</td>
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<td>3-D</td>
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in Analysin Infinitorium. What is interesting is that space curves were never discussed in class, as the topic is reserved for Multivariable Calculus. Despite this fact, it seems that Whitney developed her own understanding of parametric function as a space curve. Although Whitney’s definition seems to be higher than a pre-action definition by satisfying criteria (i)–(iii), it is not clear how to classify her definition in terms of APOS. Therefore, it is categorized simply as 3-dimensional.

In summary, excluding Whitney’s definition, four out of twelve responses were considered to be pre-action definitions, one was considered to be an action-level definition, and seven were considered to be process-level definitions. Table 4.1 summarizes the results of this section according to the criteria of the operational definition satisfied by each students’ personal definition of parametric function, as well as its corresponding APOS classification.
4.2 Students’ reasoning about real-valued functions and parametric functions given in the form \( y = f(x) \) and \( p(t) = (f(t), g(t)) \)

Step 6 of the preliminary genetic decomposition deals with the generalization of the function schema to account for parametric functions (see Section 3.3). In particular, Step 6 said to “generalize the function schema to apply to parametric functions by encapsulating the processes in Step 1 to pair \( f \) with \( g \). Then coordinate the function schema with the \( \mathbb{R}^2 \) schema to conceptualize \((f, g)\) as one function that transforms \( t \) into a unique element \((f(t), g(t))\) of \( \mathbb{R}^2 \).”

To determine if Step 6 is a reasonable model for how the function concept can be generalized to parametric functions, this study investigated students’ reasoning about real-valued functions, particularly those given in the form \( y = f(x) \), and their reasoning about parametric functions given in the form \( p(t) = (f(t), g(t)) \). This section reports on the results of the analysis of students’ responses to question 1 of the interview (see Appendix C for details). In particular, Section 4.2.1 reports on students’ responses to questions 1a–b, and Section 4.2.2 reports on students’ responses to question 1c.

4.2.1 Students’ reasoning about real-valued functions given in the form \( y = f(x) \)

On questions 1a–b of the interview, students were asked to determine whether \( \rho(\theta) = \theta \) and \( g(x) = \pm \sqrt{4 - x^2} \) represent functions (see Appendix C for details). The purpose of this was to establish a baseline, so to speak, for how the students in this study reason about functions in general, particularly real-valued functions given in the form \( y = f(x) \). At the beginning of the interviews, the students were asked to state the definition of function that they used to answer question 1. Out of fifteen students, eleven gave a definition of function in terms of the uniqueness property, correctly stating that for each input there is only one output; three students gave a definition in terms of the vertical line test; and one gave a completely incorrect definition.
This study found that the majority of the students were able to reason correctly about real-valued functions given in the form $y = f(x)$, at least in the examples on the written instrument for the interview. On question 1a, thirteen out of fifteen students affirmed with correct reasoning that $\rho$ is a function. On question 1b, twelve out of fifteen students rejected $g$ as a function with correct reasoning, although one of these twelve students required prompting from the interviewer. Section 4.2.1.1 reports on students’ responses to question 1a of the interview, and Section 4.2.1.2 reports on students’ responses to question 1b.

### 4.2.1.1 Interview question 1a

Question 1a of the interview asked students to determine whether $\rho(\theta) = \theta$ represents a function. Out of fifteen students, thirteen affirmed with correct reasoning that $\rho$ is a function. Meanwhile, one student affirmed $\rho$ as a function but gave incorrect reasoning, and one student rejected $\rho$ as a function.

Frank was the only student in this study to reject $\rho$ as a function. His incorrect response was due to a weak conception of function, as illustrated in the following excerpt:

I: What definition of function did you use to answer these questions?

F: Um I used the definition of function um that usually it’s like um more of a equation, like more of a number, and there was no equation with more than one number. It’s just a symbol for whatever. It could be a variable or a constant.

I: Did you say part (a) was or was not a function?

F: I said it wasn’t ... Um because it didn’t fit my definition. Pretty much it was no, just like it’s not a function ... It could be anything. It didn’t have a direct answer. It was just $\theta$. So I said no.

Frank’s definition of function suggests that his understanding of the concept is limited to an expression with constants and variables. According to Breidenbach et al. (1992), this is an indication of a pre-action conception of function. When Frank stated, “it didn’t have a direct answer” and “it was just $\theta$,” he appeared to think that functions must involve
performing operations on numbers, and since the output is same as the input in $\rho(\theta) = \theta$, Frank concluded that $\rho$ cannot represent a function. This seems to indicate that, for Frank, only complicated expressions can represent functions.

One student, Sam, affirmed that $\rho$ is a function, but he gave incorrect reasoning. The following excerpt contains Sam’s function definition and his reasoning for affirming $\rho$ as a function.

S: Ah, the definition I was using was just, looking at a graph, is it a one-to-one function.
I: Okay. By one-to-one, what do you mean?
S: I think, technically, I mean onto, but more graphically, just does it pass the vertical line test.
I: Okay. Okay. And so your answer?
S: So for $\rho(\theta) = \theta$, I’m pretty sure it’s a function because I believe on the uh polar coordinate system it’s just a straight line.

In the above excerpt, Sam expressed an incoherent schema for function. He struggled to give a correct definition of function and, consequently, resorted to stating vertical line test as his definition. However, Sam’s reasoning about $\rho$ did not clearly indicate that he used the vertical line test to make his conclusion. In fact, if Sam did apply the vertical line test directly, it would have been incorrect, since he was thinking in terms of the polar coordinate system. Moreover, he incorrectly believed that the graph of $\rho$ is a line in the polar coordinate system.

Unlike Sam, two students appeared to use the vertical line test correctly when reasoning about the graph of $\rho$ in the rectangular coordinate system. This, however, was not necessarily indicative of a strong conception of function. For example, Lee appeared to rely on the vertical line test in her reasoning about $\rho$, but she was not able to give a correct interpretation of it in terms of the definition of function, which is indicative of an action conception of function.

When the interviewer asked Lee what definition of function she used to answer question
1, she stated:

L: Um, I don’t really have one. I just put whether or not how I think it might look depending on what graph, what coordinate system it’s on.
I: Okay. So you’re thinking of a function more or less in terms of its picture?
L: Yeah.
I: Okay.

Although Lee explained that she thinks of an expression’s graph in order to determine whether or not it represents a function, she did not clearly explain how she uses the graph to make her conclusion. In the following excerpt, however, she appears was relying on memorized facts:

L: Um I realized it was $\rho(\theta)$ for (a). So I thought about him (the instructor) saying that for that, if ... I thought, when I did that, I was like, I pictured the function $y(x) = x$, and I think for that curve, it’s actually a spiral because something that’s linear in the rectangular function is actually a spiral in the curved one. So I was like, okay. I put yes that it’s a function because I just feel like if it’s a function in the rectangular one, then it’ll be a function in the curvy one. I can’t think of it.
I: It’s polar.
L: In polar coordinates.

Lee explained that the notation used in the equation $\rho(\theta) = \theta$ prompted her to think of its polar graph. When she said, “I thought about him saying ...,” it seems that she was remembering that the polar graph of $\rho(\theta) = \theta$ is in the shape of a spiral based on a class discussion with her instructor. This is a clear indication of an action conception of function. Lee acknowledged that $\rho$ is a function regardless of the coordinate system in which it graphed by saying, “I just feel like if it’s a function in the rectangular one, then it’ll be a function in the curvy (polar) one.” However, she did not give an indication that she knew why. Lee’s weak understanding was confirmed later, as illustrated in the following
The context of the excerpt is Lee’s discussion of question 1b and how the notation in the equation \( g(x) = \pm \sqrt{4 - x^2} \) prompted her think in terms of the rectangular coordinate system.

L: I don’t know much about polar, but when he (the instructor) was teaching it to us, we didn’t have \( g(x) \) as our function. He actually introduced, that’s when he introduced \( \rho(\theta) \). I don’t know how that goes. It’s just my first time learning it. So I’m assuming \( g(x) \) means rectangular coordinate plane. And in rectangular coordinate plane, that’s when the vertical line test comes into play, or whichever test, where you can get one \( x \) for every \( y \). And with \( \rho(\theta) \), I still don’t know nothing about polar coordinate system that much, but I assume it might be, they might let it slide in polar coordinate system.

I: They might let what slide?

L: More than one output, unless it’s with time, because if it’s time, then no, you can’t have more than one output.

I: Okay.

Despite being aware that the vertical line test can only be used in the Cartesian coordinate system, Lee’s response suggested that she did not know how the vertical line test is related to the definition of function. In fact, she misinterpreted the vertical line test to mean, “you can get one \( x \) for every \( y \).” When she referred to the polar coordinate system and said, “they might let it slide … more than one output,” she seemed to think that the definition of function changes depending on the coordinate system.

Throughout her discussion of \( \rho(\theta) = \theta \), Lee consistently indicated having an action conception of function by her inability to state a coherent function definition other than one that refers to the vertical line test or by remembering what the instructor stated in class. While students with a process conception of function may use the vertical line test in their mathematical discourse, they should be able to explain how it relates to the uniqueness property of the function definition. This was not the case with Lee. This, however, did not appear to negatively affect Lee’s ability to identify \( \rho \) as a function. On the contrary, she
showed that she was capable of coordinating several memorized facts and can apply them to obtain a correct answer.

The remaining students correctly used the uniqueness property of the function definition when determining whether $\rho$ is a function. Furthermore, their reasoning indicated a process conception of function. For example, Mary gave her definition of function by stating, “I used that every input has only output, one and only one output.” When explaining her reasoning for affirming $\rho$ as a function, Mary said:

M: Because for every $\theta$, I mean, if it’s $\rho(\theta) = \theta$, then it’s going to equal to itself.

So you’re going to have only one output for each input.

Mary’s response indicates that she has a process conception of function because she can imagine $\rho$ as a process that accepts a value of $\theta$ as an input and returns that value of $\theta$ as an output, without needing to actually plug in numbers.

4.2.1.2 Interview question 1b

Question 1b of the interview asked students to determine whether $g(x) = \pm\sqrt{4-x^2}$ represents a function. Out of fifteen students, twelve rejected $g$ as a function with correct reasoning. One of these twelve required prompting from the interviewer. Two students rejected $g$ as a function but gave incorrect reasoning. One student incorrectly affirmed $g$ as a function.

At the beginning of the interview, Ron gave a correct definition of function. He said, “okay so a function being for every input there is only one, only one output.” However, he did not apply this definition in question 1b. As a consequence, he incorrectly determined that $g$ is a function.

R: And then (b) um I saw the same thing because once you solve down, it’s the same thing. You have two different systems of square roots, but once you solve down for $x^2$ you’ll still only get one function for every uh $x$.

When Ron said, “you have two different systems of square roots,” he seems to describe distributing the radical symbol to each of the terms in $4 - x^2$. It should be mentioned
that Ron was the first student in his interview to state his definition of function and to
give a response to question 1a, which was correct. Despite this, Ron’s response to question
1b suggests that he cannot reason about more complicated functions. Therefore, he was
considered to have no higher than an action conception of function.

Recall from the previous section that Frank gave an incorrect definition of function,
which caused him to incorrectly determine that $\rho(\theta) = \theta$ in question 1a does not represent
a function. During his discussion of question 1b, he said he remembered the definition of
function, after hearing the response of another student.

F: Now that I think about it, now that she said it, I actually remember the
definition of a function that the, uh, professor gave us.
I: Is part (b) a function? Yes or no? And how does your definition relate?
F: Well, my definition does not relate to any of these now. Now that I think
about it, since it’s like, you said a function would give you just one input, uh,
output rather, um, this is a plus or minus problem. It would give you two
inputs. So it couldn’t be a function.

Frank attempted to answer question 1b based on his memory of the function definition.
He determined that $g$ is not a function, but his reasoning was not correct. He said, “it would
give you two inputs.” Twice in this excerpt, Frank confused the words “input” and “output”
which indicates that he is still at the pre-action level for function.

Griffin gave a correct definition of function by stating, “uh, a function is, every input
only has one output,” but he appeared to confuse his definition with the definition of one-to-one
in his discussion of $g(x) = \pm \sqrt{4 - x^2}$ in question 1b. The following excerpt occurs right
after another student explains how the notation “$g(x)$” made him think of the rectangular
coordinate system instead of the polar coordinate system.

G: I think also because since it was $x$, it kind of made me think of rectangular
instead of . . . I’m not sure what it would look like on a polar graph. With
that as being rectangular, then I viewed it as not being a function cause I
think since its squared that you would um there would be two . . . You can
get the same input . . . I mean you can get the same output for inputs.

I: Mmhm. So is that what makes it a function or not a function, if you can get the same output for different inputs?

G: I look at it as the vertical line test. So on a rectangular graph . . . So if you put in one um . . . if there is more than one answer for an x, then it’s not a function.

I: Okay. So if like x = 1, what do you get for (b)?

G: For (b), we get . . . plus or minus √3. And then if you put um negative, well, if you put −1 in there too, then you would get the same thing.

In the first sentence of the above excerpt, Griffin explained that the notation “g(x)” made him think in terms of the rectangular coordinate system. The rest of his response indicates that he concluded g is not a function, because of the possibility of getting the same values of g(x) for different values of x. This appears to concern him more than the multiple values of g(x) for one value of x. This is in spite of his accurate function definition and his correct interpretation of the vertical line test. According to Dubinsky and Harel (1992), confusion between the definition of function and definition of one-to-one can only be resolved with a process conception of function (see also Breidenbach et al., 1992).

One student, Whitney, required prompting on question 1b before answering correctly. Her definition of function was, “I think that if you have, um, if you plug in an input, it’ll give you one output.” The following excerpt contains her reasoning about g(x) = ±√4 − x^2.

I: Did you think it was a function? Yes or no?

W: I think it was . . . I feel like if I plug in any number, it will also give me an answer.

I: Okay. So what are you plugging into? x?

W: Yeah. If a plug a number into x, it would give me an output.

I: And how many outputs will you get for each x you plug in?

W: So, that would be two. It doesn’t represent a function [laughs].

Despite stating a correct definition of function, Whitney initially expressed that g(x) =
\[ \pm \sqrt{4 - x^2} \] represents a function because it would “give her an answer” if she plugged in a number for \( x \). It was not until the interviewer prompted her to consider the number of outputs that she realized that \( g \) is not a function. This is suggestive of an action conception of function.

The remaining eleven students correctly rejected \( g \) as a function and gave a response that indicated a process conception of function. For example, Hannah gave her definition of function by stating, “um, that each input of a function has one and only one output. So there should be only one value resulting from each input.” She reasoned that \( g \) is not a function by stating:

\[
H: I \ said \ (b) \ was \ not \ a \ function \ because \ for \ every \ x \ there \ would \ be \ both \ a \\
positive \ and \ a \ negative \ answer, \ and \ so \ that’s \ two \ outputs, \ which \ means \ it’s \\
not \ a \ function.
\]

Hannah’s definition of function and response to question 1b suggest a process conception of function because she was able to describe the getting “both a positive and a negative answer” for every \( x \) without needing to actually perform the calculations.

This section reported on students’ reasoning about real-valued functions given in the form \( y = f(x) \). The purpose of this section was to provide at least a rudimentary sense for how the students in this study reason about functions in general. The following section reports on how students’ reason about parametric functions given in the form \( p(t) = (f(t), g(t)) \).

4.2.2 Students’ reasoning about parametric functions given in the form \( p(t) = (x(t), y(t)) \)

On question 1c of the interview, students were asked to determine whether \( p(t) = (t^2, t^3) \) represents a function (see Appendix C for details). A correct response would affirm \( p \) as a function because for one value of \( t \) there is a unique ordered pair. Moreover, additional follow-up questions were asked by the interviewer to gain as much insight as possible into
students’ thinking and reasoning when answering this question (see Appendix D for the interview protocol).

Out of fifteen, three students affirmed that \( p(t) = (t^2, t^3) \) represents a function and gave correct justification, two students gave an affirmative answer with correct justification as a result of prompting from the interviewer, and ten students gave a negative answer or an affirmative answer with incorrect or unclear justification (even with prompting).

Mary is the most descriptive example of a student who without prompting gave an affirmative answer to interview question 1c with correct reasoning. She said:

M: Well, if you’re thinking of like the point, then you’re only going to have one point. So that’s why I said it was a function, because for each \( t \) like there’s only one point that it’s going to come out with.

Mary further explained:

M: You get a \( x \) output and a \( y \) output, but then I thought about it, and then it’s like a coordinate on a plane. So it’s like for every \( t \) that you plug in, you get a point on the plane. So it eventually creates a graph. So that’s kinda why I thought it’s a function.

Mary’s reasoning seems to support the constructions called for by the preliminary genetic decomposition. Her schema for \( \mathbb{R}^2 \) is apparent when she described how \( p(t) = (t^2, t^3) \) constructs a point in the plane: “You get a \( x \) output and a \( y \) output . . . and then it’s like a coordinate on a plane. So it’s like for every \( t \) that you plug in, you get a point on the plane.” When she said, “so it eventually creates a graph,” she is treating a point as an object and is imagining the process of the point tracing a curve in the plane. She concluded that \( p \) is a function by coordinating her schema for \( \mathbb{R}^2 \) with her schema for function: “So that’s why I said it was a function, because for each \( t \) like there’s only one point that it’s going to come out with.” Because Mary did not need to evaluate \( p(t) \) at particular values of \( t \) or plot points, she is considered to be coordinating her schemas for function and \( \mathbb{R}^2 \) at the process level.
Without prompting, Cindi was also considered to reason correctly in affirming $p$ as a function. The following excerpt contains her reasoning.

C: And the third one, I think it’s a function, again, because one value of $t$, when you square and cube a number, there is only one value you get.

Int: Okay. So, how many outputs to you get for each $t$ for part (c)?

Cindi: One.

Int.: One. Okay. What is the output? What do you call this?

Cindi: Uh.

Int.: Is it a number?

Cindi: A coordinate.

Cindi’s response suggests that she concluded that $p$ is a function because each of its components are single-valued. Although she said that the output is a coordinate, it seems that she meant that it is an ordered pair, since she distinguished it from being a number.

Frank was the third student who correctly affirmed $p$ as a function without prompting. The following excerpt contains his reasoning.

F: Um, I was just assuming that since it’s a coordinate, a coordinate by itself is pretty much a function, so like, since it’s like $x$ and $y$, part of the whole. So that’s one whole output.

I: What is your input for part (c)?

F: My input? Uh, whatever $t$ is.

I: Whatever $t$ is. So then when you pick a $t$, what is your output? What do you call that all together?

F: The output pretty much.

I: What do you call this? Or if I have a list of them?

F: Ordered pair.

I: Ordered pairs. So for each $t$, you’re going to get an . . .

F: Ordered pair.
In above excerpt, Frank acknowledged without prompting that \( p \) is a function. Moreover, he expressed that \( t \) is the input of the function \( p \). Although he needed a little prompting from the interviewer to say that the output is on ordered pair, he already expressed that \( x \) and \( y \) are “part of the whole. So that’s one whole output.” Therefore, he was considered to have affirmed \( p \) as a function without prompting from the interviewer.

Mary, Cindi, and Frank were the only three students who affirmed \( p \) as a function without prompting from the interviewer. However, two more students were able to correctly affirm \( p \) as a function as a result of prompting. The following section reports on the initial misconceptions of these two students, as well as the misconceptions of the remaining ten students who reasoned incorrectly about \( p \).

### 4.2.2.1 Misconceptions about parametric functions

Several misconceptions about parametric functions emerged when students were reasoning about the function given by \( p(t) = (t^2, t^3) \). These misconceptions fell into four categories: (1) misconceptions about the function value \( p(t) \), (2) misconceptions about the domain and range of \( p \), (3) misconceptions about the vertical line test, and (4) misconceptions about the input and output of \( p \). Examples of each misconception are presented below.

#### Misconception 1: Function value

The most common misconception about parametric functions observed among students in this study pertained to the function value \( p(t) \). In fact, this misconception appeared in a couple of different ways. One way, which was anticipated, pertained to the uniqueness of the function value \( p(t) \). Eight students perceived the function value as not unique because evaluating \( p(t) \) results in a value for the first component, \( t^2 \), and a value for the second component, \( t^3 \), which were viewed as two outputs instead of one output in the form of an ordered pair. Other students were not bothered by the possibility of getting different values for the components. In fact, this was their reasoning for affirming \( p \) as a function. In particular, two students believed that the values of the components had to be different in order for \( p \) to be a function.
Peggy is an example of a student who perceived the function value $p(t)$ as not unique. The following excerpt contains her reasoning for rejecting $p$ as a function:

P: And for (c), I put no because when you have um for parametric equations, you have the $x(t)$ and the $y(t)$. So I thought those are two separate equations, but then when you combine them in one, you only have the $x$. So you’ll still only have one output, and you’re losing the orientation of time when you combine them into one.

I: So what you’re talking about now is actually looking at whether $y$ is a function of $x$? Is that what you’re saying?

P: Mmhm, because since you have a $t^2$ and $t^3$. That’s two different outputs and, even, I was thinking that a parametric equation, yes, you’ll have two different outputs for $t$, but they’re in two separate functions or equations. That’s why I put no.

Peggy’s response suggests that she rejected $p$ as a function because $x$ and $y$ are in separate expressions. She explained that she converted (mentally) from vector-form, $p(t) = (t^2, t^3)$, to parametric form, $x(t) = t^2, y(t) = t^3$. Then she described the procedure of converting from parametric form to standard form so that “you’ll only have one output.” This implies that, from her perspective, $p(t) = (t^2, t^3)$ needs to “lose the orientation of time” and be expressed by a single equation in two variables in order for it to represent a function. Since $p(t) = (t^2, t^3)$ is not presented in standard form, Peggy concluded that it does not represent a function. Her reasoning was, “you have a $t^2$ and $t^3$. That’s two different outputs.” Perceiving $t^2$ and $t^3$ as two outputs suggests that Peggy did not coordinate her function schema with a schema for $\mathbb{R}^2$. In fact, her response suggests that she did not apply a schema for $\mathbb{R}^2$ at all when she was reasoning about this problem.

Griffin is an example of a student who expressed the misconception about the distinction of the two coordinates. The following excerpt contains his reasoning for affirming $p$ as a function.
G: I think it’s a function because it came out with as an ordered pair, and so it’s not really an equation. So, it’s kinda telling you whichever \( t \) you put in, then that’s, like, you use that \( t \), but you just square it to get \( x \), and then you just, um, cube it to get \( y \). So, therefore, you would never get the same answer. For every different \( t \), you would get different answers.

Griffin’s response indicates that he was able to coordinate the processes of \( t^2 \) and \( t^3 \) to imagine constructing an ordered pair. However, he seemed to believe that it is important for the value of the first coordinate to be different from the value of the second coordinate. Another possible interpretation is that Griffin requires that each ordered pair be distinct, i.e., no output is repeated. This is the same as requiring a function to be one-to-one, which is not correct. Although it is not clear which of these interpretations apply to Griffin’s reasoning, he clearly demonstrated a weak schema for function.

In the following excerpt, Lee and Bailey express the misconception about the uniqueness of \( p(t) \), while Nicole expresses the misconception about the distinction of the coordinates.

L: I put no for (c) because I remember him (the instructor) saying for time you have to have one output for every input. So it looks like for this one you have two outputs for time, which I thought made no sense.

N: I put yes because, like, other than 1 and 0, the other numbers they were different points, and I thought pretty much that they were just points.

I: Okay.

B: I put no simply because one \( t \) input and then there are two outputs.

Although Nicole affirmed \( p \) as a function, her reasoning was not correct. When she said, “other than 1 and 0, the other numbers were different points,” she was referring to the fact that \( x \) and \( y \) have different values at all values of \( t \) except 0 and 1. This implies that she viewed \( x(t) \) and \( y(t) \) as “different points” rather than viewing \((x(t), y(t))\) as one output. Furthermore, this was her justification for an affirmative answer to question 1c, indicating that she required all (or most) of the ordered pairs to have different coordinates in order for
$p$ to be a function. Lee and Bailey, on the other hand, rejected $p$ as a function, claiming that an input value $t$ gives two output values. Furthermore, nowhere in their reasoning did they consider $t^2$ and $t^3$ as components of an ordered pair, indicating the lack of coordination of schema for function and a schema for $\mathbb{R}^2$.

As a result of prompting, two of the eight students who expressed a misconception about the uniqueness of the function value $p(t)$ were able to overcome their misconception and affirm $p$ as a function with correct reasoning. Lee is an example of a student who overcame this misconception, and Bailey is an example who did not, as illustrated in the next two excerpts.

After Lee, Nicole, and Bailey’s initial elaboration about their reasoning when answering this question, the interviewer prompted them to think about the graph of the curve defined by $p$. Note that in the following excerpt, when Nicole described plugging in values for $t$ to obtain points, Lee and Bailey reconsidered their earlier answers that rejected $p$ as a function.

I: How would you go about graphing it?
N: I have no idea \[laughs\].
I: You have no idea? But you’re the only one who said it was a function \[laughs\].
N: I mean like \ldots I just put numbers in \ldots like if it was, you have (0,0), then (1,1). If you have 2, it would be 4 and 8, something like that \ldots 
L: Oh, it is a function. If you plot the points, it’s just saying, like, let’s say we choose $t$ to be 1, then the coordinates would (1,1). Then we choose $t$ to be 2. Then then coordinates would be (4,8). And keep going like that \ldots I didn’t even look at it to plot the points. I just saw that there were three $t$s \[laughs\].
I: Remind us again what your definition of a function was.
B: One input, one output.
I: If you take a single value of $t$ \ldots Let’s take 2 for you. You take 5. And $t = -1$. Do you get more than one input, I mean, more than output?
B: Yes.
L: You get two numbers, but then you’re getting one output as a ordered pair.
I: As an ordered pair. So for each \( t \) in . . .

L: There’s one ordered pair out.

I: [Speaking to Bailey] Does that satisfy your definition of a function?

B: Yes.

Lee explained that she rejected \( p \) as a function because she “saw that there were three \( ts \).” This suggests that she was not considering the output to be an ordered pair. However, when Nicole described plugging in values of \( t \) to create ordered pairs, Lee was prompted to coordinate her schemas for function and \( \mathbb{R}^2 \) and assign a unique ordered pair to each value of \( t \). As a result, Lee was able to perceive \( p \) as a function and give the correct justification, “you get two numbers, but then you’re getting one output as a ordered pair.” Furthermore, after Lee described plugging in values of \( t \), she said, “and keep going like that.” This statement indicates that she can imagine evaluating \((t^2, t^3)\) for all values of \( t \), suggestive of a process conception of parametric function.

In the previous excerpt, Bailey also changed his answer to the affirmative, but he did not give any reasoning to suggest that he understood why. In the following excerpt, Bailey changes his answer again, rejecting \( p \) as a function. His further reasoning demonstrated a weak schema for \( \mathbb{R}^2 \). This discussion took place when Bailey was trying to determine if the \( y \)-coordinate is a function of the \( x \)-coordinate in \( p(t) = (t^2, t^3) \).

B: You know what, I don’t think (c) is a function now. I’m going back to no.

I: Going back to no?

B: But this can’t be [pause] this can’t be a point.

I: Why not?

B: The variables are the same.

I: So?

B: I don’t see how it’s an ordered pair.

I: It’s in parentheses with a comma.

B: But it’s the same variable, \( t \). So how can it be . . . it’s at two places at the same . . .
I: No, it’s only at one place. When \( t \) is 0, it’s at the place \((0,0)\). When \( t \) is equal to 1, it’s at the ordered pair \((1,1)\). At that one particular spot.

B: Okay. I don’t see it, because \( t \) is our \( x \)-value \([makes\ \text{hand\ motions\ in\ the\ air}\\ \text{drawing\ a\ horizontal\ line}]\).

I: Why?

B: Because it’s \( p \) of \( t \). So if I put in 2, I get 4 and 8.

I: 4 comma 8. You get the ordered pair.

B: But they’re both \( ts \). So aren’t they both points on the line \( t \)?

I: No.

B: No? Okay.

Bailey’s response suggests that he is unable to coordinate the processes of \( x(t) = t^2 \) and \( y(t) = t^3 \) to construct an ordered pair because he cannot conceive that the components of an ordered pair could be defined by functions. He described plugging in 2 for \( t \) and getting the values 4 and 8, which are “both points on the line \( t \).” From this statement, it seems that when Bailey evaluated \( p(2) \), he constructed two separate ordered pairs, \((2,4)\) and \((2,8)\). Furthermore, referring to \( t \) as a line and making a linear hand motion when he stated, “\( t \) is our \( x \)-value,” suggests that he views \( t \) as the horizontal axis and \( t^2 \) and \( t^3 \) as values on the vertical axis. The idea of viewing \( t \) as belonging to an axis has been offered in literature to imagine \((x,y,z) = (x(t),y(t),t)\) as a point in space (Keene, 2007; Oehrtman et al., 2008). This idea also agrees with Euler’s presentation of non-planar curves in Book II of his \textit{Introductio in Analysin Infinitorum}. However, Bailey would need to first construct a schema for \( \mathbb{R}^3 \) in order to view \( t, t^2, \) and \( t^3 \) as values on mutually perpendicular axes. Such a construction was not called for by the preliminary genetic decomposition.

**Misconception 2: Domain and range**

Another misconception about parametric functions pertained to the domain and range, particularly the misconception that the domain consists of the values of the first component and the range consists of the values of the second component. Two students, Bailey and
Nicole, fell into this category. Nicole had misconceptions about the range of $p$, and Bailey had misconceptions about both the domain and the range of $p$, as illustrated in the following excerpt:

I: What’s the domain of (c)?
L: It starts at 0 because you can’t have negative time.
I: Okay. And I guess $t$ stands for time?
L: That’s what I’m guessing.
I: It’s just a letter.
B: I would say the domain for (c) is 0 to $\infty$.
I: Okay. It can’t be negative?
B: No, because your first is $t^2$.
I: Alright. Is $t^2$ the domain or the range?
L: The range.
N: Couldn’t it just be all real numbers. We’re just assuming it’s time. I mean, she said $t$ is a variable.
L: Well, if we’re going to assume that $t$ is just any variable, yes, the domain can be anything. But I like to assume $t$ is time.
I: What about the output? What does the range look like?
N: Positive and negative. Any negative cubed is still a negative.
B: $-\infty$ to $\infty$.
L: It’s always going to be positive. No matter what, it’s always going to be in the first quadrant.
N: What’s $(-1)^3$?
L: Oh, so we’re assuming that $t$ is negative? … Well, if the domain could be anything, then the graph would be positive and negative.
B: The domain is going to be positive and the range is $-\infty$ to $\infty$.
L: The graph has always got to be in the first and the fourth quadrants because the $y$-position of the point can either be positive and negative, because it’s
cubed, but the $x$-position of the point will always be positive.

B: [Nods.]

When the interviewer asked Bailey if the domain can include negative values, he said, “no, because your first is $t^2$.” This suggests that his understanding of domain is limited to values of $x$. Bailey further confirmed his misconception when he stated, “the domain is going to be positive and the range is $-\infty$ to $\infty$.” The fact that he obtained both negative and positive values for $y$ means he was considering both negative and positive values for $t$. However, his limited notion of domain prevented him from realizing that the values of $t$ make up the domain. Lee, on the other hand, understood that the range was not determined by only the $y$-coordinate. Indeed, when the interviewer asked if $t^2$ was “the domain or range,” Lee responded that it is the range. It seems that the interviewer and Lee meant that $t^2$ is one of the components that determine the range. Furthermore, when the interviewer asked what the range “looks like,” Lee described a graph in terms of quadrants. She said, “the graph has always got to be in the first and the fourth quadrants because the $y$-position of the point can either be positive and negative, because it’s cubed, but the $x$-position of the point will always be positive.” However, Nicole never clearly described a graph. Instead she only referenced values of the $y$-coordinate. Based on this it seems that she, like Bailey, had a misconception about the range.

**Misconception 3: Vertical line test**

A misconception about the vertical line test also emerged in students’ reasoning about parametric functions. In particular, two students cited the vertical line test in their reasoning for not thinking $p(t) = (t^2, t^3)$ represents a function.

For example, Alex explained, “and for the third one, um I thought of how the graph would look like and figured out by the vertical line test that it would be no.” The following discussion takes place after the interviewer tried to guide him to think about the possibility of a function having an output as an ordered pair. However, the vertical line test proved to be a significant obstacle.
I: What do you think?
A: Give me one second.
I: Yeah. Take as much as you want.
A: [Pause.]
I: Just check your definition.
A: I don’t think so.
I: So, for one input value you need to get one output value. That’s what you said that you used.
A: Right.
I: So how many output values do you have for each \( t \) that you input?
A: Um one.
I: One. And that output value would be that set of ordered pairs, right? The ordered pair.
A: Right.
I: Okay. So do you see that now or no?
A: Maybe the mistake I was doing, I was just thinking about the ordered pairs [writes a table of values].
I: So what is happening with your coordinates? For every \( t \) you have . . .
A: But then the graph that we are plotting, isn’t that the graph of \( t^2 \) versus \( t^3 \)?
I: Right.
A: So we’ll have \( t^2 \) and \( t^3 \), right? So then for each value of \( t^2 \), there is more than one . . . when \( t^2 \) is 1, \( t^3 \) can be 1 and \(-1\) . . . The graph of this, what I thought, I mean, I can be wrong, but I thought the graph looks something like this for \( t^2 \) versus \( t^3 \) [draws correct graph on paper (see Figure 4.1)].

By coordinating the processes of two functions, Alex was able to construct a table of values for \( t, x, \) and \( y \), plot points, and graph the curve defined by \( p \) (see Figure 4.1). However, he was not able to view these points as outputs of a function. According to APOS, a conception of function involves the action or process of transforming an object (the input)
into a unique object (the output). From this perspective, it is likely that Alex had not
encapsulated the process of constructing \((t^2, t^3)\) into an object to be viewed as the output
of the function \(p\). This conjecture is further supported by Alex’s reasoning about what his
graph represents. Without an object conception of \((t^2, t^3)\), Alex misinterpreted the question
in problem 1c. He said, “I thought the graph looks something like this for \(t^2\) versus \(t^3\).”
That is, instead of determining if the points on the curve are functions of \(t\), he determined
if the \(y\)-coordinate is a function of the \(x\)-coordinate. A similar misconception was reported
by Clement (2001) when precalculus students were asked if the position of a caterpillar on
a path that intersected itself is a function of time. Just as the majority of those students
could not see the curve as representing a function, Alex could not see how the points on the
curve defined by \((t^2, t^3)\) were a function of time. Perhaps this was because the value of \(t\) was
not explicitly present in his graph.

Interestingly enough, Alex was able to plot points from his table of values, draw a curve
through them, and perform the vertical line test on them (see Figure 4.1). This might be

Figure 4.1 Alex’s work for interview question 1c.
suggestive of an object conception of point, but only in a graphical context. That is, Alex does not have an object conception of point when it is represented algebraically. In the prerequisites to the preliminary genetic decomposition in Section 3.3, it was stated that an individual should have a schema for $\mathbb{R}^3$ that includes points as objects. What was not considered was the representation register. Therefore, I conjecture that constructing an object conception of point in both its algebraic and graphical representations might have helped Alex to view $p(t) = (t^2, t^3)$ as representing a function of $t$. Additionally, Alex’s misconception of the vertical line test could possibly be overcome by constructing a schema for $\mathbb{R}^3$, assigning values of $t$ to an axis perpendicular to the $x$- and $y$-axes. Then by coordinating the $\mathbb{R}^3$ schema with the schema for function, the graph of $p$ can be reinterpreted as a space curve containing the points $(x, y, z) = (t^2, t^3, t)$, which would not contradict the vertical line test as it is swept across $\mathbb{R}^3$ perpendicular to the axis corresponding to values of $t$. In fact, the vertical line test could be reinterpreted as the vertical plane test, which might be more “efficient” to use in $\mathbb{R}^3$.

**Misconception 4: Input and output**

The fourth misconception that emerged pertained to the roles of the components. One student, Hannah expressed the misconception that the first component is the input and the second component is the output.

As illustrated in the following excerpt, Hannah initially rejected $p(t) = (t^2, t^3)$ as representing a function, with the reasoning that the function value $p(t)$ is not unique (misconception 1). Then she changed her answer to affirm that $p$ is a function. However, her reasoning was incorrect.

H: Um, I also said (c) was not a function um because for um every input there was, there were two outputs um in the form of an ordered pair.

I: Ooh. Okay . . .

H: I wasn’t sure about (c), but that’s what I reasoned it to be.

I: So what function would (c) be, actually? As (indiscernible).
H: Well, (c) would be uh points, a point.
I: Uh huh. So for every time $t$ you have . . .?
H: $t^2$ and $t^3$.
I: Which represents an ordered pair, right?
H: Right. So I guess it could be a function with a parameter of $t$ [changes answer on paper].
I: And so can you just repeat once again why it would be?
H: Um because if you use $t$ as a parameter, then um then the result would be an ordered pair which would have one input and one output. So the input um based on the parameter would be $t^2$ and the output would be $t^3$.
I: Ah. What would be the input?
H: Well, if you set $t$ as a parameter, then $t$ would be two different functions in order to represent the ordered pair. So the ordered pair itself would be a function and parameterized by $t$.
I: Okay.

Hannah asserted that $t^2$ and $t^3$ are two outputs corresponding to the input $t$. After prompting from the interviewer, she began to overcome this obstacle by coordinating her function schema with her schema for $\mathbb{R}^2$. However, a new obstacle emerged: the first component is the input and the second component is the output. That is, Hannah’s reinterpretation of $p(t) = (t^2, t^3)$ suggests that she no longer considered $t$ as the input. Instead, she interpreted $t^2$ as the input “based on the parameter” and $t^3$ as the output.

4.2.3 Relation between students’ reasoning about real-valued functions and parametric functions

Sections 4.2.1 and 4.2.2 provided an overview of students reasoning about real-valued functions given in the form $y = f(x)$ and parametric functions given in the form $p(t) = (f(t), g(t))$. One of the sub-research questions (see Section 1.2) inquired whether there is a relationship between students’ reasoning about these two types of functions. The results
reported in Section 4.2.1 indicate that most of the students in this study had developed a schema for real-valued function, as evidenced by their responses to questions 1a–b of the interview. However, most students in this study did not generalize the concept of real-valued function to parametric function, evidenced by their incorrect reasoning in their responses to question 1c of the interview.

There are at least two ways of looking at these results. One way is to consider the number of students who responded correctly to all parts of question 1. The analysis of the results found that eleven students responded correctly to both parts (a) and (b) of question 1 (either with or without prompting). Of these eleven students, only three responded correctly to part (c) of question 1 (either with or without prompting). This suggests that generalizing the concept of real-valued function to parametric function is non-trivial, even for students with a previously developed function schema.

Another way to look at these results is to consider the total number of students who responded correctly to part (c) on question 1. The analysis of the results found that, overall, five students responded correctly to part (c) of question 1 with correct reasoning (either with or without prompting). Of these five students, three responded correctly to both parts (a) and (b); one responded correctly to only part (a); and one responded incorrectly to both parts (a) and (b). This indicates one discrepancy that should be mentioned.

The discrepancy was with Frank. He gave responses to questions 1a and 1b that indicated a pre-action conception of real-valued function, with an initial definition that referred to a complicated collection of symbols. However, he correctly reasoned that \( p(t) = (t^2, t^3) \) is a function. This finding was unexpected. It appeared that Frank learned (or remembered) the definition of function from another student during his interview and began to exhibit more of a conception for function. However, there is not enough data to confirm that he developed a deeper understanding of the concept.

Another thing to consider is the conceptual growth that students exhibited during the interview. Lee was a particularly notable student in this study because her conception of the function appeared to grow during her discussion of parametric functions, possibly as a
result of accommodating her function schema. Earlier in the interview, Lee demonstrated a very weak understanding of the function concept (see Section 4.2.1). It was not until her discussion of \( p(t) = (t^2, t^3) \) in question 1c that she began to demonstrate a coherent function schema. With minimal influence from the interviewer or her peers, Lee overcame a misconception about the function value \( p(t) \) and correctly affirmed \( p \) as a function. This was despite her misconception about the range being made up of values of the second component. Furthermore, her reasoning indicated a process conception of parametric function, as she described plugging in particular values for \( t \) to obtain ordered pairs and said “keep going like that” to suggest she can imagine these evaluations continuing without actually performing them. Lee’s knowledge did not seem to vanish after the interview ended. As stated in Section 4.1, on the final exam Lee gave the following definition of parametric function:

It [is] a new type of function where you have one input = \( t \) and your one output is an ordered pair or a set of values, i.e. \( (x(t), y(t)) \) or \( (v(t), h(t)) \). It is determining the position, height, or volume of something based on the value of an implicit variable [sic].

These results suggest that Lee accommodated her function schema during the interview. However, these results cannot confirm that Lee’s conception of function or parametric function lasted beyond the completion of the semester. Nevertheless, they do agree with other researchers’ findings that engaging in constructive activities involving different kinds of functions can promote a deeper understanding of the function concept (Breidenbach et al., 1992; Carlson, 1998).

4.3 Students’ performance on tasks involving parametric functions

This section reports on students’ reasoning about parametric functions on various tasks involving the concept. Section 4.3.1 reports on students’ reasoning about parametric functions when converting from parametric to standard form on the final exam, question 1a. Section 4.3.2 reports on students’ reasoning about parametric functions when sketching the graph
of a curve given parametrically on the final exam, question 1b. Section 4.3.3 reports on students' reasoning when constructing a parametric function to describe a real-world situation on question 3 during the interview.

4.3.1 Converting from parametric to standard form

Given two functions \( f \) and \( g \) such that \( x = f(t) \) and \( y = g(t) \), Step 4 of the preliminary genetic decomposition calls for reversing the process of one function, say \( f \), and composing it with the other function \( g \) to represent the relationship between \( x \) and \( y \) by a single function. This step of the genetic decomposition refers to converting from parametric to standard form.

To determine whether students made the constructions called for by Step 4 of the preliminary genetic decomposition, they were asked on the final exam, question 1a, to convert the following curve from parametric to standard form:

\[
x(t) = \frac{1}{\sqrt{t}}, \quad y(t) = \frac{1}{t^2}, \quad t > 0.
\]

See Appendix B for details. From the perspective of APOS, converting this curve from parametric to standard form can potentially be accomplished, at least partially, at the action level.\(^2\) A student with an action conception of parametric function might be able to isolate \( t \) in the first expression and substitute it into the other expression, resulting in the equation \( y = x^4 \) in rectangular coordinates. However, such a response would not be entirely correct. The curve defined by \( y = x^4 \) contains points with an \( x \)-coordinate that is less than or equal to 0, while the \( x \)-coordinate defined by \( x(t) = 1/\sqrt{t} \) is strictly greater than 0 for all \( t > 0 \). Therefore, the correct response would be \( y = x^4 \) with \( x > 0 \). These restrictions on \( x \), which make up the domain of the curve when it is defined by \( y = x^4 \), are determined by the range of the function given by \( x(t) = 1/\sqrt{t} \). According to APOS, the notions of domain and range

\(^2\)This statement refers specifically to the above curve, but I hypothesize that many curves given parametrically that are expressible in standard form can be done so partially at the action level. However, there is not enough data here to support such a hypothesis.
become accessible to an individual who has a process conception of function (Breidenbach et al., 1992; Dubinsky & Harel, 1992; Oehrtman et al., 2008). Therefore, if an individual can correctly convert a curve from parametric to standard form, with correct restrictions on the values of $x$ (or $y$), then this study considers such such a student to have a process conception of parametric function.

On the final exam, question 1a, three out of fifteen students gave responses that indicated a pre-action conception of parametric function, seven students gave responses that indicated an action conception of parametric function; and five students gave responses that indicated a process conception of parametric function. Below is the analysis of students’ responses categorized by these conceptual levels.

**Pre-action level.** According to APOS, a student is considered to have a pre-action conception of parametric function if he or she demonstrates too little understanding to be considered action level. Out of fifteen students, three gave responses to question 1a on the final exam that indicated a pre-action conception of parametric function. These students’ responses were incorrect or incoherent.

For example, in Griffin’s solution (see Figure 4.2), he appears to be unsure of what the question is asking him to do. At the top of his solution, he was able to isolate $t$ in the equation $x = 1/\sqrt{t}$, but he did not substitute this correctly into the equation $y = 1/t^2$. Below that, he differentiated (incorrectly) $x(t)$ and $y(t)$. The rest of his solution suggests he was attempting to find the equation of the line tangent to the curve at the point $(0,0)$.

![Figure 4.2 Griffin’s solution to final exam question 1a.](image-url)
Of the remaining two students who expressed a pre-action conception of parametric function on this task, one was not able to isolate $t$ in the equation for $x(t)$ or consider the restrictions on $x$. The other student constructed a linear equation based on a table of values.

**Action level.** In APOS theory, an action is a transformation of objects in response to external cues. In this study, a student’s response to question 1a on the final exam was considered to indicate an action conception of parametric function if he or she could isolate $t$ in one of the equations and correctly make a substitution into the other equation, but not consider the restrictions on $x$ or $y$ determined by their relationship with $t$. Out of fifteen students, seven gave responses to question 1a that indicated an action conception of parametric function.

For example, in Sam’s solution (see Figure 4.3), he clearly described the procedure he was performing. He wrote, “solve $x(t)$ for $t$” and “input to $y(t)$ to obtain equation in $x$ and $y$.” Sam performed this procedure correctly, but nothing he wrote suggested that he was aware of the restrictions on $x$ determined by its relationship with $t$.

![Figure 4.3 Sam’s solution to final exam question 1a.](image)

In Whitney’s solution (see Figure 4.4), she demonstrated a different approach than Sam, isolating $t$ in both of the equations and setting them equal to each other. However, she still did not determine the restrictions on $x$ or $y$. For this reason, her response was considered to an indication of an action conception of parametric function.

Similar to Sam, Oliver correctly isolated $t$ in the equation for $x(t)$ and made a correct substitution into the equation for $y(t)$ (see Figure 4.5). Unlike Sam, Oliver knew that he
Figure 4.4 Whitney’s solution to final exam question 1a.

should consider the relationship between $t$ and $x$, but he did so incorrectly. He wrote, “$x$ can be all real numbers because all numbers will result in $t > 0$ in the parametric equation.” This statement by Oliver indicates a misconception that should be addressed. Instead of finding the range of the function given by $x(t) = 1/\sqrt{t}, t > 0$, Oliver found all $x$-values that satisfy $t = 1/x^2, t > 0$.

Figure 4.5 Oliver’s solution to final exam question 1a.

Of the remaining four students whose responses indicated an action-level of understanding of parametric function, three gave solutions similar to Sam’s, without determining the restrictions on $x$. However, one of these students, Frank, mistakenly dropped the exponent in $y(t) = 1/t^2$, which led to an incorrect equation. This mistake was not considered to be indicative of a pre-action conception.

**Process level.** In APOS, when a student is able to reflect on and describe the steps of a transformation of objects without actually performing those steps, he or she is said to have an process conception of a concept. Several researchers (e.g., Breidenbach et al.,
Dubinsky & Harel, 1992; Oehrtman et al., 2008) have asserted that the notions of domain and range are more accessible to a student who has a process conception of function. Therefore, in this study, a student’s response to question 1a on the final exam was considered to indicate a process conception of parametric function if, in addition to expressing the curve by an equation in $x$ and $y$, he or she correctly determined the restrictions on $x$ based on the values of $t$. Out of fifteen students, five gave responses to this question on the final exam that indicated a process conception of parametric function.

For example, in Alex’s solution (see Figure 4.6), he correctly isolated $t$ in the equation for $x(t)$ and made a correct substitution into the equation for $y(t)$. Furthermore, beginning with the fact that $t > 0$, he found the interval on $x$ by performing the same operations on $t$ as in the equation for $x(t)$. From $t > 0$, Alex deduced that $\sqrt{t} > 0$. Then he scratched through what he wrote next, but it looks like he wrote, “$1/\sqrt{t}$.” In the remainder of his solution, Alex appeared to be unsure as to whether 0 is included in the interval for $x$. In his work, he wrote, “$0 < x < \infty$,” while in his final answer, he wrote, “$x \geq 0$.” Despite this confusion, Alex’s solution demonstrates that he carefully deduced values of $x$ based on values of $t$. For this reason, Alex’s solution was considered to be process-level.

![Figure 4.6 Alex’s solution to final exam question 1a.](image)

Alex was the only student who wrote out detailed steps for how he determined the interval on $x$. However, the other four students still determined a correct answer including the interval on $x$. As a result, they were considered to be at the process-level of understanding of parametric function.

For example, Hannah acknowledged awareness that the interval on $x$ is determined by
the restrictions on \( t \) (see Figure 4.7). In Hannah’s solution, we can see that she mistakenly dropped the exponent in \( y(t) = 1/t^2 \), which resulted in the answer equation \( y = x^2 \) instead of \( y = x^4 \). This is not considered to be indicative of poor conceptual understanding. Furthermore, indicating the correct interval on \( x \) suggests that she has more than an action conception of parametric function. Therefore, she was considered to have a process conception.

![Figure 4.7 Hannah’s solution to final exam question 1a.](image)

The overall results to question 1a on the final exam suggest that most students either did not know they need to consider restrictions on \( x \) or did not know how those restrictions are determined. These difficulties might be related to a weak understanding of inverses or compositions of functions (or both).

### 4.3.2 Sketching the graph of a curve given parametrically

Step 1 of the preliminary genetic decomposition called for the coordination of the processes of evaluating two functions \( f \) and \( g \) at values of \( t \) to imagine \( f(t) \) and \( g(t) \) changing simultaneously as \( t \) increases through values shared by \( \text{Dom}(f) \) and \( \text{Dom}(g) \). Furthermore, it was asserted that at this step, an individual can compare changes in \( f(t) \) with changes in \( g(t) \) over small intervals of \( t \). Meanwhile, Step 2 of the preliminary genetic decomposition called for the coordination of the process in Step 1 with the schema for \( \mathbb{R}^2 \) to construct elements of \( \mathbb{R}^2 \) defined by \( (x, y) = (f(t), g(t)) \). It was hypothesized that at this step, an individual with an object conception of point in \( \mathbb{R}^2 \) can describe the process of \( (f(t), g(t)) \) tracing out a curve in \( \mathbb{R}^2 \).
To determine whether students made the constructions called for by these steps of the preliminary genetic decomposition, they were asked on the final exam, question 1b, to sketch the graph of the curve given by

\[
x(t) = \frac{1}{\sqrt{t}}, \quad y(t) = \frac{1}{t^2}, \quad t > 0,
\]

which is the same curve for which they were asked to convert from parametric to standard form in question 1a on the final exam (see the previous section and Appendix B for details). There are at least two ways in which a student might be able accomplish this task at the action level. For example, he or she can sketch a graph by making a table of values, plotting points, and drawing a rough curve. A student at the action level might even be able to sketch a graph that is correct in terms of shape and quadrant, simply based on his or her answer to final exam question 1a by remembering what the graph of \( y = x^4 \) looks like. But such a student would probably rely on a table of values for \( t, x, \) and \( y \) in order to indicate the orientation of the curve. On the other hand, an individual with a process conception should be able to describe the general shape and/or orientation of the curve by imagining how the values of \( x \) and \( y \) change as \( t \) increases through its domain. That is, a student with a process conception should be able to think of a point \((x, y)\) in the first quadrant corresponding to a positive \( t \)-value. Then by recognizing that both \( x \) and \( y \) approach 0 as \( t \to \infty \), the student can treat \((x, y)\) as a dynamic point by mentally increasing \( t \) and imagining \((x, y)\) moving toward the origin (without ever reaching it), with the \( y \)-coordinate decreasing faster than the \( x \)-coordinate.

Only nine out of fifteen students responded to question 1b on the final exam. Six of them gave a response that was considered to be action-level, one student gave a response that was considered to be process-level, and two students gave a response that was considered to be in transition from action- to process-level.

An action conception. In this study, a students’ response to question 1b on the final exam was considered to be action-level if it indicated that the student was unable to
imagine in his or her mind how the curve is sketched as $t$ increases. Out of nine students, six gave a response that was considered to be action-level. These students appeared to rely on a table of values or did not consider the orientation of the curve.

For example, Peggy’s solution was considered to be action-level because she constructed a table of values (see Figure 4.8). It might be considered that her solution is actually at the process-level since she drew a correct curve in terms of shape by plotting only one point from her table of values. However, Peggy’s solution gives no clear indication that she was aware of what happens to the curve as $t$ continues to increase. That is, she did not indicate that there is a hole at the origin. On the previous exam question (question 1a), which asked students to express this curve in standard form, Peggy also gave an action-level solution by not considering the intervals of $x$. With this in mind, it is not clear how she knew to sketch the graph in only the first quadrant. Consequently, without the opportunity to probe her thinking further, Peggy’s solution was considered action-level.

![Figure 4.8 Peggy’s solution to final exam question 1b.](image)

Unlike Peggy, Kevin did not construct a table of values. Instead, he appeared to rely strictly on his answer from the previous exam question (question 1a) in which students were asked to convert the curve from parametric to standard form. His answer to question 1a was, “$y = x^4, x > 0$. So, must be in Q1.” In his answer to question 1b (see Figure 4.9), Kevin wrote, “Since $t > 0$, $x > 0$ & if $x > 0$ then it can only be in Q1.” This is correct. However, Kevin did not describe the orientation of the curve, which was an important detail
that was discussed class.\textsuperscript{3} As a result, it is unclear whether Kevin viewed the graph as a static geometrical object or a curve traced by a dynamic point. For this reason, his solution was classified as action-level.

![Kevin's solution to final exam question 1b.](image)

\textbf{A process conception.} In APOS, a process is a transformation of objects that is perceived to be completely internal. Only one student, Hannah, appeared to have made constructions similar to those called for by Steps 1 and 2 of the preliminary genetic decomposition, at least in the context of question 1b on the final exam.

In her solution (see Figure 4.10), Hannah sketched a correct curve and described its orientation by stating, “A particle traveling this curve would move from right to left because as $t$ increases $x$ decreases, and thus $y$ decreases, orienting the curve toward the hole at $x = 0$.” What Hannah described is a different process than the one called for by the preliminary genetic decomposition. Step 1 called for coordinating two functions two imagine $x$ and $y$ as changing simultaneously with respect to $t$. Hannah on the other hand described the process of function composition. That is, she determined that $x$ decreases in response to increases in $t$. Then in response to these decreases in $x$, she described that $y$ decreases. This process is viewed as an alternative to Step 1 which can be constructed in conjunction with the processes involved in converting from parametric to standard form (see Step 4 of the preliminary genetic decomposition). Despite this difference, Hannah was still considered to have constructed the process in Step 2 of the preliminary genetic decomposition, by

\textsuperscript{3}Note that the arrow on Kevin’s graph refers to the fact that $y \to \infty$ as $x \to \infty$ and does not describe the orientation of the curve as $t \to \infty$. Also note that Kevin incorrectly plotted a point at the origin.
imagining the point as a particle moving “toward the hole” at the origin.

Figure 4.10 Hannah’s solution to final exam question 1b.

Lee was one of two students who gave a solution that indicated the transition from action-level to process level. In her solution to the final exam, question 1b, Lee plotted points based on a table of values, indicative of an action concept of parametric function (see Figure 4.11). On the other hand, Lee clearly reflected on these actions by stating, “as the \( t \) value gets bigger, there are greater changes in \( y \) than in \( x \), in the fact that in 4 \( t \) values \( x \) goes from 1 \( \rightarrow \) 1/2, \( \Delta x = .5 \), \( y \) goes from 1 \( \rightarrow \) 1/16, \( \Delta y = .9375 \).” This suggests that Lee is possibly beginning to interiorize the actions of evaluating \( x(t) \) and \( y(t) \) into processes, which could then be coordinated as \( t \) increases.

Figure 4.11 Lee’s solution to final exam question 1a.

The results of this section, to some degree, support the constructions called for by the preliminary genetic decomposition in Steps 1 and 2. As we saw with Hannah, compar-
ing changes in $x$ with changes in $y$ can be accomplished through the process of function composition—by coordinating the processes of $x = f(t)$ and $y = (g \circ f^{-1})(x)$. This construction complements the constructions called for by Step 4, which are necessary for converting from parametric to standard form. The action-level responses on question 1b on the final exam by students suggest that more curve-sketching opportunities need to be offered early in the study of parametric functions.

### 4.3.3 Constructing a parametric function to describe a real-world situation

In order to construct a parametric function to describe a curve, that is, in order to parametrize a curve, Step 5 of the preliminary genetic decomposition called for the reversal of the processes of Step 4 involved in converting from parametric to standard form (see Section 3.3). The analysis of data in this section should provide more details about students’ reasoning in the process of parametrizing a curve.

To determine how students reason about parametrization, they were asked on question 3 of the interview to determine the position of a particle at a certain time as it moves along a linear path (see Figure 4.12). Part (a) asked students to find the position of the particle at 1.5 seconds, and part (b) asked students to find the position of the particle at time $t$. To solve this problem, students were expected to write separate equations for $x(t)$ and $y(t)$. One possible way to do this, as discussed in class, would involve considering two $x$-coordinates separately from the $y$-coordinates at two different values for time, particularly the initial time, $t = 0$, and the terminal time. The terminal time can be found by determining the length of the linear path (by using the Pythagorean identity or the distance formula), which is 5 units, and dividing this length by the particle’s speed to obtain 4 seconds. Then by constructing ordered pairs corresponding to the intercepts of the $tx$- and $ty$-graphs, linear equations can be determined.

Out of fifteen students, five indicated little to no understanding of this problem. These students did not know which information is crucial to solving the problem, such as the length of the segment or the terminal time, and had difficultly comprehending suggestions that were
Suppose a particle is traveling at a constant speed of 1.25 units per second along a linear course starting at the point (0, 4) and ending at the point (3, 0).

(a) What are the coordinates of the particle after 1.5 seconds?
(b) What are the coordinates of the particle after \( t \) seconds?

Figure 4.12 Question 3 of the interview.

offered by the interviewer. As a result, these students were considered to be at a pre-action level of understanding of parametric function.

Three students, Alex, Hannah, and Mary, were able to obtain at least a mostly correct solution without assistance. Among these students, there were two approaches that were observed. One approach, as discussed in class, involved constructing equations for \( x \) and \( y \) with respect to \( t \) by considering the motion of the particle in the horizontal and vertical directions separately. The other approach, which was not discussed in class, involved similar triangles.

Alex correctly solved the problem by considering the particle’s horizontal and vertical motion separately. He said:

A: When I got to this question, I started off by, I wrote down the information I had, drew a graph for it (see Figure 4.13), and I figured the particle was moving from a point on the \( y \)-axis to a point on the \( x \)-axis. I found out the total distance, and then had the distance and the speed, so I got the total time.

Alex’s work shows that he used the Pythagorean theorem to determine the length of the particle’s path to be 5 units (see Figure 4.13). Then he divided the 5 units by 1.25 units per second to determine that the particle completed the trajectory in 4 seconds.

Alex continued by explaining how he considered \( x \) and \( y \) separately:

A: I have a graph of \( x \) and \( y \). And then I just took \( x \) values separately and \( y \) values separately with respect to time. And \( x \) was going from 0 to 3, and
the duration, the total duration, which I found out was 4 seconds, and $y$ was going from 0 to 4, sorry, 4 to 0, and the total duration. And then I found out linear equations for both graphs.

In the above excerpt, Alex described something very different than what was suggested in the preliminary genetic decomposition. Instead of reversing the processes involved in converting from parametric to standard form, Alex appeared to describe the de-encapsulation of an object. In particular, when Alex said, “I have a graph of $x$ and $y$,” he was viewing the graph of $y$ with respect to $x$ as an object. Then when he said, “and then I just took $x$ values separately and $y$ values separately with respect to time,” he was de-encapsulating the object into the coordinated process of describing the particle moving horizontally and vertically (see Step 2 of the preliminary genetic decomposition).

Alex went on to draw two graphs, one for $x$ versus $t$ and one for $y$ versus $t$ (see Figure
4.13). From these graphs, he determined the equations for each of these lines in slope-intercept form by finding their slopes and recognizing that the vertical intercepts were already determined. Then Alex plugged in the time of 1.5 seconds into each of his equations to determine the $x$- and $y$-coordinates of the particle at that time.

When answering part (b), Alex observed that his equations from part (a) gave the position of the particle at any time $t$ (see Figure 4.14). Furthermore, when the interviewer asked Alex what would be the domain, he said:

A: Maybe it is giving the time limit $t$ from 0 to 4, or something like that.
I: Something like that? How would you know for sure?
A: Time cannot be negative. So, 0 and the total time it takes is 4 seconds. It stops after 4 seconds.

![Figure 4.14 Alex’s solution for question 3b of the interview.](image)

The other two students, Hannah and Mary, who were able to reason about this problem with minimal assistance, obtained a solution by using similar triangles. The following excerpt contains Hannah’s explanation of her solution (see Figure 4.15).
H: I divided um, well, I figured out first that, um, I labeled the path $h$ um between um (0, 4) and (3, 0). Um, so I divided $h$ by the speed and got that it took 4 seconds ... To figure out where it is after 1.5 seconds, um, I used similar triangles ... If you multiply 1.25 and 1.5 seconds, you get that it um traveled 1.875 um units along um along the path ... So the 1.875 relationship with 5 is the same relationship as the $x$-coordinate um as to 3, since the ending is at (3, 0), the length of that is 3. So solving for that, you get that $x$ is 1.125. And it’s the same relationship for $y$. So um the $y$-coordinate’s relationship to 4 is um the same as 1.875 to 5. So that’s how I solved for the $x$ and $y$ coordinates.

Figure 4.15 Hannah’s solution for question 3a of the interview.

In the above excerpt, it was not exactly clear what Hannah meant when she said, “I labeled the path $h$,” but based on her written work (see Figure 4.15), she was treating $h$ as both an unknown constant and a variable. On one hand, she found $h$ to be 5, the length of the particle’s path. On the other hand, when she wrote, “$dh/dt = 1.25$ units per second,” she was treating $h$ as the distance traveled by the particle at a given time $t$ (i.e., arc length). Therefore, Hannah is considered to have constructed a process conception of arc length, which she then encapsulated into an object on which the process of differentiation
was performed. By doing so, she observed that \( \frac{dh}{dt} = 1.25 \) units per second, the velocity of the particle along the path.

The remainder of Hannah’s solution appeared to involve de-encapsulating arc length back into a process. This was particularly evident in her reasoning for part (b) in which she determined the position of the particle at any time \( t \) (see Figure 4.16).

H: I used the same relationship, um, but instead of 1.875 I used, um, since \( t \) is the time and 1.25 is the speed, then um multiplying those together would give you the distance traveled over any time \( t \). So um using 1.25\( t \) instead of 1.875 gives you the same ratio but um you can use any time \( t \). So I set it up so that \( \frac{1.25t}{5} \) is the same thing as \( \frac{x}{3} \), whatever the \( x \) coordinate would be. Um then I solved for \( x \) in terms of \( t \). And I did the same relationship for \( y \). So \( \frac{y}{4} \) would be the same as 1.25\( t \) to 5. And then solving for \( y \), um, so those two together would give you \( x \) and \( y \) coordinates after any amount of time within 4 seconds.

In the above excerpt, Hannah appeared to de-encapsulate the particle’s accumulated distance, or arc length, back into a process defined by the coordination of two processes—one for the change in \( x \) with respect to \( t \) and one for the change in \( y \) with respect to \( t \), as evidenced by Hannah using two pairs of similar triangles. It should be mentioned that in Figure 4.16, Hannah initially gave an incorrect answer for the \( y \)-coordinate at time \( t \), but after prompting from the interviewer she was able to correct it. She said, “So wait. This \( y = t \) is the change in \( y \). So that’s how much \( y \) has decreased … So the actual \( y \)-coordinate is 4 – \( t \).”

Six students, who were unable to obtain a correct solution on their own, began to exhibit understanding of the problem with help from another student or the interviewer. For example, Kevin and Ron together were able to reason about this problem. The following excerpt contains Kevin’s initial explanation for how he attempted to solve this problem (see Figure 4.17).

K: It says they want the coordinates for a particle after 1.5 seconds, and so I
said that we have to find the distance. And so I had set up the equation that speed was distance over seconds. So then I rearranged and said that distance equals speed times seconds. So I did 1.25 times 1.5, and I got 1.875 for the distance. And then what I did with that ... I know I skipped a part. I like jumped the gun. I forgot to find $t$. Wait, wait, I think $t$ was 1.5 or something, and at 1.5, $x$ will be at 3, and then when you’re going up in this direction, at 1.5 or coming down, $x$ will be zero. And then when you’re going in this direction, $y$ is 0 at 1.5 and then when you’re going up, $y$ is 4 at 1.5. And so I don’t know ... That’s where I stopped at. I was a little confused in doing the problem in all honesty.

Kevin started the problem correctly by finding that the distance traveled by the particle after 1.5 seconds was 1.875 units. However, Kevin “skipped a part” by forgetting “to find $t$.” Moreover, Kevin did not know the significance of the value of $t$ that he forgot to find, evidenced by his misconception that the particle traverses the entire path in 1.5 seconds.

The following excerpt contains Ron’s initial reasoning about the problem, in which he described using proportions. However, he was unsuccessful (see Figure 4.18 for his written work).

R: I went in a kind of different direction so I’m assuming this is wrong. I said
that if it’s going to travel at 1.25, I said . . . from the start to end, $x$ is traveling only 3 units and $y$ is traveling 4 units for the whole line. Um, so I started finding what percentage .25 was of 1.5 because the difference between 1.5 and 1.25. So I found out 16.67 percent or .167, which I multiplied times 1.5—sorry 1.25—um because my line of thinking was to multiply the difference percentage between the time that we are going per second and how many seconds, [to find] the destination that were looking for in 1.5 seconds. So um then I added the two to get 1.45 seconds, or 1.45 units rather. So my line of thinking was that each, each one of these things, which it’s got to be incorrect, but my line of thinking was that . . . in 1.5 seconds, each unit will be 1.45 units over, like in their respective direction. So 1.45 units down and then . . . but that would hold true to my line of thinking if these were the same, if they were traveling the same number of units at a steady, at a constant rate for both of them, but because $x$ is traveling at 3, $y$ is traveling at 4, $x$ would reach its destination before $y$ would. So it wouldn’t actually be at this point, now that I look at it. So I think I’d have to find, I don’t know.
Ron attempted to use proportions to solve the problem, but his reasoning was incorrect, particularly when he said, “my line of thinking was to multiply the difference percentage between the time that we are going per second and how many seconds, to find the destination that were looking for in 1.5 seconds.” Instead, Ron should have found the distance traveled per second to find the “destination” after 1.5 seconds.

In the following excerpt, Ron began to exhibit more of an understanding of the problem, with help from Kevin.

I: Did you find this [points to segment]?
R: Would it be 5?
K: I said the hypotenuse was 5. I was going 4 up that way and 3 that way.
R: Yeah, just use Pythagorean theorem. I just did it.
I: Okay. So that’s one way how you can solve it. And so how long does it take?
K: [Writes on paper.] You said how long it would take, right? So we’re trying to find the seconds ... We know that this is 5 units and that, we have what the speed is. So to find seconds, you would have to do distance over speed. So then you do 5 over 125, 1.25. And then that would give you [uses calculator]
4 seconds.

I: Mmhm. Okay.

K: And so then it travels this way in 4 seconds and it travels that way in 4 seconds. Well it travels, like, these are 4 seconds. I think that’s what I remember. If I remember hard enough. I’m close.

R: If that’s correct then we’re asking at what point it will be in 1.5 seconds. Right. So going along his line of thinking, then if it takes, you said 4 seconds, right, to get to, yeah, it’s end point. And so um 1.5 is 37.5 percent of 4 seconds ... So 1.5 seconds is 37.5 percent of your overall destination. So multiplied .375 times your overall distance, and that’s where you’re going to be because in 37.5 percent of your time ...

I: So can you find it?

R: Yeah. [Enters calculations into calculator.] So, um, so x would be 1.75.

K: Would it be the x(t) and y(t)?

I: Mmhm. Yes.

K: So that means you have to find the equation of x(t) and y(t). And so x(t) in a linear perspective, I think it’s x of the initial plus, like, I dont know. The equation ... the initial x plus ... and I forget the rest, but I know, like, it’s another, like, x in there as well ... I can’t remember the full equation, like I’m close.

Kevin and Ron found that the total distance traveled by the particle is 5 units. In fact, Kevin found this originally, but he did not use it (see work in pencil in Figure 4.17). After the interviewer asked how long it takes for the particle to travel the 5 units, Kevin was able to determine the time to be 4 seconds. With this information, Ron began to exhibit more of a process conception parametric function when he adapted his reasoning from earlier in order to determine that the particle completes 37.5 percent of its path after 1.5 seconds. However, he still made an error when he calculated x to be 1.75. Kevin, on the other hand, exhibited consistently exhibited an action conception of parametric function by attempting
to solve the problem from memory.

By the end of the discussion, Kevin was able to determine the correct answers to both parts of question 3 (see his work in red pen in Figure 4.17) after the interviewer prompted him to write equations for $x(t)$ and $y(t)$ based on his knowledge of the slopes and vertical intercepts, but he required prompting from the interviewer, which is indicative of an action conception. Despite exhibiting a deeper understanding than Kevin, Ron did not obtain a correct answer (see work in red pen in Figure 4.18). Instead, he wrote that $x(t) = 4 - \frac{3}{4}t$ and $y(t) = 4 - 4t$. For this reason, Ron was also considered to be at the action-level of understanding parametric functions in this context, but it likely that he was beginning to transition to a process conception.

There is one more student, Sam, who should be mentioned, not because of his correct solution, but because of his approach. Instead of parametrizing the line segment “from scratch” like the other students, Sam started with one parametrization and attempted to adapt it to correspond to the situation.

S: I’m going to preface this by saying I know I got this one wrong, but my reasoning was, we were given initial $x$ and $y$ and we were given another point $x$ and $y$ along the uh linear path of the particle, and I took the formula from class where $x(t) = x_0 + t(x_1 - x_0)$, and I multiplied $t$ by the speed so that the units would cancel and I would be left, I would be left using units inside the equation rather than units over time. And then I plugged in the time that we’re given to find where the particle is at into each equation to give me my answer.

I: So what did you get for your equation for $x(t)$?

S: For $x(t)$, I got $3.75t \ldots$ and for $y(t)$, I got $4 - 5t$.

I: Okay.

S: The reason I know it’s wrong is because I checked with the distance formula (*slope formula*), which results in $4/3$ units as opposed to the, uh, I think it’s $1.875$ units that should be traveled.
I: Ah. So, what do you think? Where did you make a mistake?

S: I think I made a mistake multiplying. I think I should have multiplied the entire \( y(t) \) by 1.25 rather than just \( t \), but I don’t know for sure. I don’t know where I went wrong.

![Figure 4.19 Sam’s solution for question 3 of the interview.](image)

Sam’s approach, although incorrect, was potentially plausible. He began with parametrization given by \( x(t) = x_0 + t(x_1 - x_0) \) and \( y(t) = y_0 + t(y_1 - y_0) \). Then Sam attempted to manipulate the terms in the equations so that the unit of seconds would be eliminated. Sam did not realize that the original equations already satisfied this requirement, because the values of \( x_1 - x_0 \) and \( y_1 - y_0 \) refer to the change in \( x \) and \( y \) per second. Sam should have instead realized that the interval on \( t \) for his initial parametrization is \( 0 \leq t \leq 1 \). By determining that the particle traverses the segment in 4 segments (which Sam did not do), he could have then transformed the interval on \( t \) into \( 0 \leq t \leq 4 \) by dividing \( t \) by 4. This corresponds to the notion of reparametrization, which was not called for by the preliminary genetic decomposition.

The results reported in this section suggest that Step 5 of the preliminary genetic decomposition should be revised to consider an object conception of a curve defined parametrically. In particular, the process that should be encapsulated is the function for the distance traveled by the particle, or arc length in general. The action or process that induces the encapsula-
tion of arc length would be differentiation, observing that the derivative is the velocity of the particle along its path. Then in order to parameterize the path of the particle, arc length should be de-encapsulated back into a process. Another notion that should be considered is reparametrization. In order to reparametrize, an individual should encapsulate the process of generating a curve into an object to conceive of applying a transformation that results in the same curve that is generated at a different rate and/or direction. To actually perform the transformation, the individual must de-encapsulate the curve so that he or she can apply a transformation to its component functions.

4.4 Students’ reasoning about the invariant relationship between two variables

Step 2 of the preliminary genetic decomposition called for the coordination of the processes of two functions \( f \) and \( g \) with the schema for \( \mathbb{R}^2 \) to construct elements of \( \mathbb{R}^2 \) defined by \((x, y) = (f(t), g(t))\). It was hypothesized that at this step an individual with an object conception of point in \( \mathbb{R}^2 \) can describe the process of \((f(t), g(t))\) tracing out a curve in \( \mathbb{R}^2 \). Then Step 3 of the preliminary genetic decomposition called for the encapsulation of the process in Step 2 in order to compare and contrast congruent curves in \( \mathbb{R}^2 \) in terms of their points and the rate at which they are generated. It was hypothesized that at this step, the individual can perceive the relationship between \( x \) and \( y \) as being independent of \( t \). This section reports on students’ reasoning about the invariant relationship between two quantities varying simultaneously with respect to a third quantity when described by a graph (Section 4.4.1) and in words in a real-world problem (Section 4.4.2).

4.4.1 Students’ reasoning about the invariant relationship between two variables described by a graph

On question 2 of the final exam, students were asked to discriminate between two plane curves given graphically (see Figure 4.20). Part (a) asked if the relationship between \( x \) and \( y \) is the same for both graphs, and part (b) asked if both graphs can be expressed analytically in the same parametric form. In this problem, the curves are to be treated as
objects on which the actions or processes of compare and contrast should be performed. These questions can be answered at the action level by comparing these two curves point by point, focusing on the five points and their corresponding values of $t$ explicitly indicated in each of the graphs. Once an individual has interiorized the action of comparison into a process, he or she can perceive comparison as dynamic, checking the location of the particle on each of there curves in rapid succession, as if imagining the particle traveling along the curve in real time.

![Figure 4.20 Graphs for final exam, question 2.](image)

Only eleven out of fifteen students responded to question 2 on the final exam. Of those eleven, four students gave what was considered to be pre-action level response, and seven students gave what was considered to be a process-level response. No student appeared to have an action conception based on their response to question 2. This section presents the most illustrative cases of students indicating these conceptions of parametric function.

**Pre-action level.** Out of eleven students, four gave what was considered to be pre-action response. These responses illustrated an incorrect answer or incorrect justification.

For example, on question 2a, Alex wrote:

No. The graph of $x$ and $y$ look same for both but they are not since the scales are different for both graphs. The two graphs do not have the same $(x, y)$ coordinate for a specific value of $t$. 

Alex illustrated his last claim with the example of \( t = -1 \): the first graph is at \((1, -1)\) and the second graph is at \((4, -2)\). When he described the difference in scales, it seems he assigned a physical quantity to the time intervals between each indicated point, in a way as if he had in mind a 3-dimensional graph where \( t \) was on an axis perpendicular to the \( x \)- and \( y \)-axes. Because Alex did not realize that the relationship between \( x \) and \( y \) is independent of \( t \), his response was considered to be pre-action.

Kevin answered question 2a correctly, but his reasoning was incorrect.

Yes, because on each graph they have the same respect to time. When \( x = 4 \) on graph 1 \( y = 2 \) & same on graph 2. Also, each time interval on each is the same distance away.

Kevin affirmed that the relationship between \( x \) and \( y \) is the same for both of the given graphs, but he reasoned that they are the “same with respect to time.” Furthermore, he was incorrect when he stated that the time interval is the same. It seems that Kevin, like Alex above, assigned a physical quantity to each interval of time. However, he did so naively by thinking that the length of the curve was related to an interval of time. Therefore, Kevin’s response was considered to be an indication of a pre-action conception of parametric function.

**Process level.** Out of eleven students, seven gave a response that was considered to be process-level. Process-level responses discriminated between the graphs in general terms, making no reference to particular points on the graphs or their corresponding \( t \)-values.

For example, on part (a), Bailey concluded that the relation between \( x \) and \( y \) is the same for both curves by saying:

\[
\text{The relationship for } x \text{ and } y \text{ is the same. The position of the particle is different with respect to time, the } x \text{ and } y \text{ relationship does not change. } x = y^2.
\]

Then for part (b), he wrote:
The curves do not have the same analytical parametric representation because of the difference in time vs position.

Bailey’s response was considered to be process-level since it did not rely on specific points to discriminate between the curves. Furthermore, he was able to describe the whole curve by the equation \( x = y^2 \). However, his response suggests that he has not yet developed the notion of rate.

Hannah gave a similar response to part (a) by stating:

Yes. In both cases, the relationship between \( x \) and \( y \) is \( x = y^2 \), so the curves are the same.

On part (b), she wrote:

No. The particle traveling the first path is moving half as fast as the particle traveling the second path, so the parametric representations would be different.

Hannah’s response to part (a) suggests she can describe all points on the graph instead of considering a few points one by one. Her response to part (b) suggests that she has interiorized the action of checking the position of the particle at certain times of \( t \) into the process of imagining two particles passing through the same points at different speeds.

Lee was one student who demonstrated competing conceptions in her answer, but her overall reasoning seemed to indicate a process conception of parametric function (or at least in transition to a process conception). She wrote:

Yes \( t \) is changing is rate but at a certain time of \( t \) the \( x \) & \( y \) coordinates are the same in relation to each other. In graph one \( (1,1) \) & \( (4,2) \), in graph two \( (1,1) \) & \( (4,2) \). It look like \( y \) is \( \pm \sqrt{x} \) in both graphs.

Lee’s response was partially incorrect by stating, “at a certain time of \( t \) the \( x \) & \( y \) coordinates are the same in relation to each.” In fact, at any value of \( t \neq 0 \), the \( x \)- and \( y \)-coordinates are not the same in relation to each other. However, when used particular points to illustrate her reasoning, she did not reference their corresponding values of time.
This can be interpreted to mean that she was aware that the values of time played no role in determining whether there relationship between $x$ and $y$ is the same for both graphs. Then on part (b), which asked if the curves have the same analytical parametric representation, she wrote:

I don’t know what this means. I’ll say no, because the $t$ values are different? and the change in $t$ is different.

When Lee wrote, “because the $t$ values are different,” she might be indicating an action conception of parametric function, if she was considering only the few $t$-values indicated on the graphs. However, observing that the change in $t$ is different for each graph suggests more of a process conception of parametric function.

Overall, four out of eleven students had difficulty perceiving the invariant relationship between $x$ and $y$ when described by a graph. The following section reports on whether students can perceive the invariant relationship between two quantities varying simultaneously with respect to a third when the variables are described in words in a real-world problem.

4.4.2 Students’ reasoning about the invariant relationship between two variables in a real-world problem

On question 2 of the interview, students were asked to sketch graphs corresponding to a situation in which two water coolers of the same shape and size are full of water and being emptied at constant, but different rates (see Figure 4.21). Part (a) asked students to graph the relationship between time and the volume of the water for both coolers on the same coordinate plane; part (b) asked the students to graph the relationship between time and the height of the water for both coolers on the same coordinate plane; part (c) asked students to graph the relationship between the volume of the water and the height of the water for both coolers on the same coordinate plane; and part (d) asked students to indicate the orientation of their graphs (see Appendix C for details.)

The purpose of these questions was to determine if students can describe the volume and height changing independently with respect to time and, more importantly, determine if
students perceive the relationship between the volume and height as invariant between each cooler and independent of the rate at which the coolers are being emptied. Because this task gave no specific numbers for the volume and height of the coolers or for the rates at which they are being emptied, in order to sketch these graphs the student would need to designate on the appropriate axis arbitrary values for the minimum and maximum volume and height as well as the time at which the emptying process is finished. Furthermore, the student would need to recognize that his or her designated values for the minimum and maximum volume and height are constant throughout question 2. The following was expected from students on question 2a–c (see Figure 4.22 for a correct solution).

- **Part (a).** Understand that the volume is decreasing with respect to time and determines a decreasing \( tv \)-graph for each cooler. Understand that the constant rate of decrease of the volume with respect to time determines a linear graph. Understand
that Cooler 2 being emptied faster than Cooler 1 determines a graph for Cooler 2 that is steeper than the graph of Cooler 1.

- **Part (b).** Understand that the height is decreasing with respect to time and determines a decreasing \(th\)-graph for each cooler. Understand that the constant width of the top part of the cooler and the constant width of the bottom part of the cooler determine a constant rate of decrease of the height with respect to time for each part of the cooler, thereby determining a linear graph for each part of the cooler, which together form a piecewise linear graph. Understand that the graph corresponding to the top part of the cooler is steeper than the graph corresponding to the bottom part of the cooler because the top part is narrower than the bottom part, which is related to a higher rate of change in height with respect to time. Understand that Cooler 2 being emptied faster than Cooler 1 determines a graph for Cooler 2 that is steeper than the graph of Cooler 1. Understand that the seam points on the graphs for the coolers have the same vertical coordinate.

- **Part (c).** Associate small or large quantities of volume with small or large quantities of height to understand that the relationship between volume and height determines an increasing graph. Understand that the constant width of the top part of the cooler and the constant width of the bottom part of the cooler determine a constant rate of change of the height with respect to volume for each part of the cooler, thereby determining a linear graph for each part of the cooler, which together form a piecewise linear graph. Understand that the graph corresponding to the top part of the cooler is steeper than the graph corresponding to the bottom part of the cooler because the top part is narrower than the bottom part, which is related to a higher rate of change in height with respect to volume. Understand that Cooler 2 being emptied faster than Cooler 1 plays no role in the rate of change of height with respect to volume. Therefore the \(vh\)-graphs for Cooler 1 and Cooler 2 are the same for part (c).

A preliminary analysis of students’ solutions identified the following four issues, which
became the criteria that were used when further analyzing students’ responses to each parts (a)–(c) of question 2:

- Direction of the graphs (increasing, decreasing, or both);
- Shape of the graphs (e.g., linear, piecewise linear, curved, etc.);
- Domain and range (indicated by the vertical and horizontal intercepts);
- Relative rates (specifically on the interior of the graphs).

The presentation of the results is organized around these four issues. Students’ solutions to parts (a)–(b) are reported together in Section 4.4.2.1, while Section 4.4.2.2 reports on students’ solutions to part (c). Additionally, part (d) asked students to indicate the orientation of their graphs in parts (a)–(c). The presentation of the results to part (d) will focus on the relationship between volume and height in part (c) and, thus, will be reported in Section 4.4.2.2.

4.4.2.1 Interview questions 2a–b

Part (a) of question 2 asked students to graph the relationship between time and the volume of the water for both coolers on the same coordinate plane, while part (b) asked students to graph the relationship between time and the height of the water for both coolers on the same coordinate plane (see Appendix C).

In regards to direction, shape, domain and range, and relative rate, seven out of fifteen students gave completely correct solutions to part (a), either with or without prompting. All of these students associated constant rate with the slope of a line and higher rate with a steeper graph. Furthermore, their graphs for Cooler 1 and Cooler 2 had the same vertical intercept and different horizontal intercepts, with the one closer to the origin belonging to the graph corresponding to Cooler 2.

The excerpt below contains Mary’s explanation for how she correctly sketched her graphs relating time and volume. Initially, Mary overlooked the information stating that the rates
are constant, but without prompting she realized her mistake and corrected her graphs. The curves in Figure 4.23 were her original graphs, which she corrected to be lines (in red).

M: Okay. So once I corrected them, um, cause I realized it was constant, and so they’re both just straight lines starting at the same volume and starting at time is equal to 0, and then Cooler 1 has uh a slower rate. So that’s going to be kinda like the slope of the line. So the slope is going to be a little less steep. So the, um, and that will make it go out farther on the time. And then Cooler 1 (Cooler 2), the slope is going to be a little more steeper, cause the rate is faster. So even though they both start at the same volume, they’re, like, Cooler 1 is gonna take less time and Cooler 2 is going to take more time. I mean, no. Did I say that right? Yeah, no, I said that wrong. Anyways, Cooler 1 would take more time and Cooler 2 would take less time.

When Mary realized that the rates were constant, she associated the notion of constant rate with the slope of a linear function. She acknowledged the domain and range of both functions when she said that they are lines “starting at the same volume and starting at time is equal to 0” and Cooler 1 will “go out farther on the time.” Although she kept mixing up
the rates of Cooler 1 and Cooler 2, she clearly stated initially that “Cooler 1 has uh a slower rate” and “so the slope is going to be a little less steep.” In her reasoning, Mary expressed an object conception of function. First, she was able to think of one graph with specific characteristics, such as starting at \( t = 0 \) with a certain volume, having constant slope, and ending at a certain time. Then she encapsulated this process into an object, the graph, on which she transformed to obtain the other graph. This is evidenced by her statement, “so the slope is going to be a little less steep. So the, um, and that will make it go out farther on the time.”

Part (b) of this problem was similar to part (a), asking students to sketch the graph of height of the water with respect to time. Four out of fifteen students gave completely correct solutions to part (b). All of these students correctly drew graphs of piecewise linear functions, by associating the narrower part of the coolers with a line of greater slope. Each of these students drew graphs for Cooler 1 and Cooler 2 that had the same vertical intercepts and different horizontal intercepts, with the one closer to the origin belonging to the graph corresponding to Cooler 2. Furthermore, over the entire domain, the graph for Cooler 2 was more steep than the graph for Cooler 1, and the seam points for these graphs had the same vertical coordinate.

The following excerpt contains Alex’s explanation of the graphs that he drew for the relationship between the time and the height of the water.

A: This is what I drew and what I had in mind was the height starts from a specific number and eventually going to decrease to 0. However, the rate of change of height for the first part of the container would be different from the rate of change of height in the second part of the container, and since the rate of volume is same, however, the width is different, the height would decrease more rapidly in the first (upper) part than the second (lower) part. So that is why the line in the first part would be steeper than the second part. And then the same reason for \( r_1 \) and \( r_2 \) being different.

Alex explained how the shape of the coolers affected the shape of the graphs of the
functions relating time and height. In Figure 4.24 we can see that Alex carefully reasoned his way to a correct graph. Both of the graphs in Figure 4.24 were drawn prior to the discussion with interviewer. Without any prompting, Alex realized that his initial graph (left) was incorrect, marking through it and drawing a correct graph (right).

Later, the interviewer asked Alex if the seam points for these two piecewise functions have the same vertical coordinate. He responded, “Oh, this point? Right. I’m sorry,” and marked (in red) to indicate that the heights were the same. His reasoning was:

A: The height of the container is a fixed number, and this number is a fixed number as well. For example, this [time] is 10 and this height is 8. These would both be on 8. That’s what I thought.

Similar to Mary, Alex exhibited an object conception of one function when he drew the graph of the other with respect to it. However, Alex was also able to de-encapsulate that object into the process it came from when he talked about the seam points.

The majority of students had some difficulties in sketching their graphs, either in part (a) or part (b). There were four common issues identified among those students. The remainder of this section reports on these issues.

**Issue 1: Shape.** The most common issue that was observed in students’ solutions to questions 2a–b pertained to the shape of students’ graphs. In particular, on part (a), seven students did not associate the volume’s constant rate of decrease with a linear graph.
and instead drew graphs that vary in slope. Four of these students drew curves, while two students thought the graphs should be piecewise linear. The seventh student drew increasing lines and gave no indication that the lines were related to the constant rate. Therefore, he was also considered to have issues with the shape of his graphs. On part (b), ten students did not consider or incorrectly considered the effect that the shape of the coolers have on the graphs of the functions relating time and height. Instead of drawing piecewise linear graphs with two pieces, three students drew linear graphs; six students drew curved graphs; and one student drew a piecewise graph with more than two pieces.

Figure 4.25 provides an example of a student, Hannah, who drew incorrect graphs for part (a). She explained:

H: Um well I drew Cooler 1 as decreasing um over time. Um, first it decreases um more quickly and then less quickly because um the top of the cooler has, is um narrower than the rest of the cooler ... Since the rate that Cooler 2 is um being emptied is greater than Cooler 1, um, then Cooler 2 has um a steeper slope than Cooler 1. They're essentially the same graph, but Cooler 2 has a steeper slope than Cooler 1.

I: Okay. And the graphs look like? ... In two parts?

H: Yes. So it’s steeper here than it is here because this upper part of the cooler is narrower ...

I: So that would change the volume?

H: Yeah.

Hannah believed that the volume would decrease faster in the narrower part of the coolers and as a result drew a piecewise linear graph (see Figure 4.25). Despite not associating the volume’s constant rate of decrease with a linear function, her graphs were still correct in terms of direction and her graphs correctly indicated the domain and range of each function. Moreover, Hannah was able to associate the rates at which the two coolers are emptying to the different slopes of her graphs. She said, “they’re essentially the same graph, but Cooler 2 has a steeper slope than Cooler 1.” Based on her response to this question, Hannah is
considered to have a process conception, but one that is not very strong.

**Issue 2: Domain and range.** The second-most common issue that was observed on questions 2a–b pertained to the domain and range of the functions represented in students’ graphs. In particular, on part (a), seven students drew graphs that indicated incorrectly, or not all, the domain or range of the functions relating time and volume. Meanwhile on part (b), eight students had this issue.

Figure 4.26 provides an example of a student, Cindi, who appeared to need particular numbers for the intercepts. In the following excerpt contains Cindi’s discussion of her graphs for part (b).

I: Explain how you sketched your graphs.

C: Kind of like the first one. As time increases, the height is going to decrease, and it’ll be faster for the second one.

I: Uh huh.

C: Faster and steeper for the second one because the rate is more than the first one.

I: Okay. So, does the shape, in part (a), you realized that shape affected the graph. Do you think the shape will affect this graph?

C: Yeah.
I: Can you sketch on there what you think . . .
C: [Draws on paper (see red marks in Figure 4.26)].
I: So, is this where you are saying that the shape changes? This little curve that you’ve made right here?
C: Yeah. After this point, it’s a little slower.
I: Okay. Now, right here is where you have your graph, umm, beginning. Will it ever reach, when time is 0, what will height be?
C: When time is 0, height is going to be 0. No. It will be maximum.
I: Okay. So, will it actually touch the h-axis?
C: It depends on what the height of the cooler is.
I: Okay. But it will . . .
C: It will touch. Yeah.
I: Will it start at the same point?
C: Yeah, because the height is there.

Figure 4.26 Cindi’s graphs for interview question 2b.

After prompting from the interviewer, Cindi expressed that her graphs will touch the axis, but she does not know where. She said, “it depends on what the height of the cooler is.” This suggests that Cindi is not comfortable choosing an arbitrary height for the coolers and representing it on the vertical axes. Needing a particular number for the maximum
height can be possibly related to an action conception of function.

Other students were observed drawing graphs for part (b) as touching the the vertical axis, but they indicated different vertical intercepts, which is an issue concerning the range of the functions relating time and height. For example, Sam initially drew graphs that were incorrect in terms of shape and in terms of both the vertical and horizontal intercepts (see graphs drawn in pencil in Figure 4.27). His original solution is in pencil, while is corrected solution is in red. For his original graphs, we see that the vertical intercepts are different, while the horizontal intercepts are not shown. Only after another student responded did Sam change his graphs (see graphs drawn in red in Figure 4.27).

![Figure 4.27 Sam’s graphs for interview question 2b.](image)

**Issue 3: Relative rates.** The third-most common issue that was observed on questions 2a–b pertained to the relative rates of the functions represented in students’ graphs. In particular, on part (a) four students had issues representing the rates of the two functions relating time and volume. Moreover, these same four students had issues representing the rates of the two functions relating time and height in part (b).

Frank was one of the four students who demonstrated this issue (see Figure 4.28). As shown in the following excerpt, Frank’s incorrect solution can be traced back to a misconception about the rate.

F: Well, I was thinking that since $r_1$ was less than $r_2$ that they didn’t start at
the same. They were both full of water, but since, if $r_1$ is less than $r_2$, then I would assume that $r_1$ has less water than $r_2$.

I: Well, what does $r_1$ and $r_2$ represent?

F: Uh, the amount of water coming out, right? They’re being emptied at constant rates.

I: Does the amount of water coming out or the rate, affect where you initially start?

F: Ah, so I probably got that wrong, but I think, um, if $r_1$ was a slower rate than $r_2$, then um I’ll probably have to switch my notes around, but my first, top graph or whatever would like go all, um $r_1$ would take longer time for it to get all the way empty to 0 and, uh, my bottom one, it would take it a lot sooner.

![Figure 4.28 Frank’s graphs for interview question 2a.](image)

Frank incorrectly associated the rate with the amount of water in the cooler. In fact, he thought the rate and the volume were the same quantity. As a result, he concluded that Cooler 1 has less water than Cooler 2. However, this reasoning does not explain why he drew the graph for Cooler 1 to reach a volume of zero before Cooler 2 (see the two decreasing
graphs drawn in pencil in Figure 4.28). After prompting from the interviewer, he seemed to realize that his reasoning was not correct, but when he redrew his graph (in red), he still indicated that the vertical intercepts are different, while indicating the horizontal intercepts are the same. It is clear that Frank was not associating the notion of rate to the slopes of his graphs. Furthermore, Frank was considered to have a pre-action conception of function based on his incorrect reasoning about the function concepts involved in this problem.

**Issue 4: Direction.** One student, Kevin, had issues pertaining to the direction of his graphs on both questions 2a and 2b. Kevin drew increasing lines as a result of a misconception. The following excerpt contains his reasoning for his graphs in part (a). (See Figure 4.29 for his graphs.)

K: I said that $r_1$ was a little more steeper than $r_2$ cause it says assume the absolute value of $r_1$ is less than that of $r_2$. So I said that since $r_2$ is depleting faster the volume of it is smaller than the volume of $r_1$ and that it would be closer to the time than $r_1$ is. That’s what I said . . . maybe I didn’t probably draw it correctly.

I: So $r_1$ is steeper than $r_2$?

K: Because the volume, it has more volume than $r_2$ does cause $r_2$ is moving at like a faster rate cause it says it’s (indiscernable). I don’t know. I probably drew it incorrectly.

In Kevin’s solution (see Figure 4.29), we see that he initially drew (in pencil) increasing graphs for the functions relating time and volume. From his explanation above, he clearly interpreted the coolers as being emptied, but he was not able to draw his graphs correctly. Furthermore, his interpretation of rate was incorrect. It seems that he associated lower rate with higher volume. Based on this, Kevin concluded that Cooler 1 must have more volume, which is why he drew his graph for Cooler 1 above his graph for Cooler 2. Kevin eventually seemed to realize that his graphs were incorrect and changed his answer (in red). However, he never explained his reasoning. Note that even his new answer does not clearly satisfy
all of the required criteria. That is, it is not clear if his new graphs have the same vertical intercept, an issue pertaining to the range. Kevin was also considered to have a pre-action conception of function based on his reasoning.

4.4.2.2 Interview question 2c–d

On question 2c of the interview, students were asked to sketch the graph of the relationship between the volume of the water and the height of the water for each of Coolers 1 and 2 (see Appendix C). Because the coolers are identical and because time is not a factor in this graph, both coolers will have the same graph. Investigating students’ reasoning about this task was the main goal of question 2c of the interview. Because this task involves the relationship between volume and height, which is invariant between each cooler, this task is related to the concept of function by constructing a relationship between volume and height directly. Moreover, the framing of this task involved time as an explicit variable (in parts a–b). Therefore, “eliminating” the variable of time to construct the invariant relationship between volume and height in part (c) is related, at least implicitly, to the concept of parametric function. This task is further related to the concept of parametric function because on part (d) students were asked to indicate the orientation of their graph in part (c).

This problem is a variation on the bottle problem, which has appeared in several studies (e.g., Carlson et al., 2002; Hitt, 1998; Johnson, 2012a, 2012b). All of these versions of
the bottle problem focused on filling a bottle. Moreover, the studies by Johnson (2012a, 2012b) and Hitt (1998) investigated students' reasoning about how changing the shape of the bottle affects the graphs relating volume and height. However, none of these studies explicitly investigated whether students are aware that changing the rate or emptying the bottles does not determine the relationship between volume and height. The version of the bottle problem used in the current study was designed to explore this issue.

Out of the fifteen students interviewed, eleven were able to perceive, either with or without prompting, the invariant relationship between the volume and height of the water as independent of the rate at which the coolers are being emptied. The remaining four students, on the other hand, could not.

Out of the eleven students who could perceive the invariant relationship between volume and height, either with or without prompting, drew a graph for part (c) that was completely correct. A notable case was Oliver. He realized that the relationship between the volume and the height is the same for each cooler. This was despite his initial error in drawing his graph in the wrong direction (see Figure 4.30). Oliver realized his error, corrected his graph (in red pen), and gave clear and correct reasoning for the change. The following excerpt contains Oliver's reasoning.

O: Uh, so I drew this graph like this, but it doesn't make any sense, because like the more volume you put, the height would get higher. So I, you know, redrew the graph, and that point of, [makes hand motions to describe where the coolers change shape], it will fill in faster, the height will get up faster.
I: And so, um, would the rate affect the graph? Or would you have different graphs for the two containers?
O: No, because volume and height would be the same.

Oliver knew that the relationship between the volume and height is the same for both coolers, evidenced by drawing only one graph (see the graph in black ink in Figure 4.30). His reasoning for drawing only one graph was “because the volume and height would be the same.” Eventually Oliver realized that the direction of his graph did not accurately
Figure 4.30 Oliver’s graph for interview question 2c.

represent the relationship between volume and height. He said, “but it doesn’t make any sense, because like the more volume you put, the height would get higher.” Oliver’s new graph (in red ink) is completely correct by satisfying all of the required criteria: direction, shape, domain and range, and relative rates. Indeed, he considered the shape of the cooler, implying that at the narrower portion of the cooler, “it will fill in faster, the height will get up faster.” Notice that Oliver’s responses suggested that he was no longer considering that the coolers are being emptied, by describing increases in volume and height. This is an indication of a process conception of function, but not necessarily a process conception of parametric function. In fact, when Oliver was prompted to explain the orientation of his graph, he appeared to have some difficulty. He said:

O: Uh, the height is 0 when the volume is. It doesn’t—like, you know—go up with it. [Pause.] Yeah. Like, um, um, um, like the volume is decreasing. It’s supposed to decrease. So like this is like say our max and it’s supposed to decrease because we’re draining it out. So it goes to 0.

Notice that Oliver never described height as decreasing in the above excerpt. Instead, he considered only how the volume would change with respect to time. This suggests that Oliver had difficulty imagining the height and volume as decreasing simultaneously with respect to time. It is very likely that this difficulty is what led Oliver to his original incorrect solution, especially considering his graphs in parts (a) and (b). In part (a), Oliver drew
decreasing lines to describe the relationship between time and volume, and in part (b) he
drew decreasing piecewise linear graphs to describe the relationship between time and height.
Oliver’s original graph for part (c) being similar to his graphs for part (a) suggests that the
change in volume with respect to time was a predominant image for Oliver, which he was
not able to coordinate with the process of height decreasing with respect to time.

Out of the eleven students who could perceive the invariant relationship between volume
and height, seven gave solutions that were correct in terms of direction but incorrect in terms
of shape, domain, or range. For example, Hannah drew one increasing, concave up graph to
describe the relationship between the volume and height for both coolers (see Figure 4.31).
The following excerpt contains her reasoning for drawing only one graph.

H: I said that um since they’re exactly the same, they would have the same
graph. Uh, since time isn’t a factor, it doesn’t matter at what point they’re
at in relation to each other. So their height and volume ratio would be the
same. And they would start at the same um height and volume when they’re
full and steadily decrease to 0.

Figure 4.31 Hannah’s graph for interview question 2c.

When Hannah explained, “their height and volume ratio would be the same,” she was
describing that the ratio of height to volume will always be the area of a cross section of
the coolers. Notice that Hannah used the word *same* instead of *constant*. This indicates a significant observation on her part. The area of a cross section in the bottom part of the coolers is different than the area of a cross section in the top part of the coolers. Therefore, the ratio of height to volume is not *constant*. However, the ratio of height to volume is the *same* variable quantity, namely the cross-sectional area. By stating, “it doesn’t matter at what point they’re at in relation to each other,” Hannah indicated a process conception of function because she can conceive of the ratio of height to volume as determining the same variable quantity as time increases. In fact, according to Thompson’s (1994) description of ratio and rate, Hannah actually constructed a rate by realizing that the ratio continues as time increases. Furthermore, Hannah explained the orientation of her graph by stating that the coolers would start at the same height and volume and “steadily decrease to 0.” This dynamic view is indicative of a process conception of parametric function.

Two students, Bailey and Nicole, initially drew different graphs for Coolers 1 and 2, but they were able to overcome this issue with minimal prompting from the interviewer.

I: Tell me what the ordered pair (0,0) means on your graph.

B: When it’s empty. The height is 0 and the volume is 0.

I: Alright. And this is your rightmost point. You (Nicole) don’t have points together. Tell me about this. Tell me why you did that.

N: I don’t know. If the containers are the same, then their heights should be the same, and their volume should also be the same. It’s just the rate at which it’s dripping would be different. So, it really should be the same. I just wasn’t paying attention.

I: Okay . . . Why do you have different graphs?

N: Like, why do we have two lines? Or why do our graphs look different?

I: Why do you have two lines?

B: Because $r_1$ is emptying slower than $r_2$.

I: But there is no rate involved here.

B: You’re right. They should be the same.
Figure 4.32 Bailey and Nicole’s solutions to interview question 2c–d.

Initially, Bailey drew different graphs for Coolers 1 and 2 (see Figure 4.32a), but he knew that the leftmost points and rightmost points must be the same for because they represent the minimum and maximum volumes/heights for the coolers. The interviewer asked him why he drew two graphs, he expressed that he initially believed that the rate at which the coolers are emptying should be reflected in the graphs. When the interviewer pointed out that “there is no rate involved here,” Bailey corrected his answer (verbally, not on paper) by saying, “you’re right. They should be the same.”

Nicole also drew two different graphs, but her graphs did not share the same endpoints (see Figure 4.32b). With prompting from the interviewer, Nicole realized this discrepancy. She said, “if the containers are the same, then their heights should be the same, and their volume should also be the same. It’s just the rate at which it’s dripping would be different. So, it really should be the same. I just wasn’t paying attention.” From this statement, it is not clear that she realized that the points on the interior of her graphs should be the same in addition to the endpoints. However, the following excerpt confirms that she made this realization.¹

The excerpt below is a continuation of the previous one. In it, Bailey and Nicole further

¹Note that Nicole drew her graph for Cooler 1 to be more steep than her graph for Cooler 2. This was an issue that she had throughout question 2, but it was never addressed.
reason about why the graphs should be the same for both coolers. The third student in their interview, Lee, was adamant that the graphs for Cooler 1 and Cooler 2 should be different.

L: No. They’re supposed to be different because for each point, for each \( t \) value, which is what we’re determining where these points are, for Cooler 1 and Cooler 2, the height and the volume are completely, the height and the volume are different. At \( t = 2 \), the volume of Cooler 2 is lower and the height of Cooler 2 is lower than Cooler 1.

I: We’re just looking at the relationship between volume and height.

B: Yeah. They should be the same.

L: How do you look at the relationship between volume and height?

B: Because at any amount of time, the volume is determined by the height. If you look at, like, a cylinder, what is it, \( \pi r^2 \) height. So, yeah, the volume and the height, they move as one.

N: Think of it like this. Let’s say you have, like, a bowl of the same size, same everything, and you fill it up with with water and you put a little hole in one and you put a bigger hole in the other one. They’re still going to have the same height and volume. It’s just the rate, like, earlier the rate was different, but we’re not talking about that. So, in this sense, we have two identical things.

L: So, what if the height is 10 and the volume is, like, 50?

N: It could be.

L: Then it’s not the same.

N: Yeah. I mean, they’re still the same.

L: Numerically, they’re not the same though. The height is 10 and the volume is, like, 50 milliliters.

B: The two containers, they’re identical containers. So no matter what your height is, their volume is equal. No matter what their volume is, if their volume is the same, the height is the same. If their height is the same, their
volume is the same. We’re not dealing with any relation of time.

L: Okay.

Figure 4.33 Lee’s graphs for interview question 2c–d.

Bailey and Nicole appeared to resolve the issue of drawing two graphs. Bailey explained his reasoning by saying, “because at any amount of time, the volume is determined by the height.” He further explained, “no matter what their volume is, if their volume is the same, the height is the same. If their height is the same, their volume is the same.” By saying “no matter what,” Bailey indicated that this relationship continues throughout the entire time that it takes to empty the coolers, which is indicative of a process conception of function. Moreover, he said, “so, yeah, the volume and the height, they move as one,” which indicates that he paired the value of volume and height to form a multiplicative object (Thompson, 2011).

Nicole also was able to resolve the issue of drawing two graphs. She gave an example and said, “so, in this sense, we have two identical things.” Lee, on the other hand, was not able to resolve this issue. She correctly observed that the coolers have the same maximum and minimum volumes/heights, which occur at $t = 0$ and the terminal time. This was indicated by her graphs having the same endpoints. Furthermore, she correctly stated, “at $t = 2$, the volume of Cooler 2 is lower and the height of Cooler 2 is lower than Cooler 1.” However, Lee
could not view the relationship between volume and height separate from time, which was evident when she asked, “how do you look at the relationship between volume and height?” This indicates that Lee could not construct the process of height and volume varying in relation to each other.

According to Thompson (1994), a ratio is constructed by comparing two quantities multiplicatively. He further stated, “as soon as one reconceives the situation as being that the ratio applies generally outside of the phenomenal bounds in which it was originally conceived, then one has generalized that ratio to a rate (i.e., reflected it to the level of mental operations)” (Thompson, 1994, p. 192). From this perspective, it is likely that Lee constructed a ratio, indicated by her graphs having the same endpoints, but she did not generalize the ratio to a rate that applies over the entire interval of volume.

Overall on part (c), fourteen students drew graphs in the correct direction and indicated the correct orientation, even if they struggled to do so, such as the case with Oliver. Eleven students, either with or without prompting, were able to conceive of the relationship between volume and height as independent of time and invariant between the coolers. Four students, on the other hand, could not. Each of these four students was under the impression that the rate at which the coolers are being emptied somehow affects the graph of the relationship between volume and height and, therefore, drew different graphs for each cooler. This is an issue pertaining to the criteria of the relative rate of change of the height with respect to volume for each cooler. Only one of these four students, Lee, correctly indicated the domain and range by drawing her graphs to have the same endpoints. These results appear to validate Johnson’s (2012a) findings, stating that simultaneous-independent reasoning (as if two quantities are changing independently with respect to time) “does not seem to support the consideration of variation in the change in one quantity with respect to the change in another quantity” (p. 52). By not eliminating time as the variable, these students could only compare changes in volume with changes in height as they vary with respect time, rather than coordinate their changes as they vary with respect to each other.

The purpose of question 2 of the interview was to investigate how students reason about
two quantities (particularly volume and height) changing simultaneously with respect to a
third quantity (time) and to determine whether or not they can perceived the relationship
between volume and height as independent of time. Four issues were identified in students’
responses to each of parts (a)–(c) of question 2 of the interview. These issues dealt with
the shape of students’ graphs in relation to the shape of the coolers, the direction in which
students drew their graphs, the domain and range of the functions represented by the graphs,
and the relative rates\(^5\) of the function represented by their graphs. Table 4.2 provides an
overview of Section 4.4.2 by indicating the number of students who had difficulty with
each of the main issues on questions 2a–c. Note that if a student gave an initial solution
demonstrating one of these issues, he or she was excluded from this table as long as the
issue was corrected (at least verbally) and the reasoning for the correction was explained.
Therefore, this table summarizes the number of students who were not able to overcome
these issues. From the table, we can see that one student had issues with the direction of
his graphs in all three parts of question 2. This was the same student, Kevin, in each case.

Between parts (a) and (b), all students who had issues with the shape of their graphs
in part (a) had issues with the shape of their graphs in part (b), except for one student,
Hannah. Her graphs for parts (a) and (b) were identical (with part (b) completely correct).
Moreover, four students, Lee, Mary, Peggy, and Sam, had issues with the shape of their

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\(^5\)“Relative rates” for part (c) refers to the invariant relationship between volume and height for the
coolers.
graphs in part (b) but not in part (a). All seven students who had issues with the domain or range represented by their graphs for part (a) had the same issue in part (b). Moreover, one student, Sam, had an issue with the range in part (b) but not on part (a). The four students who had issues with the relative rates represented by their graphs in part (a) are the same students who had the issue in part (b).

On part (c), no additional students had issues with the shape of their graph who did not already have the same issue on at least one of parts (a) and (b). Two students, Nicole and Oliver, who had issues with the domain or range in part (b) did not have the issue in part (c), and one student, Mary, had the issue in part (c) but not in parts (a) or (b). One student, Lee, who did not have issues with the relative rates in parts (a) and (b) had the issue in part (c), while one student, Nicole, had the issue in parts (a) and (b) but not in part (c).

4.5 Overview

In summary, this chapter presented the results of the investigation into students’ reasoning about parametric functions. In particular, this chapter provided a detailed data analysis and interpretation in terms of APOS theory and with respect to each of the research questions (see Section 1.2) by identifying the constructions that the students in this study tended to make when reasoning about parametric functions. The goal of this study was not to track the progress of each student but to gain insight into the constructions that an individual should make in order to develop a deeper understanding of the concept of parametric function. The results presented in this chapter helped address this goal. In Chapter 5, these results are discussed in light of other findings reported in the literature and recent advances in undergraduate mathematics education research.
Chapter 5

CONCLUSION

This study employed APOS theory to investigate fifteen second-semester calculus students’ understanding of the concept of parametric function, as a special relation from a subset of \( \mathbb{R} \) to a subset of \( \mathbb{R}^2 \). In particular, this study explored (1) students’ personal definitions of parametric function; (2) students’ reasoning about parametric functions given in the form \( p(t) = (f(t), g(t)) \); (3) students’ reasoning about parametric functions on a variety of tasks, such as converting from parametric to standard form, sketching a plane curve defined parametrically, and constructing a parametric function to describe a real-world situation; and (4) students’ reasoning about the invariant relationship between two quantities varying simultaneously when described in both a graph and a real-world problem. The goal of this study was not to classify each student according to the constructions he or she made, but rather to determine whether the preliminary genetic decomposition proposed in Section 3.3 is a reasonable model for how an individual might construct the concept of parametric function.

Section 5.1 presents an overview of the results reported in Chapter 4 by responding to each of the research questions posed in Section 1.2. Moreover, the results of this study are discussed in light of other findings that have been reported in the literature. By doing so, this study is situated in a broader body of research that reflects current advances in research in undergraduate mathematics education. In Section 5.2 a revision of the genetic decomposition is presented based on the results of this study. Section 5.3 considers the implications for teaching the concept of parametric function, and Section 5.4 briefly notes about the limitations of this study. Finally, Section 5.5 concludes the study with suggestions for future research on the topic of teaching and learning of parametric functions.
5.1 Discussion of results

The results of this study provided answers to each of the research questions and gave insight into each of the constructions called for by the preliminary genetic decomposition. In this section, the findings from Chapter 4 are discussed in light of each of the research questions posed in Section 1.2. Moreover, these findings are situated in a larger body of related literature pertaining to the concepts of function, multivariable function, and change of coordinates.

5.1.1 Research question 1

Research question 1 asked what students’ personal definitions are for the concept of parametric function. To address this question, students were asked on the final exam to state in their own words the parametric function definition (see question 3 in Appendix B). Students’ definitions were analyzed according to whether they satisfied the following four criteria:

(i) Identify an independent variable (or parameter) \( t \).

(ii) Identify a dependent variable \( x \).

(iii) Identify a dependent variable \( y \).

(iv) Coordinate \( x \) and \( y \) to form an ordered pair \( (x, y) \).

It was considered essential for a foundational understanding of parametric function to have a personal definition that satisfies at least the first three criteria. Out of the thirteen students who responded to this question, four gave a personal definition that satisfied criteria (i)–(iii), while five students gave a personal definition that satisfied all of criteria (i)–(iv). The remaining four definitions were considered to not relate at all to the concept of parametric function and, thus, were classified as pre-action level definitions. Of the nine definitions that satisfied criteria (i)–(iii) or (i)–(iv), one was classified as an action-level definition; seven
definitions were classified as process-level definitions, one of which was close to a formal\textsuperscript{1} definition; and one definition could not be classified according to APOS theory. In particular, this student’s definition said:

A parametric equation is a function in respect to time. It’s more 3-dimensional, while a regular $y$ and $x$ function will only be a 2-dimensional graph.

Using parametric equations to describe 3-dimensional objects agrees with the historical development of the concept. In particular, Euler used parametric functions to describe the non-planar intersection of two surfaces in Book II of his 1748 *Introductio in Analysin Infinitorum*. This motivated him to first describe non-planar curves by using two equations in three variables, two of which depending on the third. Only after did Euler describe the process of converting to what we call today “standard form.” To him, eliminating the common independent variable was a way to describe the projection of the non-planar curve onto the plane corresponding to the two dependent variables. This is contrary to the way in which parametric functions are typically introduced in the modern calculus sequence, which occurs prior to the study of multivariable functions.

Note that other studies have reported cases of students demonstrating intuition about a concept that is in agreement with its historical development, despite the concept being taught differently in class (see, for example, Czarnocha et al., 2001). Several other researchers agree that an analysis of the historical development of a concept can shed light on the obstacles faced by students and suggest natural trajectories of teaching and learning the concept (Bressoud, 2010; Grugnetti, 2000; Sfard, 1992; Sierpinska, 1992).

Overall, the results pertaining to research question 1 show that most of the students in this study had a personal definition of the concept of parametric function that was different from the definition stated in class. Indeed, while the instructor defined a parametric function as a function that transforms an input into a unique output in the form of an ordered pair, only one student in this study gave a similar definition. From the perspective concept

\textsuperscript{1}By referencing the uniqueness property of the general function definition in the context of parametric functions.
image–concept definition (Tall & Vinner, 1981; Vinner, 1983; Vinner & Dreyfus, 1989), these findings suggest that the students in this study constructed (or reconstructed) their personal concept definition in response to their interpretation of the various ways in which the concept appeared (e.g., equations, curves, particle motion). In the end, the students’ personal concept definitions were at odds with the formal concept definition introduced by the instructor. This indicates that more opportunities should be created for students to develop a concept image and personal concept definition that are in agreement with the intended concept definition. As stated by Vinner (1983), “elements of the concept image that are not constantly reinforced have a good chance of being forgotten and thus the concept image is distorted” (p. 305).

5.1.2 Research question 2

Research question 2 asked how students reason about parametric functions given in the form \( p(t) = (f(t), g(t)) \). To address this question, students were asked during the interview to determine if \( p(t) = (t^2, t^3) \) represents a function (see question 1c in Appendix C). A correct answer would affirm \( p \) as a function because for one value of \( t \) there is a unique ordered pair. Students’ responses to this question provided insight into the generalization of their schema for real-valued function. Overall, only five students, with or without prompting, were able to generalize the concept of real-valued function to the concept of parametric function in order to affirm that \( p \) is a function that transforms an input into a unique output in the form of an ordered pair.

Several misconceptions emerged in students’ reasoning about the parametric function given by \( p(t) = (t^2, t^3) \), providing insight into research subquestion 2(i). Overall, twelve out of fifteen students expressed at least one misconception about parametric functions, although two students overcame their misconceptions. A total of four misconceptions were identified, which pertained to the function value \( p(t) \), the domain and range of \( p \), the vertical line test, and the input and output of \( p \).

Nine out of fifteen students expressed a misconception about parametric functions per-
tained to the function value \( p(t) \). Two of these nine students expressed a misconception by requiring the coordinates defined by the components of \( p(t) \) to be different. Another possible misconception was that all of the ordered pairs defined by the components of \( p(t) \) had to be different, i.e., \( p \) has to be one-to-one. The remaining seven students who expressed a misconception about the function value \( p(t) \) viewed \( p \) as having two outputs for one input \( t \), namely \( t^2 \) and \( t^3 \). That is, they did not perceive \( t^2 \) and \( t^3 \) as forming a unique quantity in the form of an ordered pair. Two of these seven students were able to overcome this misconception. However, the remaining five students were unsuccessful.

The second misconception about parametric functions pertained to the domain and range of \( p \). One student had misconceptions about the range of \( p \), believing it to be made up of values of the second component of \( p \), and one student had misconceptions about both the domain and the range of \( p \). Since the first component of \( p \) defines the \( x \)-coordinate and the second component defines the \( y \)-coordinate, this misconception can possibly be traced to a limited notion that the domain of a function is made up of \( x \) values and the range is made up of \( y \) values. This misconception did not appear to be related to a misconception about the input. On the contrary, Bailey clearly referred to \( t \) as the input value, while also considering the domain to be made up of values of \( t^2 \).

The third misconception about parametric functions that emerged pertained to the vertical line test. Two students believed that \( p(t) = (t^2, t^3) \) cannot represent a function because the graph of its plane curve fails the vertical line test. One of these students even knew that he was reasoning about the relationship between \( t^2 \) and \( t^3 \) and, thus, concluded that \( t^3 \) is not a function of \( t^2 \). This indicates that his existing function schema for real-valued function prevented him from comprehending the question regarding whether the ordered pair \( (t^2, t^3) \) is a function of \( t \).

The fourth misconception about parametric functions that emerged pertained to the input and output of the function \( p \). One student expressed the misconception that the value of the first component of \( p(t) = (t^2, t^3) \) is the input of the function \( p \), while the value of the second component is output of the function. This misconception appears to be related
to the usual convention of expressing the input value as the first coordinate of an ordered pair and the output value as the second coordinate. Note that misconception 4 is different from misconception 2. The student, Hannah, who expressed misconception 4 believed that $t^2$ is the input value, while Bailey who expressed misconception 2 believed that $t$ is the input value. This suggests that although misconceptions 2 and 4 might be related, they should be treated as distinct misconceptions.

Several of these issues can perhaps be traced back to the fact that, in single-variable calculus, parametric functions are used to describe plane curves, and as a result, the independent variable $t$ is not explicitly represented in the graph. For example, in order to use the vertical line test to determine if a parametric function is a function, the input value $t$ would have to be represented by an axis. This idea agrees with the historical development of the concept, as previously discussed in Section 5.1.1. Moreover, other researchers have described representing values of $t$ on an axis (Keene, 2007; Oehrtman et al., 2008). In fact, one student, Bailey, in the current study described $t$ as the horizontal axis, but this did not resolve his misconceptions, as he indicated to have a weak schema for $\mathbb{R}^2$ and no indication of a schema for $\mathbb{R}^3$.

Despite reasoning incorrectly about the parametric function given by $p(t) = (t^2, t^3)$, most of the students in this study indicated that they had previously developed, to at least some degree, a schema for real-valued function, which provided insight into research subquestion 2(ii). During the interview, the students were asked to state their definition of function. Eleven out of fifteen students gave a definition of function in terms of the uniqueness property; three students gave a definition in terms of the vertical line test; and one student gave an incorrect definition. Also during the interview, the students were asked to explain whether or not $\rho(\theta) = \theta$ and $g(x) = \pm\sqrt{4-x^2}$ represent functions. Eleven out of fifteen students\footnote{It is only a coincidence that this number aligns with the number of students who gave a function definition in terms of the uniqueness property. That is, students who reasoned correctly about both $\rho$ and $g$ had variations in their function definitions.} were able to reason correctly about both $\rho$ and $g$. However, only three of these eleven students affirmed with correct reasoning that $p(t) = (t^2, t^3)$ represents a
function. These results suggest that other constructions are involved in generalizing the schema for real-valued function to account for parametric functions. Indeed, students who affirmed $p$ as a function with correct reasoning appeared to coordinate their schemas for function and $\mathbb{R}^2$, while students who rejected $p$ as a function either had a weak function schema, a weak schema for $\mathbb{R}^2$, or were unable to coordinate their schemas for function and $\mathbb{R}^2$. Bailey was a notable case of a student who indicated having a well-developed schema for function but an underdeveloped schema for $\mathbb{R}^2$. As a result, he could not view $(t^2, t^3)$ as the output of the function $p$ at the value $t$.

Overall, the results pertaining to research question 2, illustrate the complexity in generalizing the concept of function to different types of functions and to different representations. Similar findings have been reported in literature regarding students’ understanding of multivariable functions and functions in different coordinate systems (Clement, 2001; Kabael, 2011; Martínez-Planell & Trigueros Gaisman, 2012; Montiel et al., 2008, 2009, 2011; Moore et al., 2013; Trigueros & Martínez-Planell, 2010; Weber & Thompson, 2014).

For example, other researchers have reported students having misconceptions about the vertical line test. Clement (2001) reported that out of five precalculus students, three used the vertical line test as a basis for claiming that the position of caterpillar on a path that intersects itself is not a function of time. Although Clement did not explicitly say so, this example essentially involves a parametric function. Students in Clement’s study, as well as students in the current study, did not appear to understand that the vertical line test is not suitable for curves defined parametrically. These results are also aligned with the findings of Montiel et al. (2008), who observed that students have misconceptions about the vertical line test that prevent them from seeing certain graphs, such as circles, in the polar coordinate system as representing functions. In a follow-up study, Montiel et al. (2009) provided an example of a student who knew that the vertical line test is not applicable in the polar coordinate system but could not explain why in a way that is aligned with the institutional expectation. Something similar was also observed in the current study when one student (Lee) said, “they might let it slide in polar coordinate system,” referring to
the vertical line test determining “more than one output.” Moore et al. (2013) explained students’ misconceptions about different coordinate systems by stating,

Students who develop graph and function meanings rooted in shapes and operations on these shapes are posited to solve problems within the system in which those meanings were developed. But when they move to a different system, meanings that inherently involve activity dependent on system conventions become problematic because shapes and the representational conventions are changed. (p. 471)

Furthermore, Moore et al. (2013) conjectured that engaging students in covariational reasoning while graphing in more than one coordinate system can enable students to perceive two different graphs, one in the Cartesian coordinate system and one in the polar coordinate system, as representing the same relationship.

The importance of the interplay of different schemas in developing mathematical concepts has also been reported by other researchers. The findings of Trigueros and Martínez-Planell (Martínez-Planell & Trigueros Gaisman, 2012; Trigueros & Martínez-Planell, 2010) suggest that in order to construct the notion of two-variable function, the individual should coordinate schemas for function, set, and $\mathbb{R}^3$ by assigning a unique height to each point on a given subset of $\mathbb{R}^2$. In their second study in particular (Martínez-Planell & Trigueros Gaisman, 2012), the authors found that students who were not able to coordinate these schemas had difficulty describing the domain and range of two-variable functions, and such students tended to view the domain of a two-variable function as two subsets of $\mathbb{R}$ rather than one subset of $\mathbb{R}^2$. The current study found something very related, but in the reverse order. That is, while students in the study by Martínez-Planell and Trigueros Gaisman (2012) did not view the input of a two-variable function as an ordered pair, the students in the present study did not view the output of a parametric function as an ordered pair (see also Martínez-Planell & Trigueros Gaisman, 2009). Other students in the study by Martínez-Planell and Trigueros Gaisman (2012) demonstrated the misconception that the domain refers to values on the $x$-axis, while the range refers to values on the $y$-axis. Two students in the present study expressed similar misconceptions. Kabael (2011) also reported that students have difficulty with the domain and range of two-variable functions. Furthermore, she found that
the function machine can have a positive effect on students’ generalization of the concept of single-variable function to the concept of two-variable function.

5.1.3 Research question 3

Research question 3 inquired how students reason about parametric functions on various tasks, such as (1) converting from parametric to standard form, (2) sketching the graph of a curve defined parametrically, and (3) constructing a parametric function to describe a real-world situation. While most of the students in this study did not have a formal conception of parametric function as a special relation from a subset of \( \mathbb{R} \) to a subset of \( \mathbb{R}^2 \), as discussed, many expressed a process conception or even an object conception of parametric functions when reasoning about the concept in different situations.

Question 1a on the final exam (see Appendix B) asked students to convert from parametric to standard form for the curve

\[
x(t) = \frac{1}{\sqrt{t}}, \quad y(t) = \frac{1}{t^2}, \quad t > 0.
\]

Students’ solutions provided insight into research subquestion 3(i). All fifteen students responded to this question, but only twelve were able to perform actions on the equations for \( x(t) \) and \( y(t) \) to result in a single equation in \( x \) and \( y \). Seven of these twelve students either did not know that they needed to consider restrictions on \( x \) or \( y \) or did not know how to determine those restrictions. These students were considered to have an action conception of parametric function. Meanwhile, five students gave solutions that indicated a process conception of parametric function by correctly determining the restrictions on the domain of the curve in standard form. Although these results support the constructions in Step 4 of the preliminary genetic decomposition (see Section 3.3), which called for the reversal of the process of one function and composing it with the other, it should be emphasized that this requires an object conception of inverse function. This claim is supported by APOS literature regarding compositions of functions and inverses of functions (Ayers et al., 1988;
Vidakovic, 1997). As a result, students are then more likely to understand that the range of the function given by \( x(t) \) becomes the domain of the function for the curve in standard form.

The second task involving parametric functions that was of interest to this study involved sketching the graph of a curve defined parametrically. On question 1b of the final exam, students were asked to sketch the graph of the curve that was in question 1a (see previous paragraph or Appendix B). Students’ solutions gave insight into research subquestion 3(ii). It was found that most students were limited to action conception of parametric function when sketching the graph of the curve described in the previous paragraph. Some of these students plotted points based on a table of values, while others indicated the wrong orientation of the curve (or not at all). One student, on the other hand, exhibited a process conception of parametric function by coordinating changes in \( t \) with changes in \( x \) and then coordinating those changes in \( x \) with changes in \( y \). This construction was different than what was called for by Step 1 of the preliminary genetic decomposition, which involved comparing changes in \( x \) and \( y \) as they vary simultaneously with respect to \( t \). Step 1 of the preliminary genetic decomposition describes simultaneous-independent reasoning (Johnson, 2012a, 2013; see also Thompson, 2011). On the other hand, the aforementioned student appeared to use dependent reasoning (Johnson, 2012b, 2013) by directly coordinating the changes in \( x \) and \( y \). The genetic decomposition should be revised to reflect this possible construction.

When sketching the graph of a curve defined parametrically, two students appeared to be in transition from an action to a process conception of parametric function. This was indicated by their reasoning about the changes in \( x \) and \( y \) based on what they observed in their table of values. These students compared direction or amount of change in \( x \) with the direction or amount of change in \( y \), which is simultaneous-independent reasoning (Johnson, 2012a, 2013). This seems to support the validity of Step 1 of the preliminary genetic decomposition.

Researchers have reported that covariational reasoning is important for sketching graphs of curves defined parametrically (Bishop & John, 2008; Oehrtman et al., 2008).
but related context, Weber and Thompson (2014) stated that graphing a two-variable function requires a student to construct three quantities and in order to describe the relationship between these three quantities, the student should consider the relationship between each pair of quantities. The authors found that quantitative reasoning and covariational reasoning can support the visualization of multivariable functions without relying on the memorization of prototypes. The fact that such a small number of students in the current study exhibited (or began to exhibit) a process conception of parametric function in the context of graphing parametric curves indicates that instructors should create more opportunities for students to reason about the changes in two quantities varying simultaneously with respect to a third. Furthermore, students difficulties in sketching graphs of curves given parametrically could be related to their difficulties with graphing in general. Baker, Cooley, and Trigueros (2000) reported that calculus students have difficulty on graphing tasks due to the lack of coordination of schemas that make up their calculus graphing schema. Although the current study did not investigate how students use their calculus graphing schema when sketching curves defined parametrically, such a study would be warranted, particularly to investigate how students use their calculus graphing schema to sketch graphs of more exotic curves defined parametrically, such as those that intersect themselves.

The third task involving parametric functions that was of interest to this study involved constructing a parametric function (i.e., parametrize) to describe a real-world situation. In particular, students on question 3 of the interview were asked to determine the position of a particle at a certain time as travels along a given line segment at a certain speed (see Appendix C). Students’ responses on this task gave insight into research subquestion 3(iii). Only three students were able to obtain a correct solution without a considerable amount of prompting from the interviewer. The reasoning of these students indicated a different construction than the one called for by Step 5 of the preliminary genetic decomposition (see Section 3.3). Instead of reversing the process of function composition, these students appeared to have constructed an object conception of the particle’s path, which was then de-encapsulated into the processes of $x$ and $y$ varying simultaneously and independently
with respect to $t$. Two object conceptions were observed. One student, Alex, appeared to treat the relationship between $x$ and $y$ as an object, which he de-encapsulated into the aforementioned processes. By doing so, he was able to sketch two graphs, one for $x$ versus $t$ and one for $y$ versus $t$. From these graphs, Alex reasoned about the the $x$-coordinate and the $y$-coordinate for the particle separately. Keene (2007) also observed students incorporating a $t$-axis when describing how $x$ and $y$ change with respect to $t$. The author described such behavior as *making time an explicit quantity*.

Another student, Hannah, demonstrated an object conception in a different way when parametrizing the segment in question 3 of the interview. First, she appeared to have constructed a process conception of the particle’s accumulated distance (or arc length, in general), which she encapsulated into an object by observing that differentiating the accumulated distance with respect to time results in the particle’s velocity. Then Hannah de-encapsulated the object into coordinated processes to consider the particle’s accumulated distance in both the $x$ and $y$ directions by using similar triangles. The third student (Mary) who solved the problem correctly without prompting also approached the problem using similar triangles. One other student should be mentioned. On question 3 of the interview, Sam used a formula from class to give a parametrization of the segment. To take into account the context of the problem, he then attempted to reparametrize the segment. Although he was far from successful, this is a plausible approach to the problem. Step 5 of the genetic decomposition should be revised to consider an object conception of curve, the concept of arc length, and the notion of reparametrization.

Overall, the results pertaining to research question 3 indicate that parametrizing is significantly more difficult for students than converting from parametric to standard form, which was not surprising. In terms of APOS, many students at the action level were able to obtain a partially correct solution when asked to convert a curve from parametric to standard form. Meanwhile, no students at the action level were able to obtain even a partially correct solution to the parametrization problem without help from the interviewer. Such students attempted to perform the steps of the problem based on what they remembered in class, as
with Kevin, or attempted to manipulate formulas memorized from class or the text, as with Sam. These findings agree with those of Selden, Selden, and Mason (1994) who reported that calculus students have difficulty solving non-routine problems.

5.1.4 Research question 4

Research question 4 asked how students reason about the invariant relationship between quantities changing simultaneously with respect to a third. Such reasoning is foundational for understanding that, for curve defined parametrically, the rate of change of $y$ with respect to $x$ does not depend on the rate at which the curve was generated. Furthermore, this idea has been reported to be non-trivial for students, even when the parametric curve is a line (Bishop & John, 2008). The current study investigated this reasoning in two ways.

First, this study explored students’ reasoning about the invariant relationship between quantities changing simultaneously with respect to a third when they are described in a graph, which provided insight into research subquestion 4(i). On question 2 of the final exam, students were given graphs of two congruent parabolas, each generated at different rates, as indicated by time markers. Then they were asked to determine if the relationship between $x$ and $y$ is the same for both graphs and whether they have the same parametric representation (see Appendix B). Out of the eleven students who responded to this question, seven gave a response that was at least mostly correct and indicated a process conception of parametric function. These students were able to discriminate between the graphs in general terms, making no reference to particular points on the graphs or their corresponding $t$-values. The remaining four students gave a response indicating that the relationship between $x$ and $y$ is not the same on both curves because of the difference in the values of $t$. As a result, these students were considered to have a pre-action conception of parametric function in this situation.

Second, this study explored students’ reasoning about the invariant relationship between two quantities changing simultaneously with respect to a third when they are described in words in a real-world problem, giving insight into research question 4(ii). On question 2
of the interview, students were asked to sketch graphs pertaining to a situation involving two coolers of the same shape and size that are full of water and emptying at different (but constant) rates (see Appendix C). In part (a), they were asked to graph on the same plane the relationship between time and the volume of the water in each cooler. In part (b), they were asked to graph on the same plane the relationship between time and the height of the water in each cooler. In part (c), they were asked to sketch on the same plane the relationship between the volume and the height of the water in each cooler. This question from the interview is a variation on the bottle problem that has appeared in studies by Carlson et al. (2002), Hitt (1998), and Johnson (2012a, 2012b, 2013). However, their version of the problem involved a situation in which a container was being filled. Moreover, they did not ask if increasing the rate of change of volume with respect to time would affect the graph of the relationship between volume and height.

The analysis of students’ responses to question 2 found that most students could represent graphically the direction of the change in one variable with respect to the other, whether it was volume versus time, height versus time, or height versus volume. However, few students drew graphs that reflected the amount of change in one variable with respect to the other, while even fewer students drew graphs that accurately reflected the rate-of-change in one variable with respect to the other. Similar findings were reported by Carlson et al. (2002) regarding students’ difficulty with sketching graphs that accurately reflect rate-of-change. On the other hand, the current study found that the issue of rate-of-change of one variable with respect to the other for one cooler did not negatively affect students ability to consider the relative rate-of-change of the same variables for the other cooler. For example, on the parts of question 2 corresponding to volume versus time and height versus time, students who drew a graph for Cooler 1 that was incorrect in terms of shape drew a graph of the same shape for Cooler 2, only steeper.

Part (c) of question 2 was of particular interest to this study in light of research sub-question 4(ii). In their solutions, students were expected to consider that the relationship between volume and height is an increasing function and the rate of change of height with
respect to volume is the same for both coolers. As a result, the students should have drawn only one graph, indicating that the relationship between the volume and height of the water is the same for each cooler. Out of fifteen students, eleven were able to recognize the relationship between volume and height as invariant between the two coolers. These students appeared to have paired volume and height to obtain a multiplicative object (Thompson, 2011). Meanwhile, four out of fifteen students could not perceive the invariant relationship between volume and height. Three of these four students could coordinate the direction of the change in height with respect to volume, but they were not able to view the relationship as independent of time, which caused them to draw two graphs, one for each cooler. These results agree with Johnson’s (2012a) findings, stating that simultaneous-independent reasoning (as if two quantities are changing independently with respect to time) “does not seem to support the consideration of variation in the change in one quantity with respect to the change in another quantity” (p. 52). That is, in order to view the rate of change of height with respect to volume as independent of time requires viewing changes in height as dependent on changes in volume.

5.1.5 Brief summary

Overall, this study is aligned with recent advances in research in undergraduate mathematics education regarding students’ understanding of change of coordinates and multivariable functions (Kabael, 2011; Martínez-Planell & Trigueros Gaisman, 2012; Montiel et al., 2008, 2009, 2011; Moore et al., 2013; Trigueros & Martínez-Planell, 2010; Weber & Thompson, 2014). The results presented here tend to agree with the findings of these other researchers, while also contributing to our understanding of how students’ reason about parametric functions, which is a topic that has received very little attention in earlier literature (Bishop & John, 2008; Keene, 2007). Moreover, this study naturally situates itself within a much larger body of literature regarding students’ general understanding of the concept of function (Carlson, 1998; Breidenbach et al., 1992; Carlson et al, 2002; Dubinsky & Harel, 1992; Oehrtman et al., 2008; Vinner & Dreyfus, 1989).
The results of this study reveal that the notion of parametric function is very complex. The various ways in which the concept appears in curriculum can lead to a myriad of personal definitions among students. Prior to this study, it was unknown how (or if) students generalize the concept of real-valued function to the concept of parametric function. Most of the students in this study were not able to generalize their concept of real-valued function. Furthermore, this study revealed that certain misconceptions about parametric functions, and functions in general, inhibit the generalization of the function concept, many of which have been reported in studies (Kabael, 2011; Martínez-Planell & Trigueros Gaisman, 2012; Montiel et al., 2008, 2009; Trigueros & Martínez-Planell, 2010). This study also explored the constructions involved in converting from parametric to standard form and found that students should have developed an object conception of function to in order to consider the domain restrictions on the function for the curve in standard form. Moreover, this study revealed that students who were able to reason dynamically about relationship between $x$ and $y$ changing with respect to $t$ (as well as $y$ changing with respect to $x$) were in a better position to sketch graphs of curves given parametrically. This finding validates the results and claims of other researchers (e.g., Bishop & John, 2008; Oehrtman et al., 2008). Finally, this study contributed to other literature regarding the invariance of the relationship between two quantities varying simultaneously with respect to a third.

Since this study was conducted in the context of a course in single-variable calculus, it leaves the question as to whether or not the students in the study would have developed a stronger understanding of the concept, and thereby resolve their misconceptions, as a result taking a course in multivariable calculus. The effect of multivariable calculus on students' understanding of parametric functions should be investigated in the future.

5.2 Revised genetic decomposition

Although the results of this study seem to support, to some degree, the constructions called for by the preliminary genetic decomposition (see Section 3.3), they also suggest that some steps in the preliminary genetic decomposition need to be refined, while other steps should
be added. This section presents a revision of the genetic decomposition.³

Prior to the study of parametric functions, it is hypothesized that the student should have constructed the following schemas: a schema for \(\mathbb{R}^2\) that includes points as objects. The student should have an object conception of point in both the algebraic and geometric representations. The student’s \(\mathbb{R}^2\) schema should also include curves as made up of points. Moreover, students should have developed a schema for function that includes a process conception of single-variable, real-valued function. Furthermore, the schema for function should include an object conception of inverse function and a process conception of composition of functions.

The steps that are new or revised in the following genetic decomposition are indicated in bold:

1. Coordinate the processes of evaluating two functions \(f\) and \(g\) at values of \(t\) to imagine \(f(t)\) and \(g(t)\) changing simultaneously as \(t\) increases through values shared by \(\text{Dom}(f)\) and \(\text{Dom}(g)\). At this step, the individual can compare changes in \(f(t)\) with changes in \(g(t)\) over small intervals of \(t\).

2. Coordinate a process in Step 1 with the schema for \(\mathbb{R}^2\) to construct elements of \(\mathbb{R}^2\) defined by \((x, y) = (f(t), g(t))\). At this step, an individual with an object conception of point in \(\mathbb{R}^2\) can describe the process of \((f(t), g(t))\) tracing out a curve in \(\mathbb{R}^2\).

3. Suppose that one of the functions, say \(f\), in Step 1 is invertible. Reverse the process of \(f\) and encapsulate it into an object, namely \(f^{-1}\). Then compose it with the other function \(g\) in Step 1 to describe the relationship between \(x\) and \(y\) as a real-valued function \(g \circ f^{-1}\) of a single variable, which is represented by the equation \(y = (g \circ f^{-1})(x)\). At this step, the individual should recognize that \(\text{Dom}(g \circ f^{-1}) = \text{Range}(f)\) and \(\text{Range}(g \circ f^{-1}) = \text{Range}(g)\).

4. Encapsulate the process in Step 2 to compare and contrast congruent curves in \(\mathbb{R}^2\) in terms

³Again, this genetic decomposition is for the initial learning of parametric functions in the context of a second-semester course in single-variable calculus.
of their points and the rate at which they are generated. At this step, the individual can perceive the relationship between \( x \) and \( y \) as being independent of \( t \).

5. Construct the process of imagining the length of a curve up to a value \( t \) as a dynamic quantity represented by the distance accumulated by a particle at \( t \) as it moves along the curve.

6. Encapsulate the process in Step 5 by performing differentiation to obtain the particle’s velocity along the curve.

7. Parametrizing a curve can be accomplished in the following ways:

   (i) De-encapsulate the object in Step 4 back into the processes in Steps 1 and 2.

   (ii) De-encapsulate the object in Step 6 back into the process in Step 5.

   (iii) Reparametrization: begin with a parametrization \((f(t), g(t))\) and construct a process that transforms the “desired” interval on \( t \) into the current interval on \( t \) via a function \( h \). Then encapsulate \( h \) into an object to be composed with the functions \( f \) and \( g \) to obtain the desired parametrization \(((f \circ h)(t), (g \circ h)(t))\).

8. Generalize the function schema to apply to parametric functions by encapsulating the processes in Step 1 to pair \( f \) with \( g \). Then coordinate the function schema with the \( \mathbb{R}^2 \) schema to conceptualize \((f, g)\) as one function that transforms \( t \) into a unique element \((f(t), g(t))\) of \( \mathbb{R}^2 \).

Recall that one student, Hannah, appeared to have made a construction not called for by the preliminary genetic decomposition, indicated in her reasoning about sketching the graph of a curve. In particular, she appeared to have coordinated the process of \( x \) varying with respect to \( t \) with the process of \( y \) varying with respect to \( x \), which is a composition of functions. Although this coordinated process is not explicitly stated in the genetic decomposition, it is implied considering the constructions described in Steps 1 and 3. Therefore,
Steps 1 and 3 can be coordinated to obtain the process of Step 2, just as Hannah did on the final exam.

As suggested by the APOS framework (Asiala et al., 1996), this genetic decomposition should be tested empirically. The first step to doing so is to design instructional activities that are intended to guide students through each of the steps of the genetic decomposition. Then data should be collected and analyzed to determine if the instruction led students to making these constructions (or others). Based on the data analysis, the genetic decomposition should be further refined.

5.3 Implications for teaching

Based on the results of this study, the following recommendations\(^4\) are made:

1. Prior to the study of parametric functions:
   - Create opportunities for students to discuss functions in terms of inputs and outputs, rather than strictly in terms of \(x\)-values and \(y\)-values (Oehrtman, 2008). Similarly, the notions of domain and range should be emphasized to be the set of all input and output values, respectively, instead of sets made up of \(x\)-values and \(y\)-values.
   - Create tasks in which students would be asked to identify functions and non-functions.
   - Create opportunities for students to develop their \(\mathbb{R}^2\) schema.

2. Create tasks in which students would be asked to qualitatively describe changes in \(x\) and \(y\) as they change simultaneously with respect to \(t\).

3. Create tasks in which students would be asked to graph both basic and complicated curves defined parametrically.

4. Create tasks in which students would be asked to identify individual elements that are in the domain and range of a parametric function. Furthermore, such tasks should ask students to state the entire domain and to graph the entire range of a parametric function.

\(^4\)Refer to Appendix E for a small sample of activities to address these recommendations.
5. Create opportunities for students to construct a schema for $\mathbb{R}^3$. Such a construction might help students visualize the relationship between the parameter $t$ and its dependent variables $x$ and $y$, possibly allowing students to reconcile misconceptions related to the vertical line test.

5.4 Limitations of the study

Four limitations of this study should be mentioned:

- The number of participants in this study was limited to fifteen, all of which came from one section of *Calculus of One Variable II*.

- This study did not address the longevity of students’ understanding of the concept of parametric function after the completion of the semester.

- The results of this study can only be generalized to similar situations. Further investigation is needed regarding multivariable calculus students’ conceptions of parametric functions.

- A departmental policy prevented the use of graphing calculators. Therefore, this study did not consider the possible effect of technology on students’ learning of the concept of parametric function.

5.5 Future research

As revealed in the literature review, a substantial amount research has investigated students general understanding of the concept of function, but very little is known about how students reason about parametric functions. This study aimed to fill this gap in the literature. However, further research on the topic of the teaching and learning of parametric functions is needed. This study investigated several different aspects involved in learning the concept of parametric function. It would be beneficial to expand on each of these.

1. How do students reason about the formal aspects of the concept of parametric function, such as domain, range, and uniqueness of function value?
2. How do students apply their general calculus graphing schema to the context of parametric functions in order to graph curves that cannot be expressed in standard form?

3. How do students reason about parametric functions in a multivariable context?

4. How does the study of non-planar curves contribute to students’ understanding of parametric functions?
REFERENCES


Appendix A

INSTRUCTIONAL TASK

1. Imagine that the bottle\(^1\) above is being filled with water at a constant rate.

   (a) Sketch a possible graph of the relationship between time and the volume of the water (with time on the horizontal axis and volume on the vertical axis).

   (b) Sketch a possible graph of the relationship between time and the height of the water (with time on the horizontal axis and height on the vertical axis).

   (c) Sketch a possible graph of the relationship between the volume of the water and the height of the water (with volume on the horizontal axis and height on the vertical axis).

   (d) What would your graphs for (a)–(c) look like if the bottle was being filled faster (but still at a constant rate)?

   (e) Specify the orientation of each of your graphs in (a)–(d). That is, indicate (with arrows) from where and in what direction the graphs are drawn with respect to time.

2. Let \((x(t), y(t)) = \left(\frac{1}{\sqrt{t}}, 1\right)\).

   (a) Give an analytic rectangular representation of the above parametric representation.

   (b) Sketch the graph and specify the orientation.

3. Suppose a particle is traveling at a constant speed of 2 units per second along a linear course starting at the point \((0, 8)\) and ending at the point \((4, 0)\).

   (a) What are the coordinates of the particle after 1.5 seconds?

   (b) What are the coordinates of the particle after \(t\) seconds?

\(^1\)The bottle that was pictured was the same one from Carlson et al. (2002).
Appendix B

FINAL EXAM QUESTIONS

1. Let \( x(t) = \frac{1}{\sqrt{t}} \), \( y(t) = \frac{1}{t^2} \), \( t > 0 \).

   (a) Express the above curve by an equation in \( x \) and \( y \). Explain.

   (b) Sketch the curve. Explain.

2. (a) Is the relationship between \( x \) and \( y \) the same for both of the above graphs? Explain your reasoning.

   (b) Do the curves above have the same analytic parametric representation? Explain your reasoning.

3. What is a parametric equation (or parametric function)? Explain in your own words.
Appendix C

INTERVIEW QUESTIONS

1. State whether each of the following represents a function.

(a) \( \rho(\theta) = \theta \).

(b) \( g(x) = \pm \sqrt{4 - x^2} \).

(c) \( p(t) = (t^2, t^3) \).

2. Assume that Coolers 1 and 2 above are the same size. Imagine that they are full of water and being emptied at constant rates \( r_1 \) and \( r_2 \), respectively. Assume that \( |r_1| < |r_2| \).

(a) Sketch a possible graph of the relationship between time and the volume of the water for each of the coolers. Sketch both of these graphs on the same coordinate plane with time on the horizontal axis and volume on the vertical axis.

(b) Sketch a possible graph of the relationship between time and the height of the water for each of the coolers. Sketch both of these graphs on the same coordinate plane with time on the horizontal axis and height on the vertical axis.

(c) Sketch a possible graph of the relationship between the volume of the water and the height of the water for each of the coolers. Sketch both of these graphs on the same coordinate plane with volume on the horizontal axis and height on the vertical axis.
(d) Specify the orientation of each of your graphs in (a)–(c). That is, indicate (with arrows) from where and in what direction the graphs are drawn with respect to time.

3. Suppose a particle is traveling at a constant speed of 1.25 units per second along a linear course starting at the point (0, 4) and ending at the point (3, 0).

(a) What are the coordinates of the particle after 1.5 seconds?

(b) What are the coordinates of the particle after t seconds?
Appendix D

INTERVIEW PROTOCOL

1. State whether each of the following represents a function.

   (a) \( \rho(\theta) = \theta \).
   
   (b) \( g(x) = \pm \sqrt{4 - x^2} \).
   
   (c) \( p(t) = (t^2, t^3) \).

   - What definition of function did you use to answer these questions? How does your definition relate to each of these?
   - If they answer (c) wrong, ask “For one input value \( t \), how many output values do you get? Is the output a number or something else?” (Note: the output is an ordered pair, which was emphasized in class.)
   - What is the domain?
   - What do the graphs of (a)–(c) look like? What coordinate system are you using? Does the coordinate system affect your answers to (a)–(c)?
   - What else should be taken into account when graphing (c)? (Note: the orientation must be considered. In class, I used the term curve for a graph with an orientation.)
   - In (c), is the second coordinate \( y \) a function of the first coordinate \( x \)? What restrictions can be placed on \( t \) to make \( y \) a function of \( x \).
   - Can you restrict (b) in any way so that it represents a function?

2. Assume that Coolers 1 and 2 above (not pictured) are the same size. Imagine that they are full of water and being emptied at constant rates \( r_1 \) and \( r_2 \), respectively. Assume that \( |r_1| < |r_2| \).
(a) Sketch a possible graph of the relationship between time and the volume of the water for each of the coolers. Sketch both of these graphs on the same coordinate plane with time on the horizontal axis and volume on the vertical axis.

• Explain how you sketched your graphs.
• How did you know what the shape of the graphs would be? Does the shape of the container affect the graphs? Does the rate affect the graphs?
• Is the volume of the water a function of time? (They may need reminding what “y a function of x” means in general: if x is the input variable and y is the output variable, is the relationship a function?) Why or why not?

(b) Sketch a possible graph of the relationship between time and the height of the water for each of the coolers. Sketch both of these graphs on the same coordinate plane with time on the horizontal axis and height on the vertical axis.

• Explain how you sketched your graphs.
• How did you know what the shape of the graphs would be (e.g. piecewise linear)? Does the shape of the container affect the graphs? Does the rate affect the graphs?
• If necessary, ask “Can you tell me what happens at this point [pointing to where the graphs change slope]? Why do the graphs get steeper at this point?”
• Do these points [where the graphs change slope] have the same vertical coordinate?
• Is the height of the water a function of time? Why or why not?

(c) Sketch a possible graph of the relationship between the volume of the water and the height of the water for each of the coolers. Sketch both of these graphs on the same coordinate plane with volume on the horizontal axis and height on the vertical axis.
• Explain how you sketched your graphs. Why is your graph [increasing/decreasing]? *(depending on what their graph is doing)*

• Was this problem counterintuitive? Was it difficult? Why? *(The containers are emptying, while the correct graph is increasing. Does this make the task challenging for the student?)*

• How did you know what the shape of the graphs would be? Does the shape of the container affect the graphs? Does the rate affect the graphs?

• If their solution is wrong, try to get them to reason to the correct solution: If the volume is low, what does that mean about the height? Where would that point be located? If the volume is high, what does that mean about the height? Where would that point be located?

• Is the height of the water a function of the volume of the water? Why or why not?

(d) Specify the orientation of each of your graphs in (a)–(c). That is, indicate (with arrows) from where and in what direction the graphs are drawn with respect to time.

• Explain how you determined the orientation of each of your graphs.

• If they are not able to answer, try to get them to reason to the correct solution: At $t = 0$, where is the volume/height/both *(depending on the situation you’re discussing)*. Where is that point? As $t$ increases, what happens to the volume/height/both? Where is that point? In what order (chronologically) are those points plotted?

3. Suppose a particle is traveling at a constant speed of 1.25 units per second along a linear course starting at the point $(0, 4)$ and ending at the point $(3, 0)$.

(a) What are the coordinates of the particle after 1.5 seconds?

• Explain how you found your answer.
• IF they first found a parameterization, ask why it represents the line segment.

• How long does it take for the particle to complete its course?

• While the particle is traveling, how long does it take for the particle to complete its course in the horizontal direction?

• While the particle is traveling, how long does it take for the particle to complete its course in the vertical direction?

• At $t = 0$, what is the horizontal coordinate of the particle? What is the vertical coordinate?

• At the final time, what is the horizontal coordinate of the particle? What is the vertical coordinate?

(b) What are the coordinates of the particle after $t$ seconds?

• How did you find your answer to (b).

• Are there any restrictions on $t$? What are they? Why?
Appendix E

SUGGESTED ACTIVITIES

Note to reader: The following activities are intended to be discussed in small groups in order to introduce the concept of parametric function and to generate discussion about the concept, prior to a discussion led by the instructor. These activities were developed with the consideration of implementing them at the university where this study was conducted. Since a departmental policy prohibits the use of graphing utilities, these activities do not incorporate any use of technology. However, they can easily be adapted to do so.

1. What are your definitions of function, domain, and range? How do your definitions differ from those of other members in your group?

2. Do the following represent functions? Why or why not?
   (a) \( x^2 + y = 1 \).
   (b) \( x + y^4 = 1 \).
   (c) \( x^2 + y^2 = 1 \).

3. Let \( f(t) = t - 1 \) and \( g(t) = t^3 \), with \(-3 \leq t \leq 3\). Plot a few points defined by \((x, y) = (f(2), g(2))\). Explain.

4. Consider the plane curve traced out by the point \((x, y)\), where

   \[ x = f(t) = \frac{1}{t}, \quad y = g(t) = t, \quad t > 0. \]

   (a) How do the values of \( x \) and \( y \) change in relation to each other as \( t \to \infty \)? Consider both the direction of change and the amount of change. Explain how you determined your answer.
(b) Sketch the graph (indicating orientation) of the curve. Explain how determined your graph.

5. Consider the plane curve traced out by the point \((x,y)\), where \(x = f(t) = t^2\) and \(y = g(t) = t^3\). Try to answer the following questions without making a table of values.

(a) How do the values of \(x\) and \(y\) change in relation to each other as \(t \to \infty\)? (Note that \(a\) \(t\) can be a negative value.) Consider both the direction of change and the amount of change. Explain how you determined your answer.

(b) Sketch the graph (indicating orientation) of the given curve. Explain how determined your graph.

6. Many curves are more complicated than the ones in the previous problems. Develop a strategy for sketching the graph (indicating orientation) of the plane curve defined by

\[
x = f(t) = t^2 - t, \quad y = g(t) = t^4 - t.
\]

7. Suppose \((x,y) = (f(t),g(t)) = (t^2 - 6t + 9, 2t - 6)\) gives the position of a particle at each time \(t\) from \(t = 0\) to \(t = 6\).

(a) Find the particle’s initial and terminal position. How did you find these positions?

(b) Is the particle’s position a function of time? Explain.

8. (a) Let \(p(t) = (x,y)\), where \(x = t^2\) and \(y = t^3\). Determine if \(p\) is a function. Moreover, determine if \(y\) is a function of \(x\). Explain.

(b) Let \(p(t) = (x,y)\), where \(x^2 = t\) and \(y = t\). Determine if \(p\) is a function. Moreover, determine if \(y\) is a function of \(x\). Explain.

(c) Let \(p(t) = (x,y)\), where \(x = t\) and \(y^2 = t\). Determine if \(p\) is a function. Moreover, determine if \(y\) is a function of \(x\). Explain.

9. Let \(p(t) = (f(t),g(t)) = \left(\frac{1}{\sqrt{t}}, 1\right)\).
(a) Determine the domain and range of $f$. Explain.

(b) Determine the domain and range of $g$. Explain.

(c) Determine the domain of $p$. What is the range? Explain.

(d) Graph the range of $p$. How did you sketch this graph?

(e) Sketch the plane curve (including orientation) defined by $p(t)$. Does this graph look familiar?

10. Imagine the values of $t$ as belonging to an axis perpendicular to your paper (so that there are three axes). What would the graph of the curve generated by the point $(x, y, z) = (t^2, t^3, t)$ look like? Explain.