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Mikhail Stroev

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by

MIKHAIL STROEV

Under the Direction of Frank Hall, PhD

ABSTRACT

The first part of this dissertation answers the questions posed in the article "A note on permanents and generalized complementary basic matrices", *Linear Algebra Appl.* 436 (2012), by M. Fiedler and F. Hall. Further results on permanent compounds of generalized complementary basic matrices are obtained. Most of the results are also valid for the determinant and the usual compound matrix. Determinant and permanent compound products which are intrinsic are also considered, along with extensions to total unimodularity.

The second part explores some connections of dense alternating sign matrices with total unimodularity, combined matrices, and generalized complementary basic matrices.

In the third part of the dissertation, an explicit formula for the ranks of dense alternating sign matrices is obtained. The minimum rank and the maximum rank of the sign pattern of a dense alternating sign matrix are determined. Some related results and examples are also provided.

INDEX WORDS: Generalized complementary basic matrix, Permanent, Intrinsic product, Alternating sign matrix, Dense matrix, Totally unimodular matrix, Combined matrix, Sign pattern matrix, Minimum rank, Maximum rank.

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DEDICATION

This dissertation is dedicated to my family.

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LIST OF ABBREVIATIONS

- CB-matrix - Complementary Basic matrix
- GCB-matrix - Generalized Complementary Basic matrix
- ASM - Alternating Sign Matrix

PART 1

INTRODUCTION

Complementary basic matrices were introduced by M. Fiedler in [15] in 2004 and then generalized by Fiedler and Hall in [16, 19, 20, 22, 21]. In particular, they initiated the study of permanent compounds of such matrices in [22] and raised some questions about their properties. These questions form the basis of the research discussed in part two of this dissertation. In particular, we show that permanent compounds of generalized complementary basic matrices have a block-diagonal structure if the order of indices is properly chosen. We also describe the sizes and the number of such blocks and give the values of nonzero entries for the case when the distinguish blocks of generalized complementary basic matrices are blocks of 1's.

Substantial interest in alternating sign matrices in the mathematics community originated from the alternating sign matrix conjecture of Mills et al. [30] in 1983 and has continued in several combinatorial directions. In [6] the authors initiated a study of the zero-nonzero patterns of $n \times n$ alternating sign matrices. In next two parts of this dissertation we investigate alternating sign matrices with dense structure. In part three, we first determine the connection of such matrices with total unimodularity and combined matrices. In particular, every row dense and every column dense alternating sign matrix is totally unimodular and the combined matrix of every nonsingular dense alternating sign matrix is an alternating sign matrix. Next, we consider generalized complementary basic matrices whose generators are alternating sign matrices. We show that such generalized complementary basic matrices are alternating sign matrices and make a connection between combined matrices and generalized complementary basic matrices.

In part four, we explore the ranks of alternating sign matrices and give an explicit formula for the ranks. The special dense alternating sign matrices which are rectangular shaped play a crucial role. An important part of the combinatorial matrix theory is the study of sign pattern matrices, which has been the focus of extensive research for the last 50 years ([10], [27]). In the last section of part four, we consider the sign patterns of dense alternating sign matrices and determine their minimum and maximum ranks.

Many related results and examples are also provided in this dissertation. The work in this dissertation formed a basis for the three published papers [24], [23], and [18].

PART 2

PERMANENTS, DETERMINANTS, AND GENERALIZED COMPLEMENTARY BASIC MATRICES

2.1 Introduction

In [20] and [22] the complementary basic matrices, CB-matrices for short, (see [15], [16], [19]) were extended in the following way. Let A_1, A_2, ..., A_s be matrices of respective orders $k_1,\ k_2,\ \ldots, k_s$, $k_i \geq 2$ for all i. Denote $n = \sum_{i=1}^{s} k_i - s + 1$, and form the block diagonal matrices $G_1, G_2, \ldots, G_s$ as follows:

$$G_1 = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-k_1} \end{bmatrix},\ G_2 = \begin{bmatrix} I_{k_1-1} & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & I_{n-k_1-k_2+1} \end{bmatrix}, \ldots,$$

$$G_{s-1} = \begin{bmatrix} I_{n-k_{s-1}-k_s+1} & 0 & 0 \\ 0 & A_{s-1} & 0 \\ 0 & 0 & I_{k_s-1} \end{bmatrix},\ G_s = \begin{bmatrix} I_{n-k_s} & 0 \\ 0 & A_s \end{bmatrix}.$$

Then, for any permutation $(i_1, i_2, \ldots, i_s)$ of $(1, 2, \ldots, s)$, we can consider the product

$$G_{i_1} G_{i_2} \cdots G_{i_s} \tag{2.1}$$

We call products of this form *generalized complementary basic matrices*, GCB-matrices for short. We have continued to use the notation $\prod G_k$ for these more general products. The diagonal blocks A_k are called *distinguished blocks* and the matrices G_k are called *generators* of $\prod G_k$. (In the CB-matrices, these distinguished blocks are all of order 2.) Let us also remark that strictly speaking, every square matrix can be considered as a (trivial) GCB-matrix with $s = 1$.

Let A be an $n \times n$ real matrix. Then the *permanent* of A is defined by

$$\mathrm{per}(A) = \sum a_{1i_1} a_{2i_2} \cdots a_{ni_n}$$

where the summation extends over all the $n-$permutations $(i_1, i_2, \ldots, i_n)$ of the integers $1, 2, \ldots, n$. So, $\mathrm{per(A)}$ is the same as the determinant function apart from a factor of ± 1 preceding each of the products in the summation. As pointed out in [4], certain determinantal laws have direct analogues for permanents. In particular, the Laplace expansion for determinants has a simple counterpart for permanents. But the basic law of determinants

$$\det(AB) = \det(A)\det(B) \tag{2.2}$$

is flagrantly false for permanents. The latter fact is the case even for intrinsic products (see Section 3), as was observed in [22] in the example

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

GCB-matrices have many striking properties such as permanental, graph theoretic, spectral, and inheritance properties (see for example [22], [21], and [24]). In particular, in [22], it was proved that

$$\mathrm{per}(AB) = \mathrm{per}(A)\mathrm{per}(B)$$

holds for products which are GCB-matrices.

Theorem 2.1.1. *Suppose the integers $n,\ k$ satisfy $n > k > 1$. Let A_0 be a matrix of order k, B_0 be a matrix of order $n-k+1$ (the sum of the orders of A_0 and B_0 thus exceeds n by one). Then,* *for the $n \times n$ matrix AB, where*

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & I_{n-k} \end{bmatrix} \textit{ and } B = \begin{bmatrix} I_{k-1} & 0 \\ 0 & B_0 \end{bmatrix},$$

we have that

$$\mathrm{per}(AB) = \mathrm{per}(A)\mathrm{per}(B). \tag{2.3}$$

This result was then extended to the GCB-matrices.

Corollary 2.1.2. *Independent of the ordering of the factors, for the generalized complementary basic matrix $\prod G_k$, we have that*

$$\mathrm{per}(\prod G_k) = \prod \mathrm{per}(G_k).$$

Hence, to compute the permanent of a GCB-matrix, we only need to compute the permanents of matrices of smaller sizes.

The case where each $A_k,\ k = 1, \cdots, s$, is an all 1's matrix was also considered.

Corollary 2.1.3. *Independent of the ordering of the factors, for the generalized complementary basic matrix $\prod G_k$ where each A_k matrix is an all 1's matrix, we have that*

$$\mathrm{per}(\prod G_k) = (k_1!)(k_2!)\cdots(k_s!).$$

The purpose of this part is to answer the questions posed in [22]. Further results on permanent compounds of generalized complementary basic matrices are also obtained. Most of the results are also valid for the determinant and the usual compound matrix. Determinant and permanent compound products which are intrinsic are considered as well (see Section 3).

2.2 Permanent compounds

For an $n \times n$ matrix A and index sets $\alpha, \beta \subseteq \{1, ..., n\}$, $A(\alpha, \beta)$ denotes the submatrix of A that lies at the intersection of the rows indexed by α and the columns indexed by β. We simply let $A(\alpha)$ denote the principal submatrix of A that lies in the rows and columns indexed by α. The usual h^{th} *compound matrix* of A, denoted by $C_h(A)$, is the matrix of order $\binom{n}{h}$ whose entries are $\det(A(\alpha, \beta))$, where α and β are of cardinality h. Similarly, the h^{th} *permanent compound matrix* of A, denoted by $P_h(A)$, is the matrix of order $\binom{n}{h}$ whose entries are $\operatorname{per}(A(\alpha, \beta))$, where α and β are of cardinality h. There are many possibilities for ordering the family of index sets of cardinality h. Usually, the lexicographic ordering is preferred and this will be the understood order unless otherwise specified. When a different ordering is used, we obtain a compound matrix permutationally similar to $P_h(A)$, or $C_h(A)$ (in lexicographic order).

To illustrate the latter, let us simply consider the case $h = 2$. If

$$A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$$

is a direct sum of two square matrices A_1 and A_2, then it can be seen that $C_2(A)$ is permutationally similar to

$$\begin{bmatrix} C_2(A_1) & 0 & 0 \\ 0 & A_1 \otimes A_2 & 0 \\ 0 & 0 & C_2(A_2) \end{bmatrix},$$

where $A_1 \otimes A_2$ is the Kronecker product of A_1 and A_2. Also, $P_2(A)$ is permutationally similar to

$$\begin{bmatrix} P_2(A_1) & 0 & 0 \\ 0 & A_1 \otimes A_2 & 0 \\ 0 & 0 & P_2(A_2) \end{bmatrix}.$$

We next recall the multiplicativity of the usual compound matrix:

$$C_h(AB) = C_h(A)C_h(B).$$

In contrast, we do not have the same property for permanent compounds.

In [22] a number of interesting related papers, including [1], [12], and [14], were cited. Specifically, for compound matrices, the authors in [1] show that for nonnegative $n \times n$ matrices A and B

$$P_h(AB) \geq P_h(A)P_h(B). \tag{2.4}$$

Now (2.4) implies for nonnegative matrices that we have

$$\text{per}((AB)(\alpha)) \geq \text{per}(A(\alpha))\text{per}(B(\alpha)), \tag{2.5}$$

for any index set $\alpha \subseteq \{1, ..., n\}$. The inequality (2.5) was also shown in [12].

Let the cardinality of the set α be denoted by h. As mentioned in [22] it is straightforward to show that for matrices A and B as in Theorem 2.1.1, and for $h = 1, 2$, and n, we in fact have equality in (2.5). The result for $h = n$ actually follows from Theorem 2.1.1. The following question was then raised. For GCB-matrices, to what extent can we prove equality in (2.4) and (2.5) for the other values of h, namely $h = 3, \ldots, n-1$? One of the purposes of this section is to answer this question.

Regarding (2.5), we can answer the question in the affirmative. Referring to matrices A and B in Theorem 2.1.1, let us write

$$A_0 = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ & \cdots & \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}$$

and

$$B_0 = \begin{bmatrix} b_{kk} & \cdots & b_{kn} \\ & \cdots & \\ b_{nk} & \cdots & b_{nn} \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} a_{11} & \cdots & a_{1,k-1} & a_{1k}b_{kk} & \cdots & a_{1k}b_{kn} \\ a_{21} & \cdots & a_{2,k-1} & a_{2k}b_{kk} & \cdots & a_{2k}b_{kn} \\ & \cdots & & & \cdots & \\ a_{k1} & \cdots & a_{k,k-1} & a_{kk}b_{kk} & \cdots & a_{kk}b_{kn} \\ & & & b_{k+1,k} & \cdots & b_{k+1,n} \\ & & & & \cdots & \\ & & & b_{n,k} & \cdots & b_{n,n} \end{bmatrix}. \tag{2.6}$$

Theorem 2.2.1. *In the notation of Theorem 2.1.1, for any index set* $\alpha \subseteq \{1, ..., n\}$*, we have*

$$\mathrm{per}(AB(\alpha)) = \mathrm{per}(A(\alpha))\mathrm{per}(B(\alpha)). \tag{2.7}$$

Proof. We can divide the proof into cases, each of which is easy to prove:

(i) $\alpha \subseteq \{1, ..., k\}$ with two subcases $\alpha \subseteq \{1, ..., k-1\}$ and $k \in \alpha$

(ii) $\alpha \subseteq \{k, ..., n\}$ with two subcases $\alpha \subseteq \{k+1, ..., n\}$ and $k \in \alpha$

(iii) $\alpha \cap \{1, ..., k-1\} \neq \emptyset$ and $\alpha \cap \{k+1, ..., n\} \neq \emptyset$.

Here, if $k \in \alpha$, the proof follows from the result of Theorem 2.1.1; if $k \notin \alpha$, it is very easy. The arguments for these cases can be done by analyzing the matrix in (2.6).

□

We then have a variation of Corollary 2.1.2.

Corollary 2.2.2. *Independent of the ordering of the factors, for the generalized complementary* *basic matrix* $\prod G_k$*, for any index set* $\alpha \subseteq \{1, ..., n\}$*, we have that*

$$\mathrm{per}((\prod G_k)(\alpha)) = \prod \mathrm{per}(G_k(\alpha)).$$

Proof. We use induction with respect to s. If $s = 2$, the result follows from Theorem 2.2.1. Suppose that $s > 2$ and that the result holds for $s-1$ matrices. Observe that the matrices G_i and G_k commute if $|i-k| > 1$. This means that if 1 is before 2 in the permutation $(i_1, i_2, \ldots, i_s)$, we can move G_1 into the first position without changing the product. The product Π of the remaining $s-1$ matrices G_k has the form

$$\Pi = G_{j_2} \cdots G_{j_s} = \begin{bmatrix} I_{k_1-1} & 0 \\ 0 & B_0 \end{bmatrix},$$

where $(j_2, \cdots, j_s)$ is a permutation of $(2, \cdots, s)$. By the induction hypothesis,

$$\mathrm{per}((\Pi)(\alpha)) = \mathrm{per}((G_{j_2})(\alpha)) \cdots \mathrm{per}((G_{j_s})(\alpha)),$$

where we can view $G_2, G_3, \ldots, G_s$ as $s-1$ generators of an $n \times n$ GCB-matrix. Then by Theorem 2.2.1,

$$\mathrm{per}((\prod G_k)(\alpha)) = \mathrm{per}((G_1\Pi)(\alpha))$$

$$= \mathrm{per}((G_1)(\alpha))\mathrm{per}((\Pi)(\alpha)) = \prod \mathrm{per}((G_k)(\alpha)).$$

If 1 is behind 2 in the permutation, we can move G_1 into the last position without changing the product. The previous proof then applies to the transpose of the product. Since the permanent of a matrix and its transpose are the same, the proof of this case can proceed as follows:

$$\mathrm{per}((\prod G_k)(\alpha)) = \mathrm{per}((\Pi G_1)(\alpha)) = \mathrm{per}([(\Pi G_1)(\alpha)]^T)$$

$$= \mathrm{per}((\Pi G_1)^T(\alpha)) = \mathrm{per}(({G_1}^T \Pi^T)(\alpha)) = \prod \mathrm{per}((G_{i_k}^T)(\alpha))$$

$$= \prod \mathrm{per}([(G_{i_k})(\alpha)]^T) = \prod \mathrm{per}((G_{i_k})(\alpha)) = \prod \mathrm{per}((G_k)(\alpha)).$$

□

Corollary 2.2.3. *If all the distinguished blocks A_k have positive principal permanental minors, then independent of the ordering of the factors, the generalized complementary basic matrix $\prod G_k$ has positive principal permanental minors.*

We now start the development of some further results on permanent compounds of GCB-matrices.

Lemma 2.2.4. *For $n \times n$ matrices A and B as in Theorem 2.1.1 and different index sets α, β of the same cardinality we have that*

(i.) $A(\alpha, \beta)$ *has a zero line if α and β differ by at least one index in the set $\{k+1, \ldots, n\}$, and*

(ii.) $B(\alpha, \beta)$ *has a zero line if α and β differ by at least one index in the set $\{1, \ldots, k-1\}$.*

Proof. We prove $(i.)$; the proof of $(ii.)$ is similar. By assumption, without loss of generality, there exists $i \in \alpha \cap \{k+1, \ldots, n\}$ such that $i \notin \beta$. So, $A(\alpha, \beta)$ cannot contain the 1 in the (i, i) position of A (since $i \notin \beta$). Hence, the corresponding row of $A(\alpha, \beta)$ is a zero row. □

Since two different index sets of the same cardinality differ by at least one index in the set $\{k+1, \ldots, n\}$ or in the set $\{1, \ldots, k-1\}$, we immediately obtain the following.

Theorem 2.2.5. *For $n \times n$ matrices A and B as in Theorem 2.1.1 and different index sets α, β of the same cardinality we have that*

(i.) $A(\alpha, \beta)$ *or* $B(\beta, \alpha)$ *always has a zero line, and*

(ii.) $A(\alpha, \beta)$ *or* $B(\alpha, \beta)$ *always has a zero line.*

Corollary 2.2.6. *For $n \times n$ matrices A and B as in Theorem 2.1.1 and different index sets α, β of the same cardinality we have that*

$$\mathrm{per}(A(\alpha, \beta))\mathrm{per}(B(\beta, \alpha)) = 0$$

and also

$$\mathrm{per}(A(\alpha, \beta))\mathrm{per}(B(\alpha, \beta)) = 0,$$

so that

$$P_h(A) \circ P_h(B)$$

is a diagonal matrix for any $1 \leq h \leq n$, where $\circ$ denotes the Hadamard product.

The *right normal form of the product* $\prod G_k$ is useful in many situations. This form is obtained as follows. We again use the fact that the matrices G_i and G_k commute if $|i - k| > 1$. Thus we can move G_1 in $(\prod G_k)$ to the left as far as possible without changing the product, ie., either to the first position, or until it meets G_2. In the latter case, move the pair $G_2 G_1$ as far to the left as possible, and continue until such product is in front. Then take the largest remaining index and move the corresponding G to the left, etc. In this way, we arrive at the right normal form of the product $(\prod G_k)$:

$$\prod G_k = (G_{s_1} \cdots G_2 G_1)(G_{s_2} \cdots G_{s_1+2} G_{s_1+1}) \cdots (G_s \cdots G_{s_r+2} G_{s_r+1}),$$

where the *terms* are the expressions in the parentheses. Splitting the terms at the end of some term (except the last) into two parts yields a product of two matrices A and B as in Theorem 2.1.1. We then obtain the following corollary of the previous result.

Corollary 2.2.7. *Split the terms of the right normal form of the generalized complementary basic matrix $\prod G_k$ into a product of two matrices A and B as in Theorem 2.1.1. Then for different index sets α, β of the same cardinality we have that*

$$\mathrm{per}(A(\alpha, \beta))\mathrm{per}(B(\beta, \alpha)) = 0$$

and also

$$\mathrm{per}(A(\alpha,\beta))\mathrm{per}(B(\alpha,\beta)) = 0,$$

so that

$$P_h(A) \circ P_h(B)$$

is a diagonal matrix for any $1 \leq h \leq n$*.*

We now look more specifically at how the $P_h(A)$ and $P_h(B)$ for A and B in Theorem 2.1.1 are constructed.

Theorem 2.2.8. *For* $n \times n$ *matrices* A *and* B *as in Theorem 2.1.1 and any* $1 \leq h \leq n$*, we have the following:*

(i.) $P_h(A)$ *is permutationally similar to a block diagonal matrix with* $\binom{n-k}{h-i}$ *diagonal blocks of order* $\binom{k}{i}$*, for* $i = 0, 1, \ldots, h$*, and*

(ii.) $P_h(B)$ *is permutationally similar to a block diagonal matrix with* $\binom{k-1}{i}$ *diagonal blocks of order* $\binom{n-k+1}{h-i}$*, for* $i = 0, 1, \ldots, h$*.*

(As usual, $\binom{a}{b} = 0$ *if* $b > a$ *or* $b < 0$*.)*

Proof. For the purpose of this proof, we call the indices in the set $\{1, \ldots, k-1\}$ green indices and indices in the set $\{k+1, \ldots, n\}$ red indices. We first prove $(i.)$ and fix h, $1 \leq h \leq n$. Consider index sets α, β of the same cardinality h. Observe by Lemma 2.2.4 that $A(\alpha, \beta)$ has a zero line if α and β differ by at least one red index.

Choose any $i \in \{0, 1, \ldots, h\}$, fix some $h - i$ red indices, and then make all possible $\binom{k}{i}$ choices of non-red indices. We then obtain $\binom{k}{i}$ different index sets of cardinality h where any two of them have exactly those same red indices. Keeping these index sets together yields a diagonal submatrix of order $\binom{k}{i}$.

Next, observe that in this way we then obtain $\binom{n-k}{h-i}$ diagonal blocks of order $\binom{k}{i}$, where any two of them are associated with different subsets of red indices. This completes the proof of part $(i.)$.

Note that

$$\binom{n}{h} = \sum_{i=0}^{h} \binom{k}{i}\binom{n-k}{h-i},$$

which holds for any fixed $k \in \{0, 1, \dots, n\}$ (with our matrices A and B, $k \in \{2, \dots, n-1\}$).

The proof of part $(ii.)$ is similar to the proof of $(i.)$. By Lemma 2.2.4, $B(\alpha, \beta)$ has a zero line if α and β differ by at least one green index. In this case, we choose any $i \in \{0, 1, \dots, h\}$, fix some i green indices, and then make all possible $\binom{n-k+1}{h-i}$ choices of non-green indices, thereby obtaining $\binom{n-k+1}{h-i}$ different index sets of cardinality h where any two of them have exactly those same green indices. We thus obtain $\binom{k-1}{i}$ diagonal blocks of order $\binom{n-k+1}{h-i}$, where any two of them are associated with different subsets of green indices. That completes the proof of $(ii.)$.

Observe that

$$\binom{n}{h} = \sum_{i=0}^{h} \binom{n-k+1}{h-i}\binom{k-1}{i},$$

which also holds for any fixed $k \in \{0, 1, \dots, n\}$. $\square$

Observation 2.2.9. *Since lexicographical ordering meets the requirements of the proof of part $(ii.)$ of Theorem 2.2.8, ie. for each choice of i green indices the index sets with those same green indices are grouped together, $P_h(B)$ itself is a block diagonal matrix.*

Example 2.2.10. We give an example illustrating the choice of the index sets in part (i) of Theorem 2.2.8. The corresponding block diagonal matrix should be clear. In this example $n = 5$ and $k = 3$, so that A_0 is 3×3. The green indices are in the set $\{1, 2\}$, while the red indices are in the set $\{4, 5\}$. We exhibit the collections of index sets for $P_3(A)$ (which is of order 10) in respective groups associated with the diagonal blocks in part $(i.)$ of Theorem 2.2.8 (we use parentheses for the individual index sets):

$i = 0:\ \ \emptyset$

$i = 1:\ \ \{(1,4,5), (2,4,5), (3,4,5)\}$

$i = 2:\ \ \{(1,2,4), (1,3,4), (2,3,4)\},\ \ \{(1,2,5), (1,3,5), (2,3,5)\}$

$i = 3:\ \ \{(1,2,3)\}$

The order for the four groups of indices does not matter; any order determines a block diagonal matrix permutation similar to $P_3(A)$. Also, within each group, we can place the index sets in any order.

Of special interest is the case where A_0 and B_0 are blocks of 1's. In this case the exact values of the entries in the blocks follow from Theorem 2.2.8 and its proof.

Corollary 2.2.11. *For $n \times n$ matrices A and B as in Theorem 2.1.1 where A_0 and B_0 are blocks of 1's and any $1 \leq h \leq n$, we have the following:*

(i.) *$P_h(A)$ is permutationally similar to a block diagonal matrix with $\binom{n-k}{h-i}$ diagonal blocks of order $\binom{k}{i}$, for $i = 0, 1, \ldots, h$, and where for a given value of i each entry in a corresponding diagonal block is $i!$.*

(ii.) *$P_h(B)$ is permutationally similar to a block diagonal matrix with $\binom{k-1}{i}$ diagonal blocks of order $\binom{n-k+1}{h-i}$, for $i = 0, 1, \ldots, h$, and where for a given value of i each entry in a corresponding diagonal block is $(h-i)!$.*

Proof. As in the proof of Theorem 2.2.8, for a given value of i, fixing some $h - i$ red indices contributes 1's on the diagonals of submatrices A_i of A of order h. Then the i choices of non-red indices yield order i all 1's principal submatrices of the A_i matrices. Since per$(A_i) = i!$, the result in part $(i.)$ follows. The proof of part $(ii.)$ is similar. $\square$

Example 2.2.12. Referring to Example 2.2.10 above, using A_0 and B_0 as blocks of 1's, for the corresponding block diagonal matrix for $P_3(A)$ we have one 3×3 block with all entries equal to 1, two 3×3 blocks with all entries equal to 2, and one 1×1 block with entry 6.

We note that the extensions such as for Theorem 2.2.8 and Corollary 2.2.11 to the generalized complementary basic matrix $\prod G_k$ using the right normal form should be clear.

Observation 2.2.13. *All of the above results in this section (with the exception of the ones involving all 1's in A_0, B_0) hold for the determinant and the usual compound matrix $C_h(A)$ of minors.*

Remark 2.2.14. For equality in (2.4), we have a counterexample for 3×3 matrices using 2×2 distinguished diagonal blocks, ie, using CB-matrices. Specifically, using distinguished blocks of 1's, we get

$$P_2(AB) = \begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix},$$

while

$$P_2(A)P_2(B) = \begin{bmatrix} 2 & 2 & 0 \\ 1 & 1 & 2 \\ 1 & 1 & 2 \end{bmatrix},$$

differing in only the $(1,3)-$entry.

Furthermore, note the block diagonal forms

$$P_2(A) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}, \quad P_2(B) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

We have given explicit characterizations of $P_h(A)$ and $P_h(B)$ for $n \times n$ matrices A and B as in Theorem 2.1.1, and in terms of these characterizations, we have made extensions to the generalized complementary basic matrices $\prod G_k$. In the case that A_0 and B_0 are blocks of 1's, we can give the structure of $P_h(AB)$ in terms of permutation equivalence.

Theorem 2.2.15. *For $n \times n$ matrices A and B as in Theorem 2.1.1 where A_0 and B_0 are blocks of 1's and any $1 \leq h \leq n$, $P_h(AB)$ is permutationally equivalent to the matrix, which consists of*

$$\binom{n-k}{i}\binom{k}{h-i} \times \binom{k-1}{j}\binom{n-k+1}{h-j}$$

blocks with all the entries equal to

$$(h-j)\cdots(h-j-i+1)(h-i)!$$

for a given (i,j)*-block,* $i,j \in \{0,1,\ldots,h\}$.

Proof. By (2.6) of Section 2, for A_0 and B_0 as blocks of 1's, AB is an $n \times n$ matrix of the form

$$AB = \begin{bmatrix} J_1 & J_2 \\ 0 & J_3 \end{bmatrix}$$

where the block of zeros has the size $n-k \times k-1$ and the J_i denote all 1's matrices of corresponding sizes.

As in the proof of Theorem 2.2.8, we call the indices in the set $\{1,\ldots,k-1\}$ green indices and indices in the set $\{k+1,\ldots,n\}$ red indices. Observe that the block of zeros of AB is in the rows with red indices and in the columns with green indices.

Choose any $i \in \{0,1,\ldots,h\}$, make all possible $\binom{n-k}{i}$ choices of red indices, and then for a given choice of i red indices make all possible $\binom{k}{h-i}$ choices of non-red indices. We obtain $\binom{n-k}{i}\binom{k}{h-i}$ different index sets of cardinality h, which are all the possible index sets such that the submatrix of AB has exactly i rows with zeros.

Next, choose any $j \in \{0,1,\ldots,h\}$, make all possible $\binom{k-1}{j}$ choices of green indices, and then for a given choice of j green indices make all possible $\binom{n-k+1}{h-j}$ choices of non-green indices. We obtain $\binom{k-1}{j}\binom{n-k+1}{h-j}$ different index sets of cardinality h, which are all the possible index sets such that the submatrix of AB has exactly j columns with zeros.

Thus, for all $i,j \in \{0,1,\ldots,h\}$. we obtain the

$$\binom{n-k}{i}\binom{k}{h-i} \times \binom{k-1}{j}\binom{n-k+1}{h-j}$$

block in the permuted $P_h(AB)$ such that each entry in this block is equal to

$$\operatorname{per}\begin{bmatrix} J & J \\ 0_{i\times j} & J \end{bmatrix}_{h\times h}$$

Next, by Laplace expansion

$$\operatorname{per}\begin{bmatrix} J & J \\ 0_{i\times j} & J \end{bmatrix}_{h\times h} = (h-j)\cdots(h-j-i+1)(h-i)!$$

which completes the proof. □

2.3 Intrinsic products

Following [16], we say that the product of a row vector and a column vector is *intrinsic* if there is at most one non-zero product of the corresponding coordinates. Analogously we speak about the intrinsic product of two or more matrices, as well as about *intrinsic factorizations* of matrices. The entries of the intrinsic product are products of (some) entries of the multiplied matrices. Thus there is no addition; we could also call intrinsic multiplication *sum-free multiplication*.

Observation 2.3.1. *Intrinsic multiplication is (in general) not associative.*

Indeed, if $a = (a_1, a_2, a_3)^T$, $b = (b_1, b_2, b_3)^T$ are ”full” vectors, then for the identity matrix I_3 the products $a^T I_3$ and $I_3 b$ are intrinsic but $a^T I_3 b$ is not intrinsic.

Observation 2.3.2. *Let A, B, C be matrices such that the product ABC is intrinsic in the sense that in every entry $(ABC)_{i\ell}$ (of the form $\sum_{j,k} a_{ij} b_{jk} c_{k\ell}$) there is at most one non-zero term. If A has no zero column and C no zero row, then both products AB and BC are intrinsic.*

Remark 2.3.3. In general, when ABC, AB, and BC are all intrinsic, we say that the product ABC is *completely intrinsic*, and this will be used even for more than three factors.

As was already observed in [20], independent of the ordering of the factors, the GCB-matrices $\prod G_k$ are completely intrinsic.

We now return to compound matrices.

Theorem 2.3.4. *For $n \times n$ matrices A and B as in Theorem 2.1.1 and any $1 \leq h \leq n$, the product $C_h(A)C_h(B)$ is intrinsic.*

Proof. Let

$$\alpha = \{i_1, \ldots, i_{s-1}, i_s, i_{s+1}, \ldots, i_h\},$$

where

$$\{i_1, \ldots, i_s\} \subseteq \{1, \ldots, k\}, \ \ \{i_{s+1}, \ldots, i_h\} \subseteq \{k+1, \ldots, n\}$$

and

$$\beta = \{j_1, \ldots, j_t, j_{t+1}, j_{t+2}, \ldots, j_h\},$$

where

$$\{j_1, \ldots, j_t\} \subseteq \{1, \ldots, k-1\}, \ \ \{j_{t+1}, \ldots, j_h\} \subseteq \{k, \ldots, n\}.$$

We are looking for index sets γ of cardinality h which satisfy two conditions:

(i.) $A(\alpha, \gamma)$ does not necessarily have a zero line, and

(ii.) $B(\gamma, \beta)$ does not necessarily have a zero line.

Now, by Lemma 2.2.4, $(i.)$ implies that γ and α have the same indices in the set $\{k+1, \ldots, n\}$, and $(ii.)$ implies that γ and β have the same indices in the set $\{1, \ldots, k-1\}$. Hence, $\{j_1, \ldots, j_t, i_{s+1}, \ldots, i_h\} \subseteq \gamma$ and index k may or may not be in γ.

If $k \notin \gamma$, then γ is uniquely determined as $\gamma = \{j_1, \ldots, j_t, i_{s+1}, \ldots, i_h\}$, which also implies that $t = s$.

If $k \in \gamma$, then γ is uniquely determined as $\gamma = \{j_1, \ldots, j_t, k, i_{s+1}, \ldots, i_h\}$, which implies that $t = s - 1$.

Since we cannot have both $t = s$ and $t = s-1$, there exists a unique γ which satisfies both $(i.)$ and $(ii.)$. Hence, the $(\alpha, \beta)-$entry has at most one nonzero term, namely $[C_h(A)]_{(\alpha,\gamma)}[C_h(B)]_{(\gamma,\beta)}$.

□

As in previous cases, this result can be extended to the product $\prod G_k$.

Corollary 2.3.5. *For any $1 \leq h \leq n$, independent of the ordering of the factors, for the generalized complementary basic matrix $\prod G_k$, we have that the product $\prod C_h(G_k)$ is completely intrinsic.*

Remark 2.3.6. Since square matrices which have a zero line have both determinant and permanent equal to zero, Theorem 2.3.4 also holds for permanent compounds: For $n \times n$ matrices A and B as in Theorem 2.1.1 and any $1 \leq h \leq n$, the product $P_h(A)P_h(B)$ is intrinsic.

We next formulate a generalization of intrinsic products. Let A and B be $n \times n$ matrices. We say that the product AB is *totally intrinsic* if the determinant of every square submatrix of AB is either zero, or a product of two determinants, one of a square submatrix of A, the second of a square submatrix of B.

Since $C_h(AB) = C_h(A)C_h(B)$, by Theorem 2.3.4 we immediately have the following:

Theorem 2.3.7. *For $n \times n$ matrices A and B as in Theorem 2.1.1, the product AB is totally intrinsic.*

Corollary 2.3.8. *Independent of the ordering of the factors, for the generalized complementary basic matrix $\prod G_k$, the determinant of every square submatrix of $\prod G_k$ is either zero, or a product of some determinants of submatrices of the G_k, in fact, at most one determinant from each G_k.*

Next, we recall a definition, see [4]. An $m \times n$ integer matrix A is *totally unimodular* if the determinant of every square submatrix is $0, 1$ or -1. The last corollary then implies that total unimodularity is an inherited property:

Corollary 2.3.9. *Independent of the ordering of the factors, for the generalized complementary basic matrix $\prod G_k$, if each of the distinguished blocks A_k is totally unimodular, then $\prod G_k$ is totally unimodular.*

We can note that this inheritance works in a more general sense: if all $\det(A_k)$ are in a sub-semi-group $\mathcal{S}$ of the complex numbers, then $\prod G_k$ is totally unimodular with respect to $\mathcal{S}$.

Next, we shall use Remark 2.3.6 and a version of the Cauchy-Binet theorem (see [29]) to establish a further result on permanent compounds.

Lemma 2.3.10. *If an $n \times n$ matrix A contains a $p \times q$ block of zeros with $p + q > n$, then* $\mathrm{per(A)} = 0$.

Proof. Since A has a $p \times q$ block of zeros with $p + q > n$, the minimum number of lines that cover all the nonzero enties in A is less then or equal to $n - p + n - q$, which is less than n. So, by the Theorem of Konig, see [4], the maximum number of nonzero entries in A with no two of the nonzero entries on a line is less than n. Hence, $\mathrm{per(A)} = 0$. □

Theorem 2.3.11. *For $n \times n$ matrices A and B as in Theorem 2.1.1 and any index sets α and β of the same cardinality h, where $1 \leq h \leq n$, we have the following:*

$$[P_h(A)P_h(B)]_{(\alpha,\beta)} = \begin{cases} [P_h(AB)]_{(\alpha,\beta)} = 0, & \text{if } \sigma_{\alpha,\beta} > h; \\ [P_h(AB)]_{(\alpha,\beta)}\,, & \text{if } \sigma_{\alpha,\beta} = h-1 \text{ or } h; \\ 0, & \text{if } \sigma_{\alpha,\beta} < h-1, \end{cases}$$

where $\sigma_{\alpha,\beta}$ is the number of indices in the set

$$(\alpha \cap \{k+1, \ldots, n\}) \cup (\beta \cap \{1, \ldots, k-1\}).$$

In the third case where $\sigma_{\alpha,\beta} < h-1$, $[P_h(AB)]_{(\alpha,\beta)}$ may or may not be equal to 0.

Proof.

For the proof we will use the the Binet-Cauchy Theorem for permanents (see [29]). First, we introduce a new family of index sets, $G_{h,n}$, which consists of all nondecreasing sequences of h integers chosen from $\{1,\ldots,n\}$. We will also use the previous family of strictly increasing sequences of h integers chosen from $\{1,\ldots,n\}$. We will denote this latter set by $Q_{h,n}$.

Now, since $[AB](\alpha,\beta) = A(\alpha,\{1,\ldots,n\})B(\{1,\ldots,n\},\beta)$, by the Binet-Cauchy Theorem for permanents, we get

$$[P_h(AB)]_{(\alpha,\beta)} = \sum_{\gamma\in G_{h,n}} \frac{[P_h(A)]_{(\alpha,\gamma)}\,[P_h(B)]_{(\gamma,\beta)}}{\mu(\gamma)},$$

where $\mu(\gamma)$ is the product of factorials of the multiplicities of distinct integers appearing in the sequence γ.

On the other hand, the (α,β) entry of $[P_h(A)P_h(B)]$ can be written as

$$[P_h(A)P_h(B)]_{(\alpha,\beta)} = \sum_{\gamma\in Q_{h,n}} [P_h(A)]_{(\alpha,\gamma)}\,[P_h(B)]_{(\gamma,\beta)}\,.$$

We will denote by γ^* the set of indices in $G_{h,n}$ or $Q_{h,n}$, such that both $[P_h(A)]_{(\alpha,\gamma^*)}$ and $[P_h(B)]_{(\gamma^*,\beta)}$ do not equal to zero.

Next, let $\alpha = \{i_1,\ldots,i_s,i_{s+1},\ldots,i_h\}$, where

$$\{i_{s+1},\ldots,i_h\} = \alpha\cap\{k+1,\ldots,n\}$$

and $\beta = \{j_1,\ldots,i_t,j_{t+1},\ldots,j_h\}$, where

$$\{j_1,\ldots,j_t\} = \beta\cap\{1,\ldots,k-1\},$$

which implies $\sigma_{\alpha,\beta} = h-s+t$.

Observe further that although Lemma 2.2.4 was formulated for index sets from $Q_{h,n}$, the similar assertions are true for index sequences from $G_{h,n}$, as well. Hence, γ^* must contain $\{j_1,\ldots,j_t\}$ and $\{i_{s+1},\ldots,i_h\}$ together in both cases of $Q_{h,n}$ and $G_{h,n}$.

Next, we observe that if $\gamma \in G_{h,n}$ contains a repeating index from the set $\{i_{s+1}, \dots, i_h\}$, then $A(\alpha, \gamma)$ has a $p \times q$ block of zeros with $p + q > h$. Similarly, if $\gamma \in G_{h,n}$ contains a repeating index from the set $\{j_1, \dots, j_t\}$, then $B(\gamma, \beta)$ has a $p \times q$ block of zeros with $p + q > h$. By Lemma 2.3.10, this implies that $\mathrm{per}(\mathrm{A}(\alpha, \gamma)) = 0$ or $\mathrm{per}(\mathrm{B}(\gamma, \beta)) = 0$. Hence, γ^* cannot contain repeating indices other than k.

Now, we consider all possible cases for the values of $\sigma_{\alpha,\beta}$ and exhibit the explicit form for a γ^* index sequence.

Case1. $\sigma_{\alpha,\beta} > h$. In this case there are no γ^* index sequences in either $Q_{h,n}$ or $G_{h,n}$, which implies $[P_h(A)P_h(B)]_{(\alpha,\beta)} = [P_h(AB)]_{(\alpha,\beta)} = 0$.

Case2.

Subcase 2.1 $\sigma_{\alpha,\beta} = h$. Here, γ^* is uniquely determined as

$\gamma^* = \{j_1, \dots, j_t, i_{s+1}, \dots, i_h\}$ in both $Q_{h,n}$ and $G_{h,n}$, with $\mu(\gamma^*) = 1$.

Subcase 2.2 $\sigma_{\alpha,\beta} = h - 1$. Here, γ^* is uniquely determined as

$\gamma^* = \{j_1, \dots, j_t, k, i_{s+1}, \dots, i_h\}$ in both $Q_{h,n}$ and $G_{h,n}$, with $\mu(\gamma^*) = 1$.

Hence, for any α and β which satisfy $\sigma_{\alpha,\beta} = h - 1$ or h, we get

$$[P_h(A)P_h(B)]_{(\alpha,\beta)} = [P_h(AB)]_{(\alpha,\beta)} = \mathrm{per}(\mathrm{A}(\alpha, \gamma^*))\mathrm{per}(\mathrm{B}(\gamma^*, \beta)).$$

Case3. $\sigma_{\alpha,\beta} < h - 1$. In this case there are no γ^* index sequences in $Q_{h,n}$ and there is a unique $\gamma^* = \{j_1, \dots, j_t, k, \dots, k, i_{s+1}, \dots, i_h\}$ in $G_{h,n}$ where index k appears $h - \sigma_{\alpha,\beta}$ times. Hence, $[P_h(A)P_h(B)]_{(\alpha,\beta)} = 0$ while

$$[P_h(AB)]_{(\alpha,\beta)} = \frac{\mathrm{per}(\mathrm{A}(\alpha, \gamma^*))\mathrm{per}(\mathrm{B}(\gamma^*, \beta))}{\mu(\gamma^*)}$$

which is not equal to zero in general.

□

Observation 2.3.12. *We note that Theorem 2.2.1 is a special case of Theorem 2.3.11.*

2.4 Remarks

We recall that in Remark 2.2.14, $P_2(A)P_2(B)$ and $P_2(AB)$ differed only in the second super-diagonal position. With the use of Theorem 2.3.11, one can extend this fact to $n \times n$ matrices A and B as in Theorem 2.1.1 and any $1 \leq h < n$ and obtain the following. With respect to a certain hierarchical ordering of the index sets, $P_h(A)P_h(B) - P_h(AB)$ is permutationally similar to a block upper-triangular matrix with both the block diagonal and first block super-diagonal consisting entirely of zero blocks.

An even more explicit determination of $P_h(\prod G_k)$ appears to be formidable in general, even for just three generators.

The eigenvalues of the GCB-matrix $\prod G_k$ is still an intriguing question. In [17], the eigenvectors of the usual compound matrix $C_h(A)$ are obtained as "exterior products" of the eigenvectors of the matrix A. A further research project is the determination of the eigenvalues and eigenvectors of the permanent compound $P_h(A)$.

The results of this part were presented in [24].

PART 3

DENSE ALTERNATING SIGN MATRICES

3.1 Introduction

An *alternating sign matrix*, henceforth abbreviated ASM, is an $n \times n$ $(0, +1, -1)$-matrix without zero rows and columns, such that the $+1$s and -1s alternate in each row and column, beginning and ending with a $+1$, see [6]. The substantial interest in ASMs in the mathematics community originated from the alternating sign matrix conjecture of Mills et al. [30] in 1983 and has continued in several combinatorial directions. In [6] the authors initiated a study of the zero - nonzero patterns of $n \times n$ alternating sign matrices. In this work, some connections of alternating sign matrices with total unimodularity, combined matrices, and generalized complementary basic matrices are explored.

3.2 Total Unimodularity

We define a matrix to be *dense* (*row-dense, column-dense*) if there are no zeros between two non-zero entries for every line (row, column) of this matrix.

Now, we call in a real matrix two non-zero entries in a line (i.e., in a row or in a column) *neighbors* if the only entries between them are zeros. We also say that such matrix is *completely alternating* (*row-alternating*, *column-alternating*) if any two non-zero neighbors in any line (any row, any column) of this matrix have opposite signs.

We recall that a *totally unimodular matrix* is an integer matrix in which the determinant of every square submatrix is 0, 1, or -1, see [4].

There is a close connection between totally unimodular matrices and a certain type of oriented graph. We recall that a *tree* is a connected graph that contains no cycles. Let T be a tree of order n with vertices $a_1, a_2, \ldots, a_n$ and edges $\alpha_1, \alpha_2, \ldots, \alpha_{n-1}$. We suppose that the edges of T have been oriented. Let (s_i, t_i),

$(i = 1, 2, \ldots, l)$ be l ordered pairs of vertices of T. We set $m_{ij} = 1$ if the unique (directed) path γ in T from s_i to t_i uses the edge α_j in its assigned direction, we set $m_{ij} = -1$ if γ uses the edge α_j in the direction opposite to its assigned direction, and we set $m_{ij} = 0$ if γ does not use the edge α_j. The resulting $(0, 1, -1)$-matrix

$$M = [m_{ij}], (i = 1, 2, \ldots, l; j = 1, 2, \ldots, n-1)$$

of size l by $n-1$ is called a *network matrix* (Tutte[1965]).

The following is a well-known result and is also Theorem 2.3.6 in [4].

Lemma 3.2.1. *A network matrix M corresponding to the oriented tree T is a totally unimodular matrix.*

Theorem 3.2.2.

(i.) Every row-dense $(0, 1)$-matrix M is totally unimodular.

(ii.) Every row-dense row-alternating $(0, 1, -1)$-matrix M is totally unimodular.

Proof. Let M have n columns. We show that M is a network matrix in both cases. Then the proof follows from Lemma 3.2.1.

For (i), let T be the oriented path

$$v_1 \to v_2 \to v_3 \to \cdots \to v_{n+1}.$$

Now, if some row of matrix M has 1s only in the positions from k to $k+m$, then this row corresponds to the oriented path from v_k to v_{k+m+1}.

For (ii), we define T to be the oriented path

$$v_1 \to v_2 \leftarrow v_3 \to \cdots - v_{n+1}$$

(the orientation of the last arc (v_n, v_{n+1}) depends on the parity of n). Let some row of matrix M have nonzero entries $\alpha_k, \ldots, \alpha_{k+m}$ in the positions from k to $k+m$. Then this row corresponds to the oriented path from v_k to v_{k+m+1} if $(-1)^{k+1}\alpha_k = 1$, or to the path from v_{k+m+1} to v_k if $(-1)^{k+1}\alpha_k = -1$.

$\square$

Corollary 3.2.3. *Every dense* $(0,1)$*-matrix is totally unimodular.*

Corollary 3.2.4. *Every dense ASM is totally unimodular.*

Remark 3.2.5. The theorem of Seymour[1982] asserts that a totally unimodular matrix which is not a network matrix, the transpose of a network matrix, or one of two exceptional matrices admits a "diagonal decomposition" into smaller totally unimodular matrices, [31].

We point out that in general an ASM might not be totally unimodular.

Example 3.2.6. Using the construction on pp 3-4 of [6], we found the following ASM

$$A = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

which is not totally unimodular, since the determinant of the submatrix
$A(\{2,5\},\{2,5\})$ is 2.

3.3 Combined Matrices

The goal in this section is to explore the connection between nonsingular dense alternating sign matrices and combined matrices.

Theorem 3.3.1. *The adjoint matrix of every square row-dense* $(0,1)$*-matrix is a column-alternating* $(0,1,-1)-$*matrix.*

Proof. Let A be an $n \times n$ row-dense $(0,1)-$matrix. Write A in terms of its rows $A = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$.

Denote the adjoint matrix of A by B and write it in terms of its columns $B = [b_1|\dots|b_n]$. That B is a $(0,1,-1)-$matrix follows from Theorem 3.2.2. Suppose the j^{th} column of B is not alternating; say, without loss of generality, it has two 1s as neighbors in the k^{th} and $(k+m)^{th}$ positions:

$$b_j = [\dots,1,0,\dots,0,1,\dots]^T.$$

Consider matrix A' obtained from A by substitution of row a'_j for row a_j, where a'_j is dense with 1s in the positions from the k^{th} to the $(k+m)^{th}$:

$$a'_j = [0,\dots,0,1,\dots,1,0,\dots,0].$$

Then the j^{th} column of the adjoint matrix of A' is b_j, and

$$a'_j b_j = \mathrm{detA'} = 2$$

which contradicts Theorem 3.2.2.

□

Corollary 3.3.2. *The adjoint matrix of every square dense* $(0,1)$*-matrix is a completely alternating* $(0,1,-1)-$*matrix.*

The following is straightforward.

Lemma 3.3.3. *A is a dense ASM, if and only if SAS is a dense* $(0,1)$*-matrix, where S is either* $\mathrm{diag}(1,-1,1,\dots)$ *or* $\mathrm{diag}(\mathrm{i},-\mathrm{i},\mathrm{i},\dots)$.

We will use the shorthand notation to write $(A^{-1})^T$ as A^{-T} for a nonsingular matrix A. We then recall that for a nonsingular matrix A, the *combined matrix* is $A \circ A^{-T}$ where $\circ$ denotes Hadamard product; see [25] for properties of combined matrices.

Theorem 3.3.4. *The combined matrix of every nonsingular dense (0,1)-matrix is an ASM.*

Proof. Let A be a nonsingular dense (0,1)-matrix. By Corollary 3.3.2, the inverse of A as well as A^{-T} are completely alternating. Thus also the combined matrix $A \circ A^{-T}$ is a $(0, 1, -1)$ completely alternating matrix. All its row- and column-sums are equal to one. It follows that $A \circ A^{-T}$ is an ASM since every line must begin and end with a one. $\square$

Corollary 3.3.5. *The combined matrix of every nonsingular dense ASM is an ASM.*

Proof. Let A be a nonsingular dense ASM. By Lemma 3.3.3, $SAS = B$, where B is a nonsingular dense (0,1)-matrix. So, $A = SBS$ and it is straightforward that

$$A \circ A^{-T} = B \circ B^{-T}.$$

Then the result follows by Theorem 3.3.4. $\square$

Observation 3.3.6. *For a nonsingular dense ASM, the proof of Corollary 3.3.5 shows that the combined matrix is the same as the combined matrix of the corresponding (0,1)-matrix.*

Example 3.3.7. We consider two examples of nonsingular dense ASM matrices. For D_6, the combined matrix is a permutation matrix and for $E_{9,6}$ (9×9 dense ASM with the 1 in the 6^{th} position in the 1^{st} row) the combined matrix has a more complicated structure. In these two examples, we use $+$ for 1 and $-$ for -1.

$$D_6 = \begin{bmatrix} 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & + & - & + & 0 \\ 0 & + & - & + & - & + \\ + & - & + & - & + & 0 \\ 0 & + & - & + & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 \end{bmatrix}, \quad D_6^{-1} = \begin{bmatrix} - & - & 0 & + & + & + \\ - & 0 & 0 & 0 & + & + \\ 0 & 0 & 0 & 0 & 0 & + \\ + & 0 & 0 & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 & - \\ + & + & + & 0 & - & - \end{bmatrix}$$

$$D_6 \circ D_6^{-T} = \begin{bmatrix} 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 \\ 0 & 0 & 0 & 0 & 0 & + \\ + & 0 & 0 & 0 & 0 & 0 \\ 0 & + & 0 & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 \end{bmatrix},$$

$$E_{9,6} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & + & - & + & 0 & 0 \\ 0 & 0 & 0 & + & - & + & - & + & 0 \\ 0 & 0 & + & - & + & - & + & - & + \\ 0 & + & - & + & - & + & - & + & 0 \\ + & - & + & - & + & - & + & 0 & 0 \\ 0 & + & - & + & - & + & 0 & 0 & 0 \\ 0 & 0 & + & - & + & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad E_{9,6}^{-1} = \begin{bmatrix} 0 & - & - & 0 & + & + & 0 & 0 & + \\ - & 0 & 0 & 0 & 0 & 0 & + & + & 0 \\ - & 0 & + & 0 & - & 0 & + & + & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + \\ + & 0 & - & 0 & + & 0 & - & 0 & + \\ + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & + & + & 0 & - & 0 & + & 0 & - \\ 0 & + & + & 0 & 0 & 0 & 0 & 0 & - \\ + & 0 & 0 & + & + & 0 & - & - & 0 \end{bmatrix}$$

$$E_{9,6} \circ E_{9,6}^{-T} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & + & 0 & 0 \\ 0 & 0 & 0 & 0 & + & 0 & - & + & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & + \\ 0 & 0 & + & 0 & - & 0 & + & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & + & - & 0 & + & 0 & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & + & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Remark 3.3.8. We mention that, in general, D_{2m} and its inverse have structures similar to D_6 and its inverse.

It cannot be expected that the result such as Corollary 3.3.5 holds for general ASM.

Example 3.3.9. Consider matrix $A = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$

Then

$$A^{-1} = \begin{bmatrix} -.5 & -.5 & 0 & .5 & 1 & .5 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ .5 & .5 & 0 & -.5 & 0 & .5 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ .5 & .5 & 0 & .5 & 0 & .5 \\ -.5 & .5 & 1 & .5 & 0 & -.5 \end{bmatrix}$$

and

$$A \circ A^{-T} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & .5 & 0 & .5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & .5 & 0 & .5 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Observation 3.3.10. *We observe that for the nonsingular alternating sign matrices in Examples 3.3.7 and 3.3.9, the row and column sums of the inverse are all 1. In fact, more generally, if a nonsingular matrix has equal row (column) sums $\alpha(\beta)$, it is easy to show that the inverse has equal row (column) sums $\frac{1}{\alpha}(\frac{1}{\beta})$.*

3.4 Connections with GCB-matrices

GCB-matrices were already discussed in part two; however, for completeness of this part we repeat key definitions and properties.

Let A_1, A_2, ..., A_s be matrices of respective orders $k_1, k_2, \ldots, k_s$, $k_i \geq 2$ for all i. Denote $n = \sum_{i=1}^{s} k_i - s + 1$, and form the block diagonal matrices G_1, G_2, ..., G_s as follows:

$$G_1 = \begin{bmatrix} A_1 & 0 \\ 0 & I_{n-k_1} \end{bmatrix}, \ G_2 = \begin{bmatrix} I_{k_1-1} & 0 & 0 \\ 0 & A_2 & 0 \\ 0 & 0 & I_{n-k_1-k_2+1} \end{bmatrix}, \ldots,$$

$$G_{s-1} = \begin{bmatrix} I_{n-k_{s-1}-k_s+1} & 0 & 0 \\ 0 & A_{s-1} & 0 \\ 0 & 0 & I_{k_s-1} \end{bmatrix}, G_s = \begin{bmatrix} I_{n-k_s} & 0 \\ 0 & A_s \end{bmatrix}$$. Then, for any permutation $(i_1, i_2, \ldots, i_s)$ of $(1, 2, \ldots, s)$, we can consider the product

$$G_{i_1} G_{i_2} \cdots G_{i_s}. \tag{3.1}$$

We call products of this form *generalized complementary basic matrices*, GCB-matrices for short. We use the notation $\prod G_k$ for these general products. The diagonal blocks A_k are called *distinguished blocks* and the G_k are called the *generators* of the $\prod G_k$. GCB-matrices have many striking properties such as permanental, graph theoretic, spectral, and inheritance properties (see for example [22], [21], and [24]).

Lemma 3.4.1. *Suppose the integers* $n,\ k$ *satisfy* $n > k > 1$. *Let*

$$A_0 = \begin{bmatrix} a_{11} & \cdots & a_{1k} \\ & \cdots & \\ a_{k1} & \cdots & a_{kk} \end{bmatrix}$$

be a $k \times k$ *matrix, and*

$$B_0 = \begin{bmatrix} b_{kk} & \cdots & b_{kn} \\ & \cdots & \\ b_{nk} & \cdots & b_{nn} \end{bmatrix}$$

be an $(n-k+1) \times (n-k+1)$ *matrix (the sum of the orders of* A_0 *and* B_0 *thus exceeds* n *by one). Then, for the* $n \times n$ *matrices*

$$A = \begin{bmatrix} A_0 & 0 \\ 0 & I_{n-k} \end{bmatrix} \tag{3.2}$$

and

$$B = \begin{bmatrix} I_{k-1} & 0 \\ 0 & B_0 \end{bmatrix}, \tag{3.3}$$

the product AB has the explicit form

$$AB = \begin{bmatrix} a_{11} & \cdots & a_{1,k-1} & a_{1k}b_{kk} & \cdots & a_{1k}b_{kn} \\ a_{21} & \cdots & a_{2,k-1} & a_{2k}b_{kk} & \cdots & a_{2k}b_{kn} \\ & \cdots & & & \cdots & \\ a_{k1} & \cdots & a_{k,k-1} & a_{kk}b_{kk} & \cdots & a_{kk}b_{kn} \\ 0 & \cdots & 0 & b_{k+1,k} & \cdots & b_{k+1,n} \\ & \cdots & & & \cdots & \\ 0 & \cdots & 0 & b_{n,k} & \cdots & b_{n,n} \end{bmatrix}. \tag{3.4}$$

The following result was shown in [21].

Theorem 3.4.2. *Suppose that the GCB-matrices A and B are given as in Lemma 3.4.1. If both A_0 and B_0 are ASMs, then AB and BA are ASMs.*

An extension was then made to more than two factors.

Theorem 3.4.3. *For the generalized complementary basic matrices, independent of the ordering of the factors, if each distinguished block A_k of the factors is an ASM, then $\prod G_k$ is an ASM.*

In [24] it is proved that total unimodularity is an inherited property:

Independent of the ordering of the factors, for the generalized complementary basic matrix $\prod G_k$, if each of the distinguished blocks A_k is totally unimodular, then $\prod G_k$ is totally unimodular.

So, ASM and total unimodularity are both inherited properties for GCB-matrices. We know that every dense ASM is actually a network matrix (from the proof of Theorem 3.2.2) . Although, the property of being a dense ASM is not an inherited property for GCB-matrices, we will show that the property of being a network matrix is an inherited property.

Theorem 3.4.4. *Suppose that the GCB-matrices A and B are given as in Lemma 3.4.1. If both A_0 and B_0 are network matrices, then both AB and BA are network matrices.*

Proof. We prove the assertion for AB; the proof for BA is similar. We have oriented tree T_{A_0} associated with network matrix A_0 and oriented tree T_{B_0} associated with network matrix B_0. To construct the oriented tree T associated with the matrix AB, we proceed as follows. Consider the last, ie the k^{th}, column of A_0. It is associated with some (oriented) edge α_k of T_{A_0}. Now, corresponding to the first row of B_0 is some ordered pair (s,t) of vertices of T_{B_0}. To obtain T, we simply insert the whole tree T_{B_0} into the tree T_{A_0} in the following manner: we replace the (oriented) edge α_k of T_{A_0} by the unique (oriented) path γ in T_{B_0} from vertex s to vertex t.

Notice, from (3.4) we have that

$$AB = \begin{bmatrix} A_0' & C \\ 0 & B_0' \end{bmatrix}, \tag{3.5}$$

where $A_0'(B_0')$ consists of the first (last) $k-1(n-k)$ columns (rows) of $A_0(B_0)$ and that the i^{th} row of C is

$$a_{ik}(b_{kk}b_{k,k+1}\cdots b_{kn})$$

(a_{ik} is of course 1, -1, or 0).

It is then easy to see that the matrix AB is indeed a network matrix corresponding to our constructed tree T. In particular, if the path in T_{A_0} corresponding to the i^{th} row of A_0 uses edge α_k, then the path in T corresponding to the i^{th} row of AB uses the path γ in the direction originally assigned to edge α_k. $\square$

Corollary 3.4.5. *Independent of the ordering of the factors, if each A_k is a network matrix, then the generalized complementary basic matrix $\prod G_k$ is a network matrix.*

Proof. We use induction with respect to s. If $s = 2$, the result follows from Theorem 3.4.4. Suppose that $s > 2$ and that the result holds for $s-1$ matrices. Observe that the matrices G_i and G_k commute if $|i-k| > 1$. This means that if 1 is before 2 in the permutation $(i_1, i_2, \ldots, i_s)$, we can move G_1 into the first position without changing the product. The product Π of the remaining

$s-1$ matrices G_k has the form

$$\begin{bmatrix} I_{k_1-1} & 0 \\ 0 & B_0 \end{bmatrix}.$$

By the induction hypothesis, B_0 is a network matrix. Hence, by Theorem 3.4.4, $\prod G_k = G_1\Pi$ is a network matrix.

If 1 is behind 2 in the permutation, we can move G_1 into the last position and then we have a BA product as in Theorem 3.4.4. □

We finally make a connection between combined matrices and generalized complementary basic matrices. To do this, a key preliminary property is needed.

Lemma 3.4.6. *Suppose that the GCB-matrices A and B are given as in Lemma 3.4.1, where both A_0 and B_0 are nonsingular. Then*

$$(AB) \circ (AB)^{-T} = (A \circ A^{-T})(B \circ B^{-T}).$$

Proof. Denote by X_{R_i} (X_{C_i}) the i^{th} row (column) of matrix X and by $X_{\bar{R}_i}$ ($X_{\bar{C}_i}$) the matrix obtained from X by deleting its i^{th} row (column). Then, using (3.4),

$$(AB) \circ (AB)^{-T} = \begin{bmatrix} (A_0)_{\bar{C}_k} \circ (A_0^{-T})_{\bar{C}_k} & (A_0)_{C_k}(B_0)_{R_1} \circ (A_0^{-T})_{C_k}(B_0^{-T})_{R_1} \\ \\ 0 & (B_0)_{\bar{R}_1} \circ (B_0^{-T})_{\bar{R}_1} \end{bmatrix}$$

and

$$(A \circ A^{-T})(B \circ B^{-T}) = \begin{bmatrix} (A_0 \circ A_0^{-T})_{\bar{C}_k} & (A_0 \circ A_0^{-T})_{C_k}(B_0 \circ B_0^{-T})_{R_1} \\ \\ 0 & (B_0 \circ B_0^{-T})_{\bar{R}_1} \end{bmatrix}$$

$$= \begin{bmatrix} (A_0)_{\bar{C}_k} \circ (A_0^{-T})_{\bar{C}_k} & \left((A_0)_{C_k} \circ (A_0^{-T})_{C_k}\right)\left((B_0)_{R_1} \circ (B_0^{-T})_{R_1}\right) \\ \\ 0 & (B_0)_{\bar{R}_1} \circ (B_0^{-T})_{\bar{R}_1} \end{bmatrix}.$$

Now, for columns x, y and rows v^T, w^T (each with n components), $(x \circ y)(v^T \circ w^T) = xv^T \circ yw^T$. Thus,

$$(AB) \circ (AB)^{-T} = (A \circ A^{-T})(B \circ B^{-T}).$$

□

Theorem 3.4.7. *Suppose that the GCB-matrices A and B are given as in Lemma 3.4.1, where both A_0 and B_0 are nonsingular. If $(A_0 \circ A_0^{-T})$ and $(B_0 \circ B_0^{-T})$ are ASMs, then $(AB) \circ (AB)^{-T}$ and $(BA) \circ (BA)^{-T}$ are ASMs.*

Proof. Observe that

$$A \circ A^{-T} = \begin{bmatrix} (A_0 \circ A_0^{-T}) & 0 \\ 0 & I \end{bmatrix} \text{ and } \mathrm{B} \circ \mathrm{B}^{-\mathrm{T}} = \begin{bmatrix} I & 0 \\ 0 & (B_0 \circ B_0^{-T}) \end{bmatrix}.$$

By Theorem 3.4.2, ASM is an inherited property. So, $(A \circ A^{-T})(B \circ B^{-T})$ is an ASM. Hence, by Lemma 3.4.6, $(AB) \circ (AB)^{-T}$ is an ASM. The proof that $(BA) \circ (BA)^{-T}$ is an ASM is similar. □

The proof of the following result is then similar to the proof of Corollary 3.4.5.

Corollary 3.4.8. *Independent of the ordering of the factors, if the combined matrix of each A_k is an ASM, then the combined matrix of the generalized complementary basic matrix $\prod G_k$ is an ASM.*

Now we can make a connection back to dense ASMs. Specifically, we recall that Corollary 3.3.5 tells us that the combined matrix of every nonsingular dense ASM is an ASM. Thus, we have the following.

Corollary 3.4.9. *Suppose that the GCB-matrices A and B are given as in Lemma 3.4.1. If both A_0 and B_0 are nonsingular dense ASMs, then both the combined matrices of AB and BA are ASMs.*

It is clear that the extension of Corollary 3.4.9 to more than two distinguished blocks follows directly from Corollary 3.4.8.

The results of this part were presented in [23].

PART 4

RANKS OF DENSE ALTERNATING SIGN MATRICES

4.1 Introduction

As it was discussed in the previous part, an *alternating sign matrix*, henceforth abbreviated ASM, is a square $(0, +1, -1)$-matrix without zero rows and columns, such that the $+1$s and -1s alternate in each row and column, beginning and ending with a $+1$, see [6]. Recently, it was proved in [5] that the diamond ASM D_n has the maximum spectral radius over the set of $n \times n$ ASMs, while in [11], inverses or generalized inverses of D_n are derived and the eigenvalues of D_n are considered. More recently, Brualdi and Kim have made even further progress on ASMs in [8], [9], [7].

A *sign pattern matrix*, or *sign pattern*, is a matrix whose entries are from the set $\{+, -, 0\}$. For a real matrix B, sgn(B) is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by $+$ (respectively, $-, 0$). For a sign pattern matrix A, the *qualitative class* of A, denoted $Q(A)$, is defined as

$$Q(A) = \{B : B \text{ is a real matrix and } \mathrm{sgn}(B) = A\}.$$

A square sign pattern matrix A is said to be *sign nonsingular* if every matrix in $Q(A)$ is nonsingular.

The *minimum rank* of a sign pattern matrix A, denoted mr(A), is the minimum of the ranks of the real matrices in $Q(A)$. Determination of the minimum rank of a sign pattern matrix in general is a longstanding open problem in combinatorial matrix theory. Recently, there has been a significant number of papers concerning this topic, for example [2, 3, 13, 26, 27, 28]. In particular, matrices realizing the minimum rank of a sign pattern have applications in the study of neural networks [13] and communication complexity [26].

The *maximum rank* of a sign pattern matrix A, denoted MR(A), is the maximum of the ranks of the real matrices in $Q(A)$. It should be clear that MR(A) is the maximum number of nonzero entries of A, no two of which are in the same row or in the same column. The maximum number of nonzero entries of A with no two of the nonzero entries in the same line (a row or column) is also known as the *term rank* of A ([4, 10]).

The following version of Konig's Theorem ([4]) provides another description of the maximum rank.

Theorem 4.1.1. *Let A be a sign pattern matrix. The minimal number of lines in A that cover all of the nonzero entries of A is equal to the maximal number of nonzero entries in A, no two of which are on the same line.*

In this part of the dissertation, an explicit formula for the ranks of dense alternating sign matrices is obtained. Formulas for the minimum ranks and the maximum ranks of the sign patterns of the dense alternating sign matrices are determined. Some related results and examples are also provided.

4.2 Preliminaries

As in part three, for integers n and k with $1 \leq k \leq n$, we denote by $E_{n,k}$ the $n \times n$ dense ASM whose $(1, k)$ entry is 1 and all of whose nonzero entries form a "rectangle" with vertices at the positions $(1, k)$, $(k, 1)$, $(n, n+1-k)$, and $(n+1-k, n)$. For example,

$$E_{6,4} = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix},$$

$$E_{8,3} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

By $E^+_{n,k}$ we mean the corresponding $(0,1)$-matrix obtained from $E_{n,k}$ by replacing all -1's with 1's.

Since $E_{n,k}$ and $E_{n,n-k+1}$ can be obtained from each other by arranging the columns in reverse order, for rank considerations, we may assume that $k \leq n-k+1$, namely, $k \leq \frac{n+1}{2}$, or $n \geq 2k-1$.

The following is straightforward ([23]).

Lemma 4.2.1. *$E^+_{n,k} = \pm S E_{n,k} S$, where $S = diag(1, -1, 1, -1, \ldots)$.*

It is also easily seen that every dense ASM can be obtained by replacing the 1's in a permutation matrix by matrices of the form $E_{n,k}$. For example,

$$\begin{bmatrix} 0 & E_{6,2} & 0 & 0 \\ 0 & 0 & E_{9,6} & 0 \\ E_{1,1} & 0 & 0 & 0 \\ 0 & 0 & 0 & E_{5,4} \end{bmatrix}$$

is a 21×21 dense ASM. We also note that $E_{1,1} = [1]$, so that any permutation matrix (which is a dense ASM) is obtained in this way.

In view of Lemma 4.2.1, we have the next result.

Lemma 4.2.2. *If A is a dense alternating sign matrix, then A is diagonally equivalent to a dense $(0,1)-$matrix.*

Thus, to determine the rank of a dense ASM, it suffices to know the ranks of the matrices $E^+_{n,k}$. Similarly, to determine the minimum rank or the maximum rank of the sign pattern of a dense ASM, it suffices to characterize those respective ranks of the dense $(0,+)$ sign patterns of $E^+_{n,k}$.

4.3 The Rank of the ASM $E_{n,k}$

As discussed in previous section, to determine the rank of a dense ASM, it suffices to determine the ranks of the matrices $E^+_{n,k}$. In order to simplify the notation, we will just use $\mathbf{E}_{n,k}$ for $E^+_{n,k}$.

By the rank-nullity theorem, for a matrix M with n columns, $\operatorname{rank}(M) + \nu(M) = n$, where $\nu(M)$ is the nullity of M. So, to exhibit $\operatorname{rank}(E_{n,k})$, we can equivalently present $\nu(E_{n,k})$. First, we will provide a useful tool. Denote by $\mathbf{F}_{n,k}$ the $(0,1)$ matrix of order n obtained from $\mathbf{E}_{n,k}$ by changing the zero entries $e_{1,k-1}, e_{2,k-2}, \ldots, e_{k-1,1}$ to 1's. Note that if $k = 1$, we have $\mathbf{F}_{n,1} = \mathbf{E}_{n,1} = I_n$.

Lemma 4.3.1. *For any $1 \leq k \leq n$, we have*

$$\nu(\mathbf{E}_{n,k}) = \begin{cases} \nu(\mathbf{F}_{n-k,k}), & \textit{if } n \geq 2k \\ k-1, & \textit{if } n = 2k-1 \\ \nu(\mathbf{F}_{k-1,n-k+1}), & \textit{if } n \leq 2k-2 \end{cases}$$

and

$$\nu(\mathbf{F}_{n,k}) = \begin{cases} \nu(\mathbf{E}_{n-k+1,k}), & \textit{if } n \geq 2k-1 \\ \nu(\mathbf{F}_{k-1,2k-n-1}), & \textit{if } n \leq 2k-2. \end{cases}$$

Proof. First, assume that $n \geq 2k$. Then $n-k \geq k$, so that $\mathbf{F}_{n-k,k}$ is defined. For each $1 \leq i \leq k-1$, subtract row i from ow $2k-i$ of $\mathbf{E}_{n,k}$. Then use type III elementary column operations to zero out the nonleading entries in the first k rows.

Then $\mathbf{E}_{n,k}$ is transformed to

$$\begin{bmatrix} \hat{I}_k & 0 \\ 0 & \mathbf{F}_{n-k,k} \end{bmatrix},$$

where $\hat{I}_k$ is the "backward" identity, which has full column rank. So, the first nullity result follows.

Next, consider the case $n = 2k-1$, so that $\mathbf{E}_{n,k}$ is a "diamond". Here, the first k rows are linearly independent, and the last $k-1$ rows are repetitions of the first $k-1$ rows. So, $\mathrm{rank}(\mathbf{E}_{n,k}) = k$, and $\nu(\mathbf{E}_{n,k}) = k-1$.

Finally, suppose that $n \le 2k-2$, so that $n-k+1 \le k-1$ and $\mathbf{F}_{k-1,n-k+1}$ is defined. Similarly as in the first case, by type III elementary row and column operations, $\mathbf{E}_{n,k}$ can be transformed to

$$\begin{bmatrix} 0 & I_{n-k+1} \\ \hat{\mathbf{F}}_{k-1,n-k+1} & O \end{bmatrix},$$

where $\hat{\mathbf{F}}_{k-1,n-k+1}$ is obtained from $\mathbf{F}_{k-1,n-k+1}$ by reversely arranging the columns. So, the third nullity result follows.

We now consider $\mathbf{F}_{n,k}$. Assume that $n \ge 2k-1$. For each i from 1 to $k-1$, subtract row i from row $2k-i-1$. Note that these operations zero out the lower entries in the top diamond with two vertices at $(1,k)$ and $(k,1)$ (including the entries on the horizontal axis) except the entries on the lower right edge of the diamond. Then use columns 1 to $k-1$ to zero out the nonleading nonzero entries in the first $k-1$ rows. These elementary row and column operations transform $\mathbf{F}_{n,k}$ to

$$\begin{bmatrix} \hat{I}_{k-1} & 0 \\ 0 & \mathbf{E}_{n-k+1,k} \end{bmatrix}.$$

Thus the fourth nullity result follows.

The proof of the last nullity result is similar to the proof of the third one above, and is omitted here. $\square$

We now come to the main result of this section. For integers a, b, by (a,b) we mean the greatest common divisor of a and b. Recall that $(a,b) = (a-b,b)$ and when a is even while b is odd, we have $(a,b)=(\frac{a}{2},b)$.

Theorem 4.3.2. *For any* $1\leq k\leq n$*, we have*

(i.) $\nu(\mathbf{E}_{n,k})=\dfrac{(n,2k-1)-1}{2}$;

(ii.) $\nu(\mathbf{F}_{n,k})=\dfrac{(2n+1,2k-1)-1}{2}$.

Proof. We use induction with respect to n. The base case of $n=1$ clearly holds. Next, we assume that both results hold for all positive integers less than n and use Lemma 4.3.1 to check the equalities in each case.

(i) a) $n\geq 2k$:

$$\begin{aligned}\nu(\mathbf{E}_{n,k})=\nu(\mathbf{F}_{n-k,k})=\frac{(2(n-k)+1,2k-1)-1}{2}&=\frac{(2n-(2k-1),2k-1)-1}{2}\\&=\frac{(2n,2k-1)-1}{2}=\frac{(n,2k-1)-1}{2};\end{aligned}$$

(i) b) $n=2k-1$:

$$\nu(\mathbf{E}_{n,k})=k-1=\frac{(2k-1)-1}{2}=\frac{(2k-1,2k-1)-1}{2}=\frac{(n,2k-1)-1}{2};$$

(i) c) $n\leq 2k-2$:

$$\begin{aligned}\nu(\mathbf{E}_{n,k})=\nu(\mathbf{F}_{k-1,n-k+1})=\frac{(2(k-1)+1,2(n-k+1)-1)-1}{2}&=\frac{(2k-1,2n-(2k-1))-1}{2}\\&=\frac{(2k-1,2n)-1}{2}=\frac{(n,2k-1)-1}{2};\end{aligned}$$

(ii) a) $n\geq 2k-1$:

$$\begin{aligned}\nu(\mathbf{F}_{n,k})=\nu(\mathbf{E}_{n-k+1,k})=\frac{(n-k+1,2k-1)-1}{2}&=\frac{(2(n-k+1),2k-1)-1}{2}\\&=\frac{(2n-2k+2,2k-1)-1}{2}=\frac{(2n+1,2k-1)-1}{2};\end{aligned}$$

(ii) b) $n \leq 2k-2$:

$$\nu(\mathbf{F}_{n,k}) = \nu(\mathbf{F}_{k-1,2k-n-1}) = \frac{(2(k-1)+1, 2(2k-n-1)-1)-1}{2} = \frac{(2k-1, 4k-2n-3)-1}{2}$$
$$= \frac{(2k-1, 2(2k-1)-(2n+1))-1}{2} = \frac{(2n+1, 2k-1)-1}{2}.$$

□

Thus we have determined the rank of $E_{n,k}$.

Theorem 4.3.3. *For all* $1 \leq k \leq n$,

$$\textit{rank}(E_{n,k}) = \textit{rank}(\mathbf{E}_{n,k}) = n - \frac{(n, 2k-1)-1}{2}.$$

4.4 The Minimum Rank and the Maximum Rank of $\mathcal{E}_{n,k}$

By Lemma 4.2.1, $E_{n,k}$ and $E^{+}_{n,k}$ are diagonally equivalent, so their sign patterns have the same minimum rank and the same maximum rank. We denote by $\mathcal{E}_{n,k}$ the sign pattern of $E^{+}_{n,k}$. Since $\mathcal{E}_{n,k}$ is permutation equivalent to $\mathcal{E}_{n,n-k+1}$, for minimum rank and maximum rank considerations, we may assume that $k \leq n-k+1$, namely, $n \geq 2k-1$. As before, we assume that n and k are positive integers such that $k \leq n$.

Theorem 4.4.1. *mr*$(\mathcal{E}_{n,k}) = k$ *when* $n = 2k-1$.

Proof. When $n = 2k-1$, the rectangle formed by the positive entries of $\mathcal{E}_{n,k}$ is a diamond, with vertices at the positions $(1,k), (k,1), (n,k)$, and (k,n). The submatrix $\mathcal{E}_{n,k}[\{1,2,\ldots,k\},\{1,2,\ldots,k\}]$ is sign nonsingular. Thus $mr(\mathcal{E}_{n,k}) \geq k$. On the other hand, since rank$(E^{+}_{n,k}) = k$ by Theorem 4.3.3, which ensures that $mr(\mathcal{E}_{n,k}) \leq k$. Hence, we get the desired conclusion. □

Theorem 4.4.2. $mr(\mathcal{E}_{n,k}) = n = 2k$ *when* $n = 2k$.

Proof. Note that the rectangle formed by the + entries of $\mathcal{E}_{n,k}$ has vertices at the positions $(1,k), (k,1), (n,k+1)$, and $(k+1,n)$. Let $r_1, r_2, \ldots, r_{2k}$ denote the row vectors of an arbitrary matrix $B \in Q(\mathcal{E}_{n,k})$. Let $c_i \in \mathbb{R}$ $(1 \leq i \leq 2k)$ be any scalars such that

$$c_1 r_1 + c_2 r_2 + \cdots + c_{2k} r_{2k} = 0.$$

Since the kth row is the only row whose first entry is nonzero, c_k must be zero. Similarly since the $(k+1)$th row is the only row whose last entry is nonzero, c_{k+1} must be zero.

After letting c_k and c_{k+1} be zeros and thus deleting the two corresponding terms in the above linear combination, we see that the $(k-1)$th row is the only row whose second entry is nonzero and $(k+2)$th row is the only row whose $n-1$th entry is nonzero. Thus we have $c_{k-1} = 0$ and $c_{k+2} = 0$.

Continuing in this fashion, at the end, we have $c_i = 0$ for all $i = 1, 2, \ldots, 2k$, which means that the rows $r_1, r_2, \ldots, r_{2k}$ are linearly independent. Therefore, $mr(\mathcal{E}_{n,k}) = n = 2k$. □

We remark that the preceding result can also be seen using a similar argument as in the proof of the next result.

Theorem 4.4.3. *Let $n = 2k + j$ where $k \geq 1$ and $j \geq 1$. Then*

$$mr(\mathcal{E}_{n,k}) = \begin{cases} 2k, & \text{if } j \leq k-2, \\ 2k+1, & \text{if } j = k-1, \\ k+j+1, & \text{if } j \geq k. \end{cases}$$

Proof. Let's consider the following three cases.

Case 1: $j \leq k-2$.

Note that in the rectangle formed by the + entries of $\mathcal{E}_{n,k}$, the top diamond has vertices at $(1, k), (k, 1), (2k-1, k)$ and $(k, 2k-1)$ and the bottom diamond has vertices at $(n-2k+2, n-k+1), (n-k+1, n-2k+2), (n, n-k+1)$ and $(n-k+1, n)$. Since $n = 2k+j$, the vertices of the bottom diamond can also be written as $(j+2, k+j+1), (k+j+1, j+2), (2k+j, k+j+1)$ and $(k+j+1, 2k+j)$. As $j \leq k-2$, we have $j+2 \leq k$. It follows that the leftmost vertex of the bottom diamond is to the left of the lowest vertex of the top diamond, which means that these diamonds overlap.

Consider the $(k+1)$th row of $\mathcal{E}_{n,k}$. The first $+$ of the $(k+1)$th row appears at the 2nd column, which also holds for the $(k-1)$th row. Note that these row indices are symmetric about k. Also, the last $+$ of the $(k+1)$th row appears at the $2k$ th column, which also holds for the $(k+2j+1)$th row. Note that these row indices are symmetric about $k+j+1 = n-k+1$, which is the row index of the nonzero entry in the last column. Since $j \leq k-2$, the column index of the last nonzero entry of the $(k-1)$th row, $2k-2$, is greater than or equal to the column index of the first nonzero entry of the $(k+2j+1)$th row, $2j+2$. Hence, these two rows "overlap", in the sense that both are positive at a common column index. So, we may choose the rows $k-1$ and $k+2j+1$ of a matrix $B \in Q(\mathcal{E}_{n,k})$ to be the $(0,1)$ vectors with the correct sign patterns and let the $(k+1)$th row of B be the sum of the $(k-1)$th row and the $(k+2j+1)$th row.

Similarly, for each i with $1 \leq i \leq j$, we may choose the rows $k-i$ and $k+2j-i+2$ of B to be the $(0,1)$ vectors with the correct sign patterns and let the $(k+i)$th row of B be the sum of the $(k-i)$th row and the $(k+2j-i+2)$th row.

Observe that the j triplets of rows described above are pairwise disjoint, as $k+j < k+2j-j+2$. Define the nonzero entries of $B \in Q(\mathcal{E}_{n,k})$ not in any of the triplets of rows above to be 1. Then the j linear dependence relations within the j disjoint row sets of B imply that $\text{rank}(B) \leq n-j = 2k$. Thus, $\text{mr}(\mathcal{E}_{n,k}) \leq 2k$. [We remark that this fact can also be seen using a technique employed in the proof of the next case.]

To show the reverse inequality, consider any matrix $B \in Q(\mathcal{E}_{n,k})$. Without loss of generality, we may assume that $b_{1,k} = b_{2,k-1} = \cdots = b_{k,1} = 1$ and $b_{n-k+1,n} = b_{n-k+2,n-1} = \cdots = b_{n,n-k+1} = 1$. We add suitable multiples of the first row to the lower rows to zero out all the lower entries in the kth column. We then use the second row to zero out all the lower entries in the $(k-1)$th column. Continue in this fashion until we reach the kth row. Note that the above elementary row operations only affect the entries in the top diamond of the rectangle of nonzero entries of the original matrix B. We can then use the new first k columns and elementary column operations to zero out all the nonleading nonzero entries in the first k rows.

Similarly, we can use use elementary row and column operations to zero out the entries either above or to the left of the entries $b_{n-k+1,n} = b_{n-k+2,n-1} = \cdots = b_{n,n-k+1} = 1$.

The resulting matrix $\tilde{B}$ has the form

$$\tilde{B} = \begin{bmatrix} \hat{I}_k & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & \hat{I}_k \end{bmatrix},$$

where $\hat{I}_k$ is the backward identity matrix of order k and C is a matrix of order j. It follows that rank$(B) =$ rank$(\tilde{B}) \geq 2k$. Thus we have mr$(\mathcal{E}_{n,k}) \geq 2k$.

Therefore, mr$(\mathcal{E}_{n,k}) = 2k$.

Case 2: $j = k - 1$.

Consider any matrix $B \in Q(\mathcal{E}_{n,k})$. Proceeding as in the last part of the proof of Case 1, through elementary row and column operations we can transform B into a matrix of the form

$$\tilde{B} = \begin{bmatrix} \hat{I}_k & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & \hat{I}_k \end{bmatrix},$$

where $\hat{I}_k$ is the backward identity matrix of order k and C is a matrix of order j. Since $n = 2k + j = 3k - 1$, the top diamond and the bottom diamond of the nonzero entries of B have lower left vertices at the positions $(k, 1), (2k - 1, k)$ and $(2k, k + 1), (3k - 1, 2k)$. Thus these diamonds do not overlap and the nonzero entries on the backward diagonal of B are the only nonzero entries of B not contained in either of the two diamonds. (The unconvinced reader may inspect the displayed $E_{8,3}$ in Section 2). Since the elementary row and column operations used above only affect the entries in the top or bottom diamond of the nonzero entries of B, we see that the nonzero entries on the backward diagonal of B are preserved in C. Thus rank$(C) \geq 1$. It follows that rank$(B) =$ rank$(\tilde{B}) \geq 2k + 1$. Consequently, mr$(\mathcal{E}_{n,k}) \geq 2k + 1$.

We now show the reverse inequality. Start with the matrix

$$\hat{B} = \begin{bmatrix} L_k & 0 & 0 \\ 0 & J_j & 0 \\ 0 & 0 & U_k \end{bmatrix},$$

where L_k is the $k \times k$ $(0,1)$ matrix whose nonzero entries are all the entries on or below the secondary diagonal, U_k is the $k \times k$ $(0,1)$ matrix whose nonzero entries are all the entries on or above the secondary diagonal, and J_j is the $j \times j$ matrix all of whose entries are 1's. Clearly, rank $(\hat{B}) = 2k+1$.

Add column i of $\hat{B}$ to column $2k-i$, for $i = 1, \ldots, k-1$, and then add the new row i to row $2k-i$, for $i = 1, \ldots, k-1$. Similarly, add column i of $\hat{B}$ to column $4k-i$ for $i = 2k+1, \ldots, 3k-1 = n$, and then add the new row i to row $4k-i$, for $i = 2k+1, \ldots, 3k-1 = n$. We reach a matrix $B \in Q(\mathcal{E}_{n,k})$ of rank $2k+1$. (Note that the entries in the intersection of J_j with either the top diamond or the bottom diamond are equal to 2, while all other nonzero entries of B (including those on the secondary diagonal of J_j) are 1's.) Hence, $\mathrm{mr}(\mathcal{E}_{n,k}) \leq 2k+1$.

Therefore, $\mathrm{mr}(\mathcal{E}_{n,k}) = 2k+1$.

Case 3: $j \geq k$.

We divide the rows of $B \in Q(\mathcal{E}_{n,k})$ except the kth row and the $(n-k+1)$th row into k disjoint subsets. For each $i = 1, \ldots k-1$, the ith subset S_i consists of the $(k-i)$th row, the rows whose row indices are congruent to i modulo k and are in $[k+1, n-k]$, and in case the largest of these row indices, denoted $tk+i$, falls inside $[n-2k+2, n-k]$, S_i also includes the row whose row index is symmetric with $tk+i$ about $n-k+1$. S_k includes the rows whose row indices are congruent to 0 modulo k and are in $[k+1, n-k]$, and if the largest such row index, denoted tk, falls inside $[n-2k+2, n-k]$, S_k also includes the row whose row index is symmetric with tk about $n-k+1$.

Observe that for each $i \in \{1, \ldots, k\}$, either the first two rows in S_i have leading nonzero entries in the same column (and we say these two rows are aligned on the left, which is the case for $1 \leq i \leq k-1$) or the first row in S_i has leading nonzero entry in the $(k+1)$th column (which is the case for S_k). Similarly, for each $i \in \{1, \ldots, k\}$, either the last row of S_i has its last nonzero entry in column $n-k$ or the last two rows in S_i have their last nonzero entries in the same column. Thus, for each $i \in \{1, \ldots, k\}$, we may choose the rows of B in S_i suitably so that their linear combination, with coefficients equal to 1 except that the first coefficient should be -1 when the top two rows are aligned on the left and the last coefficient should be -1 when the last two rows are aligned on the right, yields the row vector u whose first and last k components are zero and whose other entries are equal to 1. Let the kth and $(n-k+1)$th rows of B be chosen as the $(0,1)$ vectors with the correct sign patterns. The above k linear dependence relations guarantee that $\left[\begin{smallmatrix} B \\ u \end{smallmatrix}\right]$ satisfies rank$(\left[\begin{smallmatrix} B \\ u \end{smallmatrix}\right]) \leq n+1-k$. Hence, rank$(B) \leq n+1-k$, which ensures that mr$(\mathcal{E}_{n,k}) \leq n+1-k = k+j+1$.

To show the reverse inequality, observe that the submatrix $\mathcal{E}_{n,k}[\{k, k{+}1, \ldots, n\}, \{1, 2, \ldots, n-k+1\}]$ is an upper triangular sign pattern with all diagonal entries positive, so the submatrix is sign nonsingular. Hence, mr$(\mathcal{E}_{n,k}) \geq n-k+1 = k+j+1$.

Combining the above two inequalities, we obtain mr$(\mathcal{E}_{n,k}) = k+j+1$. □

We can summarize the above minimum rank results in the following way.

Theorem 4.4.4. *For all* $1 \leq k \leq n$*,*

$$mr(\mathcal{E}_{n,k}) = \begin{cases} mr(\mathcal{E}_{n,n-k+1}), & \textit{if } n \leq 2k-2, \\ k, & \textit{if } n = 2k-1, \\ 2k, & \textit{if } 2k \leq n \leq 3k-2, \\ 2k+1, & \textit{if } n = 3k-1, \\ n-k+1, & \textit{if } n \geq 3k. \end{cases}$$

The cases for sign nonsingularity now become clear.

Corollary 4.4.5. *The sign pattern* $\mathcal{E}_{n,k}$ *is sign nonsingular precisely when* $k=1$*,* $k=n, n=2k-2$*, or* $n=2k$*.*

We now turn to the maximum ranks. It turns out that MR$(\mathcal{E}_{n,k})$ can be easily determined. Of course, $\mathcal{E}_{n,1} = I_n$ and $\mathcal{E}_{n,n}$ is the “backward identity” and both of these dense patterns have minimum and maximum rank equal to n. We thus restrict ourselves to $E_{n,k}$ with $1 < k < n$. First, we claim that MR$(E_{n,k}) \geq n-1$ always holds. We show this by exhibiting at least $n-1$ nonzero entries, no two of which are on the same line. When $n = 2k-1$, the nonzero entries of $\mathcal{E}_{n,k}$ form a diamond and $n-1$ such entries can be found on opposite sides of the rectangle of + entries. When $n \neq 2k-1$, n such entries can be found by taking the entries on the shorter sides of the rectangle of nonzero entries, together with some entries on the diagonal or secondary diagonal. Next, observe that when $n = 2k-1$, the nonzero entries of $\mathcal{E}_{n,k}$ form a diamond and $n-1$ lines (such as row k and columns 2 through $n-1$) cover all the nonzero entries Thus we have now proved the following result.

Theorem 4.4.6. *For all $1 < k < n$, MR$(E_{n,k}) \geq n-1$, and MR$(E_{n,k}) = n$ if and only if $n \neq 2k-1$.*

It is well known ([27]) that for any sign pattern $\mathcal{A}$, and any integer t with mr$(\mathcal{A}) \leq t \leq$ MR$(\mathcal{A})$, there is a matrix $B \in Q(\mathcal{A})$ such that rank$(B) = t$.

In this part, we have determined the minimum ranks of the sign patterns of dense ASMs. For a general, nondense ASM, finding the minimum rank of its sign pattern is also of interest but seems to be as intractable as finding the minimum rank of a general sign pattern.

The results of this part were presented in [18].

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