Characterizations in Domination Theory

Andrew Robert Plummer

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Let $G = (V, E)$ be a graph. A set $R \subseteq V$ is a restrained dominating set (total restrained dominating set, resp.) if every vertex in $V - R$ (in $V - R$) is adjacent to a vertex in $R$ and (every vertex in $V - R$) to a vertex in $V - R$. The restrained domination number of $G$ (total restrained domination number of $G$), denoted by $\gamma_r(G)$ ($\gamma_{tr}(G)$), is the smallest cardinality of a restrained dominating set (total restrained dominating set) of $G$. If $T$ is a tree of order $n$, then $\gamma_r(T) \geq \lceil \frac{n+2}{3} \rceil$. We show that $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil$. Moreover, we show that if $n \equiv 0 \mod 4$, then $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil + 1$. We then constructively characterize the extremal trees achieving these lower bounds. Finally, if $G$ is a graph of order $n \geq 2$ such that both $G$ and $\overline{G}$ are not isomorphic to $P_3$, then $4 \leq \gamma_r(G) + \gamma_r(\overline{G}) \leq n + 2$. We provide a similar result for total restrained domination and characterize the extremal graphs $G$ of order $n$ achieving these bounds.

INDEX WORDS: Domination, Restrained Domination, Total Restrained Domination, Nordhaus-Gaddum, Dominating Set
CHARACTERIZATIONS IN DOMINATION THEORY

by

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CHARACTERIZATIONS IN DOMINATION THEORY

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The lion and the eagle, to them this thesis is dedicated.

Pasquale Visconti

and

Christopher Michael Plummer
Foremost, I wish to express my deepest feelings of appreciation for my family; my father Jude for passing on his intellectual curiosity; my mother Patricia for passing on her artistic prowess greatly needed for mathematical imagination; my sister Elizabeth for instilling in me the very essence of critical analysis; my sister Kathleen for reminding me of the world outside of school; my sister Tanya and her family, Glenn, Emily and Matthew, for their stellar example of familial perseverance and countless hours of puerile fun; my grandparents Pasquale and Mildred for inculcating me with incurable workaholism; and my uncle Robert for prompting me to chart my own course. To my extended family I wish to express my appreciation for all of the intangible yet indelible virtues they have infixed in me.

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# LIST OF FIGURES

1.1 The Queen’s Graph  
1.2 Example of a network  
1.3 Example of a dominated network  
1.4 Example of a restrained dominated network  
1.5 Example of a total restrained dominated network
Chapter 1

Proem

In the first section of this chapter we present the notation and basic definitions that will be used throughout this thesis. In Section 1.2, we précis the provenance and development of the concept of domination, and of the variants restrained domination and total restrained domination. We then give formal definitions of these concepts and state several results previously established in this field of research. Finally, in Section 1.3, we delineate the scope of the remainder of this thesis.

1.1 Definitions

A graph $G$ consists of a finite nonempty set of vertices (singular vertex) and a (possibly empty) set of unordered pairs of distinct vertices of $G$ called edges. The vertex set of $G$ is denoted by $V(G)$ (or simply $V$ if the context is clear), while the edge set of $G$ is denoted by $E(G)$ (or simply $E$). The number of vertices in $V(G)$ is denoted by $n(G)$ which is also known as the order of the graph $G$. A graph $G$ is trivial if $n(G) = 1$ and non-trivial if $n(G) \geq 2$. Unless otherwise specified, the symbol $n(G)$ (or simply $n$) will be reserved
exclusively for the order of a graph $G$. We write $G = (V, E)$ to mean that the graph $G$
has vertex set $V$ and edge set $E$.

The edge $e = uv$ is said to join the vertices $u$ and $v$. If $e = uv$ is an edge of $G$, then
$u$ and $v$ are adjacent vertices, while $u$ and $e$ are incident, as are $v$ and $e$. A graph $G$ is
called complete if every two vertices of $G$ are adjacent. We denote a complete graph of
order $n$ by $K_n$. The degree of a vertex $v$ in $G$ is the number of edges incident with $v$
and is denoted $\deg_G(v)$ (or simply $\deg(v)$ if the context is clear). The minimum degree
(respectively, maximum degree) among the vertices of $G$ is denoted by $\delta(G)$ (respectively,
$\Delta(G)$). If there is a vertex $v \in V(G)$ such that $\deg(v) = 0$, then $v$ is called an isolated
vertex, if $\deg(v) = 1$, then $v$ is called an endvertex.

A path of $G$ is a finite, alternating sequence $v_0, e_1, v_1, e_2, \ldots, v_{n-1}, e_n, v_n$ of vertices
and edges, beginning with vertex $v_0$ and ending with vertex $v_n$, such that $e_i = v_{i-1}v_i$
and $v_i \neq v_j$ for $i, j = 1, 2, \ldots, n$ and $i \neq j$. The number $n$ (the number of occurrences
of edges) is called the length of the path. For convenience, we omit the edge and comma
syntax and instead write $v_0v_1v_2\ldots v_n$ to indicate a path, unless otherwise specified. A
graph of order $n$ that is a path is denoted by $P_n$. Therefore, $P_n = v_1v_2\ldots v_n$ indicates a
path of order $n$ on the vertices $v_1, v_2, \ldots, v_n$. A cycle of $G$ is a path $v_1v_2\ldots v_n$ ($n \geq 3$)
with the additional edge $v_nv_1$. A graph of order $n$ that is a cycle is denoted by $C_n$.
Therefore, $C_n = v_1v_2\ldots v_nv_1$ indicates a cycle of order $n$ on the vertices $v_1, v_2, \ldots, v_n$.

Let $u$ and $v$ be distinct vertices of $G$. The distance between vertices $u$ and $v$, denoted
by $d_G(u, v)$ (or simply $d(u, v)$ if the context is clear) is the length of a shortest path
$u \ldots v$, if such a path exist in $G$. We call a path of maximum length in $G$ a diametrical
path in $G$. If there exists a path $u \ldots v$ in $G$ we say that $u$ is connected to $v$. The graph
$G$ is itself connected if $u$ is connected to $v$ for every pair $u, v$ of vertices of $G$. A graph
that is not connected is called disconnected. The trivial graph, then, is connected.
subgraph $H$ of a graph $G$ is a graph with $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph $H$ of $G$ is a component of $G$ if $H$ is a maximal connected subgraph of $G$.

For a graph $G = (V, E)$, let $v \in V$ and let $S \subseteq V$. The open neighborhood of $v$ is $N_G(v) = \{u \in V \mid uv \in E\}$ (or simply $N(v)$) and the closed neighborhood of $v$ is $N_G[v] = \{v\} \cup N_G(v)$ (or simply $N[v]$). A vertex $u \in N(V)$ is called a neighbor of $v$. The open neighborhood of $S$ is defined by $N_G(S) = \cup_{v \in S} N_G(v)$ (or simply $N(S)$), and the closed neighborhood of $S$ by $N_G[S] = N_G(S) \cup S$ (or simply $N[S]$). For a vertex $v$ (respectively, an edge $e$) of $G$ we denote by $G - v$ (respectively, $G - e$) the graph obtained from $G$ by deleting the vertex $v$ (respectively, the edge $e$).

A tree is a connected graph which has no cycles. We refer to a vertex of degree 1 in a tree $T$ as a leaf of $T$. A vertex adjacent to a leaf we call a remote vertex of $T$. A star is a tree of order $n$ comprising exactly $n - 1$ leaves. The trivial graph, then, is a star, and is also called the trivial star. A star of order $n \geq 2$ is called a non-trivial star. The vertex of a non-trivial star which is not a leaf is called the center of the star. For consistency, we consider $P_2$ a star on two vertices with the center chosen arbitrarily. A galaxy is a graph whose components are stars. A double star is a tree of order $n$ comprising exactly $n - 2$ leaves.

For a vertex $v$ of a tree $T$, we shall use the expression, attach a $P_m$ at $v$, to refer to the operation of taking the union of $T$ and a path $P_m$ and joining one of the ends of this path to $v$ with an edge. For $v \in V(T)$ and a leaf $\ell$ of $T$, the path $vx_1 \ldots x_k \ell$ is called a $v-L$ endpath if $\deg x_i = 2$ for each $i$. If the vertex $v$ need not be specified, a $v-L$ path is also called an endpath.
1.2 Perlustration

The concept of domination is quite natural and appears in many situations in which one desires an optimal covering of some sort. Lore has it that domination in graphs derives from strategies in the game of chess, where one desires to cover (or dominate) the squares of a chessboard using certain chess pieces. In 1862 de Jaenisch [6] considered the problem of determining the minimum number of queens (with standard movement rules) that can be placed on a chessboard such that each square is either occupied by a queen or is occupiable by a queen in a single move.

The parallel between de Jaenisch’s chessboard problem and domination in graphs is patent. Consider a standard chessboard and let the 64 squares comprise the vertex set of a graph $G$. Let two vertices (squares) be adjacent in $G$ if each square is occupiable in a single move by a queen stationed on the other square. The graph $G$ defined as such is called the queen’s graph. Choosing a set of vertices that dominates $G$ is tantamount to positioning queens on the chessboard as to either occupy or potentially occupy (in one move) each square. The domination number of $G$ is the minimum number of queens required to achieve the desired covering (see Figure 1.1).

![Figure 1.1](image)

Figure 1.1 The minimum number of queens that dominate the squares of a standard chessboard.
Domination in graphs was formalized by Berge (see [1], p. 40) in 1958, and Ore [17] in 1962. We now provide the elements of the theory. A vertex $v$ in a graph $G$ dominates itself and each of its neighbors. Hence, $v$ dominates the vertices in $N[v]$. A set $S \subseteq V$ is a dominating set if every vertex not in $S$ is adjacent to a vertex in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of cardinality $\gamma(G)$ will be called a $\gamma(G)$-set. A minimal dominating set is a dominating set that contains no dominating set as a proper subset. The concept of domination in graphs, with its many variations, is now well studied in graph theory. The recent book of Chartrand and Lesniak [3] includes a chapter on domination. A thorough study of domination appears in [10, 11]. In demonstrating the development of the theory, we give several known results.

**Theorem 1.1 (Ore [17])** Let $D$ be a dominating set of a graph $G$. Then $D$ is a minimal dominating set of $G$ if and only if each $v \in D$ has at least one of the following two properties.

$P1$: There exists a vertex $w \in V(G) - D$ such that $N(w) \cap D = \{v\}$;

$P2$: The vertex $v$ is adjacent to no other vertex of $D$.

**Theorem 1.2 (Bollobás and Cockayne [2])** If $G$ is a graph with no isolated vertex, then there exists a minimum dominating set $D$ of vertices of $G$ in which every vertex has property $P1$.

**Theorem 1.3 (Ore [17])** If $G$ is a graph with no isolated vertex and $D$ is a minimal dominating set of $G$, then $V(G) - D$ is a dominating set of $G$.

**Corollary 1.4 (Ore [17])** If $G$ is a graph of order $n$ with no isolated vertex, then $\gamma(G) \leq \frac{n}{2}$.

**Theorem 1.5 (Payan [18])** Let $G$ be a graph of order $n$ with minimum degree $\delta \geq 2$. Then $\gamma(G) \leq \frac{n(1+\ln(\delta+1))}{\delta+1}$.
The most commonly addressed application of domination in graphs is that of networking. Consider a network of transceivers and let $G$ be a graph with a vertex set comprising the transceivers in the network. Two vertices (transceivers) are adjacent in $G$ if each transceiver is capable of receiving transmissions broadcasted by the other (e.g. Figure 1.2).

![Figure 1.2](image)

**Figure 1.2** A network $N$ of transceivers and its corresponding graph $G$.

Choosing a set of vertices that dominates $G$ is tantamount to selecting transceivers in the network such that every transceiver is either broadcasting a signal or receiving a signal broadcast (see Figure 1.3). The domination number of $G$ is the minimum number of transceivers required to achieve the desired covering.

![Figure 1.3](image)

**Figure 1.3** The darkened transceivers cover $N$ with a signal broadcast, as the darkened vertices dominate $G$.

Now, suppose the network contains transceivers that broadcast and receive both a primary signal and an auxiliary signal. Moreover, no transceiver broadcasts both signals simultaneously, and a transceiver not broadcasting the primary signal must broadcast the auxiliary signal. Again, let $G$ be a graph with a vertex set comprising the transceivers in
the network with two vertices (transceivers) adjacent in $G$ if each transceiver is capable of receiving transmissions broadcasted by the other. Choosing a set of vertices that dominates $G$ is tantamount to selecting transceivers in the network such that every transceiver is either broadcasting the primary signal or receiving a primary signal broadcast. The domination number of $G$ is the minimum number of transceivers required to achieve the desired covering.

Consider the transceivers not broadcasting the primary signal. We now require that these transceivers also receive the auxiliary signal. This is the concept of restrained domination. Choosing a set of vertices that dominates $G$ with restraint is tantamount to selecting transceivers in the network such that every transceiver is either broadcasting the primary signal or receiving a primary signal broadcast, and each transceiver not broadcasting the primary signal also receives an auxiliary signal broadcast (e.g. Figure 1.4). The restrained domination number of $G$ is the minimum number of transceivers required to achieve the desired covering.

**Figure 1.4** The darkened transceivers cover $N$ with a primary signal broadcast while the remaining transceivers receive the auxiliary signal. The darkened vertices constitute a restrained dominating set of $G$.

Notice that a transceiver that receives transmissions from at most one other transceiver cannot receive both the primary signal and the auxiliary signal. Hence, each transceiver receiving transmissions from at most one other transceiver must broadcast the primary signal. That is, each vertex with degree less than or equal to one must be included in every restrained dominating set.
The concept of restrained domination was introduced by Telle and Proskurowski [19], albeit indirectly, as a vertex partitioning problem. Here conditions are imposed on a set $S$, the complementary set $V - S$ and on edges between the sets $S$ and $V - S$. For example, if we require that every vertex in $V - S$ should be adjacent to some other vertex of $V - S$ (the condition on the set $V - S$) and to some vertex in $S$ (the condition on edges between the sets $S$ and $V - S$), then $S$ is a restrained dominating set.

Restrained domination in graphs was formalized by Domke et al. [8] in 1999. We now provide the elements of the theory. Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a restrained dominating set (abbreviated RDS) if every vertex not in $S$ is adjacent to a vertex in $S$ and to a vertex in $V - S$. Every graph has a RDS, since $S = V$ is such a set. The restrained domination number of $G$, denoted by $\gamma_r(G)$, is the minimum cardinality of a RDS of $G$. A RDS of cardinality $\gamma_r(G)$ will be called a $\gamma_r(G)$-set. In demonstrating the development of the theory, we give several known results.

**Theorem 1.6 (Domke et al. [8])** Let $G$ be a graph of order $n$. Then $\gamma_r(G) = n$ if and only if $G$ is a disjoint union of stars.

Let $T_{\gamma_r} = \{T \mid T$ is obtained from $P_4, P_5$ or $P_6$ by attaching $P_1$ at $v$, where $v$ is an remote vertex of the path}. Let $C_{\gamma_r} = \{C \mid C$ is $C_4$ or $C_5$ or $C$ can be obtained from $C_3$ by attaching $P_1$ at no more than two of the vertices of the cycle\}. Finally, let $\mathcal{F} = \{F \mid F$ is one of the bad graphs described in [7], p.240\}.

**Theorem 1.7 (Domke et al. [8])** Let $G$ be a graph of order $n$. Then $\gamma_r(G) = n - 2$ if and only if exactly one of the components of $G$ is isomorphic to a graph $G' \in T_{\gamma_r} \cup C_{\gamma_r}$.

**Theorem 1.8 (Domke et al. [7])** Let $G$ be a connected graph of order $n \geq 3$ with $\delta(G) \geq 2$. If $G \notin \mathcal{F}$, then $\gamma_r(G) \leq \frac{n - 1}{2}$. 

Theorem 1.9 (Domke et al. [7]) Let $G$ be a graph of order $n$ with minimum degree $\delta \geq 2$. Then $\gamma_r(G) \leq n(1 + (\frac{1}{\delta})^{\frac{\delta}{\delta-1}} - (\frac{1}{\delta})^{\frac{1}{\delta-1}})$.

Now, consider once again the network containing transceivers that broadcast and receive both a primary signal and an auxiliary signal. Recall that no transceiver broadcasts both signals simultaneously, and a transceiver not broadcasting the primary signal must broadcast the auxiliary signal. And again, let $G$ be a graph with a vertex set comprising the transceivers in the network with two vertices (transceivers) adjacent in $G$ if each transceiver is capable of receiving transmissions broadcasted by the other.

Suppose we desire to build redundancy into the network by requiring that all transceivers in the network receive the primary signal, that is, reception of the primary signal is total among the network of transceivers. Choosing a set of vertices that totally dominates $G$ with restraint is tantamount to selecting transceivers in the network such that every transceiver is receiving a primary signal broadcast, and each transceiver not broadcasting the primary signal also receives an auxiliary signal broadcast (e.g. Figure 1.5). The total restrained domination number of $G$ is the minimum number of transceivers required to achieve the desired covering.

![Figure 1.5](image)

**Figure 1.5** The darkened transceivers totally cover $N$ with a primary signal broadcast while the remaining transceivers receive the auxiliary signal. The darkened vertices constitute a total restrained dominating set of $G$.

Notice that a transceiver that receives transmissions from no other transceiver receives neither the primary signal nor the auxiliary signal. Thus, we require that each transceiver receives transmissions from at least one other transceiver. That is, total restrained domination is well-defined only on graphs with minimum degree at least one.
Moreover, a transceiver that receives transmissions from exactly one other transceiver cannot receive both the primary signal and the auxiliary signal. Hence, each transceiver receiving transmissions from exactly one other transceiver must broadcast the primary signal, and receive the primary signal from the one other transceiver. That is, each vertex with degree equal to one, and its neighbor, must be included in every total restrained dominating set.

We note that the concept of total restrained domination was introduced by Telle and Proskurowski [19], albeit indirectly, as a vertex partitioning problem. Here conditions are imposed on a set $S$, the complementary set $V - S$ and on edges between the sets $S$ and $V - S$. For example, if we require that every vertex in $V - S$ should be adjacent to some other vertex of $V - S$ (the condition on the set $V - S$) and to some vertex in $S$ (the condition on edges between the sets $S$ and $V - S$), and every vertex in $S$ is also adjacent to some vertex in $S$ (the condition on edges among vertices of $S$), then $S$ is a total restrained dominating set.

Total restrained domination in graphs was formalized by Chen, Ma and Sun [4] in 2005, and further studied by Zelinka [20] and Maritz [13]. We now provide the elements of the theory. Let $G = (V, E)$ be a graph. A set $S \subseteq V$ is a total restrained dominating set (abbreviated TRDS) if every vertex is adjacent to a vertex in $S$ and every vertex in $V - S$ is also adjacent to a vertex in $V - S$. Every graph without isolated vertices has a total restrained dominating set, since $S = V$ is such a set. The total restrained domination number of $G$, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a TRDS of $G$. A TRDS of cardinality $\gamma_{tr}(G)$ will be called a $\gamma_{tr}(G)$-set. In demonstrating the development of the theory, we give several known results.

**Theorem 1.10 (Chen et al. [4])** Let $T$ be a tree of order $n \geq 2$. Then $\gamma_{tr}(T) \geq \Delta(T) + 1$. Furthermore, $\gamma_{tr}(T) = \Delta(T) + 1$ if and only if $T$ is a star.
Theorem 1.11 (Maritz [13]) If $G$ is a connected graph of order $n \geq 4$, maximum degree $\Delta$ where $\Delta \leq n - 2$, and minimum degree at least 2, then $\gamma_{tr}(G) \leq n - \frac{\Delta}{2} - 1$; and this bound is sharp.

Theorem 1.12 (Maritz [13]) If $G$ is a connected bipartite graph of order $n \geq 5$, maximum degree $\Delta$ where $3 \leq \Delta \leq n - 2$, and minimum degree at least 2, then $\gamma_{tr}(G) \leq n - \frac{2}{3} \Delta - \frac{2}{9} \sqrt{3\Delta - 8} - \frac{7}{9}$; and this bound is sharp.

1.3 Purview

The unifying theme of this thesis is the characterization of extremal graphs corresponding to bounds on the graphical parameters restrained domination and total restrained domination. The characterizations are novel and simple, and the proof techniques employed remain viable in other areas of domination theory. Thus, the purpose of this thesis is to further the study of restrained domination and total restrained domination in graphs by presenting original results in these fields, and by so doing, circulate the proof techniques utilized herein.

In Chapter 2, we discuss restrained domination in trees. It is established in [9] that if $T$ is a tree of order $n$, then $\gamma_{r}(T) \geq \lceil \frac{n+2}{3} \rceil$, and a characterization of trees achieving this bound is given. We recount the characterization given in [9] and conclude by giving a simpler, constructive characterization of the extremal trees $T$ of order $n$ achieving this lower bound.

In Chapter 3, we discuss total restrained domination in trees. We prove that if $T$ is a tree of order $n$, then $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil$. We then give a constructive characterization of the extremal trees $T$ of order $n$ achieving this lower bound. Next, we show that if $T$ is a tree of order $n \equiv 0 \mod 4$, then $\gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil + 1$. We again constructively characterize the extremal trees $T$ of order $n$ achieving this lower bound.
Finally, in Chapter 4, we discuss Nordhaus-Gaddum results for restrained domination and total restrained domination in graphs. We bound the sum of the total restrained domination numbers of a graph and its complement, and provide characterizations of the extremal graphs achieving these bounds. It is known (see [8]) that if $G$ is a graph of order $n \geq 2$ such that both $G$ and $\overline{G}$ are not isomorphic to $P_3$, then $4 \leq \gamma_r(G) + \gamma_r(\overline{G}) \leq n + 2$. We also provide characterizations of the extremal graphs $G$ of order $n$ achieving these bounds.
Chapter 2

Restrained Domination in Trees

2.1 Introduction

In this chapter, we continue the study of a variation of the domination theme, namely that of restrained domination [7, 8, 9, 12, 19]. Recall that a set $S \subseteq V$ is a restrained dominating set (abbreviated RDS) if every vertex not in $S$ is adjacent to a vertex in $S$ and to a vertex in $V - S$. The restrained domination number of $G$, denoted by $\gamma_r(G)$, is the minimum cardinality of a RDS of $G$. A RDS of cardinality $\gamma_r(G)$ will be called a $\gamma_r(G)$-set.

It is known [9] that if $T$ is a tree of order $n$, then $\gamma_r(T) \geq \lceil \frac{n+2}{3} \rceil$. For $n \geq 1$, let $T_n = \{T \mid T$ is a tree of order $n$ such that $\gamma_r(T) = \lceil \frac{n+2}{3} \rceil \}$. A constructive characterization of the extremal trees $T$ of order $n$ achieving this lower bound was obtained in [9]. For the purpose of stating this characterization, we define a type (1) operation on a tree $T$ as attaching a $P_2$ at $v$ where $v$ is a vertex of $T$ not belonging to some minimum RDS of $T$, and a type (2) operation as attaching a $P_3$ at $v$ where $v$ belongs to some minimum RDS of $T$. For $i = 1, 2$, let $T_i$ be the tree obtained from $K(1, 3)$ by subdividing $i$ edges once.
Let \( C_3k = \{ T \mid T \text{ is a tree of order } 3k \text{ which can be obtained from the tree } T_2 \text{ by a finite sequence of operations of type (2)} \} \). Let \( C_{3k+1} = \{ T \mid T \text{ is a tree of order } 3k + 1 \text{ which can be obtained from } P_4 \text{ by a finite sequence of operations of type (2)} \} \). Finally, let \( C_{3k+2} = \{ T \mid T \text{ is a tree of order } 3k + 2 \text{ which can be obtained from } P_5 \text{ or from the tree } T_1 \text{ by a finite sequence of operations of type (2)} \} \). Finally, let \( C_{3k+2} = \{ T \mid T \text{ is a tree of order } 3k + 2 \text{ which can be constructed from the tree } T_2 \text{ by a finite sequence of operations of type (2), followed by one operation of type (1) and then by a finite sequence of operations of type (2)} \} \). It was established in [9] that

**Theorem 2.1** For \( n \geq 4 \), \( T_n = C_n \).

The purpose of this chapter is to provide a simpler constructive characterization of the extremal trees \( T \) of order \( n \) achieving this lower bound. The technique employed in proving the characterization involves a diarametrical argument that will be utilized again in the next chapter.

### 2.2 Extremal trees \( T \) with \( \gamma_r(T) = \lceil \frac{n+2}{3} \rceil \)

Let \( T \) be the class of all trees \( T \) of order \( n \) such that \( \gamma_r(T) = \lceil \frac{n+2}{3} \rceil \). We will constructively characterize the trees in \( T \). In order to state the characterization, we define three simple operations on a tree \( T \).

**O1.** Join a leaf or a remote vertex, or a vertex \( v \) or \( x \) of \( T \) on an endpath \( vxxyz \) to a vertex of \( K_1 \), where \( n(T) \equiv 1 \) mod 3.

**O2.** Join a remote vertex, or a vertex \( v \) of \( T \) which lies on an endpath \( vxz \) to a leaf of \( P_2 \), where \( n(T) \equiv 0 \) mod 3 or \( n(T) \equiv 1 \) mod 3.

**O3.** Join a leaf of \( T \) to \( \ell \) disjoint copies of \( P_3 \) for some \( \ell \geq 1 \).
Let \( C \) be the class of all trees obtained from \( P_2 \) or \( P_4 \) by a finite sequence of Operations O1–O3. We will show that \( T \in \mathcal{T} \) if and only if \( T \in C \). Let \( S \) be a \( \gamma_r(T') \)-set of \( T' \) throughout the proofs of the following lemmas.

**Lemma 2.2** Let \( T' \in \mathcal{T} \) be a tree of order \( n \equiv 1 \mod 3 \). If \( T \) is obtained from \( T' \) by Operation O1, then \( T \in \mathcal{T} \).

**Proof.** Let \( u \) be a leaf or a remote vertex, or a vertex \( w \) or \( x \) on an endpath \( wxyz \) of \( T' \), and suppose \( T \) is formed by attaching the singleton \( v \) to \( u \). Then \( S \cup \{v\} \) is a RDS of \( T \), and so \( \lceil \frac{n+3}{3} \rceil \leq \gamma_r(T) \leq \lceil \frac{n+2}{3} \rceil + 1 \). Since \( n \equiv 1 \mod 3 \), we have \( \gamma_r(T) = \lceil \frac{n(T)+2}{3} \rceil \). Thus, \( T \in \mathcal{T} \). \( \square \)

**Lemma 2.3** Let \( T' \in \mathcal{T} \) be a tree of order \( n \equiv 0 \mod 3 \) or \( n \equiv 1 \mod 3 \). If \( T \) is obtained from \( T' \) by Operation O2, then \( T \in \mathcal{T} \).

**Proof.** Suppose \( v \) is a remote vertex or \( v \) lies on the endpath \( vxz \) and \( T \) is obtained from \( T' \) by adding the path \( vyz' \).

We show that \( v \not\in S \). First consider the case when \( v \) is a remote vertex adjacent to a leaf \( z \). Suppose \( v \in S \). Then \( S' = S - \{z\} \) is a RDS of \( T'' = T' - z \), and so \( \lceil \frac{n+1}{3} \rceil \leq \gamma_r(T'') \leq \lceil \frac{n+2}{3} \rceil - 1 \), which is a contradiction when \( n \equiv 0 \mod 3 \) or \( n \equiv 1 \mod 3 \). Thus, \( v \not\in S \). In the case when \( v \) lies on the endpath \( vxz \), one may show, as in the previous paragraph, that \( x \not\in S \). But then \( v \not\in S \), as required.

In both cases, the set \( S \cup \{z'\} \) is a RDS of \( T \), and so \( \lceil \frac{n+4}{3} \rceil \leq \gamma_r(T) \leq \lceil \frac{n+2}{3} \rceil + 1 \). However, as \( n \equiv 0 \mod 3 \) or \( n \equiv 1 \mod 3 \), we have \( \gamma_r(T) = \lceil \frac{n+4}{3} \rceil = \lceil \frac{n(T)+2}{3} \rceil \). Thus, \( T \in \mathcal{T} \). The proof is complete. \( \square \)

**Lemma 2.4** Let \( T' \in \mathcal{T} \) be a tree of order \( n \). If \( T \) is obtained from \( T' \) by the Operation O3, then \( T \in \mathcal{T} \).
Proof. Let $S$ be a $\gamma_r(T')$-set of $T'$, and suppose $v$ is a leaf of $T'$. Then $v \in S$. Let $T$ be the tree which is obtained from $T$ by adding the paths $vx_1y_1z_i$ for $i = 1, \ldots, \ell$. Then $S \cup \bigcup_{i=1}^\ell \{z_i\}$ is a RDS of $T$, and so $\lceil \frac{n + 3\ell + 2}{3} \rceil \leq \gamma_r(T) \leq \lceil \frac{n + 2}{3} \rceil + \ell$. Consequently, $\gamma_r(T) = \lceil \frac{n(T) + 2}{3} \rceil$, and so $T \in \mathcal{T}$. $\square$

We are now in a position to prove the main result of this section.

**Theorem 2.5** $T \in \mathcal{C}$ if and only if $T \in \mathcal{T}$.

**Proof.** Suppose $T \in \mathcal{C}$. We show that $T \in \mathcal{T}$, by using induction on $c(T)$, the number of operations required to construct the tree $T$. If $c(T) = 0$, then $T = P_2$ or $T = P_4$, both of which are in $\mathcal{T}$. Assume, then, for all trees $T' \in \mathcal{C}$ with $c(T') < k$, where $k \geq 1$ is an integer, that $T'$ is in $\mathcal{T}$. Let $T \in \mathcal{C}$ be a tree with $c(T) = k$. Then $T$ is obtained from some tree $T'$ by one of the Operations $O1 - O3$. But then $T' \in \mathcal{C}$ and $c(T') < k$. Applying the inductive hypothesis to $T'$, $T'$ is in $\mathcal{T}$. Hence, by Lemmas 2.2, 2.3 or 2.4, $T \in \mathcal{T}$.

To show that $T \in \mathcal{C}$ for a nontrivial $T \in \mathcal{T}$, we use induction on $n$, the order of the tree $T$. If $n = 2$, then $T = P_2 \in \mathcal{C}$. If $n = 3$, then $T \notin \mathcal{T}$. If $n = 4$, then either $T = P_4$ or $T$ is a star. If $T$ is a star then $T \notin \mathcal{T}$. If $T = P_4$ then $T \in \mathcal{C}$. Let $T \in \mathcal{T}$ be a tree of order $n \geq 5$, and assume for all trees $T' \in \mathcal{T}$ of order $4 \leq n' < n$, that $T' \in \mathcal{C}$. Since $n(T) \geq 5$ and no stars are in $\mathcal{T}$, $\text{diam}(T) \geq 3$.

If $\text{diam}(T) = 3$, then $T$ is a double star of order 5, has a remote vertex adjacent to two leaves, and is therefore constructible from $P_4$ by $O1$, whence $T \in \mathcal{C}$. Thus, we may assume $\text{diam}(T) \geq 4$. Throughout, $S$ will be used to denote a $\gamma_r(T)$-set of $T$.

**Claim 2.6** Suppose $z$ is a leaf of $T$. If $S - \{z\}$ is a RDS of $T' = T - z$, then $n(T') \equiv 1 \mod 3$ and $T' \in \mathcal{C}$.
Proof. Suppose \( S - \{z\} \) is a RDS of \( T' \). Then \( \lceil \frac{n - 1 + 2}{3} \rceil \leq \gamma_r(T') \leq \lceil \frac{n + 2}{3} \rceil - 1 \). This yields a contradiction when \( n \equiv 0 \mod 3 \) or \( n \equiv 1 \mod 3 \). Hence, \( n \equiv 2 \mod 3 \), and \( \gamma_r(T') = \frac{n + 1}{3} = \lceil \frac{n(T') + 2}{3} \rceil \). Thus, \( T' \in \mathcal{T} \), with \( n(T') = n - 1 \equiv 1 \mod 3 \). By the induction assumption, \( T' \in \mathcal{C} \). \( \diamond \)

Suppose \( vxz \) or \( vz \) is an endpath of \( T \). If \( v, x \in S \), then \( S - \{z\} \) is a RDS of \( T' = T - z \). By Claim 2.6, the tree \( T' = (T - z) \in \mathcal{C} \) and \( T \) can be constructed from \( T' \) by Operation \( O_1 \). Thus, if \( vxz \) or \( vz \) is an endpath of \( T \), we may assume \( v, x \notin S \).

Suppose \( v \) is a remote vertex adjacent to at least two leaves, and let \( z \) be a leaf adjacent to \( v \). Then \( S - \{z\} \) is a RDS of \( T' = T - z \). By Claim 2.6, the tree \( T' = (T - z) \in \mathcal{C} \) and \( T \) can be constructed from \( T' \) by Operation \( O_1 \). Thus, we may assume that every remote vertex is adjacent to exactly one leaf.

Let \( T \) be rooted at a leaf \( r \) of a longest path. Let \( v \) be any vertex on a longest path at distance \( \text{diam}(T) - 2 \) from \( r \). Suppose \( v \) lies on the endpath \( vyz' \). Then, by the above remark, \( v, y \notin S \).

Suppose \( \deg(v) \geq 3 \) and first assume \( v \) is a remote vertex adjacent to a leaf \( u \). Since \( \text{diam}(T) \geq 4 \), \( v \) has a parent vertex \( v_0 \). Suppose \( v_0 \in S \). Moreover, suppose \( \deg(v) \geq 4 \). By Claim 2.6, \( v \) is adjacent to one leaf only, \( x \) is on an endpath \( vxz \) where \( x \notin S \). Since \( v_0 \in S \), it follows that \( S' = S - \{u, z\} \) is a RDS for \( T' = T - u - x - z \). Hence, \( \lceil \frac{n - 3 + 2}{3} \rceil \leq \gamma_r(T') \leq \lceil \frac{n + 2}{3} \rceil - 2 \), which is a contradiction. Hence \( \deg(v) = 3 \).

Consider \( T' = T - u \). The vertex \( v \) in \( T' \) is on the endpath \( v_0vyz' \). Since \( v_0 \in S \), it follows that \( S' = S - \{u\} \) is a RDS for \( T' \). Thus, by Claim 2.6, \( T' \in \mathcal{C} \) and \( T \) can be constructed from \( T' \) by Operation \( O_1 \), whence \( T \in \mathcal{C} \). Therefore, we may suppose \( v_0 \notin S \). Then \( S' = S - \{z'\} \) is a RDS for \( T' = T - y - z' \). Hence, \( \lceil \frac{n - 2 + 2}{3} \rceil \leq \gamma_r(T') \leq \lceil \frac{n + 2}{3} \rceil - 1 \), which is a contradiction when \( n \equiv 1 \mod 3 \). Hence \( n \equiv 0 \mod 3 \) or \( n \equiv 2 \mod 3 \) and \( \gamma_r(T') = \lceil \frac{n}{3} \rceil = \lceil \frac{n + 2}{3} \rceil \). Thus, \( T' \in \mathcal{T} \), with \( n(T') = n - 2 \equiv 0 \mod 3 \) or
\[ n(T') = n - 2 \equiv 1 \mod 3. \]

By the induction assumption, \( T' \in \mathcal{C} \). The tree \( T \) can now be constructed from \( T' \) by applying Operation \( O2 \), whence \( T \in \mathcal{C} \).

We now assume that \( v \) is not a remote vertex. Thus, \( v \) lies on the endpaths \( vxz \) and \( vyz' \). It follows that \( S' = S - \{z'\} \) is a RDS for \( T' = T - y - z' \). Hence, by reasoning similar to that in the previous paragraph, the tree \( T \) can be constructed from \( T' \) by applying Operation \( O2 \), whence \( T \in \mathcal{C} \). Thus, we assume each vertex on a longest path at distance \( \text{diam}(T) - 2 \) or \( \text{diam}(T) - 1 \) from \( r \) has degree two.

Let \( v \) be any vertex on a longest path at distance \( \text{diam}(T) - 3 \) from \( r \). Let \( vx_1y_1z_1 \) be an endpath of \( T \). Then \( x_1, y_1 \not\in S \), and so \( v \in S \). Suppose \( \deg(v) \geq 3 \). If \( v \) is on an endpath \( vxz \), it follows that \( x, z \in S \), and by the remark following Claim 2.6, \( T \in \mathcal{C} \).

Suppose \( v \) is a remote vertex adjacent to a leaf \( u \). By Claim 2.6, \( u \) is the only leaf adjacent to \( v \). Moreover, \( S' = S - \{u\} \) is a RDS for \( T' = T - u \). Thus, by Claim 2.6, \( T' \in \mathcal{C} \) and \( T \) can be constructed from \( T' \) by Operation \( O1 \), whence \( T \in \mathcal{C} \).

We may assume that \( v \) lies only on endpaths \( vx_iz_i, \) for \( i = 1, \ldots, \ell \). Let \( e \) be the edge that joins \( v \) with its parent, and let \( T(v) \) be the component of \( T - e \) that contains \( v \). Then \( T(v) \) consists of \( \ell \) disjoint paths \( x_iz_i (i = 1, \ldots, \ell) \) with \( v \) joined to \( x_i \) for \( i = 1, \ldots, \ell \). Let \( i \in \{1, \ldots, \ell\} \). Since \( x_iz_i \) is an endpath of \( T \), we have \( x_i \not\in S \), \( y_i \not\in S \) and \( v \in S \). Then \( S - \cup_{i=1}^{\ell} \{z_i\} \) is a RDS of \( T' = T - (T(v) - \{v\}) \), and so 

\[ \left\lceil \frac{n-3\ell+2}{3} \right\rceil \leq \gamma_r(T') \leq \left\lceil \frac{n+2}{3} \right\rceil - \ell, \text{ whence } \gamma_r(T') = \left\lceil \frac{n(T') + 2}{3} \right\rceil. \]

Thus, \( T' \in \mathcal{T} \), and by the induction assumption, \( T' \in \mathcal{C} \). Note that \( v \) is a leaf of \( T' \). The tree \( T \) can now be constructed from \( T' \) by applying Operation \( O3 \), whence \( T \in \mathcal{C} \). \( \square \)
Chapter 3

Total Restrained Domination in Trees

3.1 Introduction

In this chapter, we continue the study of a variation of the domination theme, namely that of total restrained domination [4, 13, 20]. Recall that a set $S \subseteq V$ is a total restrained dominating set (abbreviated TRDS) if every vertex is adjacent to a vertex in $S$ and every vertex of $V - S$ is adjacent to a vertex in $V - S$. The total restrained domination number of $G$, denoted by $\gamma_{\text{tr}}(G)$, is the smallest cardinality of a TRDS of $G$. A TRDS of cardinality $\gamma_{\text{tr}}(G)$ will be called a $\gamma_{\text{tr}}(G)$-set.

We show that if $T$ is a tree of order $n$, then $\gamma_{\text{tr}}(T) \geq \lceil (n + 2)/2 \rceil$. Moreover, we constructively characterize the extremal trees $T$ of order $n$ achieving this lower bound. Lastly, we show that if $T$ is a tree of order $n \equiv 0 \mod 4$, then $\gamma_{\text{tr}}(T) \geq \lceil n/2 \rceil + 1$, and also constructively characterize the extremal trees $T$ of order $n$ achieving this lower bound.
3.2 The Lower Bound

The following result was established in [4], using a more cumbersome proof. As we shall see, this result will be useful in establishing a sharp lower bound on the total restrained domination number of a tree.

Proposition 3.1 If \( n \geq 2 \) is an integer, then \( \gamma_{tr}(P_n) = n - 2\lfloor \frac{n-2}{4} \rfloor \).

Proof. Suppose \( S \) is a TRDS of \( P_n \), whose vertex set is \( V = \{v_1, \ldots, v_n\} \). Note that \( v_1, v_2 \in S \). Moreover, any component of \( V - S \) is of size exactly two. Each component is adjacent to a vertex of \( S \), which, in turn, is adjacent to another vertex of \( S \). Suppose there are \( m \) such components. Then \( 2m + 2m + 2 \leq n \) and so \( m \leq \lfloor \frac{n-2}{4} \rfloor \). Thus \( |S| = n - 2m \geq n - 2\lfloor \frac{n-2}{4} \rfloor \). On the other hand, \( V - \{v_i \mid i \in \{3, 4, 7, 8, \ldots, 4\lfloor \frac{n-2}{4} \rfloor - 1, 4\lfloor \frac{n-2}{4} \rfloor \} \) is a TRDS of \( P_n \), whence \( \gamma_{tr}(P_n) = n - 2\lfloor \frac{n-2}{4} \rfloor \). \( \square \)

Corollary 3.2 If \( n \geq 2 \) is an integer, then \( \gamma_{tr}(P_n) \geq \lceil \frac{n+2}{2} \rceil \).

Proof. Since \( n - 2\lfloor \frac{n-2}{4} \rfloor \geq \lceil \frac{n+2}{2} \rceil \), the result follows from Proposition 3.1. \( \square \)

Let \( T = (V, E) \) be a tree and \( v, a, b \in V \) such that \( \deg v \geq 3 \) and \( a, b \in N(v) \). Let \( \ell_b \) be a leaf of the component of \( T - v \) that contains \( b \). Then the tree \( T' \) which arises from \( T \) by deleting the edge \( va \) and joining \( a \) to \( \ell_b \) is called a \((v, a, b)\)-pruning of \( T \).

Theorem 3.3 If \( T \) is a tree of order \( n \geq 2 \), then \( \gamma_{tr}(T) \geq \lceil \frac{n+2}{2} \rceil \).

Proof. We use induction on \( n \). It is easy to check that the result is true for all trees \( T \) of order \( n \leq 8 \). Suppose, therefore, that the result is true for all trees of order less than \( n \), where \( n \geq 9 \). Let \( \gamma_{tr} = \min\{\gamma_{tr}(T) \mid T \text{ is a tree of order } n\} \). We will show that \( \gamma_{tr} \geq \lceil \frac{n+2}{2} \rceil \).
Let $T = \{T \mid T \text{ is a tree of order } n \text{ such that } \gamma_{\text{tr}}(T) = \gamma_{\text{tr}}\}$. Among all trees in $T$, let $T$ be chosen so that the sum $s(T)$ of the degrees of its vertices of degree at least 3 is minimum. With respect to this, let $T$ be chosen such that the number of leaves of $T$ is minimum. If $s(T) = 0$, then $T \cong P_n$, and so $\gamma_{\text{tr}} = \gamma_{\text{tr}}(P_n) \geq \left\lceil \frac{n+2}{2} \right\rceil$. Suppose, therefore, that $s(T) \geq 1$. Since $s(T) \geq 1$, there exists a vertex $v$ such that $\deg(v) \geq 3$. Let $S$ be a $\gamma_{\text{tr}}(T)$-set of $T$.

**Claim 3.4** If $v$ is a vertex of degree at least 3, then

(i) $v \notin S$,
(ii) $v$ is adjacent to exactly one vertex of $S$,
(iii) $\deg(v) = 3$.

**Proof.** Suppose $v \in S$. Then there exist $a, b \in N(v)$ such that $b \in S$. Let $T'$ be a $(v, a, b)$-pruning of $T$. Then $S$ is a TRDS of $T'$, and so, by definition of $\gamma_{\text{tr}}$, we have that $\gamma_{\text{tr}} \leq \gamma_{\text{tr}}(T') \leq |S| = \gamma_{\text{tr}}$. Hence, $T' \in T$. However, as $T'$ has fewer leaves than $T$, we obtain a contradiction.

Thus, assume $v \notin S$ and let $a, b \in N(v)$ such that $a \notin S$ and $b \in S$. If $c \in N(v) - \{a, b\}$ is in $S$, then, by considering the $(v, b, c)$-pruning of $T$, we obtain a contradiction as before. We therefore assume that $b$ is the only vertex in $S$ which is adjacent to $v$.

Suppose $\deg(v) \geq 4$, let $\{c_1, \ldots, c_{\deg(v)-2}\} = N(v) - \{a, b\}$, let $c = c_1$ and let $\ell_b$ be a leaf of the component of $T - v$ that contains $b$. Let $T'$ be the tree which arises from $T$ by deleting the edges $vc_i$ for $i = 1, \ldots, \deg(v) - 2$ and joining $c$ to $\ell_b, c_2, \ldots, c_{\deg(v)-2}$. Note that $\deg_{T'}(v) = \deg_{T'}(\ell_b) = 2, \deg_{T'}(c) = \deg(c) + \deg(v) - 3 \geq \deg(c) + 1 \geq 3$, while all other vertices have the same degree in $T'$ as in $T$. On the one hand, if $\deg(c) = 2$, then $s(T') = s(T) - \deg(v) + \deg_{T'}(c) = s(T) - 1$. On the other hand, if $\deg(c) \geq 3$, then $s(T') = s(T) - \deg(v) + \deg(v) - 3 = s(T) - 3$. Then $S$ is a TRDS of $T'$. As $T' \in T$ and $s(T') < s(T)$, we obtain a contradiction in both cases. Thus, $\deg(v) = 3$. \hfill \Box
Claim 3.5 No two vertices of degree 3 are adjacent.

Proof. Using the notation employed in Claim 3.4, $b$ is the only neighbor of $v$ in $S$. By Claim 3.4, $\deg(b) \leq 2$. If $\deg(c) = 3$, then, by Claim 3.4, $c$ is adjacent to a vertex in $V - S$ (other than $v$). Let $T'$ be the $(v, c, b)$-pruning of $T$. Then $S$ is a TRDS of $T'$, and so, by definition of $\gamma_{tr}$, we have that $\gamma_{tr} \leq \gamma_{tr}(T') \leq |S| = \gamma_{tr}$. Hence, $T' \in T$. However, as $T'$ has fewer leaves than $T$, we obtain a contradiction. ◇

Using the notation employed in the proof of Claim 3.4, the vertex $b \in S$ and, as it must be adjacent to another vertex in $S$, $\deg(b) = 2$ (cf. Claim 3.4). Let $b' \in S$ be the vertex adjacent to $b$ and suppose $b'$ is not a leaf. Then, by Claim 3.4, $\deg(b') = 2$. Let $b''$ be the neighbor of $b'$ different from $b$. Then $S$ is a TRDS of a tree $T'$ obtained from $T$ by deleting the edge $b'b''$ and joining the vertex $b''$ to some leaf of the component of $T - v$ containing $c$. Thus $T' \in T$ and $b'$ is a leaf of $T'$. Hence we may assume that $b'$ is a leaf of $T$.

By Claim 3.5, $\deg(a) = \deg(c) = 2$. Let $a'(c'$, respectively) be the neighbor of $a$ ($c$, respectively) which is different from $v$. Necessarily, $a', c' \in S$. Then $\deg(a') = \deg(c') = 2$ (cf. Claim 3.4). As each vertex in $S$ is adjacent to another vertex of $S$, there exist vertices $a''$ and $c''$ in $S$ which are adjacent to $a'$ and $c'$ respectively. We may assume, as we did for $b'$, that $a''$ is a leaf of $T$.

If $n = 9$, then $\gamma_{tr}(T) = 6 = \lceil \frac{n+2}{2} \rceil$. Suppose, therefore, that $n \geq 10$. Let $T'$ be the component of $T - cc'$ containing $c'$. Then $S \cap V(T')$ is a TRDS of $T'$, so that $|S \cap V(T')| \geq \gamma_{tr}(T')$. Hence, $|S| \geq 4 + \gamma_{tr}(T')$. Applying the inductive hypothesis to the tree $T'$ of order $n - 7$, we have $\gamma_{tr}(T') \geq \lceil \frac{n-5}{2} \rceil$, and so $\gamma_{tr}(T) = |S| \geq \lceil \frac{n+3}{2} \rceil \geq \lceil \frac{n+2}{2} \rceil$. ◇
3.3 Extremal trees $T$ with $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil$

Let $\mathcal{T}$ be the class of all trees $T$ of order $n(T)$ such that $\gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil$. We will constructively characterize the trees in $\mathcal{T}$. In order to state the characterization, we define four simple operations on a tree $T$.

- **O1.** Join a leaf or a remote vertex of $T$ to a vertex of $K_1$, where $n(T)$ is even.
- **O2.** Join a vertex $v$ of $T$ which lies on an endpath $vxz$ to a leaf of $P_3$, where $n(T)$ is even.
- **O3.** Join a vertex $v$ of $T$ which lies on an endpath $vx_1x_2z$ to a leaf of $P_3$, where $n(T)$ is even.
- **O4.** Join a remote vertex or a leaf of $T$ to a leaf of each of $\ell$ disjoint copies of $P_4$ for some $\ell \geq 1$.

Let $\mathcal{C}$ be the class of all trees obtained from $P_2$ by a finite sequence of Operations O1- O4. We will show that $T \in \mathcal{T}$ if and only if $T \in \mathcal{C}$.

**Lemma 3.6** Let $T' \in \mathcal{T}$ be a tree of even order $n(T')$. If $T$ is obtained from $T'$ by one of the Operations O1-O3, then $T \in \mathcal{T}$.

**Proof.** Let $S$ be a $\gamma_{tr}(T')$-set of $T'$ throughout the proof of this result.

**Case 1.** $T$ is obtained from $T'$ by Operation O1.

Let $u$ be a leaf or a remote vertex of $T'$, and suppose $T$ is formed by attaching the singleton $v$ to $u$. Then $S \cup \{v\}$ is a TRDS set of $T$, and so $\lceil \frac{n(T')+3}{2} \rceil \leq \gamma_{tr}(T) \leq \lceil \frac{n(T')+2}{2} \rceil + 1$. Since $n(T')$ is even, we have $\gamma_{tr}(T) = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T \in \mathcal{T}$.

**Case 2.** $T$ is obtained from $T'$ by Operation O2 or Operation O3.

Suppose $v$ lies on the endpath $vxz$ or $vx_1x_2z$ and $T$ is obtained from $T'$ by adding the path $y_1y_2z'$ to $T'$ and joining $y_1$ to $v$. We show that $v \not\in S$. First consider the case
when \( v \) lies on the endpath \( vxz \). Suppose \( v \in S \). Then \( S' = S - \{z\} \) is a TRDS of \( T' = T'' - \{z\} \), and so \( \lceil \frac{n(T') + 1}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n(T') + 2}{2} \rceil - 1 \). However, as \( n(T') \) is even, we have \( \frac{n(T') + 2}{2} \leq \gamma_{tr}(T') \leq \frac{n(T') + 2}{2} - 1 \), which is a contradiction. Thus, \( v \notin S \). In the case when \( v \) lies on the endpath \( vx_1x_2z \), one may show, as in the previous paragraph, that \( x_1 \notin S \). But then \( v \notin S \), as required.

In both cases, the set \( S \cup \{y_2, z'\} \) is a TRDS of \( T'' = T' - \{z\} \), and so \( \lceil \frac{n(T') + 2}{2} \rceil \leq \gamma_{tr}(T'') \leq \lceil \frac{n(T') + 2}{2} \rceil - 1 \). However, as \( n(T') \) is even, we have \( \frac{n(T') + 2}{2} \leq \gamma_{tr}(T'') \leq \frac{n(T') + 2}{2} - 1 \), which is a contradiction. Thus, \( v \notin S \). In the case when \( v \) lies on the endpath \( vx_1x_2z \), one may show, as in the previous paragraph, that \( x_1 \notin S \). But then \( v \notin S \), as required.

Lemma 3.7 Let \( T' \in T \) be a tree of order \( n(T') \). If \( T \) is obtained from \( T' \) by the Operation \( O4 \), then \( T \in T \).

Proof. Let \( S \) be a \( \gamma_{tr}(T') \)-set of \( T' \), and suppose \( v \) is a remote vertex or a leaf of \( T' \). Then \( v \in S \). Let \( T \) be the tree which is obtained from \( T' \) by adding the paths \( u_i, y_i, z_i \) to \( T' \) and joining \( u_i \) to \( v \) for \( i = 1, \ldots, \ell \). Then \( S \cup \bigcup_{i=1}^{\ell} \{y_i, z_i\} \) is a TRDS of \( T \), and so \( \lceil \frac{n(T') + 2\ell + 2}{2} \rceil \leq \gamma_{tr}(T) \leq \lceil \frac{n(T') + 2\ell + 2}{2} \rceil + 2\ell \). Therefore, \( \gamma_{tr}(T) = \frac{n(T') + 6\ell}{2} = \lceil \frac{n(T') + 2\ell}{2} \rceil \). Thus, \( T \in T \). □

We are now in a position to prove the main result of this section.

Theorem 3.8 \( T \) is in \( C \) if and only if \( T \) is in \( T \).

Proof. Assume \( T \in C \). We show that \( T \in T \), by using induction on \( c(T) \), the number of operations required to construct the tree \( T \). If \( c(T) = 0 \), then \( T = P_2 \), which is in \( T \).

Assume, then, for all trees \( T' \in C \) with \( c(T') < k \), where \( k \geq 1 \) is an integer, that \( T' \) is in \( T \). Let \( T \in C \) be a tree with \( c(T) = k \). Then \( T \) is obtained from some tree \( T' \) by one of the Operations \( O1 - O4 \). But then \( T' \in C \) and \( c(T') < k \). Applying the inductive hypothesis to \( T' \), \( T' \) is in \( T \). Hence, by Lemma 3.6 or Lemma 3.7, \( T \in T \).

To show that \( T \in C \) for a nontrivial \( T \in T \), we use induction on \( n \), the order of the tree \( T \). If \( n = 2 \), then \( T = P_2 \in C \). Let \( T \in T \) be a tree of order \( n \geq 3 \), and assume for
all trees $T' \in \mathcal{T}$ of order $2 \leq n(T') < n$, that $T' \in \mathcal{C}$. Since $n(T) \geq 3$, $\text{diam}(T) \geq 2$. If $\text{diam}(T) = 2$, then $T$ is a star with exactly two leaves, which can be constructed from $P_2$ by applying Operation $O_1$. Thus, $T \in \mathcal{C}$. Since no double star is in $\mathcal{T}$, we may assume $\text{diam}(T) \geq 4$. Throughout $S$ will be used to denote a $\gamma_{tr}(T)$-set of $T$.

**Claim 3.9** Let $z$ be a leaf of $T$. If $S - \{z\}$ is a TRDS of $T' = T - z$, then $T \in \mathcal{C}$.

**Proof.** Assume $S - \{z\}$ is a TRDS of $T'$. Then $\lceil \frac{n-1+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 1$. This yields a contradiction when $n$ is even. Hence, $n$ is odd, and $\gamma_{tr}(T') = \frac{n+1}{2} = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{T}$, with $n(T') = n - 1$ even. By the induction assumption, $T' \in \mathcal{C}$. The tree $T$ can now be constructed from $T'$ by applying Operation $O_1$, whence $T \in \mathcal{C}$. $\diamond$

Claim 3.9 implies that if $vxz$ is an endpath of $T$, then we may assume $v \notin S$, since otherwise the tree is constructible. Claim 3.9 also implies that every remote vertex of $T$ is adjacent to exactly one leaf, since otherwise it is constructible.

**Claim 3.10** If $u$ is a leaf of $T$ and $v$ is either another leaf of $T$ or the remote vertex adjacent to $u$, then $S' = S - \{u, v\}$ is not a TRDS of $T' = T - u - v$.

**Proof.** Suppose, to the contrary, that $S'$ is a TRDS of $T'$. Then $\lceil \frac{n-2+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 2$. Thus, $\lceil \frac{n}{2} \rceil + 2 \leq \lceil \frac{n+2}{2} \rceil$, which yields a contradiction. $\diamond$

Let $T$ be rooted at a leaf $r$ of a longest path. Let $v$ be any vertex on a longest path at distance $\text{diam}(T) - 2$ from $r$. Suppose $v$ lies on the endpath $vyz'$. Then, by the remark above, $v \notin S$, which implies that $v$ is not adjacent to a leaf. If $v$ also lies on the endpath $vxz$, then $S - \{x, z\}$ is a TRDS of $T - x - z$, which is a contradiction by Claim 3.10. Thus, we assume each vertex on a longest path at distance $\text{diam}(T) - 2$ or $\text{diam}(T) - 1$ from $r$ has degree two.
Let $v$ be any vertex on a longest path at distance $\text{diam}(T) - 3$ from $r$. Let $vy_1y_2z'$ be an endpath of $T$. Then $y_1 \notin S$, and so $v \notin S$, which means all neighbors of $v$ have degree at least 2.

Assume $v$ also lies on the path $vxz$, where $z$ is a leaf. Then, since each remote vertex is adjacent to exactly one leaf, $vxz$ is an endpath. If $v$ is dominated by a vertex other than $x$, then $S - \{x, z\}$ is a TRDS of $T' = T - x - z$, which is a contradiction (cf. Claim 3.10). Hence, $v$ is dominated only by $x$. Then $S' = S - \{y_2, z'\}$ is a TRDS of $T' = T - y_1 - y_2 - z'$ and so $\lceil \frac{n-3+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 2$. This yields a contradiction when $n$ is even. Hence, $n$ is odd and $\gamma_{tr}(T') = \frac{n-1}{2} = \lceil \frac{n(T')}{2} + 2 \rceil$. Thus, $T' \in \mathcal{T}$, with $n(T') = n - 3$ even. By the induction assumption, $T' \in \mathcal{C}$. The tree $T$ can now be constructed from $T'$ by applying Operation $\text{O2}$, whence $T \in \mathcal{C}$.

Assume $v$ lies on the path $vx_1x_2z$. Since $x_1$ ($x_2$, respectively) is on a longest path at distance $\text{diam}(T) - 2$ ($\text{diam}(T) - 1$, respectively) from $r$, we have $\deg(x_1) = 2$ ($\deg(x_2) = 2$, respectively). This implies that $vx_1x_2z$ is an endpath, and so $x_1 \notin S$. But then $S' = S - \{x_2, z\}$ is a TRDS of $T' = T - x_1 - x_2 - z$. Thus, $\lceil \frac{n-3+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 2$. This yields a contradiction when $n$ is even. Hence, $n$ is odd and $\gamma_{tr}(T') = \lceil \frac{n(T')}{2} + 2 \rceil$. Thus, $T' \in \mathcal{T}$, with $n(T') = n - 3$ even. By the induction assumption, $T' \in \mathcal{C}$ and $T$ can now be constructed from $T'$ by applying Operation $\text{O3}$, whence $T \in \mathcal{C}$. Thus, we assume each vertex on a longest path at distance $\text{diam}(T) - 3$ from $r$ has degree two.

Let $v$ be any vertex on a longest path at distance $\text{diam}(T) - 4$ from $r$. As $P_5 \notin \mathcal{T}$, $v \neq r$ and $\text{diam}(T) \geq 5$. Assume $\deg_T(v) \geq 3$. Let $vy_1y_2y_3z'$ be an endpath of $T$. But then, as $y_2y_3z'$ is an endpath of $T$, it follows that $y_2 \notin S$, which implies $y_1 \notin S$ and $v \in S$. Moreover, $S' = S - \{y_3, z'\}$ is a TRDS of $T' = T - y_1 - y_2 - y_3 - z'$. Thus, $\lceil \frac{n-4+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 2$, whence $\gamma_{tr}(T') = \lceil \frac{n(T')}{2} + 2 \rceil$. We conclude that $T' \in \mathcal{T}$, and by the induction assumption, $T' \in \mathcal{C}$. If $\deg_T(v) = 2$ or when $v$ is a remote vertex, then $T$ can be constructed from $T'$ by applying Operation $\text{O4}$.
We now assume that \( \deg_T(v) \geq 3 \) and that \( v \) is not adjacent to a leaf. If \( v \) also lies on the path \( vxz \), where \( z \) is a leaf, then \( v \notin S \), which is a contradiction. We therefore assume that \( v \) lies on the path \( vx_1x_2z \), where \( z \) is a leaf. Since \( x_2 \) is a remote vertex, we have \( \deg(x_2) = 2 \). As \( x_1x_2z \) is an endpath of \( T \), it follows that \( x_1 \notin S \). As \( x_1 \) must be adjacent to another vertex in \( V - S \), vertex \( x_1 \) lies on a path \( x_1, u_1, u_2, z'' \). But then \( x_1 \), with \( \deg(x_1) \geq 3 \), is a vertex at distance \( \text{diam}(T) - 3 \) on a longest path from \( r \), which is a contradiction.

Let \( e \) be the edge that joins \( v \) with its parent, and let \( T(v) \) be the component of \( T - e \) that contains \( v \). Then \( T(v) \) consists of \( \ell \) disjoint paths \( u_i x_i y_i z_i \) \((i = 1, \ldots, \ell)\) with \( v \) joined to \( u_i \) for \( i = 1, \ldots, \ell \). Let \( i \in \{1, \ldots, \ell\} \). Since \( x_i y_i z_i \) is an endpath of \( T \), we have \( x_i \notin S \), \( u_i \notin S \) and \( v \in S \). Then \( S - \cup_{i=1}^{\ell} \{y_i, z_i\} \) is a TRDS of \( T' = T - (T(v) - v) \), and so \( \lceil \frac{n - 4k + 2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n + 2}{2} \rceil - 2\ell \), whence \( \gamma_{tr}(T') = \lceil \frac{n(T') + 2}{2} \rceil \). Thus, \( T' \in T \), and by the induction assumption, \( T' \in C \). Note that \( v \) is a leaf of \( T' \). The tree \( T \) can now be constructed from \( T' \) by applying Operation \( O_4 \), whence \( T \in C \).

**Theorem 3.11** Let \( T \) be a tree of order \( n(T) \). If \( n(T) \equiv 0 \mod 4 \), then \( \gamma_{tr}(T) \geq \lceil \frac{n(T) + 2}{2} \rceil + 1 \).

**Proof.** We will show that every tree \( T \) in \( T = C \) has \( n(T) \not\equiv 0 \mod 4 \), by using induction on \( s(T) \), the number of operations required to construct the tree \( T \). If \( s(T) = 0 \), then \( T = P_2 \), and \( 2 \not\equiv 0 \mod 4 \). Assume, then, for all trees \( T' \in C \) with \( s(T') < k \), where \( k \geq 1 \) is an integer, that \( n(T') \not\equiv 0 \mod 4 \). Let \( T \in C \) be a tree with \( s(T) = k \). Then \( T \) is obtained from some tree \( T' \) by one of the Operations \( O_1 - O_4 \). Then \( T' \in C \), and by the induction hypothesis, \( n(T') \not\equiv 0 \mod 4 \). If \( T \) is obtained from \( T' \) by one of the Operations \( O_1 - O_3 \), then \( n(T') \equiv 2 \mod 4 \), and, since either a path of order one or a path of order three is attached to \( T' \) to form \( T \), \( n(T) \not\equiv 0 \mod 4 \). Moreover, \( n(T) = n(T') + 4 \) if \( T \) is obtained from \( T' \) by Operation \( O_4 \), whence \( n(T) \not\equiv 0 \mod 4 \). The result now follows.
3.4 Extremal trees $T$ with $\gamma_{tr}(T) = \left\lceil \frac{n(T)+2}{2} \right\rceil + 1$

Let $T^* = \{ T \mid T$ is a tree of order $n(T) \equiv 0 \pmod{4}$ such that $\gamma_{tr}(T) = \left\lceil \frac{n+2}{2} \right\rceil + 1 \}$.

In order to constructively characterize the trees in $T^*$, we define the following operations on a tree $T$:

O5. Join a leaf or a remote vertex $v$ of $T$ to a vertex of $K_1$, where $n(T) \equiv 3 \pmod{4}$.

O6. Join a vertex $v$ of $T$ which lies on an endpath $vxz$ to a vertex of $K_2$, where $n(T) \equiv 2 \pmod{4}$.

O7. Join a vertex $v$ of $T$ which lies on an endpath $vx_1x_2z$ to a vertex of $K_2$, where $n(T) \equiv 2 \pmod{4}$.

O8. Join a vertex $v$ of $T$ which lies on an endpath $vxz$ to a leaf of $P_3$, where $n(T) \equiv 1 \pmod{4}$.

O9. Join a vertex $v$ of $T$ which lies on an endpath $vx_1x_2z$ to a leaf of $P_3$, where $n(T) \equiv 1 \pmod{4}$.

Let $I = \{ T \mid T$ is a tree obtained by applying one of the Operations O5 – O9 to a tree $T' \in C$ exactly once $\}$. Let $C^* = \{ T \mid T$ is a tree obtained from a tree $T' \in I$ by applying Operation O4 to $T'$ zero or more times $\}$. We will show that $T^* = C^*$.

**Lemma 3.12** Let $T' \in C$ be a tree of order $n(T') \equiv 3 \pmod{4}$. If $T$ is obtained from $T'$ by Operation O5, then $T \in T^*$.

**Proof.** Let $u$ be a leaf or a remote vertex of $T'$, and suppose $T$ is formed by attaching the singleton $v$ to $u$. Let $S$ be a $\gamma_{tr}(T')$-set of $T'$. Then $S \cup \{ v \}$ is a TRDS set of $T$, and so, since $n(T) \equiv 0 \pmod{4}$, $\left\lceil \frac{n(T')+2}{2} \right\rceil + 1 \leq \gamma_{tr}(T) \leq |S| + 1 = \left\lceil \frac{n(T')+2}{2} \right\rceil + 1 = \left\lceil \frac{n(T)+1}{2} \right\rceil + 1$.

Hence, $\gamma_{tr}(T) = \left\lceil \frac{n(T)+2}{2} \right\rceil + 1$, and so $T \in T^*$. □

**Lemma 3.13** Let $T' \in C$ be a tree of order $n(T') \equiv 2 \pmod{4}$. If $T$ is obtained from $T'$ by either Operation O6 or Operation O7, then $T \in T^*$.
Proof. Let \( \{u, v\} \) be the vertex set of \( K_2 \) and let \( S \) be a \( \gamma_{tr}(T') \)-set. The set \( S \cup \{u, v\} \) is a TRDS of \( T \), and so, since \( n(T) \equiv 0 \text{ mod } 4 \), \( \lceil \frac{n(T)+2}{2} \rceil + 1 \leq \gamma_{tr}(T) \leq |S| + 2 = \lceil \frac{n(T')}{2} \rceil + 2 = \lceil \frac{n(T) + 2}{2} \rceil + 2 \). Hence, \( \gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil + 1 \), and so \( T \in T^* \). \( \square \)

Lemma 3.14 Let \( T' \in C \) be a tree of order \( n(T') \equiv 1 \text{ mod } 4 \). If \( T \) is obtained from \( T' \) by either Operation \( O_8 \) or Operation \( O_9 \), then \( T \in T^* \).

Proof. Let \( S \) be a \( \gamma_{tr}(T') \)-set of \( T' \). Assume \( v \) lies on the endpath \( vxz \) or \( vx_1x_2z \) and \( T \) is obtained from \( T' \) by adding the path \( y_1y_2z' \) to \( T' \) and joining \( y_1 \) to \( v \). We show that \( v \not\in S \). First consider the case when \( v \) lies on the endpath \( vxz \). Suppose \( v \in S \). Then \( x, z \in S \), and \( S - \{z\} \) is TRDS of \( T'' = T' - z \). Since \( n(T'') \equiv 0 \text{ mod } 4 \), \( \lceil \frac{n(T'')+2}{2} \rceil + 1 \leq \gamma_{tr}(T'') \leq |S| - 1 = \lceil \frac{n(T')+2}{2} \rceil - 1 = \lceil \frac{n(T')+3}{2} \rceil - 1 \), and so \( \frac{n(T'')+4}{2} \leq \frac{n(T')+2}{2} \), which is a contradiction. Thus, \( v \not\in S \). In the case when \( v \) lies on the endpath \( vx_1x_2z \), one may show, as in the previous paragraph, that \( x_1 \not\in S \). But then \( v \not\in S \), as required. In both cases, the set \( S \cup \{y_2, z'\} \) forms a TRDS of \( T \), so that \( \lceil \frac{n(T)}{2} \rceil + 1 \leq \gamma_{tr}(T) \leq |S| + 2 = \lceil \frac{n(T)+2}{2} \rceil + 2 = \lceil \frac{n(T)-1}{2} \rceil + 2 \). Hence, \( \gamma_{tr}(T) = \lceil \frac{n(T)+2}{2} \rceil + 1 \), and so \( T \in T^* \). \( \square \)

The proof of the following result is similar to that of Lemma 3.7.

Lemma 3.15 If \( T \) is obtained from \( T' \in T^* \) by Operation \( O_4 \), then \( T \in T^* \).

Lemma 3.16 If \( T \) is in \( I \), then \( T \) is in \( T^* \).

Proof. Assume \( T \in I \). Then \( T \) is obtained from \( T' \in C \) by applying one of the Operations \( O_5 - O_9 \) exactly once. Then, by Lemmas 3.12, 3.13 and 3.14, \( T \in T^* \). \( \square \)

Theorem 3.17 \( T \) is in \( C^* \) if and only if \( T \) is in \( T^* \).
Proof. Assume \( T \in \mathcal{C}^* \). We show that \( T \in T^* \), by using induction on \( c(T) \), the number of operations required to construct the tree \( T \). If \( c(T) = 0 \), then \( T \in \mathcal{I} \), and the result follows from Lemma 3.16. Assume, then, for all trees \( T' \in \mathcal{C}^* \) with \( c(T') < k \), where \( k \geq 1 \) is an integer, that \( T' \) is in \( T^* \). Let \( T \in \mathcal{C}^* \) be a tree with \( c(T) = k \). Then \( T \) is obtained from some tree \( T' \) by applying Operation \( O_4 \). But then \( T' \in \mathcal{C}^* \) and \( c(T') < k \). Applying the inductive hypothesis to \( T' \), \( T' \) is in \( T^* \). Hence, by Lemma 3.15, \( T \) is in \( T^* \).

To show that \( T \in \mathcal{C}^* \) for a nontrivial \( T \in T^* \), we employ induction on \( 4n \), the order of the tree \( T \). Suppose \( n = 1 \). Then \( T \cong K_{1,3} \) or \( T \cong P_4 \), and \( T \) can be constructed from \( P_3 \in \mathcal{C} \) by applying Operation \( O_5 \). Let \( T \in T^* \) be a tree of order \( 4n \), where \( n \geq 2 \), and suppose \( T' \in \mathcal{C}^* \) for all trees \( T' \in T^* \) of order \( 4n' \) where \( n' < n \). The only trees \( T \) with \( \text{diam}(T) \leq 3 \) which are in \( T^* \) are \( K_{1,3} \) and \( P_4 \). As \( 4n \geq 8 \), it follows that \( \text{diam}(T) \geq 4 \).

Throughout \( S \) will be used to denote a \( \gamma_{tr} \)-set of \( T \), i.e. \( |S| = \lceil \frac{n+2}{2} \rceil + 1 \).

Claim 3.18 If \( u \) and \( v \) are vertices of \( T \) such that \( T' = T - u - v \) is a tree and \( S' = S - \{u, v\} \) is a TRDS of \( T' \), then \( n(T') \equiv 2 \mod 4 \) and \( T' \in \mathcal{C} \).

Proof. As \( \lceil \frac{n-2+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil + 1 - 2 \), we have \( \gamma_{tr}(T') = \lceil \frac{n-2+2}{2} \rceil = \lceil \frac{n(T')+2}{2} \rceil \), and so \( T' \in \mathcal{C} \). \( \diamond \)

Claim 3.19 Let \( z \) be a leaf of \( T \). If \( S - \{z\} \) is a TRDS of \( T' = T - z \), then \( T \in \mathcal{C}^* \).

Proof. Assume \( S - \{z\} \) is a TRDS of \( T' \). Then \( \lceil \frac{n-1+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil + 1 - 1 = \lceil \frac{n+2}{2} \rceil \). Hence, \( n - 1 \equiv 3 \mod 4 \) and \( \gamma_{tr}(T') = \lceil \frac{n+1}{2} \rceil = \lceil \frac{n(T')+2}{2} \rceil \). Thus, \( T' \in \mathcal{C} \). The tree \( T \) can now be constructed from \( T' \) by applying Operation \( O_5 \), whence \( T \in \mathcal{C}^* \). \( \diamond \)

Claim 3.19 implies that if \( vxz \) is an endpath of \( T \), then we may assume \( v \not\in S \), since otherwise the tree is constructible. Claim 3.19 also implies that every remote vertex of \( T \) is adjacent to exactly one leaf, since otherwise it is constructible.
Let $T$ be rooted at a leaf $r$ of a longest path. Let $v$ be any vertex on a longest path at distance $\text{diam}(T) - 2$ from $r$. Suppose $v$ lies on the endpath $vyz'$. Then, by the remark above, $v \not\in S$, which implies that $v$ is not adjacent to a leaf. If $v$ also lies on the endpath $vxz$, then $S - \{x, z\}$ is a TRDS of $T - x - z$ and so $T' \in \mathcal{C}$ (cf. Claim 3.18), whence $T \in \mathcal{C}^*$ (as it can be constructed from $T'$ by applying Operation O6). Thus, we assume each vertex on a longest path at distance $\text{diam}(T) - 2$ or $\text{diam}(T) - 1$ from $r$ has degree two.

Let $v$ be any vertex on a longest path at distance $\text{diam}(T) - 3$ from $r$. Let $vy_1y_2z'$ be an endpath of $T$. Then $y_1 \not\in S$, and so $v \not\in S$, which means all neighbors of $v$ have degree at least 2.

Assume $v$ also lies on the path $vxz$, where $z$ is a leaf. Then, since each remote vertex is adjacent to exactly one leaf, $vxz$ is an endpath. If $v$ is dominated by a vertex other than $x$, then $S - \{x, z\}$ is a TRDS of $T' = T - x - z$ and so $T' \in \mathcal{C}$ (cf. Claim 3.18), whence $T \in \mathcal{C}^*$ (as it can be constructed from $T'$ by applying Operation O7). Hence, $v$ is dominated only by $x$. Then $S' = S - \{y_2, z'\}$ is a TRDS of $T' = T - y_1 - y_2 - z'$ and so $\lceil \frac{n-3+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 1$. But then $\gamma_{tr}(T') = \lceil \frac{n-1}{2} \rceil = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{C}$. The tree $T$ can now be constructed from $T'$ by applying Operation O8.

Assume $v$ lies on the path $vx_1x_2z$. Since $x_1$ ($x_2$, respectively) is on a longest path at distance $\text{diam}(T) - 2$ ($\text{diam}(T) - 1$, respectively) from $r$, we have $\deg(x_1) = 2$ ($\deg(x_2) = 2$, respectively). This implies that $vx_1x_2z$ is an endpath, and so $x_1 \not\in S$. But then $S' = S - \{x_2, z\}$ is a TRDS of $T' = T - x_1 - x_2 - z$. Thus, $\lceil \frac{n-3+2}{2} \rceil \leq \gamma_{tr}(T') \leq \lceil \frac{n+2}{2} \rceil - 1$. But then $\gamma_{tr}(T') = \lceil \frac{n-1}{2} \rceil = \lceil \frac{n(T')+2}{2} \rceil$. Thus, $T' \in \mathcal{C}$ and so $T$ can now be constructed from $T'$ by applying Operation O9. Thus, we assume each vertex on a longest path at distance $\text{diam}(T) - 3$ from $r$ has degree two.

Let $v$ be any vertex on a longest path at distance $\text{diam}(T) - 4$ from $r$. As $P_5 \not\in T^*$, $v \neq r$ and $\text{diam}(T) \geq 5$. Assume $\deg_T(v) \geq 3$. Let $vy_1y_2y_3z'$ be an endpath of $T$. But
then, as $y_2y_3z'$ is an endpath of $T$, it follows that $y_2 \notin S$, which implies $y_1 \notin S$ and $v \in S$. Moreover, $S' = S - \{y_3, z'\}$ is a TRDS of $T' = T - y_1 - y_2 - y_3 - z'$. Thus, $\left\lceil \frac{n-4+2}{2} \right\rceil + 1 \leq \gamma_{tr}(T') \leq \left\lceil \frac{n+2}{2} \right\rceil - 1$, whence $\gamma_{tr}(T') = \left\lceil \frac{n(T') + 2}{2} \right\rceil + 1$. We conclude that $T' \in T^*$, and by the induction assumption, $T' \in C^*$. If $\deg_T(v) = 2$ or when $v$ is a remote vertex, then $T$ can be constructed from $T'$ by applying Operation $O4$, whence $T \in C^*$.

We therefore assume that $\deg_T(v) \geq 3$ and that $v$ is not adjacent to a leaf. If $v$ also lies on the path $vxz$, where $z$ is a leaf, then $v \notin S$, which is a contradiction. We now assume that $v$ lies on the path $vx_1x_2z$, where $z$ is a leaf. Then, since $x_2$ is a remote vertex, we have $\deg(x_2) = 2$. As $x_1x_2z$ is an endpath of $T$, it follows that $x_1 \notin S$. As $x_1$ must be adjacent to another vertex in $V - S$, vertex $x_1$ lies on a path $x_1, u_1, u_2, z''$. But then $x_1$, with $\deg(x_1) \geq 3$, is a vertex at distance $\text{diam}(T) - 3$ on a longest path from $r$, which is a contradiction.

Let $e$ be the edge that joins $v$ with its parent, and let $T(v)$ be the component of $T - e$ that contains $v$. Then $T(v)$ consists of $\ell$ disjoint paths $u_ix_iy_iz_i$ ($i = 1, \ldots, \ell$) with $v$ joined to $u_i$ for $i = 1, \ldots, \ell$. Let $i \in \{1, \ldots, \ell\}$. Since $x_iy_iz_i$ is an endpath of $T$, we have $x_i \notin S$, $u_i \notin S$ and $v \in S$. Then $S - \bigcup_{i=1}^{\ell}\{y_i, z_i\}$ is a TRDS of $T' = T - (T(v) - v)$, and so $\left\lceil \frac{n-4\ell+2}{2} \right\rceil + 1 \leq \gamma_{tr}(T') \leq \left\lceil \frac{n+2}{2} \right\rceil - 2\ell + 1$, whence $\gamma_{tr}(T') = \left\lceil \frac{n(T') + 2}{2} \right\rceil + 1$. Thus, $T' \in T^*$, and by the induction assumption, $T' \in C^*$. Note that $v$ is a leaf of $T'$. The tree $T$ can now be constructed from $T'$ by applying Operation $O4$, whence $T \in C^*$. $\square$
Chapter 4

Nordhaus-Gaddum Results for
Restrained Domination and Total
Restrained Domination in Graphs

4.1 Introduction

In this chapter, we continue the study of restrained domination and total restrained
domination in graphs. Recall that a set $S \subseteq V$ is a restrained dominating set (abbreviated
RDS), if every vertex in $V - S$ is adjacent to a vertex in $S$ and a vertex in $V - S$. The
restrained domination number of $G$, denoted by $\gamma_r(G)$, is the minimum cardinality of a
RDS of $G$. A RDS of cardinality $\gamma_r(G)$ will be called a $\gamma_r(G)$-set. A set $S \subseteq V$ is a total
restrained dominating set, (abbreviated TRDS) if every vertex is adjacent to a vertex in $S$ and every
vertex in $V - S$ is also adjacent to a vertex in $V - S$. The total restrained domination number of $G$, denoted by $\gamma_{tr}(G)$, is the minimum cardinality of a TRDS of $G$. A TRDS of cardinality $\gamma_{tr}(G)$ will be called a $\gamma_{tr}(G)$-set.
Recall that a set $S \subseteq V$ is a dominating set (abbreviated herein as DS) if every vertex not in $S$ is adjacent to a vertex in $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a DS of $G$. A DS of cardinality $\gamma(G)$ will be called a $\gamma(G)$-set. Nordhaus and Gaddum present best possible bounds on the sum of the chromatic number of a graph and its complement in [16]. The corresponding result for the domination number of a graph is presented by Jaeger and Payan in [15]: If $G$ is a graph of order $n \geq 2$, then $\gamma(G) + \gamma(\overline{G}) \leq n + 1$. A best possible bound on the sum of the restrained domination numbers of a graph and its complement is obtained in [8]:

**Theorem 4.1** If $G$ is a graph of order $n \geq 2$ such that both $G$ and $\overline{G}$ are not isomorphic to $P_3$, then $4 \leq \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 2$.

A best possible bound on the sum of the total restrained domination numbers of a graph and its complement is obtained in [4]:

**Theorem 4.2** If $G$ is a graph of order $n \geq 2$ such that neither $G$ nor $\overline{G}$ contains isolated vertices or has diameter two, then $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 4$.

Let $K$ be the graph obtained from $K_3$ by matching the vertices of $K_2$ to distinct vertices of $K_3$. Note that $K$ is self-complementary, $K$ nor $\overline{K}$ contains isolated vertices or has diameter two, while $\gamma_{tr}(K) + \gamma_{tr}(\overline{K}) = 2 \times 5 = 10 > n(K) + 4$. Thus, Theorem 4.2 is incorrect.

We will show, in Section 4.2, that if $G$ is a graph of order $n \geq 2$ such that neither $G$ nor $\overline{G}$ contains isolated vertices or is isomorphic to $K$, then $4 \leq \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 4$. Moreover, we will characterize the graphs $G$ of order $n$ for which $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$ and also characterize those graphs $G$ for which $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = 4$. In Section 4.3, we characterize the graphs $G$ of order $n$ for which $\gamma_{r}(G) + \gamma_{r}(\overline{G}) = n + 2$ as well as those graphs $G$ for which $\gamma_{r}(G) + \gamma_{r}(\overline{G}) = 4$. 


4.2 Total Restrained Domination

In this section, we provide bounds on the sum of the total restrained domination numbers of a graph and its complement, and provide characterizations of the extremal graphs achieving these bounds.

Let \( n \geq 5 \) be an integer and suppose \( \{x, y, u, v\} \) and \( X \) are disjoint sets of vertices such that \( |X| = n - 4 \). Let \( \mathcal{L} \) be the family of graphs \( G \) of order \( n \) where \( V(G) = \{x, y, u, v\} \cup X \) and with the following properties:

**P1:** \( x \) and \( y \) are non-adjacent, while \( u \) and \( v \) are adjacent,

**P2:** each vertex in \( \{x, y\} \cup X \) is adjacent to some vertex of \( \{u, v\} \),

**P3:** each vertex in \( \{u, v\} \cup X \) is non-adjacent to some vertex of \( \{x, y\} \),

**P4:** each vertex in \( \{x, y\} \cup X \) is adjacent to some vertex of \( \{x, y\} \cup X \),

**P5:** each vertex in \( \{u, v\} \cup X \) is non-adjacent to some vertex of \( \{u, v\} \cup X \).

**Theorem 4.3** If \( G \) be a graph of order \( n \geq 2 \) such that neither \( G \) nor \( \overline{G} \) contains isolated vertices, then \( \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = 4 \) if and only if \( G \in \mathcal{L} \).

**Proof.** Suppose \( G \) is a graph such that neither \( G \) nor \( \overline{G} \) contains isolated vertices, and suppose \( \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = 4 \). Then \( \gamma_{tr}(G) = \gamma_{tr}(\overline{G}) = 2 \). Let \( S = \{u, v\} \) (\( S' = \{x, y\} \), respectively) be a TRDS of \( G \) (\( \overline{G} \), respectively). Then \( x \) is non-adjacent to \( y \), while \( u \) is adjacent to \( v \), and Property **P1** holds. Clearly, \( S \neq S' \). Suppose \( u = x \) with \( v \neq y \). Since \( \{u, v\} \) is a DS of \( G \) and \( y \) is non-adjacent to \( x = u \), the vertex \( y \) must be adjacent to \( v \). But then \( v \) is not dominated by \( S' \) in \( \overline{G} \), which is a contradiction. Thus, \( S \cap S' = \emptyset \). Let \( X = V(G) - \{x, y, u, v\} \). Then \( |X| = n - 4 \), and since \( S \) (\( S' \), respectively) is a TRDS of \( G \) (\( \overline{G} \), respectively), Properties **P2** – **P5** hold for \( G \). Thus, \( G \in \mathcal{L} \). The converse clearly holds as \( \{u, v\} \) (\( \{x, y\} \), respectively) is a TRDS of \( G \) (\( \overline{G} \), respectively). □
Let \( \text{diam}(G) \) denote the diameter of \( G \), and let \( u, v \) be two vertices of \( G \) such that \( d(u, v) = \text{diam}(G) \). The set of vertices at distance \( i \) from \( u \), \( 0 \leq i \leq \text{diam}(G) \), will be denoted by \( V_i \), and the sets \( V_0, \ldots, V_{\text{diam}(G)} \) will then be called the level decomposition of \( G \) with respect to \( u \). To facilitate argumentation we use the following definition given by Cockayne, Dawes and Hedetniemi [5]. A total dominating set (abbreviated TDS) of \( G \) is a set \( S \subseteq V \) such that every vertex of \( G \) is adjacent to a vertex of \( S \).

Let \( U = \{ G \mid G \) is a graph of order \( n \) which can be obtained from a \( P_4 \) with consecutive vertices labeled \( u, v_1, v_2, v \) by joining vertices \( v_1 \) and \( v_2 \) to each vertex of \( K_{n-4} \) where \( n \geq 6 \}) \}.

**Theorem 4.4** Let \( G \) be a graph of order \( n \geq 2 \) such that neither \( G \) nor \( \overline{G} \) contains isolated vertices or is isomorphic to \( K \). Then \( \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 4 \). Moreover, \( \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4 \) if and only if \( G \in U \) or \( \overline{G} \in U \) or \( G \cong P_4 \).

**Proof.** If \( G \) is disconnected, then \( \gamma_{tr}(\overline{G}) = 2 \). Hence \( \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 2 \). Thus, without loss of generality, assume both \( G \) and \( \overline{G} \) are connected. Let \( u \) and \( v \) be vertices such that \( d(u, v) = \text{diam}(G) \) and let \( V_0, \ldots, V_{\text{diam}(G)} \) be the level decomposition of \( G \) with respect to \( u \). We consider the following cases:

**Case 1.** \( \text{diam}(G) \geq 5 \).

We claim that \( \{u, v\} \) is a \( \text{TRDS} \) of \( \overline{G} \). The vertex \( u \) is non-adjacent to all vertices in \( V_i \) where \( 2 \leq i \leq \text{diam}(G) \), while the vertex \( v \) is non-adjacent to all vertices in \( V_i \) where \( 0 \leq i \leq \text{diam}(G) - 2 \). Moreover, every vertex in \( V(G) - \{u, v\} \) is non-adjacent to some vertex of \( V(G) - \{u, v\} \). Thus, \( \gamma_{tr}(\overline{G}) = 2 \), and so \( \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 2 \).

**Case 2.** \( \text{diam}(G) = 4 \).

Suppose \( u, v_1, v_2, v_3, v \) is a diametrical path. If \( |V_4| \geq 2 \), then \( \{u, v\} \) is a \( \text{TRDS} \) of \( \overline{G} \), and the result follows. Thus, \( V_4 = \{v\} \). Let \( V_{21} = \{x \in V_2 \mid \) there exists a vertex in
$V_1 \cup V_2 \cup V_3$ that is not adjacent to $x$} and let $V_{22} = V_2 - V_{21}$. The set $\{u, v\} \cup V_{22}$ is a TRDS of $\overline{G}$. So we have that $\gamma_{tr}(G) \leq 2 + |V_{22}|$. If $|V_{22}| \leq 1$, then $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 3$. Hence $|V_{22}| \geq 2$.

Let $t \in V_{22}$ such that $t \neq v_2$. Suppose $|V_1 \cup V_{21} \cup V_3| \geq 4$. Let $s \in V_1 \cup V_{21} \cup V_3 - \{v_1, v_2, v_3\}$. Then $V_1 \cup V_{21} \cup V_3 \cup \{u, v, t\} - \{s\}$ is a TRDS of $G$ and so $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n - (|V_{22}| - 1) - 1 + |V_{22}| + 2 \leq n + 2$. Hence $|V_1| = 1$, $|V_{21}| \leq 1$ and $|V_3| = 1$. Therefore, $V(G) - V_{22}$ is a TRDS of $G$ and so $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n - |V_{22}| + 2 + |V_{22}| \leq n + 2$.

**Case 3.** $\text{diam}(G) = 3$.

Let $u, v_1, v_2, v$ be a diametrical path. Suppose $t \in V_3 - \{v\}$. We define $V_{21} = \{x \in V_2 \mid \text{there exists a vertex in } V_1 \cup V_2 \cup V_3 - \{t\} \text{ that is not adjacent to } x\}$ and let $V_{22} = V_2 - V_{21}$. The set $\{u, t\} \cup V_{22}$ is a TRDS of $G$ and so $\gamma_{tr}(G) \leq 2 + |V_{22}|$. If $|V_{22}| = 1$, then surely $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 3$. Hence $|V_{22}| \geq 2$. The vertex $t$ is adjacent to some vertex $s \in V_2$. If $s \in V_{22}$, then the set $\{u, s\} \cup V_1 \cup V_{21} \cup V_3 - \{v\}$ is a TRDS of $G$. If $s \notin V_{22}$, then the set $\{u, w\} \cup V_1 \cup V_{21} \cup V_3 - \{v\}$ is a TRDS of $G$, where $w \in V_{22}$. In both cases, $\gamma_{tr}(G) \leq n - |V_{22}|$, and so $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n - |V_{22}| + 2 + |V_{22}| = n + 2$. Thus, $V_3 = \{v\}$.

Define $V_{11} = \{x \in V_1 \mid \text{there exists a vertex in } V_1 \cup V_2 \text{ that is not adjacent to } x\}$ and let $V_{12} = V_1 - V_{11}$. Moreover, let $V_{21} = \{x \in V_2 \mid \text{there exists a vertex in } V_1 \cup V_2 \text{ that is not adjacent to } x\}$ and let $V_{22} = V_2 - V_{21}$. Then $\{u, v\} \cup V_{12} \cup V_{22}$ is a TRDS of $\overline{G}$, whence $\gamma_{tr}(\overline{G}) \leq 2 + |V_{12}| + |V_{22}|$.

**Case 3.1** $|V_{12}| + |V_{22}| \leq 2$.

Clearly $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 4$. We now investigate when, in this case, $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$. As $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$, we must have that $|V_{12}| + |V_{22}| = 2$. We first show that $\deg(u) = \deg(v) = 1$. 
Suppose, to the contrary, \( \{v_1, w\} \subseteq N(u) \), and let \( t \in V_{12} \cup V_{22} - \{w\} \). Then \( t \) is adjacent to every vertex of \( V_1 \cup V_2 \), and so \( V(G) - \{u, w\} \) is a TRDS of \( G \). It now follows that \( \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n - 2 + 4 = n + 2 \), which is a contradiction. Thus, \( \deg(u) = 1 \), and \( \deg(v) = 1 \) follows similarly. Hence \( V_1 = V_{12} = \{v_1\} \), and the set \( V_{22} \) consists of exactly one vertex, say \( w \).

Suppose \( w \neq v_2 \). If \( |V_2| = 2 \), then \( G \cong K \), which is not allowable. So, let \( w' \in V_2 - \{v_2, w\} \). Then \( w \) and \( w' \) are adjacent, and \( V(G) - \{w, w'\} \) is a TRDS of \( G \). As before, we obtain a contradiction. We conclude \( w = v_2 \). If \( V_{21} = \emptyset \), then \( G \cong P_1 \). If \( V_{21} \neq \emptyset \), then surely \( |V_{21}| \geq 2 \). If two vertices, say \( t \) and \( t' \), of \( V_{21} \) are adjacent in \( G \), then \( V(G) - \{t, t'\} \) is a TRDS of \( G \), and we obtain a contradiction as before. Thus, \( V_{21} \) is independent, and so \( \overline{G} \in \mathcal{U} \).

**Case 3.2** \(|V_{12}| + |V_{22}| \geq 3\).

If we can show that \( G \) has a TRDS of size at most \( s := n - |V_{12}| - |V_{22}| + 1 \), then \( \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n - |V_{12}| - |V_{22}| + 1 + 2 + |V_{12}| + |V_{22}| = n + 3 \). First consider the case when \( v_1 \in V_{11} \). Choose \( w = v_2 \) if \( v_2 \in V_{22} \), otherwise choose \( w \in V_{12} \cup V_{22} \). In both situations, \( \{u, v, w\} \cup V_{11} \cup V_{21} \) is a TRDS of \( G \) of size \( s \). Thus, \( v_1 \not\in V_{11} \). If \( v_2 \in V_{21} \), then \( \{u, v_1, v\} \cup V_{11} \cup V_{21} \) is a TRDS of \( G \) of size \( s \). Thus, \( v_2 \not\in V_{21} \). We conclude that \( v_1 \in V_{12} \), while \( v_2 \in V_{22} \).

Suppose \( u \) is adjacent to a vertex \( w \) which is distinct from \( v_1 \). If \( w \in V_{12} \), then \( \{v_1, v_2, v\} \cup V_{11} \cup V_{21} \) is a TRDS of size \( s \). If \( w \in V_{11} \), then \( \{v_1, v_2, v\} \cup (V_{11} - \{w\}) \cup V_{21} \) is a TRDS of size \( s - 1 \). Thus, \( \deg(u) = 1 \), and \( \deg(v) = 1 \) follows similarly.

Suppose \( V_{22} = \{v_2\} \). If \( V_{21} = \emptyset \), then \( G \cong P_4 \) and \( \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4 \). If \( V_{21} \neq \emptyset \), then surely \( |V_{21}| \geq 2 \). If two vertices, say \( t \) and \( t' \), of \( V_{21} \) are adjacent in \( G \), then \( \{u, v_1, v_2, v\} \cup (V_{21} - \{t, t'\}) \) is a TRDS of \( G \) of size \( s - 1 \). Thus, \( V_{21} \) is independent, \( \overline{G} \in \mathcal{U} \) and \( \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4 \).
Therefore, $|V_{22}| \geq 2$. If $V_{21} = \emptyset$, then $V_{22}$ induces a clique. If $|V_{22}| = 2$, then $G \cong K$, which is not allowable. If $|V_{22}| \geq 3$, then $G \in \mathcal{U}$ and $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$. Thus, $V_{21} \neq \emptyset$, and so $|V_{21}| \geq 2$. Let $\{t, t'\} \subseteq V_{21}$. Then $\{u, v_1, v_2, v\} \cup (V_{21} - \{t, t'\})$ is a TRDS of $G$ of size $s - 1$.

Case 4. $diam(G) = diam(\overline{G}) = 2$.

Note that $\delta(G) \geq 2$ and $\delta(\overline{G}) \geq 2$, since otherwise $G$ or $\overline{G}$ will have isolated vertices.

Case 4.1 $\delta(G) = 2$ or $\delta(\overline{G}) = 2$.

Without loss of generality, assume $\delta(G) = 2$ and suppose $u$ is a vertex of minimum degree in $G$. Let $N(u) = \{v, w\}$. Let $N_{v,w} = \{x \in V(G) - \{u, v, w\} | x$ is adjacent to both $v$ and $w\}$, let $N_{v,\overline{w}} = \{x \in V(G) - \{u, v, w\} | x$ is adjacent to $v$ but not to $w\}$, and let $N_{\overline{w},v} = \{x \in V(G) - \{u, v, w\} | x$ is adjacent to $w$ but not to $v\}$. Moreover, let $N_1 = \{x \in N_{u,v} | N(x) = \{v, w\}\}$ and let $N_2 = N_{v,w} - N_1$.

Now, if $N_1 = \emptyset$, then $\{u, v, w\}$ is a TRDS of $G$ and so $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 3$. Thus, $N_1 \neq \emptyset$. If $N_{v,\overline{w}} = \emptyset$ ($N_{\overline{w},v} = \emptyset$, respectively), then $\{u, w\}$ ($\{u, v\}$, respectively) is a TRDS of $G$, whence $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 2$. Thus, $N_{v,\overline{w}} \neq \emptyset$ and $N_{\overline{w},v} \neq \emptyset$.

Notice that the set $\{u, v, w\} \cup N_1$ is a TRDS of $G$. Let $Y = V(G) - \{u\} - N_1$. Since all vertices in $N_{v,\overline{w}}$ dominate all vertices in $N_1 \cup \{u\}$ in $\overline{G}$, and since $N_1 \cup \{u\}$ is a clique in $\overline{G}$, we have that $Y$ is a RDS of $\overline{G}$. If $Y$ is total, we have that $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq 3 + |N_1| + n - 1 - |N_1| = n + 2$ and we are done.

Assume, therefore, that $Y$ is not total. As $w$ ($v$, respectively) is non-adjacent to every vertex of $N(v, \overline{w})$ ($N(w, \overline{v})$, respectively), the set $N_2 \neq \emptyset$, since otherwise $Y$ is a TRDS of $\overline{G}$. Moreover, $Y$ will also be a TRDS of $\overline{G}$ if every vertex of $N_2$ is non-adjacent to some vertex of $Y$. Hence, there exists a vertex $y \in N_2$ which is adjacent to every vertex of $Y - \{y\}$.
Notice that the set \( \{v, y\} \) is a TDS of \( G \). If \( \{v, y\} \) is also a RDS, we have that \( \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 2 \). Moreover, the set \( \{w, y\} \) is also a TDS of \( G \), and if it is a RDS, we are done. Thus, there exist vertices \( v' \in N_{v,\overline{G}} \) and \( w' \in N_{w,\overline{G}} \) such that \( N(v') = \{v, y\} \) and \( N(w') = \{w, y\} \).

We now show that \( Z = \{u, v', w'\} \) is a TRDS of \( G \). Notice that \( Z \) is a TDS of \( G \). Indeed, the vertex \( v' \) dominates \( w \) in \( \overline{G} \), the vertex \( w' \) dominates \( v \) in \( \overline{G} \), while the vertex \( u \) dominates \( V(G) - \{u, v, w, v', w'\} \) in \( \overline{G} \). Moreover, the vertex \( u \) dominates \( \{v', w'\} \) in \( \overline{G} \). Now, suppose to the contrary that \( Z \) is not a RDS of \( G \). Hence, there exists a vertex \( z \notin Z \) such that \( z \) is adjacent to every vertex of \( V(G) - Z - \{z\} \) in \( G \). As \( \deg(G) \geq 2 \), the vertex \( z \) is adjacent in \( \overline{G} \) to at least two vertices of \( Z \). We consider the following cases:

Case 4.1.1 The vertex \( z \) is adjacent in \( \overline{G} \) to \( u \) and at least one of the vertices \( v' \) and \( w' \).

Without loss of generality assume that \( z \) is adjacent in \( \overline{G} \) to the vertex \( v' \). As \( z \) is non-adjacent to \( u \) in \( G \), it follows that \( z \notin \{v, w\} \). As \( z \) is adjacent to both of the vertices \( v \) and \( w \) in \( G \), we have \( z \in N_1 \cup N_2 \). If \( z \in N_1 \), then it is not adjacent to \( y \) in \( G \), which contradicts the fact that \( z \) is adjacent to every vertex of \( V(G) - Z - \{z\} \) in \( G \). If \( z \in N_2 \), then since \( N_1 \neq \emptyset \), there exists a vertex \( z' \in N_1 \) such that \( z \) is not adjacent to \( z' \) in \( G \), which is again a contradiction.

Case 4.1.2 The vertex \( z \) is adjacent in \( \overline{G} \) to \( v' \) and \( w' \), but not to \( u \).

In this case, \( z \in \{v, w\} \). Without loss of generality, assume \( z = v \). Then \( v \) is adjacent in \( \overline{G} \) to both \( v' \) and \( w' \), which is a contradiction. Therefore, the set \( Z = \{u, v', w'\} \) is a TRDS of \( \overline{G} \) and so \( \gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 3 \).

Case 4.2 \( \delta(G) \geq 3 \) and \( \delta(\overline{G}) \geq 3 \).

Let \( u \) be a vertex of minimum degree in \( G \). Suppose \( N(u) = \{u_1, \ldots, u_\delta\} \) where \( \delta = \delta(G) \). Suppose the sets \( N[u] \) and \( N[u] - \{u_i\} \) for \( i \in \{1, \ldots, \delta\} \) are not total
restrained dominating sets of $G$. Let $N_1 = \{ x \in V(G) - N[u] \mid N(x) = N(u) \}$ and let $N_2 = V(G) - N[u] - N_1$. As $N[u]$ is a TDS of $G$, but not a RDS of $G$, the set $N_1 \neq \emptyset$. If $N_2 = \emptyset$, then $\{ u, u_1 \}$ is a TRDS of $G$, whence $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq 2 + n$. Thus, $N_2 \neq \emptyset$.

Suppose $N[u] - \{ u_i \}$ is a DS for some $i \in \{ 1, \ldots, \delta \}$. If a vertex $x \in N_2$ is adjacent to vertices in $N(u) - \{ u_i \}$ only, then $\text{deg}(x) \leq \delta - 1$, which is impossible. Thus, $N[x] - \{ u_i \}$ is a TRDS of $G$, which is contrary to our assumption. Hence, for each $i \in \{ 1, \ldots, \delta \}$, there exists $u'_i \in N_2$ such that $N(u'_i) \cap N(u) = \{ u_i \}$.

We claim that $X = \{ u, u'_1, u'_2 \}$ is a TRDS of $\overline{G}$. The vertex $u'_1$ dominates all vertices in $N(u) - \{ u_1 \}$ in $\overline{G}$. Similarly, $u'_2$ dominates all vertices in $N(u) - \{ u_2 \}$ in $\overline{G}$. The vertex $u$ dominates all vertices in $V(G) - N[u]$ in $\overline{G}$, and so $X$ is a TDS. Suppose $X$ is not a RDS of $\overline{G}$. Thus, there exists a vertex $x \notin X$ such that $x$ is adjacent in $G$ to each of the vertices in $V(G) - X - \{ x \}$. As $\delta(\overline{G}) \geq 3$, the vertex $x$ is not adjacent to each of the vertices in $X$. Hence, $x \in N_1 \cup N_2$. If $x \in N_1$, then since $|N_2| \geq \delta \geq 3$, there exists a vertex $x' \in N_2 - \{ u'_1, u'_2 \} \subset V(G) - X - \{ x \}$ such that $x$ is not adjacent to $x'$ in $G$, which is a contradiction. Similarly, if $x \in N_2 - \{ u'_1, u'_2 \}$, then, since $N_1 \neq \emptyset$, there exists a vertex $x' \in N_1 \subset V(G) - X - \{ x \}$ such that $x$ is not adjacent to $x'$ in $G$, which is a contradiction. Hence $X$ is a TRDS of $\overline{G}$ and so $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq n + 3$.

We may therefore assume that $N_G[u]$ or $N_G[u] - \{ u_i \}$ is a TRDS of $G$ for some $i \in \{ 1, \ldots, \delta \}$. Similarly, if $v$ is a minimum degree vertex in $\overline{G}$ and $N_{\overline{G}}(v) = \{ v_1, \ldots, v_{\delta(\overline{G})} \}$, we assume that $N_{\overline{G}}[v]$ or $N_{\overline{G}}[v] - \{ v_j \}$ is a TRDS of $\overline{G}$ for some $j \in \{ 1, \ldots, \delta(\overline{G}) \}$. Hence $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) \leq \delta(G) + 1 + \delta(\overline{G}) + 1 = \delta(G) + 1 + n - \Delta(G) - 1 + 1 = n + \delta(G) - \Delta(G) + 1 \leq n + 1$.

Clearly, if $G \in U$ or $\overline{G} \in U$ or $G \cong P_4$, then $\gamma_{tr}(G) + \gamma_{tr}(\overline{G}) = n + 4$. □
4.3 Restrained Domination

In this section, we provide bounds on the sum of the restrained domination numbers of a graph and its complement, and provide characterizations of the extremal graphs achieving these bounds. Let $\mathcal{H}$ be the family of graphs $G$ of order $n$ where $G$ or $\overline{G}$ is one of the following four types:

**Type 1.** $V(G) = \{x, y, z\} \cup X$. Moreover:

P1.1: $x$ is adjacent to each vertex of $\{y, z\} \cup X$,

P1.2: each vertex of $\{y, z\} \cup X$ is adjacent to some vertex of $\{y, z\} \cup X$,

P1.3: each vertex of $X$ is non-adjacent to some vertex of $\{y, z\}$ and non-adjacent to some vertex in $X$.

**Type 2.** $V(G) = \{x, y\} \cup X$. Moreover:

P2.1: each vertex of $X$ is adjacent to exactly one vertex of $\{x, y\}$ and also non-adjacent to exactly one vertex of $\{x, y\}$,

P2.2: each vertex of $X$ is non-adjacent to some vertex of $X$,

P2.3: each vertex of $X$ is adjacent to some vertex of $X$.

**Type 3.** $V(G) = \{u, v, y\} \cup X$. Moreover:

P3.1: each vertex of $X \cup \{y\}$ is adjacent to some vertex of $\{u, v\}$,

P3.2: each vertex of $X \cup \{u\}$ is non-adjacent to some vertex of $\{v, y\}$,

P3.3: each vertex of $X \cup \{y\}$ is adjacent to some vertex of $X \cup \{y\}$,

P3.4: each vertex of $X \cup \{u\}$ is non-adjacent to some vertex of $X \cup \{u\}$.
Type 4. \(V(G) = \{x, y, u, v\} \cup X\). Moreover:

P4.1: each vertex in \(\{x, y\} \cup X\) is adjacent to some vertex of \(\{u, v\}\),

P4.2: each vertex in \(\{u, v\} \cup X\) is non-adjacent to some vertex of \(\{x, y\}\),

P4.3: each vertex in \(\{x, y\} \cup X\) is adjacent to some vertex of \(\{x, y\} \cup X\),

P4.4: each vertex in \(\{u, v\} \cup X\) is non-adjacent to some vertex of \(\{u, v\} \cup X\).

Theorem 4.5 If \(G\) be a graph of order \(n \geq 2\), then \(\gamma_r(G) + \gamma_r(\overline{G}) = 4\) if and only if \(G\) or \(\overline{G} \in H\).

Proof. Suppose \(G\) is a graph such that \(\gamma_r(G) + \gamma_r(\overline{G}) = 4\). Then \(\gamma_r(G) = 1\) and \(\gamma_r(\overline{G}) = 3\) or \(\gamma_r(G) = 1\) and \(\gamma_r(\overline{G}) = 1\) or \(\gamma_r(G) = \gamma_r(\overline{G}) = 2\).

Case 1. \(\gamma_r(G) = 1\) and \(\gamma_r(\overline{G}) = 3\) or \(\gamma_r(G) = 1\) and \(\gamma_r(\overline{G}) = 1\).

Suppose \(\gamma_r(G) = 1\) and \(\gamma_r(\overline{G}) = 3\). Let \(\{x\}\) be a RDS of \(G\). Then \(x\) is adjacent to every other vertex of \(G\), and so \(x\) is isolated in \(\overline{G}\) and is therefore in every RDS of \(\overline{G}\) - let \(\{x, y, z\}\) be a RDS of \(\overline{G}\). Let \(X = V(G) - \{x, y, z\}\). It now follows that Properties P1.1 - P1.3 hold for \(G\). Thus, \(G\) is a graph of Type 1. If \(\gamma_r(\overline{G}) = 1\) and \(\gamma_r(G) = 3\), then \(\overline{G}\) is also of Type 1.

Case 2. \(\gamma_r(G) = 2\) and \(\gamma_r(\overline{G}) = 2\).

Let \(\{u, v\}\) (\(\{x, y\}\), respectively) be a RDS of \(G\) (\(\overline{G}\), respectively). Let \(X = V(G) - \{u, v, x, y\}\).

Case 2.1 Suppose \(u = x\) and \(v = y\).

If some vertex \(w \in X\) is adjacent to both \(u\) and \(v\), then \(w\) is not dominated by \(\{u, v\}\) in \(\overline{G}\), which is a contradiction. As \(\{u, v\}\) is a DS of \(G\), each vertex \(w \in X\) is adjacent to at least one vertex in \(\{u, v\}\). Thus, \(G\) satisfies Property P2.1. Moreover, Properties P2.2 and P2.3 hold for \(G\). Thus, \(G\) is a graph of Type 2.
Case 2.2 Suppose \( u \neq y \) and \( x = v \).

Clearly, in this case \( G \) is a graph of Type 3.

Case 2.3 \( \{u, v\} \cap \{x, y\} = \emptyset \).

It is easy to see, that P4.1 - P4.4 hold, so \( G \) is a graph of Type 4.

For the converse, suppose \( G \in \mathcal{H} \). For a graph of Type 1 we have \( \gamma_r(G) = 1 \) and \( \gamma_r(G) \leq 3 \). For Types 2, 3 or 4 we obtain \( \gamma_r(G) \leq 2 \) and \( \gamma_r(G) \leq 2 \). Hence, in all cases \( \gamma_r(G) + \gamma_r(G) \leq 4 \). It is known (see [3]) that \( \gamma_r(G) + \gamma_r(G) \geq 4 \). Therefore, \( \gamma_r(G) + \gamma_r(G) = 4 \). \( \Box \)

We will now characterize graphs \( G \) of order \( n \) for which \( \gamma_r(G) + \gamma_r(G) = n + 2 \).

Let \( B = \{P_3, \overline{P}_3\} \), and let \( \mathcal{G} = \{G \mid G \text{ or } \overline{G} \text{ is a galaxy of non-trivial stars}\} \).

Let \( S = \{G \mid G \text{ or } \overline{G} \cong K_1 \cup S \text{ where } S \text{ is a star and } |S| \geq 3\} \).

Lastly, let \( E = G \cup S \).

Lemma 4.6 If \( G \in E - B \), then \( \gamma_r(G) + \gamma_r(G) = n + 2 \).

Proof. Suppose \( G \in \mathcal{G} \) has order \( n \) and, without loss of generality, suppose \( G \) is a galaxy of non-trivial stars \( S_1, S_2, \ldots, S_k \), for \( k \geq 2 \). Then \( \gamma_r(G) = n \). Let \( s \in V(S_1) \) and \( t \in V(S_2) \). Since \( S_i \) is non-trivial for \( i \in \{1, \ldots, k\} \), it follows that \( R = \{s, t\} \) is a RDS of \( \overline{G} \). Suppose \( \{v\} \) is a RDS of \( \overline{G} \). Then \( \deg_G(v) = 0 \), which is a contradiction. Hence \( \gamma_r(G) + \gamma_r(G) = n + 2 \). Now, suppose \( k = 1 \). That is, \( G \) is a non-trivial star \( S \) such that \( S \neq P_3 \). The result follows immediately if \( |S| = 2 \). Thus we may assume \( |S| \geq 4 \). Then \( \gamma_r(G) = n \). Let \( s \) be the center of \( S \) and let \( t \in N_G(s) \). Notice that \( \langle V(G) - \{s\} \rangle \cong K_{n-1} \) in \( \overline{G} \). Thus \( R = \{s, t\} \) is a RDS of \( \overline{G} \). Suppose \( \{v\} \) is a RDS of \( \overline{G} \). Then \( \deg_G(v) = 0 \), which is a contradiction.

Suppose \( G \in S \) and, without loss of generality, let \( G = K_1 \cup S \) where \( S \) is a star and \( |S| \geq 3 \). Then \( \gamma_r(G) = n \). Let \( s \) be the center of \( S \) and let \( \langle u \rangle \) be the second
component of $G$. Then $R = \{s, u\}$ is a RDS of $\overline{G}$. Suppose $\{v\}$ is a RDS of $\overline{G}$. Then $\deg_G(v) = 0$, and $v = u$, which is a contradiction as $\{u\}$ is not a RDS of $\overline{G}$. Hence $\gamma_r(G) + \gamma_r(\overline{G}) = n + 2$. □

**Theorem 4.7** Let $G = (V, E)$ be a graph of order $n \geq 2$ such that $G \notin B$. Then $\gamma_r(G) + \gamma_r(\overline{G}) \leq n + 2$. Moreover, $\gamma_r(G) + \gamma_r(\overline{G}) = n + 2$ if and only if $G \in E$.

**Proof.** Let $G = (V, E)$ be a graph of order $n$ such that $G \notin B$. Notice that either $G$ or $\overline{G}$ must be connected. Without loss of generality, suppose $\overline{G}$ is connected. Note that $G$ may also be connected. Let $G$ be composed of the components $G_1, G_2, \ldots, G_\ell$ with $\ell$ possibly equal to one. Without loss of generality, let $G_1$ be a component of $G$ with longest diameter.

**Claim 4.8** If $G_1$ contains a path $uv_1v_2v$ and $\ell \geq 3$, then $\gamma_r(G) + \gamma_r(\overline{G}) \leq n$.

**Proof.** Let $uv_1v_2v$ be a path in $G_1$. Notice that $V(G) - \{v_1, v_2\}$ is a RDS of $G$. Hence $\gamma_r(G) \leq n - 2$. Let $x \in V(G_1)$ and $w \in V(G_2)$. Since $\ell \geq 3$ it follows that $\{x, w\}$ is a RDS of $\overline{G}$ and $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$. ◇

**Claim 4.9** If $\ell \geq 3$ and there exists $i \in \{1, \ldots, \ell\}$ such that $G_i \cong K_1$, then $\gamma_r(G) + \gamma_r(\overline{G}) \leq n + 1$.

**Proof.** Trivial. ◇

By Claim 4.8, for cases in which $\text{diam}(G_1) \geq 3$, we may immediately assume that $\ell \leq 2$. Note that for the following two cases $V(G_2)$ may or may not be empty. Let $u$ and $v$ be vertices such that $d(u, v) = \text{diam}(G)$. As before, the sets $V_0, \ldots, V_{\text{diam}(G)}$ will denote the level decomposition of $G$ with respect to $u$.

Suppose $\text{diam}(G_1) \geq 5$. Let $uv_1v_2 \ldots v_{\text{diam}(G_1)}$ be a diametrical path in $G_1$. Notice that $V(G) - \{v_1, v_2\}$ is a RDS of $G$. Hence $\gamma_r(G) \leq n - 2$. Moreover, notice that
$R' = \{u, v_5\}$ is a RDS of $\overline{G}$, as $R'$ is clearly a dominating set of $\overline{G}$, $v_1 \in V(\overline{G}) - R'$ is adjacent to $V_3 \cup V_4 \cup \ldots \cup V_{\text{diam}(G)}$, and $v_4 \in V(\overline{G}) - R'$ is adjacent to $V_1 \cup V_2 \cup V(G_2)$. Hence $\gamma_r(\overline{G}) \leq 2$ and we have that $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$.

Now, suppose diam$(G_1) = 4$. Let $uv_1v_2v_3v_4$ be a diametrical path in $G_1$. Notice that $V(G) - \{v_1, v_2\}$ is a RDS of $G$. Hence $\gamma_r(G) \leq n - 2$. Suppose $|V_4| \geq 2$. Then there exists a vertex $t \in V_4 - \{v_4\}$. Notice that $R' = \{u, v_4\}$ is a RDS of $\overline{G}$, as $R'$ is clearly a dominating set of $\overline{G}$, $v_1 \in V(\overline{G}) - R'$ is adjacent to $V_3 \cup V_4$, and $t \in V(\overline{G}) - R'$ is adjacent to $V_1 \cup V_2 \cup V(G_2)$. Hence $\gamma_r(\overline{G}) \leq 2$ and we have that $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$.

Thus we may assume that $|V_4| = 1$. Let $V_{21} = \{x \in V_2 \mid$ there exists $y \in V_1 \cup V_2 \cup V_3$ such that $\{x, y\} \notin E(G_1)\}$ and let $V_{22} = V_2 - V_{21}$. Consider $R' = \{u, v_4\} \cup V_{22}$. Notice that $R'$ is a dominating set of $\overline{G}$, $v_1 \in V(\overline{G}) - R'$ is adjacent to $V_3$, and $v_3 \in V(\overline{G}) - R'$ is adjacent to $V_1 \cup V(G_2)$. If $V_{21} = \emptyset$, then $V_2 = V_{22} \subseteq R'$ and $R'$ is a RDS of $\overline{G}$. If $V_{21} \neq \emptyset$, then by definition, for each $x \in V_{21}$ there exists a $y \in V_1 \cup V_{21} \cup V_3$ such that $xy \notin E(G_1)$). Hence $R'$ is a RDS of $\overline{G}$. In either case we have that $\gamma_r(\overline{G}) \leq 2 + |V_{22}|$.

If $|V_{22}| \leq 1$, then $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 + |V_{22}| \leq n + 1$. Thus we may assume that $|V_{22}| \geq 2$. Hence there exists a vertex $t \in V_{22} - \{v_2\}$. Then $R = \{u, v_4, t\} \cup V(G_2)$ is a RDS of $G$, as $R$ clearly dominates $G$, and a vertex $w \in V_{22} - \{t\}$ is adjacent to every vertex of $V(G) - R$. Thus, $\gamma_r(G) \leq 3 + |V(G_2)|$ and so $\gamma_r(G) + \gamma_r(\overline{G}) \leq 3 + |V(G_2)| + 2 + |V_{22}| = 1 + (4 + |V_{22}| + |V(G_2)|) = 1 + (|\{u, v_1, v_3, v_4\}| + |V_{22}| + |V(G_2)|) = 1 + |\{u, v_1, v_3, v_4\} \cup V_{22} \cup V(G_2)| \leq 1 + |V(G)| = 1 + n$.

Now, suppose diam$(G_1) = 3$. Let $uv_1v_2v_3$ be a diametrical path in $G_1$. Notice that $V(G) - \{v_1, v_2\}$ is a RDS of $G$. Suppose that $V(G_2) \neq \emptyset$. If $V(G_2) = \{v\}$, then $\{v\}$ is a RDS of $\overline{G}$, whence $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 1 = n - 1$. Thus we may assume that $|V(G_2)| \geq 2$. Let $v \in V(G_2)$. Then $\{u, v\}$ is a RDS of $\overline{G}$ and so $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$. 

Thus \( V(G_2) = \emptyset \) and both \( G_1 = G \) and \( \overline{G} \) are connected. Suppose \(|V_3| \geq 2\) and let \( t \in V_3 - \{v_3\} \). Let \( V_{21} = \{x \in V_2 \mid \text{there exists } y \in (V_1 \cup V_2 \cup V_3) - \{t\} \text{ such that } xy \notin E(G)\} \) and let \( V_{22} = V_2 - V_{21} \). Consider \( R' = \{u, t\} \cup V_{22} \). By reasoning similar to that in the case for \( \text{diam}(G_1) = 4 \), \( R' \) is a RDS of \( \overline{G} \) and \( \gamma_r(\overline{G}) \leq 2 + |V_{22}| \). If \(|V_{22}| \leq 1\), then \( \gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 + |V_{22}| \leq n + 1 \).

Thus we may assume that \(|V_{22}| \geq 2\). Hence there exists a vertex \( z \in V_{22} - \{v_2\} \). Consider \( R = \{u, t, z\} \). By reasoning similar to that in the case for \( \text{diam}(G_1) = 4 \), \( R \) is a RDS of \( G \) and so \( \gamma_r(G) + \gamma_r(\overline{G}) \leq 3 + 2 + |V_{22}| = 1 + (4 + |V_{22}|) = 1 + (|\{u, v_1, v_3, t\}| + |V_{22}|) = 1 + |\{u, v_1, v_3, t\} \cup V_{22}| \leq 1 + |V(G)| = 1 + n \).

So we may assume that \(|V_3| = 1\). Let \( V_{11} = \{x \in V_1 \mid \text{there exists } y \in V_1 \cup V_2 \text{ such that } xy \notin E(G)\} \) and let \( V_{12} = V_1 - V_{11} \). Also, let \( V_{21} = \{x \in V_2 \mid \text{there exists } y \in V_1 \cup V_2 \text{ such that } xy \notin E(G)\} \) and let \( V_{22} = V_2 - V_{21} \). Then \( \{u, v_3\} \cup V_{12} \cup V_{22} \) is a RDS of \( \overline{G} \) and \( \gamma_r(\overline{G}) \leq 2 + |V_{12}| + |V_{22}| \).

If \(|V_{12}| + |V_{22}| \leq 1\), then \( \gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 + |V_{12}| + |V_{22}| \leq n + 1 \). So we may assume that \(|V_{12}| + |V_{22}| \geq 2\). Since \( v_1v_3uv_2 \) is a path in \( \overline{G} \), it follows that \( V(\overline{G}) - \{v_3, u\} \) is a RDS of \( \overline{G} \), whence \( \gamma_r(\overline{G}) \leq n - 2 \).

Now, suppose \(|V_{12}| \geq 2\) and let \( z \in V_{12} - \{v_1\} \). Then \( \{z, v_3\} \) is a RDS of \( G \), and so \( \gamma_r(G) + \gamma_r(\overline{G}) \leq 2 + n - 2 = n \). Thus \(|V_{12}| \leq 1\). Suppose \( V_{12} = \{z\} \). Then \( \{u, v_3, z\} \) is a RDS of \( G \) except when \( G = P_4 \), in which case \( \{u, v_3\} \) is a RDS of \( G \). In both cases \( \gamma_r(G) \leq 3 \). Hence, \( \gamma_r(G) + \gamma_r(\overline{G}) \leq 3 + n - 2 = n + 1 \). Thus \( V_{12} = \emptyset \) and so \(|V_{22}| \geq 2\). Let \( z \in V_{22} - \{v_2\} \). Then \( \{u, v_3, z\} \) is a RDS of \( G \). Therefore, \( \gamma_r(G) \leq 3 \). Hence, \( \gamma_r(G) + \gamma_r(\overline{G}) \leq 3 + n - 2 = n + 1 \).

Thus we may assume \( \text{diam}(G_1) \leq 2 \), and by a similar argument, \( \text{diam}(\overline{G}) \leq 2 \). As \( n \geq 2 \), \( \text{diam}(\overline{G}) \geq 1 \). Suppose \( \text{diam}(\overline{G}) = 1 \). Then \( \overline{G} \cong K_i \) for some \( i \geq 2 \). If \( i \geq 3 \), then \( \gamma_r(G) + \gamma_r(\overline{G}) \leq n + 1 \). Thus, \( \overline{G} \cong K_2 \), and so \( G \in \mathcal{G} \) and \( \gamma_r(G) + \gamma_r(\overline{G}) = n + 2 \).
Thus, $\text{diam}(\overline{G}) = 2$. Suppose $\text{diam}(G_1) = 0$. Then $G \cong nK_1$ and $\overline{G} \cong K_n$, which is a contradiction as $\text{diam}(\overline{G}) = 2$.

Suppose $\text{diam}(G_1) = 1$. Then $G_1 \cong K_i$ where $2 \leq i \leq n$. Since we assumed that $\overline{G}$ is connected, $\ell \neq 1$. Suppose $\ell = 2$. If $G_2 \cong K_1$, then $i \neq 2$, as $G \notin \mathcal{B}$. Thus $i \geq 3$, so $G \in \mathcal{G}$ and $\gamma_r(G) + \gamma_r(\overline{G}) = n + 2$. Thus $G_2 \cong K_j$ where $2 \leq j \leq n - i$. If $i = j = 2$, then $G \in \mathcal{G}$ and we are done. Without loss of generality, suppose $i \geq 3$. Let $V(G_1) = \{v_1, v_2, \ldots, v_i\}$ and let $z \in V(G_2)$. Since $i \geq 3$, $V(G) - \{v_2, v_3\}$ is a RDS of $G$ and $\{v_1, z\}$ is a RDS of $\overline{G}$. Hence $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$. Thus $\ell \geq 3$. By Claim 4.9, $G_k \neq K_1$ for all $k \in \{1, \ldots, \ell\}$. Suppose $G_k \cong K_2$ for all $k$. Then $G \in \mathcal{G}$ and we are done. Thus, by relabeling if necessary, we may assume that $G_1 \cong K_i$ for $i \geq 3$. Let $V(G_1) = \{v_1, v_2, \ldots, v_i\}$ and let $z \in V(G_2)$. Since $i \geq 3$, $V(G) - \{v_2, v_3\}$ is a RDS of $G$ and $\{v_1, z\}$ is a RDS of $\overline{G}$. Hence $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$.

Thus we may assume $\text{diam}(G_1) = 2$. Suppose $\ell \geq 3$. By Claim 4.9, $G_k \neq K_1$ for all $k \in \{1, \ldots, \ell\}$. If $G$ is a galaxy of non-trivial stars, then $G \in \mathcal{G}$, and we are done. Thus at least one component, say $G_1$, contains a cycle containing an edge $v_1v_2$, say. Let $z \in V(G_2)$. Then $V(G) - \{v_1, v_2\}$ is a RDS of $G$, while $\{v_1, z\}$ is a RDS of $\overline{G}$, whence $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$.

Suppose $\ell = 2$ and first suppose $G_2 \neq K_1$. If $G_1$ and $G_2$ are stars, then $G \in \mathcal{G}$ and we are done. Thus at least one component contains a cycle containing the edge $v_1v_2$. Let $z$ be an arbitrary vertex in the other component of $G$. Then $V(G) - \{v_1, v_2\}$ is a RDS of $G$, while $\{v_1, z\}$ is a RDS of $\overline{G}$, whence $\gamma_r(G) + \gamma_r(\overline{G}) \leq n - 2 + 2 = n$.

So we may assume that $G_2 \cong K_1$. Let $V(G_2) = \{z\}$. If $\Delta(G_1) \leq n - 3$, then $\{z\}$ is a RDS of $\overline{G}$ and so $\gamma_r(G) + \gamma_r(\overline{G}) \leq n + 1$. Thus $\Delta(G_1) = n - 2$, and there exists a vertex $u \in V(G_1)$ such that $\deg(u) = n - 2$. Let $L$ be the set of leaves in $G_1$ and let $X = N(u) - L$. If $L = \emptyset$, then $\{u, z\}$ is a RDS of $G$. Since $\text{diam}(G_1) = 2$, there exist nonadjacent vertices $x, y \in V(G_1)$. Then $V(\overline{G}) - \{x, y\}$ is a RDS of $\overline{G}$ and
\( \gamma_r(G) + \gamma_r(G) \leq n - 2 + 2 = n. \) Thus \( L \neq \emptyset. \) Let \( v \in L \) and consider \( \{u, v\}. \) Since \( \text{diam}(G_1) = 2, \) it follows that \( \deg(u) \geq 2. \) Thus \( \{u, v\} \) is a \textbf{RDS} of \( G. \) Suppose \( X \neq \emptyset \) and let \( s \in X. \) Since \( s \notin L, \) \( s \) is adjacent to a vertex \( t \in N(v). \) Hence \( t \notin L, \) so \( t \in X \) and thus \( |X| \geq 2. \) Moreover, \( V(G) - X \) is a \textbf{RDS} of \( G, \) and so \( \gamma_r(G) + \gamma_r(G) \leq n - 2 + 2 = n. \) Thus \( X = \emptyset \) and so \( G_1 \) is a non-trivial star of order \( n - 1 \geq 3. \) Therefore \( G \in S \) and we are done.

Thus \( G \cong G_1, \) and \( \text{diam}(G) = \text{diam}(G) = 2. \) Let \( uv_1v_2 \) be a diametrical path in \( G. \) If \( v_2 \) is a leaf of \( G, \) then every vertex \( v \in V_1 - \{v_1\} \) is adjacent to \( v_1, \) whence \( \deg(v_1) = n - 1, \) which is a contradiction as \( G \) is connected. Moreover, if some vertex \( v \in V_1 \) is a leaf, then \( \text{diam}(G) \geq d(v, v_2) = 3, \) which is a contradiction. Lastly, if \( u \) is a leaf, then \( v_1 \) is adjacent to every vertex of \( V_2, \) whence \( \deg(v_1) = n - 1, \) which is a contradiction. Thus we may assume that \( \delta(G) \geq 2. \) A similar argument shows that \( \delta(G) \geq 2. \) Let \( \mathcal{F} \) be the collection of graphs described in [7]. It is known (see [7]) that if \( G \notin \mathcal{F} \) is a connected graph with order \( n \geq 3 \) and \( \delta(G) \geq 2, \) then \( \gamma_r(G) \leq \frac{n - 1}{2}. \) It follows immediately that \( \gamma_r(G) + \gamma_r(G) \leq n - 1, \) provided that \( G, G \notin \mathcal{F}. \) Without loss of generality, suppose \( G \in \mathcal{F}. \) It is easily verified that \( \gamma_r(G) + \gamma_r(G) \leq n + 1 \) and we are done.

Finally, recounting the argument, we have that \( \gamma_r(G) + \gamma_r(G) \leq n + 1 \) in all cases, save when \( G \in \mathcal{E}. \) Hence, if \( \gamma_r(G) + \gamma_r(G) = n + 2 \) it follows that \( G \in \mathcal{E}. \) This observation together with Lemma 4.6 implies that \( \gamma_r(G) + \gamma_r(G) = n + 2 \) if and only if \( G \in \mathcal{E}. \) \( \square \)
Bibliography


