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New Extensions and Applications of Geršgorin Theory

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Under the Direction of Frank J. Hall, PhD

ABSTRACT

In this work we discover for the first time a strong relationship between Geršgorin theory and the geometric multiplicities of eigenvalues. In fact, if λ is an eigenvalue of an $n \times n$ matrix A with geometric multiplicity k, then λ is in at least k Geršgorin discs of A. Moreover, construct the matrix C by replacing, in every row, the $(k-1)$ smallest off-diagonal entries in absolute value by 0, then λ is in at least k Geršgorin discs of C. We also state and prove many new applications and consequences of these results as well as we update an improve some important existing ones.

INDEX WORDS: Geršgorin, eigenvalue, geometric multiplicity

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1 Introduction

One of the most attractive and useful results to locate the eigenvalues of a matrix is Geršgorin's theorem, which goes back to 1931. The main part of the theorem is the following.

Geršgorin Theorem Let A be an $n \times n$ real or complex matrix and let

$$R_i' = \sum_{j=1,\, j\neq i}^{n} |a_{ij}|, \quad 1 \leq i \leq n$$

denote the deleted absolute row sums of A. Every eigenvalue of A is located in the union of its n Geršgorin discs

$$\cup_{i=1}^{n} D_i$$

where

$$D_i = \{z \in C : \, |z - a_{ii}| \leq R_i'\}.$$

The proof of the theorem involves a clever idea. Let λ be an eigenvalue A, and suppose that

$$Ax = \lambda x, \; x = [x_i] \neq 0.$$

Some entry of x has largest modulus, say $|x_p| \geq |x_i|$ for all $i = 1, 2, \ldots, n$, and $x_p \neq 0$. Then

$$x_p(\lambda - a_{pp}) = \sum_{j=1,\, j\neq p}^{n} a_{pj}x_j$$

and hence

$$|x_p||\lambda - a_{pp}| \leq |x_p| \sum_{j=1,\, j\neq p}^{n} |a_{pj}| = |x_p|R'_p$$

so that $|\lambda - a_{pp}| \leq R'_p$; that is, λ lies in the pth Geršgorin disc. If λ is a simple eigenvalue of A, that is the end of the story. However, if λ is associated with several linearly independent eigenvectors, how could that fact be used to extract more information from Geršgorin's theorem?

If the *geometric multiplicity* of λ, namely the dimension of the associated eigenspace (the null space of $\lambda I - A$), is 1, then we have no control over the position of a largest modulus entry of a corresponding eigenvector: Every eigenvector is a nonzero scalar multiple of some given eigenvector, so that every eigenvector has its largest modulus entry in the same position. However, if the geometric multiplicity of λ is greater than 1, we have some flexibility in this regard. For example, let

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{1}$$

Then 0 is an eigenvalue of A with (algebraic) multiplicity 3 and geometric multiplicity 2. The vectors $[1, -2, 3]^T$ and $[0, 0, 1]^T$ are linearly independent eigenvectors associated with $\lambda = 0$, with both largest modulus entries occurring in the third position. However, $[1, -2, 0]^T$ and $[0, 0, 1]^T$

are also eigenvectors, and their largest modulus entries occur in different positions. Our proof of Geršgorin's theorem shows that the eigenvalue $\lambda = 0$ is in both the second and third Geršgorin discs; what we have observed here is no accident.

An $n \times n$ matrix A has n Geršgorin discs D_i, some of which may be duplicates, as in the trivial example of an identity matrix. We claim that if λ is an eigenvalue of A with geometric multiplicity k, then λ is in at least k of the Geršgorin discs of A, and this is what we will discus in the following section.

The work exposed in this dissertation was done originally by the author and their coauthors, and it can be found on some of the articles cited in the references section. Despite the considerable amount of work done on this theory since 1936, no one has established the relationship between the Geršgorin theory and geometric multiplicities of the eigenvalues, until our submission of the first article published in the American Mathematical Monthly journal, [3]. The main result in that article was also cited in textbook [2].

2 First extension of the Geršgorin theory

2.1 Geometric multiplicities and location of eigenvalue

The following result is proved to be an important tool in this work, [3], [8]

Lemma 2.1. *Let S be a $k-$dimensional subspace of C^n. There is a basis $\{v_1, v_2, \ldots, v_k\}$ of S with the following property: for each $i = 1, 2, \ldots, k$, there are distinct integers p_i, with $1 \leq p_i \leq n$ and $p_i \neq p_j$ for $i \neq j$, such that a largest modulus entry of each v_i is in position p_i.*

Proof. We place the vectors of a basis $\mathcal{B} = \{x_1, x_2, \ldots, x_k\}$ of S as columns of an $n \times k$ full column rank matrix $X = [x_1| \ldots |x_k]$. Let $P_1 \in M_n$ be a permutation matrix (not necessarily unique) such that a largest modulus entry of x_1 (there could be more than one) is the first entry of $P_1 x_1 = y_1$. Partition $P_1 X = [y_1 \; Y_2]$ and $y_1 = [y_{11} \; w^T]^T$. Let R_1 be an upper triangular matrix of the form

$$R_1 = \begin{bmatrix} 1 & z^* \\ 0 & I_{k-1} \end{bmatrix}$$

and choose the unique vector $z \in C^{k-1}$ such that

$$(P_1 X) R_1 = [y_1 \; Y_2] \begin{bmatrix} 1 & z^* \\ 0 & I_{k-1} \end{bmatrix} = [y_1 \; y_1 z^* + Y_2] = \begin{bmatrix} y_{11} & 0 \\ w & X^{(2)} \end{bmatrix}$$

has zero entries in the first row to the right of the $(1,1)-$entry. Now repeat this process on

$X^{(2)}, X^{(3)}, \ldots$ to obtain $(P_{k-1} \cdots P_1)X(R_1 \cdots R_{k-1}) = Z$, a matrix whose diagonal entries are largest modulus entries in their respective columns. Moreover, $Z = PXR$, in which P is a product of $k-1$ permutation matrices and R is a product of $k-1$ upper-triangular matrices with 1s on the diagonal. Thus, P is a permutation matrix, R is upper-triangular and nonsingular, and Z has full column rank. Note that the column spaces of X and XR are the same. Thus, we see that the columns of $P^T Z = XR$ have the desired property. □

We can now use Lemma 2.1 to prove our claim. In the following discussion, $A = [a_{ij}]$ is always an $n \times n$ complex matrix.

Theorem 2.2. *Let λ be an eigenvalue of A with geometric multiplicity k. Then λ is in at least k of the Geršgorin discs D_i of A.*

Proof. Lemma 2.1 ensures that there is a basis $\{x_1, x_2, \ldots, x_k\}$ of the eigenspace S of λ and distinct integers $p_1, \ldots, p_k \in \{1, \ldots, n\}$ such that each vector x_i has a largest modulus entry in position p_i. Our construction in the proof of the Geršgorin theorem shows that λ lies in Geršgorin discs $D_{p_1}, \ldots, D_{p_k}$. □

From Theorem 2.2, we see that an eigenvalue with geometric multiplicity at least $k \geq 1$ is contained in *any* union of $n-k+1$ different Geršgorin discs of A. Now that's an improvement of Geršgorin's general theorem, which is the case $k = 1$ of our assertion!

Corollary 2.3. *Let λ be an eigenvalue of A with geometric multiplicity at least $k \geq 1$. Then*

$$\lambda \in \cup_{j=1}^{n-k+1} \{z \in C : |z - a_{i_j i_j}| \leq R'_{i_j}\}$$

for any choices of indices $1 \leq i_1 < \ldots < i_{n-k+1} \leq n$. There are $\binom{n}{k-1}$ possibilities for such a union, and λ is contained in their intersection.

The *rank-nullity theorem* says that for a matrix B with n columns, the rank of B plus the dimension of the null space of B is equal to n. We apply this theorem and Theorem 2.2 to obtain the following result.

Corollary 2.4. *Let λ be an eigenvalue of A. If rank $(A - \lambda I) \leq t$, then λ is in at least $n - t$ of the Geršgorin discs D_i of A.*

Using the same two theorems, we can prove another interesting and useful result.

Corollary 2.5. *If $|a_{ii}| > R'_i$ for q different values of i, then the geometric multiplicity of $\lambda = 0$ as an eigenvalue of A is at most $n - q$ and rank $A \geq q$.*

The example (1) shows that the result in Theorem 2.2 is not valid for the *algebraic* multiplicity of an eigenvalue. Also, if λ is an eigenvalue of A with geometric multiplicity k, then λ may be in more than k of the Geršgorin discs D_i of A. For example, let

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}.$$

Then 2 is an eigenvalue of A with geometric multiplicity 1 (and algebraic multiplicity 2), but the eigenvalue 2 is contained in two Geršgorin discs of A. The same is true for the slightly more complicated example A_1 where

$$A_1 = \begin{bmatrix} 3 & -\frac{1}{2} & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{bmatrix},$$

which has an additional eigenvalue of 3.

A final example shows that an eigenvalue λ can be in t Geršgorin discs, for some t between the geometric multiplicity of λ and the algebraic multiplicity of λ. Let

$$A_2 = \begin{bmatrix} 3 & 0 & -1 \\ -2 & 4 & 6 \\ 1 & -1 & -1 \end{bmatrix}.$$

Then 2 is an eigenvalue of A_2 with geometric multiplicity 1 and algebraic multiplicity 3; the eigenvalue 2 is contained in two Geršgorin discs of A_2. The direct sum matrix

$$A_3 = \begin{bmatrix} A_2 & 0 \\ 0 & A_2 \end{bmatrix}$$

has 2 as an eigenvalue with geometric multiplicity 2 and algebraic multiplicity 6; the eigenvalue 2 is contained in four Geršgorin discs of A_2.

2.2 Open Problem

We close this section with a question.

Open Question 2.6. Let k, r, t be positive integers with $k \leq r \leq t$. Is there a square complex matrix A and an eigenvalue λ of A such that λ has geometric multiplicity k and algebraic multiplicity t, and λ is in r Geršgorin discs of A?

3 Solving the open problem

To answer the posed question in the previous section, in the affirmative, we first introduce some notation and discuss the case that λ is equal to zero, and that the geometric multiplicity of λ is 1 (i.e., that there is just one Jordan block corresponding to zero), and we also take the algebraic multiplicity of λ to be n, the order of the matrix. We thus consider the $n \times n$ Jordan block

$$J_n(0) = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix},$$

which has eigenvalue 0 with algebraic multiplicity n and geometric multiplicity 1. We shall transform $J_n(0)$ by similarity (thereby retaining the same two multiplicities) to a matrix which has the eigenvalue 0 in a number r of Geršgorin discs, for any given positive integer r, $1 \leq r \leq n$. To illustrate the idea of the proof, let us consider

$$J_4(0) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

With not yet specified ϵ_1, ϵ_2 and ϵ_3, define

$$S_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \epsilon_1 & 1 & 0 & 0 \\ 0 & \epsilon_2 & 1 & 0 \\ 0 & 0 & \epsilon_3 & 1 \end{bmatrix},$$

and transform $J_4(0)$ by similarity to $S_4^{-1}J_4(0)S_4$. Since

$$S_4^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\epsilon_1 & 1 & 0 & 0 \\ \epsilon_2\epsilon_1 & -\epsilon_2 & 1 & 0 \\ -\epsilon_3\epsilon_2\epsilon_1 & \epsilon_3\epsilon_2 & -\epsilon_3 & 1 \end{bmatrix},$$

we obtain

$$S_4^{-1}J_4(0)S_4 = \begin{bmatrix} \epsilon_1 & 1 & 0 & 0 \\ -\epsilon_1^2 & \epsilon_2 - \epsilon_1 & 1 & 0 \\ \epsilon_2\epsilon_1^2 & \epsilon_2\epsilon_1 - \epsilon_2^2 & \epsilon_3 - \epsilon_2 & 1 \\ -\epsilon_3\epsilon_2\epsilon_1^2 & \epsilon_3\epsilon_2^2 - \epsilon_3\epsilon_2\epsilon_1 & \epsilon_3\epsilon_2 - \epsilon_3^2 & -\epsilon_3 \end{bmatrix}. \quad (2)$$

The idea is to make choices for the ϵ_i so that $\lambda = 0$ is out of the first three Geršgorin discs (G-discs for short), but necessarily in the fourth since its geometric multiplicity is 1. Then, depending on the required value of r, we re-adjust some of the values of ϵ_i to zero.

We see that if we choose $\epsilon_1 > 1$, then 0 is not in the first G-disc. Next, choose $\epsilon_2 > \epsilon_1$ and $\epsilon_2 > \epsilon_1 + \epsilon_1^2 + 1$, so that 0 is not in the second G-disc. Finally, choose $\epsilon_3 > \epsilon_2 + \epsilon_2\epsilon_1^2 + |\epsilon_2\epsilon_1 - \epsilon_2^2| + 1$, so that 0 is not in the third G-disc. This covers the case $r = 1$.

Suppose $r = 2$. In this case, keep ϵ_1 and ϵ_2 as before, but set $\epsilon_3 = 0$. Then, 0 is clearly not in the first and second G-discs, but is in the fourth one. But we also have

$$\epsilon_2 < \epsilon_2\epsilon_1^2 + |\epsilon_2\epsilon_1 - \epsilon_2^2| + 1,$$

since $\epsilon_1 > 1$. So, 0 is in the third G-disc and we have covered the case $r = 2$.

For the case $r = 3$, simply set $\epsilon_1 > 1$ and $\epsilon_2 = \epsilon_3 = 0$, so that 0 is in the last three G-discs.

Of course, if $r = 4$, we set all the $\epsilon_i = 0$ and the eigenvalue 0 is in all four G-discs.

Example 3.1. Using (2) with $\epsilon_1 = 2, \epsilon_2 = 8, \epsilon_3 = 0$, we have that 0 is an eigenvalue of the matrix

$$\begin{bmatrix} 2 & 1 & 0 & 0 \\ -4 & 6 & 1 & 0 \\ 32 & -48 & -8 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

with the eigenvalue 0 having algebraic (geometric) multiplicity 4 (1), respectively. Clearly, 0 is outside the first two, but inside the last two G-discs.

Theorem 3.2. *Let k, r and n be positive integers such that $k \leq r \leq n$. Then for any complex number λ there is an $n \times n$ matrix A which has λ as an eigenvalue with geometric multiplicity k and algebraic multiplicity n, and λ is in precisely r Geršgorin discs of A.*

Proof. We first consider the case where $k = 1$ and transform $J_n(0)$ by similarity to $S_n^{-1} J_n(0) S_n$, which has eigenvalue 0 with algebraic multiplicity n and geometric multiplicity 1. We use again

$$S_n = \begin{bmatrix} 1 & & & \\ \epsilon_1 & \ddots & & \\ & \ddots & & \\ & & \epsilon_{n-1} & 1 \end{bmatrix}$$

with not yet specified entries ϵ_i. It can be verified that the (i, j) entry of S_n^{-1} is

0 for $1 \leq i < j \leq n$,

1 for $i = j, 1 \leq i \leq n$,

$(-1)^{i-j}\epsilon_j \cdots \epsilon_{i-1}$ for $1 \leq j < i \leq n$.

The matrix $A_0 = S_n^{-1} J_n(0) S_n$ has then its (i, j) entry a_{ij} equal to zero if $j > i+1$ and 1 if $j = i+1, i = 1, \ldots, n-1$; for $1 \leq i \leq n$ we have $a_{i1} = (-1)^{i-1}\epsilon_1(\Pi_{t=1}^{i-1}\epsilon_t)$; for $1 < j \leq i \leq n$, $a_{ij} = (-1)^{i-j}(\epsilon_j - \epsilon_{j-1})(\Pi_{t=j}^{i-1}\epsilon_t)$, except a_{nn}, where $a_{nn} = -\epsilon_{n-1}$; the void product is taken to be 1.

We now arrange that the eigenvalue 0 is not in the first $n - r$ G-discs of the matrix A_0 but in all the remaining r G-discs by choosing $\epsilon_{n-r+1} = \epsilon_{n-r+2} = \ldots = \epsilon_{n-1} = 0$ and if $r < n$, the numbers $\epsilon_1, \epsilon_2, \ldots, \epsilon_{n-r}$ recurrently as follows :

$$\epsilon_1 > 1, \text{ for } t \geq 1, \ \epsilon_{t+1} > \epsilon_t + \sum_{j=1}^{t} |a_{t+1,j}| + 1. \tag{3}$$

Indeed, the right-hand side of the last inequality in (3) contains only $\epsilon_1, \ldots, \epsilon_t$; if $r = 1$, the Geršgorin theorem ensures that the last G-disc contains the eigenvalue 0. In our case that $k = 1$, the matrix $A_0 + \lambda I_n$ satisfies the conditions of the theorem.

Finally, if the geometric multiplicity k is greater than one, it suffices to form a direct sum of the similarly constructed matrix $A_0 + \lambda I_{n-k+1}$ of order $n - k + 1$ and $k - 1$ matrices of order one each with the entry λ to obtain the resulting matrix A. $\square$

We now turn our attention to generalizing the above construction to more than one distinct eigenvalue. In this case, our procedure will not allow the choice of more than one resulting eigenvalue to be arbitrary. For simplicity, we first consider two distinct eigenvalues.

Theorem 3.3. *Let $k_1, k_2, r_1, r_2, n_1, n_2$, and n be positive integers such that $k_1 \leq r_1 \leq n_1$, $k_2 \leq r_2 \leq n_2$, and $n_1 + n_2 = n$. Then there is an $n \times n$ matrix A and eigenvalues λ_1, λ_2 such that λ_1 (λ_2) has geometric multiplicity k_1 (k_2) and algebraic multiplicity n_1 (n_2), and λ_1 (λ_2) is in precisely r_1(r_2) G-discs of A, respectively. In fact, the eigenvalue λ_1 can be chosen to be any complex number.*

Proof. As in Theorem 3.2, we can obtain an $n_1 \times n_1$ matrix A_{01} and an $n_2 \times n_2$ matrix A_{02} such that zero is an eigenvalue of A_{01} (A_{02}) with geometric multiplicity k_1 (k_2), algebraic multiplicity n_1 (n_2) and zero is in precisely r_1 (r_2) G-discs of A_{01}, (A_{02}), respectively.

It is then clear that for suitable λ_1 and λ_2, the requirements of the theorem are fulfilled in the matrix

$$\begin{bmatrix} A_{01} + \lambda_1 I_1 & 0 \\ 0 & A_{02} + \lambda_2 I_2 \end{bmatrix};$$

one only has to ensure that the respective G-discs are separated. $\square$

Remark 3.4. It is immediate that Theorem 3.3 can be extended to more than two distinct eigenvalues in an analogous way.

Observation 3.5. *An eigenvalue λ can, of course, be in q Geršgorin discs, where q is an arbitrary integer greater than the algebraic multiplicity of λ, but not exceeding the order of the matrix. For example, the following holds.*

Let k, t and n be positive integers such that $k \leq t \leq n$. Then there is an $n \times n$ matrix A and an eigenvalue λ of A such that λ has geometric multiplicity k and algebraic multiplicity t, and λ is in all n Geršgorin discs of A. Indeed, we can simply use the eigenvalue $\lambda = 0$ and let

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix},$$

where A_1 is a $t \times t$ matrix which is a direct sum of k Jordan blocks of the type $J_i(0)$ and A_2 is a cyclic matrix of order $n - t$, that is,

$$A_2 = \begin{bmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}.$$

Note that the eigenvalues of A_2 are the $(n-t)$th roots of 1.

Up to this point, the results that have been exhibited were contained in articles [1], [3], as well as in [8].

Consider now, an $n \times n$ matrix A which has one distinct eigenvalue λ with geometric multiplicity k and algebraic multiplicity n. We can first consider the case that λ is equal to zero, and that the geometric multiplicity of λ is 1, i.e., that there is just one Jordan block corresponding to zero in the Jordan form J_A of A. Hence, A is similar to the $n \times n$ Jordan block

$$J_n(0) = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & 0 \end{bmatrix},$$

which has eigenvalue 0 with algebraic multiplicity n and geometric multiplicity 1. We transform $J_n(0)$ by similarity (thereby retaining the same two multiplicities) to a matrix which has the eigenvalue 0 in a number r of Geršgorin discs, for any given positive integer r, $1 \leq r \leq n$. As in the proof of Theorem 3.2, this can be done by using the similarity $A_0 = S_n^{-1} J_n(0) S_n$, where

$$S_n = \begin{bmatrix} 1 & & & \\ \epsilon_1 & \ddots & & \\ & \ddots & & \\ & & \epsilon_{n-1} & 1 \end{bmatrix},$$

with appropriate choices of the ϵ_i.

In the case that $k = 1$, the matrix $A_0 + \lambda I_n$ gives a matrix similar to A for which the eigenvalue λ is in r Geršgorin discs. The Jordan Form allows us to handle the case where the geometric multiplicity of the eigenvalue is more than 1 by using a direct sum of similarities of the above type. We thus arrive at the following result, which as we have observed, was essentially proved previously in this section.

Theorem 3.6. *Let A be an $n \times n$ matrix with one distinct eigenvalue λ, which has geometric multiplicity k and algebraic multiplicity n. Then for any integer r such that $k \leq r \leq n$ there is a matrix B that is similar to A and for which λ is in precisely r Geršgorin discs of B.*

Next we extend the result in Theorem 3.6 to more than one distinct eigenvalue. A key fact is the following, which is found in [2], page 171.

Lemma 3.7. *For any complex number $\alpha \neq 0$, the $n \times n$ matrices*

$$\begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda \end{bmatrix} \quad \textit{and} \quad \begin{bmatrix} \lambda & \alpha & & \\ & \ddots & \ddots & \\ & & & \alpha \\ & & & \lambda \end{bmatrix}$$

are similar. In fact, the first matrix is the Jordan form of the second.

Proof. Both matrices have the eigenvalue λ with algebraic multiplicity n and geometric multiplicity 1. □

Theorem 3.8. *Let A be an $n \times n$ matrix with two distinct eigenvalues λ_1, λ_2, which have geometric multiplicities k_1, k_2 and algebraic multiplicities n_1, n_2, respectively. Then for any integers r_i such that $k_i \leq r_i \leq n_i$, there is a matrix B that is similar to A and for which λ_i is in precisely r_i Geršgorin discs of B, $i = 1, 2$.*

Proof. We can write the Jordan form J_A of A as

$$J_A = \begin{bmatrix} J(\lambda_1) & 0 \\ 0 & J(\lambda_2) \end{bmatrix},$$

where $J(\lambda_i)$ consists of the Jordan blocks corresponding to eigenvalue $\lambda_i,\ i = 1, 2$. Further, we can write $J(\lambda_i) = J_i(0) + \lambda_i I,\ i = 1, 2$, where $J_i(0)$ is the nilpotent part of $J(\lambda_i)$. By Lemma

3.7, for any complex number $\alpha \neq 0$, $J_i(0)$ is similar to $\alpha J_i(0)$ $i = 1, 2$, so that J_A is similar to

$$\begin{bmatrix} \alpha J_1(0) + \lambda_1 I_{n_1} & 0 \\ 0 & \alpha J_2(0) + \lambda_2 I_{n_2} \end{bmatrix}.$$

Now, by Theorem 5.16, we have nonsingular matrices M_1, M_2 such that the eigenvalue 0 is in r_i Geršgorin discs of $M_i^{-1} J_i(0) M_i$ and importantly hence in r_i Geršgorin discs of $\alpha M_i^{-1} J_i(0) M_i,\ i = 1, 2$. So, J_A is similar to

$$B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$$

where $B_i = \alpha M_i^{-1} J_i(0) M_i + \lambda_i I_{n_i} = \alpha C_i + \lambda_i I_{n_i}$, and λ_i is in r_i Geršgorin discs of $B_i,\ i = 1, 2$. We want to be able to choose α so that λ_1 is not in any Geršgorin disc of B_2 and λ_2 is not in any Geršgorin disc of B_1. To accomplish this, we can simply take α to be a sufficiently small positive real number, thereby separating the Geršgorin discs of B_1 and B_2. We point out that it is sufficient to choose

$$\alpha < \frac{|\lambda_1 - \lambda_2|}{\max\limits_{1 \leq h \leq 2} \{ \max\limits_{1 \leq i \leq n_h} \sum\limits_{j=1}^{n_h} |(C_h)_{ij}| \}},$$

where n_1 and n_2 are respectively, the orders of B_1 and B_2. $\square$

Remark 3.9. The neat use of Lemma 3.7 allowed us to start with any $n \times n$ matrix A which has two distinct eigenvalues and to find a matrix B that is similar to A for which B has certain required Geršgorin disc properties. In contrast, with Theorem 3.3 we obtain only some particular matrices with required Geršgorin disc properties.

Theorem 3.8 can be extended to more than two distinct eigenvalues in an analogous way.

Corollary 3.10. *Let A be an $n \times n$ matrix with s distinct eigenvalues $\lambda_1, \ldots, \lambda_s$, which have geometric multiplicities $k_1, \ldots, k_s$ and algebraic multiplicities $n_1, \ldots, n_s$, respectively.*
Then for any integers r_i such that $k_i \leq r_i \leq n_i$, there is a matrix B that is similar to A and for which λ_i is in precisely r_i Geršgorin discs of B, $i = 1, \ldots, s$.

4 New applications and consequences of Theorem 2.2

In this section we present some new results based entirely on Theorem 2.2 that says: $\Big($If λ is an eigenvalue of A with geometric multiplicity k then λ is in at least k Geršgorin discs of $A\Big)$. The first result concerns non-real eigenvalues of real matrices. We use the "second part" of the Geršgorin theorem: "If a union of p of the n Geršgorin discs of A forms a connected region that is disjoint from all the remaining $n-p$ discs, then there are precisely p eigenvalues of A in this region." Some of the p discs can be duplicates, as for example with the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 7 \end{bmatrix}.$$

Theorem 4.1. *Let A be an $n \times n$ real matrix with no more than q of the n Geršgorin discs of A forming a connected region. Then the geometric multiplicity of any non-real eigenvalue λ of A is less than or equal to $\frac{q}{2}$.*

Proof. Let the geometric multiplicity of a particular non-real eigenvalue λ of A be k and suppose that $k > \frac{q}{2}$, so that $2k > q$. By Theorem 2.2, λ is in at least k Geršgorin discs of A. So, we have at least k Geršgorin discs (some of which may be duplicates) of A forming a connected region $\mathcal{S}$. Now, with $S^{-1}AS = J_A$, where J_A is the Jordan form of A, we have since A is real that $(\bar{S})^{-1}A\bar{S} = \bar{J}_A$, so that $J_A = \bar{J}_A$. Hence, $\bar{\lambda}$ also has geometric multiplicity k, so that $\bar{\lambda}$ also is in at least k Geršgorin discs of A. Also since A is real, non-real eigenvalues occur in conjugate pairs, and a given Geršgorin disc D of A contains λ if and only if D contains $\bar{\lambda}$. Hence, we have at least $2k$ eigenvalues of A in the region $\mathcal{S}$, namely λ and $\bar{\lambda}$ each repeated k times. Then, by the second part of the Geršgorin Theorem, the connected region $\mathcal{S}$ must consist of at least $2k$ Geršgorin discs of A. (If $\mathcal{S}$ consisted of fewer than $2k$ Geršgorin discs of A, then $\mathcal{S}$ would contain fewer than $2k$ eigenvalues of A.) Since $2k > q$ we have a contradiction. Thus $k \leq \frac{q}{2}$. □

Example 4.2. Let

$$A = \begin{bmatrix} 1 & 2 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -2 & 1 \end{bmatrix}.$$

Then the eigenvalues of A are $1+2i, 1-2i, 1+2i, 1-2i$, with each of the eigenvalues $1+2i$ and $1-2i$ having geometric multiplicity 2. Clearly, $q = 4$ in this case, so that both of the geometric multiplicities equal to $\frac{q}{2}$.

The case $q = 1$ is known, and can be found in many references and textbooks such as [2]:Problem 6.1.P5, page 394 of [2].

Corollary 4.3. *If A is an $n \times n$ real matrix whose n Geršgorin discs are all mutually disjoint, then all the eigenvalues of A are real. More generally, if an $n \times n$ complex matrix A has real main diagonal entries, its characteristic polynomial has only real coefficients, and the n Geršgorin discs of A are all mutually disjoint, then all the eigenvalues of A are real.*

We now consider powers of a square matrix A.

Theorem 4.4. *Let A be an $n \times n$ complex matrix and suppose the non-zero complex number β has exactly s different roots $\lambda_1, \ldots, \lambda_s$ of order m, each of which is an eigenvalue of A. Then, β is an* *eigenvalue of* A^m *and letting* $h = \sum_{j=1}^{s}$ *geom mult*(λ_j)*, we have that* β *is in at least* h *Geršgorin discs of* A^m*.*

Proof. Since the eigenvalues of A^m are the m^{th} powers of the eigenvalues of A, it is the clear that β is an eigenvalue of A^m. We also observe that for a simple Jordan block $J_i(\lambda)$, $\lambda \neq 0$, the Jordan form of $J_i^m(\lambda)$ is $J_i(\lambda^m)$. So, the geometric multiplicity of β as an eigenvalue of A^m is h. Thus, from Theorem 2.2, we have that β is in at least h Geršgorin discs of A^m. $\square$

Example 4.5. Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & -1 \\ 2 & 0 & 1 & -2 \\ 1 & 0 & 1 & 0 \end{bmatrix}.$$

Then the eigenvalues of A are $-1, 1, 1, 1$, with -1 and 1 having geometric multiplicity 1 and 2, respectively. Now,

$$A^4 = \begin{bmatrix} 5 & 0 & 4 & -4 \\ 4 & 1 & 4 & -4 \\ 0 & 0 & 1 & 0 \\ 4 & 0 & 4 & -3 \end{bmatrix},$$

with the eigenvalue 1 having algebraic multiplicity 4 and geometric multiplicity 3. In the notation of Theorem 5.11, with $s = 2, \lambda_1 = -1, \lambda_2 = 1$ and $\beta = 1$ an eigenvalue of A^4, $h = 1 + 2 = 3$. In this case, β is in all four Geršgorin discs of A^4.

The next result involves the *index* of an eigenvalue of matrix A, which is the order of the largest Jordan block in J_A (the Jordan Canonical form of A) associated with the eigenvalue.

Theorem 4.6. *Let A be an $n \times n$ complex matrix, let λ be an eigenvalue of A with geometric multiplicity k and algebraic multiplicity t, and let m be the index of the eigenvalue λ. Then 0 is an eigenvalue of $(A - \lambda I)^m$ and 0 is in at least t Geršgorin discs of $(A - \lambda I)^m$. Moreover, 0 is an eigenvalue of $(A - \lambda I)^{t-k+1}$ and 0 is in at least t Geršgorin discs of $(A - \lambda I)^{t-k+1}$.*

Proof. It is clear that 0 is an eigenvalue of the matrix $(A - \lambda I)^m$ and also that we have t linearly independent eigenvectors associated with the eigenvalue 0, that is, the geometric multiplicity of 0 as an eigenvalue of $(A - \lambda I)^m$ is t. (We have t 1×1 Jordan blocks associated with eigenvalue 0.) Hence, by Theorem 2.2, 0 is in at least t Geršgorin discs of $(A - \lambda I)^m$.

Next, one can observe that $m \leq t - k + 1$. Hence, the t linearly independent eigenvectors mentioned above satisfy $(A - \lambda I)^{t-k+1}x = 0$. So, the geometric multiplicity of 0 as an eigenvalue of $(A - \lambda I)^{t-k+1}$ is at least t. Thus, by Theorem 2.2 again, 0 is in at least t Geršgorin discs of $(A - \lambda I)^{t-k+1}$. $\square$

To give a simple illustration of Theorem 4.6, if $t = 10$ and $k = 3$, then 0 is an eigenvalue of $(A - \lambda I)^8$ and 0 is in at least 10 Geršgorin discs of $(A - \lambda I)^8$.

5 Second extension of the Geršgorin theory

5.1 On the location of eigenvalues with geometric multiplicities larger than 1

We will employ the following key result, which is contained in Theorem 1.4.10 in [2].

Lemma 5.1. *Let λ be an eigenvalue of the $n \times n$ matrix A with geometric multiplicity at least k. If $\hat{A}$ is an $m \times m$ principal submatrix of A and if $m > n - k$, then λ is an eigenvalue of $\hat{A}$.*

By taking m to be $n - k + 1$ and applying Lemma 5.1 we obtain the following improvement of Theorem 2.2.

Theorem 5.2. *Let λ be an eigenvalue of the $n \times n$ matrix A with geometric multiplicity at least k. Construct the $n \times n$ matrix C_k in the following way: in every row of A, replace the smallest $k - 1$ off-diagonal entries in absolute value by zeros. Then λ is in at least k of the Geršgorin discs of C_k.*

Proof. We choose an arbitrary principal submatrix B_1 of A of order $n - k + 1$. By Lemma 5.1, λ is an eigenvalue of B_1, and hence in one of it's G-discs. In the corresponding row (say row r) of the matrix A, we can replace the smallest $k - 1$ off-diagonal entries in absolute value by zeros, so that λ is in the associated G-disc of the new matrix.

Next, delete row and column r from A to obtain a principal submatrix A_2 of A (which can be considered as A_1) of order $n - 1$. Choose an arbitrary principal submatrix B_2 of A_2 of order $n - k + 1$ and continue as above. We continue this process until we reach a principal submatrix A_k of order $n - (k - 1)$, ie $n - k + 1$, and repeat the procedure on A_k. This completes k steps in which we have replaced in k rows of A the smallest $k-1$ off-diagonal entries in absolute value by zeros, and λ is in each of the corresponding G-discs.

Finally, in each of the remaining $n-k$ rows, also replace the smallest $k-1$ off-diagonal entries in absolute value by zeros. $\square$

Example 5.3. Let

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & -1 & 2 & 0 \\ -1 & 1 & -1 & 1 & 1 \end{bmatrix}.$$

Now, 1 is an eigenvalue of A with algebraic multiplicity 5 and geometric multiplicity $k = 3$. Notice that in this case the eigenvalue is in all five G-discs. (We recall that for an $n \times n$ matrix A, an eigenvalue λ could be in any number t of G-discs where (geom mult λ) $\leq t \leq n$, see [1], [4].) We can go through the process in the proof of Theorem 5.2, or just see that the eigenvalue 1 is in all five (the last three of which are smaller) G-discs of the matrix

$$C_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Of course, other choices for the matrix C_3 are possible, such as

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 2 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

Remark 5.4. Clearly, the G-discs of the matrix C_k have radii less than or equal to the radii of the G-discs of the original matrix A, which in general gives a better inclusion region for the eigenvalues of A with geometric multiplicity $\geq k$. Furthermore, as in Example 5.3, it should be clear that the matrix C_k is not unique, since many off-diagonal entries in the same row may have the same absolute value. We also point out that the eigenvalues of A and C_k are in general not the same.

For the purposes in the sequel, we denote matrices C_i constructed from the $n \times n$ matrix A in the same way as in Theorem 5.2: in every row of A, replace the smallest $i-1$ off-diagonal entries in absolute value by zeros.

We can now give a refinement of Corollary 2.3.

Corollary 5.5. *Let λ be an eigenvalue of A with geometric multiplicity at least $k \geq 1$. Then*

$$\lambda \in \cup_{j=1}^{n-k+1} \{z \in C : |z - a_{i_j i_j}| \leq R'_{i_j}(C_k)\}$$

for any choices of indices $1 \leq i_1 < \ldots < i_{n-k+1} \leq n$. There are $\binom{n}{k-1}$ possibilities for such a union, so that λ is contained in their intersection.

The following is another interesting, immediate corollary of Theorem 2.2, based again on the matrix C_k.

Corollary 5.6. *Let A be an $n \times n$ matrix. If each collection of G-discs of the matrix C_k that is separated from the remaining G-discs of C_k consists of at most $k-1$ discs, then each eigenvalue* *of A has geometric multiplicity less than k. In particular, if this is the case for $k = 2$, then each eigenvalue has geometric multiplicity 1.*

Example 5.7. Only observing that the three G-discs of the matrix

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & -3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

form a connected region, we cannot determine the multiplicities of its eigenvalues. However, consider the matrix

$$C_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

which has three mutually disjoint G-discs. By Corollary 5.6, each eigenvalue of A has geometric multiplicity 1.

5.2 New applications and consequences of Theorem 5.2

The following result is a direct consequence of Theorem 5.2.

Theorem 5.8. *Let λ be an eigenvalue of A with geometric multiplicity k and suppose that $|a_{ij}| = \alpha$ for all $i \neq j$. Then the inequality*

$$|\lambda - a_{ii}| \leq (n-k)\alpha$$

holds for at least k values of i.

Corollary 5.9. *Let λ be an eigenvalue of A with geometric multiplicity k. Suppose that $|a_{ij}| = \alpha$ for all $i \neq j$ and that $|a_{ii}| = \beta$ for all i. Then*

$$|\lambda| \leq \beta + (n-k)\alpha.$$

Example 5.10. The above corollary is illustrated by the adjacency matrix A of the complete graph of order n, which has zeros on the diagonal and one in each off-diagonal position. The eigenvalues of A are $n-1$ and -1 with (algebraic and geometric) multiplicities 1 and $n-1$, respectively. For the eigenvalue $n-1$, Corollary 5.9 says that $n-1 \leq n-1$ and for the eigenvalue -1 that $1 \leq 1$, both of which are trivially true.

The corollary is also illustrated by the all ones matrix J_n of order n, which has eigenvalues n and 0 with (algebraic and geometric) multiplicities 1 and $n-1$, respectively.

Another application of Theorem 5.2 is the following.

Theorem 5.11. *Let A be an $n \times n$ matrix with all off-diagonal entries nonzero and let λ be an eigenvalue of A such that every G-circle of A passes through λ. Then the geometric multiplicity of λ is 1.*

Proof. We are given that all off-diagonal entries of A are nonzero and that λ is an eigenvalue of A such that every G-circle of A passes through λ. Suppose that the geometric multiplicity k of λ is greater than 1. By Theorem 5.2, λ is in at least k G-discs of C_k. However, by the construction of C_k, λ then cannot be on any of those k G-circles of A, since the G-discs of C_k are respectively properly contained in the G-discs of the original matrix A. We have a contradiction. Thus, $k = 1$. $\square$

Remark 5.12. If the $n \times n$ matrix B is irreducible and the eigenvalue λ of B satisfies

$$|\lambda - b_{ii}| \geq R_i'(B),$$

for all $i = 1, \cdots, n$, then according to Theorem 6.2.8 in [2], every G-circle of B passes through λ. By what we have proved, if also all the off-diagonal entries of B are nonzero, then in fact the geometric multiplicity of λ is 1.

Example 5.13. We use the all ones matrix J_n. Now, the eigenvalue n is on every G-circle of J_n and the geometric multiplicity of n is 1.

We can relax the condition on the off-diagonal entries in Theorem 5.11 in the following way. The proof is similar to the proof of Theorem 5.11.

Corollary 5.14. *Let A be an $n \times n$ matrix and let λ be an eigenvalue of A such that every G-circle of A passes through λ. Suppose $k > 1$ and that there are $n - k + 1$ or more rows of A with the following property: in each of these rows, the number of zero off-diagonal entries is less than $k - 1$. Then the geometric multiplicity of λ is less than k.*

When $k = 2$ we have the following.

Corollary 5.15. *Let A be an $n \times n$ matrix with all off-diagonal entries nonzero in at least $n - 1$ rows of A and let λ be an eigenvalue of A such that every G-circle of A passes through λ. Then the geometric multiplicity of λ is 1.*

This allows the matrix A to be reducible.

We present a last result which makes use of Theorem 5.2. Recall that $\|A\|_\infty$ is the maximum of the absolute row sums of A and that $\|A\|_1$ is the maximum of the absolute column sums of A.

Theorem 5.16. *Let A be an $n \times n$ matrix with all off-diagonal entries nonzero in at least $n - 1$ rows ($n - 1$ columns) and let λ be an eigenvalue of A such that $|\lambda| = \|A\|_\infty$ ($|\lambda| = \|A\|_1$). Then the geometric multiplicity of λ is 1.*

Proof. Without loss of generality, suppose that all the off-diagonal entries in the first $n - 1$ rows are nonzero, and assume that $|\lambda| = \|A\|_\infty$. Suppose that the geometric multiplicity k of λ is greater than 1. By Theorem 5.2, λ is in at least k G-discs of C_k. Letting one of these k G-discs be disc i, $i \neq n$, we then have

$$|\lambda - c_{ii}| \leq R'_i(C_k)$$

so that

$$|\lambda| \le |c_{ii}| + R_i'(C_k) \ < |a_{ii}| + R_i'(A) \le \|A\|_\infty$$

and hence

$$|\lambda| < \|A\|_\infty \,,$$

which is a contradiction. Thus, the geometric multiplicity of λ is 1.

For the proof involving $|\lambda| = \|A\|_1$, we can use the transpose of A. $\square$

Remark 5.17. We can relax the condition on the off-diagonal entries in Theorem 5.16 in a similar way as in Corollary 5.14.

Example 5.18. Theorem 5.16 is illustrated by any doubly stochastic matrix with all off-diagonal entries nonzero and its eigenvalue of 1. Such a matrix is actually a nonnegative irreducible matrix.

In Theorem 1.4.10 in [2] the authors prove the following result.

Theorem 5.19. *Let λ be an eigenvalue of the $n \times n$ matrix A with geometric multiplicity at least k. If $\hat{A}$ is an $m \times m$ principal submatrix of A and if $m > n - k$, then λ is an eigenvalue of $\hat{A}$.*

We can in fact give a lower bound on the geometric multiplicity of λ as an eigenvalue of $\hat{A}$.

Theorem 5.20. *Let λ be an eigenvalue of the $n \times n$ matrix A with geometric multiplicity at least k and let $\hat{A}$ be an $m \times m$ principal submatrix of A with $m > n - k$. Then the geometric multiplicity of λ as an eigenvalue of $\hat{A}$ is at least $m + k - n$.*

Proof. It suffices to prove the result for an eigenvalue $\lambda = 0$; for $\lambda \neq 0$ one can then use the matrix $A - \lambda I$. So, take $\lambda = 0$. Observe that since

$$\text{nullity } A \geq k,$$

we have

$$\text{rank } A \leq n - k,$$

so that

$$\text{rank } \hat{A} \leq n - k.$$

Thus,

$$\text{nullity } \hat{A} \geq m - n + k.$$

□

Corollary 5.21. *Let A be a matrix of order n and let $\hat{A}$ be a principal submatrix of A of order m with no eigenvalue of geometric multiplicity strictly larger than l with $1 \leq l \leq m$. Then no eigenvalue of A has geometric multiplicity strictly larger than $n + l - m$.*

Proof. If λ is an eigenvalue of A with geometric multiplicity k strictly larger than $n + l - m$, then $k + m > n + l - m + m = n + l > n$. By Theorem 5.20, λ is an eigenvalue of $\hat{A}$ with geometric multiplicity $\geq k + m - n$, which is greater than $n + l - m + m - n = l$. This is a contradiction. □

With the use of both Theorems 5.2 and 5.20, the following result has a similar proof.

Corollary 5.22. *Let A be a matrix of order n with $n \geq 2$, and let $\hat{A}$ be a principal submatrix of A of order m with $m \geq 2$ such that $C_2(\hat{A})$ has disjoint Geršgorin discs. Then no eigenvalue of A has geometric multiplicity strictly larger than $n + 1 - m$.*

Example 5.23. Consider the following two matrices:

$$\hat{A} = \begin{bmatrix} -2 & 1/4 & 1/2 & 1/2 \\ 1/4 & 1 & -1 & 1/2 \\ -1/2 & -1/2 & 4 & 1/2 \\ 1/2 & -1/2 & 1 & 7 \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1/4 & 1/2 & 1/2 & * & * \\ 1/4 & 1 & -1 & 1/2 & * & * \\ -1/2 & -1/2 & 4 & 1/2 & * & * \\ 1/2 & -1/2 & 1 & 7 & * & * \\ * & * & * & * & * & * \\ * & * & * & * & * & * \end{bmatrix}.$$

$C_2(\hat{A})$ has disjoint Geršgorin discs; hence it cannot have any eigenvalue with geometric multiplicity strictly larger than 1. Thus, by Corollary 5.22, the matrix A cannot have any eigenvalue with geometric multiplicity strictly larger than $3 = 6 + 1 - 4$.

Remark 5.24. Corollary 5.22 provides an algorithm that may determine an upper bound for the maximum geometric multiplicity that the eigenvalues of a given matrix A can have. If this algorithm encounters any principal submatrix $\hat{A}$ of order m such that $C_2(\hat{A})$ has disjoint Geršgorin discs, then it concludes that no eigenvalue of A has geometric multiplicity strictly larger than $n + 1 - m$. When this algorithm is conclusive, it has two advantages: it does not require any knowledge about the eigenvalues, and it is not subject to round-off error since no multiplications or divisions are required.

The next theorem is a generalization of Theorem 3.9 in [5]

Theorem 5.25. *Let A be an $n \times n$ matrix and let $\|\cdot\|$ be a matrix norm. Suppose that $\hat{A}$ is an order $n-1$ principal submatrix of A such that $\left\|\hat{A}\right\| < \|A\|$, and suppose that λ is an eigenvalue of A such that $|\lambda| = \|A\|$. Then λ has algebraic multiplicity 1.*

Proof. Since $|\lambda| = \|A\|$, Problem 5.6.P38 of [2] ensures that the eigenvalue λ is semisimple (algebraic multiplicity equals the geometric multiplicity). Thus, it suffices to show that λ has geometric multiplicity 1. If λ has geometric multiplicity greater than 1, then Theorem 1.4.10 in [2] ensures that it is an eigenvalue of $\hat{A}$. But, then $|\lambda| \leq \left\|\hat{A}\right\| < \|A\|$, which is a contradiction. $\square$

Suppose that A is an $n \times n$ matrix, and $\hat{A}$ is a principal submatrix of A of order m with a principal submatrix B of order $m-1$ such that $\|B\| < \left\|\hat{A}\right\|$. Further assume that λ is an eigenvalue of A such that $|\lambda| = \left\|\hat{A}\right\|$. If the geometric multiplicity of λ as an eigenvalue of A is strictly larger than $n - m + 1$, then according to Theorem 5.20, λ would be an eigenvalue of $\hat{A}$ with geometric multiplicity strictly larger than 1, which is in contradiction with Theorem 5.25. Thus the geometric multiplicity of λ as an eigenvalue of A must be less than or equal to $n - m + 1$.

Theorem 5.26. *Let A be an $n \times n$ matrix, let $\|\cdot\|$ be a matrix norm, and let $\hat{A}$ be a principal submatrix of A of order m with a principal submatrix B of order $m - 1$ such that $\|B\| < \left\|\hat{A}\right\|$. Further assume that λ is an eigenvalue of A such that $|\lambda| = \left\|\hat{A}\right\|$. Then the geometric multiplicity of λ as an eigenvalue of A must be less than or equal to $n - m + 1$.*

The following theorem is derived from the Theorem 5.20, along with Theorems 2.2, and 5.2:

Theorem 5.27. *Let A be a matrix of order n, and let λ be an eigenvalue of A with geometric multiplicity k. Then*

1− *λ is in at least m + k - n Geršgorin discs of every principal submatrix $\hat{A}$ of A, of order m, with $n - k < m \leq n$.*

2− *λ is in at least m + k - n Geršgorin discs of $C_{m+k-n}(\hat{A})$ for every principal submatrix $\hat{A}$ of*

A, of order m*, with* $n - k < m \leq n$.

6 More applications and consequences of Theorem 5.2

6.1 An upper bound on the absolute value of an eigenvalue

From the Geršgorin theory, it is known that if λ is an eigenvalue of the $n \times n$ matrix A, then $|\lambda| \leq \|A\|_{\infty}$. We give an extension of this result.

Theorem 6.1. *Let A be an $n \times n$ matrix. For each integer k, $1 \leq k \leq n$, construct the $n \times n$ matrix C_k in the following way: in every row of A, replace the smallest $k-1$ off-diagonal entries in absolute value by zeros. Let $\alpha_1(C_k) \geq \alpha_2(C_k) \geq ... \geq \alpha_n(C_k)$ be the absolute row-sums of C_k in non-increasing order. Let λ be an eigenvalue of A. If the geometric multiplicity of λ is k, then*

$$|\lambda| \leq \alpha_k(C_k). \tag{4}$$

Also, $\alpha_k(C_k),\ k = 1, 2, ...$ is a non-increasing sequence.

Proof. If λ has geometric multiplicity k, then according to Theorem 5.2, there exist at least k Geršgorin discs of $C_k(A)$, say $D_{i_1}(C_k), D_{i_2}(C_k), ..., D_{i_k}(C_k)$ with centers c_{i_1,i_1}, c_{i_2,i_2}, ..., c_{i_k,i_k}, and radii R'_{i_1}, R'_{i_2}, ..., R'_{i_k} such that

$$|\lambda - c_{i_j,i_j}| \leq R'_{i_j}, \quad j = 1, 2, ...k.$$

So,

$$|\lambda| \;\leq\; R'_{i_j} \;+\; |c_{i_j,i_j}|, \quad j = 1, 2, ...k.$$

Hence,

$$|\lambda| \;\leq\; \alpha_k(C_k).$$

Generally, for each i, $1 \;\leq\; i \;\leq n$, if $1 \;\leq\; p \;\leq\; q \;\leq n$, then the i^{th} disk of C_p is larger or of the same size as the i^{th} disk of C_q, so

$$q \;>\; p \;\;\Longrightarrow\;\; \alpha_q(C_q) \;\leq\; \alpha_p(C_q) \;\leq\; \alpha_p(C_p),$$

so that $\alpha_q(C_q) \;\leq\; \alpha_p(C_p)$. □

Remark 6.2. Since in each row of A, several entries may be equal, several choices for C_k may be possible, but the absolute row-sum $\alpha_i(C_k)$ is the same for all those choices.

The following corollary follows from the fact that A and its transpose matrix have the same Jordan form:

Corollary 6.3. *Let A be a matrix of order n, and let λ an eigenvalue of A with geometric multiplicity k. Let C_k and $\alpha_i(C_k)$ be defined as in Theorem 6.1, and let $\beta_i(C_k)$ be the absolute column-sum of the i^{th} column of C_k, for $1 \;\leq\; i \;\leq n$. Then*

$$|\lambda| \;\leq\; \min\left(\alpha_k(C_k), \beta_k(C_k)\right).$$

For some matrices the upper bound in the previous formula can be an explicit function of k.

Application 6.4. Let A be an $n \times n$ matrix such that

$$|a_{ii}| = \beta, \quad i = 1, 2, ..., n$$

$$|a_{ij}| = \alpha, \quad i \neq j.$$

If λ is an eigenvalue of A with geometric multiplicity k, then

$$|\lambda| \leq \alpha_k(C_k) = \beta + (n - k)\alpha.$$

Example 6.5. $n = 5$, $\beta = 0$, $\alpha = 1$

Here, $k = 5$ is impossible, since this requires that all off-diagonal entries must be zero, which is not the case. Now,

$$k = 1 \quad \Longrightarrow \quad |\lambda| \leq 4.$$

This is consistent with the fact that $|\lambda|$ cannot exceed any matrix norm of A.

If it happens that A has eigenvalue λ with geometric multiplicity $k \geq 2$, then

$$k = 2 \quad \Longrightarrow \quad |\lambda| \leq 3.$$

Example 6.6. Consider

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

Now,

$$\lambda_1 = 4, \quad k_1 = 1, \quad |\lambda_1 = 4| \leq \alpha_1(C_1) = 4$$

$$\lambda_2 = -1, \quad k_2 = 4, \quad |\lambda_2 = -1| \leq \alpha_4(C_4) = 1.$$

Application 6.7. Let $\{P_{i,1}, P_{i,2}, \ldots, P_{i,n-1}\} = \{1, 2, \ldots, n-1\}$ for each $i = 1, 2, \ldots, n$. Let $A = [a_{ij}]$ be an $n \times n$ matrix such that

$$|a_{ij}| = rP_{ij} \text{ for } i \neq j, \text{ for some positive number r,}$$

$$\{a_{11}, a_{22}, ..., a_{nn}\} = \{x_1, x_2, \ldots, x_n\}, \text{ with } |x_1| \geq |x_2| \geq ... \geq |x_n|.$$

If λ is an eigenvalue of A with geometric multiplicity k, then

$$|\lambda| \leq \alpha_k(C_k) = |x_k| + (\frac{n(n-1) - k(k-1)}{2})r.$$

Example 6.8.

$$A = C_1(A) = \begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & 1 & -3 & 1 \\ -1 & -2 & 0 & 3 \\ -1 & -2 & -3 & 0 \end{bmatrix}, \quad C_2(A) = \begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & 1 & -3 & 0 \\ 0 & -2 & 0 & 3 \\ 0 & -2 & -3 & 0 \end{bmatrix}$$

$$k = 1 \implies |\lambda| \leq 7$$

$$k = 2 \implies |\lambda| \leq 6$$

In fact $|\lambda| = 2$ is an eigenvalue of A with geometric multiplicity $k = 2$

Application 6.9. Let $\{P_{i,1}, P_{i,2}, \ldots, P_{i,n-1}\} = \{1, 2, \ldots, (n-1)\}$ for each $i = 1, 2, \ldots, n$. Let $A = [a_{ij}]$ be an $n \times n$ matrix such that

$$|a_{ij}| = rP_{ij}^2 \ for \ i \neq j, \text{ for some positive number r,}$$

$$\{a_{11}, a_{22}, ..., a_{nn}\} = \{x_1, x_2, \ldots, x_n\}, \text{ with } |x_1| \geq |x_2| \geq \ldots \geq |x_n|.$$

If λ is an eigenvalue of A with geometric multiplicity k, then

$$|\lambda| \leq \alpha_k(C_k) = |x_k| + (\frac{n(n-1)(2n-1) - k(k-1)(2k-1)}{6})r.$$

6.2 Non-real eigenvalues of real matrices

In [4], we proved the following.

Theorem 6.10. *Let A be a real matrix of order n, and let q be a positive integer with $q \geq 1$. If no more than q Geršgorin discs of A form a connected region, then no non-real number can be an* *eigenvalue of A with geometric multiplicity larger than $\frac{q}{2}$.*

An alternative to this theorem is:

Theorem 6.11. *Let A be a real matrix of order n, and let k be a positive integer such that $1 \leq k \leq n$. For a non-real number λ to be an eigenvalue of A with geometric multiplicity k or more, at least $k+1$ Geršgorin discs of $C_k(A)$ must form a connected region, and λ is in the intersection of at least k of them.*

Proof. We choose an arbitrary principal submatrix B_1 of A of order $n-k+1$. By Theorem 5.19, λ is an eigenvalue of B_1, and hence in one of it's G-discs. In the corresponding row (say row r) of the matrix A, we can replace the smallest $k-1$ off-diagonal entries in absolute value by zeros, so that λ is in the associated G-disc of the new matrix. Next, delete row and column r from A to obtain a principal submatrix A_2 of A (which can be considered as A_1) of order $n-1$. Choose an arbitrary principal submatrix B_2 of A_2 of order $n-k+1$ and continue as above. We continue this process until we reach a principal submatrix A_k of order $n-(k-1) = n-k+1$, and repeat the procedure on A_k. This completes k steps in which we have replaced in k rows of A the smallest $k-1$ off-diagonal entries in absolute value by zeros, and λ is in each of the corresponding G-discs. But, since A_k is a real matrix with a non-real eigenvalue λ, the conjugate of λ is also an eigenvalue of A_k, and so λ must be in a connected region formed by at least two Geršgorin discs of A_k. This provides another row of A in addition to the previous k rows, in which we can also replace the smallest $k-1$ off-diagonal entries in absolute value by zeros. Hence, λ is in the connected region formed by the corresponding $k+1$ discs. Finally, in each of the remaining $n-k-1$ rows, also replace the smallest $k-1$ off-diagonal entries in absolute value by zeros. $\square$

Corollary 6.12. *Let A be a real matrix of order n, and let k be an integer with $k \geq 1$. If no more than k Geršgorin discs of C_k form a connected region, then no non-real number can be an eigenvalue of A with geometric multiplicity k or larger.*

When a real matrix A is sparse, its G-discs and those of $C_k(A)$ tend to have similar sizes, so that Theorem 6.10 can be conclusive whereas Theorem 6.11 may not be.

Example 6.13.

$$\text{Let } A = \begin{bmatrix} 1 & \sqrt{2} & 0 & 0 & 0 \\ -\sqrt{2} & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & 0 & b_{21} & b_{22} \end{bmatrix}, \quad \text{and let } B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

For the matrix $A = C_2$, the complex number i is an eigenvalue and the first 3 rows are associated with 3 connected discs. Hence, Theorem 6.11 does not allow us to conclude that the geometric multiplicity of i does not exceed 1. However, assuming that the entries of the matrix B are such that the two remaining discs of A are disconnected from the 3 previous ones, Theorem 6.10 is conclusive about the fact that the geometric multiplicity of i does not exceed 1.

Also, if we replace all or some of the zeros in A by real numbers small enough in absolute values for the discs associated with the last 2 rows of A to be still disconnected from the others, then one of the eigenvalues of A may be a non-real number equal or close to the number i, since in general, the eigenvalues of a matrix are continuous functions of its entries. In this case, Theorem 6.10 is still conclusive about the fact that the geometric multiplicity of any non-real eigenvalue of A does not exceed 1, but not Theorem 6.11 since the first 3 discs are still connected in both A and C_2.

When the matrices are dense, the difference in size between the discs of A and $C_k(A)$ for some integer k can be large enough for the Theorem 6.11 to be conclusive whereas Theorem 6.10 may not be.

Example 6.14. Consider

$$A = \begin{bmatrix} 4.7 & 1 & 0.21 & 0.59 \\ 0 & 1.5 & 0 & 1.25 \\ 1 & 0.4 & 4.3 & 0.4 \\ 0 & -1 & 0 & 0.5 \end{bmatrix}.$$

The eigenvalues of A are: $\lambda_1 = 1 + i, \ \ \lambda_2 = 1 - i \ \ \lambda_3 = 4, \ \ \lambda_4 = 5.$

By looking just at the Geršgorin region of A, Theorem 6.10 does not allow us to conclude that A has no non-real eigenvalue with geometric multiplicity larger than 1. But indeed we can deduce the latter fact by looking at the Geršgorin region of C_2 in view of Theorem 6.11. See Figure 1

6.3 Connections with powers of matrices

The next theorem was proved in [4].

Theorem 6.15. *Let A be an $n \times n$ complex matrix and suppose the non-zero complex number β has exactly s different roots $\lambda_1, \ldots, \lambda_s$ of order m, each of which is an eigenvalue of A. Then, β is an eigenvalue of A^m and letting $h = \sum_{j=1}^{s}$ geom mult(λ_j), β is in at least h Geršgorin discs of A^m.*

We obtain a similar theorem using the matrix $C_h(A^m)$.

Theorem 6.16. *Let A be an $n \times n$ complex matrix and suppose the non-zero complex number β has exactly s different roots $\lambda_1, \ldots, \lambda_s$ of order m, each of which is an eigenvalue of A. Then, β is an eigenvalue of A^m and letting $h = \sum_{j=1}^{s}$ geom mult(λ_j), β is in at least h Geršgorin discs of $C_h(A^m)$.*

Proof. As in the proof of Theorem 6.15, the geometric multiplicity of β as an eigenvalue of A^m is h. Hence, the result follows by Theorem 5.2. □

Example 6.17. Consider the following matrix, which has 1 as an eigenvalue:

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 1 & -3 & 0 \\ 1 & -3 & 5 \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} 6 & -14 & 24 \\ -2 & 10 & 4 \\ 3 & -5 & 29 \end{bmatrix}.$$

Notice that -1 is also included in some of the G-discs of A, but that fact does not tell us that this number is not an eigenvalue of A. However, if -1 together with 1 are eigenvalues of A, then the geometric multiplicity of 1 as an eigenvalue of A^2 should be at least 2; thus, according to Theorem 6.15, 1 should be in at least 2 discs of A^2 which is not the case. Hence, -1 is not an eigenvalue of A.

Example 6.18. Now consider the following matrix A for which 1 is an eigenvalue:

$$A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & -2 & 0 \\ 1 & -2 & 4 \end{bmatrix}.$$

Then

$$A^2 = \begin{bmatrix} 5 & -7 & 15 \\ -1 & 5 & 3 \\ 3 & -3 & 19 \end{bmatrix}, \ \ and \ \ C_2(A^2) = \begin{bmatrix} 5 & 0 & 15 \\ 0 & 5 & 3 \\ 0 & -3 & 19 \end{bmatrix}$$

Notice that 1 is also in 2 discs of A^2, so that Theorem 6.15 may not help us to decide that -1 is not an eigenvalue of A. However 1 is only in one disc of $C_2(A^2)$; thus, according to Theorem 6.16, the number -1 cannot be an eigenvalue of A.

6.4 Connections with the rank of a matrix

We first mention a result from [2]: If A is a square matrix and $|a_{ii}| > R_i'(A)$ for more than m values of i, then the rank of A is greater than m.

Theorem 6.19. *Let A be an $n \times n$ matrix, and let k be an integer with $1 \leq k \leq n$. If $|a_{ii}| > R_i'(C_k)$ for more than $n-k$ values of i, then the rank of A is strictly greater than $n-k$, in other words, the geometric multiplicity of 0 as an eigenvalue of A cannot exceed $k-1$.*

Proof. If the geometric multiplicity of 0 as an eigenvalue of A exceeds $k-1$, then according to Theorem 5.2, 0 must be in at least k Geršgorin discs of C_k; this means that 0 is strictly outside of at most $n-k$ discs of C_k, which is not the case. $\square$

Example 6.20. Consider

$$A = C_1(A) = \begin{bmatrix} 3 & 1 & 1 & 1 \\ 1 & 3 & -1 & -1 \\ 2 & -1 & 4 & -1 \\ 1 & 1 & -2 & 2 \end{bmatrix}, \quad C_2(A) = \begin{bmatrix} 3 & 1 & 1 & 0 \\ 1 & 3 & -1 & 0 \\ 2 & -1 & 4 & 0 \\ 0 & 1 & -2 & 2 \end{bmatrix}.$$

The first column in A is a sum of the remaining columns, which indicates that the number 0 is an eigenvalue of A. However, in $C_2(A)$, the first 3 diagonal entries are strictly larger than the absolute sum of the off-diagonal entries in the corresponding rows, which allows us to conclude, according to the theorem above, that 0 cannot have geometric multiplicity larger than 1. Hence, the rank of A is equal to 3.

7 A characterization of geometric multiplicity

We now characterize when a number λ is an eigenvalue with geometric multiplicity k, for any square matrix A.

Theorem 7.1. *Let A be an $n \times n$ matrix and let λ be a complex number. Then λ is an eigenvalue of A with geometric multiplicity k if and only if for all matrices B similar to A, λ is in at least k Geršgorin discs of B, and λ is in exactly k Geršgorin discs of B for at least one matrix B similar to A.*

Proof. We have that A is similar to J_A, which for any $\alpha \neq 0$ is similar to the matrix B_α, where all the Jordan blocks

$$\begin{bmatrix} \lambda_i & 1 & & \\ & \ddots & \ddots & \\ & & & 1 \\ & & & \lambda_i \end{bmatrix}$$

in J_A are replaced by

$$\begin{bmatrix} \lambda_i & \alpha & & \\ & \ddots & \ddots & \\ & & & \alpha \\ & & & \lambda_i \end{bmatrix}$$

in B_α.

To prove the necessity, since λ is in the Geršgorin region of every matrix B_α, then for every $\alpha > 0$, there exists an integer j such that $|\lambda_j - \lambda| \leq \alpha$. This implies that $\lambda = \lambda_i$ for some i. Hence, λ is an eigenvalue of A.

Since λ is in exactly k Geršgorin discs of some matrix similar to A, Theorem 2.2 ensures that the geometric multiplicity of λ cannot be strictly greater than k. Since λ cannot be in fewer than k

Geršgorin discs of any matrix similar to A, Theorem 3.8 ensures that the geometric multiplicity of λ cannot be strictly less than k. Thus, the geometric multiplicity of λ is k.

The converse easily follows from Theorems 2.2 and 3.8. $\square$

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