Homeomorphically Irreducible Spanning Trees, Halin Graphs, and Long Cycles in 3-connected Graphs with Bounded Maximum Degrees

Songling Shan

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A tree $T$ with no vertex of degree 2 is called a \textit{homeomorphically irreducible tree} (HIT) and if $T$ is spanning in a graph, then $T$ is called a \textit{homeomorphically irreducible spanning tree} (HIST). Albertson, Berman, Hutchinson and Thomassen asked \textit{if every triangulation of at least 4 vertices has a HIST} and \textit{if every connected graph with each edge in at least two triangles contains a HIST}. These two questions were restated as two conjectures by Archdeacon in 2009. The first part of this dissertation gives a proof for each of the two
conjectures. The second part focuses on some problems about *Halin graphs*, which is a class of graphs closely related to HITs and HISTs. A *Halin graph* is obtained from a plane embedding of a HIT of at least 4 vertices by connecting its leaves into a cycle following the cyclic order determined by the embedding. And a *generalized Halin graph* is obtained from a HIT of at least 4 vertices by connecting the leaves into a cycle. Let $G$ be a sufficiently large $n$-vertex graph. Applying the Regularity Lemma and the Blow-up Lemma, it is shown that $G$ contains a spanning Halin subgraph if it has minimum degree at least $(n + 1)/2$ and $G$ contains a spanning generalized Halin subgraph if it is 3-connected and has minimum degree at least $(2n + 3)/5$. The minimum degree conditions are best possible. The last part estimates the length of longest cycles in 3-connected graphs with bounded maximum degrees. In 1993 Jackson and Wormald conjectured that for any positive integer $d \geq 4$, there exists a positive real number $\alpha$ depending only on $d$ such that if $G$ is a 3-connected $n$-vertex graph with maximum degree $d$, then $G$ has a cycle of length at least $\alpha n \log_{d-1}^2$. They showed that the exponent in the bound is best possible if the conjecture is true. The conjecture is confirmed for $d \geq 425$.

INDEX WORDS: Homeomorphically irreducible spanning tree, Halin graph, Generalized Halin graph, 3-connected graphs, Tutte decomposition.
Homeomorphically Irreducible Spanning Trees, Halin Graphs, and Long Cycles in 3-connected Graphs with Bounded Maximum Degrees

by

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Homeomorphically Irreducible Spanning Trees, Halin Graphs, and Long Cycles in 3-connected Graphs with Bounded Maximum Degrees

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DEDICATION

This dissertation is dedicated to my family and my advisor Professor Guantao Chen.
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LIST OF ABBREVIATIONS

• GSU - Georgia State University

• HIT - Homeomorphically irreducible tree

• HIST - Homeomorphically irreducible spanning tree

• SGHG - Spanning generalized Halin subgraph
PART 1

INTRODUCTION

Finding longest cycles, in particular a hamiltonian cycle in a graph, is one of a few fundamental yet very difficult problems in graph theory. In fact, to determine whether a graph is hamiltonian is a classic NP-complete problem. Moreover, Karger, Motwani, and Ramkumar [35] showed that, unless $P = NP$, it is impossible to find, in polynomial time, a path of length $n - n^\epsilon$ in an $n$-vertex hamiltonian graph for any $\epsilon < 1$. On the other hand, inspired by classic results obtained by Dirac [19] in 1954 and Tutte [51] in 1956, respectively, many sufficient conditions for hamiltonian graphs have been obtained. For examples, see [25]. Accompanying with each of these sufficient conditions, various stronger results such as being hamiltonian connected and pancyclic have also been established.

As an antithetical class to hamiltonian paths/cycles, homeomorphically irreducible graphs, graphs which have no vertex of degree 2, were introduced by graph theorists in 1970s. A homeomorphically irreducible tree is called a HIT, and a homeomorphically irreducible spanning tree of a graph is called a HIST of the graph. As graphs of at most three vertices contain no HIST, we assume the graphs in consideration are of at least four vertices when we considering HISTs. In the first part of this dissertation, we show the existence of a HIST in surface triangulations and connected graphs with each edge contained in at least two triangles. This confirms the two conjectures raised by Albertson, Berman, Hutchinson, and Thomassen [1].

Another class of graphs which is closely related to HITs and HISTs is the class of Halin graphs. Let $T$ be a HIT of at least 4 vertices. Then a Halin graph $H$ is obtained from a plane embedding of $T$ by connecting the leaves into a cycle $C$ following the cyclic order determined by the plane embedding. In this notation, we may write the Halin graph as $H = T \cup C$. A wheel is an example of a Halin graph. Since a HIT of at least 4 vertices
contains two leaves sharing the same parent, a Halin graph contains a triangle, and thus is not bipartite. Moreover, cubic Halin graphs are in one-to-one correspondence (via weak duality) with the plane triangulations of the disc. Halin constructed Halin graphs in \[27\] for the study of minimally 3-connected graphs. Lovász and Plummer named such graphs as Halin graphs in their study of planar bicritical graphs \[40\], which are planar graphs having a 1-factor after deleting any two vertices. It was conjectured by Lovász and Plummer \[40\] that \textit{every 4-connected plane triangulation contains a spanning Halin subgraph} (disproved in \[10\]). Although the conjecture is not true, it inspires new questions and problems. We may ask, can we find any other class of graphs which contain a spanning Halin subgraph or a spanning generalized Halin subgraph? The second part of this dissertation considers the existence of spanning Halin subgraphs and spanning generalized Halin subgraphs in graphs with large minimum degree. Halin graphs possess very nice hamiltonicity properties. Hence finding the existence of a spanning Halin subgraph can be viewed as a generalization of finding hamiltonian paths/cycles in graphs.

Finally, in the last part, the problem of finding longest cycles in 3-connected graphs with bounded maximum degrees is investigated. In 1993 Jackson and Wormald conjectured that for any positive integer \( d \geq 4 \), there exists a positive real number \( \alpha \) depending only on \( d \) such that if \( G \) is a 3-connected \( n \)-vertex graph with maximum degree \( d \), then \( G \) has a cycle of length at least \( \alpha n \log_{d-1} n^{2} \). They showed that the exponent in the bound is best possible if the conjecture is true. The conjecture is confirmed for \( d \geq 425 \).

Throughout this report, we limit our attention to simple and connected graphs, and further assume graphs to be finite unless we specify otherwise; and refer to Bondy and Murty \[7\] for notations and terminologies used but not defined. The vertex set and edge set of a graph \( G \) are denoted by \( V(G) \) and \( E(G) \), respectively. For \( S \subseteq V(G) \), let \( G[S] \) denote the subgraph of \( G \) induced by \( S \). Similarly, \( G[F] \) is the subgraph induced on \( F \) if \( F \subseteq E(G) \). The minimum degree and maximum degree of \( G \) are denoted by \( \delta(G) \) and \( \Delta(G) \), respectively. Other specified notations are introduced in each chapters.
PART 2

THE EXISTENCE OF HISTS IN SURFACE TRIANGULATIONS AND CONNECTED GRAPHS WITH EACH EDGE IN AT LEAST TWO TRIANGLES

Recall that a tree is called homeomorphically irreducible if it does not contain vertices of degree 2 and a homeomorphically irreducible spanning tree of a graph $G$ is called a HIST of $G$. Albertson, Berman, Hutchinson, and Thomassen [1] obtained various sufficient conditions for a graph to contain a HIST. They also showed that it is NP-complete to decide whether a graph $G$ contains a HIST. Hill [28] conjectured that every triangulation of the plane contains a HIST. Malkevitch [41] conjectured that the same result hold for near-triangulations of the plane (2-connected plane graphs such that all, but at most one, faces are triangles). Albertson, Berman, Hutchinson, and Thomassen [1] confirmed the conjecture. Furthermore, they asked whether every graph that triangulates some surface has a HIST, and more generally if every connected graph with each edge contained in two triangles contains a HIST. To establish a strategy to tackle the problem, Ellingham [20] asked whether every triangulation of a given surface with sufficiently large representativity contains a HIST. Huneke observed that every triangulation of the projective plane contains a spanning plane subgraph such that every face is a triangle with one possible exception, so every triangulation of the projective plane contains a HIST. Davidow, Hutchinson, and Huneke [18] showed that every triangulation of the torus contains a HIST. In 2009, Achdeacon [4] (Chapter 15) restated the above two questions as two conjectures.

**Conjecture 2.1.** Every surface triangulation contains a HIST.

**Conjecture 2.2.** Every connected graph with each edge in at least two triangles contains a HIST.

We confirm the two conjectures in this Chapter. The proofs can also be found in [11].
and [13].

2.1 Proof of Conjecture [2.1]

A graph $G$ is locally connected if for every vertex $v \in V(G)$, the subgraph induced by the neighborhood $N(v)$ is connected. Ringel [46] showed that every triangulation (includes orientable and nonorientable) is a connected and locally connected graph. In this section, we prove the following much more general result, which confirms the conjecture by Archdeacon and answers the first question asked by Albertson, Berman, Hutchinson, and Thomassen positively.

**Theorem 2.1.1.** Every connected and locally connected graph with order at least four contains a HIST.

**Corollary 2.1.1.** Let $\Pi$ be a surface (orientable or nonorientable). Then every triangulation of $\Pi$ with at least four vertices contains a HIST.

Let $G$ be a graph. Write $v \in G$ if $v \in V(G)$ and similarly $e \in G$ if $e \in E(G)$.

2.1.1 Proof of Theorem [2.1.1]

Let $k$ be a positive integer. A graph $G$ is called a $k$-tree if there is an ordering $v_1 \prec v_2 \prec \cdots \prec v_n$ of $V(G)$ such that (i) $G[\{v_1, v_2, \ldots, v_k\}]$ is a complete graph and (ii), for each $i > k$, $N(v_i) \cap \{v_1, v_2, \ldots, v_{i-1}\}$ induces a clique of order $k$. Clearly, 1-trees are the same as trees. Hwang, Richards, and Winter [31] proved that 2-trees are maximal series-parallel graphs. As shown in Lemma 2.2 and 2.3, we observe that every 2-tree with more than three vertices contains a HIST. However, not every connected and locally connected graph contains a 2-tree as a spanning subgraph. Let $W_n := K_1 + C_n$ be a wheel of order $n + 1$ and let $G_n$ be obtained from $W_n$ by adding $n$ new vertices such that each is adjacent to a distinct pair of two consecutive vertices on the cycle $C_n$. It is not difficult to verify that $G_n$ does not contain a spanning 2-tree. $G_4$ is depicted below.
Let $G$ be a graph and $v$ be a vertex not in $V(G)$. We write $H = G \oplus v$ if there exist two distinct vertices $u_1, u_2 \in G$ such that $V(H) = V(G) \cup \{v\}$ and $E(H) = E(G) \cup \{u_1v, u_2v, u_1u_2\}$. Note that the edge $u_1u_2$ may already exist in $G$. We let $P(v) := \{u_1, u_2\}$ and call $u_1$ and $u_2$ the parents of $v$.

**Definition 2.1.1.** A graph $T$ of order $n \geq 3$ is called a weak 2-tree (W2-tree) if there is an ordering $\prec: v_1 \prec v_2 \prec \cdots \prec v_n$ of vertices of $T$ and a sequence of graphs $G_3 \subset G_4 \subset \cdots \subset G_n = T$ such that the following properties hold.

1. $G_3 = T[\{v_1, v_2, v_3\}] \cong K_3$, and
2. for each $i = 3, 4, \ldots, n - 1$, $G_{i+1} \cong G_i \oplus v_{i+1}$.

In addition, we call the ordering $\prec$ a W2-tree ordering of $T$.

Clearly, every 2-tree is a W2-tree. However, the converse is not true, for example, the above graphs $G_n$ are W2-trees but not 2-trees.

Given a W2-tree with a W2-tree ordering $\prec$, if we shift a degree 2 vertex to the end and keep the remaining ordering unchanged, we obtain another W2-tree ordering. So, the following result holds.

**Lemma 2.1.1.** Let $G$ be a W2-tree with $n \geq 4$ vertices. Let $w \in G$ be a degree 2 vertex and $N(w) = \{u,v\}$. Then either $G - w$ or $G - w - uv$ is a W2-tree.
Lemma 2.1.2. Let $T$ be a $W_2$-tree with $n \geq 4$ vertices. Then, there exist two vertices $u$ and $v$ such that $T = (T' \oplus u) \oplus v$ and $N[u] \cap N(v) \neq \emptyset$, where $T'$ is a $W_2$-tree, $K_3$, or $K_2$. In this case, $\{u, v\}$ is called a removable pair of $T$.

Proof. We prove Lemma 2.1.2 by applying induction on $n = |V(G)|$. Since $K_4^-$ ($K_4$ minus an edge) is the unique $W_2$-tree with 4 vertices, Lemma 2.1.2 holds for $n = 4$.

Suppose $n \geq 5$ and that Lemma 2.1.2 holds for all $W_2$-trees with less than $n$ vertices. Let $T$ be a $W_2$-tree with $n$ vertices and $w$ be the last vertex in a $W_2$-ordering of $T$. Moreover, we assume that $T = T' \oplus w$, where $T'$ is a $W_2$-tree with $n - 1$ vertices. Suppose that $\{u, v\}$ is a removable pair of $T'$ and $T' = (T^* \oplus u) \oplus v$, where $T^*$ is a $W_2$-tree, $K_3$, or $K_2$. We complete the proof by considering the following five cases regarding $N(w) \cap \{u, v\}$.

- if $N(w) \cap \{u, v\} = \emptyset$, then $T = [(T^* \oplus w) \oplus u] \oplus v$, so $\{u, v\}$ is a removable pair of $T$;
- if $N(w) \cap \{u, v\} = \{v\}$, then $T = [(T^* \oplus u) \oplus v] \oplus w$, so $\{v, w\}$ is a removable pair of $T!$;
- if $N(w) \cap \{u, v\} = \{u\}$ and $uv \notin E(T')$, then $T = [(T^* \oplus v) \oplus u] \oplus w$, so $\{u, w\}$ is a removable pair of $T$;
- if $N(w) \cap \{u, v\} = \{u\}$ and $uv \in E(T')$, then $T = [(T^* \oplus u) \oplus v] \oplus w$, so $\{v, w\}$ is a removable pair of $T$;
- if $N(w) = \{u, v\}$, then $T = [(T^* \oplus u) \oplus v] \oplus w$, so $\{v, w\}$ is a removable pair of $T$.

□

Lemma 2.1.3. A $W_2$-tree with at least 4 vertices contains a HIST.

Proof. Let $G$ be a $W_2$-tree with $n \geq 4$ vertices. We proceed by induction on $n$. If $n = 4$, then $G = K_4^-$, which contains a spanning star. If $n = 5$, by case analysis, we can show that $G$ contains a spanning star, so a HIST.
Assume $n \geq 6$ and let $G$ be a W2-tree with $n$ vertices. By Lemma 2.1.2, let $\{u, v\}$ be a removable pair of $G$ and assume $G = (G' \oplus u) \oplus v$, where $G'$ is a W2-tree with $n - 2$ vertices. By the induction hypothesis, $G'$ contains a HIST, say, $T'$. Since $\{u, v\}$ is a removable pair of $G$, $N[u] \cap N(v) \neq \emptyset$. If $uv \notin E(G)$, then $N(u) \cap N(v) \neq \emptyset$; if $uv \in E(G)$, by the definition of $\oplus$, the other neighbor of $v$ is adjacent to $u$. In either case, there exists a vertex $w \in N(u) \cap N(v)$. Then, $T := T' \cup \{wu, wv\}$ is a HIST of $G$. 

**Lemma 2.1.4.** Every connected and locally connected graph with at least three vertices contains a spanning W2-tree.

**Proof.** Let $G$ be a connected and locally connected graph of order $n \geq 3$. Since every triangle is a W2-tree, $G$ contains W2-trees as subgraphs. Let $T \subseteq G$ be a W2-tree such that $|V(T)|$ is maximum. We claim that $V(T) = V(G)$. Otherwise, $W := V(G) - V(T) \neq \emptyset$. Since $G$ is connected, there is a vertex $v \in V(T)$ such that $N_W(v) \neq \emptyset$, where $N_W(v)$ is the set of neighbors of $v$ in $W$. Since $T$ is a W2-tree, $N(v) \cap V(T) \supseteq N_T(v) \neq \emptyset$. Since $G[N(v)]$ is connected, there is an edge $uw \in E(G)$ with $u \in N_T(v)$ and $w \in N_W(v)$. Then, $T \oplus w$ is a W2-tree containing more vertices than $T$, where $P(w) = \{u, v\}$. Since $uw, wv, uv \in E(G)$ and $T \subseteq G$, we have $T \oplus w \subseteq G$, which contradicts the maximality of $|V(T)|$. 

So, the proof of Theorem 2.1.1 is completed.

### 2.2 Proof of Conjecture 2.2

We now answer the second question raised by Albertson et al. positively as follows, whose proof will be given in the next section. We would like to mention that the main proof technique used in the proof is similar to that for Conjecture 2.1 in the first section. However, the induction proceeded on the spanning $\Theta-$patch graph $H$ (we will give the definition very shortly) of $G$ is not straightforward. In fact, when $H$ has property $Q_2$ (defined in subsection 2), we can not directly proceed the induction. The new approach in dealing with this case, looks easy and natural, yet really took efforts to come out.
Theorem 2.2.1. Let $G$ be a graph with every edge in at least two triangles. Then $G$ contains a HIST.

2.2.1 Proof of Theorem 2.2.1

The proof consists of three main components: (1) define a class of graphs called Θ-patch graphs (we will define this class of graphs very shortly), and show that every graph with each edge in at least two triangles contains a spanning Θ-patch graph; (2) prove a rearrangeability of Θ-patch graphs; and (3) show every Θ-patch graph contains a HIST. Throughout this section, a graph isomorphic to $K_4^-$ ($K_4$ with exactly one edge removed) is called a Θ-graph.

Definition 2.2.1. Given a graph $H$ and a vertex $v \notin V(H)$, let $H \Delta v$ be a graph with $V(H \Delta v) = V(H) \cup \{v\}$ and $E(H \Delta v) = E(H) \cup \{u_1v, u_2v, u_1u_2\}$, where $u_1$, $u_2 \in V(H)$ are two distinct vertices. That is, $H \Delta v$ is obtained from $H$ by adding a new vertex $v$ and edges $u_1v$, $u_2v$, and $u_1u_2$ if $u_1u_2 \notin E(H)$. We name such an operation $\Delta$-operation and denote by $A(v) := \{u_1, u_2\}$, the set of attachments of $v$ on $H$. Moreover, we let $A[v] := A(v) \cup \{v\}$.

Note that $u_1u_2$ may or may not be an edge of $H$.

Definition 2.2.2. Given a graph $H$ and a Θ-graph $F$ with a specified degree 3 vertex, let $H \Theta F$ be the graph obtained by identifying the specified vertex of $F$ with a vertex $u$ in $H$. Let $A(F) = \{u\}$ be the set of the attachment of $F$ on $H$. Such an operation is called a Θ-operation.

We use $\oplus$ to denote either a $\Delta$-operation or a Θ-operation.

Definition 2.2.3. A graph $G$ is called a Θ-patch graph if there exists a subgraph sequence $G_1 \subset G_2 \subset \cdots \subset G_s = G$ with $s \geq 2$ such that

1. $G_1 \cong K_3$, and

2. $G_{i+1}$ is obtained from $G_i$ by a $\oplus$-operation for each $i$ $(1 \leq i \leq s - 1)$.

By the above definition, a Θ-patch graph has at least 4 vertices, and a Θ-patch graph with exactly 4 vertices is a Θ-graph.
Lemma 2.2.1. A connected graph with every edge in at least two triangles contains a \( \Theta \)-patch graph as a spanning subgraph.

Proof. Let \( G \) be a graph such that every edge is in at least two triangles. Since two triangles sharing a common edge induce a \( \Theta \)-graph, \( G \) contains a \( \Theta \)-graph, which is also a \( \Theta \)-patch graph by Definition 2.2.3. Let \( H \subset G \) be a \( \Theta \)-patch graph such that \(|V(H)|\) is maximum. If \( V(H) = V(G) \), the proof is completed. So assume the contrary: \( W = V(G) - V(H) \neq \emptyset \). Since \( G \) is connected, there is an edge \( uw \in E(G) \) such that \( u \in V(H) \) and \( w \in W \). Let \( v_1uwv_1 \) and \( v_2uwv_2 \) be two distinct triangles containing \( uw \). If \( v_i \in V(H) \) for some \( i = 1, 2 \), then \( H\Delta w \) with \( A(w) = \{u, v_i\} \) is a \( \Theta \)-patch graph larger than \( H \), contradicting the maximality of \( H \). Hence, we have both \( v_1, v_2 \in W \). Clearly, \( G\{u, v_1, v_2, w\} \), the subgraph induced on \( \{u, v_1, v_2, w\} \), contains a \( \Theta \)-graph \( F \). So \( H\Theta F \) with \( A(F) = \{u\} \) is a \( \Theta \)-patch graph larger than \( H \), contradicting the maximality of \( H \). \( \square \)

It will be shown in the following lemma that the ordering of subgraph sequence in the definition of \( \Theta \)-patch graphs can be rearranged to preserve a nice recursive property.

Lemma 2.2.2. Let \( G \) be a \( \Theta \)-patch graph of order \( n \geq 5 \). Then there exist a subgraph \( H \) which is either a \( \Theta \)-patch graph or isomorphic to \( K_3 \) such that one of the following properties...
Case \( Q_k (0 \leq k \leq 3) \): There exist vertices \( x_1, x_2, \cdots, x_k \) such that
\[
G = (H \Theta F) \Delta x_1 \Delta x_2 \cdots \Delta x_k \quad (G = H \Theta F \text{ when } k = 0) \text{ with } A[x_i] \cap A[x_j] = \emptyset \text{ for all } i \neq j \text{ and } A(x_i) \cap (V(F) - V(H)) \neq \emptyset \text{ for } i = 1, 2, \cdots, k.
\]

**Proof.** If \( n = 5 \), from the definition of \( \Theta \)-patch graphs, there exist two vertices \( x_1 \) and \( x_2 \) such that \( G = K_3 \Delta x_1 \Delta x_2 \) and \( A(x_2) \cap A[x_1] \neq \emptyset \), so \( P \) holds. We assume that \( n \geq 6 \) and Lemma 2.2.2 holds for graphs with order \( < n \).

By the definition of \( \Theta \)-patch graphs, \( G = H^* \oplus F^* \), where \( H^* \) is a \( \Theta \)-patch graph, and \( F^* \) is either a single vertex or a \( \Theta \)-graph. If \( F^* \) is a \( \Theta \)-graph, then \( Q_0 \) holds. So, we assume \( F^* \) is a single vertex graph, and say \( V(F^*) = \{w\} \). By applying Lemma 2.2.2 to \( H^* \), we divide the remaining proof into two cases below.

**Case P.** \( H^* = (H \Delta x_1) \Delta x_2 \) and \( A(x_2) \cap A[x_1] \neq \emptyset \).

If \( A(w) \cap \{x_1, x_2\} = \emptyset \), let \( H' := H \Delta w \), which is a \( \Theta \)-patch graph and a subgraph of \( G \). Then \( G = (H' \Delta x_1) \Delta x_2 \), so \( P \) holds.

Suppose \( A(w) \cap \{x_1, x_2\} \neq \emptyset \). If \( x_1 \in A(x_2) \) or \( x_2 \in A(w) \), \( H' := H \Delta x_1 \subset G \) is a \( \Theta \)-patch graph. Then, we have \( G = (H' \Delta x_2) \Delta w \) and either \( x_1 \in A(w) \cap A[x_2] \) or \( x_2 \in A(w) \), so \( P \) holds. We may assume that \( x_1 \notin A(x_2) \) and \( x_2 \notin A(w) \). In this case, we have \( x_1 \notin A(w) \). Let \( H' = H \Delta x_2 \), which is a \( \Theta \)-patch graph and a subgraph of \( G \). Then \( G = H' \Delta x_1 \Delta w \), so \( P \) holds.

**Case Q_k.** \( H^* = (H \Theta F) \Delta x_1 \Delta x_2 \cdots \Delta x_k \), where \( F \) is a \( \Theta \)-graph and \( x_i \) is a vertex in \( H^* \).

If \( A(w) \cap ((V(F) - V(H)) \cup \{x_1, x_2, \cdots, x_k\}) = \emptyset \), then
\[
G = ((H \Delta w) \Theta F) \Delta x_1 \Delta x_2 \cdots \Delta x_k,
\]
so \( Q_k \) holds. If \( A(w) \cap A[x_i] \neq \emptyset \), w.l.o.g., say \( A(w) \cap A[x_k] \neq \emptyset \), then
\[
G = (H \Theta F \Delta x_1 \Delta x_2 \cdots \Delta x_{k-1}) \Delta x_k \Delta w,
\]
so $P$ holds. Hence, we assume $A(w) \cap (V(F) - V(H)) \neq \emptyset$ and $A(w) \cap A[x_i] = \emptyset$ for $i = 1, 2, \cdots, k$. Under this assumption together with the assumption that $A[x_i] \cap A[x_j] = \emptyset$ for $i \neq j$ and $A(x_i) \cap (V(F) - V(H)) \neq \emptyset$ for $i = 1, 2, \cdots, k$, we have $k \leq 2$. Then, we have

$$G = (H\Theta F)\Delta x_1 \Delta x_2 \cdots \Delta x_k \Delta w,$$

so $Q_{k+1}$ holds.

\[\square\]

**Lemma 2.2.3.** Every $\Theta$-patch graph contains a HIST.

**Proof.** We use induction on $n = |V(G)|$. When $n = 4$, $G \cong K_4^-$ is a $\Theta$-graph. Clearly, $G$ contains a HIST. Suppose $n \geq 5$, and assume that Lemma 2.2.3 holds for graphs of order $< n$. We divide the remaining proof into five cases according to the five properties given in Lemma 2.2.2.

If $G$ has property $Q_i$ for some $i = 0, 1, 2$ or $3$, we follow the notations given in Lemma 2.2.2 and assume that $A(F) = \{u\}$ and $V(F) - V(H) = \{v_1, v_2, v_3\}$. If $G$ has property $P$ then $u$ is a specially selected vertex in $H$. We let $T$ be a HIST of $H$ if $H$ is a $\Theta$-patch graph, and let $T \cong P_3$ with $d_T(u) = 2$ if $H \cong K_3$. The case that $G$ satisfies property $Q_2$ is the most complicated one, and we can not straightforwardly play induction on it, so we defer this case to the end.

**Property P holds.** Suppose that $G = H \Delta x_1 \Delta x_2$ and $A(x_2) \cap A[x_1] \neq \emptyset$.

In this case, we first show that $N(x_1) \cap N(x_2) \cap V(H) \neq \emptyset$. This is clearly true if $A(x_1) \cap A(x_2) \neq \emptyset$, so we may assume $x_1 \in A(x_2)$. Let $u$ be the other vertex in $A(x_2)$. Since $E(G) = E((H \Delta x_1) \Delta x_2) = E(H \Delta x_1) \cup \{x_2u, x_2x_1, ux_1\}$, we have $ux_1 \in E(G)$, that is, $u \in N(x_1) \cap N(x_2)$.

Let $u \in N(x_1) \cap N(x_2)$. Then, it is readily seen that $T \cup \{ux_1, ux_2\}$ is a HIST of $G$.

**Property Q_0 holds.** Let $G = H\Theta F$.

In this case, $T \cup \{uv_1, uv_2, uv_3\}$ is a HIST of $G$.

**Property Q_1 holds.** Let $G = (H\Theta F) \Delta x_1$, and assume, without loss of generality, $v_1 \in A(x_1) \cap (V(F) - V(H))$, and let $w_1$ be another vertex of $A(x_1)$.
In this case, \( T \cup \{w_1v_1, w_1x_1, uv_2, uv_3\} \) is a HIST of \( G \) regardless of whether \( w_1 \in V(F) \) or not.

**Property Q\(_3\) holds.** Let \( G = (H \Theta F)\Delta x_1 \Delta x_2 \Delta x_3 \) and assume that \( A(x_i) = \{v_i, w_i\} \) for each \( i = 1, 2, 3 \) with \( w_1, w_2, w_3 \in V(H) \).

By the definition of \( \Delta \)-operation, all three edges \( w_1v_1, w_2v_2, w_3v_3 \) are in \( E(G) \). Then, \( T \cup \{w_1x_1, w_1v_1, w_2x_2, w_2v_2, w_3x_3, w_3v_3\} \) is a HIST in \( G \).

**Property Q\(_2\) holds.** Let \( G = (H \Theta F)\Delta x_1 \Delta x_2 \) such that \( A(x_i) \cap (V(F) - V(H)) \neq \emptyset \) for each \( i = 1, 2 \), and \( A[x_2] \cap A[x_1] = \emptyset \). Assume that \( A(x_i) = \{v_i, w_i\} \) for \( i = 1, 2 \).

We may assume \( w_i \neq u \) for each \( i = 1, 2 \); otherwise, say \( w_1 = u \), then \( T \cup \{w_2v_2, w_2x_2, w_1, w_1x_1, uv_3\} \) is a HIST of \( G \). Since \( A[x_2] \cap A[x_1] = \emptyset \), we may assume that \( w_1 \in V(H) - \{u\} \). Moreover, under the assumption that \( w_1 \in V(H) - \{u\} \), let notation be chosen so that \( v_1 \) is the degree 2 vertex in \( F - u \) whenever it is possible, that is, if \( w_2 \in V(H) - \{u\} \) and \( v_2 \) is the degree two vertex in \( F - u \), we rename \( x_2, v_2 \) and \( w_2 \) as \( x_1, v_1 \) and \( w_1 \), and vice versa.

Let \( z \notin V(G) \) be a vertex and \( G' := H \Delta z \) with \( A(z) = \{u, w_1\} \). Clearly, \( uw_1 \in E(G') \) although \( uw_1 \) may not be in \( E(G) \). Clearly, \( G' \) is a \( \Theta \)-patch graph and \( |V(G')| < n \), so it contains a HIST \( T' \). Since \( d_{G'}(z) = 2 \), \( z \) is a degree 1 vertex of \( T' \). So, we have either \( w_1z \in E(T') \) or \( uz \in E(T') \) but not both. Let \( T_H := T' - z \).

**Subcase 1.** \( uw_1 \notin E(T') \) or \( uw_1 \in E(T') \cap E(G) \).

Note that \( d_{T'}(z) = 1 \). If \( uz \in E(T') \), let \( T^* := T_H \cup \{w_3, w_1v_1, w_1x_1, w_2v_2, w_2x_2\} \), as depicted in Figure 2.3. It is routine to check that \( T^* \) is a spanning tree of \( G \) and the following
\[\begin{align*}
  d_{T*}(u) &= d_{T'}(u) - |\{uz\}| + |\{uv_3\}| = d_{T'}(u) \neq 2 \\
  d_{T*}(w_1) &= d_{T'}(w_1) + |\{w_1v_1, w_1x_2\}| = d_{T'}(w_1) + 2 \neq 2 \\
  d_{T*}(w_2) &= \begin{cases} 
    d_{T'}(w_2) + |\{w_2v_2, w_2x_2\}| = d_{T'}(w_2) + 2 \neq 2, & \text{if } w_2 \in V(H); \\
    |\{w_2v_2, w_2x_2, uv_3\}| = 3, & \text{if } w_2 = v_3.
  \end{cases} \\
  d_{T*}(x) &= d_{T'}(x) \neq 2 \quad \text{for all other vertices } x \in V(H), \text{ and} \\
  d_{T*}(x) &\neq 2 \quad \text{for each vertex } x \in \{v_1, v_2, v_3, x_1, x_2\}.
\end{align*}\]

Consequently, \(T^*\) is a HIST of \(G\).

If \(w_1z \in E(T')\), let \(T^* := T_H \cup \{w_1x_1, uv_1, uv_3, w_2v_2, w_2x_2\}\), as depicted in Figure 2.3
\((w_2 = v_3 \text{ may occur.})\) As in the previous case, we can show that \(T^*\) is a HIST of \(G\).
Subcase 2. \(uw_1 \in E(T') - E(G)\).

In this case, \(T_1 := T_H - uw_1\) has exactly two components. We construct a HIST of \(G\) from \(T_1\) according to whether \(uz \in E(T')\) or \(w_1z \in E(T')\).

If \(uz \in E(T')\), let \(T^* = T_1 \cup \{uv_3, uw_1, v_1w_1, v_1x_1, w_2v_2, w_2x_2\}\), as depicted in Figure 2.4. It is routine to check that \(T^*\) is a spanning tree of \(G\) and the following equalities/inequalities hold.

\[
\begin{align*}
d_{T^*}(u) &= d_{T'}(u) - |\{uw_1, uz\}| + |\{uv_3, uw_3\}| = d_{T'}(u) \neq 2 \\
d_{T^*}(w_1) &= d_{T'}(w_1) - |\{uw_1\}| + |\{v_1w_1\}| = d_{T'}(w_1) \neq 2 \\
d_{T^*}(w_2) &= \begin{cases} d_{T'}(w_2) + |\{w_2v_2, w_2x_2\}| = d_{T'}(w_2) + 2 \neq 2, & \text{if } w_2 \in V(H); \\
|\{w_2v_2, w_2x_2, uw_3\}| = 3, & \text{if } w_2 = v_3.
\end{cases} \\
d_{T^*}(x) &= d_{T'}(x) \neq 2 \quad \text{for all other vertices } x \in V(H), \text{ and} \\
d_{T^*}(x) &\neq 2 \quad \text{for each vertex } x \in \{v_1, v_2, v_3, x_1, x_2\}.
\end{align*}
\]

So, \(T^*\) is a HIST of \(G\).

In the case \(w_1z \in E(T')\), if \(v_1v_3 \in E(G)\), let

\[T^* = T_1 \cup \{w_1x_1, w_1v_1, v_1u, v_1v_3, w_2v_2, w_2x_2\},\]

as depicted in Figure 2.4. As in the previous case, we can show that \(T^*\) is a HIST of \(G\). To complete the proof, we show that the vertex \(v_1\) can be chosen such that \(v_1v_3 \in E(G)\). If \(v_1v_3 \notin E(G)\), then both \(v_1\) and \(v_3\) are degree 1 vertices in \(F - u \cong P_3\). So, \(v_2\) is the degree 2 vertex in \(F - u\). If \(w_2 \in V(H)\), we would pick \(x_2\) as \(x_1\) and \(v_2\) as our \(v_1\) in the very beginning. So, \(w_2 = v_3\). In this case, we can simply swap \(v_2\) and \(v_3\) (also \(w_2\)) to ensure that \(v_1v_3 \in E(G)\).

Clearly, the combination of the above three Lemmas gives Theorem 2.2.1.
Figure (2.4) \( uw_1 \in E(T') - E(G) \)
PART 3

MINIMUM DEGREE CONDITION FOR SPANNING HALIN GRAPHS
AND SPANNING GENERALIZED HALIN GRAPHS

3.1 Notations and definitions

We consider simple and finite graphs only. Let $G$ be a graph. Denote by $e(G)$ the cardinality of $E(G)$. Let $v \in V(G)$ be a vertex and $S \subseteq V(G)$ a subset. The notation $\Gamma_G(v, S)$ denotes the set of neighbors of $v$ in $S$, and $\deg_G(v, S) = |\Gamma_G(v, S)|$. We let $\Gamma_G(v, S) = S - \Gamma_G(v, S)$ and $\deg_G(v, S) = |\Gamma_G(v, S)|$. Given another set $U \subseteq V(G)$, define $\Gamma_G(U, S) = \bigcap_{u \in U} \Gamma_G(u, S)$, $\deg_G(U, S) = |\Gamma_G(U, S)|$, and $N_G(U, S) = \cup_{u \in U} \Gamma_G(u, S)$. When $U = \{u_1, u_2, \ldots, u_k\}$, we may write $\Gamma_G(U, S)$, $\deg_G(U, S)$, and $N_G(U, S)$ as $\Gamma_G(u_1, u_2, \ldots, u_k, S)$, $\deg_G(u_1, u_2, \ldots, u_k, S)$, and $N_G(u_1, u_2, \ldots, u_k, S)$, respectively, in specifying the vertices in $U$. When $S = V(G)$, we only write $\Gamma_G(U)$, $\deg_G(U)$, and $N_G(U)$. Let $U_1, U_2 \subseteq V(G)$ be two disjoint subsets. Then $\delta_G(U_1, U_2) = \min\{\deg_G(u_1, U_2) | u_1 \in U_1\}$ and $\Delta_G(U_1, U_2) = \max\{\deg_G(u_1, U_2) | u_1 \in U_1\}$. Notice that the notations $\delta_G(U_1, U_2)$ and $\Delta_G(U_1, U_2)$ are not symmetric with respect to $U_1$ and $U_2$. We denote by $E_G(U_1, U_2)$ the set of edges with one end in $U_1$ and the other in $U_2$, the cardinality of $E_G(U_1, U_2)$ is denoted as $e_G(U_1, U_2)$. We may omit the index $G$ if there is no risk of confusion. Let $u, v \in V(G)$ be two vertices. We write $u \sim v$ if $u$ and $v$ are adjacent. A path connecting $u$ and $v$ is called a $(u, v)$-path. If $G$ is a bipartite graph with partite sets $A$ and $B$, we denote $G$ by $G(A, B)$ in emphasizing the two partite sets. A matching in $G$ is a set of independent edges; a $\wedge$-matching is a set of vertex-disjoint copies of $K_{1,2}$; and a claw-matching is a set of vertex-disjoint copies of $K_{1,3}$. The set of degree 2 vertices in a $\wedge$-matching is called the center of the $\wedge$-matching; and the set of degree 3 vertices in a claw-matching is called the center of the claw-matching. A cycle $C$ in a graph $G$ is dominating if $G - V(C)$ is an edgeless graph.
3.2 The Regularity Lemma and the Blow-up Lemma

The Regularity Lemma of Szemerédi [50] and Blow-up lemma of Komlós et al. [36] are main tools used in finding a spanning Halin subgraph or spanning generalized Halin subgraph. For any two disjoint non-empty vertex-sets $A$ and $B$ of a graph $G$, the density of $A$ and $B$ is the ratio $d(A, B) := \frac{e(A, B)}{|A||B|}$. Let $\varepsilon$ and $\delta$ be two positive real numbers. The pair $(A, B)$ is called $\varepsilon$-regular if for every $X \subseteq A$ and $Y \subseteq B$ with $|X| > \varepsilon|A|$ and $|Y| > \varepsilon|B|$, $|d(X, Y) - d(A, B)| < \varepsilon$ holds. In addition, if $\deg(a, B) > \delta|B|$ for each $a \in A$ and $\deg(b, A) > \delta|A|$ for each $b \in B$, we say $(A, B)$ an $(\varepsilon, \delta)$-super regular pair.

Lemma 3.2.1 (Regularity lemma-Degree form [50]). For every $\varepsilon > 0$ there is an $M = M(\varepsilon)$ such that if $G$ is any graph with $n$ vertices and $d \in [0, 1]$ is any real number, then there is a partition of the vertex set $V(G)$ into $l + 1$ clusters $V_0, V_1, \ldots, V_l$, and there is a spanning subgraph $G' \subseteq G$ with the following properties.

- $l \leq M$;
- $|V_0| \leq \varepsilon n$, all clusters $|V_i| = |V_j| \leq \lfloor \varepsilon n \rfloor$ for all $1 \leq i \neq j \leq l$;
- $\deg_{G'}(v) > \deg_G(v) - (d + \varepsilon)n$ for all $v \in V(G)$;
- $e(G'[V_i]) = 0$ for all $i \geq 1$;
- all pairs $(V_i, V_j)$ $(1 \leq i < j \leq l)$ are $\varepsilon$-regular, each with a density either 0 or greater than $d$.

Lemma 3.2.2 (Blow-up lemma-weak version [36]). Given a graph $R$ of order $r$ and positive parameters $\delta, \Delta$, there exists a positive $\varepsilon = \varepsilon(\delta, \Delta, r)$ such that the following holds. Let $n_1, n_2, \ldots, n_r$ be arbitrary positive integers and let us replace the vertices $v_1, v_2, \ldots, v_r$ with pairwise disjoint sets $V_1, V_2, \ldots, V_r$ of sizes $n_1, n_2, \ldots, n_r$ (blowing up). We construct two graphs on the same vertex set $V = \bigcup V_i$. The first graph $K$ is obtained by replacing each edge $v_iv_j$ of $R$ with the complete bipartite graph between the corresponding vertex sets $V_i$ and $V_j$. A sparser graph $G$ is constructed by replacing each edge $v_iv_j$ arbitrarily with an
\((\varepsilon, \delta)\)-super regular pair between \(V_i\) and \(V_i\). If a graph \(H\) with \(\Delta(H) \leq \Delta\) is embeddable into \(K\) then it is already embeddable into \(G\).

**Lemma 3.2.3** (Blow-up lemma-strengthened version [36]). Given \(c > 0\), there are positive numbers \(\varepsilon = \varepsilon(\delta, \Delta, r, c)\) and \(\gamma = \gamma(\delta, \Delta, r, c)\) such that the Blow-up lemma in the equal size case (all \(|V_i|\) are the same) remains true if for every \(i\) there are certain vertices \(x\) to be embedded into \(V_i\) whose images are a priori restricted to certain sets \(C_x \subseteq V_i\) provided that

(i) each \(C_x\) within a \(V_i\) is of size at least \(c|V_i|\);

(ii) the number of such restrictions within a \(V_i\) is not more than \(\gamma|V_i|\).

Besides the above two lemmas, we also need the two lemmas below regarding regular pairs.

**Lemma 3.2.4.** If \((A, B)\) is an \(\varepsilon\)-regular pair with density \(d\), then for any \(A' \subseteq A\) with \(|A'| > \varepsilon|A|\), there are at most \(\varepsilon|B|\) vertices \(b \in B\) such that \(\deg(b, A') \leq (d - \varepsilon)|A'|\).

**Lemma 3.2.5** (Slicing lemma). Let \((A, B)\) be an \(\varepsilon\)-regular pair with density \(d\), and for some \(\nu > \varepsilon\), let \(A' \subseteq A\) and \(B' \subseteq B\) with \(|A'| \geq \nu|A|\), \(|B'| \geq \nu|B|\). Then \((A', B')\) is an \(\varepsilon'\)-regular pair of density \(d'\), where \(\varepsilon' = \max\{\varepsilon/\nu, 2\varepsilon\}\) and \(d' > d - \varepsilon\).

The following two results on hamiltonicity are used in finding hamiltonian cycles in the proofs.

**Lemma 3.2.6** ([45]). If \(G\) is a graph of order \(n\) satisfying \(d(x) + d(y) \geq n + 1\) for every pair of nonadjacent vertices \(x, y \in V(G)\), then \(G\) is hamiltonian-connected.

**Lemma 3.2.7** ([42]). Let \(G\) be a balanced bipartite graph with \(2n\) vertices. If \(d(x) + d(v) \geq n + 1\) for any two non-adjacent vertices \(x, y \in V(G)\), then \(G\) is hamiltonian.

3.3 Dirac’s condition for spanning Halin graphs

3.3.1 Introduction

A classic theorem of Dirac [19] from 1952 asserts that every graph on \(n\) vertices with minimum degree at least \(n/2\) is hamiltonian if \(n \geq 3\). Following Dirac’s result, numerous
results on hamiltonicity properties on graphs with restricted degree conditions have been obtained (see, for instance, [26] and [25]). Traditionally, under similar conditions, results for a graph being hamiltonian, hamiltonian-connected, and pancyclic are obtained separately. We may ask, under certain conditions, if it is possible to uniformly show a graph possessing several hamiltonicity properties. The work on finding the square of a hamiltonian cycle in a graph can be seen as an attempt in this direction. However, it requires quite strong degree conditions for a graph to contain the square of a hamiltonian cycle, for examples, see [21], [22], [37], [9], and [49]. For bipartite graphs, finding the existence of a spanning ladder is a way of simultaneously showing the graph having many hamiltonicity properties (see [16] and [17]). In this paper, we introduce another approach of uniformly showing the possession of several hamiltonicity properties in a graph: we show the existence of a spanning Halin graph in a graph under given minimum degree condition.

A tree with no vertex of degree 2 is called a homeomorphically irreducible tree (HIT). A Halin graph $H$ is obtained from a HIT $T$ of at least 4 vertices embedded in the plane by connecting its leaves into a cycle $C$ following the cyclic order determined by the embedding. According to the construction, the Halin graph $H$ is denoted as $H = T \cup C$, and the HIT $T$ is called the underlying tree of $H$. A wheel graph is an example of a Halin graph, where the underlying tree is a star. Halin constructed Halin graphs in [27] for the study of minimally 3-connected graphs. Lovász and Plummer named such graphs as Halin graphs in their study of planar bicritical graphs [40], which are planar graphs having a 1-factor after deleting any two vertices. Intensive researches have been done on Halin graphs. Bondy [5] in 1975 showed that a Halin graph is hamiltonian. In the same year, Lovász and Plummer [40] showed that not only a Halin graph itself is hamiltonian, but each of the subgraph obtained by deleting a vertex is hamiltonian. In 1987, Barefoot [2] proved that Halin graphs are hamiltonian-connected, i.e., there is a hamiltonian path connecting any two vertices of the graph. Furthermore, it was proved that each edge of a Halin graph is contained in a hamiltonian cycle and is avoided by another [48]. Bondy and Lovász [6], and Skowrońska [47], independently, in 1985, showed that a Halin graph is almost pancyclic and is pancyclic if the
underlying tree has no vertex of degree 3, where an $n$-vertex graph is almost pancyclic if it contains cycles of length from 3 to $n$ with the possible exception of a single even length, and is pancyclic if it contains cycles of length from 3 to $n$. Some problems that are NP-complete for general graphs have been shown to be polynomial time solvable for Halin graphs. For example, Cornuéjols, Naddef, and Pulleyblank [15] showed that in a Halin graph, a Hamiltonian cycle can be found in polynomial time. It seems so promising to show the existence of a spanning Halin subgraph in a given graph in order to show the graph having many Hamiltonicity properties. But, nothing comes for free, it is NP-complete to determine whether a graph contains a (spanning) Halin graph [30].

Despite all these nice properties of Halin graphs mentioned above, the problem of determining whether a graph contains a spanning Halin subgraph has not yet well studied except a conjecture proposed by Lovász and Plummer [40] in 1975. The conjecture states that every 4-connected plane triangulation contains a spanning Halin subgraph (disproved recently [10]). In this paper, we investigate the minimum degree condition for implying the existence of a spanning Halin subgraph in a graph, and thereby giving another approach for uniformly showing the possession of several Hamiltonicity properties in a graph under a given minimum degree condition. We obtain the following result.

**Theorem 3.3.1.** There exists $n_0 > 0$ such that for any graph $G$ with $n \geq n_0$ vertices, if $\delta(G) \geq (n + 1)/2$, then $G$ contains a spanning Halin subgraph.

Note that an $n$-vertex graph with minimum degree at least $(n + 1)/2$ is 3-connected if $n \geq 4$. Hence, the minimum degree condition in Theorem 3.3.1 implies the 3-connectedness, which is a necessary condition for a graph to contain a spanning Halin subgraph, since every Halin graph is 3-connected. A Halin graph contains a triangle, and bipartite graphs are triangle-free. Hence, $K_{\lceil n/2 \rceil, \lceil n/2 \rceil}$ contains no spanning Halin subgraph. Immediately, we see that the minimum degree condition in Theorem 3.3.1 is best possible.
3.3.2 Ladders and “ladder-like” Halin graphs

In constructing Halin graphs, we use ladder graphs and a class of “ladder-like” graphs as substructures. We give the description of these graphs below.

**Definition 3.3.1.** An \( n \)-ladder \( L_n = L_n(A, B) \) is a balanced bipartite graph with \( A = \{a_1, a_2, \cdots, a_n\} \) and \( B = \{b_1, b_2, \cdots, b_n\} \) such that \( a_i \sim b_j \) iff \( |i - j| \leq 1 \). We call \( a_i b_i \) the \( i \)-th rung of \( L_n \). If \( 2n(\text{mod} \ 4) \equiv 0 \), we call each of the shortest \( (a_1, b_n) \)-path and \( (b_1, a_n) \)-path a side of \( L_n \); otherwise we call each of the shortest \( (a_1, a_n) \)-path and \( (b_1, b_n) \)-path a side of \( L_n \).

Let \( L \) be a ladder with \( xy \) as one of its rungs. For an edge \( gh \), we say \( xy \) and \( gh \) are adjacent if \( x \sim g, y \sim h \) or \( x \sim h, y \sim g \). Suppose \( L \) has its first rung as \( ab \) and its last rung as \( cd \). We denote \( L \) by \( ab - L - cd \) in specifying the two rungs, and we always assume that the distance between \( a \) and \( c \) is \( |V(L)|/2 \) (we make this assumption for being convenient in constructing other graphs based on ladders). Under this assumption, we denote \( L \) as \( \overrightarrow{ab} - L - \overrightarrow{cd} \). Let \( A \) and \( B \) be two disjoint vertex sets. We say the rung \( xy \) of \( L \) is contained in \( A \times B \) if either \( x \in A, y \in B \) or \( x \in B, y \in A \). Let \( L' \) be another ladder vertex-disjoint with \( L \). If the last rung of \( L \) is adjacent to the first rung of \( L' \), we write \( LL' \) for the new ladder obtained by concatenating \( L \) and \( L' \). In particular, if \( L' = gh \) is an edge, we write \( LL' \) as \( Lgh \).

We now define five types of “ladder-like” graphs, call them \( H_1, H_2, H_3, H_4 \) and \( H_5 \), respectively. Let \( L_n \) be a ladder with \( a_1 b_1 \) and \( a_n b_n \) as the first and last rung, respectively, and \( x, y, z, w, u \) five new vertices. Then each of \( H_i \) is obtained from \( L_n \) by adding some specified vertices and edges as follows. Additionally, for each \( i \) with \( 1 \leq i \leq 5 \), we define a graph \( T_i \) associated with \( H_i \).

\( H_1 \): Adding two new vertices \( x, y \) and the edges \( xa_1, xb_1, ya_n, yb_n \) and \( xy \).

Let \( T_1 = H_1[\{x, y, a_1, b_1, a_n, b_n\}] \).

\( H_2 \): Adding three new vertices \( x, y, z \) and the edges \( za_1, zb_1, xz, xb_1, ya_n, yb_n \) and \( xy \).
Let $T_2 = H_2[\{x, y, z, a_1, b_1, a_n, b_n\}]$.

$H_3$: Adding three new vertices $x, y, z$ and the edges $xa_1, xb_1, ya_n, yb_n$, either $za_i$ or $zb_i$ for some $2 \leq i \leq n-1$ and $xz, yz$.

Let $T_3 = H_3[\{x, y, z, a_1, b_1, a_n, b_n\}]$.

$H_4$: Adding four new vertices $x, y, z, w$ and the edges $wa_1, wb_1, xw, xb_1, ya_n, yb_n$, either $za_i$ or $zb_i$ for some $2 \leq i \leq n-1$ and $xz, yz$.

Let $T_4 = H_4[\{x, y, z, w, a_1, b_1, a_n, b_n\}]$.

$H_5$: Adding five new vertices $x, y, z, w, u$.

If $2(n-1)(\text{mod } 4) \equiv 2$, adding the edges $wa_1, wb_1, xw, xb_1, ua_n, ub_n, yu, yb_n$, either $za_i$ or $zb_i$ for some $2 \leq i \leq n-1$ and $xz, yz$ ;

and if $2(n-1)(\text{mod } 4) \equiv 0$, adding the edges $wa_1, wb_1, xw, xb_1, ua_n, ub_n, yu, ya_n$, either $za_i$ or $zb_i$ for some $2 \leq i \leq n-1$ and $xz, yz$.

Let $T_5 = H_5[\{x, y, z, w, u, a_1, b_1, a_n, b_n\}]$.

Let $i = 1, 2, \cdots, 5$. Notice that each of $H_i$ is a Halin graph and except $H_1$, each $H_i$ has a unique underlying tree. Notice also that $xy$ is an edge on the cycle along the leaves of any underlying tree of $H_i$. For each $H_i$, call $x$ the left end and $y$ the right end, and call a vertex of degree at least 3 in the underlying tree of $H_i$ a Halin constructible vertex. By analyzing the structure of $H_i$, we see that each of the vertices on one side of the ladder $H_i - \{x, y, z, w, u\}$ is a Halin constructible vertex. Noting that any vertex in $V(H_i) - \{x, y\}$ can be a Halin constructible vertex. In Figure 3.1 we depict a ladder $L_4$, $H_1, H_2, H_3, H_4, H_5$ constructed from $L_4$, and the graph $T_i$ associated with $H_i$. We call $a_1b_1$ the head link of $T_i$ and $a_nb_n$ the tail link of $T_i$, and for each of $T_3, T_4, T_5$, we call the vertex $z$ not contained in any triangles the pendent vertex. The notations of $H_i$ and $T_i$ are fixed hereafter.

Let $T \in \{T_1, \cdots, T_5\}$ be a subgraph of a graph $G$. Suppose that $T$ has head link $ab$, tail link $cd$, and possibly the pendent vertex $z$. It is clear that if $G - V(T)$ contains a
spanning ladder $L$ with first rung $c_1d_1$ and last rung $c_nd_n$ such that $c_1d_1$ is adjacent to $ab$, $c_nd_n$ is adjacent to $cd$, and $z$ is adjacent some vertex $z'$ on some internal rung of $L$ if $z$ exists, then $abLcd \cup T$ or $abLcd \cup T \cup \{zz'\}$ when $z$ exists is a spanning Halin subgraph of $G$. This technique is frequently used later on in constructing a Halin graph. The following proposition gives another way of constructing a Halin graph based on $H_1$ and $H_2$.

**Proposition 3.3.1.** For $i = 1, 2$, let $G_i \in \{H_1, H_2\}$ with left end $x_i$ and right end $y_i$ be defined as above, and let $u_i \in V(G_i)$ be a Halin constructible vertex, then $G_1 \cup G_2 - \{x_1y_1, x_2y_2\} \cup \{x_1x_2, y_1y_2, u_1u_2\}$ is a Halin graph spanning on $V(G_1) \cup V(G_2)$.

**Proof.** For $i = 1, 2$, let $G_i$ be embedded in the plane, and let $T_{G_i}$ be a underlying plane tree of $G_i$. Then $T' := T_{G_1} \cup T_{G_2} \cup \{u_1u_2\}$ is a homeomorphically irreducible tree spanning on $V(G_1) \cup V(G_2)$. Moreover, we can draw the edge $u_1u_2$ such that $T_{G_1} \cup T_{G_2} \cup \{u_1u_2\}$ is a plane graph. Since $G_i[E(G_i - T_{G_i}) - \{x_iy_i\}]$ is an $(x_i, y_i)$-path spanning on the leaves of $T_{G_i}$ obtained by connecting the leaves following the order determined by the embedding, we see $G_1[E(G_1 - T_{G_1}) - \{x_1y_1\}] \cup G_2[E(G_2 - T_{G_2}) - \{x_2y_2\}] \cup \{x_1x_2, y_1y_2\}$ is a cycle spanning on the leaves of $T'$ obtained by connecting the leaves following the order determined by the embedding of $T'$. Thus $G_1 \cup G_2 - \{x_1y_1, x_2y_2\} \cup \{x_1x_2, y_1y_2, u_1u_2\}$ is a Halin graph. \hfill $\Box$

3.3.3 Proof of Theorem 3.3.1

In this section, we prove Theorem 3.3.1. Following the standard setup of proofs applying the Regularity Lemma, we divide the proof into non-extremal case and extremal cases. For this purpose, we define the two extremal cases in the following.

Let $G$ be an $n$-vertex graph and $V$ its vertex set. Given $0 \leq \beta \leq 1$, the two extremal cases are defined as below.

**Extremal Case 1.** $G$ has a vertex-cut of size at most $5\beta n$.

**Extremal Case 2.** There exists a partition $V_1 \cup V_2$ of $V$ such that $|V_1| \geq (1/2 - 7\beta)n$ and $\Delta(G[V_1]) \leq \beta n$.

**Non-extremal case.** We say that an $n$-vertex graph with minimum degree at least $(n+1)/2$ is in non-extremal case if it is in neither of Extremal Case 1 and Extremal Case 2.
The following three theorems deal with the non-extremal case and the two extremal cases, respectively, and thus give a proof of Theorem 3.3.1.

**Theorem 3.3.2.** Suppose that $0 < \beta \ll 1/(20 \cdot 17^3)$ and $n$ is a sufficiently large integer. Let $G$ be a graph on $n$ vertices with $\delta(G) \geq (n + 1)/2$. If $G$ is in Extremal Case 1, then $G$ contains a spanning Halin subgraph.

**Theorem 3.3.3.** Suppose that $0 < \beta \ll 1/(20 \cdot 17^3)$ and $n$ is a sufficiently large integer. Let $G$ be a graph on $n$ vertices with $\delta(G) \geq (n + 1)/2$. If $G$ is in Extremal Case 2, then $G$ contains a spanning Halin subgraph.

**Theorem 3.3.4.** Let $n$ be a sufficiently large integer and $G$ an $n$-vertex graph with $\delta(G) \geq (n + 1)/2$. If $G$ is in the Non-extremal case, then $G$ has a spanning Halin subgraph.

We need the following “Absorbing Lemma” in each of the proofs of Theorems 3.3.2 - 3.3.4 in dealing with “garbage” vertices.

**Lemma 3.3.1 (Absorbing Lemma).** Let $F$ be a graph such that $V(F)$ is partitioned as $S \cup R$. Suppose that (i) $\delta(R, S) \geq 3|R|$, (ii) for any two vertices $u, v \in N(R, S)$, $\deg(u, v, S) \geq 6|R|$,
and (iii) for any three vertices $u,v,w \in N(N(R,S),S)$, $\deg(u,v,w,S) \geq 7|R|$. Then there is a ladder spanning on $R$ and some other $7|R| - 2$ vertices from $S$.

**Proof.** Let $R = \{w_1, w_2, \ldots, w_r\}$. Consider first that $|r| = 1$. Choose $x_{i1}, x_{i2}, x_{i3} \in \Gamma(w_i, S)$. By (ii), there are distinct vertices $y_{i2}^1 \in \Gamma(x_{i1}, x_{i2}, S)$ and $y_{23}^1 \in \Gamma(x_{i2}, x_{i3}, S)$. Then the graph $L$ on $\{w_1, x_{i1}, x_{i2}, x_{i3}, y_{i2}^1, y_{23}^1\}$ with edges in

$$\{w_1 x_{i1}, w_1 x_{i2}, w_1 x_{i3}, y_{i2}^1 x_{i1}, y_{i2}^1 x_{i2}, y_{23}^1 x_{i2}, y_{23}^1 x_{i3}\}$$

is a ladder covering $R$ with $|V(L)| = 6$. Suppose now $r \geq 2$. For each $i$ with $1 \leq i \leq r$, choose distinct (and unchosen) vertices $x_{i1}, x_{i2}, x_{i3} \in \Gamma(w_i, S)$. This is possible since $\deg(x, S) \geq 3|R|$ for each $x \in R$. By (ii), we choose distinct vertices $y_{i2}^1, y_{23}^1, \ldots, y_{i2}^r, y_{23}^r$ different from the existing vertices already chosen such that $y_{i2}^1 \in \Gamma(x_{i1}, x_{i2}, S)$ and $y_{23}^1 \in \Gamma(x_{i2}, x_{i3}, S)$ for each $i$, and at the same time, we chose distinct vertices $z_1, z_2, \ldots, z_{r-1}$ from the unchosen vertices in $S$ such that $z_i \in \Gamma(x_{i3}, x_{(i+1),1}, S)$ for each $1 \leq i \leq r - 1$. Finally, by (iii), choose distinct vertices $u_1, u_2, \ldots, u_{r-1}$ from the unchosen vertices in $S$ such that $u_i \in \Gamma(x_{i3}, x_{i+1,1}, z_i, S)$. Let $L$ be the graph with

$$V(L) = R \cup \{x_{i1}, x_{i2}, x_{i3}, y_{i2}^1, y_{23}^1, z_i, u_i, x_{r1}, x_{r2}, x_{r3}, y_{i2}^r, y_{23}^r | 1 \leq i \leq r - 1\}$$

and

$E(L)$ consisting of the edges $w, x_{r1}, w, x_{r2}, w, x_{r3}, y_{i2}^1 x_{r1}, y_{i2}^r x_{r2}, y_{23}^r x_{r2}, y_{23}^r x_{r3}$ and the edges indicated below for each $1 \leq i \leq r - 1$:

$$w_i \sim x_{i1}, x_{i2}, x_{i3}; y_{i2}^1 \sim x_{i1}, x_{i2}; y_{23}^1 \sim x_{i2}, x_{i3}; z_i \sim x_{i3}, x_{i+1,1}; u_i \sim x_{i3}, x_{i+1,1}, z_i.$$

It is easy to check that $L$ is a ladder covering $R$ with $|V(L)| = 8r - 2$. Figure 3.2 gives a depiction of $L$ for $|R| = 2$. \qed

The following simple observation is heavily used in the proofs explicitly or implicitly.

**Lemma 3.3.2.** Let $U = \{u_1, u_2, \ldots, u_k\}, S \subseteq V(G)$ be subsets. Then $\deg(u_1, u_2, \ldots, u_k, S) \geq |S| - (\deg_G(u_1, S) + \cdots + \deg_G(u_k, S)) \geq |S| - k(|S| - \delta(U, S))$. 


Extremal Case 1 is relatively easy among the three cases, therefore we prove Theorem 3.3.2 first below.

3.3.3.1 Proof of Theorem 3.3.2 We assume that $G$ has a vertex-cut $W$ such that $|W| \leq 5\beta n$. As $\delta(G) \geq (n+1)/2$, by simply counting degrees we see $G-W$ has exactly two components. Let $V_1$ and $V_2$ be the vertex set of the two components, respectively. Then $(1/2 - 5\beta)n \leq |V_i| \leq (1/2 + 5\beta)n$. We partition $W$ into two subsets as follows:

$$W_1 = \{w \in W \mid \deg(w, V_1) \geq (n+1)/4 - 2.5\beta n\} \quad \text{and} \quad W_2 = W - W_1.$$ 

As $\delta(G) \geq (n+1)/2$, we have $\deg(w, V_2) \geq (n+1)/4 - 2.5\beta n$ for any $w \in W_2$. Since $G$ is 3-connected and $(1/2 - 5\beta)n > 3$, there are three independent edges $p_1p_2, q_1q_2,$ and $r_1r_2$ between $G[V_1 \cup W_1]$ and $G[V_2 \cup W_2]$ with $p_1, q_1, r_1 \in V_1 \cup W_1$ and $p_2, q_2, r_2 \in V_2 \cup W_2$.

For $i = 1, 2$, by the partition of $W_i$, we see that $\delta(W_i, V_i) \geq 3|W_i| + 3$. As $\delta(G) \geq (n+1)/2$, we have $\delta(G[V_i]) \geq (1/2 - 5\beta)n$. Then, as $|V_i| \leq (1/2 + 5\beta)n$, for any $u, v \in V_i$, $\deg(u, v, V_i) \geq (1/2 - 25\beta)n \geq 6|W_i| + 2$, and for any $u, v, w \in V_i$, $\deg(u, v, w, V_i) \geq (1/2 - 35\beta)n \geq 7|W_i| + 2$. By Lemma 3.4.2 we can find a ladder $L_i$ spans $W_i - \{p_i, q_i\}$ and another $7|W_i - \{p_i, q_i\}| - 2$ vertices from $V_i - \{p_i, q_i\}$ if $W_i - \{p_i, q_i\} \neq \emptyset$. Denote $a_ib_i$ and $c_id_i$ the
first and last rung of $L_i$ (if $L_i$ exists), respectively. Let
\[ G_i = G[V_i - V(L_i)] \text{ and } n_i = |V(G_i)|. \]

Then for $i = 1, 2,$
\[ n_i \geq (n+1)/2 - 5\beta n - 7|W_i| \geq (n+1)/2 - 40\beta n \quad \text{and} \quad \delta(G_i) \geq \delta(G[V_i]) - 7|W_i| \geq (n+1)/2 - 40\beta n. \]

Let $i = 1, 2.$ We now show that $G_i$ contains a spanning subgraph isomorphic to either $H_1$ or $H_2$ as defined in the beginning of this section. Since $|n_i| \leq (1/2 + 5\beta)n$ and $\delta(G_i) \geq (n+1)/2 - 40\beta n,$ any subgraph of $G_i$ induced on at least $(1/4 - 40\beta)n$ vertices has minimum degree at least $(1/4 - 85\beta)n,$ and thus has a matching of size at least 2. Hence, when $n_i$ is even, we can choose independent edges $e_i = x_iy_i$ and $f_i = z_iw_i$ with
\[ x_i, y_i \in \Gamma_{G_i}(p_i) - \{q_i\} \quad \text{and} \quad z_i, w_i \in \Gamma_{G_i}(q_i) - \{p_i\}. \]

(Notice that $p_i$ or $q_i$ may be contained in $W_i,$ and in this case we have $\text{deg}_{G_i}(p_i), \text{deg}_{G_i}(q_i) \geq (1/4 - 40\beta)n.$) And if $n_i$ is odd, we can choose independent edges $g_iy_i$ and $f_i = z_iw_i$ with
\[ g_i, x_i \in \Gamma_{G_i}(p_i) - \{q_i\}, x_i \in \Gamma_{G_i}(g_i, y_i) - \{p_i, q_i\} \quad \text{and} \quad z_i, w_i \in \Gamma_{G_i}(q_i) - \{x_i, p_i\}, \]

where the existence of the vertex $x_i$ is possible since the subgraph of $G_i$ induced on $\Gamma_{G_i}(p_i)$ has minimum degree at least $(1/2 - 40\beta)n - ((1/2 + 5\beta)n - |\Gamma_{G_i}(p_i)|) \geq |\Gamma_{G_i}(p_i)| - 45\beta n,$ and hence contains a triangle. In this case, again, denote $e_i = x_iy_i.$ Let
\[
\begin{cases}
G'_i = G_i - \{p_i, q_i\}, & \text{if } n_i \text{ is even;} \\
G'_i = G_i - \{p_i, q_i, g_i\}, & \text{if } n_i \text{ is odd.}
\end{cases}
\]

By the definition above, $|V(G'_i)|$ is even.

The following claim is a modification of (1) of Lemma 2.2 in [17].
Claim 3.3.1. For $i = 1, 2$, let $a'_ib'_i, c'_di'_i \in E(G'_i)$ be two independent edges. Then $G'_i$ contains two vertex disjoint ladders $Q_{i1}$ and $Q_{i2}$ spanning on $V(G'_i)$ such that $Q_{i1}$ has $e_i = x_iy_i$ as its first rung, $a'_ib'_i$ as its last rung, and $Q_{i2}$ has $c'_di'_i$ as its first rung and $f_i = z_iw_i$ as its last rung, where $e_i$ and $f_i$ are defined prior to this claim.

Proof. We only show the claim for $i = 1$ as the case for $i = 2$ is similar. Notice that by the definition of $G'_1$, $|V(G'_1)|$ is even. Since $|V(G'_1)| \leq (1/2 + 5\beta)n$ and $\delta(G'_1) \geq (n+1)/2 - 40\beta n - 2 \geq |V(G'_1)|/2 + 8$, $G'_1$ has a perfect matching $M$ containing $e_1, f_1, a'_ib'_i, c'_di'_i$. We identify $a'_i$ and $c'_i$ into a vertex called $s'$, and identify $b'_i$ and $d'_i$ into a vertex called $t'$. Denote $G''_1$ as the resulting graph and let $s't' \in E(G''_1)$ if the two vertices are not adjacent. Partition $V(G''_1)$ arbitrarily into $U$ and $V$ with $|U| = |V|$ such that $x_1, z_1, s' \in U$, $y_1, w_1, t' \in V$, and let $M' := M - \{a'_ib'_i, c'_di'_i\} \cup \{s't'\} \subseteq E_G(U, V)$. Define an auxiliary graph $H'$ with vertex set $M'$ and edge set defined as follows. If $xy, uv \in M'$ with $x, u \in U$ then $xy \sim_{H'} uv$ if and only if $x \sim_{G'_1} v$ and $y \sim_{G'_1} u$ (we do not include the case that $x \sim_{G'_1} u$ and $y \sim_{G'_1} v$ as we defined a bipartition here). Particularly, for any $pq \in M' - \{s't'\}$ with $p \in U$, $pq \sim_{H'} s't'$ if and only if $p \sim_{G'_1} b'_i, d'_i$ and $q \sim_{G'_1} a'_i, c'_i$. Notice that a ladder with rungs in $M'$ is corresponding to a path in $H'$ and vice versa. Since $(1/2 - 40\beta)n - 2 \leq |V(G'_1)| \leq (1/2 + 5\beta)n - 2$ and $\delta(G'_1) \geq (n+1)/2 - 40\beta n - 2$, any two vertices in $G'_1$ has at least $(1/2 - 130\beta)n$ common neighbors. This together with the fact that $|U| = |V| \leq |V(G''_1)|/2 \leq (1/4 + 2.5\beta)n$ gives that $\delta(U, V), \delta(V, U) \geq (1/4 - 132.5\beta)n$. Hence

$$\delta(H') \geq (1/4 - 132.5\beta)n - ((1/4 + 2.5\beta)n - (1/4 - 132.5\beta)n) = (1/4 - 267.5\beta)n \geq |V(H')|/2 + 1,$$

since $\beta < 1/2200$ and $n$ is very large. Hence $H'$ has a hamiltonian path starting with $e_1$, ending with $f_1$, and having $s't'$ as an internal vertex. The path with $s't'$ replaced by $a'_ib'_i$ and $c'_di'_i$ is corresponding to the required ladders in $G'_1$. \hfill \Box

We may assume $n_1$ is even and $n_2$ is odd and construct a spanning Halin subgraph of $G$ (the construction for the other three cases follow a similar argument). Recall that $p_1p_2, q_1q_2, r_1r_2$ are the three prescribed independent edges between $G[V_1 \cup W_1]$ and $G[V_2 \cup W_2]$. 

where \( p_1, q_1, r_1 \in V_1 \cup W_1 \) and \( p_2, q_2, g_2, r_2 \in V_2 \cup W_2 \). For a uniform discussion, we may assume that both of the ladders \( L_1 \) and \( L_2 \) exist. Let \( i = 1, 2 \). Recall that \( L_i \) has \( a_ib_i \) as its first rung and \( c_id_i \) as its last rung. Choose \( a'_i \in \Gamma_{G'_i}(a_i), b'_i \in \Gamma_{G'_i}(b_i) \) such that \( a'_ib'_i \in E(G) \) and \( c'_i \in \Gamma_{G'_i}(c_i), d'_i \in \Gamma_{G'_i}(d_i) \) such that \( c'_id'_i \in E(G) \). This is possible as \( \delta(G'_i) \geq (n+1)/2 - 40\beta n - 2 \). Let \( Q_{1i} \) and \( Q_{2i} \) be the ladders of \( G'_i \) given by Claim 3.3.1. Set \( H_a = Q_{11}L_1Q_{12} \cup \{p_1x_1, p_1y_1, q_1z_1, q_1w_1\} \). Assume \( Q_{21}L_2Q_{22} \) is a ladder can be denoted as \( \overrightarrow{x_2y_2} - Q_{21}L_2Q_{22} - \overrightarrow{z_2w_2} \). To make \( r_2 \) a Halin constructible vertex, we let \( H_b = Q_{21}L_2Q_{22} \cup \{g_2x_2, g_2y_2, p_2g_2, p_2y_2, q_2z_2, q_2w_2\} \) if \( r_2 \) is on the shortest \((y_2, w_2)\)-path in \( Q_{21}L_2Q_{22} \), and let \( H_b = Q_{21}L_2Q_{22} \cup \{g_2x_2, g_2y_2, p_2g_2, p_2x_2, q_2z_2, q_2w_2\} \) if \( r_2 \) is on the shortest \((x_2, z_2)\)-path (recall that \( g_1, x_1, y_1 \in \Gamma_{G_1}(p_1) \)). Let \( H = H_a \cup H_b \cup \{p_1p_2, r_1r_2, q_1q_2\} \). Then \( H \) is a spanning Halin subgraph of \( G \) by Proposition 3.3.1 as \( H_a \cup \{p_1q_1\} \cong H_1 \) and \( H_b \cup \{p_2q_2\} \cong H_2 \). Figure 3.3 gives a construction of \( H \) for the above case when \( r_2 \) is on the shortest \((y_2, w_2)\)-path in \( Q_{21}L_2Q_{22} \).

Figure (3.3) A Halin graph \( H \)
3.3.3.2 Proof of Theorem 3.3.3  Recall Extremal Case 2: There exists a partition $V_1 \cup V_2$ of $V$ such that $|V_1| \geq (1/2 - 7\beta)n$ and $\Delta(G[V_1]) \leq \beta n$. Since $\delta(G) \geq (n+1)/2$, the assumptions imply that

$$(1/2 - 7\beta)n \leq |V_1| \leq (1/2 + \beta)n \quad \text{and} \quad (1/2 - \beta)n \leq |V_2| \leq (1/2 + 7\beta)n.$$ 

Let $\beta$ and $\alpha$ be real numbers satisfying $\beta \leq \alpha/20$ and $\alpha \leq (1/17)^3$. Set $\alpha_1 = \alpha^{1/3}$ and $\alpha_2 = \alpha^{2/3}$. We first repartition $V(G)$ as follows.

$$V'_2 = \{v \in V_2 \mid deg(v, V_1) \geq (1 - \alpha_1)|V_1|\}, V_{01} = \{v \in V_2 - V'_2 \mid deg(v, V'_2) \geq (1 - \alpha_1)|V'_2|\},$$

$$V'_1 = V_1 \cup V_{01}, \quad \text{and} \quad V_0 = V_2 - V'_2 - V_{01}.$$

Claim 3.3.2. $|V_{01}|, |V_0| \leq |V_2 - V'_2| \leq \alpha_2 |V_2|$. 

**Proof:** Notice that $e(V_1, V_2) \geq (1/2 - 7\beta)n|V_2| \geq \frac{1/2 - 7\beta}{1/2 + \beta}|V_1||V_2| \geq (1 - \alpha)|V_1||V_2|$ as $\beta \leq \alpha/20$. Hence,

$$(1 - \alpha)|V_1||V_2| \leq e(V_1, V_2) \leq e(V_1, V'_2) + e(V_1, V_2 - V'_2) \leq |V_1||V'_2| + (1 - \alpha_1)|V_1||V_2 - V'_2|.$$ 

This gives that $|V_2 - V'_2| \leq \alpha_2 |V_2|$, and thus $|V_{01}|, |V_0| \leq |V_2 - V'_2| \leq \alpha_2 |V_2|$. \hfill \Box

As a result of moving vertices from $V_2$ to $V_1$ and by Claim 3.3.2, we have the following.

$$\Delta(G[V'_1]) \leq \beta n + |V_{01}| \leq \beta n + \alpha_2 |V_2|,$$

$$\delta(V'_1, V'_2) \geq (1/2 - \beta)n - |V_2 - V'_2| \geq (1/2 - \beta)n - \alpha_2 |V_2|,$$

$$\delta(V'_2, V'_1) \geq (1 - \alpha_1)|V'_1| \geq (1 - \alpha_1)(1/2 - 7\beta)n, \quad (3.1)$$

$$\delta(V_0, V'_1) \geq (n + 1)/2 - (1 - \alpha_1)|V'_2| - |V_0| \geq 3\alpha_2 n + 8 \geq 3|V_0| + 10,$$

$$\delta(V_0, V'_2) \geq (n + 1)/2 - (1 - \alpha_1)|V_1| - |V_0| \geq 3\alpha_2 n + 8 \geq 3|V_0| + 10,$$

where the last two inequalities hold because we have $7\beta + 10/n \leq \alpha$, and $\alpha \leq (1/8)^3$. 
Claim 3.3.3. We may assume that $\Delta(G) < n - 1$.

Proof. Suppose on the contrary and let $w \in V(G)$ such that $\deg(w) = n - 1$. Then by $\delta(G) \geq (n + 1)/2$ we have $\delta(G - w) \geq (n - 1)/2$, and thus $G - w$ has a Hamiltonian cycle. This implies that $G$ has a spanning wheel subgraph, in particular, a spanning Halin subgraph of $G$. □

Claim 3.3.4. There exists a subgraph $T \subseteq G$ such that $|V(T)| \equiv n \pmod{2}$, where $T$ is isomorphic to some graph in $\{T_1, T_2, \ldots, T_5\}$. Assume that $T$ has head link $x_1x_2$ and tail link $y_1y_2$. Let $m = n - |V(T)|$. Then $G - V(T)$ contains a balanced spanning bipartite graph $G'$ with partite sets $U_1$ and $U_2$ and a subset $W$ of $U_1 \cup U_2$ with at most $\alpha_2n$ vertices such that the following holds:

(i) $\deg_{G'}(x, V(G') - W) \geq (1 - \alpha_1 - 2\alpha_2)m$ for all $x \notin W$;

(ii) There exists $x'_1x'_2, y'_1y'_2 \in E(G')$ such that $x'_i, y'_i \in U_i - W$, $x'_{3-i} \sim x_i$, and $y'_{3-i} \sim y_i$, for $i = 1, 2$; and if $T$ has a pendant vertex, then the vertex is contained in $V'_1 \cup V'_2 - W$.

(iii) There are $|W|$ vertex-disjoint $3$-stars ($K_{1,3}$s) in $G'' - \{x'_1, x'_2, y'_1, y'_2\}$ with the vertices in $W$ as their centers.

Proof. By (3.1), for $i = 1, 2$, we notice that for any $u, v, w \in V'_i$,

$$\deg(u, v, w, V'_{3-i}) \geq |V'_{3-i}| - 3(|V'_{3-i}| - \delta(V'_i, V'_{3-i})) \geq (1/2 - 28\beta - 3\alpha_1)n > n/4.$$(3.2)

We now separate the proof into two cases according to the parity of $n$.

Case 1. $n$ is even.

Suppose first that $\max\{|V'_1|, |V'_2|\} \leq n/2$. We arbitrarily partition $V_0$ into $V_{10}$ and $V_{20}$ such that $|V'_1 \cup V_{10}| = |V'_2 \cup V_{20}| = n/2$. Suppose $G[V'_1]$ contains an edge $x_1u_1$ and there is a vertex $u_2 \in \Gamma(u_1, V'_2)$ such that $u_2$ is adjacent to a vertex $y_2 \in V'_2$. By (3.2), there exist distinct vertices $x_2 \in \Gamma(x_1, u_1, V'_2) - \{y_2, u_1\}, y_1 \in \Gamma(y_2, u_2, V'_1) - \{x_1, u_1\}$. Then $G[\{x_1, u_1, x_2, y_1, u_1, y_2\}]$ contains a subgraph $T$ isomorphic to $T_1$. So we assume $G[V'_1]$...
contains an edge $x_1u_1$ and no vertex in $\Gamma(u_1,V_2')$ is adjacent to any vertex in $V_2'$. As $\delta(G) \geq (n + 1)/2$, $\delta(G[V_2' \cup V_2]) \geq 1$. Let $u_2 \in \Gamma(u_1,V_2')$ and $u_2y_2 \in E(G[V_2' \cup V_2])$.

Since $\deg(u_2,V_1') \geq (n + 1)/2 - |V_0| > |V_1' \cup V_{10}| - |V_0|$ and $\deg(y_2,V_1') \geq 3|V_0| + 10$, $\deg(u_2,y_2,V_1' \cup V_{10}) \geq 2|V_0| + 10$. Let $x_2 \in \Gamma(x_1,u_1,V_2') - \{y_2,u_2\}, y_1 \in \Gamma(y_2,u_2,V_1') - \{x_1,u_1\}$.

Then $G[\{x_1,u_1,x_2,y_1,u_2,y_2\}]$ contains a subgraph $T$ isomorphic to $T_1$. By symmetry, we can find $T \cong T_1$ if $G[V_2']$ contains an edge. Hence we assume that both $V_1'$ and $V_2'$ are independent sets. Again, as $\delta(G) \geq (n + 1)/2$, $\delta(G[V_1' \cup V_{10}]), \delta(G[V_2' \cup V_2]) \geq 1$. Let $x_1u_1 \in E(G[V_1' \cup V_{10}])$ and $y_2u_2 \in E(G[V_2' \cup V_2])$ such that $x_1 \in V_1'$ and $u_2 \in \Gamma(u_1,V_2')$.

Since $\deg(x_1,V_2') \geq (n + 1)/2 - |V_0| > |V_2' \cup V_2| - |V_0|$ and $\deg(u_1,V_2') \geq 3|V_0| + 10$, we have $\deg(x_1,u_1,V_2') \geq 2|V_0| + 10$. Hence, there exists $x_2 \in \Gamma(x_1,u_1,V_2') - \{y_2,u_2\}$. Similarly, there exists $y_1 \in \Gamma(y_2,u_2,V_1') - \{x_1,u_1\}$. Then $G[\{x_1,u_1,x_2,y_1,u_2,y_2\}]$ contains a subgraph $T$ isomorphic to $T_1$. Let $m = (n - 6)/2, U_1 = (V_1' - V(T)) \cup V_{10}$ and $U_2 = (V_2' - V(T)) \cup V_20$, and $W = V_0 - V(T)$. We then have $|U_1| = |U_2| = m$.

Let $G' = (V(G) - V(T), E_G(U_1,U_2))$ be the bipartite graph with partite sets $U_1$ and $U_2$. Notice that $|W| \leq |V_0| \leq \alpha_2|V_2| < \alpha_2 n$. By (3.1), we have $\deg_{G'}(x,V(G') - W) \geq (1 - \alpha_1 - 2\alpha_2)m$ for all $x \notin W$. This shows (i). By the construction of $T$ above, we have $x_1,y_1 \in V_1'$. Let $i = 1, 2$. By (3.1), we have $\delta(V_0, U_i - W) \geq 3|V_0| + 6$. Applying statement (i), we have $e_{G'}(\Gamma_G(x_1,U_2 - W), \Gamma_{G'}(x_2,U_1 - W)), e_{G'}(\Gamma_G(y_1,U_2 - W), \Gamma_{G'}(y_2,U_1 - W)) \geq (3|V_0| + 4)(1 - 2\alpha_1 - 4\alpha_2)m > 2m$. Hence, we can find independent edges $x'_1x'_2$ and $y'_1y'_2$ such that $x'_1, y'_1 \in U_i - W$, $x'_{3-i} \sim x_i$, and $y'_{3-i} \sim y_i$. This gives statement (ii). Finally, as $\delta(V_0, U_i - W) \geq 3|V_0| + 6$, we have $\delta(V_0, U_i - W - \{x'_1, x'_2, y'_1, y'_2\}) \geq 3|V_0| + 2$. Hence, there are $|W|$ vertex-disjoint 3-stars with their centers in $W$.

Otherwise we have $\max\{|V_1'|, |V_2'|\} > n/2$. Assume, w.l.o.g., that $|V_1'| \geq n/2 + 1$. Then $\delta(G[V_1']) \geq 2$ and thus $G[V_1']$ contains two vertex-disjoint paths isomorphic to $P_3$ and $P_2$, respectively. Let $m = (n - 8)/2$. We consider three cases here. Case (a): $|V_1'| - 5 \leq m$. Then let $x_1u_1w_1y_1v_1 \in G[V_1']$ be two vertex-disjoint paths, and let $x_2 \in \Gamma(x_1,u_1,V_2'), y_2 \in \Gamma(y_1,v_1,V_2')$ and $z \in \Gamma(w_1,v_1,V_2')$ be three distinct vertices.

Then $G[\{x_1,u_1,w_1,x_2,z,y_1,v_1,y_2\}]$ contains a subgraph $T$ isomorphic to $T_4$. Notice that
\(|V'_2 - V(T)| \leq m\). We arbitrarily partition \(V_0\) into \(V_{10}\) and \(V_{20}\) such that \(|V'_1 \cup V_{10}| = |V'_2 \cup V_{20}| = m\). Let \(U_1 = (V'_1 - V(T)) \cup V_{10},\) \(U_2 = (V'_2 - V(T)) \cup V_{20},\) and \(W = V_0\). Hence we assume \(|V'_1| - 5 = m + t_1\) for some \(t_1 \geq 1\). This implies that \(|V'_1| \geq n/2 + t_1 + 1\) and thus \(\delta(G[V'_1]) \geq t_1 + 2\). Let \(V'_1^0\) be the set of vertices \(u \in V'_1\) such that \(\text{deg}(u, V'_1) \geq \alpha_1 m\). Case (b): \(|V'_1^0| \geq |V'_1| - 5 - m\). Then we form a set \(W\) with \(|V'_1| - 5 - m\) vertices from \(V'_1^0\) and all the vertices of \(V_0\). Then \(|V'_1 - W| = m + 5 + t_1 - (|V'_1| - 5 - m) = m + 5 = n/2 + 1\), and hence \(\delta(G[V'_1 - W]) \geq 2\). Similarly as in Case (a), we can find a subgraph \(T\) of \(G\) contained in \(G[V'_1 - W]\) isomorphic to \(T_4\). Let \(U_1 = V'_1 - V(T) - W, U_2 = (V'_2 - V(T)) \cup W\). Then \(|U_1| = |U_2| = m\). Thus we have Case (c): \(|V'_1^0| < |V'_1| - 5 - m\). Suppose that \(|V'_1 - V'_1^0| = m + 5 + t'_1 = n/2 + t'_1 + 1\) for some \(t'_1 \geq 1\). This implies that \(\delta(G[V'_1 - V'_1^0]) \geq t'_1 + 2\). We show that \(G[V'_1 - V'_1^0]\) contains \(t'_1 + 2\) vertex-disjoint 3-stars. To see this, suppose \(G[V'_1 - V'_1^0]\) contains a subgraph \(M\) of at most \(s < t'_1 + 2\) 3-stars. By counting the number of edges between \(V(M)\) and \(V'_1 - V'_1^0 - V(M)\) in two ways, we get that \(t'_1 |V'_1 - V'_1^0 - V(M)| \leq e_{G-V'_1^0}(V(M), V'_1 - V'_1^0 - V(M)) \leq 4s\Delta(G[V'_1 - V'_1^0]) \leq 4s\alpha_1 m\). Since \(|V'_1 - V'_1^0| = m + 5 + t'_1 = n/2 + t'_1 + 1, |V'_1 - V'_1^0 - V(M)| \geq m - 3t'_1 \geq m - 6\alpha_2 m\), where the last inequality holds as \(|V'_1| \leq (1/2 + \beta)n + \alpha_2 |V'_2|\) implying that \(t'_1 \leq |V'_1| - m - 5 \leq 2\alpha_2 m\). This, together with the assumption that \(\alpha \leq (1/8)^3\) gives that \(s \geq t'_1 + 2\), showing a contradiction. Hence we have \(s \geq t'_1 + 2\). Let \(x_1 u_1 w_1\) and \(y_1 v_1\) be two paths taken from two 3-stars in \(M\). Then we can find a subgraph \(T\) of \(G\) isomorphic to \(T_4\) the same way as in Case (a). We take exactly \(t'_1\) 3-stars from the remaining ones in \(M\) and denote the centers of these stars by \(W'\). Let \(U_1 = V'_1 - V'_1^0 - V(T) - W', W = W' \cup V'_1^0 \cup V_0,\) and \(U_2 = (V'_2 - V(T)) \cup W\). Then \(|U_1| = |U_2| = m\).

For the partition of \(U_1\) and \(U_2\) in all the cases discussed in the paragraph above, we let \(G' = (V(G) - V(T), E_G(U_1, U_2))\) be the bipartite graph with partite sets \(U_1\) and \(U_2\). Notice that \(|W| \leq |V_0| \leq \alpha_2 n\) if Case (a) occurs, \(|W| \leq |V_0| + |V'_1| - m - 5 \leq (1/2 + \beta)n + |V_0| - n/2 \leq \alpha_2 n\) if Case (b) occurs, and \(|W| = |W' \cup V'_1^0 \cup V_0| = |V'_1 - U_1 - V(T)| + |V_0| \leq (1/2 + \beta)n - (1/2 - 4)n + |V_0| \leq \alpha_2 n\) if Case (c) occurs. Since \(\delta(V'_2, V'_1) \geq (1 - \alpha_1)|V_1|\) from (3.1) and \(|V'_1 - U_1| \leq 2\alpha_2 m\), we have \(\delta(U_2 - W, U_1 - W) \geq (1 - \alpha_1 - 2\alpha_2)m\). On the other hand, from (3.1),
\[ \delta(V'_1, V'_2) \geq (1/2 - \beta)n - \alpha_2|V_2|. \] This gives that \( \delta(U_1 - W, U_2 - W) \geq (1 - \alpha_1 - 2\alpha_2)m. \) Hence, we have \( \deg_{G'}(x, V(G') - W) \geq (1 - \alpha_1 - 2\alpha_2)m \) for all \( x \notin W. \) Applying statement (i), we have \( e_{G'}(\Gamma_{G'}(x_1, U_2 - W), \Gamma_{G'}(x_2, U_1 - W)), e_{G'}(\Gamma_{G'}(y_1, U_2 - W), \Gamma_{G'}(y_2, U_1 - W)) \geq (3|V_0| + 4)(1 - 2\alpha_1 - 4\alpha_2)m > 2m. \) Hence, we can find independent edges \( x'_i, y'_j \) such that \( x'_i, y'_j \in U_i - W, x'_{3-i} \sim x_i, \) and \( y'_{3-i} \sim y_i. \) By the construction of \( T, T \) is isomorphic to \( T_4, \) and the pendent vertex \( z \in V'_2 \subseteq V_1 \cup V'_2 - W. \) This gives statement (ii). Finally, as \( \delta(V_0, U_1 - W) \geq 3\alpha_2 n + 5 \geq 3|W| + 5, \) we have \( \delta(V_0, U_1 - W - \{x'_1, x'_2, y'_1, y'_2\}) \geq 3|W| + 1. \) By the definition of \( \alpha \), we have \( \delta(V_0, V'_1 - W - \{x'_1, x'_2, y'_1, y'_2\}) \geq \alpha_1 m - \alpha_2 n - 4 \geq 3|W|. \) For the vertices in \( W' \) in Case (c), we already know that there are vertex-disjoint 3-stars in \( G' \) with centers in \( W'. \) Hence, regardless of the construction of \( W, \) we can always find \( |W| \) vertex-disjoint 3-stars with their centers in \( W. \)

**Case 2.** \( n \) is odd.

Suppose first that \( \max\{|V'_1|, |V'_2|\} \leq (n + 1)/2 \) and let \( m = (n - 7)/2. \) We arbitrarily partition \( V_0 \) into \( V_{10} \) and \( V_{20} \) such that, w.l.o.g., say \( |V'_1 \cup V_{10}| = (n + 1)/2 \) and \( |V'_2 \cup V_{20}| = (n - 1)/2. \) We show that \( G[V'_1 \cup V_{10}] \) either contains two independent edges or is isomorphic to \( K_{1, (n-1)/2}. \) As \( \delta(G) \geq (n + 1)/2, \) we have \( \delta(G[V'_1 \cup V_{10}]) \geq 1. \) Since \( n \) is sufficiently large, \((n + 1)/2 > 3. \) Then it is easy to see that if \( G[V'_1 \cup V_{10}] \not\cong K_{1, (n-1)/2}, \) then \( G[V'_1 \cup V_{10}] \) contains two independent edges. Furthermore, we can choose two independent edges \( u_1 v_1 \) and \( v_1 v_1 \) such that \( u_1, v_1 \in V'_1. \) This is obvious if \( |V_{10}| \leq 1. \) So we assume \( |V_{10}| \geq 2. \) As \( \delta(V_0, V'_1) \geq 3|V_0| + 10, \) by choosing \( x_1, y_1 \in V_{10}, \) we can choose distinct vertices \( u_1 \in \Gamma(x_1, V'_1) \) and \( v_1 \in \Gamma(y_1, V'_1). \) Let \( x_2 \in \Gamma(x_1, u_1, V'_2), y_2 \in \Gamma(y_1, v_1, V'_2) \) and \( z \in \Gamma(u_1, v_1, V'_2). \) Then \( G[\{x_1, u_1, x_2, y_1, v_1, y_2, z\}] \) contains a subgraph \( T \) isomorphic to \( T_3. \)

We assume now that \( G[V'_1 \cup V_{10}] \) is isomorphic to \( K_{1, (n-1)/2}. \) Let \( u_1 \) be the center of the star \( K_{1, (n-1)/2}. \) Then each leave of the star has at least \((n-1)/2\) neighbors in \( V'_2 \cup V_{20}. \) Since \( |V'_2 \cup V_{20}| = (n - 1)/2, \) we have \( \Gamma(v, V'_2 \cup V_{20}) = V'_2 \cup V_{20} \) if \( v \in V'_1 \cup V_{10} - \{u_1\}. \) By the definition of \( V_0, \) \( \Delta(V_0, V'_1) < (1 - \alpha_1)|V_1| \) and \( \Delta(V_0, V'_2) < (1 - \alpha_1)|V'_2|, \) and so \( u_1 \in V'_1, V_{10} = \emptyset \) and \( V_{20} = \emptyset. \) We claim that \( V'_2 \) is not an independent set. Otherwise, by \( \delta(G) \geq (n + 1)/2, \) for each \( v \in V'_2, \) \( \Gamma(v, V'_1) = V'_1. \) This in turn shows that \( u_1 \) has degree
n − 1, showing a contradiction to Claim 2.3.3. So let \( y_2v_2 \in E(G[V'_2]) \) be an edge. Let \( w_1 \in \Gamma(v_2, V'_1) - \{u_1\} \) and \( w_1u_1x_1 \) the path containing \( w_1 \). Choose \( y_1 \in \Gamma(y_2, v_2, V'_1) - \{w_1, u_1, x_1\} \) and \( x_2 \in \Gamma(x_1, u_1, w_1, V'_2) - \{y_1, v_1\} \). Then \( G[\{x_1, u_1, x_2, w_1, v_2, y_1\}] \) contains a subgraph \( T \) isomorphic to \( T_2 \). Let \( U_1 = (V'_1 - V(T)) \cup V_0 \) and \( U_2 = (V'_2 - V(T)) \cup V_0 \) and \( W = V_0 - V(T) \). We have \( |U_1| = |U_2| = m \) and \( |W| \leq |V_0| \leq \alpha_2 n \).

Otherwise we have \( \max\{|V'_1|, |V'_2|\} \geq (n + 1)/2 + 1 \). Assume, w.l.o.g., that \( |V'_1| \geq (n + 1)/2 + 1 \). Then \( \delta(G[V'_1]) \geq 2 \) and thus \( G[V'_1] \) contains two independent edges. Let \( m = (n - 7)/2 \) and \( V'_0 \) be the set of vertices \( u \in V'_1 \) such that \( deg(u, V'_1) \geq \alpha_1 m \). We consider three cases here. Since \( |V'_1| \geq (n + 1)/2 + 1 > m + 4 \), we assume \( |V'_1| = m + 4 + t_1 \) for some \( t_1 \geq 1 \). Case (a): \( |V'_1| \geq |V'_2| - m - 4 \). Then we form a set \( W \) with \( |V'_1| - 4 - m \) vertices from \( V'_0 \) and all the vertices of \( V_0 \). Then \( |V'_1 - W| = m + 4 + t_1 - (|V'_1| - 4 - m) = m + 4 = (n + 1)/2 + 1 \). Then we have \( \delta(G[V'_1 - W]) \geq 2 \). Hence \( G[V'_1 - W] \) contains two independent edges. Let \( x_1u_1, y_1v_1 \subseteq E(G[V'_1 - W]) \) be two independent edges, and let \( x_2 \in \Gamma(x_1, u_1, V'_2), y_2 \in \Gamma(y_1, v_1, V'_2) \) and \( z \in \Gamma(w, u_1, V'_2) \) be three distinct vertices. Then \( G[\{x_1, x_2, u, y_1, v, y_2\}] \) contains a subgraph \( T \) isomorphic to \( T_3 \). Let \( U_1 = V'_1 - V(T) - W, U_2 = (V'_2 - V(T)) \cup W \). Then \( |U_1| = |U_2| = m \) and \( |W| \leq |V_0| + |V'_1 - U_1| \leq |V_2 - V'_2| + \beta n + 4\alpha_2 n \). Thus we have \( |V'_0| < |V'_1| - 4 - m \). Suppose that \( |V'_1 - V'_0| = m + 4 + t'_1 = (n + 1)/2 + t'_1 \) for some \( t'_1 \geq 1 \). This implies that \( \delta(G[V'_1 - V'_0]) \geq t'_1 + 1 \). Case (b): \( t'_1 \geq 2 \). We show that \( G[V'_1 - V'_0] \) contains \( t'_1 + 2 \) vertex-disjoint 3-stars. To see this, suppose \( G[V'_1 - V'_0] \) contains a subgraph \( M \) of at most \( s \) vertex disjoint 3-stars. We may assume that \( s < t'_1 + 2 \). Then we have \( (t_1 - 1)|V'_1 - V'_0 - V(M)| \leq e_{G - V'_2}(V(M), V'_1 - V'_0 - V(M)) \leq 4s\Delta(G[V'_1 - V'_0]) \). Since \( |V'_1 - V'_0| = m + 5 + t'_1 = (n + 1)/2 + t'_1, |V'_1 - V'_0 - V(M)| \geq m - 3t'_1 \geq m - 6\alpha_2 m, \) where the last inequality holds as \( |V'_1| \leq (1/2 + \beta)n + \alpha_2 |V_2| \) implying that \( t'_1 \leq |V'_1| - m - 5 \leq 2\alpha_2 m \). This, together with the assumption that \( \alpha \leq (1/8)^3 \) gives that \( s \geq t'_1 + 2 \), showing a contradiction. Hence we have \( s \geq t'_1 + 2 \). Let \( x_1u_1 \) and \( y_1v_1 \) be two paths taken from two 3-stars in \( M \), and we can find a subgraph \( T \) of \( G \) isomorphic to \( T_3 \) the same way as in Case (a). We take exactly \( t'_1 + 3 \) stars from the remaining ones in \( M \) and denote the centers of these stars by \( W' \). Let \( U_1 = V'_1 - V'_0 - V(T) - W', W = W' \cup V'_0 \cup V_0, \) and \( U_2 = (V'_2 - V(T)) \cup W \). Then
$|U_1| = |U_2| = m$. Case (c): $t'_1 = 1$. In this case, we let $m = (n - 9)/2$. If $G[V'_1 - V_1^0]$ contains a vertex adjacent to all other vertices in $V'_1 - V_1^0$, we take this vertex to $V'_2$. This gets back to Case (a). Hence, we assume that $G[V'_1 - V_1^0]$ has no vertex adjacent to all other vertices in $V'_1 - V_1^0$. Then by the assumptions that $\delta(G) \geq (n + 1)/2$ and $|V'_1 - V_1^0| = (n + 1)/2 + 1$, we can find two copies of vertex disjoint $P_3$s in $G[V'_1 - V_1^0]$. Let $x_1 u_1 w_1$ and $y_1 v_1 z_1$ be two $P_3$s in $G[V'_1]$. There exist distinct vertices $x_2 \in \Gamma(x_1, u_1, w_1, V'_2)$, $y_2 \in \Gamma(y_1, v_1, z_1, V'_2)$ and $z \in \Gamma(w_1, z_1, V'_2)$. Then $G\{x_1, u_1, w_1, y_1, v_1, z_1, y_2, z\}$ contains a subgraph $T$ isomorphic to $T_5$. Let $U_1 = V'_1 - V_1^0 - V(T)$, $W = V_1^0 \cup V_0$, and $U_2 = V'_2 - V(T)$. Then $|U_1| = |U_2| = m$.

For the partition of $U_1$ and $U_2$ in all the cases discussed in Case 2, we let $G' = (V(G) - V(T), E_G(U_1, U_2))$ be the bipartite graph with partite sets $U_1$ and $U_2$. Similarly as in Case 1, we can show that all the statements (i)-(iii) hold.

Let $W_1 = U_1 \cap W$ and $W_2 = U_2 \cap W$. For $i = 1, 2$, by the definition of $W$, we see that $\delta(W_i, U_i - \{x'_1, y'_1, x'_2, y'_2\}) \geq 3|W_i|$. And for any $u, v \in U_i$, $\Gamma(u, v, U_{3-i}) \geq 6|W_i|$, and for any $u, v, w \in U_i$, $\Gamma(u, v, w, U_{3-i}) \geq 7|W_i|$. By Lemma 3.4.2, we can find ladder $L_i$ spanning on $W_i$ and another $7|W_i| - 2$ vertices from $U_i - \{x'_1, x'_2, y'_1, y'_2\}$ if $W_i \neq \emptyset$. Denote $a_{1i}, a_{2i}$ and $b_{1i}, b_{2i}$ the first and last rungs of $L_i$ (if $L_i$ exists), respectively, where $a_{1i}, b_{1i} \in U_1$. Let

$$U'_1 = U_i - V(L_i), \quad m' = |U'_1| = |U'_2|, \quad \text{and} \quad G'' = G''(U'_1 \cup U'_2, E_G(U'_1, U'_2)).$$

Since $|W| \leq \alpha_2 n$, $m \geq (n - 9)/2$, and $n$ is sufficiently large, we have $1/n + 7|W| \leq 15\alpha_2 m$. As $\delta(G' - W) \geq (1 - \alpha_1 - 2\alpha_2)m$ and $\alpha \leq (1/17)^3$, we obtain the following:

$$\delta(G'') \geq 7m'/8 + 1.$$

Let $a_{2i}' \in \Gamma(a_{1i}, U'_2)$, $a_{1i}' \in \Gamma(a_{2i}, U'_1)$ such that $a_{1i}' a_{2i}' \in E(G)$; and $b_{2i}' \in \Gamma(b_{1i}, U'_2)$, $b_{1i}' \in \Gamma(b_{2i}, U'_1)$ such that $b_{1i}' b_{2i}' \in E(G)$. We have the claim below.

**Claim 3.3.5.** The balanced bipartite graph $G''$ contains three vertex-disjoint ladders $Q_1$, $Q_2$, and $Q_3$ spanning on $V(G'')$ such that the first rung of $Q_1$ is $x'_1 x'_2$ and the last rung of $Q_1$ is
the first rung of $Q_2$ is $b'_1 b_{21}$ and the last rung of $Q_2$ is $a'_1 a'_{22}$, the first rung of $Q_3$ is $b'_1 b'_{22}$ and the last rung of $Q_3$ is $y'_1 y'_2$.

**Proof.** Since $\delta(G'') \geq 7m'/8 + 5$, $G''$ has a perfect matching $M$ containing the following edges: $x'_1 x'_2, a'_1 a'_{21}, b'_1 b'_{21}, a'_1 a'_{22}, b'_1 b'_{22}, y'_1 y'_2$. We identify $a'_1$ and $b'_1$, $a'_{21}$ and $b'_{21}$, $a'_{12}$ and $b'_{12}$, and $a'_{22}$ and $b'_{22}$ as vertices called $c'_{11}$, $c'_{21}$, $c'_{12}$, and $c'_{22}$, respectively. Denote $G^* = G^*(U^*_1, U^*_2)$ as the resulting graph and let $c'_{11} c'_{21}, c'_{12} c'_{22} \in E(G^*)$ if they do not exist in $E(G^*)$. Denote $M' := M - \{a'_1 a'_{21}, b'_1 b'_{21}, a'_1 a'_{22}, b'_1 b'_{22}\} \cup \{c'_{11} c'_{21}, c'_{12} c'_{22}\}$. Define an auxiliary graph $H'$ on $M'$ as follows. If $xy, uv \in M'$ with $x, u \in U'_1$ then $xy \sim_{H'} uv$ if and only if $x \sim_{G^*} u$ and $y \sim_{G^*} u$. Particularly, for any $pq \in M' - \{c'_{11} c'_{21}, c'_{12} c'_{22}\}$ with $p \in U'_2$, $pq \sim_{H'} c'_{11} c'_{21}$ (resp. $pq \sim_{H'} c'_{12} c'_{22}$) if and only if $p \sim_{G^*} a'_{11}, b'_{11}$ and $q \sim_{G^*} a'_{21}, b'_{21}$ (resp. $p \sim_{G^*} a'_{12}, b'_{12}$ and $q \sim_{G^*} a'_{22}, b'_{22}$). Notice that there is a natural one-to-one correspondence between ladders with rungs in $M'$ and paths in $H'$. Since $\delta_{G^*}(U^*_1, U^*_2), \delta_{G^*}(U^*_2, U^*_1) \geq 3m'/4 + 1$, we get $\delta(H') \geq m'/2 + 1$. Hence $H'$ has a hamiltonian path starting with $x'_1 x'_2$, ending with $y'_1 y'_2$, and having $c'_{11} c'_{21}$ and $c'_{12} c'_{22}$ as two internal vertices. The path with the vertex $c'_{11} c'_{21}$ replaced by $a'_{11} a'_{21}$ and $b'_{11} b'_{21}$, and with the vertex $c'_{12} c'_{22}$ replaced by $a'_{12} a'_{22}$ and $b'_{12} b'_{22}$ is corresponding to the required ladders in $G''$. 

If $T \in \{T_1, T_2\}$, then

$$H = x_1 x_2 Q_1 L_1 Q_2 L_2 Q_3 y_1 y_2 \cup T.$$ 

is a spanning Halin subgraph of $G$. Suppose now that $T \in \{T_3, T_4, T_5\}$ and $z$ is the pendent vertex. Then $z \in V'_1 \cup V'_2 - W$ by Claim \[3.3.4\] and the definition of $U'_1$ and $U'_2$, we get $deg_G(z, U'_1), deg_G(z, U'_2) \geq (1 - \alpha_1 - 9\alpha_2)m > m'/2$. So $z$ has a neighbor on each side of the ladder $Q_1 L_1 Q_2 L_2 Q_3$. Let $H'$ be obtained from $x_1 x_2 Q_1 L_1 Q_2 L_2 Q_3 y_1 y_2 \cup T$ by suppressing the degree 2 vertex $z$. Then $H'$ is a Halin graph such that each vertex on one side of $Q_1 L_1 Q_2 L_2 Q_3$ is a degree 3 vertex on its underlying tree. Let $z'$ be a neighbor of $z$ such that $z'$ has degree 3 in the underlying tree of $H'$. Then

$$H = x_1 x_2 Q_1 L_1 Q_2 L_2 Q_3 y_1 y_2 \cup T \cup \{zz'\},$$
3.3.3.3 Proof of Theorem 3.4.3 We first show that $G$ contains a subgraph $T$ isomorphic to $T_1$ if $n$ is even and to $T_2$ if $n$ is odd. Then by showing that $G - V(T)$ contains a spanning ladder $L$ with its first rung adjacent to the head link of $T$ and its last rung adjacent to the tail link of $T$, we get a spanning Halin subgraph $H$ of $G$ formed by $L \cup T$.

Finding a subgraph $T$

Claim 3.3.6. Let $n$ be a sufficient large integer and $G$ an $n$-vertex graph with $\delta(G) \geq (n + 1)/2$. If $G$ is not in Extremal Case 2, then $G$ contains a subgraph $T$ isomorphic to $T_1$ if $n$ is even and to $T_2$ if $n$ is odd.

Proof. Suppose first that $n$ is even. Let $xy \in E(G)$ be an edge. We show that $G[N(x) - \{y\}]$ contains an edge $x_1x_2$ and $G[N(y) - \{x\}]$ contains an edge $y_1y_2$ such that the two edges are independent. Since $G$ is not in Extremal Case 2, it has no independent set of size at least $(1/2-7\beta)n$. Hence, we can find the two desired edges, and $G[\{x, y, x_1, x_2, y_1, y_2\}]$ contains a subgraph $T$ isomorphic to $T_1$. Then assume that $n$ is odd. We show in the first step that $G$ contains a subgraph isomorphic to $K_4^-$ ($K_4$ with one edge removed). Let $yz \in E(G)$. As $\delta(G) \geq (n + 1)/2$, there exists $y_1 \in \Gamma(y, z)$. If there exists $y_2 \in \Gamma(y, z) - \{y_1\}$, we are done. Otherwise, $(\Gamma(y) - \{y_1, y\}) \cap (\Gamma(z) - \{y_1, y\}) = \emptyset$. As $\delta(G) \geq (n + 1)/2$, $y_1$ is adjacent to a vertex $y_2 \in \Gamma(y) \cup \Gamma(z) - \{y_1, y, z\}$. Assume $y_2 \in \Gamma(z) - \{y_1, y\}$. Then $G[\{y, y_1, z, y_2\}]$ contains a copy of $K_4^-$. Choose $x \in \Gamma(y) - \{z, y_1, y_2\}$ and choose an edge $x_1x_2 \in G[\Gamma(x) - \{y, y_1, y_2, z\}]$. Then $G[\{y, y_1, z, y_2, x, x_1, x_2\}]$ contains a subgraph $T$ isomorphic to $T_2$. \(\square\)

Let $T$ be a subgraph of $G$ as given by Claim 3.3.6. Suppose the head link of $T$ is $x_1x_2$ and the tail link of $T$ is $y_1y_2$. Let $G' = G - V(T)$. We show in next section that $G'$ contains a spanning ladder with first rung adjacent to $x_1x_2$ and its last rung adjacent to $y_1y_2$. Let $n' = |V(G')|$. Then we have $\delta(G') \geq (n + 1)/2 - 7 \geq n'/2 - 4 \geq (1/2 - \beta)n'$.

Finding a spanning ladder of $G'$ with prescribed end rungs
Theorem 3.3.5. Let $n'$ be a sufficiently large even integer and $G'$ the subgraph of $G$ obtained by removing vertices in $T$. Suppose that $\delta(G') \geq (1/2 - \beta)n'$ and $G = G[V(G') \cup V(T)]$ is in Non-extremal case, then $G'$ contains a spanning ladder with first rung adjacent to $x_1x_2$ and its last rung adjacent to $y_1y_2$.

**Proof.** We fix the following sequence of parameters

$$0 < \varepsilon \ll d \ll \beta \ll 1$$

and specify their dependence as the proof proceeds.

Let $\beta$ be the parameter defined in the two extremal cases. Then we choose $d \ll \beta$ and choose

$$\varepsilon = \frac{1}{4} \varepsilon(d/2, 3, 2, d/4)$$

following the definition of $\varepsilon$ in the Blow-up Lemma.

Applying the Regularity Lemma to $G'$ with parameters $\varepsilon$ and $d$, we obtain a partition of $V(G')$ into $\ell + 1$ clusters $V_0, V_1, \cdots, V_\ell$ for some $\ell \leq M \leq M(\varepsilon)$, and a spanning subgraph $G''$ of $G'$ with all described properties in the Regularity Lemma. In particular, for all $v \in V(G')$,

$$\deg_{G''}(v) > \deg_{G'}(v) - (d + \varepsilon)n' \geq (1/2 - \beta - \varepsilon - d)n' \geq (1/2 - 2\beta)n' \quad (3.3)$$

provided that $\varepsilon + d \leq \beta$. On the other hand,

$$e(G'') \geq e(G') - \frac{(d + \varepsilon)^2}{2}(n')^2 > e(G') - d(n')^2$$

by $\varepsilon < d$.

We further assume that $\ell = 2k$ is even; otherwise, we eliminate the last cluster $V_\ell$ by removing all the vertices in this cluster to $V_0$. As a result, $|V_0| \leq 2\varepsilon n'$, and

$$ (1 - 2\varepsilon)n' \leq \ell N = 2kN \leq n', \quad (3.4)$$
where \( N = |V_i| \) for \( 1 \leq i \leq \ell \).

For each pair \( i \) and \( j \) with \( 1 \leq i \neq j \leq \ell \), we write \( V_i \sim V_j \) if \( d(V_i, V_j) \geq d \). As in other applications of the Regularity Lemma, we consider the reduced graph \( G_r \), whose vertex set is \( \{1, 2, \cdots, r \} \) and two vertices \( i \) and \( j \) are adjacent if and only if \( V_i \sim V_j \). From \( \delta(G') > (1/2 - 2\beta)n' \), we claim that \( \delta(G_r) \geq (1/2 - 2\beta)\ell \). Suppose not, and let \( i_0 \in V(G_r) \) be a vertex with \( \text{deg}_{G_r}(i_0) < (1/2 - 2\beta)\ell \). Let \( V_{i_0} \) be the cluster in \( G \) corresponding to \( i_0 \). Then we have

\[
(1/2 - \beta)n'|V_{i_0}| \leq |E_{G'}(V_{i_0}, V - V_{i_0})| < (1/2 - 2\beta)\ell N|V_{i_0}| + 2\varepsilon n'|V_{i_0}| < (1/2 - \beta)n'|V_{i_0}|
\]

This gives a contradiction by \( \ell N \leq n' \) from inequality (3.4).

Let \( x \in V(G') \) be a vertex and \( A \) a cluster. We say \( x \) is typical to \( A \) if \( \text{deg}(x, A) \geq (d - \varepsilon)|A| \), and in this case, we write \( x \sim A \).

**Claim 3.3.7.** Each vertex in \( \{x_1, x_2, y_1, y_2\} \) is typical to at least \( (1/2 - 2\beta)\ell \) clusters in \( \{V_1, \cdots, V_\ell \} \).

**Proof.** Suppose on the contrary that there exists \( x \in \{x_1, x_2, y_1, y_2\} \) such that \( x \) is typical to less than \( (1/2 - 2\beta)\ell \) clusters in \( \{V_1, \cdots, V_\ell \} \). Then we have \( \text{deg}_{G'}(x) < (1/2 - 2\beta)\ell N + (d + \varepsilon)n' \leq (1/2 - \beta)n' \) by \( \ell N \leq n' \) and \( d + \varepsilon \leq \beta \).

Let \( x \in V(G') \) be a vertex. Denote by \( \mathcal{V}_x \) the set of clusters to which \( x \) typical.

**Claim 3.3.8.** There exist \( V_{x_1} \in \mathcal{V}_{x_1} \) and \( V_{x_2} \in \mathcal{V}_{x_2} \) such that \( d(V_{x_1}, V_{x_2}) \geq d \).

**Proof.** We show the claim by considering two cases based on the size of \( |\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}| \).

Case 1. \( |\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}| \leq 2\beta l \).

Then we have \( |\mathcal{V}_{x_1} \setminus \mathcal{V}_{x_2}| \geq (1/2 - 4\beta)l \) and \( |\mathcal{V}_{x_2} \setminus \mathcal{V}_{x_1}| \geq (1/2 - 4\beta)l \). We conclude that there is an edge between \( \mathcal{V}_{x_1} \setminus \mathcal{V}_{x_2} \) and \( \mathcal{V}_{x_2} \setminus \mathcal{V}_{x_1} \) in \( G_r \). For otherwise, let \( \mathcal{U} \) be the union of clusters in \( \mathcal{V}_{x_1} \cap \mathcal{V}_{x_2} \). Then \( |V_0 \cup \mathcal{U} \cup V(T)| \leq 5\beta n \) is a vertex-cut of \( G \), implying that \( G \) is in Extremal Case 1.

Case 2. \( |\mathcal{V}_{x_1} \cap \mathcal{V}_{x_2}| > 2\beta l \).
We may assume that \( V_{x_1} \cap V_{x_2} \) is an independent set in \( G_r \). For otherwise, we are done by finding an edge within \( V_{x_1} \cap V_{x_2} \). Also we may assume that \( E_G(V_{x_1} \cap V_{x_2}, V_{x_1} - V_{x_2}) = \emptyset \) and \( E_G(V_{x_1} \cap V_{x_2}, V_{x_2} - V_{x_1}) = \emptyset \). Since \( \delta(G_r) \geq (1/2 - 2\beta)l \) and \( \delta(G_r)(V_{x_1} \cap V_{x_2}, V_{x_1} \cup V_{x_2}) = 0 \), we know that \( l - |V_{x_1} \cup V_{x_2}| \geq (1/2 - 2\beta)l \). Hence, \( |V_{x_1} \cup V_{x_2}| = |V_{x_1}| + |V_{x_2}| - |V_{x_1} \cap V_{x_2}| \leq (1/2 + 2\beta)l \). This gives that \( |V_{x_1} \cap V_{x_2}| \geq |V_{x_1}| + |V_{x_2}| - (1/2 + 2\beta)l \geq (1/2 - 2\beta)l + (1/2 - 2\beta)l - (1/2 + 2\beta)l \geq (1/2 - 6\beta)l \). Let \( \mathcal{U} \) be the union of clusters in \( V_{x_1} \cap V_{x_2} \). Then \( |\mathcal{U}| \geq (1/2 - 7\beta)n \) and \( \Delta(G[\mathcal{U}]) \leq (d + \varepsilon)n' \leq \beta n \). This shows that \( G \) is in Extremal Case 2.

Similarly, we have the following claim.

**Claim 3.3.9.** There exist \( V_{y_1} \in V_{y_1} - \{V_{x_1}, V_{x_2}\} \) and \( V_{y_2} \in V_{y_2} - \{V_{x_1}, V_{x_2}\} \) such that \( d(V_{y_1}, V_{y_2}) \geq d \).

**Claim 3.3.10.** The reduced graph \( G_r \) has a hamiltonian path \( X_1Y_1 \cdots X_kY_k \) such that \( \{X_1, Y_1\} = \{V_{x_1}, V_{x_2}\} \) and \( \{X_k, Y_k\} = \{V_{y_1}, V_{y_2}\} \).

**Proof:** We contract the edges \( V_{x_1}V_{x_2} \) and \( V_{y_1}V_{y_2} \) in \( G_r \). Denote the two new vertices as \( V'_x \) and \( V'_y \) respectively, and denote the resulting graph as \( G'_r \). Then we show that \( G'_r \) contains a hamiltonian \( (V'_x, V'_y) \)-path. This path is corresponding to a required hamiltonian path in \( G_r \).

To show \( G'_r \) has a hamiltonian \( (V'_x, V'_y) \)-path, we need the following generalized version of a result due to Nash-Williams [44]: Let \( Q \) be a 2-connected graph of order \( m \). If \( \delta(Q) \geq \max\{(m+2)/3+1, \alpha(Q)+1\} \), then \( Q \) is hamiltonian connected, where \( \alpha(Q) \) is the size of a largest independent set of \( Q \).

We claim that \( G'_r \) is \( 2\beta l \)-connected. For otherwise, let \( S \) be a vertex-cut of \( G'_r \) with \( |S| < 2\beta l \) and \( S \) the vertex set corresponding to \( S \) in \( G \). Then \( |S \cup V_0 \cup V(T)| \leq 2\beta n' + 2\varepsilon n' < 5\beta n \), showing that \( G \) is in Extremal Case 1. Since \( n' = NL + |V_0| \leq (l+2)\varepsilon n' \), we have \( l \geq 1/\varepsilon - 2 \geq 1/\beta \). Hence, \( G'_r \) is 2-connected. As \( G \) is not in Extremal Case 2, \( \alpha(G'_r) \leq (1/2 - 7\beta)l \). By \( \delta(G_r) \geq (1/2 - 2\beta)l \), we have \( \delta(G'_r) \geq (1/2 - 2\beta)l - 2 \geq \max\{(l+2)/3+1, (1/2 - 7\beta)l + 1\} \). Thus, by the result on hamiltonian connectedness given above, we know that \( G'_r \) contains a hamiltonian \( (V'_x, V'_y) \)-path.
Following the order of the clusters on the hamiltonian path given in Claim 3.3.10 for $i = 1, 2, \cdots, k$, we call $X_i, Y_i$ partners of each other and write $P(X_i) = Y_i$ and $P(Y_i) = X_i$. 

Claim 3.3.11. For each $1 \leq i \leq k$, there exist $X'_i \subseteq X_i$ and $Y'_i \subseteq Y_i$ such that $(X'_i, Y'_i)$ is $(2\varepsilon, d-3\varepsilon)$-super-regular, $|Y'_i| = |X'_i| + 1$, $|Y'_k| = |X'_k| + 1$, and $|X'_i| = |Y'_i|$ for $2 \leq i \leq k-1$. Additionally, each pair $(Y'_i, X'_{i+1})$ is $2\varepsilon$-regular with density at least $d-\varepsilon$ for $i = 1, 2, \cdots, k$, where $X'_{k+1} = X'_1$.

Proof. For each $1 \leq i \leq k$, let

$$X''_i = \{x \in X_i \mid \text{deg}(x, Y_i) \geq (d-\varepsilon)N\},$$

$$Y''_i = \{y \in Y_i \mid \text{deg}(y, X_i) \geq (d-\varepsilon)N\}.$$

If necessary, we either take a subset $X'_i$ of $X''_i$ or take a subset $Y'_i$ of $Y''_i$ such that $|Y'_i| = |X'_i| + 1$, $|Y'_k| = |X'_k| + 1$, and $|X'_i| = |Y'_i|$ for $2 \leq i \leq k-1$. Since $(X_i, Y_i)$ is $\varepsilon$-regular, we have $|X'_i|, |Y''_i| \geq (1-\varepsilon)N$. This gives that $|X'_i|, |X'_k| \geq (1-\varepsilon)N - 1$ and $|X'_i| = |Y'_i| \geq (1-\varepsilon)N$ for $2 \leq i \leq k-1$. As a result, we have $\text{deg}(x, Y'_i) \geq (d-2\varepsilon)N$ for each $x \in X'_i$ and $\text{deg}(y, X'_i) \geq (d-2\varepsilon)N - 1 \geq (d-3\varepsilon)N$ for each $y \in Y'_i$. By Slicing lemma (Lemma 3.2.5), $(X'_i, Y'_i)$ is $2\varepsilon$-regular. Hence $(X'_i, Y'_i)$ is $(2\varepsilon, d-3\varepsilon)$-super-regular for each $1 \leq i \leq k$. By Slicing lemma again, we know that $(X'_i, Y'_{i+1})$ is $2\varepsilon$-regular with density at least $d-\varepsilon$. 

For $1 \leq i \leq k$, we call $(X'_i, Y'_i)$ a super-regularized cluster (sr-cluster). Denote $R = V_0 \cup \bigcup_{i=1}^{k} (X_i \cup Y_i - (X'_i \cup Y'_i))$. Since $|(X_i \cup Y_i) - (X'_i \cup Y'_i)| \leq 2\varepsilon N$ for $2 \leq i \leq k-1$ and $|(X_1 \cup Y_1) - (X'_1 \cup Y'_1)|, |(X_k \cup Y_k) - (X'_k \cup Y'_k)| \leq 2\varepsilon N + 1$, we have $|R| \leq 2\varepsilon n + 2k\varepsilon N + 2 \leq 3\varepsilon n'$. As $n'$ is even and $|X'_1| + |Y'_1| + \cdots + |X'_k| + |Y'_k|$ is even, we know $|R|$ is even. We arbitrarily group vertices in $R$ into $|R|/2$ pairs. Given two vertices $u, v \in R$, we define a $(u, v)$-chain of length $2t$ as distinct clusters $A_1, B_1, \cdots, A_t, B_t$ such that $u \sim A_1 \sim B_1 \sim \cdots \sim A_t \sim B_t \sim v$ and each $A_j$ and $B_j$ are partners, in other words, $\{A_j, B_j\} = X_{i_j}, Y_{i_j}$ for some $i_j \in \{1, \cdots, k\}$. We call such a chain of length $2t$ a $2t$-chain.

Claim 3.3.12. For each pair $(u, v)$ in $R$, we can find a $(u, v)$-chain of length at most 4 such that every sr-cluster is used in at most $d^2N/5$ chains.
Proof. Suppose we have found chains for the first $m < 2\varepsilon n'$ pairs of vertices in $R$ such that no sr-cluster is contained in more than $d^2 N/5$ chains. Let $\Omega$ be the set of all sr-clusters that are used exactly by $d^2 N/5$ chains. Then

$$\frac{d^2 N}{5} |\Omega| \leq 4m < 8\varepsilon n' \leq 8\varepsilon \frac{2kN}{1 - 2\varepsilon},$$

where the last inequality follows from (3.4). Therefore,

$$|\Omega| \leq \frac{80k\varepsilon}{d^2 (1 - 2\varepsilon)} \leq \frac{80\varepsilon}{d^2} \leq \beta l/2,$$

provided that $1 - 2\varepsilon \geq 1/2$ and $80\varepsilon \leq d^2 \beta / 2$.

Consider now a pair $(w, z)$ of vertices in $R$ which does not have a chain found so far, we want to find a $(w, z)$-chain using sr-clusters not in $\Omega$. Let $\mathcal{U}$ be the set of all sr-clusters adjacent to $w$ but not in $\Omega$, and let $\mathcal{V}$ be the set of all sr-clusters adjacent to $z$ but not in $\Omega$. We claim that $|\mathcal{U}|, |\mathcal{V}| \geq (1/2 - 2\beta)l$. To see this, we first observe that any vertex $x \in R$ is adjacent to at least $(1/2 - 3\beta/2)l$ sr-clusters. For instead,

$$(1/2 - \beta)n' \leq \deg_G(x) < (1/2 - 3\beta/2)lN + (d - 2\varepsilon)lN + 3\varepsilon n',$$

$$\leq (1/2 - 3\beta/2 + d + 2\varepsilon)n'$$

$$< (1/2 - \beta)n' \text{ (provided that } d + 2\varepsilon < \beta/2),$$

showing a contradiction. Since $|\Omega| \leq \beta l/2$, we have $|\mathcal{U}|, |\mathcal{V}| \geq (1/2 - 2\beta)l$. Let $P(\mathcal{U})$ and $P(\mathcal{V})$ be the set of the partners of clusters in $\mathcal{U}$ and $\mathcal{V}$, respectively. By the definition of the chains, a cluster $A \in \Omega$ if and only its partner $P(A) \in \Omega$. Hence, $(P(\mathcal{U}) \cup P(\mathcal{V})) \cap \Omega = \emptyset$. Notice also that each cluster has a unique partner, and so we have $|P(\mathcal{U})| = |\mathcal{U}| \geq (1/2 - 2\beta)l$ and $|P(\mathcal{V})| = |\mathcal{V}| \geq (1/2 - 2\beta)l$.

If $E_G(P(\mathcal{U}), P(\mathcal{V})) \neq \emptyset$, then there exist two adjacent clusters $B_1 \in P(\mathcal{U}), A_2 \in P(\mathcal{V})$. If $B_1$ and $A_2$ are partners of each other, then $w \sim A_2 \sim B_1 \sim z$ gives a $(w, z)$-chain of length 2. Otherwise, assume $A_1 = P(B_1)$ and $B_2 = P(A_2)$, then $w \sim A_1 \sim B_1 \sim A_2 \sim B_2 \sim z$. If their neighborhood is not contained in more than $d^2 N/5$ chains, then $A_1 \sim B_1 \sim A_2 \sim B_2 \sim z$ gives a $(w, z)$-chain of length 2.
Claim 3.3.13. holds.

Given by Claim 3.3.12. By the definition of 2-chains and 4-chains, we have the following.

For each $i$, $j$ such that $i, j \leq k$, let $R_i$ denote the set of vertices in $R$ using $Z$ in the 2-chains and $R_4$ denote the set of vertices in $R$ using $Z$ in the 4-chains given by Claim 3.3.12. By the definition of 2-chains and 4-chains, we have the following holds.

**Claim 3.3.13.** For each $i = 1, 2, \ldots, k$, if $R_2(X_i^t) \neq \emptyset$, then $|R_2(X_i^t)| = |R_2(Y_i^t)|$; and if $R_4(X_i^t) \neq \emptyset$, then $|R_4(X_i^t)| = |R_4(Y_i^t)|$.

**Claim 3.3.14.** For each $i = 1, 2, \ldots, k$, if $R_2(X_i^t) \neq \emptyset$, then there exist vertex-disjoint ladders $L_{2x}^i$ and $L_{2y}^i$ covering all vertices in $R_2(X_i^t) \cup R_2(Y_i^t)$ such that $|X_i^t \cap V(L_{2x}^i \cup L_{2y}^i)| = |Y_i^t \cap V(L_{2x}^i \cup L_{2y}^i)|$; and if $R_4(X_i^t) \neq \emptyset$, then there exist three vertex disjoint ladders $L_{4x}^i, L_{4xy}^i, L_{4y}^i$ covering all vertices in $R_4(X_i^t) \cup R_4(Y_i^t)$ such that $V(L_{4x}^i) \subseteq X_i^t \cup Y_i^t$, $V(L_{4xy}^i) \subseteq Y_i^t \cup X_{i+1}^t$, and $V(L_{4y}^i) \subseteq X_{i+1}^t \cup Y_{i+1}^t$, and that $|X_i^t \cap V(L_{4x}^i \cup L_{4xy}^i)| = |Y_i^t \cap V(L_{4x}^i \cup L_{4xy}^i \cup L_{4y}^i)| = |Y_{i+1}^t \cap V(L_{4x}^i \cup L_{4xy}^i \cup L_{4y}^i)|$.

**Proof.** Notice that by Claim 3.3.11 (3.3.12) $(X_i^t, Y_i^t)$ is $(2\varepsilon, d - 3\varepsilon)$-super-regular and $(Y_i^t, X_{i+1}^t)$ is $2\varepsilon$-regular. Assume $R_2(X_i^t) \neq \emptyset$. By Claim 3.3.12 and Claim 3.3.13, we have $|R_2(X_i^t)| = |R_2(Y_i^t)| \leq d^2 N/5$. Let $R_2(X_i^t) = \{x_1, \ldots, x_r\}$. For each $j = 1, \ldots, r$, since $|\Gamma(x_j, X_i^t)| \geq (d - 2\varepsilon)|X_i^t| > 2\varepsilon|X_i^t|$, by Lemma 3.2.4, there exists a vertex set $B_j \subseteq Y_i^t$ with $|B_j| \geq (1 - 2\varepsilon)|Y_i^t|$ such that $B_j$ is atypical to $\Gamma(x_j, X_i^t)$. If $r \geq 2$, for
such that \( b = 1 \) when \( r | u \) distinct vertices \( y \cdot b \) vertex and it is easy to check that \( L \). Let \( X_i \) be the graph with

\[
V(L_{2x}) = R_2(X'_i) \cup \{x_1, x_2, x_3, y_{12}', y_{23}', z, u, x_1, x_2, x_3, y_{12}', y_{23}' \mid 1 \leq i \leq r-1 \}
\]

and

\[
E(L_{2x}) \text{ consisting of the edges } x_r x_1, x_r x_2, x_r x_3, y_{12}' x_r, y_{12}' x_r, y_{23}' x_r, y_{23}' x_r\text{ and the edges indicated below for each } 1 \leq i \leq r-1:
\]

\[
x_i \sim x_{i1}, x_{i2}, x_{i3}; y_{12}' \sim x_{i1}, x_{i2}; y_{23}' \sim x_{i2}, x_{i3}; z_i \sim x_{i3}, x_{i+1,1}; u_i \sim x_{i3}, x_{i+1,1}, z_i.
\]

It is easy to check that \( L_{2x}^i \) is a ladder spanning on \( R_2(X'_i) \), \( 4|R_2(X'_i)| - 1 \) vertices from \( X'_i \) and \( 3|R_2(X'_i)| - 1 \) vertices from \( Y'_i \). Similarly, we can find a ladder \( L_{2y}^i \) spanning on \( R_2(Y'_i) \), \( 4|R_2(Y'_i)| - 1 \) vertices from \( X'_i \) and \( 3|R_2(X'_i)| - 1 \) vertices from \( X'_i \). Clearly, we have

\[
|X'_i \cap V(L_{2x}^i \cup L_{2y}^i)| = |Y'_i \cap V(L_{2x}^i \cup L_{2y}^i)|.
\]

Assume now that \( R_4(X'_i) \neq \emptyset \). Then by Claim 3.3.12, we have \( |R_4(X'_i)| = |R_4(Y'_i)| \).

By the similar argument as above, we can find ladder \( L_{4x}^i, L_{4y}^{i+1} \) such that \( R_4(X'_i) \subseteq \)
$V(L_{4x}^i), R_4(Y_{i+1}^*) \subseteq V(L_{4y}^{i+1})$. Furthermore, we have

\begin{align*}
|X_i' \cap V(L_{4x}^i)| &= 4|R_4(X_i')| - 1, \quad |Y_i' \cap V(L_{4x}^i)| = 3|R_4(X_i')| - 1; \\
|Y_{i+1}' \cap V(L_{4y}^{i+1})| &= 4|R_4(Y_{i+1}')| - 1, \quad |X_{i+1}' \cap V(L_{4y}^{i+1})| = 3|R_4(Y_{i+1}')| - 1.
\end{align*}

Finally, we claim that we can find a ladder $L_{4xy}^i$ between $(Y_i', X_{i+1}')$ such that $|Y_i' \cap V(L_{4xy}^i)| = |X_{i+1}' \cap V(L_{4xy}^i)| = |R_4(Y_{i+1}')|$ and is vertex-disjoint from $L_{4x}^i \cup L_{4y}^{i+1}$. Since $3|R_4(Y_{i+1}')| \leq 3d^2N/5$ and $(Y_i', X_{i+1}')$ is $2\varepsilon$-regular with density at least $d - \varepsilon$ by Claim 3.3.11, a similar argument as in the proof of Lemma 3.3.11, we can find $Y_i'' \subseteq Y_i' - V(L_{4x}^i)$ and $X_{i+1}'' \subseteq X_{i+1}' - V(L_{4y}^{i+1})$ such that $(Y_i'', X_{i+1}'')$ is $(4\varepsilon, d - 5\varepsilon)$-super-regular and $|Y_i''| = |X_{i+1}''|$, and thus is $(4\varepsilon, d/2$)-super-regular (provided that $\varepsilon \leq d/10$). Notice that there are at least $(d - 9\varepsilon)|Y_i''| \geq d|Y_i''|/4$ vertices typical to $X_{i+1}''$, and there are at least $(d - 9\varepsilon)|X_{i+1}''| \geq d^2|X_{i+1}''|/4$ vertices typical to $Y_i''$. Applying the Below-up Lemma (Lemma 3.2.2), we can find a ladder $L_{4xy}^i$ within $(Y_i'', X_{i+1}'')$ such that $|Y_i'' \cap V(L_{4xy}^i)| = |X_{i+1}'' \cap V(L_{4xy}^i)| = |R_4(Y_{i+1}')|$. It is routine to check that $L_{4x}^i, L_{4y}^{i+1}, L_{4xy}^i$ are the desired ladders.

For each $i = 1, 2, \cdots, k$, let $X_i^{**} = X_i' - V(L_{2x}^i \cup L_{2y}^i \cup L_{4x}^i \cup L_{4y}^i \cup L_{4xy}^i)$ and $Y_i^{**} = Y_i' - V(L_{2x}^i \cup L_{2y}^i \cup L_{4x}^i \cup L_{4y}^i \cup L_{4xy}^i)$. Using Lemma 3.2.3 for $i \in \{1, \cdots, k-1\}$, choose $y_i^* \in Y_i^{**}$ such that $|A_{i+1}| \geq dN/4$, where $A_{i+1} := X_{i+1}^{**} \cap \Gamma(y_i^*)$. This is possible, as $(Y_i^{**}, X_{i+1}^{**})$ is $4\varepsilon$-regular (applying Slicing lemma based on $(Y_i', X_{i+1}')$). Similarly, choose $x_{i+1}^* \in A_{i+1}$ such that $|B_i| \geq dN/4$, where $B_i := Y_i^{**} \cap \Gamma(x_{i+1}^*)$. Let $S = \{y_i^*, x_{i+1}^* \mid 1 \leq i \leq k-1\}$, and let $X_i^* = X_i^{**} - S$ and $Y_i^* = Y_i^{**} - S$. We have the following holds.

Claim 3.3.15. For each $i = 1, 2, \cdots, k$, $(X_i^*, Y_i^*)$ is $(4\varepsilon, d/2$)-super-regular such that $|Y_i^*| = |X_i^*| + 1$, $|Y_k^*| = |X_k^*| + 1$, and $|X_i^*| = |Y_i^*|$ for $2 \leq i \leq k - 1$.

Proof. Since $|R_2(X_i')|, |R_4(Y_{i+1}')| \leq d^2N/5$ for each $i$, we have $|X_i^*|, |Y_i^*| \geq (1 - \varepsilon - d^2)N - 1$. As $\varepsilon, d \ll 1$, we can assume that $1 - \varepsilon - d^2 - 1/N < 1/2$. Thus, by Slicing lemma based on the $2\varepsilon$-regular pair $(X_i', Y_i')$, we know that $(X_i^*, Y_i^*)$ is $4\varepsilon$-regular. Recall from Claim 3.3.11 that $(X_i', Y_i')$ is $(2\varepsilon, d - 3\varepsilon$)-super-regular, as $4|R_2(X_i')|, 4|R_4(Y_{i+1}')| < d^2|Y_i^*|$, we know that for each $x \in X_i^*$, $\deg(x, Y_i^*) \geq (d - 3\varepsilon - d^2)|Y_i^*| > d|Y_i^*|/2$. Similarly, we
have for each $y \in Y^*_i$, $\deg(y, X^*_i) \geq d|X^*_i|/2$. Thus $(X^*_i, Y^*_i)$ is $(4\varepsilon, d/2)$-super-regular. Finally, Combining Claims 3.3.11 and 3.3.14, we have $|Y^*_i| = |X^*_i| + 1$, $|Y^*_k| = |X^*_k| + 1$, and $|X^*_i| = |Y^*_i|$ for $2 \leq i \leq k - 1$.

For each $i = 1, 2, \ldots, k - 1$, now set $B_{i+1} := Y^*_i \cap \Gamma(x^*_{i+1})$ and $C_i := X^*_i \cap \Gamma(y^*_i)$. Since $(X^*_i, Y^*_i)$ is $(4\varepsilon, d/2)$-super-regular, we have $|B_{i+1}|, |C_i| \geq d|X^*_i|/2 > d|X^*_i|/4$. Recall from Claim 3.3.10 that $\{X_1, Y_1\} = \{V_{x_1}, V_{x_2}\}$ and $\{X_k, Y_k\} = \{V_{y_1}, V_{y_2}\}$. We assume, w.l.o.g., that $X_1 = V_{x_1}$ and $X_k = V_{y_1}$. Let $A_1 = X^*_1 \cap \Gamma(x_1)$, $B_1 = Y^*_1 \cap \Gamma(x_2)$, $C_k = X^*_k \cap \Gamma(y_1)$, and $D_k = Y^*_k \cap \Gamma(y_2)$. Since $\deg(x_1, X_1) \geq (d - \varepsilon)N$, we have $\deg(x_1, X^*_1) \geq (d - \varepsilon - 2\varepsilon - d^2)N \geq d|X^*_1|/4$, and thus $|A_1| \geq d|X^*_1|/4$. Similarly, we have $|B_1|, |C_k|, |D_k| \geq d|X^*_1|/4$.

For each $1 \leq i \leq k$, we assume that $L^i_{2x} = a^i_1b^i_1 - L^i_{2x} - c^i_1d^i_1$, $L^i_{2y} = a^i_2b^i_2 - L^i_{2y} - c^i_2d^i_2$, $L^i_{4x} = a^i_3b^i_3 - L^i_{4x} - c^i_3d^i_3$, $L^i_{4y} = a^i_4b^i_4 - L^i_{4y} - c^i_4d^i_4$, and $L^i_{4y} = a^i_5b^i_5 - L^i_{4y} - d^i_5d^i_5$, where $a^i_j, c^i_j \in Y^*_i \subseteq Y_i$ for $j = 1, 2, \ldots, 5$. For $j = 1, 2, \ldots, 5$, let $A^i_j = X^*_i \cap \Gamma(a^i_j), C^i_j = X^*_i \cap \Gamma(c^i_j)$, $B^i_j = Y^*_i \cap \Gamma(b^i_j)$, and $D^i_j = Y^*_i \cap \Gamma(d^i_j)$. Since $(X^*_i, Y^*_i)$ is $(2\varepsilon, d - 3\varepsilon)$-super-regular, for $j = 1, 2, 3, 5$, we have $|\Gamma(a^i_j, X^*_i)|, |\Gamma(c^i_j, X^*_i)| \geq (d - 3\varepsilon)|X^*_i|$ and $|\Gamma(b^i_j, Y^*_i)|, |\Gamma(d^i_j, Y^*_i)| \geq (d - 3\varepsilon)|Y^*_i|$. From the proof of Claim 3.3.14, the pair $(Y^*_i, X^*_{i+1})$ is $(4\varepsilon, d - 5\varepsilon)$-super-regular. Hence, $|\Gamma(a^i_j, X^*_{i+1})|, |\Gamma(c^i_j, X^*_{i+1})| \geq (d - 4\varepsilon)|X^*_{i+1}|$ and $|\Gamma(b^i_j, Y^*_i)|, |\Gamma(d^i_j, Y^*_i)| \geq (d - 4\varepsilon)|Y^*_i|$. Thus, we have $|A^i_j|, |B^i_j|, |C^i_j|, |D^i_j| \geq (d - 4\varepsilon)|X^*_i| - d^2N \geq d|X^*_i|/4 = d|Y^*_i|/4$.

We now apply the Blow-up lemma on $(X^*_i, Y^*_i)$ to find a spanning ladder $L^i$ with its first and last rungs being contained in $A_i \times B_i$ and $C_i \times D_i$, respectively, and for $j = 1, 2, \ldots, 5$, its $(2j)$-th and $(2j + 1)$-th rungs being contained in $A^i_j \times B^i_j$ and $C^i_j \times D^i_j$, respectively. We can then insert $L^i_{2x}$ between the 2nd and 3rd rungs of $L^i$, $L^i_{2y}$ between the 4th and 5th rungs of $L^i$, $L^i_{4x}$ between the 6th and 7th rungs of $L^i$, $L^i_{4y}$ between the 8th and 9th rungs of $L^i$, and $L^i_{4y}$ between the 10th and 11th rungs of $L^i$ to obtain a ladder $L^i$ spanning on $X_i \cup Y_i - S$. Finally, $L^i \cdot y^*_1 \cdot x^*_2 \cdot L^2 \cdot \cdots \cdot y^*_{k-1} \cdot x^*_k \cdot L^k$ is a spanning ladder of $G''$ with its first rung adjacent to $x_1x_2$ and its last rung adjacent to $y_1y_2$.

The proof is then complete.
3.4 Minimum degree condition for spanning generalized Halin graphs

3.4.1 Introduction

A tree with no vertex of degree 2 is called a *homeomorphically irreducible tree (HIT)*, and a spanning tree with no vertex of degree 2 is a *homeomorphically irreducible spanning tree (HIST)*. A *Halin graph*, constructed by Halin in 1971 [27], is a graph formed from a plane embedding of a HIT $T$ of at least 4 vertices by connecting its leaves into a cycle following the cyclic order determined by the embedding. Relaxing the planarity requirement, a generalized Halin graph is obtained from a HIT $T$ of at least 4 vertices by connecting its leaves into a cycle. We call the HIT $T$ the *underlying tree* or *underlying HIST* of the resulting (generalized) Halin graph.

Halin graphs possess many hamiltonicity properties. For examples, Halin graphs are hamiltonian [5], hamiltonian-connected [2] (there is a hamiltonian path between any two distinct vertices), and almost pancyclic [6] (contains all possible cycle lengths with one possible exception of a single even length). Compared to Halin graphs, generalized Halin graphs are less studied. Kaiser et al. in [34] showed that a generalized Halin graph is *prism hamiltonian*; that is, the Cartesian product of a generalized Halin graph and $K_2$ is hamiltonian. Since a tree with no degree 2 vertices has more leaves than the non-leaves, a generalized Halin graph contains a cycle of length at least half of its order. Also, one can notice that by contracting the non-leaves of the underlying tree of a generalized Halin graph into a single vertex, a wheel graph is resulted with the contracted vertex as the hub, where a minor of a graph is obtained from the graph by deleting edges/contracting edges, or deleting vertices. Therefore, a generalized Halin graph contains a wheel-minor of order at least half of its order. Although a generalized Halin graph may not be hamiltonian, we conjecture that the lengths of a longest cycle in a generalized Halin graph is large.

**Conjecture 3.1.** Let $G$ be an $n$-vertex generalized Halin graph. Then the length of a longest cycle of $G$ is at least $4n/5$.

It was shown by Horton, Parker, and Borie [30] that it is NP-complete to determine
whether a graph contains a (spanning) Halin graph. For generalized Halin graphs we obtain the following.

**Theorem 3.4.1.** It is NP-hard to determine whether a graph contains a spanning generalized Halin graph.

A classic theorem of Dirac [19] from 1952 asserts that every graph on \( n \) vertices with minimum degree at least \( n/2 \) is hamiltonian if \( n \geq 3 \). As a continuous “generalization” of Dirac’s Theorem as well as an approach of showing many hamiltonicity properties simultaneously in a graph, the existence of a spanning Halin graph in graphs with large minimum degree was investigated in the previous section, and it was shown that any sufficiently large \( n \)-vertex graph with minimum degree at least \((n + 1)/2\) contains a spanning Halin graph. We here determine the minimum degree threshold for a graph to contain a spanning generalized Halin graph.

**Theorem 3.4.2.** There exists a positive integer \( n_0 \) such that every 3-connected graph with \( n \geq n_0 \) vertices and minimum degree at least \((2n + 3)/5\) contains a spanning generalized Halin graph. The result is best possible in the sense of the connectivity and minimum degree constraints.

Since a generalized Halin graph of order \( n \) contains a wheel-minor of order at least \( n/2 \), we get the following corollary.

**Corollary 3.4.1.** There exists a positive integer \( n_0 \) such that every 3-connected graph with \( n \geq n_0 \) vertices and minimum degree at least \((2n + 3)/5\) contains a wheel-minor of order at least \( n/2 \).

For notational convenience, for a graph \( T \), we denote by \( L(T) \) the set of degree 1 vertices of \( T \) and \( S(T) = V(T) - L(T) \). Also we abbreviate spanning generalized Halin graph as \( SGHG \) in what follows, and denote a generalized Halin graph as \( H = T \cup C \), where \( T \) is the underlying HIST of \( H \) and \( C \) is the cycle spanning on \( L(T) \).
3.4.2 Proof of Theorem 3.4.1 and the sharpness of Theorem 3.4.2

Proof of Theorem 3.4.1. To show the problem is NP-hard we assume the existence of a polynomial algorithm to test for an SGHG and use it to create a polynomial algorithm to test for a hamiltonian path between two vertices in an arbitrary graph. The decision problem for such hamiltonian paths is a classic NP-complete problem [24].

Let $G$ be a graph and $x, y \in V(G)$. We want to determine whether there exists a hamiltonian path connecting $x$ and $y$. We first construct a new graph $G'$ and show that $G$ contains a hamiltonian path between $x$ and $y$ if and only if $G'$ contains a HIST (the proof of this part is the same as the proof of Albertson et al. in [1]). Then based on $G'$, we construct a graph $G''$ and show that $G'$ contains a HIST if and only if $G''$ contains an SGHG.

Let $\{z_1, z_2, \ldots, z_t\} = V(G) - \{x, y\}$. Then $G'$ is formed by adding new vertices $\{z'_1, z'_2, \ldots, z'_t\}$ and new edges $\{z_i z'_i : 1 \leq i \leq t\}$. It is clear that if $P$ is a hamiltonian path between $x$ and $y$, then $P \cup \{z_i z'_i : 1 \leq i \leq t\}$ is a HIST of $G'$. Conversely, let $T$ be a HIST of $G'$. Since $1 \leq d_T(z'_i) \leq d_{G'}(z'_i) = 1$, we get $d_T(z'_i) = 1$ for each $i$. Since $N_{G'}(z'_i) = \{z_i\}$ and $T$ is a HIST, we have $d_T(z_i) \geq 3$. Hence $T - \{z'_1, z'_2, \ldots, z'_t\}$ is a tree with leaves possibly in $\{x, y\}$. Since each tree has at least 2 leaves and a tree with exactly two leaves is a path, we conclude that $T - \{z'_1, z'_2, \ldots, z'_t\}$ is a path between $x$ and $y$.

Then based on $G'$, we construct a graph $G''$. First, for each $i$ with $1 \leq i \leq t$, we add new vertices $z'_{i1}, z'_{i2}, z'_{i3}$ and new edges $z'_iz'_{i1}, z'_iz'_{i2}, z'_iz'_{i3}, z'_iz'_{i1}, z'_iz'_{i2}, z'_iz'_{i3}$. Then we connect all vertices in $\{x, y\} \cup \{z'_{i1}, z'_{i2}, z'_{i3} : 1 \leq i \leq t\}$ into a cycle $C''$ such that $\{z'_{i1}z'_{i2}, z'_{i2}z'_{i3} : 1 \leq i \leq t\} \subseteq E(C'')$. If $T'$ is a HIST of $G'$, then $T'' := T' \cup \{z'_{i1}z'_{i1}, z'_{i2}z'_{i2}, z'_{i3}z'_{i3} : 1 \leq i \leq t\}$ is a HIST of $G''$ and $T'' \cup C''$ is an SGHG of $G''$. Conversely, suppose $H = T \cup C$ is an SGHG of $G''$. We claim that $C = C''$. This in turn gives that $T = T''$ and therefore $T'' - \{z'_{i1}, z'_{i2}, z'_{i3} : 1 \leq i \leq t\}$ is a HIST of $G'$. To show that $C = C''$, we first show that $z'_{i2} \in L(T)$ for each $i$. Suppose on the contrary and assume, without loss of generality, that $z'_{i2} \in S(T)$. Then as $N_{G''}(z'_{i2}) = \{z'_1, z'_1, z'_{i3}\}$, we get $\{z'_{i2}z'_1, z'_{i2}z'_{i1}, z'_{i2}z'_{i3}\} \subseteq E(T)$. Since $T$ is acyclic, $z'_1z'_{i1}, z'_1z'_{i3} \notin E(T)$. This in turn shows that $\{z'_1, z'_1, z'_{i3}\} \subseteq L(T)$. However, $\{z'_{i2}z'_1, z'_{i2}z'_{i1}, z'_{i2}z'_{i3}\}$ forms a component of $T$, showing a contradiction. Then we show that
Thus, by Proposition 3.4.1, it is easy to see that the complete bipartite graphs as $z'_1 \in L(T)$ for each $i$. Suppose on the contrary and assume, without loss of generality, that $z'_{11} \in S(T)$. By the previous argument, we have $z'_{12} \in L(T)$. Then $z'_1, z'_3 \in L(T)$ as $z'_{12}$ is on $C$ and $z'_1$ and $z'_3$ are the only two neighbors of $z'_{12}$ which can be on the cycle $C$. As $d_{G'}(z'_{11}) = 3$ and $\{z'_{12}, z'_1\} \subseteq N_{G'}(z'_{11})$, $z'_{11}z'_{12}, z'_{11}z'_1 \in E(T)$. Since $z'_{12} \in L(T)$ and $z'_1, z'_3 \in L(T)$, we get $z'_{12}z'_3, z'_{12}z'_1, z'_1z'_3 \notin E(T)$. Since $d_{G'}(z'_{12}) = d_{G'}(z'_{13}) = 3$, we have $z'_{12}z'_3, z'_{12}z'_1, z'_1z'_3 \in E(C)$. However, $z'_{12}z'_3, z'_{12}z'_1, z'_1z'_3$ forms a triangle but $|V(C)| \geq 4$, showing a contradiction. So we have shown that $\{z'_{i1}, z'_{i2}, z'_{i3} : 1 \leq i \leq t\} \subseteq L(T)$, $\{z'_{12}, z'_{11}, z'_{13}\}$ is on a cycle, and $\{z'_{i1}, z'_{i2}, z'_{i3} \in L(T) \leq 1 \leq t\}$ is a HIST of $G'$.

Combining the arguments in the two paragraphs above, we see that $G$ has a hamiltonian path between $x$ and $y$ if and only if $G''$ has an SGHG. Hence a polynomial SGHG-tester becomes a polynomial path-tester. 

Since a generalized Halin graph is 3-connected, the connectivity requirement in Theorem [3.4.2] is necessary. To show that the minimum degree requirement is best possible, we show the following proposition.

**Proposition 3.4.1.** Let $G(A, B) = K_{a,b}$ be a complete bipartite graph with $|A| = a$ and $|B| = b$. Then $G(A, B)$ has no HIST $T$ with $|L(T) \cap A| = |L(T) \cap B|$ if $b > \frac{3(a-1)}{2}$.

If a bipartite graph $G(A, B)$ contains an SGHG $H = T \cup C$, then $|L(T) \cap A| = |L(T) \cap B|$. Thus, by Proposition 3.4.1, it is easy to see that the complete bipartite graphs $K_{a,b}$ with $b = \frac{3a-1}{2}$ when $a$ is odd and $b = \frac{3a-2}{2}$ when $a$ is even does not have an SGHG. Let $n = a + b$. By direct computation, we get $\delta(K_{a,b}) = \frac{2a+1}{5}$ when $b = \frac{3a-1}{2}$ and $\delta(K_{a,b}) = \frac{2a+2}{5}$ when $b = \frac{3a-2}{2}$. We now prove Proposition 3.4.1.

**Proof of Proposition 3.4.1.** Suppose on the contrary that $G(A, B)$ contains a HIST $T$ such that $|L(T) \cap A| = |L(T) \cap B|$. Then

$$
|S(T) \cap B| - |S(T) \cap A| = |B| - |L(T) \cap B| - (|A| - |L(T) \cap A|)
= |B| - |A| + \frac{3(a-1)}{2} - a = \frac{a-3}{2}.
$$
Since $G(A, B)$ is bipartite and $T$ is a HIST of $G(A, B)$, we have $|S(T) \cap A| \geq 1$. Thus, from the inequalities above, we obtain $|S(T) \cap B| > (a - 1)/2$. Since $T$ is a HIST, we have $d_T(y) \geq 3$ for each $y \in S(T) \cap B$. Let $E_B = \{ e \in E(T) : e$ is incident to a vertex in $S(T) \cap B \}$. Denote by $T'$ the subgraph of $T$ induced on $E_B$. Notice that $T'$ is a forest of at least $3|S(T) \cap B|$ edges. Hence $T'$ has at least $3|S(T) \cap B| + 1$ vertices. As $T'$ is a bipartite graph with one partite set as $S(T) \cap B$, and another as a subset of $A$, we conclude that $|V(T) \cap A| = |V(T)| - |S(T) \cap B| \geq 2|S(T) \cap B| + 1$. Since $|S(T) \cap B| > (a - 1)/2$, we then have $|V(T) \cap A| > a$. This gives a contradiction to the assumption $|A| = a$. $\square$

3.4.3 Proof of Theorem 3.4.2

Given $0 \leq \beta \ll \alpha \ll 1$, we define the two extremal cases with parameters $\alpha$ and $\beta$ as follows.

**Extremal Case 1.** There exists a partition of $V(G)$ into $V_1$ and $V_2$ such that $|V_i| \geq (2/5 - 4\beta)n$ and $d(V_1, V_2) < \alpha$. Furthermore, $deg(v_1, V_2) \leq 2\beta n$ for each $v_1 \in V_1$.

**Extremal Case 2.** There exists a partition of $V(G)$ into $V_1$ and $V_2$ such that $|V_1| > (3/5 - \alpha)n$ and $d(V_1, V_2) \geq 1 - 3\alpha$. Furthermore, $deg(v_1, V_2) \geq (2n + 3)/5 - 2\beta n$ for each $v_1 \in V_1$.

Then Theorem 3.4.2 is shown through the following three theorems.

**Theorem 3.4.3 (Non-extremal Case).** For every $\alpha > 0$, there exists $\beta > 0$ and a positive integer $n_0$ such that if $G$ is a 3-connected graph with $n \geq n_0$ vertices and $\delta(G) \geq (2n + 3)/5 - \beta n$, then $G$ contains an SGHG or $G$ is in one of the two extremal cases.

**Theorem 3.4.4 (Extremal Case 1).** Suppose that $0 < \beta \ll \alpha \ll 1$ and $n$ is a sufficiently large integer. Let $G$ be a 3-connected graph on $n$ vertices with $\delta(G) \geq (2n + 3)/5$. If $G$ is in Extremal Case 1, then $G$ contains an SGHG.

**Theorem 3.4.5 (Extremal Case 2).** Suppose that $0 < \beta \ll \alpha \ll 1$ and $n$ is a sufficiently large integer. Let $G$ be a 3-connected graph on $n$ vertices with $\delta(G) \geq (2n + 3)/5$. If $G$ is in Extremal Case 2, then $G$ contains an SGHG.
We show Theorems 3.4.3-3.4.5 separately in the following three subsections.

### 3.4.3.1 Proof of Theorem 3.4.3

We fix the following sequence of parameters,

\[ 0 < \varepsilon \ll d \ll \beta \ll \alpha < 1, \tag{3.5} \]

and specify their dependence as the proof proceeds. We let \( \beta \ll \alpha \) be the same \( \alpha \) and \( \beta \) as defined in the two extremal cases. Then we choose \( d \ll \beta \). Finally we choose \( \varepsilon = \min \left\{ \frac{1}{4} \varepsilon \left( \frac{d}{2}, \left\lceil \frac{2}{d^2} \right\rceil, \frac{d}{2} \right), \frac{1}{9} \varepsilon \left( \frac{d}{2}, \left\lceil \frac{3}{d^2} \right\rceil, 3 \right), \frac{1}{4} \varepsilon \left( \frac{d}{2}, 2, 2, \frac{d}{2} \right) \right\}, \tag{3.6} \]

where \( \varepsilon \left( \frac{d}{2}, \left\lceil \frac{3}{d^2} \right\rceil, 3 \right) \) follows from the definition of the \( \varepsilon \) in the weak version of the Blow-up lemma and \( \varepsilon \left( \frac{d}{2}, \left\lceil \frac{2}{d^2} \right\rceil, 2, \frac{d}{2} \right) \) and \( \varepsilon \left( \frac{d}{2}, 2, 2, \frac{d}{2} \right) \) follow from the definition of the \( \varepsilon \) in the strengthened version of the Blow-up lemma. Choose \( n \) to be sufficiently large. In the proof, we omit non-necessary ceiling and floor functions.

Let \( G \) be a graph of order \( n \) such that \( \delta(G) \geq \frac{(2n+3)}{5} - \beta n \) and suppose that \( G \) is not in any of the two extremal cases. Applying the regularity lemma to \( G \) with parameters \( \varepsilon \) and \( d \), we obtain a partition of \( V(G) \) into \( l+1 \) clusters \( V_0, V_1, \ldots, V_l \) for some \( l \leq M = M(\varepsilon) \), and a spanning subgraph \( G' \) of \( G \) with all described properties in Lemma 3.2.1 (the Regularity lemma). In particular, for all \( v \in V \),

\[ \deg_{G'}(v) \geq \deg_G(v) - (d + \varepsilon)n \geq \left( \frac{2}{5} - \beta - d - \varepsilon \right)n \geq \left( \frac{2}{5} - 2\beta \right)n \] (provided that \( \varepsilon + d \leq \beta \)), \tag{3.7} \]

and

\[ e(G') \geq e(G) - \frac{(d + \varepsilon)}{2}n^2 \geq e(G) - dn^2, \]

by using \( \varepsilon < d \).

We further assume that \( l = 2k \) is even; otherwise, we eliminate the last cluster \( V_l \) by
removing all the vertices in this cluster to \( V_0 \). As a result, \(|V_0| \leq 2\varepsilon n\) and

\[
(1 - 2\varepsilon)n \leq lN = 2kN \leq n, \tag{3.8}
\]

here we assume that \(|V_i| = N\) for \( i \geq 1 \).

For each pair \( i \) and \( j \) with \( 1 \leq i < j \leq l \), we write \( V_i \sim V_j \) if \( d(V_i, V_j) \geq d \). We now consider the reduced graph \( G_r \), whose vertex set is \( \{1, 2, \ldots, l\} \), and two vertices \( i \) and \( j \) are adjacent if and only if \( V_i \sim V_j \). We claim that \( \delta(G_r) \geq (2/5 - 2\beta)l \). Suppose not, and let \( i_0 \in V(G_r) \) such that \( \deg(i_0, V(G_r)) < (2/5 - 2\beta)l \). Then, for the corresponding cluster \( V_{i_0} \) we have \( e_{G'}(V_{i_0}, V(G') - V_{i_0}) < |V_{i_0}|(2/5 - 2\beta)lN \). On the other hand, by using (3.7), we have \( e_{G'}(V_{i_0}, V(G') - V_{i_0}) \geq |V_{i_0}|(2/5 - 2\beta)n \). As \( lN \leq n \) from (3.8), we obtain a contradiction. The rest of the proof consists of the following steps.

**Step 1.** Show that \( G_r \) contains a dominating cycle \( C \) and there is a \( \wedge \)-matching in \( G_r \) with all vertices in \( V(G_r) - V(C) \) as its center. We distinguish two cases in Step 1, and each of the other steps will be separated into two cases correspondingly.

**Case A.** \( C = X_1Y_1X_2Y_2\cdots X_tY_t \) is an even cycle for some \( t \leq k \).

**Case B.** \( C = X_0X_1Y_1X_2Y_2\cdots X_tY_t \) is an odd cycle for some \( t < k \).

Notice that in Case B there is at least one vertex in \( V(G_r) - V(C) \) by the assumption that \(|V(G_r)| = l\) is even. In what follows, if we denote a vertex of \( G_r \) by a capital letter, it means either a vertex of \( G_r \) or the corresponding cluster in \( G \), but the exact meaning will be clear from the context. For \( 1 \leq i \leq t \), we call \( X_i \) and \( Y_i \) the partners of each other, and write as \( P(X_i) = Y_i \) and \( P(Y_i) = X_i \).

Since \( C \) is not necessarily hamiltonian in \( G_r \), we need to take care of the clusters of \( G \) which are not represented on \( C \). For each vertex \( F \in V(G_r) - V(C) \), we partition the corresponding cluster \( F \) into two small clusters \( F_1 \) and \( F_2 \) such that \(-1 \leq |F_1| - |F_2| \leq 1 \). We call each \( F_1 \) and \( F_2 \) a half-cluster. Then we group all the original clusters and the partitioned clusters into pairs \((A, B)\) and triples \((C, D, F)\) with \( F \) as a half-cluster such that each pair \((A, B)\) and \((C, D)\) is still \( \varepsilon \)-regular with density \( d \) and the pair \((D, F)\) is \( 2.1\varepsilon \)-regular with
density \( d - \varepsilon \). Having the cluster groups like this, in the end, we will find “small” HITs within each pair \((A, B)\) or among each triple \((C, D, F)\).

**Step 2.** For each \(1 \leq i \leq t - 1\), initiate two independent edges connecting \(Y_i\) and \(X_{i+1}\). In Case A, also initiate two independent edges connecting \(X_1\) and \(Y_i\); and in Case B, initiate two independent edges connecting the clusters in each pair of \(X_0\) and \(X_1\), and \(X_0\) and \(Y_i\).

**Step 3.** Make each regular pair in the new grouped pairs and triples given in Step 1 super-regular.

**Step 4.** Construct HITs covering all vertices in \(V_0\) using vertices from the super-regular pairs obtained from Step 3, and obtain new super-regular pairs.

**Step 5.** Apply the Blow-up lemma to find a HIT between a super-regular pair resulted from Step 4 or among a triple \((A, B, F)\), where both \((A, F)\) and \((A, B)\) are super-regular pairs resulted from Step 4, and \(F\) is a half cluster. In addition, in the construction, for each triple \((A, B, F)\), we require the HIT to use as many vertices as possible from \(F\) as non-leaves.

**Step 6.** Apply the Blow-up Lemma again on the regular-pairs induced on the leaves of each HIT obtained in Step 5 to find two disjoint paths covering all the leaves. Then connect all the HITs into a HIST of \(G\) using edges guaranteed by the regularity and connect the disjoint paths into a cycle using the edges initiated in Step 2. The union of the HIST and the cycle gives an SGHG of \(G\).

We now give details of each step. The assumption that \(G\) is not in any of the two extremal cases leads to the following claim, which will be used in Step 1.

**Claim 3.4.1.** Each of the following holds for \(G_r\).

(a) \(G_r\) contains no cut-vertex set of size at most \(\beta l\);

(b) \(G_r\) contains no independent set of size more than \(\left(\frac{3}{5} - \frac{\alpha}{2}\right)l\).

**Proof.** (a) Suppose instead that \(G_r\) contains a vertex-cut \(W\) of size at most \(\beta l\). As \(\delta(G_r) \geq (2/5 - 2\beta)l\), then each component of \(G_r - W\) has at least \((2/5 - 3\beta)l\) vertices. Let \(U\) be the vertex set of one of the components of \(G_r - W\), \(A = \bigcup_{i \in U} V_i\), and \(B = V(G) - A\).
We see that $|A|, |B| \geq (2/5 - 3\beta)lN \geq (2/5 - 4\beta)n$, and since $e(G) \leq e(G') + dn^2$, we have

\[
e_{G}(A, B) \leq e_{G'}(A, B) + dn^2 \leq |W||A| + dn^2
\]

\[
\leq \beta lN(3/5 + 3\beta)lN + dn^2 \leq (3\beta/5 + 3\beta^2 + d)n^2 \quad \text{(as } |A| \leq (3/5 + 3\beta)lN \text{ and } ln \leq n)\]

\[
\leq \frac{25}{3}(3\beta/5 + 3\beta^2 + d)|A||B| \quad \text{(since } |A||B| \geq 3n^2/25)\]

\[
< \alpha|A||B| \quad \text{(provided that } \frac{25}{3}(3\beta/5 + 3\beta^2 + d) < \alpha).\]

This shows that $d(A, B) < \alpha$. Since $\text{deg}_{G_r}(u, V(G_r) - U) = \text{deg}_{G_r}(u, W) \leq \beta l$ for each $u \in U$, we see that $\text{deg}(a, B) \leq \beta lN + (d + \varepsilon)n \leq 2\beta n$ for each $a \in A$ provided that $d + \varepsilon \leq \beta$.

However, the above argument shows that $G$ is in Extremal Case 1, showing a contradiction.

(b) Suppose instead that $G_r$ contains an independent set $U$ of size larger than $(3/5 - \alpha/2)l$. Let $U' = V(G_r) - U$, $A = \bigcup_{i \in U} V_i$, and $B = V(G) - A$. Then $|A| \geq (3/5 - \alpha/2)lN \geq (3/5 - \alpha)n$. For each vertex $v \in A$, since $\text{deg}(v, A) \leq \text{deg}(v, A) + (d + \varepsilon)n \leq \beta n$, we have $\text{deg}(v, B) \geq (2n + 3)/5 - \beta n - \beta n \geq (2n + 3)/5 - 2\beta n$. This gives that

\[
d(A, B) \geq \frac{(2/5 - 2\beta)n}{|B|} \geq \frac{(2/5 - 2\beta)n}{(2/5 + \alpha)n} \geq 1 - 3\alpha,
\]

provided that $\beta \leq \alpha/10 + 3\alpha^2/2$. We see that $G$ is in Extremal Case 2. \qed

Step 1. Show that $G_r$ contains a dominating cycle $C$, and there is a $\Lambda$-matching in $G_r$ with all vertices in $V(G_r) - V(C)$ as its center.

We need some results on longest cycles and paths as follows.

**Lemma 3.4.1** ([24]). Let $G$ be a 2-connected graph on $n$ vertices with $\delta(G) \geq (n + 2)/3$. Then every longest cycle in $G$ is a dominating cycle.

**Lemma 3.4.2** ([3]). Let $G$ be a 2-connected graph on $n$ vertices with $\delta(G) \geq (n + 2)/3$. Then $G$ contains a cycle of length at least $\min\{n, n + \delta(G) - \alpha(G)\}$, where $\alpha(G)$ is the size of a largest independent set in $G$.

**Lemma 3.4.3** ([8]). If $G$ is a 3-connected graph of order $n$ such that the degree sum of any
four independent vertices is at least $3n/2 + 1$, then the number of vertices on a longest path and that on a longest cycle differs at most by 1.

By (a) of Claim 3.4.1, $G_r$ is $\beta l$-connected. Since $n = Nl + |V_0| \leq (l + 2)\varepsilon n$, we get $l \geq 1/\varepsilon - 2$. Since $1/\varepsilon - 2 \geq 3/\beta$ (provided that $\beta \geq 3\varepsilon/(1 - 2\varepsilon)$), we then have $\beta l \geq 3$. So $G_r$ is 3-connected. By Claim 3.4.1 (b), $G_r$ has no independent set of size more than $(3/5 - \alpha/2)l$.

Notice that $\delta(G_r) \geq (2/5 - 2\beta)l > (l + 2)/3$. Applying Lemma 3.4.1 and Lemma 3.4.2 on $G_r$, we see that there is a cycle $C$ in $G_r$ which is longest, dominating, and has length at least $(4/5 + \alpha/2 - 2\beta)l$. Let $W = V(G_r) - V(C)$. In Case B, we order and label the vertices of $C$ such that $X_0$ is adjacent to a vertex, say $Y_0 \in W$ (recall that $W \neq \emptyset$ in this case). We fix $(X_0, Y_0)$ as a pair at the first place $(X_0Y_0 \in E(G_r)$, as cluster in $G$, $(X_0, Y_0)$ is an $\varepsilon$-regular pair with density $d$). Let

$$W' = \begin{cases} W, & \text{if in Case A;} \\ W - \{Y_0\}, & \text{if in Case B.} \end{cases}$$

We have $|W'| \leq (1/5 - \alpha/2 + 2\beta)l$ if in Case A and $|W'| \leq (1/5 - \alpha/2 + 2\beta)l - 1$ if in Case B. So $2|W'| \leq (2/5 - \alpha + 4\beta)l < (2/5 - 2\beta)l$ (provided that $\beta < \alpha/6$) if in Case A and $2|W'| \leq (2/5 - \alpha + 4\beta)l - 2 < (2/5 - 2\beta)l - 1$ (provided that $\beta < \alpha/6$) if in Case B. Thus there is a $\wedge$-matching centered in all vertices in $W'$; furthermore, if in Case B, we can choose the matching such that $X_0$ is not covered by it. Let $M_\wedge$ be such a matching. For a vertex $X \in W'$, denote by $M_\wedge(X)$ the two vertices from $V(C)$ to which $X$ is adjacent in $M_\wedge$. Then we have two facts about vertices in $M_\wedge(X)$.

**Fact 3.4.1.** Let $X \in W'$. Then the two vertices in $M_\wedge(X)$ are non-consecutive on $C$. (By the assumption that $C$ is longest.)

**Fact 3.4.2.** Let $X \neq Y \in W'$. Then no two vertices from $M_\wedge(X) \cup M_\wedge(Y)$ are adjacent on $C$. (By applying Lemma 3.4.3)

For a complete bipartite graph, if it contains an SGHG, then the ratio of the cardinalities of the two partite sets should be greater than $2/3$ as shown in Proposition 3.4.1. Since a
longest dominating cycle in $G_r$ is not necessarily hamiltonian, we need to take care of the clusters of $G$ which are not represented by the vertices on $C$. One possible consideration is that for each $F \in V(G_r) - V(C)$, suppose $F$ is adjacent to $A \in V(C)$, recall $P(A)$ is the partner of $A$. Then as clusters, we consider the bipartite graph of $G$ with partite sets $A$ and $P(A) \cup F$. However, $|A|/|P(A) \cup F|$ is about 1/2, which is less than $2/3$. For this reason, we partition $F \in V(G_r) - V(C)$ into two parts to attain the right ratio in the corresponding bipartite graphs. Suppose $M_\lambda(F) = \{D_1, D_2\} \subseteq V(C)$. As a cluster of $G$, we partition $F$ into $F_1$ and $F_2$ arbitrarily such that

$$|F_1| = \left\lfloor \frac{|F|}{2} \right\rfloor = \left\lfloor \frac{N}{2} \right\rfloor \quad \text{and} \quad |F_2| = \left\lceil \frac{|F|}{2} \right\rceil = \left\lceil \frac{N}{2} \right\rceil.$$  

We call each $F_i$ a half-cluster of $G$. Then we create two pairs $(D_i, F_i)$, and call $D_i$ the dominator of $F_i$, and $F_i$ the follower of $D_i$, and $(D_i, F_i)$ a DF-pair, for $i = 1, 2$. We have the following fact about a DF-pair.

**Fact 3.4.3.** Each DF-pair $(D, F)$ is $2.1\varepsilon$-regular with density at least $d - \varepsilon$. (By Slicing lemma.)

Also, by Fact 3.4.1 and Fact 3.4.2 if $D \in V(C)$ is a dominator, then $P(D)$, the partner of $D$, is not a dominator for any followers. As $X_0 \not\in V(W')$, we know that $X_0$ is not a dominator for any half-clusters. We group the clusters and half-clusters of $G$ into $H$-pairs and $H$-triples in a way below. For each pair $(X_i, Y_i)$ on $C$, if $\{X_i, Y_i\} \cap V(M_\lambda) = \emptyset$, we take $(X_i, Y_i)$ as an H-pair. Otherwise, $|\{X_i, Y_i\} \cap V(M_\lambda)| = 1$ by Fact 3.4.1 and Fact 3.4.2. Since there is no difference for the proof for the case that $X_i \in V(M_\lambda)$ or the case that $Y_i \in V(M_\lambda)$, throughout the remaining proof, we always assume that $Y_i \in V(M_\lambda)$ if $\{X_i, Y_i\} \cap V(M_\lambda) \neq \emptyset$. In this case, there is a unique half-cluster $F$ with $Y_i$ as its dominator. Then we take $(X_i, Y_i, F)$ as an H-triple. We assign $(X_0, Y_0)$ as an $H$-pair.

**Step 2.** Initiating connecting edges.

Given an $\varepsilon$-regular pair $(A, B)$ of density $d$ and a subset $B' \subseteq B$, we say a vertex $a \in A$ typical to $B'$ if $\deg(a, B') \geq (d - \varepsilon)|B'|$. Then by the regularity of $(A, B)$, the fact below
holds.

**Fact 3.4.4.** If \((A, B)\) is an \(\varepsilon\)-regular pair, then at most \(\varepsilon|A|\) vertices of \(A\) are not typical to \(B' \subseteq B\) whenever \(|B'| > \varepsilon|B|\).

For each \(1 \leq i \leq t - 1\), choose \(y_i^* \in Y_i\) typical to both \(X_i\) and \(X_{i+1}\), and \(y_i^{**} \in Y_i\) typical to each of \(X_i\), \(X_{i+1}\), and \(\Gamma(y_i^*, X_i)\). Correspondingly, choose \(x_{i+1}^* \in \Gamma(y_i^*, X_{i+1})\) typical to \(Y_{i+1}\), and \(x_{i+1}^{**} \in \Gamma(y_i^{**}, X_{i+1})\) typical to both \(Y_{i+1}\) and \(\Gamma(x_{i+1}^*, Y_{i+1})\). For \(i = t\), we choose \(y_t^*\) and \(y_t^{**}\) the same way as for \(i < t\), but if in Case A, choose \(x_1^* \in \Gamma(y_t^*, X_1)\) typical to \(Y_1\), and \(x_1^{**} \in \Gamma(y_t^{**}, X_1)\) typical to both \(Y_1\) and \(\Gamma(x_1^*, Y_1)\); and if in Case B, choose \(x_0^* \in \Gamma(y_t^{**}, X_0)\) typical to \(X_1\), and \(x_0^{**} \in \Gamma(y_t^*, X_0)\) typical to both \(X_1\) and \(\Gamma(x_0^*, Y_0)\). Furthermore, in Case B, we choose \(y_{t+1}^* \in X_0\) typical to both \(Y_0\) and \(X_1\), and \(y_{t+1}^{**} \in X_0\) typical to each of \(Y_0\), \(X_1\), and \(\Gamma(y_{t+1}^*, Y_0)\). Correspondingly, choose \(x_{t+1}^* \in \Gamma(y_{t+1}^*, X_{t+1})\) typical to \(Y_{t+1}\) and \(x_{t+1}^{**} \in \Gamma(y_{t+1}^{**}, X_{t+1})\) typical to both \(Y_{t+1}\) and \(\Gamma(x_{t+1}^*, Y_{t+1})\). Additionally, we choose \(y_0^* \in \Gamma(y_{t+1}^*, Y_0)\) such that \(y_0^*\) is typical to \(X_0\), and choose \(y_0^{**} \in \Gamma(y_{t+1}^{**}, Y_0)\) such that \(y_0^{**}\) is typical to \(X_0\). Notice that by the choice of these vertices above, we have the following.

\[
\begin{cases}
  y_i^* x_{i+1}^*, y_i^{**} x_{i+1}^{**} \in E(G), & \text{for } 1 \leq i \leq t - 1; \\
  x_1^* y_1^{**}, x_1^* y_t^* \in E(G), & \text{in Case A}; \\
  x_0^* y_0^{**}, x_0^* y_1^*, x_1^* y_{t+1}^*, x_1^{**} y_{t+1}^{**}, y_0^* y_{t+1}^*, y_0^{**} y_{t+1}^{**} \in E(G), & \text{in Case B}.
\end{cases}
\]

By Fact 3.4.4 for each \(0 \leq i \leq t\), we have

\[
|\Gamma(x_i^*, Y_i) \cap \Gamma(x_i^{**}, Y_i)|, |\Gamma(y_i^*, X_i) \cap \Gamma(y_i^{**}, X_i)| \geq (d - \varepsilon)^2 N, \quad \text{and} \quad |\Gamma(y_{t+1}^*, Y_0) \cap \Gamma(y_{t+1}^{**}, Y_0)| \geq (d - \varepsilon)^2 N.
\]

**Step 3.** Super-regularizing the regular pairs in each H-pair and H-triple given in Step 1.

For each \(0 \leq i \leq t\), if \((X_i, Y_i)\) is an H-pair, let

\[
X_i' = \{ x \in X_i : \deg(x, Y_i) \geq (d - \varepsilon)N \} \quad \text{and} \quad Y_i' = \{ y \in Y_i : \deg(y, X_i) \geq (d - \varepsilon)N \}.
\]

By Fact 3.4.4 we have \(|X_i'|, |Y_i'| \geq (1 - \varepsilon)N\). Recall that \(x_i^*, x_i^{**} \in X_i\) and \(y_i^*, y_i^{**} \in Y_i\) are the initiated vertices in Step 2. For \(1 \leq i \leq t\), if \(|X_i' - \{x_i^*, x_i^{**}\}| \neq |Y_i' - \{y_i^*, y_i^{**}\}|\), say \(|X_i' - \{x_i^*, x_i^{**}\}| > |Y_i' - \{y_i^*, y_i^{**}\}|\), we then remove \(|X_i' - \{x_i^*, x_i^{**}\}| - |Y_i' - \{y_i^*, y_i^{**}\}|\) vertices
out from $X_i' - \{x^*_i, x^{**}_i\}$, and denote the remaining set still as $X_i'$. Denote $Y_i' - \{y^*_i, y^{**}_i\}$ still as $Y_i'$. We see that $|X_i'| = |Y_i'|$. As $|Y_i'| \geq (1 - \varepsilon)N$ (to be precise, the lower bound should be $(1 - \varepsilon)N - 2$, however, the constant 2 can be made vanished by adjusting the $\varepsilon$ factor, we ignore the slight different of the $\varepsilon$-factor here), we have that $|X_i \cup Y_i - (X_i' \cup Y_i')| \leq 2\varepsilon N$.

For $i = 0$, if $|X_i' - \{x^*_i, x^{**}_i, y^*_{i+1}, y^{**}_{i+1}\}| \neq |Y_i' - \{y^*_i, y^{**}_i\}|$, say $|X_i' - \{x^*_i, x^{**}_i, y^*_{i+1}, y^{**}_{i+1}\}| > |Y_i' - \{y^*_i, y^{**}_i\}|$, then we remove $|X_i' - \{x^*_i, x^{**}_i, y^*_{i+1}, y^{**}_{i+1}\}| - |Y_i' - \{y^*_i, y^{**}_i\}|$ vertices out from $X_i' - \{x^*_i, x^{**}_i, y^*_{i+1}, y^{**}_{i+1}\}$ and denote the remaining set still as $X_i'$. Denote $Y_i' - \{y^*_i, y^{**}_i\}$ still as $Y_i'$. We see that $|X_i'| = |Y_i'|$. We call the resulting H-pairs \textit{supper-regularized H-pairs}.

By Slicing lemma (Lemma 3.2.5) and the definitions of $X_i', Y_i'$, we see that

\textbf{Fact 3.4.5}. Each \textit{supper-regularized H-pair} $(X_i', Y_i')$ is a $(2\varepsilon, d - 2\varepsilon)$-\textit{super-regular pair}.

For each H-triple $(X_i, Y_i, F)$, by Fact 3.4.3 $(Y_i, F)$ is $2.1\varepsilon$-regular with density at least $d - \varepsilon$. Let

\begin{align*}
X_i' &= \{x \in X_i : \deg(x, Y_i) \geq (d - \varepsilon)N\}, \\
Y_i' &= \{y \in Y_i : \deg(y, X_i) \geq (d - \varepsilon)N, \deg(y, F) \geq (d - 3.1\varepsilon)|F|\}, \text{ and} \\
F' &= \{f \in F : \deg(f, Y_i) \geq (d - 3.1\varepsilon)N\}.
\end{align*}

Recall that $x^*_i, x^{**}_i \in X_i$ and $y^*_i, y^{**}_i \in Y_i$ are the initiated vertices in Step 2. We remove $x^*_i, x^{**}_i$ out from $X_i'$, and remove $y^*_i, y^{**}_i$ out from $Y_i'$. Still denote the resulted clusters as $X_i'$ and $Y_i'$, respectively. Remove $\lceil d^3 N \rceil$ vertices out from $F$, which consists of all vertices in $F - F'$ and any $\lceil d^3 N \rceil - |F - F'|$ vertices from $F'$ (we need to increase the ratio $|Y_i'|/|X_i' \cup F'|$ a little as later on we may use vertices in $Y_i'$ in constructing HITs covering vertices in $V_0$). Denote the resulting set still by $F'$. Then we see that $|X_i'| \geq (1 - \varepsilon)N$, $|Y_i'| \geq (1 - 3.1\varepsilon)N$, and $|F'| \geq (1 - 2.1\varepsilon)|F| - d^3 N \geq (1 - 2.1\varepsilon - 2d^3)|F|$. We call the resulted H-triples \textit{supper-regularized H-triples}. By the Slicing Lemma and the definitions above, the following is true.

\textbf{Fact 3.4.6}. For each \textit{supper-regularized H-triple} $(X_i', Y_i', F')$, $(X_i, Y_i)$ is $(2\varepsilon, d - 3.1\varepsilon)$-\textit{super-regular}, and $(Y_i', F')$ is $(4.2\varepsilon, d - 3.1\varepsilon - 2d^3)$-\textit{super-regular}. 
Let \( V_0^1 \) be the union of the set of vertices from each \((X_i \cup Y_i) - (X_i' \cup Y_i')\) - \(\{x_i^*,x_i'^*,y_i^*,y_i'^*\} - \{y_{i+1}^*,y_{i+1}'^*\}\) \((\{y_{i+1}^*,y_{i+1}'^*\}\) exists only if in Case B), where \((X_i,Y_i)\) is an H-pair, and let \( V_0^2 \) be the union of the set of vertices from each \((X_i \cup Y_i \cup F - (X_i' \cup Y_i' \cup F'))\) - \(\{x_i^*,x_i'^*,y_i^*,y_i'^*,y_t^*,y_t'^*\}\), where \((X_i,Y_i,F)\) is an H-triple. Notice that for each H-pair \((X_i,Y_i)\), we have \(|X_i \cup Y_i - (X_i' \cup Y_i')| \leq 2\varepsilon N\); and for each H-triple \((X_i,Y_i,F)\), we have \(|X_i - X_i'| \leq \varepsilon N\), \(|Y_i - Y_i'| \leq (\varepsilon + 2.1\varepsilon)N\), and \(|F - F'| \leq d^3N\). Hence by using the facts that \(|W'| \leq (1/5 - \alpha/2 + 2\beta)l\), \(l = l/2\), and \(Nl \leq n\) from inequality (3.8), we get

\[
|V_0^1| + |V_0^2| \leq 2\varepsilon Nl/2 + 2(1/5 - \alpha + 2\beta)l(d^3N + 2.1\varepsilon N) \leq 2d^3Nl/5 + 2\varepsilon Nl \leq 2d^3n/5 + 2\varepsilon n.
\]

Let \( V_0' = V_0 \cup V_0^1 \cup V_0^2 \). Then

\[
|V_0'| \leq 2\varepsilon n + 2d^3n/5 + 2\varepsilon n \leq d^3n/2 \quad \text{ (provided that } \varepsilon \leq d^3/40) \tag{3.9}.
\]

**Step 4.** Construct small HITs covering all vertices in \( V_0' \).

Consider a vertex \( x \in V_0' \) and a cluster or a half-cluster \( A \), we say that \( x \) is adjacent to \( A \), denoted by \( x \sim A \), if \( \deg(x,A) \geq (d - \varepsilon)|A| \). We call \( A \) the partner of \( x \).

**Claim 3.4.2.** For each vertex \( x \in V_0' \), there is a cluster or a half-cluster \( A \) such that \( x \sim A \), where \( A \) is not a dominator, and we can assign all vertices in \( V_0' \) to their partners which are not dominators such that each of the cluster or half-cluster is used by at most \( d^2N/20 \) vertices from \( V_0' \).

**Proof.** Suppose we have found partners for the first \( m < d^3n/2 \) (recall that \(|V_0'| \leq d^3n/2\)) vertices of \( V_0' \) such that no cluster or half-cluster is used by at most \( d^2N/20 \) vertices. Let \( \Omega \) be the set of all clusters and half-clusters that are used exactly by \( d^2N/20 \) vertices. Then

\[
\frac{d^2N}{20}|\Omega| \leq m < d^3n/2 \leq d^3(2kN + 2\varepsilon n)/2
\]

\[
\leq d^3kN + d^3 \frac{2kN}{1 - 2\varepsilon}.
\]
by inequality (3.8). Therefore,

\[ |\Omega| \leq \frac{20d^3k}{d^2} + \frac{20d^3l}{d^2(1 - 2\varepsilon)} \]

\[ \leq 10dl + 40dl \quad (\text{provided that } 1 - 2\varepsilon \geq 1/2) \]

\[ \leq \beta l \quad (\text{provided that } 50d \leq \beta). \]

Consider now a vertex \( v \in V'_0 \) not having a partner found so far. Let \( \mathcal{U} \) be the set of all non-dominator clusters and half-clusters adjacent to \( v \) not contained in \( \Omega \). We claim that \( |\mathcal{U}| \geq (\alpha - 7\beta)l \). To see this, we first observe that any vertex \( v \in V'_0 \) is adjacent to at least \( (\alpha - 6\beta)l \) non-dominator clusters and half-clusters. For instead, as \( v \) may adjacent to \( 2|W'| \) dominators, vertices in \( V'_0 \), or clusters \( A \) with \( \deg(v) < (d - \varepsilon)|A| \), we have

\[
\frac{2}{5} - \beta)n \leq \deg_G(v) < (\alpha - 6\beta)lN + (2/5 + 4\beta - \alpha)lN + d^3n/2 + (d - \varepsilon)lN
\]

\[
\leq (2/5 - 2\beta + d^3/2 + d - \varepsilon)n
\]

\[
< (2/5 - 3\beta/2)n \quad (\text{provided that } d - \varepsilon + d^3/2 < \beta/2),
\]

showing a contradiction. Since \( |\Omega| \leq \beta l \), we have \( |\mathcal{U}| \geq (2\alpha - 7\beta)l \).

Now for each non-dominator cluster \( A \) (\( A \) is either a cluster \( X'_i \), \( Y'_i \), or a half cluster \( F' \)), let \( I(A) \) be the set of vertices from \( V'_0 \) such that each of them has \( A \) as its partner. By Claim 3.4.2, we have \( |I(A)| \leq \frac{d^2n}{26} \).

We need three operations below for constructing small HITs covering vertices in \( V'_0 \).

**Operation I** Let \((A, B)\) be an \((\varepsilon', \delta)\)-super-regular pair, and \( I \) a set of vertices disjoint from \( A \cup B \). Suppose that (i) \( \deg(x, B) \geq d'|B| > \varepsilon'|B| \), and \( \deg(x, B) \geq d'|B| \geq 3|I| \) for any \( x \in I \); (ii) \( (\delta - \varepsilon')d'|B| \geq 3|I| \); (iii) \( (\delta - \varepsilon')|A| > |I| \); and (iv) \( \delta|A| > 4|I| \). Then we can do the following operations on \((A, B)\) and \( I \).

Let \( I = \{x_1, x_2, \ldots, x_{|I|}\} \). We first assume that \( |I| \geq 2 \).

Since \((A, B)\) is \((\varepsilon', \delta)\)-super-regular, for each \( v \in \Gamma(x_i, B) \), \( |\Gamma(v, A)| \geq \delta|A| \). By condition (i), we have \( |\Gamma(x_i, B)| > \varepsilon'|B| \) for each \( i \). Applying Fact 3.4.3, we then know that there are
at least \((\delta - \varepsilon')|A| > |I|\) vertices from \(\Gamma(v, A)\) typical to \(\Gamma(x_{i+1}, B)\) for each \(1 \leq i \leq |I| - 1\). That is, there exists \(A_1 \subseteq \Gamma(v, A)\) with \(|A_1| \geq (\delta - \varepsilon')|A| > |I|\) such that for each \(a_1 \in A_1\), 

\[ |\Gamma(a_1, \Gamma(x_{i+1}, B))| \geq (\delta - \varepsilon')d'|B| \geq 3|I|. \]

As \(\deg(x, B) \geq d'|B| \geq 3|I|\) for any \(x \in I\) and 

\[ (\delta - \varepsilon')d'|B| \geq 3|I|, \]

combining the above argument, we know there is a claw-matching \(M_I\) from \(I\) to \(B\) centered in \(I\) such that one vertex from \(\Gamma(x_i, V(M_I))\) and one vertex from \(\Gamma(x_{i+1}, V(M_I))\) have at least \((\delta - \varepsilon')|A| > |I|\) common neighbors in \(A\). Let \(x_{i1}, x_{i2}, x_{i3}\) be the three neighbors of \(x_i\) in \(M_I\) (in fact in \(B\)) and suppose that \(|\Gamma(x_{i3}, A) \cap \Gamma(x_{i+1,1}, A)| \geq |I|\). For \(1 \leq i \leq |I| - 1\), we then choose distinct vertices \(y_i \in \Gamma(x_{i3}, A) \cap \Gamma(x_{i+1,1}, A)\). By condition (iv), there is a \(\wedge\)-matching \(M_2\) between the vertex set \(\{x_{i3} : 1 \leq i \leq |I| - 1\}\) and the vertex set \(A - \{y_i : 1 \leq i \leq |I| - 1\}\) centered in the first set, a matching \(M_3\) between \(\{x_{i+1,1} : 1 \leq i \leq |I| - 1\}\) and \(A - \{y_i : 1 \leq i \leq |I| - 1\} - V(M_2)\) covering the first set, and a matching \(M_4\) between the vertex set \(\{y_i : 1 \leq i \leq |I| - 1\}\) and \(B - V(M_I)\) covering the first set. Finally, by using (iv) again, we can find three distinct vertices \(y_{31}, y_{32}, y_{33} \in \Gamma(x_{13}, A) - \{y_i : 1 \leq i \leq |I| - 1\} - V(M_2) - V(M_3)\). Let \(T_B\) be the graph with

\[ V(T_B) = V(M_I) \cup \{y_i : 1 \leq i \leq |I| - 1\} \cup V(M_2) \cap V(M_3) \cup V(M_4) \cup \{y_{31}, y_{32}, y_{33}\} \]

and

\[ E(T_B) = M_I \cup \{y_ix_{i3}, y_ix_{i+1,1} : 1 \leq i \leq |I| - 1\} \cup M_2 \cup M_3 \cup M_4 \cup \{x_{13}y_{31}, x_{13}y_{32}, x_{13}y_{33}\}. \]

If \(|I| = 1\), we choose \(x_{11}, x_{12}, x_{13} \in \Gamma(x_1, B)\) and \(y_{31}, y_{32}, y_{33} \in \Gamma(x_{13}, A)\). Then let \(T_B\) be the graph with

\[ V(T_B) = \{x_1, x_{11}, x_{12}, x_{13}, y_{31}, y_{32}, y_{33}\} \]

and

\[ E(T_B) = \{x_1x_{11}, x_1x_{12}, x_1x_{13}, x_{13}y_{31}, x_{13}y_{32}, x_{13}y_{33}\}. \]
In any case, we see that $T_B$ is a HIT satisfying

\[
|V(T_B) \cap B| = |V(T_B) \cap A| = 4|I| - 1,
\]

\[
|L(T_B) \cap B| = \min\{2|I| + 1, 3|I| - 1\}, |L(T_B) \cap A| = 3|I|.
\]

(3.10)

We call $T_B$ the insertion HIT associated with $B$. Figure 3.4 gives a depiction of $T_B$ for $|I| = 1, 3$, respectively.

\[\text{Figure (3.4) The HIT } T_B\]

**Operation II** Let $(A, B)$ be an $(\varepsilon', \delta)$-super-regular pair, and $I$ a set of vertices disjoint from $A \cup B$. Suppose that (i) $\deg(x, A) \geq d' |A| > \varepsilon' |A|$ and $\deg(x, A) \geq d' |A| \geq 3|I|$ for any $x \in I$; (ii) $(\delta - \varepsilon')d'|A| \geq 3|I|$; (iii) $(\delta - 2\varepsilon')|B| > |I|$; and (iv) $\delta |B| > 3|I|$. Then we can do the following operations on $(A, B)$ and $I$.

Let $I = \{x_1, x_2, \ldots, x_{|I|}\}$. We first assume that $|I| \geq 3$.

Since $(A, B)$ is $(\varepsilon', \delta)$-super-regular, for each $v \in \Gamma(x_i, A)$, $|\Gamma(v, B)| \geq \delta |B|$. By condition (i), we have $|\Gamma(x_i, A)| > \varepsilon' |A|$ for each $i$. Applying Fact 3.4.4 we then know that there are at least $(\delta - 2\varepsilon')|B| > |I|$ vertices from $\Gamma(v, B)$ typical to both $\Gamma(x_{i+1}, A)$ and $\Gamma(x_{i+2}, A)$ for each $1 \leq i \leq |I| - 2$. That is, there exists $B_1 \subseteq \Gamma(v, B)$ with $|B_1| \geq (\delta - 2\varepsilon')|B| > |I|$ such that for each $b_1 \in B_1$, $|\Gamma(b_1, \Gamma(x_{i+1}, A))|, |\Gamma(b_1, \Gamma(x_{i+2}, A))| \geq (\delta - \varepsilon')d'|A| \geq 3|I|$. As $\deg(x, A) \geq d'|A| \geq 3|I|$ for any $x \in I$ and $(\delta - \varepsilon')d'|A| \geq 3|I|$, combining the above argument, we know there is a claw-matching $M_I$ from $I$ to $A$ centered in $I$ such that any one vertex from $\Gamma(x_i, V(M_I))$, any one vertex from $\Gamma(x_{i+1}, V(M_I))$, and any one vertex from $\Gamma(x_{i+2}, V(M_I))$ have at least $|I|$ common neighbors in $B$. Let $x_{i1}, x_{i2}, x_{i3}$ be the three neighbors of $x_i$ in $M_I$ (in
fact in $A$). For $i = 1$, choose $y_0 \in \Gamma(x_{13}, A) \cap \Gamma(x_{23}, A) \cap \Gamma(x_{33}, A)$. Let $h = \lceil (|I| - 3)/2 \rceil$. For $1 \leq k \leq h$, we then choose distinct vertices $y_k \in \Gamma(x_{1+2k,2}, A) \cap \Gamma(x_{2+2k,3}, A) \cap \Gamma(x_{3+2k,3}, A)$ (if $|I| = 2 + 2k$, let $\Gamma(x_{3+2k,3}, A) = A$). By condition (iv), there is a matching $M$ between the vertex set $\{x_{i3}, x_{i+2k,2} : 1 \leq i \leq |I|, 1 \leq k \leq h\}$ and the vertex set $B - \{y_0, y_k : 1 \leq k \leq h\}$ covering the first set. If $|I|$ is even, choose $y_{31}, y_{32} \in \Gamma(x_{13}, B)$ such that they have not been chosen before; if $|I|$ is odd, choose $y_{31}, y_{32}, y_{33} \in \Gamma(x_{13}, B)$ such that they have not been chosen before. Let $T_A$ be the graph with

$$V(T_A) = \begin{cases} V(M_I) \cup V(M) \cup \{y_0, y_k : 1 \leq k \leq h\} \cup \{y_{31}, y_{32}\}, & \text{if } |I| \text{ is even;} \\ V(M_I) \cup V(M) \cup \{y_0, y_k : 1 \leq k \leq h\} \cup \{y_{31}, y_{32}, y_{33}\}, & \text{if } |I| \text{ is odd;} \end{cases}$$

and $E(T_A)$ containing all edges in $M_I \cup M \cup \{y_0 x_{13}, y_0 x_{23}, y_0 x_{33}\}$ and all edges in

$$\begin{cases} \{x_{1+2k,2} y_k, x_{2+2k,2} y_k, x_{3+2k,2} y_k, x_{1+2k,2} y_h, x_{2+2k,2} y_h : 1 \leq k \leq h - 1\} \cup \{y_{31}, y_{32}\}, & \text{if } |I| \text{ is even;} \\ \{x_{1+2k,2} y_k, x_{2+2k,2} y_k, x_{3+2k,2} y_k : 1 \leq k \leq h\} \cup \{y_{31}, y_{32}, y_{33}\}, & \text{if } |I| \text{ is odd.} \end{cases}$$

If $|I| = 1$, we choose $x_{11}, x_{12}, x_{13} \in \Gamma(x_1, A)$ and $y_{31}, y_{32} \in \Gamma(x_{13}, B)$, and then let $T_A$ be the graph with

$$V(T_B) = \{x_1, x_{11}, x_{12}, x_{13}, y_{31}, y_{32}\} \text{ and } E(T_B) = \{x_1 x_{11}, x_1 x_{12}, x_1 x_{13}, x_{13} y_{31}, x_{13} y_{32}\}.$$ 

If $|I| = 2$, we choose $x_{11}, x_{12}, x_{13} \in \Gamma(x_1, A), x_{11}, x_{12}, x_{13} \in \Gamma(x_2, A), y \in \Gamma(x_{13}, B) \cap \Gamma(x_{21}, B), y_{11}, y_{12} \in \Gamma(x_{13}, B), \text{ and } y_{21}, y_{22} \in \Gamma(x_{21}, B)$ such that they are all distinct, then let $T_A$ be the graph with

$$V(T_B) = \{x_i, x_{i1}, x_{i2}, x_{i3}, y, y_{i1}, y_{i2} : i = 1, 2\} \text{ and }$$

$$E(T_B) = \{x_i x_{i1}, x_i x_{i2}, x_i x_{i3}, x_{13} y, x_{21} y, x_{13} y_{11}, x_{13} y_{12}, x_{21} y_{21}, x_{21} y_{22}\}.$$ 

We see that $T_A$ is a tree which has a degree 2 vertex $y$ only if $|I| = 2$ and a degree 2 vertex
only if \( |I| > 2 \) and \( |I| \) is even. In addition, \( T_A \) satisfies the following.

\[
|V(T_A) \cap A| = 3|I| \quad \text{and} \quad |L(T_A) \cap A| = \begin{cases} 2|I|, & \text{if } |I| = 1, 2; \\ 2|I| - \left\lceil \frac{|I|-3}{2} \right\rceil, & \text{if } |I| \geq 3; \end{cases}
\]

\[
|V(T_A) \cap B| = \begin{cases} 2, & \text{if } |I| = 1; \\ 2|I| + 1, & \text{if } |I| \geq 2; \end{cases} \quad \text{and}
\]

\[
|L(T_A) \cap B| = \begin{cases} 2|I|, & \text{if } |I| = 1, 2; \\ 2|I| - \left\lceil \frac{|I|-3}{2} \right\rceil, & \text{if } |I| \geq 3. \end{cases}
\] (3.11)

In this case, we call \( T_A \) the insertion tree associated with \( A \). Notice that \( |L(T_A) \cap A| = |L(T_A) \cap B| \) always holds. Figure [3.4] gives a depiction of \( T_A \) for \( |I| = 1, 2, 5, 6 \), respectively.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{diagram.png}
\caption{The tree \( T_A \)}
\end{figure}

**Operation III** Let \((B, F)\) be an \((\varepsilon', \delta)\)-super-regular pair, and \( I \) a set of vertices disjoint from \( B \cup F \). Suppose that \( \deg(x, F) \geq d'|F| \geq 3|I| \) for any \( x \in I \) and \( \delta|B| \geq 6|I| \). Then we can do the following operations on \((A, B)\) and \( I \).

Let \( I = \{ x_1, x_2, \ldots, x_{|I|} \} \). Since \( \deg(x, B) \geq d'|B| \geq 3|I| \) for any \( x \in I \), there is a claw-matching \( M_I \) from \( I \) to \( F \) centered in \( I \). Then as \( \delta|B| \geq 6|I| \), there is a \( \wedge \)-matching
$M_\lambda$ from $V(M_I) \cap F$ to $B$ centered in $V(M_I) \cap F$. Let $T_F$ be the graph with

$$V(T_B) = V(M_I) \cup V(M_\lambda) \quad \text{and} \quad E(T_B) = M_I \cup M_\lambda.$$ 

We see that $T_F$ is a forest with no vertex of degree 2 satisfying

$$|V(T_F) \cap F| = |S(T_F) \cap F| = 3|I| \quad \text{and} \quad |V(T_F) \cap B| = |L(T_F) \cap B| = 6|I|. \quad (3.12)$$

We call $T_F$ the insertion forest associated with $F$.

Now for each H-pair $(X'_i, Y'_i)$, we may assume that $I(X'_i) \neq \emptyset$ and $I(Y'_i) \neq \emptyset$ for a uniform discussion, as the consequent argument is independent of the assumptions. Recall that $(X'_i, Y'_i)$ is $(2\varepsilon, d-2\varepsilon)$-super-regular by Fact 3.4.5. Notice that $\text{deg}(x, X'_i) \geq (d-\varepsilon)|X'_i|$ for each $x \in I(X'_i)$, $|I(X'_i)| \leq \frac{d^4N}{20}$, and $|X'_i|, |Y'_i| \geq (1-\varepsilon)N$. By simple calculations, we see that (i) $\text{deg}(x, X'_i) \geq (d-\varepsilon)|X'_i| > 2\varepsilon|X'_i|$ and $(d-\varepsilon)|X'_i| \geq 3d^2N/20$ for each $x \in I(X'_i)$; (ii) $(d-2\varepsilon-2\varepsilon)(d-\varepsilon)|X'_i| > 3d^2N/20$; (iii) $(d-4\varepsilon)|Y'_i| > d^2N/20$; and (iv) $(d-2\varepsilon)|Y'_i| > d^2N/5 \geq 4I(X'_i)$. Thus all the conditions in Operation I are satisfied. So we can find a HIT $T_{X'_i}$ associated with $X'_i$. As $|V(T_{X'_i}) \cap X'_i| = |V(T_{X'_i}) \cap Y'_i| \leq 4|I(X'_i)| \leq \frac{d^4N}{5}$, we know that $(X'_i - V(T_{X'_i}), Y'_i - V(T_{X'_i}))$ is $(4\varepsilon, d-2\varepsilon-d^2N/5)$-super regular. Since $\text{deg}(y, Y'_i) \geq (d-\varepsilon)|Y'_i|$ for each $y \in I(Y'_i)$, we get $\text{deg}(y, Y'_i - V(T_{X'_i})) \geq (d-\varepsilon - d^2/5)|Y'_i|$ for each $y \in I(Y'_i)$. By direct checking, conditions (i) $\sim$ (iv) of Operation I are satisfied by the pair $(X'_i - V(T_{X'_i}), Y'_i - V(T_{X'_i}))$ and $I(Y'_i)$. Then we use Operation I on $(X'_i - V(T_{X'_i}), Y'_i - V(T_{X'_i}))$ and $I(Y'_i)$ to get a HIT $T_{Y'_i}$ associated with $Y'_i - V(T_{X'_i})$. Denote

$$X'_i^* = X'_i - V(T_{X'_i}) - V(T_{Y'_i}) \quad \text{and} \quad Y'_i^* = Y'_i - V(T_{X'_i}) - V(T_{Y'_i}).$$

By using $(3.10)$ in Operation I, we have $|X'_i^*| = |Y'_i^*| \geq (1 - 2d^2/5 - \varepsilon)N \geq N/2$. By Slicing lemma (Lemma 3.2.5) and Fact 3.4.5 we have the following.

**Fact 3.4.7.** For each H-pair $(X_i, Y_i)$, $(X'_i, Y'_i)$ is $(4\varepsilon, d-2\varepsilon-2d^2/5)$-super-regular with $|X'_i| = |Y'_i|$. We call $(X'_i, Y'_i)$ a ready H-pair.
Then for each H-triple \((X'_i, Y'_i, F')\), we may assume that \(I(X'_i) \neq \emptyset\) and \(I(F') \neq \emptyset\) (recall that \(Y_i\) is assumed to be the dominator of \(F\), so \(I(Y'_i) = \emptyset\) by the distribution principle of vertices in \(V'_0\) from Claim 3.4.2). By Fact 3.4.6 we know that \((X'_i, Y'_i)\) is \((2\varepsilon, d - 3.1\varepsilon)\)-super-regular and \((Y'_i, F')\) is \((4.2\varepsilon, d - 3.1\varepsilon - 2d^3)\)-super-regular. Notice also that \(|X'_i| \geq (1 - \varepsilon)N, |Y'_i| \geq (1 - 3.1\varepsilon)N, |F'| \geq (1 - 2.1\varepsilon - 2d^3)N/2\), and \(\deg(x, X'_i) \geq (d - \varepsilon)|X'_i|\) and \(\deg(y, F') \geq (d - \varepsilon)|F'|\) for each \(x \in I(X'_i)\) and each \(y \in I(F')\). Since \(|I(X'_i)|, |I(F')| \leq \frac{d^3N}{20}\) and \(\varepsilon \ll d \ll 1\), the conditions of Operation III are satisfied by \((Y'_i, F')\) and \(I(F')\) by direct calculations. Let \(T_{F'}\) be the insertion forest associated with \(F'\). Then we use Operation II on \((X'_i, Y'_i - V(T_{F'}))\) and \(I(X'_i)\) to get a tree \(T_{X'_i}\) associated with \(X'_i\). Denote

\[
X^*_i = X'_i - V(T_{X'_i}), Y^*_i = Y'_i - V(T_{F'}) - V(T_{X'_i}), \quad \text{and} \quad F^* = F' - V(T_{F'}). 
\]

By using (3.11) and (3.12) in Operation II and Operation III, respectively, we have \(|X^*_i|, |Y^*_i| \geq (1 - 3.1\varepsilon - 9d^2/20)N \geq N/2\) and \(|F^*| \geq (1 - 2.1\varepsilon - 2d^3)N/2 - 3d^2N/20 \geq (1 - 2.1\varepsilon - 2d^3 - 3d^2/10)N/2\). By Slicing lemma and Fact 3.4.6 we have the following.

**Fact 3.4.8.** For each H-triple \((X_i, Y_i, F)\), \((X^*_i, Y^*_i)\) is \((4\varepsilon, d - 3.1\varepsilon - 9d^2/20)\)-super-regular and \((Y^*_i, F^*)\) is \((8.4\varepsilon, d - 2.1\varepsilon - 3d^2/10 - 2d^3)\)-super-regular. We call \((X^*_i, Y^*_i, F^*)\) a ready H-triple.

**Step 5.** Apply the Blow-up lemma to find a HIT within each ready H-pair and among each ready H-triple.

In order to apply the Blow-up Lemma, we first give two lemmas which assure the existence of a given subgraph in a complete bipartite graph.

**Lemma 3.4.4.** Suppose \(0 < \varepsilon \ll d \ll 1\) and \(N\) is a large integer. If \(G(A, B)\) is a balanced complete bipartite graph with \((1 - \varepsilon - d^2/2)N \leq |A| = |B| \leq N\), then \(G(A, B)\) contains a HIST \(T_{\text{pair}}\) with \(\Delta(T_{\text{pair}}) \leq \lceil 2/d^3 \rceil\) and \(||L(T_{\text{pair}}) \cap A| - |L(T_{\text{pair}}) \cap B|| = \ell\) for any given non-negative integer \(\ell\) with \(\ell \leq d^2N\).

**Proof.** By the symmetry, we only show that we can construct a HIST \(T\) such that
\[|L(T) \cap A| - |L(T) \cap B| = \ell.\] Let \(\Delta' = \lceil d^3N \rceil.\) We choose distinct \(a_1, a_2, \ldots, a_{\Delta'} \in A\) and distinct \(b_1, b_2, \ldots, b_{\Delta' - 1} \in B.\) Then we decompose all vertices in \(B\) into \(B_1, B_2, \ldots, B_{\Delta'}\) such that \(3 \leq |B_i| \leq 1/d^3, B_i \cap B_{i+1} = \{b_i\}\) for \(1 \leq i \leq \Delta' - 1,\) and \(B_i \cap B_j = \emptyset\) for \(|i - j| > 1.\)

Now we choose \(\ell + 1\) distinct vertices \(b_{\Delta'}, b_{\Delta' + 1}, \ldots, b_{\Delta' + \ell}\) from \(B - \{b_i : 1 \leq i \leq \Delta' - 1\}.\)

As \(\Delta' = \lceil d^3N \rceil, \ell + \Delta' \leq (d^2 + d^3)N + 1,\) and thus

\[2(\ell + \Delta') \leq (2d^2 + 2d^3)N + 2 \leq (1 - d^2/2 - \varepsilon)N - \lceil d^3N \rceil \leq |A| - \lceil d^3N \rceil.\]

Thus we can use all of the vertices in \(\{b_i : 1 \leq i \leq \Delta' + \ell\}\) to cover all vertices in \(A - \{a_i \mid 1 \leq i \leq \Delta' - 1\}\) such that each \(b_i\) can be adjacent to at least two distinct vertices. We partition \(A - \{a_i \mid 1 \leq i \leq \Delta' - 1\}\) arbitrarily into \(A_1, A_2, \ldots, A_{\ell + \Delta'}\) such that \(2 \leq |A_i| \leq 1/d^3.\)

Now let \(T\) be a spanning subgraph of \(G(A, B)\) such that

\[E(T) = \{a_ib \mid b \in B_i, 1 \leq i \leq \Delta'\} \cup \{b_ja \mid a \in A_j, 1 \leq j \leq \Delta' + \ell\}.\]

Clearly, \(\Delta(T) \leq \lceil 2/d^3 \rceil.\) As \(|A| = |B|, |S(T) \cap A| = \Delta',\) and \(|S(T) \cap B| = \Delta' + \ell,\) we then have that \(|L(T) \cap A| - |L(T) \cap B| = \ell.\) We denote \(T\) as \(T_{\text{pair}}.\) \(\square\)

**Lemma 3.4.5.** Suppose \(0 < \varepsilon \ll d \ll 1\) and \(N\) is a large integer. Let \(G = G(A, B, F)\) be a tripartite graph with \(V(G)\) partitioned into \(A \cup B \cup F\) such that both \(G[A \cup B]\) and \(G[B \cup F]\) are complete bipartite graphs. If (i) \((1 - 4\varepsilon - d^2/2)N \leq |A|, |B| \leq N,\) (ii) \((1/2 - 2.1\varepsilon - 3d^2/20 - d^3)N \leq |F| \leq (1/2 - d^3)N,\) and (iii) for any given non-negative integer \(l \leq 3d^2N/10,\) we have \(|B| - 2(|A \cup F| - |B| - l) \geq 3d^3N/2\) holds, then \(G\) contains a \(HIST T_{\text{triple}}\) and a path \(P_{\text{triple}}\) spanning on a subset of \(L(T_{\text{triple}})\) such that

(a) \(T_{\text{triple}}\) is a \(HIST\) of \(G\) with \(\Delta(T_{\text{triple}}) \leq \lceil 3/d^3 \rceil;\)

(b) \(|L(T_{\text{triple}}) \cap B| = |L(T_{\text{triple}}) \cap (A \cup F)| - l.\)

(c) \(P_{\text{triple}}\) is a \((b, f)\)-path on \(L(T_{\text{triple}}) \cap F\) and any \(|L(T_{\text{triple}}) \cap F|\) vertices from \(L(T_{\text{triple}}) \cap B,\) and \(|V(P_{\text{triple}}) \cap F| \leq 5d^2N/6.\)
Proof. Let \( \Delta' = [d^3N/2] \). We choose distinct \( b_1, b_2, \ldots, b_{\Delta'} \in B \) and partition all vertices in \( F \) into \( F_1, F_2, \ldots, F_{\Delta'} \) such that \( 3 \leq |F_i| \leq 1/d^3 \). Then we choose distinct \( a_1, a_2, \ldots, a_{\Delta'-1} \in A \) and decompose all vertices in \( A \) into \( A_1, A_2, \ldots, A_{\Delta'} \) such that \( 3 \leq |A_i| \leq 2/d^3 \), \( A_i \cap A_{i+1} = \{a_i\} \) for \( 1 \leq i \leq \Delta' - 1 \), and \( A_i \cap A_j = \emptyset \) for \( |i - j| > 1 \). Choose one more vertex, say \( a_{\Delta'} \in A - \{a_i | 1 \leq i \leq \Delta' - 1\} \). Let \( l' = |A \cup F| - |B| - l \). Notice that \( l' > 0 \). Now we choose \( l' \) distinct vertices \( f_1, f_2, \ldots, f_l \) from \( A - \{a_i : 1 \leq i \leq \Delta'\} \cup F \) (choose as many as possible from \( F \) first) and partition any \( 2l' \) vertices of \( B - \{b_i : 1 \leq i \leq \Delta'\} \) into \( B_1, B_2, \ldots, B_{l'} \) such that \( |B_i| = 2 \). By (iii), we see that there are at least \( |d^3N| \) vertices left in \( B' = B - \{b_i : 1 \leq i \leq \Delta'\} - \bigcup_{i=1}^{l'} \{B_i\} \). Hence we can partition \( B' = B'_1 \cup B'_2 \cup \cdots \cup B'_{\Delta'} \) such that \( |B'_{\Delta'}| \geq 2 \) and \( |B'_j| \geq 1 \) for \( j \neq \Delta' \). We let \( T \) be a subgraph of \( G \) on \( A \cup B \cup F \) with

\[
E(T) = \{b_if, b_ifa_i b'_i : f \in F_i, a \in A_i, b'_i \in B_i', 1 \leq i \leq \Delta'\} \cup \{f_b : b \in B_i, 1 \leq i \leq l'\}.
\]

By the construction, \( T \) is a HIST of \( G \), which clearly satisfies (a). Since \( |S(T) \cap B| = \Delta' \) and \( |S(T) \cap (A \cup F)| = \Delta' + l' = \Delta' + |A \cup F| - |B| - l \), we then see that \( T \) satisfies (b). If \( L(T) \cap F \neq \emptyset \), let \( f \in L(T) \cap F \) and \( b \in L(T) \cap B \), we can then take a \((b, f)\)-path \( P \) with \( V(P) \cap F = L(T) \cap F \) and \( |V(P)| = 2|L(T) \cap F| \). By (i) and (ii), we see that \( l' = |A \cup F| - |B| - l \geq (1/2 - 6.1\e - 4d^2/5 - d^3)N \). Hence \( |V(P) \cap F| = |F| - l' \leq 5d^2N/6 \).

Denote \( T \) as \( T_{\text{triple}} \) and \( P \) as \( P_{\text{triple}} \). \( \square \)

Now for \( 1 \leq i \leq t \) and for each ready H-pair \((X^*_i, Y^*_i)\), suppose, without of loss generality, that \( |(L(T_{X^*_i}) \cap Y^*_i) \cup ((L(T_{Y^*_i}) \cap Y^*_i)) - |((L(T_{X^*_i}) \cap X^*_i) \cup ((L(T_{Y^*_i}) \cap X^*_i))| \geq l' \), where \( T_{X^*_i} \) is the insertion HIT associated with \( X^*_i \) and \( T_{Y^*_i} \) is the insertion HIT associated with \( Y^*_i \). Notice that \( l' \leq d^2N \) from (3.10) and (3.11). Let \( x_a \in S(T_{X^*_i}) \cap X^*_i \) be a non-leaf of \( T_{X^*_i} \) and \( y_b \in S(T_{Y^*_i}) \cap Y^*_i \) a non-leaf of \( T_{Y^*_i} \). Since \((X^*_i, Y^*_i)\) is \((2\e, d - 2\e)\)-super-regular by Fact 3.4.3 and \( |Y^*_i - Y^*_i*| \leq 2d^2N/5 \), we have \( \deg(x_a, Y^*_i) \geq (d - 2\e - d^2/2)N \geq dN/2 \). Similarly, \( \deg(y_b, X^*_i) \geq (d - 2\e - d^2/2)N \geq dN/2 \). Also, from Step 2, we have \( \Gamma(x^*_i, Y_i), \Gamma(x^*_i, Y_i) \geq (d - 3\e)N \). So, \( \Gamma(x^*_i, Y_i), \Gamma(x^*_i, Y_i) \geq (d - 3\e - d^2/2)N \geq dN/2 \). Similarly, we have
\(\Gamma(y_i^*, X_i^*) \geq (d-3\varepsilon - d^2/2)N \geq dN/2\). Recall that \((X_i^*, Y_i^*)\) is \((4\varepsilon, d-2\varepsilon-8d^2/20)\)-super-regular by Fact 3.4.7 and therefore \((X_i^*, Y_i^*)\) is \((4\varepsilon, d/2)\)-super-regular. By the the strengthened version of the Blow-up lemma and Lemma 3.4.4 (the conditions are certainly satisfied by \(X_i^*\) and \(Y_i^*\)), we can find a HIST \(T_i^i \cong T_{\text{pair}}\) on \(X_i^* \cup Y_i^*\) such that there exist \(y_a \in S(T_i^i) \cap \Gamma(x_a, Y_i^*), x_b \in S(T_i^i) \cap \Gamma(y_b, X_i^*), y_i' \in S(T_i^i) \cap \Gamma(x_i^*, Y_i), y_i'' \in S(T_i^i) \cap \Gamma(x_i^*, Y_i),\)
and \(x_i' \in S(T_i^i) \cap \Gamma(y_i^*, X_i), x_i'' \in S(T_i^i) \cap \Gamma(y_i^*, X_i)\) such that \(|L(T_i^i) \cap X_i^*| - |L(T_i^i) \cap Y_i^*| = l'\).
Hence \(|L(T_i^i) \cap X_i^*| + |L(T_{X_i^*}) \cap X_i^*| = |L(T_i^i) \cap Y_i^*| + |L(T_{Y_i^*}) \cap Y_i^*| + |L(T_{T_i^i}) \cap Y_i^*|\).
Let \(T^i = T_i^i \cup T_{X_i^*} \cup T_{Y_i^*} \cup \{x_a y_a, y_b x_b\} \cup \{x_i^* y_i', x_i^* y_i'', y_i^* x_i', y_i^* x_i''\}\). It is clear that \(T^i\) is a HIST on \(X_i^* \cup Y_i^* \cup I(X_i^*) \cup I(Y_i^*)\) such that
\[
\{x_i^*, x_i^{**}, y_i^*, y_i^{**}\} \subseteq L(T^i) \quad \text{and} \quad |L(T^i) \cap X_i^*| = |L(T^i) \cap Y_i^*|.
\]
For the ready H-pair \((X_0^*, Y_0^*)\), let \(x_a \in S(T_{X_0^*}) \cap X_0^*\) be a non-leaf of \(T_{X_0^*}\) and \(y_b \in S(T_{Y_0^*}) \cap Y_0^*\) a non-leaf of \(T_{Y_0^*}\). By the the strengthened version of the Blow-up lemma and Lemma 3.4.4 (the conditions are certainly satisfied by \(X_0^*\) and \(Y_0^*\)), we can find a HIST \(T_1^0 \cong T_{\text{pair}}\) on \(X_0^* \cup Y_0^*\) such that there exist \(y_0 \in S(T_1^0) \cap \Gamma(x_0^*, Y_0), y_0'' \in S(T_1^0) \cap \Gamma(x_0^{**}, Y_0), x_0' \in S(T_1^0) \cap \Gamma(y_0^*, X_0), x_0'' \in S(T_1^0) \cap \Gamma(y_0^{**}, X_0),\)
and \(x_0'' \in S(T_1^0) \cap \Gamma(y_0^{**}, X_0)\) such that
\[
|L(T_1^0) \cap X_0^*| + |L(T_{X_0^*}) \cap X_0^*| = |L(T_1^0) \cap Y_0^*| + |L(T_{Y_0^*}) \cap Y_0^*| + 2.
\]
Let \(T^0 = T_1^0 \cup T_{X_0^*} \cup T_{Y_0^*} \cup \{x_a y_a, y_b x_b\} \cup \{x_0^* y_0', x_0^{**} y_0', y_0^* x_0', y_0^{**} x_0', y_0' x_0' + 1, y_0'' x_0' + 1\}\). It is clear that \(T^0\) is a HIST on \(X_0^* \cup Y_0^* \cup I(X_0^*) \cup I(Y_0^*)\) such that
\[
\{x_0^*, x_0^{**}, y_0^*, y_0^{**}, y_0' x_0', y_0'' x_0', y_0' x_0' + 1, y_0'' x_0' + 1\} \subseteq L(T^0) \quad \text{and} \quad |L(T^0) \cap X_0^*| = |L(T^0) \cap Y_0^*| + 2.
\]
For each ready triple \((X_i^*, Y_i^*, F_i^*)\), we know that \((X_i^*, Y_i^*)\) is \((4\varepsilon, d - 3.1\varepsilon - 9d^2/20)\)-super-regular and \((Y_i^*, F_i^*)\) is \((8.4\varepsilon, d - 2.1\varepsilon - 3d^2/10 - 2d^3)\)-super-regular by Fact 3.4.8. Notice that \((1 - 4\varepsilon - 9d^2/20)N \leq |X_i^*|, |Y_i^*| \leq N\) and \((1/2 - 2.1\varepsilon - 3d^2/30 - d^3)N \leq |F_i^*| \leq (1/2 - d^3)N\). Let \(|I(X_i^*)| = l'\) and \(|I(F_i^*)| = l/6\) for some integer \(l\). By Operation II we have \(|V(T_{X_i^*}) \cap X_i^*| \leq 3l'\) and \(|V(T_{Y_i^*}) \cap Y_i^*| \leq 2l' + 1\). By Operation III we have \(|V(T_{F_i^*}) \cap F_i^*| = l/2\).
and $|V(T_{F'}) \cap Y_i'| = l$. Notice that $|L(T_{X_i'}) \cap X_i'| = |l(T_{X_i'}) \cap Y_i'|$. Hence,

$$|Y_i'| - 2(|X_i^* \cup F^*| - |Y_i^*| - l) \geq 3(|Y_i'| - 2l' - l - 1) - 2(|X_i'| - 3l') - 2(|F'| - l/2) + 2l = 3|Y_i'| - 2|X_i'| - 2|F'| - 3 \geq 3(1 - 3.1\varepsilon)N - 2N - N + 2d^3N - 3 > 3d^3N/2.$$

By the weak version of the Blow-up lemma (Lemma 3.2.2) and Lemma 3.4.5, we then can find a HIT $T_i \cong T_{\text{triple}}$ on $X_i^* \cup Y_i^* \cup F^*$ and a path $P_i \cong P_{\text{triple}}$ spanning on $L(T_i) \cap F^*$ and other $|L(T_i^*) \cap F^*|$ vertices from $Y_i^*$. Let $y_a \in S(T_{X_i'}) \cap Y_i'$ be a non-leaf of $T_{X_i'}$ (take $y_a$ as the degree 2 vertex if $T_{X_i'}$ has one) and $y'_a \in S(T_{F'}) \cap Y_i'$ a non-leaf of $T_{F'}$. Then as $(Y_i', F')$ is $(4.1\varepsilon, d - 2.1\varepsilon - 2d^3)$-super-regular, we have $|\Gamma(y_a, F')|, |\Gamma(y'_a, F')| \geq (d - 2.1\varepsilon - 2d^3)N/2$. Since $|F' - F^*| \leq 3d^2N/20$, we then know that $|\Gamma(y_a, F^*)|, |\Gamma(y'_a, F^*)| \geq (d - 2.1\varepsilon - 3d^2/10 - 2d^3)N/2$. Since $|F^* \cap L(T_i)| = |V(P_i) \cap F^*| \leq 5d^2N/6 < (d - 2.1\varepsilon - 3d^2/10 - 2d^3)N/2$, there exist $f_a \in (S(T_i) \cap F^*) \cap \Gamma(y_a, F^*)$ and $f'_a \in (S(T_i') \cap F^*) \cap \Gamma(y'_a, F^*)$. For each $x \in I(F')$, since $\deg(x, F') \geq (d - \varepsilon)|F'| \geq (d - \varepsilon)(1 - 2.1\varepsilon - d^3)N/2$, we know there exists $f' \in (S(T_i') \cap F^*) \cap \Gamma(x, F^*)$. From Step 2, we have $|\Gamma(x_i^*, Y_i) \cap \Gamma(x_i^{**}, Y_i)| \geq (d - \varepsilon)^2N$ and $|\Gamma(y_i^*, X_i) \cap \Gamma(y_i^{**}, X_i)| \geq (d - \varepsilon)^2N$. Hence $|\Gamma(x_i^*, Y_i) \cap \Gamma(x_i^{**}, Y_i')| \geq ((d - \varepsilon)^2 - 3.1\varepsilon)N$. Since $|S(T_i \cup T_{X_i} \cup T_{F'}) \cap X_i'| < d^2N/2$, we see that there exists $y' \in \Gamma(x_i^*, Y_i) \cap \Gamma(x_i^{**}, Y_i) \cap L(T_i \cup T_{X_i} \cup T_{F'})$. Similarly, there exists $x' \in \Gamma(y_i^*, X_i) \cap \Gamma(y_i^{**}, X_i) \cap L(T_i \cup T_{X_i} \cup T_{F'})$. Let $T^i = T_i \cup T_{X_i} \cup T_{F'} \cup \{xf' : x \in I(F') \cap \Gamma(x, F^*) \} \cup \{yaf_a, y'_af'_a\} \cup \{y'x_i^*, y'i^{**}, x'y_i^*, x'y_i^{**}\}$. It is clear that $T^i$ is a HIST on $X_i^* \cup Y_i^* \cup F^* \cup I(X_i') \cup I(F')$ such that

$$\{x_i^*, x_i^{**}, y_i^*, y_i^{**}\} \subseteq L(T^i) \quad \text{and} \quad |L(T^i) \cap X_i'| = |L(T^i) \cap Y_i'|.$$  

Let $H^i = T^i \cup P_i$. We call $P_i$ the accompany path of $T^i$.

**Step 6.** Apply the Blow-up Lemma again on the regular-pairs induced on the leaves of each HIT obtained in Step 5 to find two vertex-disjoint paths covering all the leaves. Then connect all the HITs into a HIST of $G$ and connect the disjoint paths into a cycle using the edges initiated in Step 2.
Suppose \(1 \leq i \leq t\). For each H-pair \((X_i, Y_i)\), let \(X_i^L = X_i^L \cap L(T^i) - \{x_i^*, x_i^{**}\}\) and \(Y_i^L = Y_i^L \cap L(T^i) - \{y_i^*, y_i^{**}\}\), and for each H-triple \((X_i, Y_i, F)\), let \(X_i^L = X_i^L \cap L(T^i \cup P_i) - \{x_i^*, x_i^{**}\}\) and \(Y_i^L = Y_i^L \cap L(T^i \cup P_i) - \{y_i^*, y_i^{**}\}\), where \(T^i\) is the HIST found in Step 5, and \(P_i\) is the accompanying path of \(T^i\). By Operations I, II and III, and the proofs of the Lemmas 3.4.4 and 3.4.5, we have \(I(X_i^L) \cup I(Y_i^L) \subseteq S(T_i)\) and \(F^i \cup F(\varrho) \subseteq S(T_i \cup P_i)\). Thus, \(X_i^L \cup Y_i^L = L(T^i) - \{x_i^*, x_i^{**}, y_i^*, y_i^{**}\}\) for each H-pair and \(X_i^L \cup Y_i^L = L(T^i \cup P_i) - \{x_i^*, x_i^{**}, y_i^*, y_i^{**}\}\) for each H-triple. Furthermore, we have \(|X_i^L| = |Y_i^L|\). For the H-pair \((X_0, Y_0)\), let \(X_0^L = X_0^L \cap L(T^0) - \{x_0^*, x_0^{**}, y_{t+1}^*, y_{t+1}^{**}\}\) and \(Y_0^L = Y_0^L \cap L(T^0) - \{y_0^*, y_0^{**}\}\). We have \(X_0^L \cup Y_0^L = L(T^0) - \{x_0^*, x_0^{**}, y_{t+1}^*, y_{t+1}^{**}\}\) and \(|X_0^L| = |Y_0^L|\) since from Step 5 we have \(|L(T^0) \cap X_0^L| = |L(T^0) \cap Y_0^L| + 2\). By the construction of \(T_{pair}\) and \(H_{triple}\), we see that \(|S(T_i) \cap X_i^L|, |S(T_i) \cap Y_i^L| \leq d^2 N\). Since each H-pair \((X_i', Y_i')\) is \((2\varepsilon, d - 2\varepsilon)\)-super-regular, and each pair \((X_i^L, Y_i^L)\) from an H-triple \((X_i', Y_i', F')\) is \((2\varepsilon, d - 3.1\varepsilon)\)-super-regular, by Slicing Lemma, we then know that \((X_i^L, Y_i^L)\) is \((4\varepsilon, d - 3.1\varepsilon - d^2)\)-super-regular and hence is \((4\varepsilon, d/2)\)-super-regular.

For each \(1 \leq i \leq t\), by the choice of \(x_i^*, x_i^{**}, y_i^*, y_i^{**}\), we have \(|\Gamma(x_i^*, Y_i)|, |\Gamma(x_i^{**}, Y_i)| \geq (d - \varepsilon) N\) and \(|\Gamma(y_i^*, X_i)|, |\Gamma(y_i^{**}, X_i)| \geq (d - \varepsilon) N\). Hence, \(|\Gamma(x_i^*, Y_i^L)|, |\Gamma(x_i^{**}, Y_i^L)| \geq (d - \varepsilon - 3.1\varepsilon - d^2) N > dN/2\) and \(|\Gamma(y_i^*, X_i^L)|, |\Gamma(y_i^{**}, X_i^L)| \geq (d - \varepsilon - 3.1\varepsilon - d^2) N > dN/2\). Similar results hold for the vertices \(x_0^*, x_0^{**}, y_{t+1}^*, y_{t+1}^{**}\). For each \(0 \leq i \leq t\), we choose distinct vertices \(y_i' \in \Gamma(x_i^*, Y_i^L), y_i'' \in \Gamma(x_i^{**}, Y_i^L)\) and \(x_i' \in \Gamma(y_i^*, X_i^L), x_i'' \in \Gamma(y_i^{**}, X_i^L)\). If \(T^i\) does not have the accompanying path, then by the strengthened version of the Blow-up lemma, we can find an \((x_i', y_i')\)-path \(P_1^i\) and an \((x_i'', y_i'')\)-path \(P_2^i\) such that \(P_1^i \cup P_2^i\) is spanning on \(X_i^L \cup Y_i^L\). If \(T^i\) has the accompanying \((b, f)\)-path \(P_i\), we see that \(\deg(b, X_i^L), \deg(f, Y_i^L) \geq dN/2\) as \((X_i', Y_i')\) is \((2\varepsilon, d - 3.1\varepsilon)\)-super-regular, and \((Y_i', F')\) is \((4\varepsilon, d - 3.1\varepsilon - 2d^3)\)-super-regular. Applying the strengthened version of the Blow-up lemma, we can find an \((x_i', y_i')\)-path \(P_{11}^i\) and an \((x_i'', y_i'')\)-path \(P_{12}^i\) such that \(P_{11}^i \cup P_{12}^i\) is spanning on \(X_i^L \cup Y_i^L\), and two consecutive internal vertices \(a', b' \in V(P_{11}^i)\) with \(b' \in \Gamma(f, Y_i^L)\), and \(a' \in \Gamma(b, X_i^L)\). Let \(P_i = P_{11}^i \cup P_i \cup \{fb', ba'\} - \{a'b'\}\). Notice that for the H-pair \((X_0, Y_0)\), the two vertices \(y_0^*, y_0^{**}\) are not used in this step, but we will connect them to \(y_0^*\) and \(y_0^{**}\), respectively, in next step.

We now connect the small HITs and paths together to find an SGHG of \(G\). In Case A,
for $1 \leq i \leq t - 1$, we have $|S(T^i) \cap Y_i| \geq d^3 N/2 > \varepsilon N$ and $|S(T^{i+1}) \cap X_{i+1}| \geq d^3 N/2 > \varepsilon N$. Since $(Y_i, X_{i+1})$ is an $\varepsilon$-regular pair with density $d$, we see that there is an edge $e_i$ connecting $S(T^{i+1}) \cap X_{i+1}$ and $S(T^{i+1}) \cap X_{i+1}$. Let

$$T = \bigcup_{i=1}^{t} T^i \cup \{e_i \mid 1 \leq i \leq t - 1\}.$$ 

Then $T$ is a HIST of $G$. Let $C$ be the cycle formed by all the paths in $\bigcup_{i=1}^{t} (P^i \cup P_i^2)$ and all edges in the following set

$$\{x^i y^i, x^i y^i, y^i x^i, y^i x^i : 1 \leq i \leq t\} \cup \{y^i x^i, y^i x^i : 1 \leq i \leq t - 1\} \cup \{y^i x^i, y^i x^i\},$$

notices that the edges in $\{y^i x^i, y^i x^i : 1 \leq i \leq t - 1\} \cup \{y^i x^i, y^i x^i\}$ above are guaranteed in Step 2. It is easy to see that $C$ is a cycle on $L(T)$. Hence $H = T \cup C$ is an SGHG of $G$.

In Case B, for $1 \leq i \leq t - 1$, we have $|S(T^i) \cap Y_i| \geq d^3 N/2 > \varepsilon N$ and $|S(T^{i+1}) \cap X_{i+1}| \geq d^3 N/2 > \varepsilon N$. Since $(Y_i, X_{i+1})$ is an $\varepsilon$-regular pair with density $d$, we see that there is an edge $e_i$ connecting $S(T^{i+1}) \cap X_{i+1}$ and $S(T^{i+1}) \cap X_{i+1}$. Similarly, there is an edge $e_0$ connecting $S(T_0) \cap X_0$ and $S(T_1) \cap X_1$. Let

$$T = \bigcup_{i=1}^{t} T^i \cup \{e_i \mid 0 \leq i \leq t - 1\}.$$ 

Then $T$ is a HIST of $G$. Let $C$ be the cycle formed by all paths in $\bigcup_{i=1}^{t} (P^i \cup P_i^2)$ and all edges in the set $\{y_0 y^i_{i+1}, y_0 y^i_{i+1}, y^i_{i+1} x^i_{i+1}, y^i_{i+1} x^i_{i+1}, x^i_{0} y^i_{t}, x^i_{0} y^i_{t}\}$ and in the following set

$$\{x^i y^i, x^i y^i, y^i x^i, y^i x^i : 0 \leq i \leq t\} \cup \{y^i x^i, y^i x^i : 1 \leq i \leq t - 1\}.$$ 

It is easy to see that $C$ is a cycle on $L(T)$. Hence $H = T \cup C$ is an SGHG of $G$.

The proof of Theorem \ref{3.4.3} is now finished. \qed
3.4.3.2 Proof of Theorem 3.4.4

By the assumption that \( \text{deg}(v_1, V_2) \leq 2\beta n \) for each \( v_1 \in V_1 \) and the assumption that \( \delta(G) \geq (2n + 3)/5 \) in Extremal Case 1, we see that

\[
\delta(G[V_1]) \geq (2n + 3)/5 - 2\beta n. \tag{3.13}
\]

Then (3.13) implies that

\[
|V_1| \geq (2n + 3)/5 - 2\beta n \quad \text{and} \quad |V_2| \leq 3n/5 + 2\beta n. \tag{3.14}
\]

Also, by \( |V_2| \geq (2/5 - 4\beta)n \) in the assumption,

\[
|V_1| \leq (3/5 + 4\beta)n. \tag{3.15}
\]

We will construct an SGHG of \( G \) following several steps below.

Step 1. Repartitioning

Set \( \alpha_1 = \alpha^{1/3} \) and \( \alpha_2 = \alpha^{2/3} \). Let

\[
V'_1 = V_1 \quad \text{and} \quad V'_2 = \{ v \in V_2 \mid \text{deg}(v, V_1) \leq \alpha_1|V_1| \}.
\]

Then by \( d(V_1, V_2) \leq \alpha \), we have

\[
\alpha_1|V_1||V_2 - V'_2| \leq e(V_1, V'_2) + e(V_1, V_2 - V'_2) = e(V_1, V_2) \leq \alpha|V_1||V_2|.
\]

This gives that

\[
|V_2 - V'_2| \leq \alpha_2|V_2|. \tag{3.16}
\]

Denote \( V''_{12} = V_2 - V'_2 \). Then by the definition of \( V'_2 \), we have

\[
\delta(V''_{12}, V'_1) > \alpha_1|V'_1| \quad \text{and} \quad \delta(G[V'_2]) \geq (2n + 3)/5 - \alpha_1|V'_1| \geq (2/5 - \alpha_1(3/5 + 4\beta))n, \tag{3.17}
\]
where the last inequality follows from (3.15).

Let \( n_i = |V'_i| \) for \( i = 1, 2 \). Then by (3.13) and (3.15),

\[
\delta(G[V'_1]) \geq \left( \frac{2n + 3}{5} - 2\beta n \right) \geq \frac{2/5 - 2\beta}{3/5 + 4\beta} n_1 \geq (2/3 - 8\beta)n_1, \tag{3.18}
\]

and by (3.14) and the second inequality in (3.17),

\[
\delta(G[V'_2]) \geq (2/5 - \alpha_1(3/5 + 4\beta))n \geq \left( \frac{2/5 - \alpha_1(3/5 + 4\beta)}{3/5 + 2\beta} \right) n_2 \geq (2/3 - 1.1\alpha_1)n_2,
\]

provided that \( \beta \leq \frac{0.3\alpha_1}{0.9\alpha_1 + 20/3} \).

**Step 2. Finding three connecting edges**

AS \( G \) is 3-connected, there are 3 independent edges \( x_1^L y_L, x_2^L y_L \) and \( x_N y_N \) connecting \( V'_1 \cup V'_2 \) such that \( x_1^L, x_2^L, x_N \in V'_1 \cup V'_2 \) and \( y_1^L, y_2^L, y_N \in V'_2 \). In the remaining steps, we will find a HIST \( T_1 \) in \( G[V'_1 \cup V'_2] \) with \( x_N \) as a non-leaf and \( x_1^L, x_2^L \) as leaves, and a HIST \( T_2 \) of \( G[V'_2] \) with \( y_N \) as a non-leaf and \( y_1^L, y_2^L \) as leaves. Then \( T = T_1 \cup T_2 \cup \{x_N y_N\} \) is a HIST of \( G \). By finding a hamiltonian \((x_1^L, x_2^L)\)-path \( P_1 \) on \( L(T_1) \), and a hamiltonian \((y_1^L, y_2^L)\)-path on \( L(T_2) \), we see that

\[
C := P_1 \cup P_2 \cup \{x_1^L y_1^L, x_2^L y_2^L\}
\]

forms a cycle on \( L(T) \). Hence \( H := T \cup C \) is an SGHG of \( G \).

**Step 3. Initiating two HITs**

In this step, we first initiate a HIT in \( G[V'_1 \cup V'_2] \) containing \( X_N \) as a non-leaf and \( x_1^L \) and \( x_2^L \) as leaves. Then, we initiate a HIT in \( G[V'_2] \) containing \( y_N \) as a non-leaf and \( y_1^L \) and \( y_2^L \) as leaves.

For \( x_1^L, x_2^L, x_N \in V'_1 \cup V'_2 \), by (3.13) and (3.17), each of them has at least \( \alpha_1 |V'_1| \geq 9 \) neighbors in \( V'_1 \). Thus, we choose distinct \( z_1^L, z_1^L, z_2^L, z^2, z_4^L, z_N^L, z_N^L \in V'_1 \) such that

\[
x_1^L \sim z_1^L, z^1, \quad x_2^L \sim z_2^L, z^2, \quad x_N \sim z_4^L, z_N^L, z_N^L.
\]

(Note that \( x_1^L \) and \( x_2^L \) may be from \( V'_2 \), and therefore they may not have too many neighbors...
in $V'_1$, we then choose $z_L^1$ and $z_L^2$ from $V'_1$ as their neighbors, respectively.)

By (3.18), we see that any two vertices in $G[V'_1]$ have at least $(1/3 - 16\beta)n_1 \geq 14$ neighbors in common. Thus, we can choose distinct vertices $z^{11}, z^{22}, z^{12}, v^R_1 \in V'_1 - \{x_L^1, x_L^2, x_N, z_L^1, z_L^2, z_N^1, z_N^2, z_N^3\}$ such that

$$z^{11} \sim z_L^1, z^1, z^{22} \sim z_L^2, z^2, z^{12} \sim z^{11}, z^{22}, v^R_1 \sim z^{12}, z_N^1.$$  

Furthermore, by (3.18) again, we have $\delta(G[V'_1]) \geq (2/3 - 8\beta)n_1 \geq 17$. Choose $z_1^1, z_2^2, z_N^1 \in V'_1$ not chosen above such that

$$z_1^1 \sim z^1, z_2^2 \sim z^2, z_N^1 \sim z_N^1.$$  

Let $T_{11}$ be the graph with

$$V(T_{11}) = \{x_L^1, x_L^2, x_N, z_L^1, z_N^1, z_L^2, z_N^2, z_L^1, z_N^1, z_L^2, z_N^2, z_L^2, z_N^3, v^R_1, z_1^1, z_2^2, z_N^1\}$$

and with edges indicated above except the edges $x_L^1z_L^1$ and $x_L^2z_L^2$. We see that $T_{11}$ is a tree with $v^R_1$ as the only degree 2 vertex, and $|V(T_{11})| = 17$ and $|L(T_{11})| = 9$. Notice that in $T_{11}$, $z_L^1, x_L^1$ and $z_N^1, x_N^2$ are leaves, and $x_N$ is a non-leaf. Figure 3.6 gives a depiction of $T_{11}$.

![Figure (3.6) The tree $T_{11}$](image)

Notice that the edges $x_L^1z_L^1$ and $x_L^2z_L^2$ are not used in $T_{11}$. We will first construct a HIST $T_1$ in $G[V'_1 \cup V_{12}^0]$ containing $T_{11}$ as a subgraph, then find a hamiltonian $(z_L^1, z_L^2)$-path on $L(T_1) - \{x_L^1, x_L^2\}$ by Lemma 3.2.6 finally by adding $x_L^1z_L^1$ and $x_L^2z_L^2$ to the path, we get a
hamiltonian \((x_1^L, x_2^L)\)-path on \(L(T_1)\). The reason that we avoid using \(x_1^L\) and \(x_2^L\) is that when \(x_1^L, x_2^L \in V_{12}^0\), we may not be able to have the condition of Lemma 3.2.6 on \(G[L(T_1)]\) in our final construction.

Then we initiate a HIT in \(G[V'_2]\) containing \(y_1^L, y_2^L\) as leaves, and \(y_N\) as a non-leaf.

As \(y_1^L, y_2^L, y_N \in V'_2\), by (3.19) and the fact that each two vertices from \(V'_2\) have at least \((1/3 - 2\alpha_1)n_2 \geq 7\) common neighbors implied from (3.19), we can choose distinct vertices

\[y_{12}, y_N, y_N, y_N, y_N, v_2^R \in V'_2 - \{y_1^L, y_2^L, y_N\}\]

such that

\[y_{12} \sim y_1^L, y_2^L, y_N \sim y_N, y_N, y_N, v_2^R \sim y_{12}^L, y_N.\]  

(3.19)

Let \(T_{21}\) be the graph with \(V(T_{21}) = \{y_1^L, y_2^L, y_N, y_{12}^L, y_N^3, y_N, v_2^R\}\) and with \(E(T_{21})\) described as in (3.19).

We see that \(T_{21}\) is a tree with \(v_2^R\) the only degree 2 vertex and \(y_1^L, y_2^L \in L(T_{21}), y_N \in S(T_{21})\) and

\[|V(T_{21}) \cap V'_2| = 8, \quad |L(T_{21}) \cap V'_2| = 5.\]

(3.20)

Denote

\[U_1 = V'_1 - V(T_{11}), \quad U_2 = V'_2 - V(T_{21}), \quad \text{and} \quad V_{12} = V_{12}^0 - V(T_{11}).\]

**Step 4. Absorbing vertices in \(V_{12}^0\)**

We may assume that \(V_{12}^0 \neq \emptyset\). For otherwise, we skip this step. Let \(|V_{12}| = n_{12}\) and \(V_{12}^0 = \{x_1, x_2, \cdots, x_{n_{12}}\}\).

Since \(|V(T_{11})| = 17\), by (3.17), we get

\[\delta(V_{12}^0, U_1) > \alpha_1|V'_1| - 17 \geq 3\alpha_2|V_2| \geq 3|V_2 - V'_2| \geq 3|V_{12}^0|.\]

Thus, there is a claw-matching \(M_c\) from \(V_{12}^0\) to \(U_1\) centered in \(V_{12}^0\). For \(i = 1, 2, \cdots, n_{12}\), let \(x_{i1}, x_{i2}\) and \(x_{i3}\) be the three neighbors of \(x_i\) in \(M_c\). If \(n_{12} = 1\), let \(T_a = M_c\), and we finish
this step. Thus we assume $n_{12} \geq 2$.

By (3.18), each two vertices in in $V'_1$ have at least

$$(1/3 - 16\beta)n_1 \geq 6\alpha_2 |V'_{12}| + 17 \tag{3.21}$$

neighbors in common. The above inequality holds as $n_1 \geq 2n/5 - 2\beta n$, $|V_2| \leq 3n/5 + 2\beta n$ by (3.14), and we can assume that $18\alpha_2/5 + 106\beta/15 + 12\alpha_2\beta + 18/n - 32\beta^2 \leq 2/15$.

Thus, for each $i = 1, 2, \cdots, n_{12} - 1$, we can find distinct vertices $x^i_{13}, x^i_{23}, x^i_{13}, x^i_{1+1,1}$ in $U_1 - V(M_c)$ such that

$$x^i_{13} \sim x_{i3}, x_{i+1,1}, \quad x^i_{23} \sim x^i_{13}, \quad x^i_{13} \sim x_{i3}, \quad x^i_{1+1,1} \sim x_{i+1,1}. \tag{3.22}$$

Let $T_a$ be the graph with $V(T_a) = V(M_c) \cup \{x^i_{13}, x^i_{23}, x^i_{13}, x_{i+1,1} : 1 \leq i \leq n_{12} - 1\}$, and $E(T_a)$ including all edges indicated in (3.22) for all $i$ and all edges in $M_c$. It is easy to see, by the construction, that $T_a$ is a HIT with

$$|V(T_a) \cap U_1| = 7n_{12} - 4 \quad \text{and} \quad |L(T_a) \cap U_1| = 4n_{12} - 1.$$

Using (3.21) again, we can find $x^1_{11} \in U_1 - V(T_a)$ such that $x^1_{11} \sim v^1_{11}, x_{11}$, where $v^1_{11} \in V(T_{11})$ and $x_{11} \in V(T_a)$. By (3.18),

$$\delta[G[V'_1]] \geq (2n + 3)/5 - 2\beta n \geq 6\alpha_2 |V'_{12}| + 20,$$

since $|V_2| \leq 2n/5 + 2\beta n$, and we can assume that $2\beta - 12\alpha_2\beta - 18\alpha_2/5 - 21/n \leq 2/5$. So we can find distinct vertices $x^2_{11}, x_{11} \in U_1 - V(T_a) - \{x^1_{11}\}$ such that $x^2_{11} \sim x^1_{11}, x^1_{11} \sim x_{11}$.

Let $T^1_1$ be the graph with

$$V(T^1_1) = V(T_{11}) \cup V(T_a) \cup \{x^1_{11}, x^2_{11}, x_{11}\} \quad \text{and} \quad E(T^1_1) = E(T_{11}) \cup E(T_a) \cup \{x^1_{11}v^1_{11}, x^1_{11}x_{11}, x^2_{11}x_{11}, x_{11}x_{11}\}.$$
Then $T_1^1$ is a HIT such that

$$|V(T_1^1) \cap U_1| = 7n_{12} + 16 \quad \text{and} \quad |S(T_1^1) \cap U_1| = 3n_{12} + 7. \quad (3.23)$$

Denote $U'_1 = U_1 - V(T_1^2)$ and $U'_2 = U_2 - V(T_1^2)$.

**Step 5. Completion of HITs $T_1$ and $T_2$**

In this step, we construct a HIST $T_1$ in $G[V'_i](i = 1, 2)$ containing $T_1^i$ as an induced subgraph.

The following lemma guarantees the existence of a specified HIST in a graph with $n$ vertices and minimum degree at least $(2/3 - \alpha')n$ for some $0 < \alpha' \ll 1$.

**Lemma 3.4.6.** Let $H$ be an $n$-vertex graph with $\delta(H) \geq (2/3 - \alpha')n$ for some constant $0 < \alpha' \ll 1$. Then $H$ has a HIST $T_H$ satisfying

(i) $T_H$ has a vertex $v_R$ of degree at least $(2/3 - \alpha')n - 1$, and $v_R$ can be chosen arbitrarily from $V(H)$;

(ii) $|S(T_H)| \leq (1/6 + \alpha'/2)n + 2$.

**Proof.** Let $v_R \in V(H)$ be an arbitrary vertex. If $n(mod 2) \equiv deg(v_R) + 1(mod 2)$, then we let $N_R = N_H(v_R)$. For otherwise, let $N_R$ be a subset of $N(v_R)$ with $|N_H(v_R)| - 1$ elements. Let $T_{v_R}$ be the star with $V(T_{v_R}) = \{v_R\} \cup N_R$ and $E(T_R) = E(\{v_R\}, N_R)$. Let $V_0 = V(H) - V(T_{v_R})$. By $\delta(H) \geq (2/3 - \alpha')n$, we have $|V_0| \leq (1/3 + \alpha')n + 1$. By the choice of $N_R$, we have $|V_0| \equiv 0(mod 2)$. If $V_0 = \emptyset$, then let $T_H = T_{v_R}$. For otherwise, we claim as follows.

**Claim 3.4.3.** Let $V_1 \subseteq V(H)$ be a subset with $|V_1| \geq (2/3 - \alpha')n - 1$ and $|V_1|(mod 2) \equiv n(mod 2)$. Then there exist two vertices from $V_0 = V(H) - V_1$ such that they have a common neighbor in $V_1$.

**Proof of Claim 3.4.3.** We assume that $|V_1| \leq (2/3 + 2\alpha')n$. For otherwise, $|V_0| < (1/3 - 2\alpha')n$. Since $\delta(H) \geq (2/3 - \alpha')n$, any two vertices of $H$ have at least $(1/3 - 2\alpha')n$ neighbors in common. By $|V_0| < (1/3 - 2\alpha')n$, any two vertices from $V_0$ have a common neighbor from
$V_1$. We are done. Thus $|V_1| \leq (2/3 + 2\alpha')n$, and hence $|V_0| \geq (1/3 - 2\alpha')n \geq 3$. By the assumption that $|V_1| \geq (2/3 - \alpha')n - 1$, we have $|V_0| \leq (1/3 + \alpha')n + 1$. This implies that $\deg(v_0, V_1) \geq (1/3 - 2\alpha')n - 2$ for each $v_0 \in V_0$. As $|V_0| \geq 3$ and $3((1/3 - 2\alpha')n - 2) > (2/3 + 2\alpha')n \geq |V_1|$ (provided that $8\alpha' + 6/n < 1/3$), we see that there must be two vertices from $V_0$ such that they have a neighbor in common in $V_1$. \hfill\qed

By Claim 3.4.3 there exist two vertices $v_0^{11}, v_0^{12} \in V_0$ such that they have a common neighbor in $T_{v_R}$. Adding $v_0^{11}$ and $v_0^{12}$ to $T_{v_R}$ and two edges connecting them to one of their common neighbor in $V(T_{v_R})$. Let $T_{v_R}^1$ be the resulting graph. Then we see that $T_{v_R}^1$ is a HIT with $|V(T_{v_R}^1)| = |V(T_{v_R})| + 2$, and hence $(|V(T_{v_R})| + 2)(mod 2) \equiv n(mod 2)$. Also $|V(T_{v_R}^1)| \geq |V(T_{v_R})| \geq (2/3 - \alpha')n - 1$. So we can use Claim 3.4.3 again to find another pair of vertices from $V_0 - \{v_0^{11}, v_0^{12}\}$ such that they have a common neighbor in $V(T_{v_R}^1)$. Adding the new pair of vertices and two edges connecting them to one of their common neighbor in $V(T_{v_R}^1)$ into $T_{v_R}^1$, we get a new HIT $T_{v_R}^2$. By repeating the above process another $l_0 = (|V_0| - 4)/2$ times, we get a HIT $T_{v_R}^{l_0}$. Let $T_H = T_{v_R}^{l_0}$. We claim that $T_H$ has the required properties in Lemma 3.4.6. Notice first that $d_{T_H}(v_R) \geq (2/3 - \alpha')n - 1$. Then since $T_H$ has $v_R$ and at most $|V_0|/2$ distinct vertices as non-leaves and $|V_0| \leq (1/3 + \alpha')n + 1$, we see that $|S(T_H)| \leq (1/6 + \alpha'/2)n + 2$. \hfill\qed

Let $H_1 = G[U_1^1 \cup \{v_1^{1}\}]$. Recall that $v_1^{1}$ is a non-leaf in $T_1^1$. By (3.18) and (3.23), and by noticing that $n_{12} \leq |V_2 - V_2'| \leq \alpha_2|V_2| \leq 3\alpha_2n_1/2$ (by (3.14)), we see that

$$\delta(H_1) \geq (2/3 - 8\beta)n_1 - (7n_{12} + 19) \geq (2/3 - 8\beta)n_1 - 21\alpha_2n_1/2 - 19 \geq (2/3 - 11\alpha_2)|V(H_1)|.$$

(3.24)

Let $\alpha' = 11\alpha_2 \ll 1$ (by assuming $\alpha \ll 1$). By Lemma 3.4.6, we can find a HIT $T_1'$ in $H_1$ with $v_R^1$ as the prescribed vertex in condition(i). It is easy to see that $T_1 := T_1^1 \cup T_1'$ is a
HIST of $G[V'_1 \cup V'_2]$ and

\[
s_1 = |S(T_1) \cap V'_1| = |S(T'_1) \cap V'_1| + |S(T'_1) \cap V'_1|
\leq 3n_{12} + 7 + (1/6 + 5.5\alpha_2)|V(H_1)| + 2 \text{ (by (3.23) and Lemma 3.4.6)}
\leq 3n_{12} + 9 + (1/6 + 5.5\alpha_2)n_1
\leq (1/6 + 10.5\alpha_2)n_1 \text{ (by } n_{12} \leq 3\alpha_2n_1/2) .
\tag{3.25}
\]

Let $H_2 = G[U'_2 \cup \{v'_2\}]$. By (3.19) and (3.20), we see that

\[
\delta(H_2) \geq (2/3 - 1.1\alpha_1)n_2 - 8 \geq (2/3 - 1.2\alpha_1)|V(H_2)|.
\]

By letting $\alpha' = 1.2\alpha_1$, we can find a HIT $T'_2$ in $H_2$ with $v'_2$ as the prescribed vertex in condition (i) of Lemma 3.4.6. Then $T'_2 := T'_2 \cup T'_2$ is a HIST of $G[V'_2]$. Also, notice that

\[
s_2 = |S(T'_2) \cap V'_2| = |S(T''_2) \cap V'_2| + |S(T''_2) \cap V'_2|
\leq 3 + (1/6 + 0.6\alpha_1)|V(H_2)| + 2
\leq (1/6 + 0.7\alpha_2)n_2 ,
\tag{3.26}
\]

where the last inequality holds by assuming $5/n_2 \leq 0.1\alpha_2$.

**Step 6. Finding two long paths**

In this step, we first find a hamiltonian $(z'_1, z'_2)$-path $P'_1$ in $G[L(T_1) - \{x'_1, x'_2\}]$; then find a hamiltonian $(y'_1, y'_2)$-path $P_2$ in $G[L(T_2)]$. Let $G_{11} = G[L(T_1) - \{x'_1, x'_2\}]$ and $n_{11} = |V(G_{11})|$. We will show that $\delta(G_{11}) > \frac{1}{2}n_{11}$. We may assume $s_1 \geq (1/6 - 8\beta)n_1 - 2$. For otherwise, if $s_1 < (1/6 - 8\beta)n_1 - 2$, then by (3.18), we get

\[
\delta(G_{11}) \geq \delta(G[V'_1]) - s_1 - 2
\geq (2/3 - 8\beta)n_1 - ((1/6 - 8\beta)n_1 - 1 - 2) - 2
\geq \frac{1}{2}n_1 + 1 \geq \frac{1}{2}n_{11} + 1 .
\]
Hence, \( s_1 \geq (1/6 - 8\beta)n_1 - 2 \), implying that

\[
n_{11} \leq (5/6 + 8\beta)n_1 + 2 \quad \text{and thus} \quad n_1 \geq \frac{n_{11} - 2}{5/6 + 8\beta},
\]

(3.27)

Hence, by (3.25)

\[
\delta(G_{11}) \geq \delta(G[V_1']) - s_1 - 2 \geq (2/3 - 8\beta)n_1 - (1/6 + 10.5\alpha_2)n_1 - 2 \\
\geq (1/2 - 8\beta - 11\alpha_2)n_1 \geq \frac{1/2 - 2\beta - 11\alpha_2}{5/6 + 2\beta}(n_{11} - 2) > n_{11}/2,
\]

the last inequality holds by assuming \( 3\beta + 11\alpha_2 + 2/n_{11} < 1/12 \). By applying Lemma 3.4.6 on \( G_{11} \), we find a hamiltonian \((z_1^L, z_2^L)\)-path \( P_1^1 \) in \( G_{11} \). Let \( P_1 = P_1^1 \cup \{z_1^Lx_1^L, z_2^Lx_2^L\} \). We see that \( P_1 \) is a hamiltonian \((x_1^L, x_2^L)\)-path on \( L(T_1) \).

Let \( G_{22} = G[L(T_2)] \) and \( n_{22} = |V(G_{22})| \). We will show that \( \delta(G_{22}) > n_{22}/2 \). We may assume that \( s_2 \geq (1/6 - 1.1\alpha_1)n_2 - 2 \). For otherwise, if \( s_2 < (1/6 - 1.1\alpha_1)n_2 - 2 \), then by (3.19), we see that

\[
\delta(G_{22}) \geq \delta(G[V_2']) - s_2 - 2 \\
> (2/3 - 1.1\alpha_1)n_2 - ((1/6 - 1.1\alpha_1)n_2 - 2) - 2 \\
> n_2/2 \geq n_{22}/2.
\]

Thus, \( s_2 \geq (1/6 - 1.1\alpha_1)n_2 - 2 \), implying that

\[
n_{22} \leq n_1 - s_2 \leq (5/6 + 1.1\alpha_1)n_2 + 2 \quad \text{gives that} \quad n_2 \geq \frac{n_{22} - 2}{5/6 + 1.1\alpha_1}.
\]
By (3.19) and (3.26),

\[ \delta(G_{22}) \geq \delta(G[V'_2]) - s_2 - 2 \]
\[ \geq (2/3 - 1.1\alpha_1)n_2 - (1/6 + 0.7\alpha_1)n_2 - 2 \]
\[ \geq (1/2 - 1.9\alpha_1)n_2 \geq \frac{1/2 - 1.9\alpha_2}{5/6 + 1.1\alpha_2}(n_{22} - 2) \]
\[ > n_{22}/2. \]

The last inequality follows by assuming that \(2.45\alpha_1 + 2/n_{11} < 1/12\). Hence, by Lemma 3.4.6 there is a hamiltonian \((y_1^L, y_2^L)\)-path \(P_2\) in \(G_{22}\).

**Step 7. Forming an SGHG**

Let \(T = T_1 \cup T_2 \cup \{x_Ny_N\}\) and \(C = P_1 \cup P_2 \cup \{x_1^Ly_1^L, x_2^Ly_2^L\}\). We see that \(T\) is a HIST of \(G\) with \(L(T) = V(P_1) \cup V(P_2)\) and \(C\) is a cycle spanning on \(L(T)\). Hence \(H = T \cup C\) is an SGHG of \(G\).

**3.4.3.3 Proof of Theorem 3.4.5** Notice that the assumption of Extremal Case 2 implies that
\[ |V_1| > (3/5 - \alpha)n \quad \text{and} \quad |V_2| \geq (2/5 - 2\beta)n. \]

We may assume that the graph \(G\) is minimal with respective to the number of edges. This implies that no two adjacent vertices both have degree larger than \((2n + 3)/5\). (For otherwise, we could delete any edges incident to two vertices both with degree larger than \((2n + 3)/5\).) We construct an SGHG in \(G\) step by step.

**Step 1. Repartitioning**

Set \(\alpha_1 = \alpha^{1/3}\) and \(\alpha_2 = \alpha^{2/3}\). Let

\[ V'_2 = \{v \in V_2 | \deg(v, V_1) \geq (1 - 3\alpha_1)|V_1|\}, \]
\[ V'_0 = \{v \in V_2 - V'_2 | \deg(v, V_1) \leq \alpha_1|V_2|/6\}, \]
\[ V'_1 = V_1 \cup V'_0, \quad V'_{12} = V_2 - V'_2 - V'_0. \]
As $d(V_1, V_2) \geq 1 - 3\alpha$, the following holds,

$$(1 - 3\alpha)|V_1||V_2| \leq e_G(V_1, V_2) = e_G(V_1, V_2') + e_G(V_1, V_2 - V_2')$$

$$\leq |V_1||V_2'| + (1 - 3\alpha)|V_1||V_2 - V_2'|.$$

The inequality implies that

$$|V_2 - V_2'| \leq \alpha|V_2|.$$  (3.28)

As a consequence of moving vertices in $V_2 - V_2'$ out from $V_2$, by (3.28) we get

$$\delta(V_1, V_2') \geq (2n + 3)/5 - 2\beta|V_2|$$

$$\geq (2n + 3)/5 - 2\beta|V_2| - \alpha|V_2|$$

$$\geq (2n + 3)/5 - \alpha|V_2|,$$  (3.29)

provided that $6\beta \leq \alpha_2$. And as a consequence of moving vertices in $V_0'$ to $V_1$,

$$\delta(V_0', V_2') \geq \delta(G) - \Delta(V_0', V_1) - \Delta(V_0', V_2 - V_2')$$

$$\geq (2n + 3)/5 - \alpha_1|V_2|/6 - \alpha_2|V_2|$$

$$\geq (2n + 3)/5 - \alpha_1|V_2|/3 \text{ (provided that } \alpha_2 \leq \alpha_1/6),$$  (3.30)

and

$$\alpha_1|V_2|/6 < \delta(V_0', V_1') < (1 - 3\alpha_1)|V_1|. \text{ (3.31)}$$

From (3.29) and (3.30), we have

$$\delta(V_1', V_2') \geq (2n + 3)/5 - \alpha_1|V_2|/3.$$  (3.32)

As

$$\delta(V_2', V_1') \geq (1 - 3\alpha_1)|V_1| \geq (1 - 3\alpha_1)(3/5 - \alpha)n > [(2n + 3)/5].$$  (3.33)
we get that
\[ \deg(v_1') = \lceil (2n + 3)/5 \rceil \] (3.34)
for each \( v_1' \in V_1' \), by the minimality assumption of \( e(G) \). Hence (3.32) and (3.34) give that
\[ \Delta(G[V_1']) \leq \alpha_1 |V_2|/3. \] (3.35)

**Step 2. Finding a vertex \( v_2^* \) from \( V_2' \) with large degree in \( V_1' \)**

Let
\[ e_{in} = e(G[V_1']) \] (3.36)
be the number of edges within \( V_1' \), notice that \( e_{in} \) maybe 0. Then
\[ e_G(V_1', V_2' \cup V_{12}^0) = |V_1'| \lceil (2n + 3)/5 \rceil - 2e_{in}. \] (3.37)

Let
\[ d_{in} = e_{in}/|V_1'| \text{ and } |n_0| = |V_2' \cup V_{12}^0| - \lceil (2n + 3)/5 \rceil. \] (3.38)

By (3.35) and the definition of \( d_{in} \) in (3.38), we have
\[ |d_{in}| \leq \alpha_1 |V_2|/6. \]

In fact, since \( \Delta(V_1, V_1') \leq \Delta(V_1, V_1) + \Delta(V_1, V_0') \leq 2\beta n + |V_0'| \leq 2\beta n + \alpha_2 |V_2| \), and \( \Delta(V_0', V_1') \leq \alpha_1 |V_2|/6 + \alpha_2 |V_2| \), more precisely, we have
\[ 2d_{in} = 2e_{in}/|V_1'| \leq (2\beta n + \alpha_2 |V_2|)/|V_1'| + (\alpha_1 |V_2|/6 + \alpha_2 |V_2|)/|V_0'|/|V_1'| \]
\[ \leq (2\beta n + \alpha_2 |V_2|) + 2(\alpha_1 |V_2|/6 + \alpha_2 |V_2|) \quad \text{(as } |V_0'| \leq \alpha_2 |V_2| \text{ and } |V_1|, |V_2| \leq |V_1'|) \]
\[ \leq (6\beta + \alpha_2 + \alpha/6 + \alpha_2^2) |V_2| \quad \text{(as } \beta n \leq 3\beta |V_2|) \]
\[ \leq 2\alpha_2 |V_2| \quad \text{(provided that } 6\beta + \alpha/6 + \alpha_2^2 \leq \alpha_2) \] (3.39)

Set
Case A. \([ (2n + 3)/5] - |V'_2 \cup V^0_{12}| = n_0 \geq 0;\)

Case B. \([ V'_2 \cup V^0_{12}] - [ (2n + 3)/5] = n_0 \geq 1.\)

We have

\[
n_0 = \begin{cases} 
[ (2n + 3)/5] - |V'_2 \cup V^0_{12}| \leq 2\beta n + \alpha_2 |V_2| \leq (6\beta + \alpha_2) |V_2| \leq 2\alpha_2 |V_2|, & \text{Case A}, \\
|V'_2 \cup V^0_{12}| - [ (2n + 3)/5] \leq (2/5 + \alpha)n - [ (2n + 3)/5] \leq \alpha n, & \text{Case B}. 
\end{cases}
\] (3.40)

Then in case A,

\[
e_G(V'_1, V'_2 \cup V^0_{12}) = |V'_1|[ (2n + 3)/5] - 2e_{in} \quad \text{by (3.34)}
\]

\[
= |V'_1|( |V'_2 \cup V^0_{12}| + n_0 - 2d_{in}) \\
\geq |V'_2 \cup V^0_{12}| (|V'_1| + 1.4n_0 - 3.2d_{in}),
\]

as \(1.4|V'_2 \cup V^0_{12}| \leq 1.4((2n + 3)/5 + \alpha n) \leq (3/5 - \alpha)n < |V'_1|\) and \(1.6|V'_2 \cup V^0_{12}| \geq 1.6((2n + 3)/5 - 2\beta - \alpha_2)n \geq (3/5 + 2\beta + \alpha_2)n > |V'_1|\) provided that \(2.4\alpha < 1/25\) and \(5.2\beta + 2.6\alpha_2 \leq 1/25\) respectively. Since \(e_G(V'_1, V'_2 \cup V^0_{12}) \leq |V'_2 \cup V^0_{12}| |V'_1|\), we have \(|V'_1| + 1.4n_0 - 3.2d_{in} \leq |V'_1|\), and thus \(1.4n_0 \leq 3.2d_{in}\).

In Case B,

\[
e_G(V'_1, V'_2 \cup V^0_{12}) = |V'_1|[ (2n + 3)/5] - 2e_{in} \quad \text{by (3.34)}
\]

\[
= |V'_1|( |V'_2 \cup V^0_{12}| - n_0 - 2d_{in}) \\
\geq |V'_2 \cup V^0_{12}| (|V'_1| - 1.6n_0 - 3.2d_{in}),
\]

as \(1.6|V'_2 \cup V^0_{12}| \geq 1.6((2n + 3)/5 - 2\beta - \alpha_2)n \geq (3/5 + 2\beta + \alpha_2)n > |V'_1|\) provided that \(5.2\beta + 2.6\alpha_2 \leq 1/25\).

Let

\[
d_l = \begin{cases} 
[3.2d_{in} - 1.4n_0], & \text{if Case A}, \\
[1.6n_0 + 3.2d_{in}], & \text{if Case B}. 
\end{cases}
\] (3.41)
By (3.39) and (3.40), we see that

\[ d_i \leq \begin{cases} 3.2\alpha_2 |V_2|, & \text{if Case A,} \\ 6.4\alpha_2 |V_2|, & \text{if Case B.} \end{cases} \tag{3.42} \]

Then there is a vertex \( v_2^* \) in \( V'_2 \cup V^0_{12} \) of degree at least \( |V'_1| - d_i \). We will fix this vertex in what follows. In fact, such a vertex \( v_2^* \) is in \( V'_2 \) by the facts that

\[ \delta(V^0_{12}, V'_1) < (1 - 3\alpha_1)|V_1| \quad \text{and} \quad |V'_1| - d_i \geq (1 - 3\alpha_1)|V_1|, \tag{3.43} \]

where \( |V'_1| - d_i \geq (1 - 3\alpha_1)|V_1| \) holds because of (3.42).

**Step 3. Finding a matching \( M \) within \( G[\Gamma(v_2^*, V'_1)] \)**

In this step, if \( e_{in} \geq 1 \), we first find a matching within \( G[V'_1] \) of size at least \( e_{in}/(2\Delta(G[V'_1])) \). We assume this by giving the following lemma.

**Lemma 3.4.7.** If \( G \) is a graph with maximum degree \( \Delta \), then \( G \) contains a matching of size at least \( \frac{|E(G)|}{2\Delta} \).

**Proof.** We use induction on \( |V(G)| \). We may assume that the graph is connected. For otherwise, we are done by the induction hypothesis. Let \( e = xy \in E(G) \) be an edge and \( G' = G - \{x, y\} \). Since \( |N_G(x) \cup N_G(y)| - |\{x, y\}| \leq 2(\Delta - 1) \), we have

\[ e(G') \geq e(G) - 2(\Delta - 1) - 1 \geq e(G) - 2\Delta. \]

Hence, by the induction hypothesis, \( G' \) has a matching of size at least \( \frac{e(G') - 2\Delta}{2\Delta} = \frac{e(G) - 2\Delta}{2\Delta} - 1 \). Adding \( e \) to the matching obtained in \( G' \) gives a matching of size at least \( \frac{e(G)}{2\Delta} \) in \( G \). \( \Box \)

In case A, we take a matching in \( G[V'_1] \) of size at least \( \max\{11d_{in}, 11n_0\} \). This is possible because

\[ \frac{e_{in}}{2\Delta(G[V'_1])} \geq \frac{e_{in}}{2\alpha_1|V'_1|/3} = \frac{3d_{in}}{2\alpha_1} \geq 11d_{in} \]
provided that \(\alpha \leq (\frac{3}{12})^3\), and

\[
2e_{in} \geq |V'_1|[(2n+3)/5] - |V'_1||V'_2| - (1-3\alpha_1)|V_1||V'_{12}|
\]

\[
\geq |V'_1|[(2n+3)/5] - |V'_1|((2n+3)/5) - n_0 - |V'_{02}| - |V_1||V'_{12}|
\]

\[
\geq |V'_1|n_0 + 3\alpha_1|V_1||V'_{12}|
\]

(3.44)

implying that

\[
\frac{e_{in}}{2 \Delta (G[V'_1])} \geq \frac{e_{in}}{2\alpha_1|V'_1|/3} \geq \frac{|V'_1|n_0/2}{2\alpha_1|V'_1|/3} \geq \frac{3n_0}{4\alpha_1} \geq 11n_0
\]

provided that \(\alpha \leq (\frac{3}{12})^3\).

By (3.41), \(|V'_1| - \Gamma(v^*_2, V'_1) \leq d_l \leq \lfloor 3.2d_{in} \rfloor\), we can then choose a matching \(M\) from \(\Gamma(v^*_2, V'_1)\) such that

\[
|M| = \max\{\lfloor 7d_{in} \rfloor, 7n_0\}. \tag{3.45}
\]

In case B, we take a matching in \(G[V'_1]\) of size at least \(\lfloor 8d_{in} \rfloor\). This is possible as

\[
\frac{e_{in}}{\Delta(G[V'_1])} \geq \frac{e_{in}}{2\alpha_1|V'_1|/3} = \frac{3d_{in}}{2\alpha_1} \geq \lfloor 8d_{in} \rfloor
\]

provided that \(\alpha \leq (\frac{3}{16})^3\).

By the second equality of (3.41), \(|V'_1| - \Gamma(v^*_2, V'_1) \leq [3.2d_{in} + 1.6n_0]\). If \(n_0 < 2d_{in}\), then \([3.2d_{in} + 1.6n_0] \leq \lfloor 7d_{in} \rfloor\). Thus, there is a matching \(M\) within \(\Gamma(v^*_2, V'_1)\) such that

\[
|M| = \begin{cases} \lfloor d_{in} \rfloor, & \text{if } n_0 < 2d_{in}, \\ 0, & \text{if } n_0 \geq 2d_{in}. \end{cases} \tag{3.46}
\]

We fix \(M\) for denoting the matching we defined in this step hereafter.

**Step 4. Insertion**

In this step, we insert vertices in \(V'_{12}\) into \(V'_1 - V(M)\). Let \(I = V'_{12} = \{x_1, x_2, \ldots, x_I\}, \)
\( U_1 = \Gamma(v_2^*, V'_1) - V(M) \), and \( U_2 = V'_2 \). Then (i) 
\[
\delta(I, U_1) \geq \delta(I, V'_1) - |V(M)| - |V'_1 - \Gamma(v_2^*, V'_1)| \\
\geq \alpha_1|V_2|/6 - \max\{\lfloor 7d_{in} \rfloor, 7n_0 \} - 1.6n_0 + 3.2d_{in}, \\
\geq \alpha_1|V_2|/6 - 20.4\alpha_2|V_2| \quad \text{(by (3.39) and (3.40))} \\
\geq 3\alpha_2|V_2| \geq 3|I| \quad \text{(provided that } 23.4\alpha_2 \leq \alpha_1/6),
\]
and from (3.32), we have (ii) 
\[
\delta(U_1, U_2 - \{v_2^*\}) \geq \lceil (2n + 3)/5 \rceil - \alpha_1|V_2|/3 - 1 > \alpha_2|V_2| \geq |I|.
\]

By condition (i), there is a claw-matching \( M_1 \) between \( I \) and \( U_1 \) centered in \( I \). Suppose that \( \Gamma(x_i, M_1) = \{x_{i0}, x_{i1}, x_{i2}\} \). We denote by \( P_{x_i} \) the path \( x_{i1}x_{i2} \). By (ii), there is a matching \( M_2 \) between \( \{x_{i0} : 1 \leq i \leq |I|\} \) and \( U_2 - \{v_2^*\} \) covering \( \{x_{i0} : 1 \leq i \leq |I|\} \). So far, we get two matchings \( M_1 \) and \( M_2 \).

Next we delete three types of edges not contained in
\[
\bigcup_{i=1}^{|I|} E(P_{x_i}) \cup \{x_i x_{i0} : 1 \leq i \leq |I|\}.
\]
Those edges include edges incident to a vertex in \( I \), edges incident to a vertex in
\[
\bigcup_{i=1}^{|I|} (\Gamma(x_{i1}) - \Gamma(x_{i2})) \cup (\Gamma(x_{i2}) - \Gamma(x_{i1}))
\]
and one edge from the two edges connecting a vertex in \( \Gamma(x_{i1}) \cap \Gamma(x_{i2}) \) to both \( x_{i1} \) and \( x_{i2} \), for each \( i = 1, 2, \ldots, |I| \).

For the resulting graph after the deletion of edges above, we contract each path \( P_{x_i} (1 \leq i \leq |I|) \) into a single vertex \( v_{x_i} \). We call each \( v_{x_i} \) a \textit{wrapped vertex} and call \( P_{x_i} \) the \textit{preimage} of \( v_{x_i} \). Denote by \( G^* \) the graph obtained by deleting and contracting the same edges as above, and let \( U_2^* = V'_2 \) and \( U_1^* = V(G^*) - U_2^* \). (We will need the following degree condition
in the end of this proof.) Since \( |U_2^*| = |V_2'| \leq (2/5 + \alpha)n \), combining with (3.32), we have

\[
deg(v_{x_i}, U_2^*) \geq |\Gamma(x_{i1}, U_2^*) \cap \Gamma(x_{i2}, U_2^*)| - 1 \geq 2n/5 - \alpha_1|V_2|.
\]

By the above inequality and (3.32), we get the first inequality below in (3.47). Since one edge from the two edges connecting a vertex in \( \Gamma(x_{i1}) \cap \Gamma(x_{i2}) \) to both \( x_{i1} \) and \( x_{i2} \) is deleted in \( G^* \) for each \( i = 1, 2, \ldots, |I| \), combining with (3.33), we have the second inequality as follows.

\[
\begin{align*}
\delta(U_1^*, U_2^*) &\geq 2n/5 - \alpha_1|V_2|, \\
\delta(U_2^*, U_1^*) &\geq \delta(V_1', V_1') - 1 \geq (1 - 3\alpha_1)|V_1| - 1.
\end{align*}
\]

(3.47)

Let \( U_1' \) and \( U_2' \) be the corresponding sets of \( U_1 \) and \( U_2 \), respectively, after the contraction. Let \( T_W \) be the graph with

\[
V(T_W) = \{x_{i0}, v_{x_i} : 1 \leq i \leq |I|\} \cup (V(M_2) \cap U_2) \quad \text{and} \quad E(T_W) = \{x_{i0}v_{x_i} : 1 \leq i \leq |I|\} \cup E(M_2).
\]

By the construction,

\[
|V(T_W) \cap U_1'| = |\{x_{i0}, v_{x_i} : 1 \leq i \leq |I|\}| = 2|I|, \quad |L(T_W) \cap U_1'| = |\{v_{x_i} : 1 \leq i \leq |I|\}| = |I|, \quad \text{and}
\]

\[
|V(T_W) \cap U_2'| = |L(T_I) \cap U_2'| = |V(M_2) \cap U_2'| = |I|.
\]

Notice that \( T_W \) is a forest with \( |I| \) components and each vertex \( x_{i0} \) (\( 1 \leq i \leq |I| \)) has degree 2 in \( T_W \). (We will make \( T_W \) connected in the end by connecting each \( x_{i0} \) to \( v_{2}^* \).) See a depiction of this operation with \( |I| = 1 \) in Figure 3.7 below.

Let \( U_1^I = (V_1' - U_1) \cup U_1' - V(T_W), U_2^I = U_2' - V(T_W), \) and \( G_I \) the resulting graph with
Figure (3.7) $T_W$ with $|I| = 1$

$V(G_I) = U_I^1 \cup U_I^2$. We have that

$$\begin{align*}
|U_I^1| &= |V_I^1| - 3|I| = |V_I^1| - 3n_{12}^0, \quad |U_I^2| = |V_I^2| - |I| = |V_I^2| - n_{12}^0, \\
\delta(U_I^1, U_I^2) &\geq (2n + 3)/5 - \alpha_1|V_2|/3 - n_{12}^0, \\
\delta(U_I^2, U_I^1) &\geq (1 - \alpha_1)|V_1| - 3n_{12}^0. 
\end{align*}$$

(3.48)

Step 5. Matching Extension

In this step, in the graph $G_I$, we do some operation on the matching $M$ found in Step 3. Notice that the vertices in $M$ are unused in Step 4. Recall that $|M| \leq \max\{7n_0, \lceil 7d_{in} \rceil\}$.

By $|d_{in}| \leq \alpha_2|V_2|$ from (3.39) and $n_0 \leq 2\alpha_2|V_2|$ from (3.40), we get

$$|M| \leq 14\alpha_2|V_2|. \quad (3.49)$$

Hence, $\delta(U_I^1, U_I^2 - \{v_2^*\}) \geq [(2n + 3)/5] - \alpha_1|V_2|/3 - n_{12}^0$ \geq |M|. Let $V_M$ be the set of vertices containing exactly one end of each edge in $M$. Then there is a matching $M'$ between $V_M$ and $U_2 - \{v_2^*\}$ covering $V_M$. Let $F_M$ be a forest with

$$V(F_M) = V(M) \cup (V(M') \cap U_2) \quad \text{and} \quad E(F_M) = E(M) \cup E(M').$$

Notice that

$$|V(F_M) \cap U_1| = 2|M|, \quad |L(F_M) \cap U_1| = |V(M) - V_M| = |M|, \quad |V(F_M) \cap U_2| = |L(F_M) \cap U_2| = |M|.$$
Notice that $F_M$ has $|M|$ components, and all vertices in $V_M$ has degree 2. (We will make $F_M$ a HIT later on by connecting each vertex in $V_M$ to the vertex $v^*_2 \in U_2$) See Figure 3.8 for a depiction of $F_M$ with $|M| = 3$.

\[
\begin{array}{c}
U_1 \\
U_2
\end{array}
\]

Figure (3.8) $F_M$ with $|M| = 3$

Let

\[
U^1_M = U^1_I - V(F_M) \quad \text{and} \quad U^2_M = U^2_I - V(F_M).
\]

Notice that

\[
\begin{align*}
|U^1_M| &= |U^1_I| - 2|M| = |V'_1| - 3n^0_{12} - 2|M|, \\
|U^2_M| &= |U^2_I| - |M| = |V'_2| - n^0_{12} - |M|,
\end{align*}
\]

and

\[
\begin{align*}
\delta(U^1_M, U^2_M) &= \left[ (2n + 3)/5 \right] - \alpha_1|V_2|/3 - n^0_{12} - |M|, \\
\delta(U^2_M, U^1_M) &\geq (1 - 3\alpha_1)|V_1| - 3n^0_{12} - 2|M|.
\end{align*}
\]

**Step 6. Distribute Remaining vertices in $U^1_M - \Gamma(v^*_2, V'_1)$**

Let

We may assume $n_l \geq 1$. For otherwise, we skip this step. By (3.42), we have

\[
n_l \leq \begin{cases} 
3.2\alpha_2|V_2|, & \text{Case A}, \\
6.4\alpha_2|V_2|, & \text{Case B}.
\end{cases}
\]
By $n_{12}^0 \leq \alpha_2|V_2|$ from (3.28) and $|M| \leq 14\alpha_2|V_2|$ from (3.49), we have (i)

\[
\delta(U^1_M, U^2_M) \geq \left( \frac{2n + 3}{5} \right) - \alpha_1|V_2|/3 - n_{12}^0 - |M| \geq \left( \frac{2n + 3}{5} \right) - \alpha_1|V_2|/3 - 15\alpha_2|V_2|
\]
\[
\geq (1 - 3\alpha)|V_2| - \alpha_1|V_2|/3 - 15\alpha_2|V_2| \quad \text{(as } \left( \frac{2n + 3}{5} \right) \geq (1 - 3\alpha)(2/5 + \alpha)n)\]
\[
\geq (1 - 3\alpha - \alpha_1/3 - 15\alpha_2)|V_2|) \geq (1 - \alpha_1)|V_2| \quad \text{(provided } 3\alpha + 15\alpha_2 \leq 2\alpha_1/3)\]
\[
\geq (1 - \alpha_1)|U^2_M|.
\]  

(3.53)

By (3.50) and (3.52), we have (ii)

\[
|U^2_M| - 10\alpha_1|V_2| - \left[ n_l/10 \right] - 1 \geq |V_2'| - n_{12}^0 - |M| - 16\alpha_1|V_2| - 0.64\alpha_2|V_2| - 2
\]
\[
\geq (1 - \alpha_2 - 14\alpha_2 - 10\alpha_1 - 0.64\alpha_2 - |V_2|/2)|V_2|
\]
\[
\geq (1 - 11\alpha_1)|V_2| \quad \text{(provided } 15.64\alpha_2 + |V_2|/2 \leq \alpha_1)\]
\[> 0 \quad \text{(provided } 11\alpha_1 < 1)\].

Let $U_R = U^1_M - \Gamma(v_2^*, V_1')$ and denote $\left\lceil \frac{|U_R|}{10} \right\rceil = l$. Suppose first that $|U_R| \geq 2$. We partition $U_R = U_{R_1} \cup U_{R_2} \cup \cdots \cup U_{R_l}$ arbitrary such that each set contains at least 2 and at most $|U_R|/10$ vertices. Then by the conditions (i) and (ii), for each $1 \leq i \leq l$, there is a vertex $y_i \in U_2 - \{v_2^*\}$ which is common to all vertices in $U_{R_i}$, and is not used by any other $U_{R_j}$ with $j \neq i$. Let $T_R$ be the graph with

$$V(T_R) = U_R \cup \{y_i : 1 \leq i \leq l\} \quad \text{and} \quad E(T_R) = \{xy_i : x \in U_{R_i}, 1 \leq i \leq l\}.$$  

Suppose now $|U_R| = 1$, let $U_R = \{x_R\}$. Choose $x_R' \in U^1_M - U_R$ and $y_R \in U^2_M - \{v_2^*\}$ be a vertex common to $x_R$ and $x_R'$. Let $T_R$ be a tree with

$$V(T_R) = \{x_R, x_R', y_R\} \quad \text{and} \quad E(T_R) = \{x_Ry_R, x_R'y_Ry_R\}.$$
By the construction,

\[ |V(T_R) \cap U^1_M| = |L(T_R \cap U^1_M)| = \max\{|U_R|, 2\}, \quad |V(T_R) \cap U^2_M| = l, \quad \text{and} \quad |L(T_R \cap U^2_M)| = 0. \]

Notice that \( T_R \) is not connected when \( |U_R| \geq 17 \) and that \( T_R \) may have degree 2 vertices in \( V(T_R) \cap U^2_M \). Later on, by joining each vertex in \( T_R \cap U^2_M \) to a vertex of a tree, we will make the resulting graph connected, and thereby eliminating the possible degree 2 vertices in \( T_R \).

Let

\[ U_R^1 = U_M^1 - V(T_R) \quad \text{and} \quad U_R^2 = U_M^2 - V(T_R). \]

Then we have

\[
\begin{align*}
|U_R^1| &= |U_M^1| - n_t = |V'_1| - 3n_1^0 - 2M - \max\{2, n_t\}, \\
|U_R^2| &= |U_M^2| - \lceil n_t/10 \rceil = |V'_2| - n_1^0 - |M| - \lceil n_t/10 \rceil,
\end{align*}
\]

and

\[
\begin{align*}
\delta(U_R^1, U_R^2) &\geq \left\lceil (2n + 3)/5 \right\rceil - \alpha_1|V'_2|/3 - n_1^0 - |M| - \lceil n_t/10 \rceil, \\
\delta(U_R^2, U_R^1) &= (1 - 3\alpha_1)|V'_1| - 3n_1^0 - 2|M| - \max\{2, n_t\}.
\end{align*}
\]

Let \( G_R \) be the subgraph of \( G \) induced on \( U^1_R \cup U^2_R \).

**Step 7. Completion of a HIST in \( G_R \)**

In this step, we find a HIST \( T_{main} \) in \( G_R \) such that

\[
\begin{align*}
|L(T_{main}) \cap U^1_R| &= |L(T_W)|/2 + |L(F_M) \cap U^1_I| + |L(T_R) \cap U^1_M| = \\
|L(T_{main} \cap U^2_R)| &= |L(T_W)|/2 + |L(F_M) \cap U^2_I| + |L(T_R) \cap U^2_M|.
\end{align*}
\]

By the construction of \( F_M \) and \( T_R \), we have \( |L(F_M) \cap U^1_I| = |L(F_M) \cap U^2_I| \) and \( |L(T_R) \cap \)
\[ U_M^1 - |L(T_R) \cap U_M^2| = \max\{2, n_l\}, \text{ respectively. So} \]

\[ |L(T_{main}) \cap U_R^2| - |L(T_{main}) \cap U_R^1| = \max\{2, n_l\}. \quad (3.56) \]

Notice that \( v_2^* \in U_R^2 \), \( v_2^* \) is adjacent to each vertex in \( U_R^1 \), and \( V_1' - \Gamma(v_2^*, V_1') \subseteq V(U_R^1) \).

We now construct \( T_{main} \) step by step.

**Step 7.1**

Let \( T_{main}^1 \) be the graph with

\[
V(T_{main}^1) = \{v_2^*\} \cup U_R^1 \quad \text{and} \quad E(T_{main}^1) = \{v_2^*x \mid x \in U_R^1\}.
\]

To make the requirement of (3.56) possible, we need to make at least

\[
d_3 = |U_R^1| - |U_R^2| + \max\{2, n_l\},
\]

\[
= |V_1'| - |V_2'| - 2n_0^1 - |M| + \lceil n_l/10 \rceil (3.57)
\]

vertices in \( U_R^1 \) with degree at least 3 in \( T_{main} \), where the last inequality above follows from (3.54). Hereinafter, we assume that \( \max\{2, n_l\} = n_l \). Since the proof for \( \max\{2, n_l\} = 2 \) follows the same idea, we skip the details.

Since all vertices in \( U_R^1 \) are included in \( T_{main}^1 \) and \( T_{main}^1 \) is connected, each vertex in \( T_{main}^1 \) needs to join to at least two distinct vertices from \( U_R^2 - \{v_2^*\} \) to have degree no less than 3. Hence, to make a desired HIST \( T_{main} \), it is necessary that

\[
d_{f^*} = |U_R^2| - 1 - 2d_3
\]

\[
= |V_2'| - n_0^1 - e_M - \lceil n_l/10 \rceil - 1 - 2d_3
\]

\[
= 3|V_2'| - 2|V_1'| + 3n_0^1 + |M| - 3\lceil n_l/10 \rceil - 1
\]

\[
\geq 0. \quad (3.58)
\]
We show (3.58) is true, separately, for each of Case A and Case B. For Case A, notice that

\[ |V'_1| = n - [(2n + 3)/5] + n_0 \quad \text{and} \quad |V'_2| = [(2n + 3)/5] - n_0 - n_{12}^0. \]

Hence,

\[ 3|V'_2| = 3[(2n + 3)/5] - 3n_0 - 3n_{12}^0 \quad \text{and} \quad 2|V'_1| = 2n - 2[(2n + 3)/5] + 2n_0. \]

Thus,

\[
d_{f^*} = 5[(2n + 3)/5] - 2n - 5n_0 - 3n_{12}^0 + 3n_{12}^0 + |M| - 3[n_l/10] - 1 \\
\geq 2 - 5n_0 + |M| - 3[3.2d_{in}] / 10 \quad \text{(by} \ n_l \leq d_l[3.2d_{in} - 1.4n_0] \text{from (3.41))} \\
= 2 - 5n_0 + \max\{7n_0, [7d_{in}]\} - 3[3.2d_{in}] / 10 \\
\geq 0.
\]

Now we show (3.58) is true for case B. Notice that

\[ |V'_1| = n - [(2n + 3)/5] + n_0 \quad \text{and} \quad |V'_2| = [(2n + 3)/5] + n_0 - n_{12}^0. \]

So

\[ 3|V'_2| = 3[(2n + 3)/5] + 3n_0 - 3n_{12}^0 \quad \text{and} \quad 2|V'_1| = 2n - 2[(2n + 3)/5] + 2n_0. \]

Recall that \( n_0 \geq 1 \) in this case. We have

\[
d_{f^*} = 5[(2n + 3)/5] - 2n + n_0 - 3n_{12}^0 + 3n_{12}^0 + |M| - 3[n_l/10] - 1 \\
\geq 2 + n_0 + |M| - 3[n_l/10] \\
= 2 + n_0 + |M| - 3[3.2d_{in} + 1.6n_0] / 10 \quad \text{(by} \ n_l \leq [3.2d_{in} + 1.6n_0] \text{from (3.41))} \\
\geq \begin{cases} 
2 + n_0 + [d_{in}] - [9.2d_{in}/10] - [4.8n_0/10] - 1 \geq 0, & \text{if} \ n_0 < 2d_{in}; \\
2 + n_0 - [9.2d_{in}/10] - [4.8n_0/10] - 1 \geq 0, & \text{if} \ n_0 \geq 2d_{in}. 
\end{cases}
\]
We now in Step 2 below show that there is a way to make exactly $d_f$ vertices in $T_{\text{main}}^{1}$ with degree 3 by joining each to two distinct vertices from $U_R^2 - \{v^*_2\}$.

**Step 7.2**

We first take $2d_3$ vertices from $U_R^2 - \{v^*_2\}$. For those $2d_3$ vertices, pair them up into $d_3$ pairs. We show that for each pair of vertices, they have at least $d_3$ common neighbors in $U_R^1$.

Using (3.55), $|M| \leq 14\alpha_2|V_2|$ from (3.49), $n_t \leq d_t \leq 6.4\alpha_2|V_2|$ from (3.42), we have

$$
\delta(U_R^2, U_R^1) \geq (1 - 3\alpha_1)|V_1| - 3n_1 - 2|M| - \max\{2, n_t\} \\
\geq |V_1| - 3\alpha_1|V_1| - 3\alpha_2|V_2| - 28\alpha_2|V_2| - 6.4\alpha_2|V_2| \\
\geq |V_1'| - |V_1 - V_1| - 37.4\alpha_2|V_2| - 3\alpha_1|V_1|. \quad (3.59)
$$

Since $|U_R^1| \leq |V_1'|$, we know that any two vertices in $U_R^2$ have at least

$$
n_c = |V_1| - 2|V_1'| - |V_1| - 74.8\alpha_2|V_2| - 6\alpha_1|V_1| \\
\geq (3/5 - \alpha)n - 76.8\alpha_2|V_2| - 6\alpha_1|V_1| \quad \text{(by } |V_1' - V_1| = |V_0'| \leq |V_2 - V_2'| \leq \alpha_2|V_2|\text{)} \\
\geq 3n/5 - 10\alpha_1|V_1| \quad \text{(provided that } 76.8\alpha_2 + 3\alpha \leq 4\alpha_1\text{)}
$$

common neighbors in $U_R^1$. On the other hand,

$$
d_3 = |V_1'| - |V_2'| - 2\alpha_1^0 - |M| - \lfloor n_t/10 \rfloor \\
\leq (3/5 - \alpha)n - (2n/5 - 2\beta n - |V_2 - V_2'|) + (1.6n_0 + 3.2|d_{\text{in}}|)/10 + 1 \\
= n/5 - \alpha n + 2\beta n + |V_2 - V_2'| + (3.2\alpha_2|V_2| + 3.2\alpha_2|V_2|)/10 \quad \text{(by } 3.39 \text{ and } 3.40) \\
\leq n/5 - \alpha n + 2\beta n + \alpha_2|V_2| + 0.64\alpha_2|V_2| \\
\leq n/5 + 2\alpha_1|V_2| < n_c \quad \text{(provided } 12\alpha_1 < 2/5\text{)}.
$$

Denote by $\{u_1^1, u_1^2\}, \{u_2^1, u_2^2\}, \ldots, \{u_{d_3}^1, u_{d_3}^2\}$ the $d_3$ pairs of vertices from $U_R^2 - \{v^*_2\}$. Then by the above argument, we can choose $d_3$ distinct vertices say $v_1, v_2, \ldots, v_{d_3}$ from $L(T_{\text{main}}^{1})$ such that $v_i \sim u_i^1, u_i^2$ for all $1 \leq i \leq d_3$. 
Let $T_{\text{main}}^2$ be the graph with

$$V(T_{\text{main}}^2) = V(T_{\text{main}}^1) \cup \{u_i^1, u_i^2 : 1 \leq i \leq d_3\} \quad \text{and} \quad E(T_{\text{main}}^2) = E(T_{\text{main}}^1) \cup \{v_i u_i^1, v_i u_i^2 : 1 \leq i \leq d_3\}.$$ 

If $V(G_R) - V(T_{\text{main}}^2) = \emptyset$, we let $T_{\text{main}} = T_{\text{main}}^2$. For otherwise, we need one more step to finish constructing $T_{\text{main}}$.

**Step 7.3**

For the remaining vertices in $U_R^2 - V(T_{\text{main}}^2)$, we show that each of them has a neighbor in $S(T_{\text{main}}^2) \cap U_R^1$, that is, a neighbor in $U_R^1$ of degree 3 in $V(T_{\text{main}}^2)$. This is clear, as by (3.59), we have

$$\delta(U_R^2, U_R^1) \geq |V_1'| - |V_1' - V_1| - 37.4\alpha_2|V_2| - 3\alpha_1|V_1|$$

$$\geq |U_R^1| - 38.4\alpha_2|V_2| - 3\alpha_1|V_1| \quad \text{(by} |V_1' - V_1| \leq |V_2 - V_2'| \leq \alpha_2|V_2|).$$

Since $|S(T_{\text{main}}^2) \cap U_R^1| = d_3$, and

$$d_3 = |V_1| - |V_2'| - 2n_2^0 - 2|M| + \lceil n/10 \rceil$$

$$\geq (3/5 - \alpha)n - (2/5 + \alpha)n - 2\alpha_2|V_2| - 28\alpha_2|V_2| + 0.64\alpha_2|V_2|$$

$$\geq n/5 - 2\alpha n - 29.36\alpha_2|V_2|$$

$$> 38.4\alpha_2|V_2| + 3\alpha_1|V_1| \quad \text{(provided} 2\alpha + 67.76\alpha_2 + 3\alpha_1 < 1/5).$$

Now, we join an edge between each vertex in $U_R^2 - V(T_{\text{main}}^2)$ and a neighbor of the vertex in $S(T_{\text{main}}^2) \cap U_R^1$. Let $T_{\text{main}}$ be the resulting tree. By the construction procedure, it is easy to verify that $T_{\text{main}}$ is a HIST of $G_R$.

**Step 8. Connecting $T_W$, $F_M$, $T_R$, and $V(T_{\text{main}})$ into a connected graph**

In this step, we connect $T_W$, $F_M$, $T_R$, and $V(T_{\text{main}})$ into a connected graph. Recall that each degree 2 vertex in $T_W$ and $F_M$ is a neighbor of $v_2^*$. We join an edge connecting $v_2^*$ in $V(T_{\text{main}})$ and each degree 2 vertex in $T_W$ and $F_M$. By the argument in step 7.3 above, we know each vertex in $V(T_R) \cap U_M^2$ has a neighbor in $S(T_{\text{main}}) \cap U_R^1$. Thus, we join an edge
between each vertex in $V(T_R) \cap U_M^2$ to exactly one of its neighbor in $S(T_{\text{main}}) \cap U_R^1$. Let $T^*$ be the final resulting graph. Notice that $I = V_{12}^0 = \{x_1, x_2, \cdots, x_I\} \subseteq L(T^*)$ is the set of the wrapped vertices from Step 4. Recall that $G^*$ is the graph obtained from $G$ be deleting and contracting edges from Step 4. Then by the constructions of $T_W$, $F_M$, $T_R$, and $T_{\text{main}}$, we see that $T^*$ is a HIST of $G^*$ with $|L(T^*) \cap U^*_1| = |L(T^*) \cap U^*_2|$.

**Step 9. Finding a cycle on $L(T^*)$**

Denote

$$U^*_L = L(T^*) \cap U^*_1, \quad U^*_L^2 = L(T^*) \cap U^*_2 \quad \text{and} \quad G_L = G[E_G(U^*_L, U^*_L^2)].$$

Notice that $G_L$ is a balanced bipartite graph. And

$$|S(T^*) \cap U^*_1| = d_3 \leq n/5 + 2\alpha_1|V_2| \quad \text{(by (3.60))}$$
$$|S(T^*) \cap U^*_2| = 1 + \left\lfloor n_l/10 \right\rfloor \leq 2 + 0.64\alpha_2|V_2| \quad \text{(by $n_l \leq d_l \leq 6.4\alpha_2|V_2|$ from (3.42)).}$$

Thus by (3.47),

$$\delta_{G^*}(U^*_L, U^*_L^2) \geq 2n/5 - \alpha_1|V_2| - (2 + 0.64\alpha_2|V_2|) > 3n/10 > |U^*_L|^2/2 + 1,$$
$$\delta_{G^*}(U^*_L^2, U^*_L) \geq (1 - 3\alpha_1)|V_1| - 1 - (n/5 + 2\alpha_1|V_2|) > n/3 > |U^*_L|^2/2 + 1.$$

By Lemma 3.2.7, $G_L$ contains a hamiltonian cycle $C'$.

**Step 10. Unwrap vertices in $V(C') \cap \{v_{x_1}, v_{x_2}, \cdots, v_{x_{|I|}}\}$**

On $C'$, replace each vertex $v_{x_i}$ with its preimage $P_{x_i} = x_{i_1}x_{i_2}$ for each $i = 1, 2, \cdots, |I|$. Denote the resulting cycle by $C$. Recall that $x_{i_1}, x_{i_2} \in \Gamma(v^*_2)$ by the choice of $x_{i_1}$ and $x_{i_2}$. Let $T$ be the graph on $V(G)$ with

$$E(T) = E(T^*) \cup \{v^*_2x_{i_1}, v^*_2x_{i_2} : i = 1, 2, \cdots, |I|\}.$$

Then $T$ is a HIST of $G$. Let $H = T \cup C$. Then $H$ is an SGHG of $G$. 
The proof of Extremal Case 2 is finished.
4.1 Introduction

In 1980, Bondy and Simonovits [8] showed that the best general lower bound on the length of a longest cycle in an $n$-vertex 3-connected cubic graph is between $\exp(c_1 \sqrt{\log n})$ and $n^{c_2}$ for some positive constants $c_1$ and $c_2$, and they also obtained similar bounds for 3-connected graphs with bounded degrees. The lower bound $\exp(c_1 \sqrt{\log n})$ for cubic graphs was improved to $n^{0.69}$ by Jackson [32] and was further improved to $n^{0.8}$ by Liu, Yu and Zhang [39]. In 1993, Jackson and Wormald [33] proved that every 3-connected $n$-vertex graph with maximum degree at most $d$ has a cycle of length at least $\frac{1}{2}n^{\log_d 2} + 1$ with $b = 6d^2$. They also conjectured that for $d \geq 4$ the correct value for $b$ should be $d - 1$, and they gave an infinite class of graphs showing that $b = d - 1$ is the best possible value that one can hope for.

Recently there has been considerable interest in approximating longest cycles in 3-connected graphs with bounded degrees. Karger, Motwani, and Ramkumar [35] showed that unless $P = NP$, it is impossible to find, in polynomial time, a path of length $n - n^\epsilon$ (for any $\epsilon < 1$) in an $n$-vertex Hamiltonian graph. They conjectured that it is hard even for graphs with bounded degrees. On the positive side, Feder, Motwani, and Subi [23] showed that there is a polynomial time algorithm for finding a cycle of length at least $n^{(\log_3 2)/2}$ in any 3-connected cubic $n$-vertex graph, and they asked the same question for 3-connected graphs with bounded degrees. Chen, Xu, and Yu [14] provided a cubic-time algorithm that, given a 3-connected $n$-vertex graph with maximum degree at most $d$, finds a cycle of length
at least $n^{\log_2^2} + 3$ with $b = 2(d - 1)^2 + 1$. This result was improved to $b = 4d + 1$ by Chen, Gao, Yu, and Zang \[12\].

Before stating the main result, we introduce some notation. For any graph $G$, we denote by $|G|$ the number of vertices of $G$, $G - z$ the graph obtained from $G$ by deleting the vertex $z \in V(G)$, and $N_G(z)$ the set of neighbors of $z$ in $G$. If $G$ is a path or cycle, then $\ell(G)$ denotes the length of $G$. Let $S_1, S_2 \subseteq V(G)$ be two disjoint sets. An $(S_1, S_2)$-path is a path $P$ connecting one vertex in $S_1$ and one vertex in $S_2$ such that $|V(P) \cap S_1| = |V(P) \cap S_2| = 1$. When $S_1 = \{x\}$ is a singleton, we simply write as $(x, S_2)$-path. The main result of this paper is the following:

**Theorem (4.1.1).** Let $d \geq 425$ be an integer, and $r = \log_{d-1} 2$. Let $G$ be either a cycle or a 3-connected graph and $e = xy \in E(G)$ be an edge.

(a) If $G$ is 3-connected, then for any $z \in V(G) - \{x, y\}$ such that $\Delta(G - z) \leq d$ and $z$ has at most $t$ neighbors distinct from $x$ and $y$, there is a cycle $C$ in $G - z$ through $xy$ such that $\ell(C) \geq \frac{1}{4} \left( \frac{d - 2.1 |G|}{d - 1} \right)^r + 2$.

(b) If $\Delta(G) \leq d$, then for any $f \in E(G) - \{e\}$, there is a cycle $C$ in $G$ through $e$ and $f$ such that $\ell(C) \geq \frac{1}{4} \left( \frac{d - 2.1 |G|}{(d - 1)^2} \right)^r + 2$.

(c) If $\Delta(G) \leq d$, then there is a cycle $C$ through $e$ in $G$ such that $\ell(C) \geq \frac{1}{4} |G|^r + 2$.

We first note that Theorem (4.1.1) holds trivially when $|G| \leq d$; hence, throughout the rest of this part, we assume $|G| \geq d + 1 \geq 426$. Also note that Theorem (4.1.1) holds trivially when $G$ is a cycle. However, we include cycles in the statement of Theorem (4.1.1) for the following reason: cycles occur in our inductive arguments, and their inclusion makes many arguments less cumbersome.

The rest of this part is organized as follows. In Section 2, we recall Tutte decomposition \[52\] for decomposing a 2-connected graph into 3-connected components and some results
from [12] concerning paths in 2-connected graphs. In Section 3, we prove a useful inequality about the function \( f(x) = x^{\log_2 2} \). In Section 4, we state lemmas concerning paths in a chain of 3-connected components, and in Section 5, we inductively prove Theorem (1.1) (a) and (b). Section 6 is the most significant part of the paper, where we prove Theorem (4.1.1)(c) inductively.

4.2 Paths in block-chains

We recall Tutte decomposition for decomposing a 2-connected graph into 3-connected components. A detailed description can be found in [14] and [29]. Let \( \mathcal{D} \) denote the set of all 3-connected (simple) graphs, \( \mathcal{C} \) denote the set of cycles (with at least three vertices), and \( \mathcal{B} \) denote the set of bonds (a bond is a multigraph with two vertices and at least three edges between them). Tutte [52] proved that every 2-connected graph \( G \) can be uniquely decomposed into 3-connected components, which belong to \( \mathcal{B} \cup \mathcal{C} \cup \mathcal{D} \). We call such a decomposition as the Tutte decomposition. Those 3-connected components are linked together by virtual edges to form a tree-like structure. More precisely, if we define a graph whose vertices are the 3-connected components of \( G \) obtained from the Tutte decomposition and two vertices are adjacent if the corresponding two 3-connected components share a common virtual edge, then such a graph is a tree, which we call the block-bond tree of \( G \). Hopcroft and Tarjan [29] gave a linear time algorithm for decomposing any 2-connected graph into 3-connected components.

Recall that in the block-cut tree of a connected graph there is a cut-vertex between two consecutive blocks. However, in a block-bond tree, it is not necessarily true that there is a bond between any two 3-connected components. For example, let \( G_1 \) and \( G_2 \) be two 3-connected graphs such that each \( G_i \) contains two nonadjacent vertices \( u_i \) and \( v_i \) for each \( i = 1, 2 \). Let \( G \) be obtained from \( G_1 \) and \( G_2 \) by identifying \( u_1 \) with \( v_1 \) and \( u_2 \) with \( v_2 \), respectively. According to Tutte's decomposing algorithm, \( G_1 + u_1v_1 \) and \( G_2 + u_2v_2 \) are the
only two 3-connected components of $G$. Clearly, in the block-bond tree, they are adjacent but there is no bond between them.

For convenience, 3-connected components that are not bonds are called 3-blocks, consisting of cycles and simple 3-connected graphs. An extreme 3-block is a 3-block that contains at most one virtual edge. That is, either it is the only 3-connected component (in which case $G$ is either a cycle or a 3-connected simple graph), or it corresponds to a degree one vertex in the block-bond tree.

A block-chain in $G$ is a sequence $H_1H_2 \ldots H_h$ of 3-blocks of $G$ for which there exist $B_1, B_2, \ldots, B_{h-1}$ such that for each $1 \leq j \leq h-1$, $B_j = \emptyset$ or $B_j$ is a bond, and $H_1B_1H_2B_2 \ldots B_{h-1}H_h$ is a path in the block-bond tree of $G$. A detailed description with examples can be found in [14]. For convenience, we sometimes write $\mathcal{H} := H_1H_2 \ldots H_h$ for this situation. In the rest of the paper, unless stated otherwise, we will always assume that each virtual edge in $E(H_i \cap H_{i+1})$ is deleted from $\mathcal{H}$ if at least one of $H_i$ and $H_{i+1}$ is 3-connected and there are exactly two components in $G - V(H_i \cap H_{i+1})$. Because in this case it is not possible to replace the virtual edge by a path in $G$ outside of $\mathcal{H}$. However, if both $H_i$ and $H_{i+1}$ are cycles, then the virtual edge shared by $H_i$ and $H_{i+1}$ can always be replaced by a path outside of $\mathcal{H}$. Throughout this section, we adopt the convention that an object is empty if it is not defined. For example, if $\mathcal{H} = H_1H_2 \ldots H_h$ is a block-chain under consideration, then $H_0$ and $H_{h+1}$ are both empty graphs.

The following result is proved in the proof of Lemma (3.6) in [14], which will be used to link together long paths from different block-chains. We note that the path stated in the lemma can be found in linear time by using a result from [43].

**Lemma (4.2.1).** Let $\mathcal{H} = H_1H_2 \ldots H_h$ be a block-chain in a 2-connected graph $G$, $x \in V(H_1) - V(H_2)$, and $f \in E(H_h) - E(H_{h-1})$ such that $f$ is not incident with $x$, and let $pq, vw$ be two distinct edges in $E(\mathcal{H}) - \{f\}$. Then there is a path $P$ in $\mathcal{H}$ through $f$ from $x$ to some $z \in \{p, q\} \cup \{v, w\}$ such that if $z \in \{p, q\}$ then $pq \notin E(P)$ and $vw \in E(P)$ and if $z \in \{v, w\}$
then \( vw \notin E(P) \) and \( pq \in E(P) \).

**Lemma (4.2.2).** Let \( \mathcal{H} = H_1H_2 \ldots H_h \) be a block-chain in a 2-connected graph \( G \), \( x \in V(H_1) - V(H_2) \), and \( pq, f \in E(H_h) - E(H_{h-1}) \) be distinct such that neither \( pq \) nor \( f \) is incident with \( x \). If \( H_h \) is 3-connected and \( q \) is not incident with \( f \), then in \( \mathcal{H} - q \), there is an \( (x,p) \)-path through \( f \).

**Proof.** We use induction on \( h \). If \( h = 1 \), then \( H_1 \) is 3-connected and \( H_1 - q \) is 2-connected, so \( H_1 - q \) contains an \( (x,p) \)-path through \( f \). Suppose \( h \geq 2 \) and let \( \{a,b\} = V(H_{h-1} \cap H_h) \). Since \( pq \notin E(H_{h-1}) \), we may assume \( a \notin \{p,q\} \). Let \( P_h \) be an \( (a,p) \)-path through \( f \) in \( H_h - q \).

If \( ab \notin P_h \), let \( P_1 \) be an \( (x,a) \)-path in \( H_1H_2 \ldots H_{h-1} - b \); then \( P_h := P_1 \cup P_h \) is the desired path. If \( ab \in P_h \), let \( P_1 \) be an \( (x,b) \)-path in \( H_1H_2 \ldots H_{h-1} - a \); then \( P := P_1 \cup (P_h - ab) \) is the desired path. \( \square \)

**Lemma (4.2.3).** Let \( \mathcal{H} = H_1H_2 \ldots H_h \) be a block-chain. Let \( xy, pq, uv \) be three edges such that \( xy \in E(H_1) - E(H_2) \) and \( pq, uv \in E(H_h) - E(H_{h-1}) \), where \( pq \neq xy \neq uv \) but it is possible that \( pq = uv \). Then there is a path \( P \) in \( \mathcal{H} \) from some \( z \in \{x,y\} \) to \( w \in \{p,q\} \cup \{u,v\} \) such that \( \{x,y\} \not\subseteq V(P) \), \( uv \in E(P) \) if \( w \in \{p,q\} \), and \( pq \in E(P) \) if \( w \in \{u,v\} \).

**Proof.** We first consider \( h = 1 \), that is \( \mathcal{H} = H_1 \). The result is trivial if \( H_1 \) is a cycle. Suppose that \( H_1 \) is 3-connected. Then \( H_1 - y \) is 2-connected, and thus contains an \( (x,\{p,q\}) \)-path \( P \) through \( uv \).

We now assume \( h \geq 2 \). Let \( \{a,b\} = V(H_h) \cap V(H_{h-1}) \). By the same argument as for the case where \( h = 1 \), there is a path \( P_H \) from \( z^* \in \{a,b\} \) to \( w \in \{p,q\} \cup \{u,v\} \) such that \( \{a,b\} \not\subseteq V(P_H) \), \( uv \in E(P_H) \) if \( w \in \{p,q\} \) and \( pq \in E(P_H) \) if \( w \in \{u,v\} \). Clearly, there is a path \( Q \) in \( H_1 \ldots H_{h-1} - ab \) from some \( z \in \{x,y\} \) to \( z^* \) such that \( \{x,y\} \not\subseteq V(Q) \). Then \( Q \cup P_H \) is the desired path. \( \square \)
Lemma (4.2.4). Let $\mathcal{H} = H_1H_2\ldots H_h$ be a block-chain, $u, v \in V(H_h)$ be distinct, and $x \in V(\mathcal{H} - v)$. Then there is a path from $x$ to $u$ avoiding $v$.

Proof. We use induction on $h$. The result is clearly true when $h = 1$. Assume the claim holds for block-chains with fewer than $h$ blocks. Let $\{a, b\} = V(H_{h-1} \cap H_h)$. If $x \in V(H_h)$, let $P_h$ be a path in $H_h$ from $x$ to $u$ avoiding $v$. If $ab \notin E(P_h)$, let $P := P_h$; if $ab \in E(P_h)$, let $P$ be obtained from $P_h$ by replacing $ab$ by a path in $H_1\ldots H_{h-1}$ from $a$ to $b$.

Suppose $x \notin V(H_h)$. Assume, without loss of generality, that $a \neq v$, and let $P_h$ be a path in $H_h$ from $a$ to $u$ avoiding $v$. By induction, let $P_1$ be an $(x, a)$-path avoiding $b$ in $H_1\ldots H_{h-1}$ if $ab \notin E(P_h)$, and let $P_1$ be an $(x, b)$-path avoiding $a$ in $H_1\ldots H_{h-1}$ if $ab \in E(P_h)$. Then $P := P_1 \cup (P_h - ab)$ is the desired path.

Lemma (4.2.5). Let $\mathcal{H} = H_1H_2\ldots H_h$ be a block-chain, and let $xx' \in E(H_1) - E(H_2)$ and $uv \in E(H_h)$ be two edges in $\mathcal{H}$. Then there is an $(x', \{u, v\})$-path in $\mathcal{H} - x$.

Proof. We use induction on $h$. The statement is clearly true when $h = 1$ as $H_1 - x$ is connected. So we assume $h \geq 2$, and let $V(H_1) \cap V(H_2) = \{a, b\}$. Suppose first that $x \notin \{a, b\}$. By induction, we let $P_1$ be an $(x', \{a, b\})$-path, say $(x', a)$-path, in $H_1 - x$, and let $P_2$ be an $(a, \{u, v\})$-path in $H_2H_2\ldots H_h - b$. Then $P_1 \cup P_2$ is the desired path. Then, suppose, without loss of generality, that $x = a$. Let $P_1$ be an $(x', b)$-path in $H_1 - x$, and by induction, let $P_2$ be a $(b, \{u, v\})$-path in $H_2H_2\ldots H_h - a$. Then $P_1 \cup P_2$ is the desired path.

We conclude this section by recalling two graph operations from [14]. Let $G$ be a graph and let $e, f$ be distinct edges of $G$. An $H$-transform of $G$ at $\{e, f\}$ is an operation that subdivides $e$ and $f$ by vertices $x$ and $y$, respectively, and then adds the edge $xy$. Let $x \in V(G)$ such that $x$ is not incident with $e$. A $T$-transform of $G$ at $\{x, e\}$ is an operation that subdivides $e$ with a vertex $y$ and then adds the edge $xy$. Let $G'$ be a graph obtained
from a 3-connected graph $G$ by an $H$-transform or a $T$-transform. It is easy to see that $G'$ is also 3-connected (see, e.g., Lemma (3.3) in [14] for a proof).

4.3 Lower bounds of $m^{\log_b 2} + n^{\log_b 2}$

Fix $b = d - 1$ hereinafter, where $d \geq 4$ is an integer and let $r = \log_b 2$. Clearly, $0 < r < 1$, which in turn gives that $m^r + n^r \geq (m + n)^r$. In this section we improve this inequality under different situations. The new inequalities will be used to show that the union of some long paths has the desired length. The first one is a strengthening of Lemmas (3.1) and (3.2) in [12].

Lemma (4.3.1). Let $m$ and $n$ be two positive real numbers such that $m \geq b^\beta n > 0$ for some real number $\beta$. Then,

$$m^r + n^r \geq \left( m + b^\beta \left( b^{\log_2(1+2^{-\beta})} - 1 \right) n \right)^r \geq \left( b^\beta b^{\log_2(1+2^{-\beta})} n \right)^r. \quad (4.1)$$

Proof. Define $f(t) = \frac{1}{r} \left( (1 + t^r)^{1/r} - 1 \right)$. It is easy to verify that

$$m^r + n^r = (m + f(n/m)n)^r \quad \text{and} \quad f'(t) = \frac{1}{t^2} \left( 1 - (1 + t^r)^{(1-r)/r} \right).$$

Since $b \geq 3$, we have $0 < r < 1$, and hence $f'(t) < 0$ when $t > 0$. Therefore $f(t)$ is a decreasing function on the interval $(0, \infty)$. For $m \geq b^\beta n > 0$, we have $n/m \leq b^{-\beta}$, and so (since $b^r = 2$)

$$f(n/m) \geq f(b^{-\beta}) = b^\beta \left( (1 + 2^{-\beta})^{1/r} - 1 \right) = b^\beta \left( b^{\log_2(1+2^{-\beta})} - 1 \right).$$

Here, the first inequality in (4.1) follows from $m^r + n^r \geq (m + f(n/m)n)^r$, and the 2nd inequality in (4.1) follows from $m \geq b^\beta n.$

Taking $\beta = 0, \log_b 1.1, 1.2, -1, -2$, we get the following inequalities from (4.1). It is
straightforward to verify (4.1a) - (4.1f)

\[ m^r + n^r \geq \begin{cases} 
(m + (b - 1)n)^r, & \text{if } m \geq n; \\
(m + 1.1(b^{\log_2(1+2^{-\log_b1.1})} - 1)n)^r, & \text{if } m \geq 1.1n; \\
(m + b(b^{\log_2 3/2} - 1)n)^r, & \text{if } m \geq bn; \\
(m + b^2(b^{\log_2 5/4} - 1)n)^r, & \text{if } m \geq b^2n; \\
(b^{\log_2 3/2}n)^r, & \text{if } m \geq n/b; \\
(b^{\log_2 5/4}n)^r, & \text{if } m \geq n/b^2. 
\end{cases} \tag{4.1a-4.1f} \]

In the proofs, the following elementary inequality will be used frequently for any two positive real numbers \( x \) and \( y \),

\[ x^r + y^r \geq 2\sqrt{x^r y^r} = ((d - 1)^2xy)^{r/2}. \tag{4.2} \]

**Lemma (4.3.2).** The following inequalities hold:

\[ x^r + 1 \geq (x + d - 1)^r \quad \text{provided } x \geq 1. \]

**Lemma (4.3.3).** Let \( b \geq 23 \) be an integer. If \( m \) and \( n \) are two positive real numbers such that \( m \geq 1.1n \), then \( m^r + n^r \geq (m + bn)^r \).

**Proof.** Applying Lemma (4.3.1) for \( \beta = \log_b 1.1 \leq \log_{23} 1.1 \), we have \( m^r + n^r \geq (m + 1.1(b^{\log_2(1+2^{-\log_b1.1})} - 1)n)^r \). So, we only need to show that \( 1.1(b^{\log_2(1+2^{-\log_b1.1})} - 1) \geq b \) provided \( b \geq 23 \). For any \( x \geq 1.1 \), let \( \tau := \tau(x) = \log_x 1.1 \), \( \phi := \phi(\tau) = \log_2(1 + 2^{-\tau}) \) and \( f(x) = x^{\phi(\tau(x))} \). It is clearly that \( \lim_{x \to \infty}(1.1f(x) - x) = \infty \). It is sufficient to show that \( 1.1f(x) - x \) is an increasing function for \( x \geq 23 \), which is equivalent to \( \frac{df}{dx}f(x) \geq 10/11 \) for \( x \geq 23 \).
Simple calculations show that \( \frac{d}{dx} \tau(x) = -\frac{\ln 1.1}{x \ln x} = -\frac{\tau(x)}{x \ln x} \) and \( \frac{d}{d\tau} \phi(\tau) = -\frac{1}{1 + 2^{2\tau}} \). So,

\[
\frac{d}{dx} f(x) = f(x) \left( \ln x \frac{d\phi d\tau}{d\tau dx} + \phi \frac{d\ln x}{dx} \right) = f(x) \left( \frac{\tau}{1 + 2^{\tau} + \phi(\tau)} \right).
\]

Since \( \lim_{x \to \infty} \tau(x) = 0 \) and \( \lim_{\tau \to 0} \phi(\tau) = 1 \), \( \lim_{x \to \infty} \frac{d}{dx} f(x) = 1 \). It is sufficient to show that \( \frac{d}{dx} f(x) \) is decreasing as \( x \) increasing. Writing \( \frac{d}{dx} f(x) \) in terms of \( \tau \), we have

\[
\frac{d}{dx} f(x) = x^{\phi(\tau) - 1} \left( \frac{\tau}{1 + 2^{\tau}} + \phi(\tau) \right) = e^{(\phi(\tau) - 1) \ln 1.1} \left( \frac{\tau}{1 + 2^{\tau} + \phi(\tau)} \right).
\]

We only need to show \( \frac{d}{dx} f(x) \) is increasing as \( \tau \) increasing when \( \tau \leq \tau(23) \), which is equivalent to \( \frac{d}{d\tau} \frac{df}{dx} > 0 \). Taking derivative, we obtain

\[
\frac{d}{d\tau} \frac{df}{dx} = e^{(\phi(\tau) - 1) \ln 1.1} \left( \frac{\ln 1.1 \left( 1 - \frac{\tau}{1 + 2^{2\tau}} \right) \left( \phi + \frac{\tau}{1 + 2^{\tau}} \right) - \frac{\tau^{2}}{2} \ln 2}{(1 + 2^{\tau})^{2}} \right).
\]

So, we only need to show that

\[
g(\tau) = \left( \frac{\ln 1.1 \left( 1 - \frac{\tau}{1 + 2^{2\tau}} \right) \left( \phi + \frac{\tau}{1 + 2^{\tau}} \right) - \frac{\tau^{2}}{2} \ln 2}{(1 + 2^{\tau})^{2}} \right) > 0 \quad \text{if} \quad \tau \leq \tau(23).
\]

We define the value of \( g(\tau) \) at \( \tau = 0 \) as

\[
g(0) = \lim_{\tau \to 0} g(\tau) = \frac{\ln 1.1 \ln 2}{8}.
\]

Then \( g(\tau) \) is a continuous function on the closed interval \([0, \tau(23)]\). To show \( g(\tau) > 0 \) within \([0, \tau(23)]\), as \( g(0) > 0 \), by the intermediate zero theorem, instead, we show that \( g(\tau) \) has no zero in \([0, \tau(23)]\). To do so, using the bisection method, with tolerance as \( 1 \times 10^{-10} \), a numerical search within \([0, 1] \) interval gives 0.04765221 as the root of \( g(\tau) \). Since \( \tau(23) < 0.0304 < 0.04765221 \), we conclude that \( g(\tau) > 0 \) when \( 0 \leq \tau \leq \tau(23) \). The proof is completed.
4.4 Long paths in block-chains

In this section, we will give a few lower bounds of long paths connecting special vertices in a block-chain. Throughout this section, we assume that \( n \geq 4 \), Theorem (4.1.1) holds for graphs with at most \( n - 1 \) vertices, and \( \mathcal{H} := H_1H_2\ldots H_h \) is a block-chain such that \( |\mathcal{H}| \leq n - 1 \) such that

- \( \Delta(H_i) \leq d \) for each \( 1 \leq i \leq h \).

- As of a subgraph of \( G \), \( \mathcal{H} \) contains at most \( 2d - 1 \) vertices of degree 2.

Recall, for convention, we also denote by \( \mathcal{H} \) the graph with vertex set \( \bigcup V(H_i) \) and edge set \( \bigcup E(H_i) \) with the deletion of virtual edges.

**Lemma (4.4.1).** For any edge \( uv \in E(H_1) - E(H_2) \), there is a \((u,v)\)-path \( P \) in \( \mathcal{H} \) such that

\[
\ell(P) \geq \frac{1}{4} \left( \frac{d - 2.1}{d - 1}(|\mathcal{H}| + 1) \right)^r + 1,
\]

provided that \( d \geq 23 \).

*Proof.* Since \( |\mathcal{H}| \leq n - 1 \), it follows from our assumption that Theorem (4.1.1) holds for each \( H_i \). We proceed with induction on \( h \). Suppose \( h = 1 \). Then \( \mathcal{H} \) is either a cycle or a 3-connected graph. Since the case \( |\mathcal{H}| \leq d \) is trivial, we may assume \( |\mathcal{H}| \geq d + 1 \), and hence

\[
|\mathcal{H}| > \frac{d - 2.1}{d - 1}(|\mathcal{H}| + 1).
\]

By Theorem (4.1.1)(c), \( \mathcal{H} = H_1 \) contains a cycle \( C \) through \( uv \) such that

\[
\ell(C) \geq \frac{1}{4} |\mathcal{H}|^r + 2 \geq \frac{1}{4} \left( \frac{d - 2.1}{d - 1}(|\mathcal{H}| + 1) \right)^r + 2.
\]

Hence \( P := C - \{uv\} \) gives the desired path.
Therefore, assume $h \geq 2$. Let $\mathcal{H}' := H_2 \cdots H_h$ and $\{a_1, b_1\} := V(H_1 \cap H_2)$. We consider two cases.

First, assume $|H_1| \geq \frac{d-2.1}{d-1}(|\mathcal{H}| + 1)$. By Theorem (4.1.1)(c), we can find a cycle $C_1$ in $H_1$ through $uv$ such that
\[
\ell(C_1) \geq \frac{1}{4} |H_1|^r + 2 \geq \frac{1}{4} \left( \frac{d-2.1}{d-1}(|\mathcal{H}| + 1) \right)^r + 2.
\]

If $C_1$ does not contain $a_1b_1$, then $P := C_1 - \{uv\}$ is the desired path. If $C_1$ contains $a_1b_1$, then let $C_2$ be a cycle in $\mathcal{H}'$ through $a_1b_1$. It is clear that $P := (C_1 \cup C_2) - \{a_1b_1, uv\}$ is the desired path.

Now assume $|H_1| < \frac{d-2.1}{d-1}(|\mathcal{H}| + 1)$. Then
\[
|\mathcal{H}'| + 1 = |\mathcal{H}| - |H_1| + 3 > \frac{d-1}{d-2.1}|H_1| - |H_1| + 2 > \frac{1.1|H_1|}{d-2.1} > \frac{1.1|H_1|}{d-1}.
\]

Applying Theorem (4.1.1)(b), we find a cycle $C_1$ in $H_1$ through $uv$ and $a_1b_1$ such that
\[
|C_1| \geq \frac{1}{4} \left( \frac{(d-2.1)|H_1|}{(d-1)^2} \right)^r + 2.
\]

By induction, we find a path $P'$ in $\mathcal{H}'$ between $a_1$ and $b_1$ such that
\[
\ell(P') \geq \frac{1}{4} \left( \frac{d-2.1}{d-1}(|\mathcal{H}'| + 1) \right)^r + 1.
\]

Hence $P := (C_1 \cup P') - \{uv, a_1b_1\}$ is a path between $u$ and $v$ in $\mathcal{H}$ such that
\[
\ell(P) \geq \frac{1}{4} \left( \frac{(d-2.1)|H_1|}{(d-1)^2} \right)^r + \frac{1}{4} \left( \frac{d-2.1}{d-1}(|\mathcal{H}'| + 1) \right)^r + 1
\]
\[> \frac{1}{4} \left( \frac{d-2.1}{d-1} \left( |\mathcal{H}'| + 1 + (d-1)\frac{|H_1|}{d-1} \right) \right)^r + 1 \quad \text{(by Lemma (4.3.3))}
\]
\[> \frac{1}{4} \left( \frac{d-2.1}{d-1}(|\mathcal{H}| + 1) \right)^r + 1.
\]
where in the 2nd inequality above, the inequality $|\mathcal{H}'| + 1 \geq \frac{1}{d-1} |H_1|$ is used.

Lemma (4.4.2). Let $x \in V(H_h) - V(H_{h-1})$ such that $d_{H_h}(x) = d_{H_h}(x) \leq d - 1$ and $uv \in E(H_1) - E(H_2)$ such that $x \notin \{u, v\}$ when $h = 1$. Then there exists a path $P$ in $\mathcal{H} - v$ from $u$ to $x$ such that

$$\ell(P) \geq \frac{1}{4} \sum_{i=1}^{h} \left( \frac{d - 2.1}{(d - 1)^2} |H_i| \right)^r + \frac{1}{2} \geq \frac{1}{4} \left( \frac{d - 2.1}{(d - 1)^2} |\mathcal{H}| \right)^r + \frac{1}{2}.$$

Moreover, if $H_1$ is 3-connected, we can improve the constant $1/2$ to $1$:

$$\ell(P) \geq \frac{1}{4} \sum_{i=1}^{h} \left( \frac{d - 2.1}{(d - 1)^2} |H_i| \right)^r + 1 \geq \frac{1}{4} \left( \frac{d - 2.1}{(d - 1)^2} |\mathcal{H}| \right)^r + 1.$$

Proof. Note that the second inequality in each of the lower bounds above for $\ell(P)$ is a simple application of Lemma (4.3.1). So we only show the first part of the lower bounds.

We apply induction on $h$. Suppose $h = 1$. If $H_1$ is a cycle, then $|H_1| \leq 2d - 1$, which in turn gives $\frac{1}{4}(\frac{d - 2.1}{(d - 1)^2})^r < 1/2$ and $\frac{1}{4}(\frac{d - 2.1}{(d - 1)^2}|H_1|)^r + \frac{1}{2} < 1$. On the other hand, since $x \notin \{u, v\}$, there is an $(x, u)$-path $P$ in $H_1 - v$ with $\ell(P) \geq 1$. Hence, the assertion holds.

Now assume $H_1 = \mathcal{H}$ is 3-connected. Note that $\Delta(H_1 + xu - v) \leq d$ and $v$ has at most $d - 1$ neighbors in $H_1 - v$. By applying Theorem (4.1.1)(a) to $H_1 + xu$ we find a cycle $C$ through $xu$ in $(H_1 + xu) - v$ such that $\ell(C) \geq \frac{1}{4} ((d - 2.1)|H_1|/(d - 1)^2)^r + 2$. Hence $P := C - \{xu\}$ gives the desired path.

Now we assume $h \geq 2$. Let $\{a_1, b_1\} := V(H_1 \cap H_2)$ and $I := H_2 H_3 \cdots H_h$. By induction, in $H_1 - v$, there exists a path $P_1$ from $u$ to some vertices in $\{a_1, b_1\}$, say to $a_1$ (notice that $a_1 b_1$ may be on $P_1$) such that $\ell(P_1) \geq \frac{1}{4} ((d - 2.1)|H_1|/(d - 1)^2)^r + \frac{1}{2}$ unless $H_1$ is a cycle and $u \notin \{a_1, b_1\}$ (in this case, $P_1$ may only contain one vertex). Moreover, $\ell(P_1) \geq \frac{1}{4} ((d - 2.1)|H_1|/(d - 1)^2)^r + 1$ when $H_1$ is 3-connected. We will consider the case that $H_1$ is a cycle and $u \in \{a_1, b_1\}$ at the end.
Applying induction again we find a path $P_2$ in $\mathcal{I} - b_1$ from $x$ to $a_1$ such that

$$\ell(P_2) \geq \frac{1}{4} \sum_{i=2}^{h} \left( \frac{(d-2.1)|H_i|}{(d-1)^2} \right)^r + \frac{1}{2}.$$ 

Moreover, when $H_2$ is 3-connected,

$$\ell(P_2) \geq \frac{1}{4} \sum_{i=2}^{h} \left( \frac{(d-2.1)|H_i|}{(d-1)^2} \right)^r + 1.$$ 

If $a_1b_1 \in E(\mathcal{H})$ or $a_1b_1 \notin E(P_1)$, $P := P_1 \cup P_2$ is the desired path. Thus, we may assume that $a_1b_1$ is a virtual edge in $H_1$, $a_1b_1 \in E(P_1)$ and $a_1b_1 \notin E(\mathcal{H})$.

If $H_1$ is a cycle, then $H_2$ must be 3-connected since $a_1b_1 \notin E(\mathcal{H})$. Let $P_2$ in $\mathcal{I} - a_1$ from $x$ to $b_1$ such that

$$\ell(P_2) \geq \frac{1}{4} \sum_{i=2}^{h} \left( \frac{(d-2.1)|H_i|}{(d-1)^2} \right)^r + 1.$$ 

Then, $P := (P_1 - a_1) \cup P_2$ satisfying

$$\ell(P) \geq \ell(P_2) \geq \frac{1}{4} \sum_{i=2}^{h} \left( \frac{(d-2.1)|H_i|}{(d-1)^2} \right)^r + 1 \geq \frac{1}{4} \sum_{i=1}^{h} \left( \frac{(d-2.1)|H_i|}{(d-1)^2} \right)^r + \frac{1}{2},$$ 

so $P$ is the desired path.

We may assume that $H_1$ is 3-connected. In this case, we have $\ell(P_1) \geq \frac{1}{4} \left( \frac{(d-2.1)|H_1|}{(d-1)^2} \right)^r + 1$. Let $P_2$ be an $(x, b_1)$-path in $\mathcal{I} - a_1$ such that $\ell(P_2) \geq \frac{1}{4} \sum_{i=2}^{h} \left( \frac{(d-2.1)|H_i|}{(d-1)^2} \right)^r + \frac{1}{2}$. Moreover, in this case, we can find the desired path if $H_2$ is 3-connected. So, we may additionally assume that $H_2$ is a cycle. Since $P_1$ is a $(u, a_1)$-path, $a_1b_1 \in E(P_1)$, and $\ell(P_1) > 1$, $u \notin \{a_1, b_1\}$.

We now, under the assumption that $H_1$ is 3-connected, $H_2$ is a cycle, $u \notin \{a_1, b_1\}$, and $a_1b_1 \in E(P_1)$, construct a path $P$ according to the following two cases.

Suppose first that $h = 2$. If $H_2$ is a triangle, that is, $V(H_2) = \{a_1, b_1, x\}$. Applying
Theorem 1.1) a) to \((\mathcal{H} + u)x - v\), we obtain the desired \((x, u)\)-path \(P\) in \(\mathcal{H}−v\). So we may assume that \(|V(H_2)| \geq 4\), which in turn shows that \(H_2\) contains a path \(P_2\) with \(\ell(P_2) \geq 2\), which is either an \((x, a_1)\)-path or an \((x, b_1)\)-path. Assume, without loss of generality, \(P_2\) is an \((x, a_1)\)-path and \(Q_2\) is the \((x, b_1)\)-path in \(H_2 - a_1b_1\). Let \(Q_1\) be a \((u, b_1)\)-path in \(H_1 - v\) such that \(\ell(Q_1) \geq \frac{1}{4}\left(\frac{(d-2.1)|H_1|}{(d-1)^2}\right)^r + 1\). If \(a_1b_1 \notin E(Q_1)\), then \(P := Q_2 \cup Q_1\) is the desired path. If \(a_1b_1 \in E(Q_1)\), then \(P_2 \cup (Q_1 - a_1)\) is the desired path since \(|H_2| \leq 2d - 1\).

We now assume that \(h \geq 3\) and let \(\{a_2, b_2\} := V(H_2 \cap H_3)\) and \(\mathcal{H}'' := H_3H_4\cdots H_h\). Applying induction, there is a \((u, a_2)\)-path \(P_1\) in \(H_1H_2\) avoiding \(v\) such that

\[
\ell(P_1) \geq \frac{1}{4} \sum_{i=1}^{2} \left(\frac{(d-2.1)|H_i|}{(d-1)^2}\right)^r + 1.
\]

If \(a_2b_2 \notin E(P_1)\), by the induction hypothesis, we find an \((a_2, x)\)-path \(P'\) in \(\mathcal{H}'' - b_2\) such that

\[
\ell(P') \geq \frac{1}{4} \sum_{i=3}^{h} \left(\frac{(d-2.1)|H_i|}{(d-1)^2}\right)^r + \frac{1}{2}.
\]

Then \(P := P_1 \cup P'\) gives the desired path. Thus, we assume \(a_2b_2 \in E(P_1)\). If \(H_3\) is 3-connected, then by induction there is a \((b_2, x)\)-path \(P'\) avoiding \(a_2\) in \(\mathcal{H}''\) such that \(\ell(P') \geq \frac{1}{4} \sum_{i=3}^{h} \left(\frac{(d-2.1)|H_i|}{(d-1)^2}\right)^r + 1\) in \(\mathcal{H}''\). Hence, \(P := (P_1 - a_2) \cup P'\) gives the desired path. Thus, we have \(a_2b_2 \in E(P_1)\) and \(H_3\) is a cycle. Since both \(H_2\) and \(H_3\) are cycles, \(a_2b_2 \in E(\mathcal{H})\). We find an \((a_2, x)\)-path \(P'\) in \(\mathcal{H}'' - b_2\) such that \(\ell(P') \geq \frac{1}{4} \sum_{i=3}^{h} \left(\frac{(d-2.1)|H_i|}{(d-1)^2}\right)^r + \frac{1}{2}\). Then, \(P := P_1 \cup P'\) is the desired path.

Let \(U\) and \(W\) be two vertex sets. By definition, an \((U, W)\)-path \(P\) is a \((u, w)\)-path for some \(u \in U\) and \(w \in W\), and \(|V(P) \cap U| = 1\) and \(|V(P) \cap W| = 1\). We call \(P\) a path from \(U\) to \(W\) if \(P\) is a \((u, v)\)-path from some \(u \in U\) and \(w \in W\) while \(V(P) \cap U\) or \(V(P) \cap W\) may not be singleton.

**Lemma (4.4.3).** Suppose that \(|\mathcal{H}| \leq n - 2\) and Theorem 1.1 holds for graphs with less than \(n\) vertices. Let \(x \in V(H_h) - V(H_{h-1})\) such that \(d_\mathcal{H}(x) \leq d - 1\), \(f \in E(H_1) - E(H_2)\)
and \( pq \in E \left( \bigcup_{i=1}^{h} H_i \right) - \{ f \} \). Then there exists a path \( P \) in \( \mathcal{H} \) from \( x \) to \( z \in \{ p, q \} \) through \( f \) such that \( pq \notin E(P) \) and \( \ell(P) \geq \frac{1}{4} \left( \frac{d-2.1}{(d-1)^2} \right)^r |\mathcal{H}| \). \( r \)

**Proof.** We use induction on \( h \) and consider the base case \( h = 1 \) first. In this case, if \( H_1 \) is a cycle, then there exists a path \( P \) from \( x \) to \( \{ p, q \} \) through \( f \). Since \( |H_1| < 2d - 1 \), \( \ell(P) \geq 1 \geq \frac{1}{4} \left( \frac{d-2.1}{(d-1)^2} \right)^r |\mathcal{H}| \). We may assume that \( H_1 \) is 3-connected and consider two cases according to whether \( x \in \{ p, q \} \).

If \( x \notin \{ p, q \} \), let \( H'_1 \) be the graph obtained from \( H_1 \) by a \( T \)-transform at \( \{ x, pq \} \), and let \( x' \) be the new vertex. Since \( |V(H'_1)| \leq |V(\mathcal{H})| + 1 \leq n - 1 \), we use Theorem (1.1)(b) to find a cycle \( C \) in \( H'_1 \) through \( xx' \) and \( f \) such that \( \ell(C) \geq \frac{1}{4} ((d-2.1)|H'_1|/(d-1)^2) + 2 \). It is clear that \( C - x' \) gives a desired path. If \( x \in \{ p, q \} \), we use Theorem (4.1.1)(b) to find a cycle in \( H_1 \) through \( pq \) and \( f \). Then \( C - \{ pq \} \) is the desired path.

Assume \( h \geq 2 \). Let \( \mathcal{H}' = H_2 H_3 \cdots H_h \) and \( \{ a_1, b_1 \} := V(H_1) \cap V(H_2) \). We consider the following three cases.

**Case 1.** \( pq \notin E(H_1) \). We use Theorem (4.1.1)(b) to find a cycle \( C \) in \( H_1 \) through \( f \) and \( a_1 b_1 \) such that

\[
\ell(C) \geq \frac{1}{4} \left( \frac{(d-2.1)|H_1|}{(d-1)^2} \right)^r + 2.
\]

We apply induction to find a path \( P' \) in \( \mathcal{H}' \) from \( x \) to \( \{ p, q \} \) such that \( a_1 b_1 \in E(P') \), \( pq \notin E(P') \), and

\[
\ell(P') \geq \frac{1}{4} \left( \frac{(d-2.1)|H'|}{(d-1)^2} \right)^r.
\]

Then \( (C - \{ a_1 b_1 \}) \cup P' \) is also the desired path.

**Case 2.** \( \{ p, q \} = \{ a_1, b_1 \} \). We use Lemma (4.4.2) to find a path \( P' \) in \( \mathcal{H}' - q \) from \( x \) to \( p \) avoiding \( q \) such that \( \ell(P') \geq \frac{1}{4} \left( \frac{d-2.1}{(d-1)^2} |\mathcal{H}'| \right)^r + \frac{1}{2} \) and let \( C \) be the cycle in \( H_1 \) as in Case 1. Then \( P := P' \cup (C - \{ pq \}) \) is the desired path.
**Case 3.** \(pq \in E(H_1) - \{a_1b_1\}\). We assume, without loss of generality, that \(a_1 \notin V(\cup_{i \geq 3} H_i)\). By induction, there is a path \(P_1\) in \(H_1\) from \(a_1\) to \(\{p, q\}\) such that \(f \in E(P_1)\), \(pq \notin E(P_1)\), and \(\ell(P_1) \geq \left(\frac{d-2.1}{(d-1)^2}|H_1|\right)^r\). Since \(x \in V(H_h) - V(H_{h-1})\) and \(\{a_1, b_1\} = V(H_1) \cap V(H_2)\), we have \(x \notin \{a_1, b_1\}\). By Lemma 4.4.2 there is a path \(P'\) in \(\mathcal{H}' - b_1\) from \(x\) to \(a_1\) such that

\[
\ell(P') \geq \frac{1}{4} \left(\frac{d-2.1}{(d-1)^2}|\mathcal{H}'|\right)^r + \frac{1}{2}.
\]

Moreover, if \(H_2\) is 3-connected then

\[
\ell(P') \geq \frac{1}{4} \left(\frac{d-2.1}{(d-1)^2}|\mathcal{H}'|\right)^r + 1.
\]

Hence \(P := P_1 \cup P'\) is the desired path in \(\mathcal{H}\) if \(a_1b_1 \notin E(P_1)\), \(a_1b_1 \in E(\mathcal{H})\), or \(H_2\) is 3-connected. So, we assume \(a_1b_1 \in E(P_1)\), \(a_1b_1\) is a virtual edge in \(H_1\), \(a_1b_1 \notin E(\mathcal{H})\), and \(H_2\) is a cycle.

If \(h = 2\) and \(|V(H_2)| \geq 4\), then in \(H_2\), we can find an \((x, \{a_1, b_1\})\)-path \(P'\) such that \(\ell(P') \geq 2\). If \(P'\) is an \((x, b_1)\)-path, then \(P := (P_1 - a_1) \cup P'\) is the desired path by noting that \(|H_2| \leq 2d - 1\). So assume that \(P'\) is an \((x, a_1)\)-path. In \(H_1\), let \(P_1\) be a path from \(b_1\) to \(\{p, q\}\) such that \(f \in E(P_1)\), \(pq \notin E(P_1)\) and \(\ell(P_1) \geq \frac{1}{4}((d-2.1)|H_1|/(d-1)^2)^r\). We may assume that \(a_1b_1 \in E(P_1)\) (otherwise, let \(P'\) be an \((x, b_1)\)-path in \(H_2 - a_1\), then \(P := P_1 \cup P'\) is the desired path, as \(x \in V(H_2) - V(H_1)\), \(\ell(P') \geq 1\)). Then \(P := (P_1 - b_1) \cup P'\) is the desired path. So we assume \(|V(H_2)| = 3\) or \(V(H_2) - V(H_1) = \{x\}\). Let \(H^*\) be the graph obtained from \(H_1H_2\) by a \(T\)-transform at \(\{x, pq\}\), and let \(x'\) be the new vertex. Since \(|V(H^*)| \leq n - 1\), we can then apply Theorem 1.1(b) to find a cycle \(C\) in \(H^*\) through \(xx'\) and \(f\) such that \(\ell(C) \geq \frac{1}{4}((d-2.1)|H_1'|/(d-1)^2)^r + 2\). It is clear that \(C - x'\) gives the desired path.

If \(h \geq 3\), let \(\{a_2, b_2\} := V(H_2) \cap V(H_3)\) and \(\mathcal{H}'' := H_3H_4 \cdots H_h\). Assume, without loss of generality, that \(a_2 \in V(H_2) - V(H_1)\). Applying induction, there is a path \(P_1\) in \(H_1H_2\)
from $a_2$ to $\{p, q\}$ such that $f \in E(P_1)$ and $pq \notin E(P_1)$, and

$$\ell(P_1) \geq \frac{1}{4}\sum_{i=1}^{2} \left( \frac{(d - 2.1)|H_i|}{(d - 1)^2} \right)^r.$$  

If $a_2b_2 \notin E(P_1)$, by Lemma (4.4.2) we find an $(a_2, x)$-path $P'$ in $\mathcal{H}'' - b_2$ such that $\ell(P') \geq \frac{1}{4}\sum_{i=3}^{h} \left( \frac{(d - 2.1)|H_i|}{(d - 1)^2} \right)^r + \frac{1}{2}$. Then $P := P_1 \cup P'$ gives the desired path. Thus, we assume $a_2b_2 \in E(P_1)$. If $H_3$ is 3-connected, then there is a $(b_2, x)$-path $P'$ in $\mathcal{H}''$ such that $a_2 \notin V(P')$ and $\ell(P') \geq \frac{1}{4}\sum_{i=3}^{h} \left( \frac{(d - 2.1)|H_i|}{(d - 1)^2} \right)^r + 1$ by Lemma (4.4.2). Hence, $P := (P_1 - a_2) \cup P'$ gives the desired path. Thus, we have $a_2b_2 \in E(P_1)$ and $H_3$ is a cycle. Recall that $H_2$ is a cycle by our earlier assumption. We use Lemma (4.4.2) to find an $(a_2, x)$-path $P'$ of desired length in $\mathcal{H}'' - b_2$. Let $P := P_1 \cup P'$ (since $a_2b_2 \in E(\mathcal{H})$ in this case). Then $P$ is the desired path. □

**Lemma (4.4.4).** Assume that Theorem (4.1.1) holds for graphs with less than $n$ vertices. Let $\mathcal{H} = H_1H_2 \ldots H_h$ be a block-chain in $G - y$ such that $|\mathcal{H}| < n$, $x \in V(H_1) - V(H_2)$ with $d_\mathcal{H}(x) \leq d - 1$, $w \in V(H_k) - V(H_{k-1}) - \{x\}$ for some $k$ with $d_\mathcal{H}(w) \leq d - 1$, and let $1 \leq m \leq h$ be fixed. Then there is a $(w, x)$-path $P$ in $\mathcal{H}$ such that

$$\ell(P) \geq \frac{1}{4}|H_m|^r + \frac{1}{4}\sum_{i=1, \neq m}^{\max\{k, m\}} \left( \frac{d - 2.1|H_i|}{(d - 1)^2} \right)^r;$$  

(4.3)

particularly, when $k = h$,

$$\ell(P) \geq \frac{1}{4}|H_m|^r + \frac{1}{4}\sum_{i \neq m} \left( \frac{d - 2.1|H_i|}{(d - 1)^2} \right)^r.$$  

(4.4)

**Proof.** Let $V(H_i \cap H_{i+1}) = \{a_i, b_i\}$ for $i = 1, 2, \ldots, h - 1$ such that each $H_i - a_ib_i - a_{i-1}b_{i-1}$ contains two vertex-disjoint paths connecting $a_{i-1}$ to $a_i$ and $b_{i-1}$ to $b_i$, respectively. We consider the following cases.

**Case 1.** $m = k = 1$
If $H_m$ is 3-connected, then by Theorem \((4.1.1)\) (c), $H_m + wx$ contains a cycle $C_m$ through $wx$ such that $|C_m| \geq \frac{1}{4}|H_m|^r + 2$. So, $C_m - \{wx\}$ is the desired path. If $H_m$ is a cycle, the $(x, w)$-path in $H_m - a_m b_m$ with length at least 1 is the desired path (as $x \neq w$).

**Case 2. $m = 1$ and $k > 1$**

If $H_m$ is 3-connected, we perform a $T$-transform on $(x, a_m b_m)$ and let $z$ be the resulting new vertex. Then, by Theorem \((4.1.1)\) (c), there is a cycle $C_m$ in the $T$-transformation through $xz$ such that $|C_m| \geq \frac{1}{4}|H_m|^r + 2$. Let $P_m = C_m - z$. Clearly, $a_m b_m \notin E(P_m)$ and $\ell(P_m) \geq \frac{1}{4}|H_m|^r$.

Assume $P_m$ is from $x$ to $a_m$. If $H_m$ is a cycle, let $P_m$ be a path from $x$ to $\{a_m, b_m\}$, say to $a_m$, which has length at least 1. Let $Q$ be a $(w, a_m)$-path in $H_{m+1} H_{m+2} \ldots H_k - b_m$ given by Lemma \((4.4.2)\). Then $P := P_m \cup Q$ is the desired path.

**Case 3. $m > 1$ and $k = m$**

Then $w \in H_m$ and $w \notin \{a_{m-1}, b_{m-1}\}$. If $H_m$ is 3-connected, we do a $T$-transformation on $(w, a_{m-1} b_{m-1})$ and, use Theorem \((4.1.1)\) (c) to obtain a path $P_m$ from $\{a_{m-1}, b_{m-1}\}$, say $a_{m-1}$, to $w$ with $\ell(P_m) \geq \frac{1}{4}|H_m|^r$; if $H_m$ is a cycle, let $P_m$ be a path from $w$ to $\{a_{m-1}, b_{m-1}\}$, say $a_{m-1}$ of length at least 1. Let $P_1$ be an $(x, a_{m-1})$-path in $H_1 \ldots H_{m-1} - b_{m-1}$ with $\ell(P_1) \geq \frac{1}{4} \sum_{i < m} \left(\frac{d-2}{(d-1)^r} |H_i| \right)^r$ given by Lemma \((4.4.2)\). Then, $P := P_1 \cup P_m$ is the desired path.

**Case 4. $m > 1$ and $k < m$**

Applying Theorem \((4.1.1)\)(c), we find an $(a_{m-1}, b_{m-1})$-path $P_m$ in $H_m$ with $\ell(P_m) \geq \frac{1}{4}|H_m|^r + 1$. In case that $a_m b_m \in E(P_m)$ and $a_m b_m \notin E(G)$, the edge $a_m b_m$ on $P_m$ is replaced by an $(a_m, b_m)$-path in $H_{m+1} H_{m+2} \ldots H_h$.

For each $i$ with $k < i < m$, we use Theorem \((4.1.1)\)(b) to find a cycle $C_i$ in $H_i$ through $a_{i-1} b_{i-1}$, and $a_i b_i$ such that $\ell(C_i) \geq \frac{1}{4} \left(\frac{(d-2)}{(d-1)^r} |H_i| \right)^r + 2$. Let $P_1$ and $Q_i$ be the two components of $C_i - \{a_{i-1} b_{i-1}, a_i b_i\}$.
If \( w \in \{a_k, b_k\} \), say \( w = a_k \), applying Lemma (4.4.2) we find an \((x, b_k)\)-path \( P_1 \) in \( H_1 \ldots H_k - a_k \) with \( \ell(P_1) \geq \frac{1}{4} \sum_{i<k} \left( \frac{(d-2)}{(d-1)^2} |H_i| \right)^r \). Clearly, \( P_1 \cup (\cup_{i=k}^{m-1} (P_i \cup Q_i)) \cup P_m \) gives the desired path. So assume \( w \notin \{a_k, b_k\} \). If \( H_k \) is 3-connected, we do a \( T \)-transformation on \((w, a_{k-1}b_{k-1})\) and let \( w' \) denote the new vertex. Applying Theorem (4.1.1) (b), we find a cycle \( C_k \) in \( H_k \) through \( w'w \) and \( a_kb_k \) such that \( \ell(C_k) \geq \frac{1}{4} \left( \frac{(d-2)}{(d-1)^2} |H_k| \right)^r + 2 \). Let \( P_k \) and \( Q_k \) be the two components of \( C_k - \{w', a_kb_k\} \). Without loss of generality, we may assume that \( P_k \) is a \((w, a_{k-1})\)-path. Note that, \( \ell(P_k) + \ell(Q_k) \geq \frac{1}{4} \left( \frac{(d-2)}{(d-1)^2} |H_k| \right)^r - 1 \). If \( H_k \) is a cycle, let \( P' \) be the path from \( w \) to \( \{a_{k-1}, b_{k-1}\} \), say \( a_{k-1} \), in \( H_k - a_{k-1}b_{k-1} \) through \( a_kb_k \). Let \( P_k \) and \( Q_k \) be the two components of \( P' - \{a_kb_k\} \). Then \( \ell(P_k) + \ell(Q_k) \geq \frac{1}{4} \left( \frac{(d-2)}{(d-1)^2} |H_k| \right)^r \) as \( w \notin \{a_{k-1}, a_{k-1}, b_{k-1}, b_k\} \) and \( |H_k| \leq 2d - 1 \).

Applying Lemma (4.4.2) we find an \((x, a_{k-1})\)-path \( P_1 \) in \( H_1 \ldots H_{k-1} - b_{k-1} \) with \( \ell(P_1) \geq \frac{1}{4} \sum_{i<k} \left( \frac{(d-2)}{(d-1)^2} |H_i| \right)^r \). Clearly, \( P_1 \cup (\cup_{i=k}^{m-1} (P_i \cup Q_i)) \cup P_m \) gives the desired path.

**Case 5.** \( k > m > 1 \)

We start by finding a desired path in \( H_m \) and first consider the case that \( H_m \) is 3-connected. Let \( H'_m \) be obtained by an \( H \)-transform of \( H_m \) over \((a_{m-1}b_{m-1}, a_mb_m)\) and let \( c_{m-1} \) and \( c_m \) be new vertices. By Theorem (4.1.1) (c), we find a cycle \( C_m \) in \( H'_m \) through \( c_{m-1}c_m \) such that \( |C_m| \geq \frac{1}{4} |H_m|^r + 2 \). Then \( C_m - \{c_m, c_{m-1}\} \) gives a path \( P_m \) from \( \{a_{m-1}, b_{m-1}\} \) to \( \{a_m, b_m\} \), say from \( a_{m-1} \) to \( b_m \), such that \( \ell(P_m) \geq \frac{1}{4} |H_m|^r - 1 \). If \( H_m \) is a cycle, let \( P_m \) be a nontrivial path from \( \{a_{m-1}, b_{m-1}\} \) to \( \{a_m, b_m\} \), say from \( a_{m-1} \) to \( b_m \), not containing the edges \( a_{m-1}b_{m-1} \) and \( a_mb_m \).

Applying Lemma (4.4.2) we find an \((x, a_{m-1})\)-path \( P_1 \) in \( H_1H_2 \ldots H_{m-1} - b_{m-1} \) with \( \ell(P_1) \geq \frac{1}{4} \sum_{i<m} \left( \frac{(d-2)}{(d-1)^2} |H_i| \right)^r + \frac{1}{2} ; \) and find a \((b_m, w)\)-path in \( H_{m+1}H_{m+2} \ldots H_k - a_m \) with \( \ell(P_2) \geq \frac{1}{4} \sum_{m<i<k} \left( \frac{(d-2)}{(d-1)^2} |H_i| \right)^r + \frac{1}{2} \). Clearly, \( P_1 \cup P_m \cup Q \) is the desired path. 

Let \( H \) be a 3-connected graph and \( ab \in E(H) \). Then \( H - ab \) is a block-chain, and by a simple argument, each of \( a \) and \( b \) belongs to exactly one block in \( H - ab \). In the following
lemma, we will say $H - ab$ is a block-chain even in the case that $H$ is a cycle. We include this trivial case just for notational convenience.

**Lemma (4.4.5).** Let $\mathcal{H} = H_1H_2 \ldots H_h$ be a block-chain in $G - y$ such that $|\mathcal{H}| \leq n - 1$, and $xx' \in E(H_1) - E(H_2)$ and $y' \in V(H_h) - V(H_{h-1}) - \{x, x'\}$. Suppose $H_k = \max\{|H_i| : H_i \in \mathcal{H}\}$. Let $\{a, b\} = V(H_k) \cap V(H_{k-1})$, where $a = x'$ and $b = x$ if $k = 1$. Let $H_0 := H_{k_1}H_{k_2} \cdots H_{k_{k_0}} \cdots H_{k_{k_1}}$ be the block-chain $H_k - ab$ (when $H_k$ is a cycle, $H_{k_i}$ is a copy of $K_2$ for each $1 \leq i \leq k_1$) such that $a \in H_{k_1}$, $b \in H_{k_{k_0}}$, and $|H_{k_{k_0}}| = \max\{|H_{k_i}| : 1 \leq i \leq k_1\}$. Then there is a path $P$ in $\mathcal{H} - x$ from $x'$ to $y'$ with

$$\ell(P) \geq \frac{1}{4} |H_{k_{k_0}}|^r + \frac{1}{4} \sum_{i=1}^{k_0-1} \left( \frac{d-2}{(d-1)^2} |H_{k_i}| \right)^r + \frac{1}{4} \sum_{i=k+1}^h \left( \frac{d-2}{(d-1)^2} |H_i| \right)^r - \frac{1}{2}, \quad (4.5)$$

and a path $Q$ in $\mathcal{H}$ from $x$ to $x'$ with

$$\ell(Q) \geq \frac{1}{4} |H_{k_{k_0}}|^r + \frac{1}{4} \sum_{i \neq k_0} \left( \frac{d-2}{(d-1)^2} |H_{k_i}| \right)^r + \frac{1}{4} \sum_{i<k} \left( \frac{d-2}{(d-1)^2} |H_i| \right)^r; \quad (4.6)$$

moreover, if $H_1$ is a cycle,

$$\ell(Q) \geq \frac{1}{4} |H_{k_{k_0}}|^r + \frac{1}{4} \sum_{i \neq k_0} \left( \frac{d-2}{(d-1)^2} |H_{k_i}| \right)^r + \frac{1}{4} \sum_{i<k} \left( \frac{d-2}{(d-1)^2} |H_i| \right)^r + \frac{1}{2}. \quad (4.7)$$

**Proof.** We prove the first statement first.

**Case 1.** $h = k = 1$

If $H_1$ is a cycle, then since $y' \notin \{x, x'\}$, we can find an $(x', y')$-path $P$ in $H_1 - x$ such that

$$\ell(P) \geq 1 \geq \frac{1}{4} |H_{k_{k_0}}|^r + \frac{1}{4} \sum_{i=1}^{k_0-1} \left( \frac{d-2}{(d-1)^2} |H_{k_i}| \right)^r \quad \text{(using } |H_1| \leq 2d - 1\text{).} \quad (4.4)$$

So assume $H_1$ is 3-connected, then apply Lemma 4.4.4] on $H_0 = H_k - ab$, there is a path $P$ from $x'$ to $y'$ such that

$$\ell(P) \geq \frac{1}{4} |H_{k_{k_0}}|^r + \frac{1}{4} \sum_{i=1}^{k_0-1} \left( \frac{d-2}{(d-1)^2} |H_{k_i}| \right)^r.$$

**Case 2.** $h > 1$ and $k < h$
Let \( \{a_k, b_k\} = V(H_k \cap H_{k+1}) \). Suppose \( k = 1 \). If \( H_1 \) is a cycle, then let \( P_1 \) be a path in \( H_1 - x \) from \( x' \) to \( \{a_k, b_k\} \), say to \( a_k \), such that \( \ell(P_1) \geq 1 \) (notice that \( a_kb_k \) may be on \( P_1 \)). If \( a_kb_k \not\in E(P_1) \), then let \( P_2 \) be an \((a_k, y')\)-path in \( H_2H_3 \cdots H_h - b_k \) given by Lemma (4.2.5) such that \( \ell(P_2) \geq \frac{1}{4} \sum_{i \geq 2} \left( \frac{d-2}{(d-1)^2} |H_i| \right)^r + \frac{1}{2} \). Then \( P := P_1 \cup P_2 \) gives the desired path. Hence we assume that \( a_kb_k \in E(P_1) \). If \( H_2 \) is also a cycle, then \( a_kb_k \in E(G) \). We let \( P := P_1 \cup P_2 \) as in the previous case. So assume \( H_2 \) is 3-connected. Let \( P_2 \) be a \((b_k, y')\)-path in \( H_2H_3 \cdots H_h - a_k \) given by Lemma (4.4.2) such that \( \ell(P_2) \geq \frac{1}{4} \sum_{i \geq 2} \left( \frac{d-2}{(d-1)^2} |H_i| \right)^r + 1 \). Then \( (P_1 - \{a_k, b_k\}) \cup P_2 \) gives the desired path. Suppose \( H_1 \) is 3-connected. Since \( \{a, b\} = V(H_k) \cap V(H_{k+1}) \) and \( \{a_k, b_k\} = V(H_k) \cap V(H_{k+1}) \), we have \( \{a, b\} \neq \{a_k, b_k\} \). Assume that \( a \neq a_k \). By applying Lemma (4.4.4) on \( H_0 = H_1 - ab \), there is a path \( P_1 \) from \( a \) to \( a_k \) such that \( \ell(P_2) \geq \frac{1}{4} |H_{kk_0}|^r + \frac{1}{4} \sum_{i=1}^{k_0-1} \left( \frac{d-2}{(d-1)^2} |H_{kl}^i| \right)^r \). If \( a_kb_k \not\in E(P_1) \), let \( P_2 \) be an \((a_k, y')\)-path in \( H_2H_3 \cdots H_h - b_k \) given by Lemma (4.4.2) such that \( \ell(P_2) \geq \frac{1}{4} \sum_{i \geq 2} \left( \frac{d-2}{(d-1)^2} |H_i| \right)^r + \frac{1}{2} \). Then \( P := P_1 \cup P_2 \) gives the desired path. Hence we assume that \( a_kb_k \in E(P_1) \). Let \( P_2 \) be a \((b_k, y')\)-path in \( H_2H_3 \cdots H_h - a_k \) given by Lemma (4.4.2) such that \( \ell(P_2) \geq \frac{1}{4} \sum_{i \geq 2} \left( \frac{d-2}{(d-1)^2} |H_i| \right)^r + \frac{1}{2} \). Then \( (P_1 - \{a_k, b_k\}) \cup P_2 \) gives the desired path.

If \( k \geq 2 \), let \( P_1 \) be an \((x', \{a, b\})\)-path, say \((x', a)\)-path, in \( H_1H_2 \cdots H_{k-1} - x \) given by Lemma (4.2.5). Again assume that \( a \neq a_k \). By applying Lemma (4.4.4) on \( H_0 = H_k - ab \), there is a path \( P_2 \) from \( a \) to \( a_k \) such that \( \ell(P_2) \geq \frac{1}{4} |H_{kk_0}|^r + \frac{1}{4} \sum_{i=1}^{k_0-1} \left( \frac{d-2}{(d-1)^2} |H_{kl}^i| \right)^r \). If \( a_kb_k \not\in E(P_2) \), then by Lemma (4.4.2) we find a path \( P_3 \) in \( H_{k+1}H_{k+2} \cdots H_h - b_k \) from \( a_k \) to \( y' \) such that \( \ell(P_3) \geq \frac{1}{4} \sum_{i \geq k+1} \left( \frac{d-2}{(d-1)^2} |H_{i}| \right)^r + \frac{1}{2} \). Then \( P := P_1 \cup P_2 \cup P_3 \) is the desired path. Hence assume \( a_kb_k \in E(P_2) \). By Lemma (4.4.2) let \( P_3 \) in \( H_{k+1}H_{k+2} \cdots H_h - a_k \) from \( b_k \) to \( y' \) such that \( \ell(P_3) \geq \frac{1}{4} \sum_{i \geq k+1} \left( \frac{d-2}{(d-1)^2} |H_{i}| \right)^r + \frac{1}{2} \). Then \( P := P_1 \cup (P_2 - a_kb_k) \cup P_3 \) is the desired path.

Case 3. \( h = k > 1 \)
Since \( y' \in V(H_{h}) - V(H_{h-1}) \) and \( \{a, b\} = V(H_k) \cap V(H_{k-1}) \), we have \( y' \not\in \{a, b\} \). Applying Lemma (4.4.4) on \( H_0 = H_k - ab \), we obtain an \((a, y')\)-path \( P_2 \) such that \( \ell(P_2) \geq \frac{1}{4} |H_{kk_0}|^r + \frac{1}{2} \)
\[
\frac{1}{4} \sum_{i=1}^{k_0-1} \left( \frac{d-2}{(d-1)^2} |H_{ki}| \right)^r. \]
Then \(P := P_1 \cup P_2\) is the desired path.

For the second statement, we first apply Lemma \((4.4.4)\) to \(H_0 = H_k - ab\) to find an \((a, b)\)-path \(P_k\) of length at least
\[
\frac{1}{4} |H_{kk_0}|^r + \frac{1}{4} \sum_{i \neq k_0} \left( \frac{d-2}{(d-1)^2} |H_{ki}| \right)^r \quad \text{the virtual edge may contained in}
\end{equation}
\(E(H_k \cap H_{k+1})\) will be replaced by a path in \(H_{k+1} H_{k+2} \cdots H_h\). Denote \(\{a_0, b_0\} := \{x, x'\}\). Then for each \(1 \leq i \leq k - 1\) we use Theorem \((4.1.1)\) (b) to find a cycle \(C_i\) in \(H_i\) through \(a_{i-1} b_{i-1}\), and \(a_i b_i\) such that \(\ell(C_i) \geq \frac{1}{4} \left( \frac{(d-2)|H_1|}{d-1} \right)^r + 2\). Let \(P_i\) and \(Q_i\) be the two components of \(C_i - \{a_{i-1} b_{i-1}, a_i b_i\}\). Then \(Q := \bigcup_{i=1}^{k-1} (P_i \cup Q_i) \cup P_k\) gives the desired path.

Particularly, suppose \(H_1\) is a cycle. If \(k = 1\), we can find an \((x, x')\)-path \(Q\) in \(H_1\) such that \(\ell(Q) \geq 2\) (the virtual edge may contained in \(E(H_1 \cap H_2)\) will be replaced by a path in \(H_2 H_2 \cdots H_h\)). Then \(\ell(Q) \geq 2 \geq \frac{1}{4} |H_{kk_0}|^r + \frac{1}{4} \sum_{i \neq k_0} \left( \frac{d-2}{(d-1)^2} |H_{ki}| \right)^r + 1/2\). If \(k > 1\), then we apply Lemma \((4.4.4)\) to \(H_0 = H_k - ab\) to find an \((a, b)\)-path \(P_k\) of length at least
\[
\frac{1}{4} |H_{kk_0}|^r + \frac{1}{4} \sum_{i \neq k_0} \left( \frac{d-2}{(d-1)^2} |H_{ki}| \right)^r \quad \text{the virtual edge may contained in}
\end{equation}
\(E(H_k \cap H_{k+1})\) will be replaced by a path in \(H_{k+1} H_{k+2} \cdots H_h\). Denote \(\{a_0, b_0\} := \{x, x'\}\). Then for each \(1 \leq i \leq k - 1\) we use Theorem \((4.1.1)\) (b) to find a cycle \(C_i\) in \(H_i\) through \(a_{i-1} b_{i-1}\), and \(a_i b_i\) such that \(\ell(C_i) \geq \frac{1}{4} \left( \frac{(d-2)|H_1|}{d-1} \right)^r + 2\). Let \(P_i\) and \(Q_i\) be the two components of \(C_i - \{a_{i-1} b_{i-1}, a_i b_i\}\). As \(H_1\) is a cycle, in particular, \(\ell(P_1) + \ell(Q_1) \geq 1 \geq \frac{1}{4} \left( \frac{d-2}{(d-1)^2} |H_1| \right)^r + 1/2\). Then \(Q := \bigcup_{i=1}^{k-1} (P_i \cup Q_i) \cup P_k\) gives the desired path.

\textbf{Lemma (4.4.6).} Assume that Theorem \((4.1.1)\) holds for graphs with less than \(n\) vertices. Let \(G\) be a \(3\)-connected graph with \(\Delta(G) \leq d\), \(|G| < n\) and \(xy \in E(G)\). Suppose \(\mathcal{H} = H_1 H_2 \cdots H_h\) and \(\mathcal{L} = L_1 L_2 \cdots L_\ell\) are two block-chains in \(G - y\) such that \((a)\) \(x \in (V(H_1) - V(H_2)) \cap (V(L_1) - V(L_2))\); \((b)\) \(xw \in E(H_1) - E(H_2)\) and \(xw' \in E(L_1) - E(L_2)\); and \((c)\) \(\{x\} = V(\mathcal{H}) \cap V(\mathcal{L})\) when \(w \neq w'\), and \(\{x, w\} = V(\mathcal{H}) \cap V(\mathcal{L})\) otherwise. Let \(y' \in V(H_h) - V(H_{h-1}) - \{x, w\}\) and \(y'' \in V(L_\ell) - V(L_{\ell-1}) - \{x, w'\}\). Then, provided that \(d \geq 25\), either there is a path \(P_H\) from \(w\) to \(y'\) in \(\mathcal{H} - x\), and a path \(P_L\) from \(w'\) to \(x\) in \(\mathcal{L}\) or there is a path \(P_H\) from \(w\) to \(x\) in \(\mathcal{H}\), and a path \(P_L\) from \(w'\) to \(y'\) in \(\mathcal{L} - x\) such that \(\ell(P_H) + \ell(P_L) \geq \frac{1}{d} |\mathcal{H}|^r + \frac{1}{d} |\mathcal{L}|^r - 1/2\); moreover, if \(H_1\) is a cycle and \(L_1\) is a cycle, we can
have $\ell(P_H) + \ell(P_L) \geq \frac{1}{4}|\mathcal{H}|^r + \frac{1}{4}|\mathcal{L}|^r$.

Proof. Let $1 \leq k \leq h$ and $1 \leq p \leq \ell$ such that $|H_k| = \max\{|H_i| : 1 \leq i \leq h\}$ and $|L_p| = \max\{|L_i| : 1 \leq i \leq \ell\}$. Let $(a, b) = V(H_k) \cap V(H_{k-1})$, where $(a, b) = \{x, w\}$ when $k = 1$; and $(c, d) = V(L_p) \cap V(L_{p-1})$, where $(c, d) = \{x, w'\}$ when $p = 1$. Let $H_0 := H_{k1}H_{k2} \cdots H_{kk_0} \cdots H_{kk_1}$ be the block-chain $H - ab$ and $L_0 := L_{p1}L_{p2} \cdots L_{pp_0} \cdots P_{pp_1}$ be the block-chain $L_p - cd$, such that (i) $|H_{kk_0}| = \max\{|H_{ki}| : H_{ki} / H_0\}$ and $|L_{pp_0}| = \max\{|L_{pi}| : L_{pi} \in L_0\}$, and (ii) $a \in H_{k1}, b \in H_{kk_1}$ and $c \in L_{p1}, d \in L_{pp_1}$ be distinct. Denote

- $h^+ = \sum_{i > k} \left(\frac{d-1}{d-1}\right)^r |H_i|$, $h^- = \sum_{i < k} \left(\frac{d-1}{d-1}\right)^r |H_i|$;
- $h_0^+ = \sum_{i > k_0} \left(\frac{d-1}{d-1}\right)^r |H_{ki}|$, $h_0^- = \sum_{i < k_0} \left(\frac{d-1}{d-1}\right)^r |H_{ki}|$;
- $l^+ = \sum_{i > p} \left(\frac{d-1}{d-1}\right)^r |L_i|$, $l^- = \sum_{i < p} \left(\frac{d-1}{d-1}\right)^r |L_i|$;
- $l_0^+ = \sum_{i > p_0} \left(\frac{d-1}{d-1}\right)^r |L_{pi}|$, $l_0^- = \sum_{i < p_0} \left(\frac{d-1}{d-1}\right)^r |L_{pi}|$.

By symmetry between $\mathcal{H}$ and $\mathcal{L}$ (both $H_1$ and $L_1$ are cycles for the statement in the “moreover” part), we may assume

$$h^+ + l^- + l_0^+ \geq l^+ + h^- + h_0^+.$$  

Let $P_H$ be a path in $\mathcal{H} - x$ from $w$ to $y'$ given by Lemma (4.4.5) (see (4.5)) such that

$$\ell(P_H) \geq \frac{1}{4}|H_{kk_0}|^r + \frac{1}{4}h^+ + \frac{1}{4}h_0^- - 1/2,$$

and $P_L$ be a path in $\mathcal{L}$ from $w'$ to $x$ given by Lemma (4.4.5) (see (4.6)) such that

$$\ell(P_L) \geq \frac{1}{4}|L_{pp_0}|^r + \frac{1}{4}l^+ + \frac{1}{4}l_0^- + \frac{1}{4}l^-.$$
In particular, if $L_1$ is a cycle, then

$$\ell(P_L) \geq \frac{1}{4} |L_{pp0}|^r + \frac{1}{4} l_0^+ + \frac{1}{4} l_0^- + \frac{1}{4} l_0^- + 1/2.$$  

Since $h^+ + l^- + l_0^+ \geq \ell^+ + h^- + h_0^+$,

$$h^+ + l^- + l_0^+ \geq 1/2(h^+ + l^- + l_0^+) + 1/2(l^+ + h^- + h_0^+).$$

Using $(d-1)^{\log_2 5/4} - 1 \geq \frac{d-1}{d-2}$ when $d \geq 25$ and $x^r + y^r \geq (x + (d-1)^{2((d-1)^{\log_2 5/4} - 1)y})^r$ if $x \geq (d-1)^2y$ (equality (1d)), we have

$$\frac{1}{4} |H_{kk0}|^r + \frac{1}{4} |L_{pp0}|^r + \frac{1}{4} l_0^+ + \frac{1}{4} l_0^- + \frac{1}{4} h^- + \frac{1}{4} h_0^+ \geq \frac{1}{4} |H|^r,$$

and,

$$\frac{1}{4} |L_{pp0}|^r + \frac{1}{4} |L_{pp0}|^r + \frac{1}{4} l_0^+ + \frac{1}{4} l_0^- + \frac{1}{4} l_0^- + \frac{1}{4} l_0^- \geq \frac{1}{4} |L|^r.$$  

Hence,

$$\ell(P_H) + \ell(P_L) \geq \frac{1}{4} |H_{kk0}|^r + \frac{1}{4} h^+ + \frac{1}{4} h^- + \frac{1}{4} l_0^+ + \frac{1}{4} l_0^- + \frac{1}{4} l_0^- + \frac{1}{4} l_0^- - \frac{1}{2}$$

$$= \frac{1}{4} |H_{kk0}|^r + \frac{1}{4} h^- + \frac{1}{4} |L_{pp0}|^r + \frac{1}{4} l_0^- + \frac{1}{4} h^+ + \frac{1}{4} h^- + \frac{1}{4} l_0^- - \frac{1}{2}$$

$$\geq \frac{1}{4} |H_{kk0}|^r + \frac{1}{4} h^- + \frac{1}{4} |L_{pp0}|^r + \frac{1}{4} l_0^- + \frac{1}{4} (h^+ + l^- + l_0^+ + l_0^- + h^- + h_0^+) - \frac{1}{2}$$

$$\geq \frac{1}{4} |H_{kk0}|^r + \frac{1}{4} h^- + \frac{1}{4} |L_{pp0}|^r + \frac{1}{4} l_0^- + \frac{1}{4} (h^+ + l^- + l_0^+ + l_0^- + h^- + h_0^+) - \frac{1}{2}$$

$$+ \frac{1}{4} \left(\frac{1}{d-1}\right)^r l_0^+ + \frac{1}{4} \left(\frac{1}{d-1}\right)^r h_0^+ + \frac{1}{4} \left(\frac{1}{d-1}\right)^r l_0^- + \frac{1}{4} \left(\frac{1}{d-1}\right)^r l_0^- - \frac{1}{2}$$  

(since $\frac{1}{d-1} = \frac{1}{2}$)

$$= \frac{1}{4} |H_{kk0}|^r + \frac{1}{4} \left(\frac{1}{d-1}\right)^r h^+ + \frac{1}{4} h^- + \frac{1}{4} \left(\frac{1}{d-1}\right)^r h^- + \frac{1}{4} \left(\frac{1}{d-1}\right)^r h_0^+ +$$

$$\frac{1}{4} |L_{pp0}|^r + \frac{1}{4} \left(\frac{1}{d-1}\right)^r l^+ + \frac{1}{4} l_0^+ + \frac{1}{4} \left(\frac{1}{d-1}\right)^r l^- + \frac{1}{4} \left(\frac{1}{d-1}\right)^r l_0^- - \frac{1}{2}$$

$$\geq \frac{1}{4} |H|^r + \frac{1}{4} |L|^r - \frac{1}{2};$$
and when both $H_1$ and $L_1$ are cycles, $\ell(P_H) + \ell(P_L) \geq \frac{1}{4} |H|^r + \frac{1}{4} |L|^r$. \hfill \square

4.5 Proofs of Theorem (4.1.1) (a) and (b)

The following two lemmas state that parts (a) and (b) of Theorem (4.1.1) can be reduced to Theorem (4.1.1) for smaller graphs. The proof of (a) is essentially the same as that in [12], but the proof of (b) needs more work.

**Lemma (4.5.1).** Let $n \geq 4$ be an integer. If Theorem (4.1.1) holds for graphs with at most $n - 1$ vertices, then Theorem (4.1.1)(a) holds for graphs with $n$ vertices.

*Proof.* Let $G$ be an arbitrary 3-connected graph with $n$ vertices, let $xy \in E(G)$ and $z \in V(G) - \{x, y\}$, and assume that $\Delta(G - z) \leq d$. Let $t$ denote the number of neighbors of $z$ in $G - \{x, y\}$. Since $G$ is 3-connected, $t \geq 1$.

Let $\mathcal{H} = H_1 \ldots H_h$ be a block-chain in $G - z$ such that $xy \in E(H_1) - E(H_2)$ and, subject to this, $|\mathcal{H}|$ is maximum. Therefore, $H_h$ is an extreme 3-block of $G - z$. Since each extreme 3-block of $G - z$ must contain a neighbor of $z$, there are at most $t - 1$ extreme 3-blocks of $G - z$ different from $H_h$, and hence $V(G - z)$ is covered by at most $t$ block-chains starting from $H_1$ and ending with an extreme 3-block of $G - z$. It then follows that $|\mathcal{H}| \geq (n - 1)/t$.

Note that $\Delta(G - z) \leq d$ implies that $\Delta(H_i) \leq d$ for $1 \leq i \leq h$. By Lemma (4.4.1), there is a path $P$ in $\mathcal{H}$ from $x$ to $y$ such that

$$\ell(P) \geq \frac{1}{4} \left( \frac{d - 2.1}{d - 1} (|\mathcal{H}| + 1) \right)^r + 1 \geq \frac{1}{4} \left( \frac{d - 2.1}{d - 1} \cdot \frac{n}{t} \right)^r + 1.$$ 

Then $C := P + xy$ is a cycle through $xy$ in $G - z$ with $\ell(C) = \ell(P) + 1$, giving the desired cycle. \hfill \square

**Lemma (4.5.2).** Let $n \geq 4$ be an integer. Suppose Theorem (4.1.1) holds for graphs with
at most \( n - 1 \) vertices. Then Theorem (4.1.1)(b) holds for graphs with \( n \) vertices.

Proof. We note that by Lemma (4.5.1), Theorem (4.1.1)(a) holds for graphs with \( n \) vertices. Also note that Theorem (4.1.1)(b) holds trivially for cycles. So it suffices to show that Theorem (4.1.1)(b) holds for 3-connected graphs. Let \( G \) be a 3-connected graph with \( n \) vertices, let \( e = xy, f \) be two distinct edges of \( G \), and assume \( \Delta(G) \leq d \).

Suppose \( e \) and \( f \) share a common vertex. Let \( f = yz \). We note that \( G' := G + xz \) is 3-connected, \( \Delta(G' - y) \leq d \), and that \( y \) has at most \( d - 2 \) neighbors distinct from \( x \) and \( z \). By employing Theorem (4.1.1)(a) to \( G' \), which has \( n \) vertices, there is a cycle \( C' \) through \( xz \) in \( G' - y \) such that

\[
\ell(C') \geq \frac{1}{4} \left( \frac{(d - 2.1)n}{(d - 1)(d - 2)} \right)^r + 2 \geq \frac{1}{4} \left( \frac{(d - 2.1)n}{(d - 1)^2} \right)^r + 2.
\]

Then \( C := (C' - \{xz\}) \cup \{e, f\} \) gives a cycle through \( e \) and \( f \) such that

\[
\ell(C) \geq \frac{1}{4} \left( \frac{(d - 2.1)n}{(d - 1)^2} \right)^r + 3.
\]

Therefore, we may assume that \( e \) and \( f \) are not adjacent. Let \( \mathcal{H} := H_1 \ldots H_h \) be a block-chain in \( G - y \) such that \( x \in V(H_1) - V(H_2) \) and \( f \in E(H_h) - E(H_{h-1}) \). Note that the degree of \( x \) is at most \( d - 1 \) in \( \mathcal{H} \) and \( \Delta(H_i) \leq d \) for all \( 1 \leq i \leq h \). Suppose \( V(\mathcal{H}) = V(G - y) \). If \( \mathcal{H} \) is a cycle, then every vertex of \( \mathcal{H} \) is adjacent to \( y \). Let \( x' \) be a neighbor of \( x \) in \( \mathcal{H} \) and \( P \) be the path in \( \mathcal{H} \) from \( x \) to \( x' \) through \( f \). Then \( P \cup \{yx, yx'\} \) is the desired cycle for Theorem (4.1.1)(b).

Now assume that \( \mathcal{H} \) is not a cycle. If \( H_h \) is a cycle, we choose \( x' \) to be an endvertex of \( f \) which has degree 2 in \( \mathcal{H} \); otherwise, let \( x' \in (V(H_h) - V(H_{h-1})) \cap N_G(y) \) such that \( x' \) is incident with \( f \) whenever possible (for the choice of \( f' \) in the following). Let \( H' \) be the graph obtained from \( \mathcal{H} \) by joining \( x \) to \( x' \), and then suppressing all the remaining degree
2 vertices. It is clear that $H'$ is 3-connected, $|H'| \geq n - 1 - (d - 1)$, and $\Delta(H') \leq d$. Let $f' = f$ if $f \in E(H')$, otherwise let $f'$ denote the new edge incident with $xx'$ in $H'$.

We use Theorem (4.1.1)(b) to find a cycle $C'$ in $H'$ through $xx'$ and $f'$ such that $\ell(C') \geq \frac{1}{4} \left( \frac{d - 2}{(d - 1)^2} |H'| \right)^r + 2$. Then $(C' - \{xx'\}) \cup \{yx, yx'\}$ (adding back the suppressed vertices if necessary) gives a cycle $C$ in $G$ through $xy$ and $f$ such that

$$\ell(C) \geq \frac{1}{4} \left( \frac{d - 2.1}{(d - 1)^2} |H'| \right)^r + 3 \geq \frac{1}{4} \left( \frac{d - 2.1}{(d - 1)^2} n \right)^r + 2 \quad (\text{by Lemma (4.3.2)})$$

So we may assume that $V(H) \neq V(G - y)$. Then there is a 3-block $B$ of $G - y$ such that $|V(B) \cap V(H)| = 2$. Let $\{p, q\} := V(B) \cap V(H)$ and $G_1$ be the graph obtained from $G$ by deleting those components of $G - \{y, p, q\}$ containing a vertex of $H$. We choose $\{p, q\}$ so that $|G_1|$ is maximum. Then,

$$(d - 1)|G_1| + |H| \geq n. \quad (4.8)$$

If $V(G) = V(G_1 \cup H)$, we let $G_2 = \emptyset$. Otherwise, there is a 3-block $B'$ of $G - y$ such that $V(B') \cap V(H \cup G_1) = \{v, w\}$ for some $\{v, w\} \neq \{p, q\}$. (Note that $\{v, w\} \subseteq V(H)$.) Define $G_2$ as the graph obtained from $G$ by deleting those components of $G - \{y, v, w\}$ containing a vertex of $G_1 \cup H$. We choose $G_2$ such that $|G_2|$ is maximum. Then

$$(d - 2)|G_2| + |G_1| + |H| \geq n. \quad (4.9)$$

Clearly, $|G_1| \geq |G_2|$. Let $G_1'$ be the graph obtained from $G_1$ by adding the edges $yp, yq$, and $pq$ if they are not already in $G_1$. Define $G_2'$ similarly from $G_2$. We note that $G_1'$ and $G_2'$ (if nonempty) are both 3-connected. We shall find the desired cycle for Theorem (4.1.1)(b) by combining long paths in the two largest graphs among $H$, $G_1$, and $G_2$. Let $t_i := |N(y) \cap V(G_i) - (\{p, q\} \cup \{v, w\})|$ for $i = 1, 2$, respectively. We divide the remaining proof into two cases.
4.5.1 **Case 1:** \( t_1 \geq 2 \) or \( t_2 \geq 2 \).

In this case, inequalities (4.8) and (4.9) can be improved (by exactly the same reasons) to

\[
(d - 2)|G_1| + |\mathcal{H}| \geq n \quad \text{and} \quad (4.10)
\]

\[
(d - 3)|G_2| + |G_1| + |\mathcal{H}| \geq n.
\] (4.11)

Suppose \( |\mathcal{H}| \geq |G_2| \). Then from (4.11), we have

\[
|G_1| + (d - 2)|\mathcal{H}| \geq n. \tag{4.12}
\]

If \( pq \neq f \), we use Lemma (4.4.3) to find a path \( P \) in \( \mathcal{H} \) from \( x \) to \( z \in \{p, q\} \), say \( z = p \), through \( f \) such that \( pq \notin E(P) \), and

\[
\ell(P) \geq \frac{1}{4} \left( \frac{(d - 2.1)|\mathcal{H}|}{(d - 1)^2} \right)^r.
\]

Since \( e = xy \) is not adjacent to \( f \), \( x \notin \{p, q\} \). Hence if \( pq = f \), we can apply Lemma (4.4.2) to find a path \( P' \) in \( \mathcal{H} - p \) from \( x \) to \( q \) such that

\[
\ell(P') \geq \frac{1}{4} \left( \frac{(d - 2.1)|\mathcal{H}|}{(d - 1)^2} \right)^r + \frac{1}{2},
\]

and set \( P := P' \cup \{pq\} \) in this case. Since \( \Delta(G_1' - q) \leq d \), we may use Theorem (4.1.1)(a)
to find a cycle $C_1$ through $py$ in $G'_1 - q$ such that

$$
\ell(C_1) \geq \frac{1}{4} \left( \frac{d - 2.1}{(d - 1)^2} |G_1| \right)^r + 2.
$$

Then $P \cup (C_1 - \{py\}) \cup \{xy\}$ gives a cycle $C$ in $G$ through $xy$ and $f$ such that

$$
\ell(C) \geq \frac{1}{4} \left( \frac{d - 2.1}{(d - 1)^2} |G_1| \right)^r + \frac{1}{4} \left( \frac{d - 2.1}{(d - 1)^2} |\mathcal{H}| \right)^r + 2
\geq \begin{cases} 
\frac{1}{4} \left( \frac{d - 2.1}{(d - 1)^2} ((d - 2)|G_1| + |\mathcal{H}|) \right)^r + 2, & \text{if } |\mathcal{H}| \geq |G_1|; \\
\frac{1}{4} \left( \frac{d - 2.1}{(d - 1)^2} (|G_1| + (d - 2)|\mathcal{H}|) \right)^r + 2, & \text{if } |\mathcal{H}| < |G_1|;
\end{cases}
\geq \frac{1}{4} \left( \frac{(d - 2.1)n}{(d - 1)^2} \right)^r + 2 \quad (\text{by (1a) and (1.10) and (1.12)}).
$$

Now assume $|\mathcal{H}| < |G_2|$, and hence $G_2 \neq \emptyset$. Let $P_1$ be a path in $\mathcal{H}$ from $x$ to $z \in \{p, q\} \cup \{v, w\}$ through $f$ as given by Lemma 4.2.1 such that (i) exactly one of $pq$ and $vw$ is in $E(P_1)$; (ii) if $pq \in E(P_1)$ then $z \in \{v, w\}$; and (iii) if $vw \in E(P_1)$ then $z \in \{p, q\}$. Assume, without loss of generality, that $pq \in E(P_1)$ and $z = v$. Let $P_2$ be a $(p, q)$-path in $G'_1 - v$ given by Theorem 4.1.1(a), and let $P_3$ be a $(v, y)$-path in $G'_2 - w$ given by Theorem 4.1.1(a). Then $\ell(P_2) \geq \frac{1}{4} \left( \frac{(d - 2.1)}{(d - 1)^2} |G_1| \right)^r + 1$ and $\ell(P_3) \geq \frac{1}{4} \left( \frac{(d - 2.1)}{(d - 1)^2} |G_2| \right)^r + 1$. Now $C := (P_1 - \{pq\}) \cup P_2 \cup P_3 \cup \{xy\}$ is a cycle through $xy$ and $f$ in $G$ such that

$$
\ell(C) \geq \frac{1}{4} \left( \frac{(d - 2.1)}{(d - 1)^2} |G_1| \right)^r + \frac{1}{4} \left( \frac{(d - 2.1)}{(d - 1)^2} |G_2| \right)^r + 2
\geq \frac{1}{4} \left( \frac{d - 2.1}{(d - 1)^2} (|G_1| + (d - 2)|G_2|) \right)^r + 2 \quad (\text{by (1a) and } |G_1| \geq |G_2|)
\geq \frac{1}{4} \left( \frac{d - 2.1}{(d - 1)^2} (|G_1| + (d - 3)|G_2| + |\mathcal{H}|) \right)^r + 2 \quad (\text{since } |G_2| > |\mathcal{H}|)
\geq \frac{1}{4} \left( \frac{(d - 2.1)n}{(d - 1)^2} \right)^r + 2 \quad (\text{by (10)}).
$$
4.5.2 Case 2. \( t_1 = t_2 = 1 \).

Let \( P \) be a path in \( \mathcal{H} \) from \( x \) to \( z \in \{p, q\} \cup \{v, w\} \) through \( f \) as given by Lemma (4.2.1) such that if \( pq \in E(P) \) then \( z \in \{v, w\} \) and if \( vw \in E(P) \) then \( z \in \{p, q\} \).

Suppose \(|\mathcal{H}| \leq (d - 3)|G|\). If \( vw \in E(P) \) and \( z \in \{p, q\} \), say \( z = p \), then let \( P_1 \) be a longest path in \( G'_1 - q \) from \( p \) to \( y \) and \( P_2 \) a longest \((v, w)\)-path in \( G'_2 - y \). By Theorem (4.1.1) (a) for \( P_1 \) and Lemma (4.4.1) for \( P_2 \) (using \( t_2 = 1 \), in this case \( G'_2 - y \) is a block-chain), we have the following lower bounds for \( \ell(P_1) \) and \( \ell(P_2) \).

\[
\ell(P_1) \geq \frac{1}{4} \left( \frac{(d - 2.1)|G_1|}{(d - 1)^2} \right)^r + 2, \quad (4.13) \\
\ell(P_2) \geq \frac{1}{4} \left( \frac{(d - 2.1)|G_2|}{d - 1} \right)^r + 1. \quad (4.14)
\]

Let \( C := (P - \{vw\}) \cup P_1 \cup P_2 \cup \{xy\} \). Then \( C \) is a cycle in \( G \) though \( xy \) and \( f \) such that

\[
\ell(C) \geq \begin{cases} 
\frac{1}{4} \left( \frac{(d - 2.1)}{(d - 1)^2} \right)^r ((d - 2)|G_1| + (d - 1)|G_2|) + 3 \geq \frac{1}{4} \left( \frac{(d - 2.1)n}{(d - 1)^2} \right)^r + 3, & \text{if } (d - 1)|G_2| \geq |G_1| \\
\frac{1}{4} \left( \frac{(d - 2.1)}{(d - 1)^2} \right)^r (|G_1| + (d - 2)(d - 1)|G_2|) + 3 \geq \frac{1}{4} \left( \frac{(d - 2.1)n}{(d - 1)^2} \right)^r + 3, & \text{if } (d - 1)|G_2| < |G_1|;
\end{cases}
\]

If \( pq \in E(P) \), \( z \in \{v, w\} \), say \( z = w \), let \( P_2 \) be a longest path in \( G'_2 - v \) from \( w \) to \( y \), and \( P_1 \) be a longest \((p, q)\)-path in \( G'_1 - y \). Using Theorem (4.1.1) (a) for \( P_2 \) and Lemmas (4.4.1) for \( P_1 \) (using \( t_1 = 1 \), then in this case \( G'_1 - y \) is a block-chain), we have the following lower bounds for \( \ell(P_1) \) and \( \ell(P_2) \).

\[
\ell(P_1) \geq \frac{1}{4} \left( \frac{(d - 2.1)|G_1|}{d - 1} \right)^r + 1 \quad (4.16) \\
\ell(P_2) \geq \frac{1}{4} \left( \frac{(d - 2.1)|G_2|}{(d - 1)^2} \right)^r + 2. \quad (4.17)
\]
Let $C := (P - \{pq\}) \cup P_1 \cup P_2 \cup \{xy\}$. Since $(d - 2)|G_2| \geq |\mathcal{H}|$, by (7), $(d - 1)|G_1| + (d - 2)|G_2| \geq n$. Then as $(d - 1)|G_1| \geq |G_2|$ always holds, by (1a) we have

$$\ell(C) \geq \frac{1}{4} \left(\frac{(d - 2.1)}{(d - 1)^2} ((d - 1)|G_1| + (d - 2)|G_2|) \right)^r + 3 \geq \frac{1}{4} \left(\frac{(d - 2.1)}{(d - 1)^2} n \right)^r + 3.$$

So we may assume $|\mathcal{H}| > (d - 3)|G_2|$. If $pq \neq f$, we use Lemma (4.4.3) to find a path $P$ in $\mathcal{H}$ from $x$ to $z \in \{p, q\}$, say $z = p$, through $f$ such that $pq \notin E(P)$ and

$$\ell(P) \geq \frac{1}{4} \left(\frac{(d - 2.1)|\mathcal{H}|}{(d - 1)^2} \right)^r.$$

Again $x \notin \{p, q\}$. Hence if $pq = f$, we can apply Lemma (4.4.2) to find a path $P'$ in $\mathcal{H} - p$ from $x$ to $q$ such that

$$\ell(P') \geq \frac{1}{4} \left(\frac{(d - 2.1)|\mathcal{H}|}{(d - 1)^2} \right)^r,$$

and set $P := P' \cup \{pq\}$. Since $\Delta(G_1' - q) \leq d$, we may use Theorem (4.1.1)a) to find a cycle $C_1$ through $py$ in $G_1' - q$ such that

$$\ell(C_1) \geq \frac{1}{4} \left(\frac{d - 2.1}{(d - 1)^2} |G_1| \right)^r + 2.$$

Then $P \cup (C_1 - \{py\}) \cup \{xy\}$ gives a cycle $C$ in $G$ through $xy$ and $f$. If $|\mathcal{H}| \geq (d - 4)|G_1|$, then by inequality (1c),

$$|C| \geq \frac{1}{4} \left(\frac{d - 2.1}{(d - 1)^2} (|\mathcal{H}| + (d - 4)((d - 1)^{\log_2(3/2)} - 1)|G_1|) \right)^r + 2 \geq \frac{1}{4} \left(\frac{(d - 2.1)}{(d - 1)^2} (|\mathcal{H}| + (d - 1)|G_1|) \right)^r + 2 \text{ (when } d \geq 8, (d - 1)^{\log_2(3/2)} \geq 3 \text{ and } 2(d - 4) \geq (d - 1)) \geq \frac{1}{4} \left(\frac{(d - 2.1)}{(d - 1)^2} n \right)^r + 2 \text{ (by (4.8))}.$$

So $(d - 3)|G_2| < |\mathcal{H}| < (d - 4)|G_1|$. Then from (4.8) and (4.9), we have

$$|G_1| + (d - 2)|\mathcal{H}| \geq n \text{ and } (d - 2)|G_1| + |\mathcal{H}| \geq |G_1| + (d - 2)|G_2| + |\mathcal{H}| \geq n,$$
where the second inequality follows from \((d-3)|G_1| > (d-2)|G_2|\) as \((d-4)|G_1| > (d-3)|G_2|\).

By using the inequalities,

\[ |G_1|^r + |H|^r \geq \min\{(|G_1| + (d-2)|H|)^r, ((d-2)|G_1| + |H|)^r\}, \]

we obtain that \(\ell(C) \geq \frac{1}{4}((d-2.1)n/(d-1)^2)^r + 2. \)

\[ \square \]

### 4.6 Reduction of Theorem (4.1.1)(c)

In this section, we prove the following result which reduces Theorem (4.1.1)(c) to Theorem (4.1.1) for smaller graphs. The part of proof is long and tedious, but contains a few crucial new ideas in estimating the lower bound of special paths.

**Lemma (4.6.1).** Let \(n \geq 4\) be an integer. If Theorem (4.1.1) holds for graphs with at most \(n-1\) vertices, then Theorem (4.1.1)(c) holds for graphs with \(n\) vertices.

To prove Lemma (4.6.1) let \(G\) be a 3-connected graph with \(n\) vertices and \(\Delta(G) \leq d\), and let \(xy \in E(G)\). It is easy to see that when \(n \geq 5\), \(G\) contains a cycle through \(xy\) of length at least \(5 = \frac{1}{2}((d-1)^{\log_2 6})^r + 2.\) Hence, Theorem (4.1.1)(c) holds when \(n \leq (d-1)^{\log_2 6}\). So we assume \(n > (d-1)^{\log_2 6}\) hereafter.

Let \(\mathcal{H} := H_1H_2\cdots H_h\) be a block-chain in \(G - y\) such that \(x \in V(H_1) - V(H_2)\) and subject to this, \(|\mathcal{H}|\) is maximum. We note that \(\mathcal{H}\) may contain only one block \(H_1\). In this case, all 3-blocks attached to \(H_1\) contain \(x\) and \(H_1\) may not be an extreme block. However, when \(h \geq 2\), \(H_h\) must be an extreme 3-block in \(G - y\) and there is a vertex \(x' \in (V(H_h) - V(H_{h-1})) \cap N_G(y)\).

**Claim (4.6.0.1).** We may assume \(V(G - y) \neq V(\mathcal{H})\).

**Proof.** Suppose \(V(G - y) = V(\mathcal{H})\). Since \(G\) is 3-connected, there exists a vertex \(x' \in \)
\((V(H_h) - V(H_{h-1})) \cap (N_G(y) - x)\). Let \(H'\) be obtained from \(H\) by joining \(x'\) and \(x\) and then suppressing all remaining degree 2 vertices. Clearly, \(H'\) is 3-connected with \(\Delta(H') \leq d\) and \(n > |H'| \geq (n - 1) - (d - 2) = n - d + 1\). Let \(C'\) be a longest cycle in \(H'\) through \(xx'\). By Theorem (4.1.1) (c), we have \(|C'| \geq \frac{1}{4}|H'|^r + 2\). Let \(C = (C' - \{xx'\}) \cup \{xy, x'y\}\). Then by Lemma (4.3.2), we have \(|C| \geq |C'| + 1 \geq \frac{1}{4}(n - d + 1)^r + 3 \geq \frac{1}{4}n^r + 2\), so \(C\) is the desired cycle.

A block-chain \(L := L_1L_2 \ldots L_\ell\) different from \(H\) is called an \(H\)-leg if \(H \cap L \subseteq L_1 - L_2\) and \(L_\ell\) is an extreme block. Note that, for each extreme block \(L\) not in \(H\), there is a unique \(H\)-leg containing \(L\).

Since \(H \neq G - y\) and \(G\) is 3-connected, there are \(H\)-legs. Let \(L := L_1L_2 \ldots L_\ell\) be an \(H\)-leg with \(|L|\) maximum. Suppose further that \(V(H) \cap V(L) = V(H_t) - V(H_{t-1}) \cap V(L_1) = \{p, q\}\) for some \(1 \leq t \leq h\). Since each \(H\)-leg contains an extreme block and each extreme block contains a neighbor of \(y\), there are at most \(d - 1\) \(H\)-legs. Hence, \((d - 1)(|L| - 2) + |H| \geq n - 1\), that is,

\[
(d - 1)|L| + |H| \geq n + 2d - 3. \tag{4.18}
\]

We will use the following parameters (which approach 0 as \(d \to \infty\)):

\[
\epsilon_1 := \frac{d - 1}{(d - 2.1)((d - 1)^{5/2} - 1)}, \quad \epsilon_2 := \frac{1}{(d - 1)^{5/4} - 1}.
\]

Claim (4.6.0.2). We may assume \(|H| \leq (\epsilon_1 + \epsilon_2)n\).

Proof. Suppose \(|H| > (\epsilon_1 + \epsilon_2)n\). Let \(H_m \in \mathcal{H}\) such that \(|H_m|\) is maximum, \(\mathcal{H}' := H_1H_2 \ldots H_{m-1}\) and \(\mathcal{H}'' := H_mH_{m+1}H_{m+2} \ldots H_h\). For each \(2 \leq i \leq h\), let \(\{a_i, b_i\} = V(H_i) \cap V(H_{i-1})\).

If the vertex \(x' \in (V(H_h) - V(H_{h-1})) \cap (N_G(y) - x)\) is well defined, we let \(P\) be a longest \((x, x')\)-path in \(\mathcal{H}\) and \(C := P \cup \{xy, x'y\}\). Clearly, \(C\) is a cycle containing edge \(xy\).
We will show that \( C \) is the desired cycle by estimating lower bounds of \(|C|\) in different cases accordingly.

Suppose \(|H'\) + \(|H''|\) ≥ \( \epsilon_1 n > 0 \). Then, \( h \geq 2 \) and \( H_h \) is an extreme block of \( G - y \), so the vertex \( x' \in (V(H_h) - V(H_{h-1})) \cap (N_G(y) - x) \) is well defined. Applying Lemma (4.4.4) (see (4)), we obtain a lower bound of \(|C|\) below.

\[
|C| \geq 1 \left( |H_m| + ((d - 1)^{\log_2(5/4)} - 1) |H'| \right) + 1 \left( |H''| + ((d - 1)^{\log_2(5/4)} - 1) |H'| \right) + 2 \]  

Then, using \(|H_m| \geq |H_i|\) for each \( 1 \leq i \leq m - 1 \) and (4.1c), we have a lower bound of \(|C|\) below.

\[
|C| \geq 1 \left( |H_m| + ((d - 1)^{\log_2(5/4)} - 1) |H'| \right) + 1 \left( |H''| + ((d - 1)^{\log_2(5/4)} - 1) |H'| \right) + 2 \]  

Thus, we assume \(|H'| + |H''| < \epsilon_1 n\). Then, \(|H_m| > \epsilon_2 n\) as \(|H| > (\epsilon_1 + \epsilon_2)n\).

We distinguish two cases by considering which one is bigger between \(|L|\) and \(|H''|\).

Suppose first that \(|L| \leq |H''|\). Then, using \( |H_m| \geq |H_i| = \frac{(d - 1)^{\log_2(5/4)} - 1}{(d - 1)^2} |H_i| \) for each \( 1 \leq i \leq m - 1 \) and (4.1d), we have a lower bound of \(|C|\) below.

\[
|C| \geq 1 \left( |H_m| + ((d - 1)^{\log_2(5/4)} - 1) |H'| \right) + 1 \left( |H''| + ((d - 1)^{\log_2(5/4)} - 1) |H'| \right) + 2 \]  

Thus we may assume \( (d - 1)^{\log_2(5/4)} \geq \frac{|H_m| + ((d - 1)^{\log_2(5/4)} - 1) |H'|}{(d - 1)^2} \). Using inequality (4.4.4) in
Lemma (4.3.1), by noting that \((d - 1)^{\log_2(5/4)} > 2\) if \(d > 10\), we have

\[
|C| \geq \frac{1}{4} \left( |H_m| + ((d - 1)^{\log_2(5/4)} - 1) |\mathcal{H}'| + ((d - 1)^{\log_2(5/4)} - 1) \frac{(d-1)^2(d-2.1) |\mathcal{H}''|}{(d-1)^2} \right)^r + 2,
\]

\[
\geq \frac{1}{4} \left( |H_m| + |\mathcal{H}'| + |\mathcal{H}''| + (d - 4)((d - 1)^{\log_2(5/4)} - 1) |\mathcal{H}''| \right)^r + 2
\]

\[
\geq \frac{1}{4} (|\mathcal{H}| + (d - 1) |\mathcal{L}|)^r + 2 \quad \text{(since } |\mathcal{H}''| \geq |\mathcal{L}| \text{ and } (d - 4)((d - 1)^{\log_2(5/4)} - 1) \geq d - 1)\]

\[
\geq \frac{1}{4} n^r + 2 \quad \text{(by } (4.18)\text{)}.
\]

We now consider the case \(|\mathcal{L}| > |\mathcal{H}''|\). Since \(|\mathcal{H}|\) is maximum subject to \(x \in V(H_1) - V(H_2)\), we have either \(m \geq 2\) and \(1 \leq t \leq m - 1\) or \(m = t = 1\) and \(x \in \{p, q\}\).

If \(m \geq 2\), let \(P_m\) be a path in \(H_m\) between \(a_{m-1}\) and \(b_{m-1}\) as given by Theorem (4.1.1)(c).

If \(\{a_{m-1}, b_{m-1}\} \neq \{p, q\}\), let \(P'\) be a path in \(\mathcal{H}'\) through \(a_{m-1}b_{m-1}\) from \(x\) to \(\{p, q\}\) as given by Lemma (4.4.3), and let the notation be chosen so that \(P'\) is from \(x\) to \(p\); otherwise, \(\{a_{m-1}, b_{m-1}\} = \{p, q\}\), let \(P''\) be a path in \(\mathcal{H}' - p\) from \(x\) to \(q\) as given by Lemma (4.4.2), and let \(P' := P'' \cup \{pq\}\). Let \(P_L\) be a path in \(\mathcal{L} - q\) from \(p\) to \(y' \in (V(L) - V(L_{l-1})) \cap N_G(y)\) as given by Lemma (4.4.2). Then \(C := P_L \cup P' \cup P_m \cup \{yx, yy'\} - \{a_{m-1}b_{m-1}\}\) is a cycle through \(xy\) in \(G\) such that

\[
|C| \geq \frac{1}{4} |H_m|^r + \frac{1}{4} \left( \frac{(d - 2.1) |\mathcal{H}'|}{(d - 1)^2} \right)^r + \frac{1}{4} \left( \frac{(d - 2.1) |\mathcal{L}|}{(d - 1)^2} \right)^r + 2.
\]

By the same argument as above and using (4.1d) and (4.1f) depending on whether \(\frac{(d - 2.1) |\mathcal{H}'|}{(d - 1)^2} \geq \frac{|H_m|}{(d - 1)^r}\), we have

\[
|C| \geq \frac{1}{4} \left( (d - 1)^{\log_2(5/4)} |H_m| \right)^r + \frac{1}{4} \left( \frac{(d - 2.1) |\mathcal{L}|}{(d - 1)^2} \right)^r + 2 \geq \frac{1}{4} n^r + 2 \quad \text{or}
\]
\[ |C| \geq \frac{1}{4} (|H_m| + (d-1)^2 ((d-1)^{\log_2(5/4)} - 1) \frac{(d-2.1)|H'|}{(d-1)^2})^r + \frac{1}{4} \left( \frac{(d-2.1)|L|}{(d-1)^2} \right)^r + 2. \]

If \( |H_m| + |H'| \geq (d-1)^2 \frac{(d-2.1)|L|}{(d-1)^2}, \) then

\[ \frac{1}{4} \left( |H_m| + |H'| \right)^r + \frac{1}{4} \left( \frac{(d-2.1)|L|}{(d-1)^2} \right)^r + 2 \]
\[ \geq \frac{1}{4} \left( |H_m| + |H'| \right)^r + \frac{1}{4} \left( \frac{(d-2.1)|L|}{(d-1)^2} \right)^r + 2 \]
\[ \geq \frac{1}{4} \left( |H_m| + |H'| \right)^r + |H'''| + (d-1)|L| \geq 2 \quad \text{(since } |L| > |H'''|) \]
\[ \geq \frac{1}{4} n^r + 2. \]

Otherwise, \( |H_m| + |H'| < (d-1)^2 \frac{(d-2.1)|L|}{(d-1)^2}, \) then we get

\[ \frac{1}{4} \left( |H_m| + |H'| \right)^r + \frac{1}{4} \left( \frac{(d-2.1)|L|}{(d-1)^2} \right)^r + 2 \]
\[ \geq \frac{1}{4} \left( (d-1)^{\log_2(5/4)} - 1 \right) \left( |H_m| + |H'| \right)^r + 2 \]
\[ \geq \frac{1}{4} n^r + 2 \quad \text{(since } |H_m| > \epsilon_2 n). \]

We now assume \( m = t = 1 \) and, without loss of generality, \( x = p. \) Let \( P_m \) be a longest path from \( x \) to \( q \) in \( \mathcal{H} \) given by Lemma (4.4.4) and \( P_L \) a longest path in \( \mathcal{L} - p \) from \( q \) to \( y' \in (V(L_t) - V(L_{t-1})) \cap N_G(y) \) given by Lemma (4.4.2). Then \( C := P_m \cup P_L \cup \{xy, y'y\} \) is a cycle of length \( \ell(C) \geq \frac{1}{4} |H_m|^r + \frac{1}{2} \left( \frac{(d-2.1)|L|}{(d-1)^2} \right)^r + 2 \geq \frac{1}{4} n^r + 2, \) where the last inequality follows from a similar argument as above for \( m \geq 2 \) and \( |L| > |H'''|. \)

An \( \mathcal{H} \)-leg \( \mathcal{M} \) is called a minor-leg of \( \mathcal{H} \) if \( V(\mathcal{M} \cap \mathcal{H}) \left\{ \begin{array}{ll} \neq \{p, q\} & \text{if } x \notin \{p, q\} \\ \neq x & \text{if } x \in \{p, q\} \end{array} \right. \); or there is another \( \mathcal{H} \)-leg \( \mathcal{L}' \) such that both \( \mathcal{M} \) and \( \mathcal{L}' \) intersect \( \mathcal{H} \) on \( \{p, q\}, V(\mathcal{L}') \cap (\mathcal{M} \neq \{p, q\}), \) and \( |\mathcal{L}' - \mathcal{M}| \leq \epsilon_2 n/(d-2.1). \) We call the minor-leg \( \mathcal{M} \) defined in the first case an A-type
Let $P$ have $g < h$

Claim (4.6.0.3). We have the following claim.

Let $\{g\} \subseteq H$. Similarly, let $s$ be the smallest integer such that $V(\mathcal{M}) \cap V(H) = V(\mathcal{M}) \cap V(H_s)$. Let $\{u, v\} = V(H_s) \cap V(\mathcal{M})$ and $g := \max\{s, t\}$. Moreover, let $\mathcal{H}' = H_1H_2\cdots H_g$ and $\mathcal{H}'' = H_{g+1}H_{g+2}\cdots H_h$. By the maximality of $|\mathcal{H}|$ and $|\mathcal{L}|$ and the existence of $\mathcal{M}$, we have $g < h$, $|\mathcal{H}''| \geq |\mathcal{M}|$, and $|\mathcal{L}| \geq |\mathcal{M}|$. Recall $V(H_g) \cap V(H_{g+1}) = \{a_g, b_g\}$. Since $x \in V(H_1) - V(H_2)$ and $a_g, b_g \in V(H_g) \cap V(H_{g+1})$, we know $x$ is not incident to $a_gb_g$. Hence, we can apply Lemma (4.4.1) to find a path $P'$ in $\mathcal{H}'$ through $a_gb_g$ from $x$ to $z \in \{p, q\} \cup \{u, v\}$. Let $P''$ be a path in $\mathcal{H}''$ between $a_g$ and $b_g$ as given by Lemma (4.4.1) such that $\ell(P'') \geq \frac{1}{4}(\frac{d-2.1}{d-1}(|\mathcal{H}'| + 1))r + 1$. If $z \in \{u, v\}$ (say, $z = u$), let $P_\mathcal{M}$ be a path in $\mathcal{M} - v$ from $u$ to $y' \in N_G(y) \cap (V(M_m) - V(M_{m-1}) - \{u, v\})$ (the vertex $y'$ exists by the 3-connectivity of $G$) as given by Lemma (4.4.2) with $\ell(P_\mathcal{M}) \geq \frac{1}{4}(\frac{d-2.1}{d-1}(|\mathcal{M}|)^r + \frac{1}{2}$, and $P_\mathcal{L}$ be a path in $\mathcal{L}$ between $p$ and $q$ as given by Lemma (4.4.1) with $\ell(P_\mathcal{L}) \geq \frac{1}{4}(\frac{d-2.1}{d-1}(|\mathcal{L}| + 1)^r + 1$. The case $z \in \{p, q\}$ is treated similarly. Since $|\mathcal{M}| \leq |\mathcal{L}|$ and $|\mathcal{M}| \leq |\mathcal{H}''|$, the paths $P', P'', P_\mathcal{L}, P_\mathcal{M}$, and edges $yx, yy'$ give rise to a cycle $C$ in $G$ through $xy$ such that

$$|C| \geq \frac{1}{4} \left( \frac{d - 2.1}{d - 1} |\mathcal{M}| \right)^r + \frac{1}{4} \left( \frac{d - 2.1}{d - 1} |\mathcal{M}| \right)^r + \frac{1}{4} \left( \frac{d - 2.1}{(d - 1)^2} |\mathcal{M}| \right)^r + 2 \quad (4.19)$$

$$= \frac{1}{4} \left( \frac{d - 2.1}{d - 1} |\mathcal{M}| \right)^r + \frac{1}{4} \left( \frac{d - 2.1}{(d - 1)^2} |\mathcal{M}| \right)^r + 2 \quad (4.20)$$

$$\geq \frac{1}{4} \left( \frac{d - 1}{4} |\mathcal{M}| \right)^r + \frac{1}{4} \left( \frac{d - 2.1}{(d - 1)^2} |\mathcal{M}| \right)^r \quad \text{((1f) in Lemma (4.3.1))}$$

$$\geq \frac{1}{4} r^r + 2.$$

$\square$
Recall that \( t \) is the minimum positive integer such that \( \{p, q\} \subseteq V(H_t) \). Let \( \mathcal{I} := H_1H_2\cdots H_t \) and \( \mathcal{J} := H_{t+1}H_{t+2}\cdots H_h \). We have a similar claim for a B-type \( \mathcal{H} \)-minor-leg.

**Claim (4.6.0.4).** Let \( \mathcal{L}^* \) and \( \mathcal{L}^{**} \) be two \( \mathcal{H} \)-legs with attachments \( \{p, q\} \). If \( \mathcal{L}^* \cap \mathcal{L}^{**} \neq \{p, q\} \), then we may assume that one of \( \mathcal{L}^* \) and \( \mathcal{L}^{**} \) is a B-type \( \mathcal{H} \)-minor-leg.

**Proof.** Assume, without loss of generality, that \( |\mathcal{L}^*| \leq |\mathcal{L}^{**}| \). Let \( \mathcal{L}_0 = \mathcal{L}^* \cap \mathcal{L}^{**} \). Let \( \{u^*, v^*\} = V(\mathcal{L}_0) \cap V(L_1^*) \) and \( \{u^{**}, v^{**}\} = V(\mathcal{L}_0) \cap V(L_1^{**}) \).

By Lemma (4.2.3), we may assume that there is a \((p, u^*)\)-path \( P_0 \) through edge \( u^{**}v^{**} \) in \( \mathcal{L}_0 \), but not the edge \( u^*v^* \). Let \( \omega^* = |\mathcal{L}^* - \mathcal{L}_0| + 2 \). Let \( P_I \) be a path in \( \mathcal{I} - p \) from \( x \) to \( q \) such that \( \ell(P_I) \geq \frac{1}{4}\left((d - 2.1)|\mathcal{I}|\right)^r \) given by Lemma (4.4.2). Let \( P_J \) a \((q, p)\)-path in \( \mathcal{J} \) such that \( \ell(P_J) \geq \frac{1}{4}\left((d - 2.1)|\mathcal{J}|\right)^r \) given by Lemma (4.4.1) (note that \( |\mathcal{J}| \geq |\mathcal{L}| \geq \omega^* \)), \( P^{**} \) be a path in \( \mathcal{L}^{**} - (\mathcal{L}_0 - \{u^{**}, v^{**}\}) \) from \( u^{**} \) to \( v^{**} \) given by Lemma (4.4.1) with \( \ell(P^{**}) \geq \frac{1}{4}\left((d - 2.1)|\mathcal{L}^{**}|\right)^r + 1 \), and \( P^* \) a path in \( G[\mathcal{L}^* - (\mathcal{L}_0 - \{u^*, v^*\})] \) from \( x' \) to \( u^* \) avoiding \( v^* \) given by Lemma (4.4.2) where \( x' \) is a neighbor of \( y \) in the last block of \( \mathcal{L}^* \). Then we obtain a cycle \( C := P^* \cup (P_0 - \{u^{**}v^{**}\}) \cup P^{**} \cup P_J \cup P_I \cup \{xy', xy\} \) through \( xy \) such that

\[
\ell(C) \geq \frac{1}{4}\left(2\left(\frac{d - 2.1}{d - 1}\omega^*\right)^r + \left(\frac{d - 2.1}{d - 1}\omega^r\right)^r\right) + 2 \geq \frac{1}{4}(d - 2.1)(d - 1)^{\log_25/4}\omega^r + 2,
\]

where the last inequality follows from (4.11). Noticing \( \epsilon_2 = \frac{1}{(d - 1)^{\log_2(5/4)} - 1} \), we have \((d - 2.1)(d - 1)^{\log_25/4}\omega^r \geq n \) if \( \omega^* \geq \epsilon_2n/(d - 2.1) \). So, we may assume \( \omega^* < \epsilon_2n/(d - 2.1) \). \( \square \)

Let \( G_0 \) be the subgraph of \( G \) obtained by deleting the components of \( G - \{y, p, q\} \) that contain a vertex in \( \mathcal{H} \). By adding a few special edges to \( G_0 \), we define \( G'_0 \) as follows:

\[
G'_0 := \begin{cases} 
G_0 \cup \{py, qy\} & \text{if } H_t \text{ is a cycle and } \{p, q\} \neq \{a_t, b_t\}; \\
G_0 \cup \{y\} \cup \{py, qy, pq\} & \text{if } pq \notin E(G) \text{ and the above case false.} 
\end{cases}
\]

Note that the difference is whether the edge \( pq \) is forced to be added.
Suppose that there are exactly $\varsigma$ $H$-minor-legs. Then $0 \leq \varsigma \leq d - 3$ (as $y$ is adjacent to at least two vertices in $H$ and at least one vertex in $L$, there are at most $d - 3$ neighbors of $y$ contained in $H$-minor-legs). Let $M$ be one of the largest minor-legs if there is one. Then, the following inequalities hold.

$$|G_0| \geq n - |H| - \varsigma |M| \geq n - |H| - \frac{\varsigma \epsilon_2 n}{d - 2,1} \geq (1 - \epsilon_1 - 2 \epsilon_2)n,$$

$$|H| \geq \frac{n - \varsigma |M|}{d - 1 - \varsigma}, \quad \text{and}$$

$$|J| \geq |L| \geq |M|.$$ 

Since $|H| < (\epsilon_1 + \epsilon_2)n$, we have $|L| \geq \frac{n - |H|}{d - 1} > \frac{(1 - \epsilon_1 - \epsilon_2)n}{d - 1}$. To complete our proof of Lemma (4.6.1), we consider two cases according to whether $x \in \{p, q\}$ or not.

### 4.6.1 Case 1 $x \notin \{p, q\}$.

In this case, by the maximality of $|H|$, we have $|H| \geq |J| \geq |L|$ and $1 \leq t \leq h - 1$. Consequently, we have $h \geq 2$ and the vertex $x' \in (V(H_h) - V(H_{h-1})) \cap (N_G(y) - x)$ is well defined.

**Claim (4.6.1.1).** $\Delta(G_0') \leq d$ and $G_0'$ is 3-connected.

**Proof.** Since $d_{G_0'}(v) = d_G(v)$ for every $v \in V(G_0) - \{y, p, q\}$, we only need to verify that degrees $d_{G_0'}(p)$, $d_{G_0'}(q)$, and $d_{G_0'}(y)$ are not bigger than $d$. Since $|J| \geq |L| > 0$, $H_{t+1}$ exists. Then both $p$ and $q$ have at least two neighbors in $G - V(G_0)$, and thus $d_{G_0}(p) \leq d$ and $d_{G_0}(q) \leq d$. Furthermore, $d_{G_0'}(y) \leq d_{G}(y) + |\{p, q\}| - |\{x, x'\}| \leq d$.

For the connectivity, it is clear that if there exist at least three internally vertex-disjoint $(p, q)$-path, then $G_0'$ is 3-connected. As $G_0$ is connected, there is a $(p, q)$-path using only vertices of $G_0'$; $pyq$ is another $(p, q)$-path which intersects $V(G_0)$ only on $\{p, q\}$. If $pq \in E(G_0')$, the edge $pq$ gives the third $(p, q)$-path. Hence $G_0'$ is 3-connected if $pq \in E(G_0')$. So, we only
need to show that \( G'_0 \) is 3-connected when \( G'_0 = G_0 \cup \{yp, yq\} \) and \( pq \notin E(G'_0) \). We suppose on the contrary that \( G'_0 \) has exactly two internally vertex-disjoint \((p, q)\)-paths (as \( G'_0 + pq \) is 3-connected). As \( y \) connecting \( p \) and \( q \), \( G_0 \) contains exactly one \((p, q)\)-path. Denote by \( P[p, q] \) a shortest \((p, q)\)-path in \( G_0 \). Then in \( G - y \), \((H_t - pq) \cup P[p, q] \) is an induced cycle. According to Tutte’s decomposition algorithm, \((H_t - pq) \cup P[p, q] \) forms a 3-block. This gives a contradiction to that \( H_t \) is a 3-block.

Claim (4.6.1.2). There is a path \( P_0 \) in \( G'_0 \) with two endvertices in \{\( y, p, q \)\} such that \( \ell(P_0) \geq \frac{1}{4}(|G_0| + 1)^r + 1 \) by Theorem (4.1.1)(c). If \( qy \in P'_0 \), then \( P_0 := P'_0 - y \) is the desired \((p, q)\)-path. Since \( G_0 \) contains exactly one \((p, q)\)-path (as \( G'_0 + pq \) is 3-connected), \( G_0 \) contains a \((p, y)\)-path \( P'_0 \) such that \( \ell(P'_0) \geq \frac{1}{4}(|G_0| + 1)^r + 1 \) by Theorem (4.1.1)(c). If \( qy \in P'_0 \), then \( P_0 := P'_0 - y \) is the desired \((p, q)\)-path. Since \( |G_0| \geq (1 - \epsilon_1 - 2\epsilon_2)n \geq (1 - \epsilon_1 - 2\epsilon_2)(d - 1)^{\log_2 6} > (d - 1)^2 \), we have \( \ell(P'_0) \geq 3 \). Hence if \( pq \in P'_0 \), then \( qy \notin E(P'_0) \). So \( P_0 := P'_0 - p \) is the desired path.

When \( H_t \) is a cycle, we use Theorem (4.1.1)(c) to find a \((z, y)\)-path \( P_0 \) in \( G'_0 \) such that \( \ell(P_0) \geq \frac{1}{4}(|G_0| + 1)^r + 1 \). If \( V(P_0) \cap \{p, q\} = \emptyset \), then \( P_0 \) itself is the desired path. So assume \( \{p, q\} - z \subseteq V(P_0) \). If \( pq \notin E(P_0) \), then \( P_0 - y \) is the desired path. Hence, assume that \( pq \in E(P_0) \), and so \( pq \in E(G'_0) \). We may assume \( pq \notin E(G) \); otherwise \( P_0 \) is the desired path. By the definition of \( G'_0 \), we have \( \{p, q\} = \{a_t, b_t\} \) in this case. Let \( P_\mathcal{J} \) be an \((a_t, b_t)\)-path in \( \mathcal{J} \) given by Lemma (4.4.1). Then \( P_0 := (P_0 - \{pq\}) \cup P_\mathcal{J} \) is the desired path with \( \ell(P_0) \geq \frac{1}{4}(|G_0| + 1)^r + \frac{1}{4}(\frac{(d - 2)\mathcal{J}}{d - 1})^r + 1 \).

4.6.1.1 Subcase 1.1. \( \{p, q\} \neq \{a_t, b_t\} = V(H_t \cap H_{t+1}) \). Using the inequalities \( \max \{|I|, |\mathcal{J}| \} \geq |\mathcal{I}| \geq |\mathcal{J}| \geq |\mathcal{L}| \frac{n - |I|}{d - 1} \geq \frac{(1 - \epsilon_1 - \epsilon_2)n}{d - 1} \), we will consider a few cases to show that
there exists a cycle $C$ through $xy$ such that

$$|C| \geq \frac{1}{4} \left( (|G_0| + 1)^r + \frac{(d-2.1) \cdot \max\{|I|, |J|\}^r}{(d-1)^2} \right) + 2$$

$$\geq \frac{1}{4} \left( (1 - \epsilon_1 - 2\epsilon_2)^r + \frac{(1 - \epsilon_1 - \epsilon_2)(d-2.1)}{(d-1)^3} \right)^r n^r + 2$$

$$\geq \frac{1}{4} \left( (1 - \epsilon_1 - 2\epsilon_2) + (1 - \epsilon_1 - 2\epsilon_2)((d-1)^{\log_2(1+2^{-\beta})} - 1) \right)^r n^r + 2$$

$$\geq \frac{1}{4} n^r + 2,$$

where we let $\beta = \log_{d-1} \left( \frac{(d-1)^3(1-\epsilon_1-2\epsilon_2)}{(d-2.1)(1-\epsilon_1-\epsilon_2)} \right)$ for Lemma (4.3.1), which is greater than 1 but less than 2 when $d \geq 42$. We also use the inequalities $1 - \epsilon_1 - 2\epsilon_2 > 0$ when $d \geq 43$, and $(1 - \epsilon_1 - 2\epsilon_2) + (1 - \epsilon_1 - 2\epsilon_2)((d-1)^{\log_2(1+2^{-\beta})} - 1) > 1$ when $d \geq 68$.

We first consider the case that there is a $(p,q)$-path $P_0$ in $G_0 - y$ such that $\ell(P_0) \geq \frac{1}{4} (|G_0| + 1)^r$. Let $P_I$ be an $(\{a_t,b_t\}, x)$, say $(a_t, x)$-path in $I$ through $pq$ given by Lemma (4.4.3) such that $\ell(P_I) \geq \frac{1}{4} \left( \frac{(d-2.1)|I|}{(d-1)^2} \right)^r$ (as $\{p, q\} \neq \{a_t, b_t\}$), and $P_J$ be an $(a_t, x')$ path in $J - b_t$ given by Lemma (4.4.2) such that $\ell(P_J) \geq \frac{1}{4} \left( \frac{(d-2.1)|J|}{(d-1)^2} \right)^r$, where $x' \in (V(H_h) - V(H_{h-1})) \cap N_G(y)$ (as $H_{t+1}$ exists and $G$ is 3-connected, $x' \notin \{a_t, b_t\}$). Then, $C := (P_I - \{pq\}) \cup P_J \cup P_0 \cup \{xy\}$ is the desired path.

Suppose that $H_t$ is 3-connected. By Claim (4.6.1.2) and the discussion above, we may assume that there is a path $P_0$ in $G_0$ from $p$ to $y$ avoiding $q$ such that $\ell(P_0) \geq \frac{1}{4} (|G_0| + 1)^r$. If $|I| \geq |J|$, by Lemma (4.4.2), let $P_H$ be a path in $I - q$ from $x$ to $p$ such that $\ell(P_H) \geq \frac{1}{4} \left( \frac{(d-2.1)|I|}{(d-1)^2} \right)^r + 1$. If $a_tb_t \in E(P_H)$, then we replace $a_tb_t$ by a path in $J$ from $a_t$ to $b_t$. Then, $C := P_H \cup P_0 \cup \{xy\}$ is the desired cycle. If $|I| \leq |J|$, let $P_I$ be a path in $I - q$ from $x$ to $p$ through $a_t b_t$ given by Lemma (4.4.3) and $P_J$ be a path in $J$ from $a_t$ to $b_t$ such that $\ell(P_J) \geq \frac{1}{4} \left( \frac{(d-2.1)|J|}{(d-1)^2} \right)^r + 1$ given by Lemma (4.4.1). Then $C := (P_I - \{a_t b_t\}) \cup P_J \cup P_0 \cup \{xy\}$ is the desired cycle.

Finally, we assume that $H_t$ is a cycle and $P_0$ given by Claim (4.6.1.2) is a $(p,y)$-path. Since $|J| \geq |L|$ in this case, the edge $a_t b_t$ exists. As $\{a_t, b_t\} \neq \{p, q\}$, we can assume,
without loss of generality, that \( a_{t-1} \ldots a_t b_t \ldots b_{t-1} \) lie in this order along \( H_t - a_{t-1} b_{t-1} \).

Let \( \mathcal{I}^* := H_1 H_2 \ldots H_{t-1} \). Applying Lemma \((4.4.2)\) we find a path \( P_H^* \) in \( \mathcal{I}^* - b_{t-1} \) from \( x \) to \( a_{t-1} \) such that \( \ell(P_H^*) \geq \frac{1}{4} \left( \frac{(d-2.1)|I|}{d-1} \right)^r + \frac{1}{2} \). Extending this path along \( H_t \) and \( J \), we obtain a path \( P_H \) in \( \mathcal{H} \) from \( x \) to \( p \) avoiding \( q \) such that \( \ell(P_H) \geq \ell(P_{H}^*) + 2 \). Since the number of degree 2 vertices in \( \mathcal{H} \) is no more than \( 2d - 1 \), we have \( \ell(P_H) \geq \frac{1}{4} \left( \frac{(d-2.1)|J|}{d-1} \right)^r + 1 \). Let \( P_J \) be a path in \( J \) from \( a_t \) to \( b_t \) such that \( \ell(P_J) \geq \frac{1}{4} \left( \frac{(d-2.1)|J|}{d-1} \right)^r + 1 \) given by Lemma \((4.4.1)\). Note that in this case, we can choose \( P_0 \) to be a \((p, y)\)-path such that \( \ell(P_0) \geq \frac{1}{4}(|G_0| + 1)^r \). Then, \( C := (P_H - \{a_t b_t\}) \cup P_J \cup P_0 \cup \{xy\} \) is the desired cycle.

4.6.1.2 Case \( \{p, q\} = \{a_t, b_t\} = V(H_t \cap H_{t+1}) \) In this case, \( G'_0 := G_0 \cup \{py, qy, pq\} \).

Assume, without loss of generality, that \( d_{G_0}(p) \leq d_{G_0}(q) \). Let \( t_p = |N_{G_0}(p) - \{q, y\}| \). Clearly, \( t_p \leq d(G(p)) - 2 \leq d - 2 \). Let \( P_I \) be an \((x, p)\)-path in \( I - q \) given by Lemma \((4.4.2)\) \( P_J \) a \((p, q)\)-path in \( J \) given by Lemma \((4.4.1)\), and \( P_0 \) a \((q, y)\)-path in \( G'_0 - p \) given by Theorem \((4.1.1)\) (a). Let \( C := P_I \cup P_J \cup P_0 \cup \{xy\} \). Then we have

\[
\ell(C) \geq \frac{1}{4} \left( \left( \frac{(d-2.1)|G_0|}{d-1} \right)^r + \left( \frac{(d-2.1)|J|}{d-1} \right)^r + \left( \frac{(d-2.1)|I|}{d-1} \right)^r \right) + 2. \tag{4.21}
\]

Claim \((4.6.1.3)\). \(|J|^r + \left( \frac{|I|}{d-1} \right)^r \geq \left( |J| + |I| \right)^r \geq |H|^r \) provided \( d \geq 61 \).

**Proof.** By Lemma \((4.3.3)\), we only need to show that \( |J| \geq \frac{1}{4} |J| \). Otherwise, using \( |L| \geq \frac{(1-\epsilon_1-\epsilon_2)n}{d-1} \), we have

\[
(\epsilon_1 + \epsilon_2)n \geq |H| \geq |I| \geq \frac{(d-1)|J|}{1.1} \geq \frac{(d-1)|L|}{1.1} \geq \frac{(1-\epsilon_1-\epsilon_2)n}{1.1}.
\]

However, when \( d \geq 61 \), \( \epsilon_1 + \epsilon_2 \leq 0.47 \) and \( \frac{(1-\epsilon_1-\epsilon_2)}{1.1} \geq 0.47 \), showing a contradiction. \( \square \)
Consequently, by Lemma 4.22, we have

\[
\ell(C) \geq \frac{1}{4} \left( \left( \frac{(d-2.1) |G_0|}{(d-1) t_p} \right)^r + \left( \frac{(d-2.1) |\mathcal{H}|}{d-1} \right)^r \right) + 2 \tag{4.22}
\]

\[
\geq \frac{1}{4} \left( \frac{(d-2.1)^2 t_p}{(d-2.1)/t_p} \right)^{r/2} (1 - |\mathcal{H}|/n - \zeta \epsilon_2/(d-2.1)(|\mathcal{H}|/n)^r) n^r + 2.
\]

Clearly, \(C\) is the desired cycle if

\[
\frac{(d-2.1)^2 (1 - |\mathcal{H}|/n - \zeta \epsilon_2/(d-2.1)(|\mathcal{H}|/n)}{t_p} \geq 1. \tag{4.23}
\]

Assuming this is not the case, we will show that there are very few minor-legs of \(\mathcal{H}\), which reveals some properties of \(\mathcal{H}\).

**Claim (4.6.1.4).** We may assume \(\zeta \leq 3\) (provided \(d \geq 195\)).

**Proof.** By plugging \(|\mathcal{H}|/n \geq \frac{1-\epsilon \epsilon_2/(d-2.1)}{d-1-\zeta}\) and \(t_p \leq d-2\) in (4.23), we get

\[
\frac{(d-2.1)^2 (1 - |\mathcal{H}|/n - \zeta \epsilon_2/(d-2.1)(|\mathcal{H}|/n)}{t_p} \geq 1,
\]

provided \(d \geq 195\) and \(\zeta \geq 4\).

We now refine the legs of \(\mathcal{H}\) contained in \(G'_0\). Let \(\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_\ell\) attach \(\mathcal{H}\) at \(\{p, q\}\) such that \(V(\mathcal{L}_i) \cap V(\mathcal{L}_j) = \{p, q\}\) for any \(i \neq j\) and, subject to this constraint, \(\sum_i |V(\mathcal{L}_i)|\) is maximum. We name them major-legs of \(\mathcal{H}\). Clearly, all other \(\mathcal{H}\)-legs remained in \(G'_0\) are \(B\)-type minor-legs.

**Claim (4.6.1.5).** We may assume \(\ell \geq d - 5\) provided that \(d \geq 195\).
Proof. Since $\mathcal{L}_1, \mathcal{L}_2, \ldots, \mathcal{L}_\ell$ are all possible non-minor-legs, $|\mathcal{H}| \geq (1 - \frac{c_2}{d - 2.1})n/(\ell + 1)$. Plugging this inequality in (4.28) and assuming $\ell \leq d - 6$, we get the following

$$\ell(C) \geq \frac{1}{4} \left( \frac{(d - 2.1)^2}{t} - \frac{1 - \frac{c_2}{d - 2.1} - \zeta \epsilon_2/(d - 2.1)}{\ell + 1} \right)^{r/2} n^r + 2$$

$$\geq \frac{1}{4} \left( \frac{\ell(d - 2.1 - \zeta \epsilon_2)^2}{(\ell + 1)^2 t} \right)^{r/2} n^r + 2$$

$$\geq \frac{1}{4} n^r + 2,$$

for each $\zeta = 0, 1, 2, 3$ when $d \geq 195$; where we used $t \leq d - 2$ and $\ell \leq d - 6$.

For each $\mathcal{L}_i$, let $G_i$ be induced by the component of $G - \{p, q, y\}$ containing $\mathcal{L}_i - \{p, q\}$, and including two vertices $p$ and $q$, that is, the union of $\mathcal{L}_i$ and all $B$-type minor-legs sharing a vertex with $\mathcal{L}_i - \{p, q\}$. Let $G'_i = G[V(G_i) \cup \{y\}] \cup \{pq, py, qy\}$. Clearly, each $G'_i$ is 3-connected with $\Delta(G'_i) \leq d$. Let $t_i(p) = d_{G'_i}(p) - 2$, $t_i(q) = d_{G'_i}(q) - 2$, and $t_i = \frac{1}{2}(t_i(p) + t_i(q))$. By counting the neighbors of $p$ and $q$, respectively, we have

$$t_1(p) + t_2(p) + \cdots + t_\ell(p) \leq d_G(p) - |N_H(p) - \{q\}| \leq d - 2,$$

$$t_1(q) + t_2(q) + \cdots + t_\ell(q) \leq d_G(q) - |N_H(q) - \{p\}| \leq d - 2,$$

$$t_1 + t_2 + \cdots + t_\ell = \sum_{1 \leq i \leq \ell} (t_i(p) + t_i(q)) \leq d - 2.$$

We note that, for each $i$, $t_i(p) \geq 1$, $t_i(q) \geq 1$, and $t_i \geq 1$. Assume, without loss of generality, $|G_i|/t_i = \max_{1 \leq i \leq \ell} |G_i|/t_i$.

Claim (4.6.1.6). We may assume $t_1 = 1$ provided that $d \geq 194$.

Proof. Otherwise, we have $t_1 \geq 3/2$ since both $t_1(p)$ and $t_1(q)$ are positive integers. Then either $p$ or $q$ has degree at least 2 in $G_1$; and consequently, $\ell \leq d - 2 - 1 - \zeta$ (each $\mathcal{L}_i$ has a neighbor of $p$ and a neighbor of $q$). By Claim (4.6.1.5) that $\ell \geq d - 5$, we may assume that $\zeta \leq 2$ in this case.
Let $T_1 := \{ i : t_i = t_i(p) = t_i(q) = 1 \}$ and $T_2 := \{ i : t_i > 1 \}$ and let $\ell_1 := |T_1|$ and $\ell_2 := |T_2|$. By Claim (4.6.1.5) we have $\ell_1 + \ell_2 = \ell \geq d - 5$. On the other hand, we have $\ell_1 + 3/2\ell_2 \leq \sum_{1 \leq i \leq \ell} t_i \leq d - 2$, which in turn gives $\ell_2 \leq 12$. So, $\ell_1 \geq d - 17$. As $d \geq 180$, $2\ell_1/3 \geq \ell_2$.

For each $i = 1, 2, \ldots, \ell$, let $\omega_i = |V(G_i) - V(L_i)|$. Clearly, $\sum_{i=1}^{\ell} \omega_i < \sum_{M \text{ is an } H\text{-minor-leg}} |M| \leq \varsigma n / (d-1)$.

For each $i \in T_1$, by the maximality of $|G_1|/t_1$, we have

$$|G_i| = |G_i|/t_i \leq |G_1|/t_1 \leq (2/3)|G_1| \leq (2/3)(|H| + \omega_1) \text{ (since } |L_1| \leq |H| \text{ when } x \notin \{p, q\}).$$

For each $i \in T_2$, we have

$$|G_i| = |L_i| + \omega_i \leq |H| + \omega_i.$$ 

A simple calculation gives the following inequalities.

$$\sum_{1 \leq i \leq \ell} |G_i| \leq (\frac{2\ell_1}{3} + \ell_2)|H| + \frac{2\ell_1}{3}\omega_1 + \sum_{i \in T_2} \omega_i \leq (\frac{2\ell}{3} + \frac{\ell_2}{3})|H| + \frac{2\ell_1}{3}\omega_1 + \max_{i \in T_2} \{\omega_i\} \text{ (since } \ell_2 \leq 2\ell_1/3)$$

$$\leq (\frac{2\ell}{3} + \frac{\ell_2}{3})|H| + \frac{2\ell_1}{3}\omega_1 \leq \frac{2\ell_1}{3}\omega_1 \leq \frac{\varsigma n}{d-2.1} \text{ (since } \sum_{i} \omega_i \leq \frac{\varsigma n}{d-2.1} \text{) }$$

$$\leq \frac{2(d+3)}{3}|H| + \frac{2\varsigma n}{3(d-2.1)} \text{ (since } \ell \leq d - 3, \ell_2 \leq 12 \text{) }$$

Since $|H| + \sum_{1 \leq i \leq \ell} |G_i| \geq n - \frac{\varsigma n}{d-2.1}$, we get the following inequality

$$|H| \geq \frac{3 - \frac{2(d-1.5)\varsigma n}{d-2.1}}{2(d+4.5)}n.$$

When $d \geq 194$, for each $\varsigma = 0, 1, 2$, $\frac{3 - \frac{2(d-1.5)\varsigma n}{2(d+4.5)}}{2(d+4.5)}n > \frac{n - \frac{\varsigma n}{d-5}}{d-5}$. Recall that $\frac{n - \frac{\varsigma n}{d-5}}{d-5}$ is the lower bound on $|H|$ used in the proofs of both Claim (4.6.1.4) and Claim (4.6.1.5), and so we are done by the previous conclusions. \qed
Let $H_k$ be a block of $J$ with maximum number of vertices, that is, $|H_k| = \max\{|H_i| : t + 1 \leq i \leq h\}$. Let $L_1 := L_1 L_2 \ldots L_s$ and let $L_m$ be a block of $L_1$ with maximum number of vertices. Since $t_1 = 1$, $L_1$ is a cycle.

**Claim (4.6.1.7).** Let $z' \in (V(L_s) - V(L_{s-1})) \cap N_G(y)$. We may assume that there is a $(p, z')$-path $P_1$ in $L_1 - q$ such that

$$\ell(P) \geq \frac{1}{4} \left( |L_m|^r + \sum_{i \neq m} \left( \frac{d - 2.1}{(d - 1)^2} |L_i|^r \right) \right) - 1 \geq \frac{1}{4} |L_1|^r - 1.$$ 

**Proof.** Assume $V(L_1 \cap L_2) = \{a, b\}$. In $L_1$, we replace $L_1$ by a triangle $zab$ and apply the particular part of Lemma (4.4.4) to get a $(z, z')$-path. Replacing either the edge $za$ or $zb$ by a path from $p$ to $\{a, b\}$ (we can fix $p$ as $L_1$ is a cycle), and denote the resulted path by $P$. We obtain the desired path; in case that $p \in \{a, b\}$, we may have the lower bound $\ell(P)$ above 1 unit less than the bound given in Lemma (4.4.4).

**Claim (4.6.1.8).** $d_H(p) - 1 \leq 2$, so both $H_t$ and $H_{t+1}$ are cycles.

**Proof.** Otherwise, we have $\ell \leq d - 3$, which in turn shows that

$$|H| \geq \frac{n - \zeta \epsilon n / (d - 2.1)}{\ell + 1}.$$ 

Let $P_I$ be an $(x, q)$-path in $I - p$ given by Lemma (4.4.2), $P_J$ be a $(p, q)$-path in $J$ given by Lemma (4.4.1), and $P_1$ be a $(z', p)$-path given by Claim (4.6.1.7). Let $C := P_I \cup P_J \cup P_1 \cup$
\[\{yz', xy\}. Then \(C\) is a cycle through \(xy\) and
\[
\ell(C) \geq \frac{1}{4} |\mathcal{L}| - 1 + \frac{1}{4} \left(\frac{(d - 2.1)|\mathcal{I}|}{d} \right)^r + \frac{1}{4} \left(\frac{(d - 2.1)|\mathcal{J}|}{d} \right)^r + 1 + 2
\]
\[
\geq \frac{1}{4} \left(\frac{|\mathcal{L}| - (d - 2.1)|\mathcal{H}|}{d - 1} \right)^r + 2 \quad \text{(by Claim (4.6.1.3))}
\]
\[
\geq \frac{1}{4} \left(\frac{n - |\mathcal{H}| - \frac{\mathcal{e}^2 n}{d - 2.1}}{\ell} \right)^r + \left(\frac{(d - 2.1)|\mathcal{H}|}{d - 1} \right)^r + 2
\]
\[
\geq \frac{1}{4} \left(\frac{(d - 1)^2(n - \frac{n - \mathcal{e}^2 n}{d - 1} - \frac{\mathcal{e}^2 n}{d - 2.1})(d - 2.1)(n - \frac{\mathcal{e}^2 n}{d - 2.1})}{\ell(d - 1)(\ell + 1)} \right)^{r/2}
\]
\[
= \frac{1}{4} \left(\frac{(d - 1)(d - 2.1 - \mathcal{e}^2)}{(d - 2.1)(\ell + 1)^2} \right)^{r/2} n^r + 2
\]
\[
\geq \frac{1}{4} \left(\frac{(d - 1)(d - 2.1 - \mathcal{e}^2)}{(d - 2.1)(d - \mathcal{e} - 1)^2} \right)^{r/2} n^r + 2 \quad \text{(by } \ell + \mathcal{e} \leq d - 2\text{)}
\]
\[
\geq \frac{1}{4} n^r + 2,
\]
when \(d \geq 41\) and \(\mathcal{e} \geq 1\). Thus, we assume \(\mathcal{e} = 0\). Then by \(\ell \leq d - 3\), we get
\[
\ell(C) \geq \frac{1}{4} \left(\frac{(d - 1)(d - 2.1 - \mathcal{e}^2)}{(d - 2.1)(\ell + 1)^2} \right)^{r/2} n^r + 2
\]
\[
\geq \frac{1}{4} \left(\frac{(d - 1)(d - 2.1)^2}{(d - 2.1)(\ell + 1)^2} \right)^{r/2} n^r + 2
\]
\[
\geq \frac{1}{4} n^r + 2.
\]

\[\square\]

Let \(P_I\) be an \((x, p)\)-path in \(\mathcal{I} - q\) given by Lemma (4.4.2) such that \(\ell(P_I) \geq \frac{1}{4} \left(\frac{(d - 2.1)|\mathcal{I}|}{d} \right)^r + \frac{1}{4} \frac{|\mathcal{I}|}{d - 1} = \frac{1}{4}\). Applying Lemma (4.4.6) on \(\mathcal{J}\) and \(\mathcal{L}\), with \(\mathcal{J}\) taking the role of \(\mathcal{H}\), \(p\) taking the role of \(x\) and \(q\) taking the role of both \(w\) and \(w'\) in the lemma, respectively. Let \(y' \in (V(H_h) - V(H_{h-1})) \cap N_G(y)\). Then we can find a \((q, y')\)-path \(P_J\) in \(\mathcal{J} - p\) and a \((p, q)\)-path \(P_L\) in \(\mathcal{L}\) such that \(\ell(P_J) + \ell(P_L) \geq \frac{1}{4} |\mathcal{J}|^r + \frac{1}{4} |\mathcal{L}|^r - 1/2\). Then \(C := P_I \cup P_J \cup P_L \cup \{xy, yy'\}\) is a cycle.
through $xy$ such that

$$
\ell(C) \geq \frac{1}{4}\left(\frac{d - 2.1}{d - 1}\right)^2 + \frac{1}{2}d - 2 + \frac{1}{4}|J|^r + \frac{1}{4}|L|^r - \frac{1}{2} + 2
$$

$$
\geq \frac{1}{4}|H|^r + \frac{1}{4}|L|^r + 2 \quad \text{(By Claim (4.6.1.3))}
$$

$$
\geq \frac{1}{4}\left(\frac{d - 1}{d - 1}\right)^2 \cdot \frac{n}{d - 1} \cdot \frac{n - n/(d - 1)}{d - 2} \right)^{r/2} + 2
$$

$$
= \frac{1}{4}n^r + 2,
$$

since in this case, $|L| \geq \frac{n - |H|}{d - 2}$, and as $|H| > |J| \geq |L|$ gives that $|H| \geq \frac{n}{d - 1}$.

### 4.6.2 Case 2 $x \in \{p, q\}$.

Let the notation be chosen so that $x = p$. In this case, the notation $L = L_1 L_2 \cdots L_m$ is used to indicate an arbitrary $H$-leg. We note that $|H| \geq |L|$ may no longer hold because it is possible that $x \in V(L_2)$. An $H$-leg $L$ is proper if $x \in V(L_1) - V(L_2)$. For a proper $H$-leg $L$, $|L| \leq |H|$ still holds.

If $\{x, v\}$ is a 2-cut of $G - y$ for some $v \in V(H_1)$, let $G_v$ be obtained from $G - y$ by deleting all components of $G - \{y, x, v\}$ containing a vertex of $H$ and adding the edge $xv$ when $xv \notin E(G)$. Let $v_0 \in V(H_1)$ such that $\frac{|G_{v_0}|}{d_{G_{v_0}}(x) - 1}$ is maximum. Let $G'_v = G_v \cup \{xy, vy\}$.

**Claim (4.6.2.1).** $\frac{|G_{v_0}|}{d_{G_{v_0}}(x) - 1} > \frac{\epsilon_2 n}{d - 2.1}$ provided $d \geq 93$.

**Proof.** Notice that all $H$-legs not containing $x$ are $A$-type minor-legs, and there are at most three $H$-minor-legs by Claim (4.6.1.4). Hence $|G_{v_0}| + \sum_{v' \neq v_0} |G_{v'}| + |H| + \frac{3\epsilon_2 n}{d - 2.1} \geq n$. Thus we
have
\[
\frac{|G_{v_0}|}{d_{G_{v_0}}(x) - 1} \geq \frac{\sum_v |G_v|}{\sum_v (d_{G_v}(x) - 1)} \geq \frac{n - |H| - \frac{3\epsilon_2 n}{d - 2}}{d - 2} \geq \frac{n - (\epsilon_1 + \epsilon_2)n - \frac{3\epsilon_2 n}{d - 2}}{d - 2.1} > \frac{\epsilon_2 n}{d - 2.1}.
\]

Let \( T \) denote the block-bond tree resulted in from \( G - y + xv_0 \) by the Tutte decomposition. We treat \( T \) as a rooted tree with the root at the bond \( B_0 \) containing the edge \( xv_0 \) (notice that as \( xv_0 \in E(G - y + xv_0) \), and \( G - y + xv_0 - \{x, v_0\} \) has at least two components, the bond \( B_0 \) exists). Except \( B_0 \), we assume that all bonds are removed from \( T \) and two 3-blocks are adjacent if either one is the parent of the other one in the original tree or there is a bond \( B \) between them such that one is the parent of \( B \) and the other one is a child of \( B \). We will follow the partial order \( \prec \) of \( T \) generalized naturally by the parent-child relationship of the tree, that is \( B_1 \prec B_2 \) if \( B_1 \) is a descendant of \( B_2 \).

A block of \( T \) is called an \( x \)-block if it contains \( x \). Let \( X \) be the union of all \( x \)-blocks. A block-chain \( Y_1Y_2\ldots Y_m \) such that \( Y_1 \cap X \) is a virtual edge not incident with \( x \) is called a \( y \)-chain. The following definition will play a key role in our proof.

**Claim (4.6.2.2).** If \( Y_1 \) and \( Y_2 \) are two distinct \( y \)-chains attached to the same \( x \)-block \( B \notin H \) with \( |Y_1| \geq |Y_2| \), then \( |Y_2| < \frac{\epsilon_2 n}{d - 2.1} \).

**Proof.** Suppose to the contrary that \( |Y_2| \geq \frac{\epsilon_2 n}{d - 2.1} \). Suppose there are \( x \)-blocks \( B_1, B_2, \ldots, B_t \) such that \( BB_1B_2\ldots B_t \) is a block-chain and \( B_i \cap H_1 = \{x, v\}, x_iy_i = E(Y_i \cap B) \) for each \( i = 1, 2, \ldots, t \) and \( B \cap B_1 = \{x, u\} \).

By Lemma \( (4.2.1) \) we may assume that there is a path \((u, x_1)\)-path \( P_B \) in \( B \) containing edge \( x_2y_2 \). Let \( P_1 \) be a longest \((x_1, z')\)-path in \( Y_1 - y_1 \) given by Lemma \( (4.4.2) \) where \( z' \) is a
vertex in the extreme block in $Y_1$ which is adjacent to $y$. Let $P_2$ be a longest $(x_2, y_2)$-path in $Y_2$ given by Lemma (4.4.1) and $P_H$ be a longest $(x, v)$-path in $H$ given by Lemma (4.4.1). Since $BB_1B_2 \cdots B_t$ is 2-connected, $BB_1B_2 \cdots B_t - x$ is connected. Let $P_3$ be a $(u, v)$-path in $BB_1B_2 \cdots B_t - x$. Let $C = P_H \cup P_3 \cup (P_B - x_2y_2) \cup P_2 \cup P_1 \cup \{xy, yz\}$. Since $x \notin Y_1$ and $x \notin Y_2$, $|H| \geq |Y_1| \geq |Y_2|$. Then

$$\ell(C) \geq \frac{1}{4} \left( \frac{d - 2.1}{d - 1} |H| \right)^r + \frac{1}{4} \left( \frac{d - 2.1}{d - 1} |Y_2| \right)^r + \frac{1}{4} \left( \frac{d - 2.1}{(d - 1)^2} |Y_1| \right)^r + 2 \geq \frac{1}{4} \cdot 2.5 \left( \frac{d - 2.1}{d - 1} |Y_2| \right)^r + 2 \geq \frac{1}{4} n^r + 2,$$

since $2.5 = ((d - 1)^{\log_2 5})^r$.

We call a $y$-chain $Y$ a small chain if $|Y| < \frac{c_2 n}{d - 2.1}$.

**Definition (4.6.2).** A block $B \in \mathcal{X}$ is called a giant block (GB) if

- $\frac{|B|}{d_B(x) - 1} \geq \frac{c_2 n}{d - 2.1}$, or if there is a $y$-chain $Y$ attached to $B$ such that $|Y| \geq \frac{c_2 n}{d - 2.1}$,
- $\frac{|BY|}{d_{BY}(x) - 1} \geq \frac{2c_2 n}{d - 2.1}$.

If $B$ is not a GB, we call $B$ a small block (SB).

Let $B$ be an $x$-block. If there exist $y$-chains attached to $B$, let $Y$ be one of the $y$-chains with largest cardinality. Then $BY$ is called a $y$-extension of $B$. Notice that $BY$ is a proper $H$-leg, and so $|BY| \leq |H|$.

Following the notation in the above definition, we have the following observation.

**Claim (4.6.2.3).** Let $B$ be an $x$-block and $BY$ a $y$-extension of $B$. Suppose that $xb$ and $xb'$ are the virtual edges of $B$ corresponding to its parent and one of its children, respectively.
Then, there is a $(b, b')$-path $P$ in $BY - x$ of length $\ell(P) \geq \frac{1}{4}(\frac{d-2}{d-1}|Y|)^r + 1$ and there is a $(b, y)$-path $Q$ in $G[V(BY) \cup \{y\}] - x$ of length $\ell(Q) \geq \frac{1}{4}(\frac{d-2}{d-1}|Y|)^r + 1$.

Proof. Let $\{u, v\} = V(B) \cap V(Y)$. If $B$ is a cycle, let $P_1$ be the unique path from $b$ to $b'$ through $uv$. If $B$ is 3-connected, then $B - x$ is 2-connected. There is a path $P_1$ from $b$ to $b'$ through $uv$. By Lemma (4.4.1) there is a $(u, v)$-path $P_2$ in $Y$ such that $\ell(P_2) \geq \frac{1}{4}(\frac{d-2}{d-1}|Y|)^r + 1$. Then $P := (P_1 - \{uv\}) \cup P_2$ gives the desired $(b, b')$-path.

To prove the second statement, if $B$ is a cycle, let $P_1$ be the unique path in $B - x$ from $b$ to $z \in \{u, v\}$, say $u$, avoiding $v$; if $B$ is 3-connected, $B - x - v$ is connected, there is a path $P_1$ from $b$ to $u$. We may assume that $d_Y(v) \geq 3$. For otherwise, let $Y^*$ be the graph obtained from $G[(V(Y) \cup \{y\}) \cup \{yu, yv, uv\}]$ by suppressing all degree 2 vertices. Then, applying Theorem (4.1.1) (a) to $Y^* - v$, we can find a $(u, y)$-path $P_2$ not containing $uv$ such that $\ell(P_2) \geq \frac{1}{4}(\frac{d-2}{d-1}|Y|)^r + 1$. Thus, $P_2 \cup P_1$ is the desired path. Hence, $d_Y(v) \geq 3$. This implies that the first block of $Y$ is 3-connected. Let $Y^*$ be the graph obtained from $G[(V(Y) \cup \{y\}) \cup \{yu, uv\}]$ by suppressing all degree 2 vertices. If $d_Y^*(y) \geq 3$, then $Y^*$ is 3-connected. Applying Theorem (4.1.1) (c) on $Y^*$, we find a $(u, y)$-path $P_2$ of $\ell(P_2) \geq \frac{1}{4}(\frac{d-2}{d-1}|Y|)^r + 2$. (We may assume that $uv \notin E(P_2)$. As otherwise we can choose $P_1$ to be a $(b, v)$-path avoiding $u$ and $P := P_1 \cup (P_2 - \{uv\})$ gives the desired path.) Otherwise, $d_Y^*(y) = 2$; and thus $Y^*$ is a block-chain with edge $uv$ in one end-block and $v$ at the other end. Applying Lemma (4.4.1) on $Y^*$, we find a $(u, y)$-path $P_2$ of $\ell(P_2) \geq \frac{1}{4}(\frac{d-2}{d-1}|Y|)^r + 1$. In any case, $P_2 \cup P_1$ is the desired path.

We need to distinguish three different types of degrees of $x$ in $B$ for each $x$-block: $d_B(x)$ is the degree of $x$ in $B$, $d_{(G, B)}(x)$ is the number of edges of $G$ incident with $x$ in $B$, and $d_{(V, B)}(x)$ is the number of virtual edges in $B$, that is, the degree of $B$, as a vertex in the subtree of $T$ induced by all $x$-blocks. We have $d_B(x) \leq d_{(G, B)}(x) + d_{(V, B)}(x)$ and the strick inequality may hold (for example, an edge may be counted in both $d_{(G, B)}(x)$ and $d_{(V, B)}(x)$).
For each $x$-block, we now associate it with a number $t(B)$:

$$t(B) = d_B(x) - 2.$$  

This number is in correspondence of the parameter $t$ in Theorem \((4.1.1)\) (a), and it is used when applying Theorem \((4.1.1)\) (a) on $B$. We will consider the ration $|B|/t(B)$. For convention, we define $|B|/t(B) = 0$ when $t(B) = 0$. (In this case $B$ is a cycle.)

**Claim (4.6.2.4).** All GBs form an anti-chain (a set of vertices in the block-bond tree forms an anti-chain if, pairwise, they don’t have the parent-child relationship) in $T$.

**Proof.** Suppose there is an $H$-leg $B = B_1 \cdots B_{k-1} B_k \mathcal{L}$, where each $B_j$, $1 \leq j \leq k$, is a 3-block containing $x$, and $\mathcal{L}$ is the largest block-chain attached to $B_k$ (so $B_k \mathcal{L}$ contains an extreme block, and so has a neighbor of $y$). Let \( \{x, b_0\} := V(H_1) \cap V(B_1) \), and $V(B_i) \cap V(B_{i+1}) = \{x, b_i\}$ for $1 \leq i \leq k - 1$. Suppose there are indices $i$ and $m$ with $i < m$ such that both $B_i$ and $B_m$ are GBs, and $m = k$ if $B_m$ is an external GB.

For each $1 \leq j \neq m \leq k - 1$, if $B_j$ is a cycle then let $P_j$ be the path in $B_j - x$ from $b_{j-1}$ to $b_j$; otherwise let $P_j$ be a path in $B_j - x$ from $b_{j-1}$ to $b_j$ as given by Theorem \((4.1.1)\) (a). Let $Y_i$ and $Y_m$ be the largest $y$-chains (if exist) attached to $B_i$ and $B_m$, respectively.

In the case $k \neq m$, let $P_m$ be a longest $(b_{m-1}, b_m)$-path in $B_m - x$ if $\frac{|B_m|}{d_{B_m}(x) - 2} \geq \frac{\epsilon_{m}}{d-2.1}$, and let $P_m$ be a longest $(b_{m-1}, b_m)$-path as guaranteed by Claim \((4.6.2.3)\) if $|Y_m| \geq \frac{\epsilon_{m}}{d-2.1}$; let $P_k$ be a longest path in $G[V(B_k \mathcal{L}) \cup \{y\}] + b_{k-1} y - x$ from $b_{k-1}$ to $y$ as given by Theorem \((4.1.1)\) (a) (as $B_k \mathcal{L}$ has an extreme block which contains a neighbor of $y$, $G[V(B_k \mathcal{L}) \cup \{y\}] + b_{k-1} y$ is 3-connected).

If $k = m$, we pick $P_k$ as in the previous case if $\frac{|B_k|}{t(B_k)} \geq \frac{2\epsilon_{m}}{d-2}$, or $\frac{|B_k|}{d_{B_k}(x) - 2} \geq \frac{\epsilon_{m}}{d-2.1}$; if $|Y_k| \geq \frac{\epsilon_{m}}{d-2.1}$, let $P_k$ be the $(b_{k-1}, y)$-path as guaranteed by Claim \((4.6.2.3)\). Let $P_0$ be a path in $H$ from $x$ to $b_0$ as given by Lemma \((4.4.1)\). So, we have in either case that
\( \ell(P_k) \geq \frac{1}{4} \left( \frac{d-2.1}{d-1} \frac{\epsilon_2 n}{d-2.1} \right)^r. \)

Then \( C := P_0 \cup (\bigcup_{i=1}^{k-1} P_i) \cup P_k \cup \{xy\} \) gives a cycle in \( G \) through \( xy \). Noting \( |H| \geq |B_j| \) and \( |H| \geq |B_k Y| \), thus

\[
\ell(C) \geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |H| \right)^r + \frac{1}{4} \left( \frac{d-2.1}{d-1} \frac{|B_i|}{d_{B_i}(x)} - 2 \right)^r + \frac{1}{4} \left( \frac{d-2.1}{d-1} \frac{\epsilon_2 n}{d-2.1} \right)^r + 2 \\
\geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |B_i| \right)^r + \frac{1}{4} \left( \frac{d-2.1}{d-1} \frac{|B_i|}{d_{B_i}(x)} - 2 \right)^r + \frac{1}{4} \left( \frac{d-2.1}{d-1} \frac{\epsilon_2 n}{d-2.1} \right)^r + 2 \\
= \frac{1}{4} n^r + 2. \quad \text{(Provided \( \frac{|B_i|}{d_{B_i}(x)} - 2 \geq \frac{\epsilon_2 n}{d-2.1} \))}
\]

So, we may assume \( \frac{|B_i|}{d_{B_i}(x)} - 2 < \frac{\epsilon_2 n}{d-2.1} \). Since \( B_i \) is a GB, by the definition, it has a y-chain \( Y_i \) such that \( |Y_i| \geq \frac{\epsilon_2 n}{d-2.1} \). Let \( P_i \) be a path in \( B_i - x \) from \( b_{i-1} \) to \( b_i \) as guaranteed by Claim [4.6.2.3] such that \( \ell(P_i) \geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |Y_i| \right)^r + 1 \). All other paths are as defined in the previous argument, we obtain a cycle \( C \) in \( G \) through \( xy \) such that

\[
\ell(C) \geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |H| \right)^r + \frac{1}{4} \left( \frac{d-2.1}{d-1} |Y_i| \right)^r + \frac{1}{4} \left( \frac{d-2.1}{d-1} \frac{\epsilon_2 n}{d-2.1} \right)^r + 2 \\
\geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |Y_i| \right)^r + \frac{1}{4} \left( \frac{d-2.1}{d-1} \frac{|Y_i|}{d_{B_i}(x)} - 2 \right)^r + \frac{1}{4} \left( \frac{d-2.1}{d-1} \frac{\epsilon_2 n}{d-2.1} \right)^r + 2 \\
= \frac{1}{4} n^r + 2. \quad \text{(Provided \( |Y_i| \geq \frac{\epsilon_2 n}{d-2.1} \))}
\]

\[\square\]

Claim (4.6.2.5). We may assume \( |H| \geq \frac{2\epsilon_2 n}{d-2.1} \) provided that \( d \geq 123 \).

Proof. Suppose that \( |H| < \frac{2\epsilon_2 n}{d-2.1} \). Then for each maximal proper \( H \)-leg \( \mathcal{L} \), we have \( |\mathcal{L}| \leq |H| < \frac{2\epsilon_2 n}{d-2.1} \). As each maximal proper \( H \)-leg either contains an extreme block (and thus has a neighbor of \( y \)), or it is an x-block (and thus has a neighbor of \( x \)), we then have at most \( 2(d-1) \) maximal proper \( H \)-legs. All those \( H \)-legs, together with \( H \), cover all the vertices of \( V(G) - y \). However, \( \frac{4(d-1)\epsilon_2 n}{d-2.1} < n - 0.1 \) when \( d \geq 123 \), showing a contradiction. \[\square\]
Claim (4.6.2.6). For any virtual edge $xy$ with $v \neq v_0$, if $\mathcal{L}$ is an $\mathcal{H}$-leg such that $\mathcal{L} \cap \mathcal{H} = xy$, then $|\mathcal{L}| \leq \frac{\epsilon_2 n}{d-2.1}$ provided $d \geq 123$.

Proof. We consider two cases according to whether $|H_1| > |H_2 H_2 \ldots H_h|$. If $|H_1| \geq |H_2 H_3 \ldots H_h|$, then $H_1 \geq \frac{\epsilon_2 n}{d-2.1}$ by Claim (4.6.2.5). Let $P_L$ be a longest $(x, v)$-path in $\mathcal{L}$, $P_0$ be a longest $(v_0, y)$-path in $G'_{v_0} - x$ given by Theorem (4.1.1)(a), $C_H$ be a longest cycle in $H_1$ through two edges $xy$ and $xv_0$ given by Theorem (4.1.1)(b), and let $P_H = C_H - \{xv_0\}$. From them, we can obtain a cycle $C$ through $xy$ such that $\ell(C) \geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |\mathcal{L}| \right)^r + \frac{1}{4} \left( \frac{d-2.1}{d-1} \frac{|G'_{v_0}|}{d_{G'_{v_0}}(x)-2} \right)^r + \frac{1}{4} \left( \frac{d-2.1}{(d-1)^2} |H_1| \right)^r + 2 \geq \frac{1}{4} n^r + 2$, provided $\min\{|\mathcal{L}|, \frac{|G'_{v_0}|}{d_{G'_{v_0}}(x)-2}, |H_1|\} \geq \frac{\epsilon_2 n}{d-2.1}$. Since the other two already do by Claims (4.6.2.1) and (4.6.2.5), we may assume $|\mathcal{L}|$ does not.

If $|H_1| < |H_2 H_3 \ldots H_h|$, then $|H_2 H_3 \ldots H_h| > \frac{\epsilon_2 n}{d-2.1}$. Define $P_L$ and $P_0$ the same way as above. Let $\mathcal{H}' = H_2 H_3 \ldots H_h$. Let $C_1$ be a cycle in $H_1$ through edges $xy, xv_0$, and $ab$, where $ab = H_1 \cap \mathcal{H}'$, and $P_1 = C_1 - \{xv_0\}$. Let $P_H'$ be a longest $(a, b)$-path in $\mathcal{H}'$ given by Lemma (4.4.1). Then we have $\ell(P_H') \geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |\mathcal{H}'| \right)^r + 1$. Similarly, we can show that the cycle $C := (P_1 - \{ab, xv\}) \cup P_H' \cup P_L \cup \{xy\}$ passes through $xy$, and $\ell(C) \geq \frac{1}{4} n^r + 2$ if $|\mathcal{L}| \geq \frac{\epsilon_2 n}{d-2.1}$.

Notice that there are at most $d - 2$ (as $G_{v_0}$ has an extreme block, and thus has a neighbor of $y$) $\mathcal{H}$-legs $\mathcal{M}$ with $\mathcal{M} \cap \mathcal{H} \neq xy_0$. By Claim (4.6.0.3) and Claim (4.6.2.6), $|\mathcal{M}| \leq \frac{\epsilon_2 n}{d-2.1}$ for each such $\mathcal{H}$-leg, which gives $\sum_{\mathcal{M}} |\mathcal{M}| \leq \frac{d-2}{d-2.1} \epsilon_2 n$. An immediate consequence is that

$$\frac{|G'_{v_0}|}{d_{G'_{v_0}}(x)} \geq \frac{n - |\mathcal{H}| - \frac{d-2}{d-2.1} \epsilon_2 n}{d - 2}.$$  

(4.24)

Claim (4.6.2.7). We may assume that $|\mathcal{H}| < \frac{(1+1.5\epsilon_2) n}{d-1}$ provided $d \geq 125$. 


Proof. Let \( P_0 \) be a longest \((v_0, y)\)-path in \( G'_{v_0} - x \) given by Lemma \([4.1.1]\) (a), and \( P_H \) be a longest \((x, v_0)\)-path in \( H \) given by Lemma \([4.4.1]\). Then \( C := P_0 \cup P_H \cup \{xy\} \) gives a cycle through \( xy \) such that

\[
\ell(C) \geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |H| \right)^r + \frac{1}{4} \left( \frac{d-2.1}{d-1} \frac{|G'_{v_0}|}{d_{G'_{v_0}}(x) - 2} \right)^r + 2
\]

\[
\geq \frac{1}{4} \left( (d-1)^2 \cdot \frac{(d-2.1)^2 |H|}{(d-1)^2(d-2)} \left( 1 - \frac{|H|}{n} - \frac{d-2}{d-2.1} \epsilon_2 n \right) \right)^{r/2} n^r + 2 \geq \frac{1}{4} n^r + 2,
\]

provided \(|H| \geq \frac{(1+1.5\epsilon_2)n}{d-1}\) and \( d \geq 125\).

Since all GBs form an anti-chain of \( T \) and the root \( B_0 \) is not a GB, there is a subtree \( T_0 \) of \( T \) containing the root such that it contains no GB and each branch of \( T - T_0 \) contains at most one GB. (The subtree containing \( B_0 \) obtained from \( T \) by deleting all of the GBs has the described property.) We may assume \( T_0 \) has this property with maximum cardinality. Let \( T_1, T_2, \ldots, T_m \) be the block-trees corresponding to branches of \( T - T_0 \). For each \( T_i \), we call the block of \( T_i \) which is an immediate child of the \( x \)-block to which \( T_i \) attaching in \( T \) the first block of \( T_i \). By the maximality of \( T_0 \) and the fact that all GBs form an anti-chain in \( T \), we have the following observations:

- Excluding \( B_0 \), every \( x \)-block which is a leaf of \( T_0 \) is adjacent to a GB, which is the first block of some \( T_i \) (as if not, we can make \( T_0 \) larger by adding the first block of the \( T_i \) to that leaf). Conversely, each GB is attached to an \( x \)-block which is a leaf of \( T_0 \);

- Each virtual edge in \( B \in T_0 \) which is adjacent to neither the parent of \( B \) nor the child of \( B \) is corresponding to at least one some branch \( T_i \).

For each \( x \)-block \( B \), let \( \overline{B} := B\mathcal{L} \) be a maximal block-chain containing \( B \) as the first block such that \( \overline{B} \) has the largest cardinality among all of such block-chains. Then by the maximality of \( \overline{B} \), it contains an extreme block.
For each $i$, let $B_i$ be the GB contained in $T_i$, we let $L_i := \overline{B_i}$. Then by Claim (4.6.0.3) and Claim (4.6.0.4) we know each leg of $L_i$ is contained in an $\mathcal{H}$-minor-leg, and hence has cardinality less than $\frac{\epsilon n}{d - 2 \epsilon}$. By the construction of $L_i$, it contains an extreme block. Note that each $L_i$ and the branch $T_i$ containing it, have exactly the same predecessors, that is they are connected to $B_0$ through exactly a same block-chain in $T_0$. Also, notice that the first block of each $L_i$ is an $x$-block. Let

$$L = \{L_1, L_2, \ldots, L_m\}.$$ 

Correspondingly, for each $L_i \in L$, we let $M_i \subset T_0$ be the chain of $x$-blocks which connects $L_i$ to $B_0$. Thus, $M_i \mathcal{L}_i$ is a block-chain. Notice that $M_i$ may be empty in case that the first block of $L_i$ is an immediate child of $B_0$.

We use the partial order $\prec$ generalized by $T$ naturally, i.e., if $B_1$ is a child of $B_0$, we have $B_1 \prec B_0$. For each $L_i$, if $M_i \neq \emptyset$, let $\eta(L_i) = \sum_{B \in M_i} \frac{|B|}{t(B)}$, and

$$\omega(L_i) = |L_i| + \eta(L_i).$$

Note that by introducing $\eta(L_i)$, $(d_{(V,B)}(x) - 2) \cdot \frac{|B|}{t(B)}$ vertices in $B$ are distributed into $L_i$ when $d_{(V,B)}(x) \geq 3$ ( $B$ is not a cycle). As each virtual edge incident to $x$ in $B$ which is not incident to the parent or the child of $B$ is contained also in some $T_i$, and there are $(d_{(V,B)}(x) - 2)$ of such virtual edges. Let us see now which portion of vertices of $G$ are not considered into $\sum_i \omega(L_i)$.

(i) On a cycle-block $B \in T_0$, degree 2 vertices which are neighbors of $y$;

(ii) Small $y$-chains and legs of $L_i$ contained in the branch $T_i$;

(iii) For each $B \in T_0$, we have $(d_{(V,B)}(x) - 2) \cdot \frac{|B|}{t(B)}$ vertices in $|B|$ are distributed into $\omega(L_i)$.

So, there are at most $d_{(G,B)}(x) \cdot \frac{|B|}{t(B)}$ vertices in $B$ remained.
We estimate the number of vertices in the above three cases.

• If (i), let $\delta_2$ be the number of such degree 2 vertices.

• If (ii), as each maximal block-chain has an extreme block, and each block-chain can be extended to a maximal one, we suppose there are in total exactly $s$ extreme blocks which are contained in some branch $T_i$, but not in any one of $L_j$. Then,

$$\sum_{i} |T_i| \leq \sum_{i} |L_i| + \frac{s\epsilon_2n}{d-21}.$$

• For each $B \in T_0$, which is not in case (i), $|B| \leq (d_{V,B}(x) - 2) \cdot \frac{|B|}{t(B)} + d_{G,B}(x) \cdot \frac{|B|}{t(B)}$.

As each $B \in T_0$ is a SB, we have $|B| \leq \left((d_{V,B}(x) - 2) \cdot \frac{|B|}{t(B)} + d_{G,B}(x)\right) \frac{\epsilon_2n}{d-21}$.

Let

$$s' = \delta_2 + \sum_{B \in T_0} d_{G,B}(x).$$

For each $L \in L$, let $\tau_x(L) = |N_L(x)| - 1$, $\tau_y(L) = |N_G(y) \cap L|$, and $\tau(L) = \frac{1}{2}(\tau_x(L) + \tau_y(L))$. Note that the definition for $\tau_x(L)$ is different from that for $\tau_y(L)$, as when we remove legs of $L_i$ in $T_i$, it may be possible that in $L_i$, $x$ is only incident to virtual edges. However, each virtual edge incident to $x$ corresponds to at least one real edge incident to $x$ in some legs of $L_i$, so we let $\tau_x(L) = |N_L(x)| - 1$. The following inequalities hold.

$$\sum_{L \in L} \tau_x(L) \leq d(x) - 1 \leq d - 1,$$

$$\sum_{L \in L} \tau_y(L) \leq d(y) - 1 \leq d - 1,$$

and

$$\sum_{L \in L} \tau(L) \leq \frac{1}{2}(d(x) + d(y) - 2) \leq d - 1.$$

We note that for each $L \in L$, we have $\tau_x(L) \geq 1$, $\tau_y(L) \geq 1$, and $\tau(L) \geq 1$. By
relabeling the branches $\mathcal{L} \in \mathcal{L} - \mathcal{H}$, suppose we have

\[
\frac{\omega(\mathcal{L})}{\tau(\mathcal{L})} \geq \frac{\omega(\mathcal{L}')}{\tau(\mathcal{L}')} \geq \cdots \geq \frac{\omega(\mathcal{L}_m)}{\tau(\mathcal{L}_m)}.
\]

As $xy \in E(G)$, and also by noticing that $|N_H(y) - \{x\}| \geq 1$ and $d_H(x) - 1 \geq 1$, when $d \geq 425$,

\[
\frac{\omega(\mathcal{L})}{\tau(\mathcal{L})} \geq \frac{\sum \omega(\mathcal{L}_i)}{\sum \tau(\mathcal{L}_i)} \geq \frac{n - |H| - \frac{(s + s')k2n}{d - 2}}{d - 1 - \frac{1}{d}(\tau_x(H) + \tau_y(H)) - (s + s')/2} \geq \frac{n - |H|}{d - 2},
\]

(4.25)

since under the assumption that $|H| \leq \frac{(1 + 1.5k^2)n}{d - 1}$ and $s + s' \leq 2(d - 3) (xy \in E(G)$, and both $x$ and $y$ have at least one neighbor in each of $\mathcal{L}$ and $\mathcal{H}$), $\frac{n - |H| - \frac{(s + s')k2n}{d - 2}}{d - 2 - (s + s')/2}$ is an increasing function of $s + s'$. The notations $\mathcal{L}$ and $\mathcal{L}'$ will be fixed for the above definition hereafter.

Claim (4.6.2.8). Let $\mathcal{M} := M_1M_2 \cdots M_m$ be the block-chain connecting some $\mathcal{L}'' \in \mathcal{L}$ to $B_0$. Suppose $V(\mathcal{M} \cap \mathcal{L}) = V(M_m \cap \mathcal{L}) = \{x, v_m\}$ and $xv_0 \in E(M_1)$. Then in $\mathcal{M}$, there is a $(v_0, v_m)$-path $P_M$ with $\ell(P_M) \geq \frac{1}{4} \sum \left(\frac{d - 2}{d - 1} \frac{|M_i|}{\tau(M_i)}\right)^r$.

Proof. For each $i = 1, 2, \cdots, m - 1$, let $M_i \cap M_{i+1} = \{x, v_i\}$. Let $P_i$ be an $(m_{i-1}, m_i)$-path in $M_i - x$ given by Theorem (4.1.1) (a) (when $M_i$ is a cycle, the assertion trivially holds) such that $\ell(P_i) \geq \frac{1}{4} \left(\frac{d - 2}{d - 1} \frac{|M_i|}{\tau(M_i)}\right)^r$. If $P_i$ contains some virtual edges, which are supposed to be replaced by a path connecting the two ends of the virtual edge in a $y$-chain of $M_i$ with the ends as attachments (notice that this $y$-chain is a small-chain; and thus is not contained in any other block-chain in $\mathcal{L} - \{\mathcal{L}''\}$ by the construction of $\mathcal{L}''$). Let $P_M := \cup_i P_i$, which is the desired path.

Claim (4.6.2.9). We have $\mathcal{L}$ satisfies $\tau(\mathcal{L}) = 1$. In particular, if let $\mathcal{L} = L_1L_2 \cdots L_i$, then $x \in V(L_1) - V(L_2)$ and $L_1$ is a cycle provided that $d \geq 85$. 

\[\square\]
Proof. In the proof, we let \( \alpha = \tau(\mathcal{L}) \). Suppose on the contrary that \( \alpha > 1 \). Then \( \alpha \geq 1.5 \) from the definition. So, by (4.25) we have

\[
\omega(\mathcal{L}) \geq \frac{1.5(n - |\mathcal{H}|)}{d - 2}.
\]

Let \( \mathcal{M} := M_1M_2\ldots M_m \) be the block chain connecting the root \( B_0 \) of \( T \) and the block \( L_1 \) in \( \mathcal{L} \), and suppose \( L_1 \cap M_m = \{x, v_m\} \). Let \( G' \) be a graph obtained from \( G[V(\mathcal{L}) \cup \{y\}] \cup \{yx, yv_m\} \) by suppressing all degree 2 vertices. Then it is 3-connected, and then by Theorem (4.1.1)(a), there is an \((v_m, y')\)-path \( P_L \) in \( G' - x \) such that \( \ell(P_L) \geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |\mathcal{L}| \right)^r + 2 \), where \( y' \) is a neighbor of \( y \) in the last block of \( \mathcal{L} \). Let \( P_H \) be an \((x, v_0)\)-path in \( \mathcal{H} \) given by Lemma (4.4.1), and \( P_M \) be a \((v_0, v_m)\)-path in \( \mathcal{M} - x \) such that \( \ell(P_M) \geq \frac{1}{4} \sum_i \left( \frac{d-2.1}{d-1} |M_i| \right)^{r_i} \) given by Claim (4.6.2.8).

Set \( C := P_H \cup P_M \cup P_L \cup \{yy', xy\} \), which is a cycle through \( xy \) such that

\[
\ell(C) \geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |\mathcal{H}| \right)^r + \frac{1}{4} \sum_i \left( \frac{d-2.1}{d-1} |M_i| \right)^{r_i} + \frac{1}{4} \left( \frac{d-2.1}{d-1} |\mathcal{L}| \right)^r + 2.
\]

As \( |\mathcal{H}| \geq \frac{n - \frac{(d-2)_e \cdot n}{d-1}}{d-1} \) and \( |\mathcal{L}| \geq \frac{|M_i|}{t(M_i)} \) for each \( M_i \) (as \( \mathcal{L} \) contains a GB and each \( M_i \) is not a GB), by using (4.1c),

\[
\ell(C) \geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |\mathcal{H}| \right)^r + \frac{1}{4} \left( \frac{d-2.1}{d-1} (|\mathcal{L}| + (d-1)\eta(\mathcal{L})) \right)^r + 2
\]

\[
\geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |\mathcal{H}| \right)^r + \frac{1}{4} \left( \frac{d-2.1}{d-1} \omega(\mathcal{L}) \right)^r + 2
\]

\[
\geq \frac{1}{4} \left( \frac{(d-1)^2(d-2.1)^2 |\mathcal{H}| \omega(\mathcal{L})}{(d-1)^2} \right)^{r/2} n^r + 2
\]

\[
= \frac{1}{4} \left( \frac{1.5(d-2.1) - (d-2)\epsilon_2)(d-2.1 + \epsilon_2)}{(d-1)^2} \right)^{r/2} n^r + 2
\]

\[
\geq \frac{1}{4} n^r + 2,
\]
when \( d \geq 85. \)

**Claim (4.6.2.10).** We may assume that \( \omega(L) < \frac{n}{d-2} \) provided \( d \geq 25. \)

**Proof.** Suppose not. Then we have also \( |H| + \eta(L) \geq \frac{n}{d-2}. \) Let \( \mathcal{M} := M_1M_2\ldots M_m \) be the block chain connecting the root \( B_0 \) of \( T \) and the block \( L_1 \) in \( L \), and suppose \( L_1 \cap M_m = \{ x, x' \} \).

Let \( G' \) be a graph obtained from \( G[V(L) \cup \{ y \}] + \{yx, yx'\} \) by suppressing all degree 2 vertices. Then it is 3-connected, and by Theorem (4.1.1) (a), there is an \((\ell_1, \ell_2)\) such that \( \ell(P_L) \geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |L| \right)^r + 2, \) as \( |N_G(x) \cap L| = 2, \) where \( y' \) is a neighbor of \( y \) in the last block of \( L \). Let \( P_H \) be an \((x, v_0)\)-path in \( H \) given by Lemma (4.4.1), and \( P_M \) be a \((v_0, x')\)-path in \( M - x \) such that \( \ell(P_M) \geq \frac{1}{4} \sum_i \left( \frac{d-2.1}{d-1} |M_i| \right)^r. \)

Set \( C := P_H \cup P_M \cup P_L \cup \{yy', xy\} \), which is a cycle through \( xy \) such that

\[
\ell(C) \geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |H| \right)^r + \frac{1}{4} \sum_i \left( \frac{d-2.1}{d-1} |M_i| \right)^r + \frac{1}{4} \left( \frac{d-2.1}{d-1} |L| \right)^r + 2.
\]

As \( \frac{1}{4} \sum_i \left( \frac{d-2.1}{d-1} |M_i| \right)^r = \frac{1}{4} \sum_i \left( \frac{d-2.1}{(d-1)^2 t(M_i)} \right)^r + \frac{1}{4} \sum_i \left( \frac{d-2.1}{(d-1)^2 t(M_i)} \right)^r \), and \( |H| \geq \frac{|M_i|}{t(M_i)} \) and \( |L| \geq \frac{|M_i|}{t(M_i)} \) for each \( M_i \), by using (4.1C), and the fact that \( \frac{(d-2.1)(d-1)^{\log_2(3/2)}}{(d-1)} \geq 1 \) when \( d \geq 25, \) we have

\[
\ell(C) \geq \frac{1}{4} \left( \frac{d-2.1}{d-1} (|H| + \eta(L)) \right)^r + \frac{1}{4} \left( \frac{d-2.1}{d-1} (|L| + \eta(L)) \right)^r + 2 \\
\geq \frac{1}{4} \left( \frac{(d-1)^2(d-2.1)^2}{(d-1)^2} \right)^{r/2} n^r + 2 \\
= \frac{1}{4} n^r + 2.
\]

We now show that there is a cycle \( C \) through \( xy \) in \( G \) such that \( \ell(C) \geq \frac{1}{4} n^r + 2. \)
Case 1. $H_1 \in \mathcal{H}$ is a cycle.

Recall that $|\mathcal{L}| + \eta(\mathcal{L}) \geq \frac{n - |H|}{d - 2}$. This gives that $|\mathcal{H}| + \eta(\mathcal{L}) \geq \frac{n}{d - 1}$ by $|\mathcal{H}| \geq |\mathcal{L}|$ (as $\mathcal{L}$ is a proper $\mathcal{H}$-leg).

Let $w := v_0$, and let $\mathcal{M} := M_1M_2 \ldots M_m$ be the block-chain connecting the root $B_0$ of $T$ and the block $L_1$ in $\mathcal{L}$. Suppose that $V(\mathcal{M} \cap \mathcal{L}) = V(M_m \cap L_1) = \{x, w'\}$. Let $y'$ be a neighbor of $y$ in $\mathcal{H}$ different from $x$ ($y'$ exists by the 3-connectivity of $G$). Let $P_M$ be a path in $\mathcal{M} - x$ from $w$ to $w'$ such that $\ell(P_M) \geq \frac{1}{4} \sum_i \left(\frac{d - 2.1}{d - 1}\right)^r |M_i|^r$ given by Claim 4.6.2.8. As both $H_1$ and $L_1$ are cycles, apply the particular part of Lemma 4.4.6 on $H$ and $L$, without loss of generality, assume that we find a $(w, y')$-path $P_{H}$ in $\mathcal{H} - x$, and an $(x, w')$-path $P_{L}$ in $\mathcal{L}$ such that $\ell(P_{H}) + \ell(P_{L}) \geq \frac{1}{4}(|\mathcal{H}|^r + |\mathcal{L}|^r)$. Let $C := P_{H} \cup P_{M} \cup P_{L} \cup \{yy', xy\}$. Then $C$ is a cycle through $xy$ such that $\ell(C) \geq \frac{1}{4}|\mathcal{H}|^r + \frac{1}{4}|\mathcal{L}|^r + \ell(P_M) + 2$. By splitting the value $\ell(P_M)$, we have

\[
\ell(C) \geq \frac{1}{4}|\mathcal{H}|^r + \frac{1}{4}|\mathcal{L}|^r + \ell(P_M) + 2 \\
\geq \frac{1}{4}(|\mathcal{H}| + \eta(\mathcal{L}))^r + \frac{1}{4}(|\mathcal{L}| + \eta(\mathcal{L}))^r + 2 \\
\geq \frac{1}{4} \left( (d - 1)^2 \cdot \frac{n}{(d - 1)} \cdot \frac{n - n/(d - 1)}{d - 2} \right)^{r/2} + 2 \\
= \frac{1}{4}n^r + 2.
\]

Case 2. $H_1 \in \mathcal{H}$ is 3-connected.

In this case, we have $\tau_x(\mathcal{H}) \geq 2$. Hence by 4.25, we have

\[
\frac{\omega(\mathcal{L})}{\tau(\mathcal{L})} \geq \frac{n - |\mathcal{H}|}{d - 2.5}.
\] (4.26)
As \( \tau(\mathcal{L}) \geq 1 \), a similar argument as in (4.25) gives that

\[
\frac{\omega(\mathcal{L}')}{\tau(\mathcal{L}')} \geq \frac{n - |\mathcal{H}| - \omega(\mathcal{L})}{d - 3.5}.
\] (4.27)

If \( \eta(\mathcal{H}) \geq \eta(\mathcal{L}') \), then we construct a cycle \( C_1 \) through \( xy \) using \( \mathcal{H} \) and \( \mathcal{L} \) and a cycle \( C_2 \) through \( xy \) using \( \mathcal{L} \) and \( \mathcal{L}' \), and show that \( \ell(C_1) + \ell(C_2) \geq 2(\frac{1}{4}n^r + 2) \). If \( \eta(\mathcal{H}) < \eta(\mathcal{L}') \), then we construct a cycle \( C_1 \) through \( xy \) using \( \mathcal{H} \) and \( \mathcal{L}' \) and a cycle \( C_2 \) through \( xy \) using \( \mathcal{L} \) and \( \mathcal{L}' \), and show that \( \ell(C_1) + \ell(C_2) \geq 2(\frac{1}{4}n^r + 2) \). Assume, without loss of generality, that \( \eta(\mathcal{H}) \geq \eta(\mathcal{L}') \). Suppose \( \mathcal{L} = L_1L_2\cdots L_l \) and \( \mathcal{L}' = L'_1L'_2\cdots L'_{l'} \). Let \( \mathcal{M} := M_1M_2\cdots M_m \) be the block-chain connecting the root \( B_0 \) of \( T \) and the first block \( L_1 \) in \( \mathcal{L} \), and let \( \mathcal{M}' := M'_1M'_2\cdots M'_{m'} \) be the block-chain connecting the root \( B_0 \) of \( T \) and the first block \( L'_1 \) in \( \mathcal{L}' \). Furthermore, we suppose

- \( M_m \cap L_1 = \{x, b\} \) and \( M'_m \cap L'_1 = \{x', b'\} \);
- \( L_k = \max\{L_i : L_i \in \mathcal{L}_1\} \) and \( L'_p = \max\{L'_i : L'_i \in \mathcal{L}_2\} \);
- \( L_k \cap L_{k+1} = \{a, b\}, L_k \cap L_{k+1} = \{a_k, b_k\}, L'_p \cap L'_{p+1} = \{c, d\}, \) and \( L'_p \cap L'_{p+1} = \{c_k, d_k\} \);
- \( L_0 := L_{k1}L_{k2}\cdots L_{kk_0}\cdots L_{kk_1} \) is the block-chain \( L_k - ab \), and
- \( L'_0 := L'_{p1}L'_{p2}\cdots L'_{pp_0}\cdots L'_{pp_1} \) is the block-chain \( L'_p - cd \) such that
  - (i) \( L_{kk_0} = \max\{L_{ki} : L_{ki} \in L_0\} \) and \( L'_{pp_0} = \max\{L'_{pi} : L'_{pi} \in L'_0\} \),
  - (ii) \( a \in L_{k1}, b \in L_{kk_1}, c \in L'_{p1}, \) and \( d \in L'_{pp_1} \), and
  - (iii) given by Lemma \( (4.2.5) \) \( P_{L1} \) is a path in \( L_1L_2\cdots L_{k-1} - x \) from \( b \) to \( a \), and \( P_{L'1} \) is a path in \( L'_1L'_2\cdots L'_{k-1} - x \) from \( b' \) to \( c' \).

We include the trivial case that \( L_k \) or \( L'_p \) is a cycle in the above notations. Denote

- \( l^+ = \sum_{i > k} \left( \frac{d-2}{(d-1)^r} |L_i| \right)^r \), \( l^- = \sum_{i < k} \left( \frac{d-2}{(d-1)^r} |L_i| \right)^r \).
\[ l_0^+ = \sum_{i > k_0} \left( \frac{d-2.1}{d-1} |L_{ki}| \right)^r, \quad l_0^- = \sum_{i < k_0} \left( \frac{d-2.1}{d-1} |L_{ki}| \right)^r; \]
\[ w_0^+ = \sum_{i > p_0} \left( \frac{d-2.1}{d-1} |L_i'| \right)^r, \quad w_0^- = \sum_{i < p_0} \left( \frac{d-2.1}{d-1} |L_i'| \right)^r; \]
\[ w_0'^+ = \sum_{i > p_0} \left( \frac{d-2.1}{d-1} |L'_{pi}| \right)^r, \quad w_0'^- = \sum_{i < p_0} \left( \frac{d-2.1}{d-1} |L'_{pi}| \right)^r. \]

Let \( y' \) be a neighbor of \( y \) in the last block of \( L \). We now construct a cycle \( C_1 \) through \( xy \) by using paths in \( H \) and \( L \) as follows:

- Let \( P_H \) be a path in \( H \) from \( x \) to \( v_0 \) given by Lemma \((4.4.1)\) such that \( \ell(P_H) \geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |H| \right)^r + 1, \)
- \( P_M \) be a path from \( v_0 \) to \( b \) in \( M - x \) such that \( \ell(P_M) \geq \frac{1}{4} \sum_i \left( \frac{d-2.1}{d-1} \frac{|M_i|}{r(M_i)} \right)^r \) given by Claim \((4.6.2.8)\), and
- \( P_L \) be a path in \( L - x \) from \( b \) to \( y' \) given by Lemma \((4.4.5)\) such that \( \ell(P_L) \geq \frac{1}{4} |L_{kk_0}|^r + \frac{1}{4} \ell_0^- + \frac{1}{4} \ell^+ - \frac{1}{2}. \)

Then \( C_1 := P_H \cup P_M \cup P_L \cup \{yy', xy\} \) is a cycle through \( xy \). Now we construct a cycle \( C_2 \) in \( L \) and \( L' \). Assume, without loss of generality, that the following inequality holds.

\[ l^+ + w^- + w_0'^+ \geq w^+ + l^- + l_0'^+. \]

Let \( P_L \) be a path in \( L - x \) from \( b \) to \( y' \) given by Lemma \((4.4.5)\) such that

\[ \ell(P_L) \geq \frac{1}{4} |L_{kk_0}|^r + \frac{1}{4} l^+ + \frac{1}{4} l_0^- - \frac{1}{2}, \]

and \( P_{L'} \) be a path in \( L' - x \) from \( b' \) to \( x \) given by \((4.6)\) of Lemma \((4.4.5)\) such that

\[ \ell(P_{L'}) \geq \frac{1}{4} |L'_{pp_0}|^r + \frac{1}{4} w_0'^+ + \frac{1}{4} w_0^- + \frac{1}{4} w^- . \]
Let $P_M$ be a path in $\mathcal{M} - x$ from $b$ to $v_0$ such that $\ell(P_M) \geq \frac{1}{4} \sum_i \left( \frac{d-2.1 \cdot |M_i|}{t(M_i)} \right)^r$ given by Claim (4.6.2.8) and let $P_{M'}$ be a path in $\mathcal{M}' - x$ from $b'$ to $v_0$ such that $\ell(P_{M'}) \geq \frac{1}{4} \sum_i \left( \frac{d-2.1 \cdot |M_i|}{t(M_i')} \right)^r$ given by Claim (4.6.2.8). Then $C_2 := P_L \cup P_M \cup P_{M'} \cup P_L' \cup \{xy, yy'\}$ contains a cycle through $xy$ of length at least $\ell(P_L) + \ell(P_M) + 2 - \frac{1}{2}$ (notice that $P_M$ and $P_{M'}$ may intersect). Then,

$$
\ell(P_H) + \frac{1}{4} \ell(P_M) = \frac{1}{4} \left( \frac{d-2.1}{d-1} |\mathcal{H}| \right) + \frac{1}{4} \sum_i \left( \frac{d-2.1 \cdot |M_i|}{(d-1)t(M_i)} \right)^r + 1 \\
\geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |\mathcal{H}| \right) + \frac{1}{4} \sum_i \left( \frac{d-2.1}{(d-1)^3} \cdot \frac{|M_i|}{t(M_i)} \right)^r + 1 \\
\geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |\mathcal{H}| + \eta(\mathcal{L}_1) \right)^r + 1,
$$

as when $d \geq 12$, $\frac{(d-2.1)(d-1)^{\log_2(8/9)} - 1}{d-1} > 1$ and

$$
\ell(P_L) + \frac{1}{4} \cdot (L_0^+ + L^- + \ell(P_M)) \\
\geq \frac{1}{4} |L_{pp0}|^r + \frac{1}{4} \sum_j \left( \frac{d-2.1}{(d-1)^r} \cdot |L_{pj}| \right)^r + \frac{1}{4} \sum_{i \neq p} \left( \frac{d-2.1}{(d-1)^r} \cdot |L_i| \right)^r + \frac{1}{4} \sum_i \left( \frac{d-2.1}{(d-1)^r} \cdot \frac{|M_i|}{t(M_i)} \right)^r - \frac{1}{2} \\
\geq \frac{1}{4} |L_{pp0}| + \sum_j \left( \frac{d-2.1}{(d-1)^{\log_2(9/8)} - 1} \cdot |L_{pj}| \right)^r + \sum_{i \neq p} \left( \frac{d-2.1}{(d-1)^{\log_2(9/8)} - 1} \cdot |L_i| \right)^r \\
+ \sum_i \left( \frac{d-2.1}{(d-1)^{\log_2(8/9)} - 1} \cdot \frac{|M_i|}{t(M_i)} \right)^r - \frac{1}{2} \\
\geq \frac{1}{4} \omega(\mathcal{L})^r - \frac{1}{2},
$$

as $(d-2.1)(d-1)^{\log_2(9/8)} - 1)/(d-1) > 1$ when $d \geq 64$. Therefore,

$$
\ell(C_1) + \frac{1}{4} \cdot (L_0^+ + L^-) - \frac{1}{2} \ell(P_M) \geq \frac{1}{4} \left( \frac{d-2.1}{d-1} |\mathcal{H}| + \eta(\mathcal{L}) \right)^r + \frac{1}{4} \omega(\mathcal{L})^r + 2 + \frac{1}{2} \\
\geq \frac{1}{4} \left( \frac{(d-1)(d-2-1.5\varepsilon_2)}{(d-1.5)(d-2.5)} \right)^{r/2} n^r + 2 + \frac{1}{2} \\
\geq \frac{1}{4} n^r + 2 + \frac{1}{2},
$$

where the last inequality is obtained by using inequality (4.2), $\omega(\mathcal{L}) \geq \frac{n - |\mathcal{H}|}{d-2.5}$ from (4.26), $|\mathcal{H}| + \eta(\mathcal{L}) \geq \frac{n}{d-1.5}$ following from $|\mathcal{H}| + \eta(\mathcal{L}) \geq \omega(\mathcal{L})$, and $|\mathcal{H}| \leq \frac{(1+1.5\varepsilon_2)n}{d-1}$ from Claim.
Similarly,

$$
\ell(P_L) + \frac{1}{4} \ell(P_M) + \ell(P_{L'}) + \frac{1}{4} \ell(P_M) - \frac{1}{4} (l_0^+ + l^-)
\geq \frac{1}{4} (|L| + \eta(L))^r + \frac{1}{4} (|L'| + \eta(L'))^r - \frac{1}{2},
$$

Thus

$$
\ell(C_2) + \frac{1}{4} \cdot (\ell(P_M) + \ell(P_M)) - \frac{1}{4} (l_0^+ + l^-) \geq \frac{1}{4} (|L_1| + \eta(L_1))^r + \frac{1}{4} (|L_2| + \eta(L_2))^r + 2 - \frac{1}{2}
\geq \frac{1}{4} \left( \frac{(d - 2 - 1.5\epsilon_2)(d - 2 - 1.5\epsilon_2 - \frac{1}{d - 2 - 1})}{(d - 2.5)(d - 3.5)} \right)^{r/2} + 2 - \frac{1}{2}
\geq \frac{1}{4} n^r + 2 - \frac{1}{2},
$$

provided that $d \geq 125$, where the conditions that $\omega(L) \geq \frac{n - |H|}{d - 2.5}$, $\omega(L') \geq \frac{n - |H| - \omega(L)}{d - 3.5}$ from (4.27), $\omega(L) \leq \frac{n}{d - 2.1}$ from Claim (4.6.2.10), and $|H| \leq \frac{(1 + 1.5\epsilon_2)n}{d - 1}$ from Claim (4.6.2.7) are used.

From above, we now can see $\ell(C_1) + \ell(C_2) \geq 2 \cdot (\frac{1}{4} n^r + 2)$, this implies that at least one of $\ell(C_1)$ and $\ell(C_2)$ is at least $\frac{1}{4} n^r + 2$. The proof is then completed. \qed
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