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Spanning Halin Subgraphs Involving Forbidden Subgraphs

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by

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Under the Direction of Guantao Chen, PhD

ABSTRACT

In structural graph theory, connectivity is an important notation with a lot of applications. Tutte, in 1961, showed that a simple graph is 3-connected if and only if it can be generated from a wheel graph by repeatedly adding edges between nonadjacent vertices and applying vertex splitting. In 1971, Halin constructed a class of edge-minimal 3-connected planar graphs, which are a generalization of wheel graphs and later were named "Halin graphs" by Lovász and Plummer. A Halin graph is obtained from a plane embedding of a tree with no stems having degree 2 by adding a cycle through its leaves in the natural order determined according to the embedding. Since Halin graphs were introduced, many useful properties, such as hamiltonian, hamiltonian-connected and pancyclic, have been discovered. Hence, it will reveal many properties of a graph if we know the graph contains a spanning Halin subgraph. But unfortunately, until now, there is no positive result showing under which conditions a graph contains a spanning Halin subgraph. In this thesis, we characterize all forbidden pairs implying graphs containing spanning Halin subgraphs. Consequently, we provide a complete proof conjecture of Chen et al. Our proofs are based on Chudnovsky and Seymour's decomposition theorem of claw-free graphs, which were published recently in a series of papers.

INDEX WORDS: Forbidden pairs, Spanning subgraph, Halin graph, Strong spanning Halin subgraph, 3-connected graph, Claw-free, Z_3-free, $B_{1,2}$-free.

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DEDICATION

This dissertation is dedicated to my parents.

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LIST OF ABBREVIATIONS

- GSU - Georgia State University
- HIST - Homeomorphically Irreducible Spanning Tree
- SHS - Spanning Halin Subgraph
- SSHS - Strong Spanning Halin Subgraph

Chapter 1

INTRODUCTION

1.1 Halin graphs

A graph is called planar if it can be embedded in the plane without edge-crossings. Such an embedded graph is called a plane graph. A tree T is a connected acyclic graph. Every vertex in a tree with degree 1 is called a leaf and all others are called stems of the tree. In particular, if a graph G has a spanning tree T with no stems of degree 2, T is called a homeomorphically irreducible spanning tree (HIST) of G. A wheel graph is a graph obtained from a tree with exactly one stem by adding a cycle along its leaves. A graph is k-connected(resp. k-edge connected) if the removal of any vertex(resp. edge) set of size at most $k-1$ results in a connected graph. In 1961, Tutte [41] showed that a simple graph is 3-connected if and only if it can be generated from a wheel graph by repeatedly adding edges between nonadjacent vertices and applying vertex splitting. This result led in a direction to investigate the minimal k-edge connected graphs. In 1971, Halin [36] constructed a class of minimal 3-edge connected graphs, which was later named Halin graph by Lovász and Plummer. A Halin graph is a plane graph $H = T \cup C$ such that T is a HIST of H with $|T| \geq 4$ and C is a cycle obtained by connecting all leaves of T following the order given by the plane embedding. According to the definition, we can see that a Halin graph is a natural generalization of a wheel graph.

Since the Halin graph was introduced, massive research has been done and many interesting properties were obtained. In 1973, Bondy [32] showed that Halin graphs are hamiltonian (there exists a spanning cycle in H). Later, Barefoot [1] pointed out that Halin graphs are also hamiltonian connected (there exists a hamiltonian path between every pair of vertices in H). In [34], Lovász and Plummer illustrated that Halin graphs are 1-hamiltonian (both H and the graph obtained from H by deleting a vertex are hamiltonian). In 1983, Cornuéjols, Naddef and Pulleyblank [22] proved that Halin graphs are 1-edge-hamiltonian (each edge of H belongs to a hamiltonian cycle). Later, Skupien [39] discovered that Halin graphs are uniformly hamiltonian (each edge of H is contained in some hamiltonian cycles and avoided by some others) and Bondy and Lovász [4], independently, Skowrońska [38] showed that Halin graphs are almost pancyclic (the graph H contains cycles of length from 3 to $|V(H)|$ with the possible exception of a single even length) and is pancyclic if the underlying tree has no vertex of degree 3. In addition, in [22], Cornuéjols, Naddef and Pulleyblank showed that the traveling salesman problem (TSP) on Halin graphs was solvable in polynomial time.

Since Halin graphs have many interesting properties, it is natural to ask under what conditions a graph contains a Halin graph as a spanning subgraph. Horton, Parker and Borie [30] showed that deciding whether a graph has a spanning Halin subgraph is NP-complete, which indicates this is a very hard problem. Since Halin graphs are hamiltonian and almost pancyclic, the natural candidates are sufficient conditions for graphs to be hamiltonian. The majority sufficient conditions for hamiltonian graphs are results: 1) Tutte [40] showed that every 4-connected planar graph contains a hamiltonian cycle, 2) Dirac [23], in 1952, proved that a simple graph with n vertices is hamiltonian if every vertex has degree at least $\frac{n}{2}$, 3) In term of forbidden subgraphs, Bedrossian [2], Faudree and Gould [25] completely determined the forbidden pairs for 2-connected graphs containing a hamiltonian cycle.

In 1975, Lovász and Plummer [34] conjectured that: Every 4-connected plane triangulation has a spanning Halin subgraph. Unfortunately, this conjecture was recently disproved by Chen et al. [9]. Moreover, Chen et al. also pointed out that this conjecture does not hold even if the graph is 5-connected, see [8]. Looking for the degree condition, Chen, Shan and I [27] showed that: There exists $n_0 > 0$ such that for any graph G with $n \geq n_0$ vertices, if the minimum degree of G is at least $\frac{n+1}{2}$, then G contains a spanning Halin subgraph. In this thesis, it is natural to consider the sufficient conditions in term of “forbidden subgraphs”.

1.2 Forbidden subgraphs

Let H be a graph. A graph H is an induced subgraph of G if there exists a set $A \subseteq V(G)$ such that $\langle A \rangle$ isomorphic to H. A graph G is said to be H-free if G does not contain H as an induced subgraph. More generally, given a family of connected graphs $\mathcal{H} = \{H_1, H_2, \cdots, H_k\}$, we say that a graph G is $\mathcal{H}$-*free* if G contains no induced subgraph isomorphic to any graph in $\mathcal{H}$. In this case, we call the graphs in $\mathcal{H}$ forbidden subgraphs of G. We call $\mathcal{H}$ a forbidden pair if $\mathcal{H} = \{H_1, H_2\}$.

The research in this area has been mainly driven by the conjecture of Matthew and Sumner [35] that: Every 4-connected claw-free graph is hamiltonian. Although Matthew and Sumners conjecture is still open, Kaiser and Vrána [33] showed that every 6-connected claw-free graph is hamiltonian, and Hu, Tian and Wei [31] showed that every 8-connected claw-free graph is hamiltonian connected. On the other hand, Brandt, Favaron and Ryjácek [6] showed that for any positive integer k, there are infinitely many k-connected claw-free graphs which are not pancylic. Although in their examples only one even cycle is missing, I observed that by modifying their examples, we can construct infinitely many k-connected claw-free graphs without a spanning Halin subgraph. So, it makes sense to investigate what pair of graphs $\{H_1, H_2\}$ such that a k-connected $\{H_1, H_2\}$-free graph possesses some hamiltonian properties.

In 1991, Bedrossian [2] completely determined all forbidden pairs for when a 2-connected graph is hamiltonian. Later, in 1995, Faudree, Gould, Ryjáček and Schiermeyer [24] proved that every 2-connected $\{Claw, Z_3\}$-free graph of order at least 10 is hamiltonian. In [25], Faudree and Gould also completely characterized the forbidden pairs for hamiltonicity of 2-connected graphs with large order by allowing a finite number of exceptions. In the same paper, they also determined the forbidden pairs for the class of traceable graphs. In 2004, Gould, Luczak and Pfender [29] got positive results about forbidden pairs for pancyclic graphs. In 1997, Faudree and Gould [25]; in 2000, Chen and Gould [12]; and in 2002, Broersma et al. [7] separately gave some necessary conditions in terms of forbidden pairs for the class of graphs that are hamiltonian connected.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two sets of forbidden subgraphs; we write $\mathcal{H}_1 \leq \mathcal{H}_2$ if for every $H_2 \in \mathcal{H}_2$, there exists $H_1 \in \mathcal{H}_1$ such that H_1 is an induced subgraph of H_2. Clearly, if $\mathcal{H}_1 \leq \mathcal{H}_2$, then every $\mathcal{H}_1$-free graph is also $\mathcal{H}_2$-free. A pair $\mathcal{H}$ of graphs is called a forbidden pair for spanning Halin subgraph if every 3-connected $\mathcal{H}$-free graph contains a spanning Halin subgraph. The reason that we add connectivity condition is that Halin graphs are 3-connected. In [11], Chen et al. investigated forbidden pairs for graphs containing a spanning Halin subgraph, made the following conjecture and showed that the necessary condition holds for graphs with large size.

Conjecture 1.2.1. *Let $\mathcal{H}$ be a pair of connected graphs. Then every 3-connected H-free graph has a spanning Halin subgraph (of sufficiently large order) if and only if $\mathcal{H} \leq \{Claw, Z_3\}$ or $\mathcal{H} \leq \{Claw, B_{1,2}\}$.*

In 2014, Chen et al. [11] showed that: Every 3-connected $\{Claw, P_5\}$-free graph has a spanning Halin subgraph. Later, in [10], Chen et al. showed that: There exists a spanning Halin subgraph in 3-connected $\{Claw, Z_2\}$-free graphs or 3-connected $\{Claw, B_{1,1}\}$-free graphs.

In a series of papers([13, 14, 15, 16, 17, 18, 19, 20]), Chudnovsky and Seymour give a decomposition theorem for claw-free graphs. Roughly speaking, a claw-free graph is a line graph, or a long circular interval graph, or an antiprismatic graph, or several additional classes of graphs, or could be decomposed into some smaller graphs by a few specified operations, named “joins”. Since their theorem is much more involved in this thesis, in the following, we only state the theorem and will give specific definitions in Chapter 2.

Theorem 1. [17](Decomposition Theorem For Claw-Free Graphs)
Let G be a claw-free trigraph. Then either

- *$G \in \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_7$, or*
- *G admits either a 0-join, a 1-join, a generalized 2-join, a hex-join, a nondominating W-join or twins.*

In this thesis, we will apply decomposition theorem for claw-free graphs to show that the sufficient condition of Conjecture 1.2.1 holds by following two theorems.

Theorem 2. *Every 3-connected* $\{Claw, Z_3\}$*-free graph has a spanning Halin subgraph.*

Theorem 3. *Every 3-connected* $\{Claw, B_{1,2}\}$*-free graph has a spanning Halin subgraph.*

The specific plan for this thesis is given below. In Chapter 2, we will introduce some definitions, notations, the decomposition theorem for claw-free trigraphs and illustrate that some families of trigraphs mentioned in Theorem 1 are indeed some families of graphs(for example, line trigraphs are also line graphs). In Chapter 3, we will show that every trigraph in $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_4$ contains both Z_3 and $B_{1,2}$ as induced subgraphs, every graph in $\mathcal{F}_5$ is not 3-connected and both Theorem 2 and Theorem 3 hold for near-antiprismatic trigraphs($\mathcal{F}_6$). In Chapter 4, we will prove some elementary lemmas which will be used repeatedly in later chapters and will introduce the definition of "strong spanning Halin subgraph" and point out the relations between strong spanning Halin subgraphs and spanning Halin subgraphs. From Chapter 5 to Chapter 7, we will show that Theorem 2 and Theorem 3 hold for line graphs, long circular interval graphs and antiprismatic graphs. From Chapter 8 to Chapter 11, we will discuss that a graph admits a 1-join, a (generalized) 2-join, a hex join or a nondominating W-join. In Chapter 12, we will talk about twins.

Chapter 2

PRELIMINARIES

2.1 Definitions and notations

In this thesis, we consider simple and finite graphs only. The notations and definitions not defined here can be found in [5]. A *graph* G is an ordered pair $(V(G), E(G))$ that consists of a set $V(G)$ of *vertices* and a set $E(G)$, disjoint from $V(G)$, of *edges*, together with an incidence function ψ_G that associates with each edge of G an unordered pair of vertices of G. If e is an edge and u and v are vertices such that $\psi_G(e) = uv$, then e is said to *incident* u and v, and the vertices u and v are called *adjacent* to each other and nonadjacent otherwise. A graph is *complete* if any two vertices are adjacent. A graph is called *planar* if it can be embedded in the plane without edge-crossings. Such an embedded graph is called a *plane graph*. A graph G is called *finite* if both $V(G)$ and $E(G)$ are finite. A graph is *simple* if there exists at most one edge between every two vertices.

Two graphs G and H are *isomorphic*, written $G \cong H$, if there are bijections ϕ: $V(G) \rightarrow V(H)$ and φ: $E(G) \rightarrow E(H)$ such that $\psi_G(e) = uv$ if and only if $\psi_H(\varphi(e)) = \phi(u)\phi(v)$. Given two graphs H and G, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$, then H is called a *subgraph* of G and denoted as $H \subseteq G$. A subgraph obtained from G by deleting edges only is called an *edge-deleted subgraph*, and a subgraph obtained from G by deleting some vertices and all their incident edges is called a *vertex-deleted subgraph*. A *spanning subgraph* of a graph G is a subgraph obtained by edge deletions only. A subgraph obtained by vertex deletions only is called an *induced subgraph*; in such cases, the induced subgraph is denoted by $\langle S \rangle$ and referred to as the subgraph of G induced by S. A *clique* of a graph is a set of pairwise adjacent vertices. A graph G is called a *line graph* of H if $V(G) = E(H)$ and for every distinct $e, f \in E(H)$, e and f are adjacent in G if and only if they share a common vertex in H.

A *path* is a simple graph whose vertices can be arranged in a linear sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. Likewise, a *cycle* on three or more vertices is a simple graph whose vertices can be arranged in a cyclic sequence in such a way that two vertices are adjacent if they are consecutive in the sequence, and are nonadjacent otherwise. A *hole* is a cycle with at least 4 vertices. A path or cycle which contains every vertex of a graph is called a *hamiltonian path* or *hamiltonian cycle* of the graph. A graph is *traceable* if it contains a hamiltonian path, and *hamiltonian* if it contains a hamiltonian cycle. A graph is *hamiltonian connected* if any two vertices are connected by a hamiltonian path. A simple graph on n vertices is *pancyclic* if it contains at least one cycle of each length l, where $3 \leq l \leq n$.

A graph is *connected* if, for every partition of its vertex set into two nonempty sets X and Y, there is an edge with one end in X and the other in Y; otherwise the graph is called *disconnected*. Every graph G may be expressed uniquely (up to order) as a disjoint union of connected graphs; these graphs are called the *connected components*, or simply the *components*, of G. A graph is k*-connected*(resp. k*-edge connected*) if the removal of any vertex(resp. edge) set of size at most $k-1$ results in a connected graph. A *vertex cut* in a graph G is a subset X of $V(G)$ such that if we delete X from G and all edges incident to X, then $\langle G \setminus X \rangle$ has more than one components. In this thesis, vertex cuts of size one, two and three are called *cut vertex*, 2-*cut* and 3-*cut*, respectively.

An *acyclic* graph is one that contains no cycles. A connected acyclic graph is called a *tree*. A vertex of a tree having degree exactly one is called a *leaf* of and all others are called *stems*. A *star*, denoted by $\langle u; v_1, v_2, \cdots, v_t \rangle$, is a tree with exactly one stem. The stem of a star is also called a *center*. A *homeomorphically irreducible tree(HIT)* is a tree with no stems having degree 2. In particular, if the homeomorphically irreducible tree is a spanning subgraph of G, it is called a *homeomorphically irreducible spanning tree* (HIST) of G. A *Halin graph* is a plane graph, denoted by $H = T \cup C$, such that T is a HIST of H with $|T| \geq 4$ and C is a cycle obtained by connecting all leaves of T following the order given by the plane embedding. Following the definition of H, we notice that for any stem x, there are $deg_T(x)$ many faces F_x containing x and an edge on C. We call such a face an x-face. A spanning Halin subgraph $H = T \cup C$ of G is called a *strong spanning Halin subgraph* of G if for any stem $x \in V(T)$, there is an x-face F_x such that x is adjacent in G to two end vertices x_1, x_2 of the edge of F_x on C, for any two different stems x and x^*, we have $x_1x_2 \neq x_1^*x_2^*$ if x and x^* are not adjacent in T.

Let $\mathcal{P}$ be a hamiltonian property (such as traceable, hamiltonian, pancyclic and so on), and k be the least connectivity possible in a graph with property $\mathcal{P}$. Thus, for example, if $\mathcal{P}$ is traceability, then $k = 1$, while if $\mathcal{P}$ is hamiltonicity, then $k = 2$. A set of graphs $\mathcal{H} = \{H_1, H_2, \cdots, H_k\}$ is called *forbidden subgraphs* of $\mathcal{P}$ if every k-connected $\mathcal{H}$-free graph has property $\mathcal{P}$. In particular, we call $\mathcal{H}$ a *forbidden pair* of $\mathcal{P}$ if $\mathcal{H} = \{H_1, H_2\}$.

A *claw*, denoted by $\langle v; u_1, u_2, u_3 \rangle$, is a star with exactly three leaves. For three nonnegative integers k, l, m, let $N_{k,l,m}$, be obtained from a triangle K_3 by attaching three disjoint paths with length k, l, m to three distinct vertices of K_3, respectively. Commonly, $N_{k,l,0}$ is usually denoted by $B_{k,l}$, where B stands for "Bull", and $N_{k,0,0}$ is denoted by Z_k. In particular, we denote Z_3 by $\langle x, y, z; uvw \rangle$, $B_{1,2}$ by $\langle u; x, y, z; vw \rangle$, and $N_{1,1,1}$ by $\langle x, y, z; u; v; w \rangle$, where $\langle x, y, z \rangle$ is the triangle K_3. Please refer to the following figures for examples.

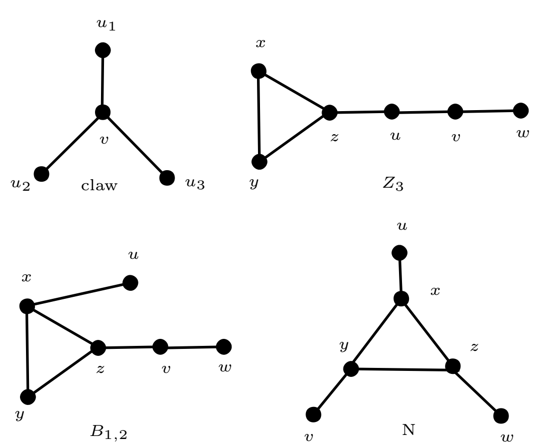

Figure 2.1. Example of some forbidden graphs

For any two subgraphs H and K of G, $E(H, K)$ denotes the set of edges in G with one end in H and the other in K. Let $v \in V(G)$; the *degree* of v in G, denoted by $deg_G(v)$, is the number of edges that are incident with v. For any $u, v \in V(G)$, the *distance* between u and v in G is the number of edges in a shortest path connecting them, denoted by $dist(u, v)$. Similarly, $dist(v, A)$ is the number of edges in a shortest path from v to A, where $A \subseteq V(G)$ and $v \in V(G)$. For any vertex sets $A, B \in V(G)$, $N_A^i(B) = \{v \in B | \, dist(v, A) = i\}$ and we denote $N_A^1(B)$ by $N_A(B)$ for simplicity.

2.2 Decomposition theorem for claw-free graphs

Because the main tool for proving Theorem 2 and Theorem 3 in this thesis is Chudnovsky and Seymour's decomposition theorem for claw-free graphs, in this section, we will quote all definitions related to decomposition theorem for claw-free graphs from their paper directly. We will also give some explanations for these definitions and convert some families of trigraphs into graphs if necessary and possible.

In a graph, every pair of vertices is either adjacent or nonadjacent, but in a trigraph, some pairs may be undecided. In [17], Page 1, Chudnovsky and Seymour defined a trigraph as following. A *trigraph* G consists of a finite set $V(G)$ of vertices, and a map $\theta_G : V(G)^2 \rightarrow \{1, 0, -1\}$, satisfying:

- for all $v \in V(G)$, $\theta_G(v, v) = 0$,
- for all distinct $u, v \in V(G)$, $\theta_G(u, v) = \theta_G(v, u)$,
- for all distinct $u, v, w \in V(G)$, at most one of $\theta_G(u, v)$, $\theta_G(v, w) = 0$.

θ_G is called the *adjacency function* of G. For distinct u, v in $V(G)$, we say that u, v are *strongly adjacent* if $\theta_G(u, v) = 1$, *strongly antiadjacent* if $\theta_G(u, v) = -1$, and *semiadjacent* if $\theta_G(u, v) = 0$. We say that u, v are *adjacent* if they are either strongly adjacent or semiadjacent, and *antiadjacent* if they are either strongly antiadjacent or semiadjacent. If we denote by $F(G)$ the set of all pairs $\{u, v\}$ such that $u, v \in V(G)$ are distinct and semiadjacent, then a trigraph G is a graph if $F(G) = \emptyset$. By the definition of trigraph, we can easily see that the set of undecided pairs $F(G)$ is a matching. If H is a graph and G is a trigraph, we say that G is an *H-trigraph* if $V(G) = V(H)$, and for all distinct $u, v \in V(H)$, if u, v are adjacent in H then they are adjacent in G, and if u, v are nonadjacent in H then they are antiadjacent in G.

Based on "trigraphs", Chudnovsky and Seymour defined some basic classes of trigraphs for claw-free trigraphs and also introduced some decomposition operations, named as "joins". Although they did not explicitly state in their paper, some families of trigraphs defined there are indeed families of graphs. For example, every line trigraph is indeed a line graph. Here we strictly follow and directly quote their definitions (see [17], Page 3 to 6) for easily understanding.

Some basic classes of claw-free trigraphs:

$\mathcal{F}_0$: Line trigraphs. Let H be a graph and G be a trigraph with $V(G) = E(H)$. We say that G is a *line trigraph* of H if for all distinct $e, f \in E(H)$:

- if e, f have a common end in H then they are adjacent in G, and if they have a common end of degree at least three in H, then they are strongly adjacent in G;
- if e, f have no common end in H then they are strongly antiadjacent in G.

We say that $G \in \mathcal{F}_0$ if G is isomorphic to a line trigraph of some graph. It is easy to check that any line trigraph is claw-free. The following lemma shows that a line trigraph is also a line graph.

Lemma 2.2.1. *Let H be a graph. If G is a line trigraph of a graph H in $\mathcal{F}_0$, then there exists a graph H^* such that G is the line graph of H^*.*

Proof. Let G be a line trigraph of a graph H. For every two vertices e and f in $V(G)$, if e and f share a common vertex with degree at least 3 in H or e and f do not share a common vertex, then ef is determined (strongly adjacent or strongly antiadjacent) in $E(G)$. Thus we assume e and f share a common vertex u with degree 2 in H. Let $e = v_1u$ and $f = v_2u$. If ef exists in G, let $H' = H$; if ef does not exist in G, then split u into two vertices, say u_1 and u_2, and let $e = v_1u_1$, $f = v_2u_2$ and denote the new graph by H'. Then G is a line trigraph of H' with ef is determined. If we perform this operation to all undetermined edges of G and denote the final graph obtained from H by splitting degree 2 vertices, if necessary, as H^*, then G is a line graph of H^*. ■

$\mathcal{F}_1$**: Trigraphs from the icosahedron**. The icosahedron is a planar graph with twelve vertices and each vertex has degree exactly five (See Figure 2.2(1)). For $k \in [0, 2]$, $icosa(-k)$ denotes the graph obtained from the icosahedron by deleting k pairwise adjacent vertices. We say $G \in \mathcal{F}_1$ if G is a claw-free $icosa(0)$-trigraph, $icosa(-1)$-trigraph or $icosa(-2)$-trigraph.

In Section 5.1 and 5.2 of paper [17], Chudnovsky and Seymour showed that every claw-free icosa(0)-trigraph G and every claw-free icosa(-1)-trigraph G satisfies $F(G) = \emptyset$, and therefore they are graphs. Every claw-free icosa(-2)-trigraph G satisfies $|F(G)| \leq 2$ and the two undetermined edges do not exist in the corresponding icosa(-2)-graph.

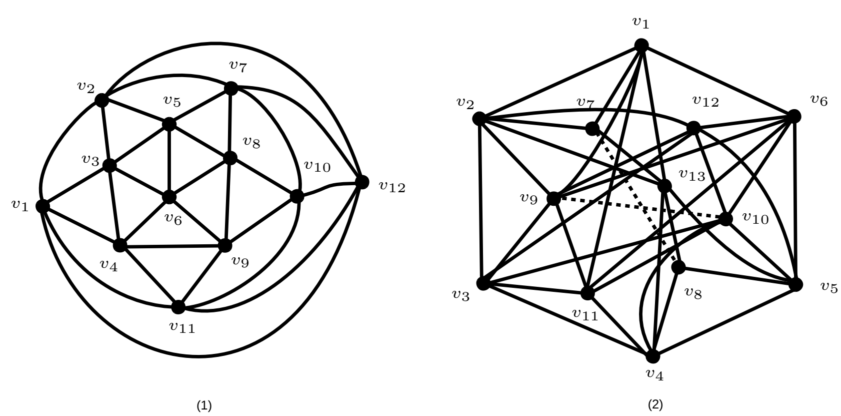

Figure 2.2. Examples of trigraphs in $\mathcal{F}_1$ or $\mathcal{F}_2$.

$\mathcal{F}_2$**: The trigraphs.** Let G be the trigraph with vertex set $\{v_1, \cdots, v_{13}\}$, with adjacency as follows. $v_1 \cdots v_6v_1$ is a hole in G of length 6. Next, v_7 is adjacent to v_1, v_2; v_8 is adjacent to v_4, v_5 and possibly to v_7; v_9 is adjacent to v_6, v_1, v_2, v_3; v_{10} is adjacent to v_3, v_4, v_5, v_6, v_9; v_{11} is adjacent to v_3, v_4, v_6, v_1, v_9, v_{10}; v_{12} is adjacent to v_2, v_3, v_5, v_6, v_9, v_{10}; and v_{13} is adjacent to v_1, v_2, v_4, v_5, v_7, v_8. No other pairs are adjacent, and all adjacent pairs are strongly adjacent except possibly for v_7, v_8 and v_9, v_{10}. (Thus the pair v_7v_8 may be strongly adjacent, semiadjacent or strongly antiadjacent; the pair v_9v_{10} is either strongly adjacent or semiadjacent, see Figure 2.2(2)). We say $H \in \mathcal{F}_2$ if H is isomorphic to $G \setminus X$, where $X \subseteq \{v_7, v_{11}, v_{12}, v_{13}\}$.

$\mathcal{F}_3$**: Long circular interval trigraphs.** Let Σ be a circle, and let $F_1, \cdots, F_k \subseteq \Sigma$ be homeomorphic to the interval $[0,1]$. Assume that no three of $F_1, \cdots, F_k$ have union Σ, and no two of $F_1, \cdots, F_k$ share an end-point. Now let $V \subseteq \Sigma$ be finite, and let G be a trigraph with vertex set V, in which, for distinct $u, v \in V$,

- if $u, v \in F_i$ for some i then u, v are adjacent, and if also at least one of u, v belongs to the interior of F_i then u, v are strongly adjacent;
- if there is no i such that $u, v \in F_i$ then u, v are strongly antiadjacent.

Such a trigraph G is called a *long circular interval trigraph*. We write $G \in \mathcal{F}_3$ if G is a long circular interval trigraph. In particular, a graph G with vertex set V is called a long circular interval graph if every distinct $u, v \in V$, u, v are adjacent if and only if there exists i such that $u, v \in F_i$.

Lemma 2.2.2. *Every long circular interval trigraph in* $\mathcal{F}_3$ *is also a long circular interval graph.*

Proof. If every edge in G is determined, then the conclusion is clearly true. Thus, we can assume there exists an undetermined edge uv in G. By definition, there exists an interval F_i containing both u and v as end-points.

Since no two of $F_1, \cdots, F_k$ share an end-point and V is finite, if uv exists in G, we can find two extremal small close intervals, say $[a, u]$ and $[v, b]$, such that $[a, u], [v, b] \subseteq \Sigma \setminus \cup_{i=1}^{k} F_k$, and for any vertices $w \in V \setminus \{u, v\}$, $w \notin [a, u] \cup [v, b]$. Let $F_i' = [a, u] \cup F_i \cup [v, b]$, then F_i' possesses the same properties as F_i. If uv does not exist in G, we can also find two extremal small intervals $[u, c)$ and $(d, v]$ such that $[u, c), (d, v] \subseteq F_i$ and for any vertices $w \in V \setminus \{u, v\}$, $w \notin [u, c] \cup [d, v]$. Let $F_i' = F_i \setminus ([u, c) \cup (d, v])$, then F_i' possesses the same properties as F_i.

If we perform this operation to all undetermined edges of G and denote the final intervals as $F_1', \cdots, F_k'$(note that we have $F_j = F_j'$ if one end of F_j is not a vertex of G), then $F_1', \cdots, F_k'$

have the same properties of $F_1, \cdots, F_k$ and G is a long circular interval graph based on $F'_1, \cdots, F'_k$. ■

$\mathcal{F}_4$**: Modifications of** $L(K_6)$**.** Let H be a graph with seven vertices $v_1, \cdots, v_7$, in which v_7 is adjacent to v_6 and to no other vertex, v_6 is adjacent to at least three of $v_1, \cdots, v_5$, and there is a cycle with vertices $v_1 v_2 \cdots v_5 v_1$ in order. Let $J(H)$ be the graph obtained from the line graph of H by adding one new vertex, adjacent precisely to those members of $E(H)$ that are not incident with v_6 in H (See Figure 2.3(1)). Then $J(H)$ is a claw-free graph. Let G be either $J(H)$ (regarded as a trigraph), or (in the case when v_4, v_5 both have degree two in H), the trigraph obtained from $J(H)$ by making the vertices v_3v_4, $v_1v_5 \in V(J(H))$ semiadjacent. Let $\mathcal{F}_4$ be the class of all such trigraphs G.

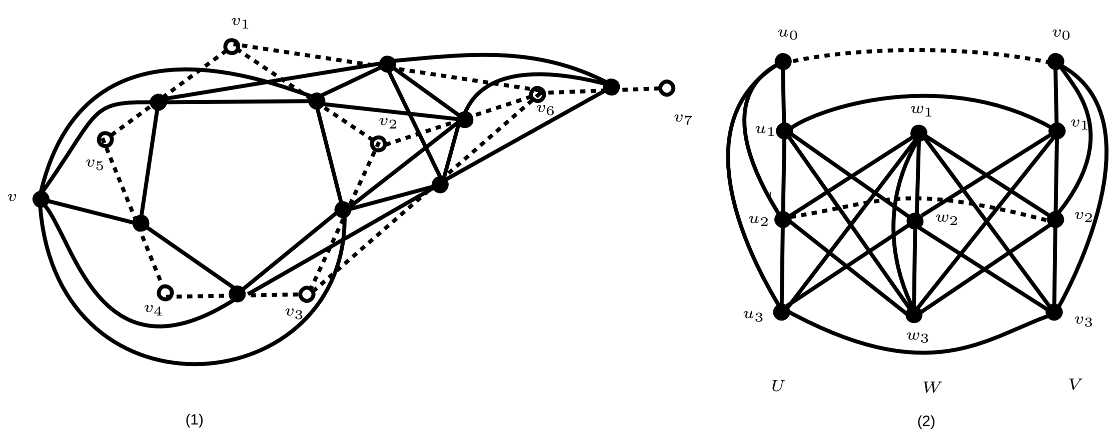

Figure 2.3. Trigraphs in $\mathcal{F}_4$ or $\mathcal{F}_6$.

$\mathcal{F}_5$**: The trigraphs.** Let $n \geq 2$. Construct a trigraph G as follows. Its vertex set is the disjoint union of four sets U, V, W and $\{x_1, \cdots, x_5\}$, where $|U| = |V| = |W| = n$, say $U = \{u_1, \cdots, u_n\}$, $V = \{v_1, \cdots, v_n\}$ and $W = \{w_1, \cdots, w_n\}$. Let $A \subseteq U \cup V \cup W$ with $|A \cap U|$, $|A \cap V|$, $|A \cap W| \leq |W| - 1$. Adjacency is as follows: U, V, W are strong cliques. For $1 \leq i, j \leq n$, u_i, v_j are adjacent if and only if $i = j$. And w_i is strongly adjacent to u_j if and only if $i \neq j$, and w_i is strongly adjacent to v_j if and only if $i \neq j$. Moreover

- u_i is semiadjacent to w_i for at most one value of $i \in [1, n]$, and if so then $v_i \in A$,

- v_i is semiadjacent to w_i for at most one value of $i \in [1, n]$, and if so then $u_i \in A$,
- u_i is semiadjacent to v_i for at most one value of $i \in [1, n]$, and if so then $w_i \in A$,
- no two of $U \setminus A$, $V \setminus A$, $W \setminus A$ are strongly complete to each other.

Also, x_1 is strongly $U \cup V \cup W$-complete; x_2 is strongly complete to $U \cup V$, and either semiadjacent or strongly adjacent to x_1; x_3 is strongly complete to $U \cup \{x_2\}$; x_4 is strongly complete to $V \cup \{x_2, x_3\}$; x_5 is strongly adjacent to x_3, x_4; and all other pairs are strongly antiadjacent. Let the trigraph just constructed be G, and let $H = \langle V(G) \setminus A \rangle$. Then H is claw-free; let $\mathcal{F}_5$ be the class of all such trigraphs H.

$\mathcal{F}_6$**: Near-antiprismatic trigraphs.** Let $n \geq 2$. Construct a trigraph as follows. Its vertex set is the disjoint union of three sets U, V, W, where $|U| = |V| = n + 1$ and $|W| = n$, say $U = \{u_0, u_1, \cdots, u_n\}$, $V = \{v_0, v_1, \cdots, v_n\}$ and $W = \{w_1, \cdots, w_n\}$. Adjacency is as follows: U, V, W are strong cliques. For $0 \leq i, j \leq n$ with $(i, j) \neq (0, 0)$, let u_i, v_j be adjacent if and only if $i = j$, and for $1 \leq i \leq n$ and $0 \leq j \leq n$ let w_i be adjacent to u_j, v_j if and only if $i \neq j \neq 0$; u_0, v_0 may be semiadjacent or strongly antiadjacent. All other pairs not mentioned so far are strongly antiadjacent. Now let $A \subseteq U \cup V \cup W \setminus \{u_0, v_0\}$ with $|W \setminus A| \geq 2$. Let all adjacent pairs be strongly adjacent except: u_i is semiadjacent to v_i for at most one value of $i \in [1, n]$, and if so then $w_i \in A$ (See Figure 2.3(2)).

Let the trigraph just constructed be G, and let $H =\langle V(G) \setminus A \rangle$. Then H is claw-free; let $\mathcal{F}_6$ be the class of all such trigraphs H. We call such a trigraph H *near-antiprismatic*, since making u_0, v_0 strongly adjacent would produce an antiprismatic trigraph.

$\mathcal{F}_7$**: Antiprismatic trigraphs**. Let us say a trigraph is *antiprismatic* if for every $X \subseteq V(G)$ with $|X| = 4$, X is not a claw and there are at least two pairs of vertices in X that are strongly adjacent. Let $\mathcal{F}_7$ be the class of all antiprismatic trigraphs.

A graph G is called *antiprismatic graph* if for every $X \subseteq V(G)$ with $|X| = 4$, X is not a claw and there are at least two pairs of vertices in X that are adjacent. It is clearly that every antiprismatic trigraph is also an antiprismatic graph.

Now we want to introduce some classes of graphs admit certain decomposition operations, which are named as "joins":

0-join: Suppose that W_1, W_2 is a partition of $V(G)$ such that W_1, W_2 are nonempty and W_1 is strongly anticomplete to W_2. We call the pair (W_1, W_2) a *0-join* in G. If a tirgraph G admits a 0-join, then G is disconnected. Since the spanning Halin subgraph exists in a graph with connectivity at least 3, we do not need to consider about the family of trigraphs admit 0-join.

1-join: Suppose that W_1, W_2 is a partition of $V(G)$, and for $i = 1, 2$ there is a subset $A_i \subseteq W_i$ such that:

- A_i, $W_i \setminus A_i \neq \emptyset$ for $i = 1, 2$,
- $A_1 \cup A_2$ is a strong clique, and
- $W_1 \setminus A_1$ is strongly anticomplete to W_2, and W_1 is strongly anticomplete to $W_2 \setminus A_2$.

In these circumstances, we say that (W_1, W_2) is a *1-join*. If G admits a 1-join, all edges between W_1 and W_2 are determined (strongly adjacent or strongly antiadjacent), and all edges in W_1 and W_2 are not known.

Generalized 2-join: Suppose that W_0, W_1, W_2 are disjoint subsets with union $V(G)$, and for $i = 1, 2$ there are subsets A_i, B_i of W_i satisfying the following:

- $W_0 \cup A_1 \cup A_2$ and $W_0 \cup B_1 \cup B_2$ are strong cliques, and W_0 is strongly anticomplete to $W_i \setminus (A_i \cup B_i)$ for $i = 1, 2$;
- for $i = 1, 2$, $A_i \cap B_i = \emptyset$ and A_i ,B_i and $W_i \setminus (A_i \cup B_i)$ are all nonempty; and
- for all $v \in W_1$ and $w \in W_2$, either v is strongly antiadjacent to w, or $v \in A_1$ and $w \in A_2$, or $v \in B_1$ and $w \in B_2$.

We call the triple ($W_0 \cup W_1 \cup W_2$) a *generalized 2-join*, and if $W_0 = \emptyset$ we call the pair (W_1, W_2) a *2-join*. Note that every edge in (generalized) 2-join is either determined (strongly adjacent or strongly antiadjacent) or not known.

Hex-join: Let (W_1, W_2) be a partition of $V(G)$, such that for $i = 1, 2$ there are strong cliques $A_i, B_i, C_i \subseteq W_i$ with the following properties:

- W_1, W_2 are both nonempty;
- for $i = 1, 2$ the sets A_i , B_i , C_i are pairwise disjoint and have union W_i;
- if $v_1 \in W_1$ and $v_2 \in W_2$, then v_1 is strongly adjacent to v_2 unless either $v_1 \in A_1$ and $v_2 \in A_2$, or $v_1 \in B_1$ and $v_2 \in B_2$, or $v_1 \in C_1$ and $v_2 \in C_2$; and in these cases v_1, v_2 are strongly antiadjacent.

In these circumstances, we say that G is a *hex-join* of $\langle W_1 \rangle$ and $\langle W_2 \rangle$. The edges between W_1 and W_2 are all determined and the edges between A_i, B_i and C_i are not known.

Nondominating W-join: The pair (A, B) is called a *homogeneous pair* in G if both A and B are strong cliques, and for every vertex $v \in V(G) \setminus (A \cup B)$, either $N_A(v) = A$ or $N_A(v) \cap A = \emptyset$ and either $N_B(v) = B$ or $N_B(v) \cap B = \emptyset$. Let (A, B) be a homogeneous pair, such that $N_B(A) \cap B \neq \emptyset$ and $N_B(A) \cap B \neq B$, and at least one of A, B has at least two members. In these circumstances, we call (A, B) a *W-join*. The W-join is called a *nondominating W-join* if there exist a vertex of $G \setminus (A \cup B)$ has no neighbor in $A \cup B$.

Twins: We call u, v are *twins* if $uv \in E(G)$ and $N_{G\setminus\{u,v\}}(u) = N_{G\setminus\{u,v\}}(v)$.

In [17], Chudnovsky and Seymour introduced decomposition theorem for claw-free trigraphs as follows.

Theorem 1:(Decomposition Theorem For Claw-Free Trigraphs)

Let G be a claw-free trigraph. Then either

- $G \in \mathcal{F}_0 \cup \cdots \cup \mathcal{F}_7$, or
- G admits either a 0-join, a 1-join, a generalized 2-join, a hex-join, a nondominating W-join or twins.

In Chapter 3, we will show that every trigraph in $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_4$ contains both Z_3 and $B_{1,2}$ as induced subgraphs on their strongly adjacent pairs; every trigraph in $\mathcal{F}_5$ is not 3-connected; and there exists a strong spanning Halin subgraph in any near-antiprismatic trigraph by only using strongly adjacent pairs. Since all pairs are mentioned in a trigraph admits a 1-join, a (generalized) 2-join, a hex-join or a nondominating W-join are either determined (strongly adjacent or strongly antiadjacent) or not known, if G is a trigraph admits these joins, there exists a graph H such that $V(G) = V(H)$ and for any pair $\{u, v\}$, uv is adjacent in H if and only if uv is strongly adjacent in G and uv is nonadjacent in H if and only if uv is strongly antiadjacent in G. To find a spanning Halin subgraph in G, we only need to find a spanning Halin subgraph in H. By Lemma 2.2.1, Lemma 2.2.2 and the definition of antiprismatic graphs, we only need to consider graphs, instead of trigraphs, when we are searching a spanning Halin subgraph in G if $G \in \mathcal{F}_0 \cup \mathcal{F}_3 \cup \mathcal{F}_7$ or G admits a 0-join, a 1-join, a generalized 2-join, a hex-join, a nondominating W-join or twins.

Chapter 3

GRAPHS IN $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_4 \cup \mathcal{F}_5 \cup \mathcal{F}_6$.

In this chapter, we will show that every trigraph in $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_4$ contains both Z_3 and $B_{1,2}$ as induced subgraphs by only using strongly adjacent pairs, every trigraph in $\mathcal{F}_5$ is not 3-connected and every trigraph in $\mathcal{F}_6$ contains a strong spanning Halin subgraph.

3.1 Trigraphs in $\mathcal{F}_1 \cup \mathcal{F}_2 \cup \mathcal{F}_4$ are neither Z_3-free nor $B_{1,2}$-free.

Proposition 1. *Every trigraph in $\mathcal{F}_1$ contains both Z_3 and $B_{1,2}$ as induced subgraphs.*

Proof. Let G be a trigraph in $\mathcal{F}_1$, v_{11}, v_{12} be the two possible deleted vertices, and v_2v_4, v_7v_9 be two semiadjacent edges in $F(G)$ if both v_{11} and v_{12} are deleted. By Lemma 5.1 and 5.2 in Chudnovsky and Seymour's paper, we know every edge, except for v_2v_4 and v_7v_9, id strongly adjacent. So $\langle v_1, v_2, v_3; v_6v_8v_{10}\rangle$ is an induced Z_3 and $\langle v_6; v_8, v_{10}, v_7; v_2v_1\rangle$ is an induced $B_{1,2}$. ■

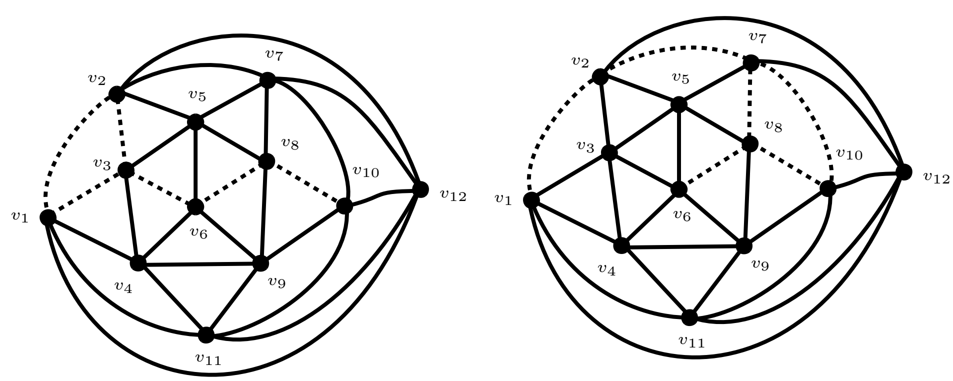

Figure 3.1. Every trigraph in $\mathcal{F}_1$ contains both Z_3 and $B_{1,2}$ as induced subgraphs.

Proposition 2. *Every trigraph in $\mathcal{F}_2$ contains both Z_3 and $B_{1,2}$ as induced subgraphs.*

Proof. By the definition, it is easy to see that $\langle v_8, v_4, v_5; v_6v_1v_2\rangle$ is an induced Z_3 and $\langle v_3; v_4, v_8, v_5; v_6v_1\rangle$ is an induced $B_{1,2}$. ■

Proposition 3. *Every trigraph in $\mathcal{F}_4$ contains both Z_3 and $B_{1,2}$ as induced subgraphs.*

Proof. We may assume $v_1v_6, v_3v_6 \in E(H)$ and denote the new vertex of $J(H)$ as v. Then $\langle v_1v_6, v_6v_7, v_3v_6; v_3v_2, v, v_4v_5\rangle$ is an induced Z_3 and $\langle v_1v_2; v_1v_6, v_6v_7, v_3v_6; v_3v_4, v_4v_5\rangle$ is an induced $B_{1,2}$(See Figure 3.2). ■

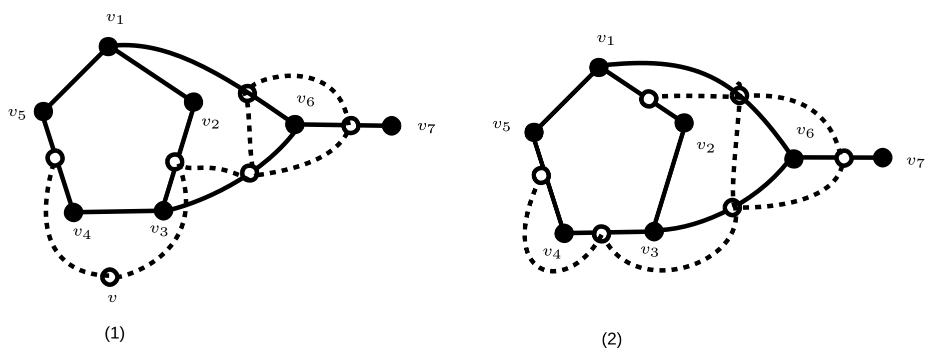

Figure 3.2. Every trigraph in $\mathcal{F}_4$ contains both Z_3 and $B_{1,2}$ as induced subgraphs.

3.2 Connectivity of trigraphs in $\mathcal{F}_5$

Proposition 4. *Every trigraph in $\mathcal{F}_5$ is not 3-connected.*

Proof. If G is a trigraph in $\mathcal{F}_5$, then $N_G(x_5) = \{x_3, x_4\}$. Thus G is not 3-connected. ■

3.3 Near-antiprismatic trigraphs contain strong spanning Halin subgraphs

Since we will show that every antiprismatic trigraph contains a strong spanning Halin subgraph in Chapter 7, here, by the definition of near-antiprismatic trigraphs, we can assume $u_0v_0 \notin E(G)$. We will prove the following proposition in this section.

Proposition 5. *If G is a 3-connected near-antiprismatic trigraph, then G contains a strong spanning Halin subgraph on its strongly adjacent pairs.*

We want to introduce the following two claims about near-antiprismatic trigraphs before searching a strong spanning Halin subgraph in G.

Claim 3.3.1. $|U \setminus A| \geq 4$ *and* $|V \setminus A| \geq 4$.

Proof. Since G is 3-connected and $u_0w_i, u_0v_j, v_0w_i, v_0u_j \notin E(G)$ for all $w_i \in W$, $u_j \in U$ and $v_j \in V$, we have $deg_{U\setminus A}(u_0) \geq 3$ and $deg_{V\setminus A}(v_0) \geq 3$. Thus $|U \setminus A| \geq 4$ and $|V \setminus A| \geq 4$. ■

Claim 3.3.2. *For any* $w_i \in W \setminus A$, $|N_{U\setminus A}(w_i)| \geq |U \setminus A| - 2 \geq 2$ *and* $|N_{V\setminus A}(w_i)| \geq |V \setminus A| - 2 \geq 2$.

Proof. Since for any $w_i \in W \setminus A$, it is strongly antiadjacent to at most two vertices, u_0 and u_i, in $U \setminus A$, we have $|N_{U\setminus A}(w_i)| \geq |U \setminus A| - 2 \geq 2$. Similarly, we can show that $|N_{V\setminus A}(w_i)| \geq |V \setminus A| - 2 \geq 2$. ■

According to Claim 3.3.1 and 3.3.2, we can find a strong spanning Halin subgraph in G as follows.

Case 1: $|W \setminus A| \geq 3$.

We let $W \setminus A = \{w_1, w_2, \cdots, w_t\}$ and assume $u_iw_1, u_jw_2, u_kw_3, v_sw_1, u_jw_1, v_tw_2, v_lw_t \in E(G)$, where $i, j, k, s, t, l \in [1, n]$. Since $U \setminus (A \cup \{u_i\})$, $V \setminus (A \cup \{v_s\})$ and $W \setminus (A \cup \{w_1, w_2\})$ are strong cliques, there exist strong hamiltonian paths, say $P_1 = u_jP_1u_k$, $P_2 = v_tP_2v_l$ and $P_3 = w_3P_3w_t$ in them, respectively. Let $W = P_1P_3P_2 \cup \{v_tw_2, w_2u_j\}$ be a strong cycle and all vertices on the strong path $u_iw_1v_s$ be stems of T with $N_W(u_i) = V(P_1)$, $N_W(w_1) = V(P_3) \cup \{w_2\}$ and $N_W(v_s) = V(P_2)$. Let $H = T \cup W$, since both u_i and v_s are adjacent to two consecutive vertices of C^* in T, $u_jw_1, w_2w_1 \in E(G)$ and $u_jw_2 \in E(C)$, we know H is a strong spanning Halin subgraph of G (See Figure 3.3(1)).

Case 2: $W \setminus A = \{w_1, w_2\}$.

Since G is 3-connected, there must exist $k \in [1, n]$ such that $u_k b_k \in E(G)$. By Claim 3.3.1 and 3.3.2, we may assume that $u_i w_1, u_j w_2, u_j w_1, v_s w_1, v_t w_2 \in E(G)$. Since both $U \setminus (x \cup \{u_i\})$ and $V \setminus \{x \cup \{v_s\}\}$ are strong cliques, there exist hamiltonian strong paths $P_1 = u_j P_1 u_k$ and $P_2 = v_t P_2 v_k$ in them respectively. Let $W = P_1 P_2 \cup \{v_t w_2, w_2 u_j\}$ be a strong cycle and all vertices on the strong path $u_i w_1 v_s$ be stems of T with $N_W(u_i) = V(P_1)$, $N_W(w_1) = \cup\{w_2\}$ and $N_W(v_s) = V(P_2)$. Let $H = T \cup W$, similar as seen as Case 1, we know H is a strong spanning Halin subgraph of G (See Figure 3.3(2)).

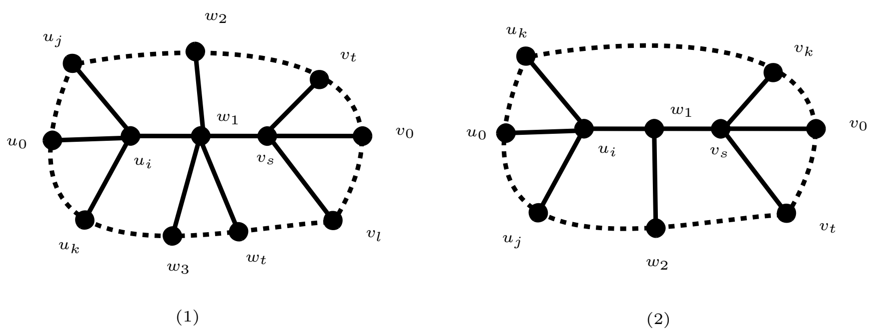

Figure 3.3. G is a 3-connected near-antiprismatic trigraph.

Chapter 4

SOME LEMMAS

From this chapter and after, we only consider graphs. We introduce several lemmas in this chapter, which will be used repeatedly in later chapters.

4.1 Properties of minimum vertex cut in a claw-free graph

In the remaining part of this thesis, we always assume G is a non-complete claw-free graph and reserve S as a minimum vertex cut of G.

Lemma 4.1.1. *Let G be a claw-free graph, then*

1. *For any $x \in S$ and any component D of $G \setminus S$, $N(x) \cap D \neq \emptyset$*
2. *$G \setminus S$ has exact two components, say G_1 and G_2.*

Proof. The first statement holds for minimum cuts in any graph. The second one follows the first one and that G is claw-free. ■

Let $V_i = V(G_i)$ for $i \in [1, 2]$, and assume, without loss of generality, that $|V_1| \leq |V_2|$. In addition, we assume $|V_1|$ is a minimum subject to $|S|$ being minimum.

Lemma 4.1.2. *Following the above definitions of G, S and V_1, we have the following statements.*

1. *$|N_{V_1}(x)| \geq 2$ for each $x \in S$ if $|V_1| \geq 2$. In general, for each $A \subseteq S$, we have $|N_{V_1}(A)| > |A|$ or $N_{V_1}(A) = V_1$ and $|N_{V_2}(A)| > |A| - 1$ or $N_{V_1}(A) = V_2$.*
2. *$N_{V_i}(x)$ is a clique for each $x \in S$, where $i \in [1, 2]$.*

Proof. Suppose to the contrary, there is a vertex $x \in S$ such that $deg_{V_1}(x) = 1$. Let w be the unique neighbor of v in V_1. Then, $S^* = (S \setminus \{v\}) \cup \{w\}$ is also a cut of G, and

$V_1 \setminus \{w\}$ is not adjacent to any other vertices in $G \setminus S^*$. Thus $G \setminus S^*$ has a component smaller than V_1, giving a contradiction.

The second conclusion is true since G is claw-free. ■

4.2 Strong spanning Halin subgraph and spanning Halin subgraph

Let G be a graph, and x, y are two vertices of G. x and y are called twins of G if $xy \in E(G)$ and $N_G(x) = N_G(y)$. Twins emerge when we duplicate vertices. For any vertices $v \in V(G)$, by duplicating a vertex v, we mean that we add a new vertex w to G and let w be adjacent to all neighbors of v and v itself. Clearly, v and w are twins in a new graph. If G is a claw-free graph, then duplicating any vertex of G results in a claw-free graph. A 3-connected claw-free graph is called twins-free if every pair of twins appears only in the 3-cut. Recall that a Halin graph $H = T \cup C$ is a plane graph. Following the definition of H, we notice that for any stem x of T, there are $deg_T(x)$ many faces F_x containing x and an edge on C. We call such a face an x-face.

Let G be a graph. A spanning Halin subgraph $H = T \cup C$ of G is called a *strong spanning Halin subgraph* of G if for any stem $x \in V(T)$, there is an x-face F_x such that x is adjacent (in G) to two end vertices x_1, x_2 of the edge of F_x on C. For any two different stems x and x^*, we have $x_1x_2 \neq x_1^*x_2^*$ if x and x^* are not adjacent in T.

Note that the induced subgraph on all stems of T is still a tree, denoted by T'. Every leaf of T' is adjacent to at least two consecutive vertices of $V(C)$ in T. So to check whether H is a strong spanning Halin subgraph, we only need to investigate that for every stem x of T', there is a x-face F_x that contains an edge uv of C such that both u and v are adjacent to x in G.

Lemma 4.2.1. *Let G be a graph and $\{x, y\}$ be a pair of twins of G. If $G \setminus \{y\}$ has a strong spanning Halin subgraph, then G also has one.*

Proof. Let $H = T \cup C$ be a strong spanning Halin subgraph of $G \setminus \{y\}$. We will insert y to C to form a strong spanning Halin subgraph of G by considering either x in

$V(C)$ or x is a stem of T.

Case 1: Assume that $x \in V(C)$.

Let x^+ and x^- be the successor and the processor of x on C. Set v as the stem of T is adjacent to x. After replacing x by x, y, we insert y in C to obtain a cycle $C^* = (C \setminus \{x^+x, xx^-\}) \cup \{x^+x, xy, yx^-\}$ and a tree $T^* = T \cup \{vy\}$.

We are now checking that $H^* = T^* \cup C^*$ is a strong spanning Halin subgraph of G. For any stem $w \in V(T)$, if $w = v$, then w has two neighbors x, y in G^* and xy is on C^*; if $w \neq v$, then $N_G(w)$ contains two consecutive vertices u and u^+. If $uu^+ \neq x^-x$, then u, u^+ are also consecutive on C^*; if $uu^+ = xx^-$, then y, x^- are two consecutive neighbors of w on C^*. In any case, we are done.

Case 2: Suppose that x is a stem of T.

In this case, we insert y between x_1, x_2 by letting $C^* = (C \setminus \{x_1x_2\}) \cup \{x_1y, x_2y\}$ and a tree $T^* = T \cup \{xy\}$. Clearly, $x_1, y \in N_G(x)$ are two consecutive vertices on C^*. For each stem z with $z \neq x$, let $z_1, z_2 \in N_G(z)$ be two associated neighbors of z, which are consecutive on C. If $z_1z_2 = x_1x_2$, which implies x and z are adjacent in T, then $yz \in E(G)$ and y, x_2 are two consecutive vertices associate with z on C^*. If $z_1z_2 \neq x_1x_2$, then z_1, z_2 are still two consecutive vertices associate with z on C^*. Clearly, each stem is associated with a distinct edge of C^*. ■

In Chapter 12, we will show that if G is 3-connected $\{claw, Z_3\}$-free or $\{claw, B_{1,2}\}$-free and admits a twins in its 3-cut, then G contains a spanning Halin subgraph. Thus, from the next chapter on (except Chapter 12), we always assume G is twins-free and try to find a strong spanning Halin subgraph, instead of a spanning Halin subgraph, in G.

4.3 One special case of 3-connected $\{claw, Z_3\}$-free graph

In this section, we assume G is a 3-connected $\{claw, Z_3\}$-free graph with $|V_1| \geq 2$, and at least one of V_1 and V_2 is not a clique. Then the following proposition is showing that G contains a strong spanning Halin subgraph.

Proposition 6. *Let G be an n-vertex 3-connected $\{claw, Z_3\}$-free graph with $|V_1| \geq 2$, and at least one of V_1 and V_2 is not a clique, then G contains a strong spanning Halin subgraph.*

Proof. Let G be a graph satisfying the condition of Proposition 6. We will find a strong spanning Halin subgraph through the following sequence of claims.

Claim 4.3.1. *For any $x \in S$, the induced subgraph $\langle V_2 \cup \{x\} \rangle$ does not contain an induced path $xw_1w_2w_3$ and $|N_2(x)| \geq 2$.*

Proof: To prove the first part of the claim, suppose on the contrary that there is an $x \in S$ and an induced path $xw_1w_2w_3$ with $w_1, w_2, w_3 \in V_2$. Since $|V_1| \geq 2$, there are two vertices $u_1, u_2 \in N_1(x)$, then $\langle u_1, u_2, x; w_1w_2w_3 \rangle$ is an induced Z_3, a contradiction.

We now prove the second part of the claim. If there exist $x \in S$ such that $|N_2(x)| = 1$, let $N_2(x) = \{w\}$. By the above statement, we know $V_2 = \{w\} \cup N_2^2(x)$. Since G is claw-free, $N_2(w) \setminus \{v\}$ is a clique, as well as V_2. This in turn implies V_1 is not a clique. Let $w', w'' \in N_2^2(x)$. Since $N_1(x)$ is a clique, $V_1 \setminus N_1(x) \neq \emptyset$. Let xu_1u_3 be an induced path with $u_1, u_3 \in V_1$. Then $\langle w', w'', w; xu_1u_3 \rangle$ is an induced Z_3, giving a contradiction. ■

For any vertex $x \in S$, since $|V_1| \geq 2$, we have $|N_1(x)| \geq 2$. Following Claim 3.1, we have $|N_2(x)| \geq 2$. So $|N_i(x)| \geq 2$ for both $i \in [1, 2]$. Thus V_1 and V_2 are symmetric if we only use the property $|N_i(x)| \geq 2$. We assume, without loss of generality, that V_2 is not a clique. Moreover, in the following proof, we let x be a vertex in S and $W_1, W_2, \cdots, W_k$ be the vertex sets of components of $\langle N_2^2(x) \rangle$. Let $W_i' = N(W_i)$ for each $i \in [1, k]$. Clearly $W_i' \subseteq N_2(x)$, otherwise, Z_3 will be found.

Claim 4.3.2. *$W_i \cup W_i'$ is a clique for all $i \in [1, k]$.*

Proof: For any $w \in W_i'$, we have $N(w) \supseteq W_i$, otherwise there is an induced path xww_1w_2 with $w_1, w_2 \in W$, giving a contradiction to Claim 4.3.1. Then $E(W_i, w_i')$ contains every possible edge between W_i and W_i'. Since G is claw-free, W_i is a clique. Since $W_i' \in N_2(w)$, it is a clique. So $W_i \cup W_i'$ is a clique. ■

Claim 4.3.3. *If $N_1(x) \neq V_1$, then $N_2^2(x)$ is an independent set. Thus $|W_1| = |W_2| = \cdots = |W_k| = 1$.*

Proof: Since $V_1 \backslash N_1(x) \neq \emptyset$, there exists an induced path xu_1u_3 with $u_1, u_3 \in V_1$. If there exists $i \in [1, k]$ such that $|W_i| \geq 2$, denote $w_1, w_2 \in W_i$ and $w' \in W_i'$, then $\langle w_1, w_2, w'; xu_1u_3 \rangle$ is a Z_3, giving a contradiction. ■

Claim 4.3.4. *For any $y \in S \setminus \{x\}$, we have*

1) $N_2(x) = N_2(y)$ and $xy \in E(G)$ if $N_2(y) \cap N_2^2(x) = \emptyset$.

2) $N_2^2(x) = W_1$ and $N_2(y) = W_1 \cup W_1'$ if $N_2(y) \cap N_2^2(x) \neq \emptyset$.

Proof 1) Suppose that $N_2(y) \cap N_2^2(x) = \emptyset$. Since V_2 is not a clique, $N_2^2(x) \neq \emptyset$. Thus, there exists an induced path xw_1w_2 with $w_1, w_2 \in V_2$. We can show that $N(y) \cap N(w_2) \neq \emptyset$. Otherwise, let $w \in N(y) \setminus N(w_2)$. Then yww_1w_2 is an induced P_4. This path and two neighbors of y in V_1 induce a Z_3, showing a contradiction. Similarly, we can prove that $N(w_2) \subseteq N(y)$. We now claim that $N_2(x) \setminus N(w_2) \subseteq N(y)$. Otherwise, there exists a vertex $w'' \in N(x) \setminus N(y)$, then $\langle w_1; y, w_2, w'' \rangle$ is a claw.

2) Suppose that $N_2(y) \cap N_2^2(x) \neq \emptyset$. Recall that $N_2^2(x)$ is a union of disjoint cliques $W_1, W_2, \cdots, W_k$. We assume, without loss of generality, $N_2(y) \cap W_1 \neq \emptyset$. Let $w_1 \in W_1 \cap N(y)$ and $w_1' \in N(w_1) \cap N(x)$. Since $N_2(y)$ is a clique, $N_2(y) \subseteq W_1 \cup W_1'$. If $N_2^2(x) \setminus W_1 \neq \emptyset$, then $N_2^2(x) \backslash W_1 \nsubseteq N_2(y) \cup N_2^2(y)$, which contradicts Claim 4.3.1. Thus $N_2^2(x) = W_1$. We now show that $N_2(y) = W_1 \cup W_1'$. Since V_2 is not a clique, there exists a vertex $w' \in N_2(x) \backslash W_1 \cup W_1'$. If there exists a vertex $w'' \in W_1' \setminus N(y)$, then there is an induced path $yw_1w''w'$, together with the two neighbors of y in V_1, it will induce a Z_3. Thus $W_1' \subseteq N_2(y)$. For any $w''' \in W_1 \backslash \{w_1\}$, since $w''', y, w' \in N(w_1')$ and G is claw-free, $w'''y \in E(G)$, which implies $W_1 \subseteq N_2(y)$. ■

Following Claim 4.3.4, S could be divided into two subsets, S_1 and S_2, such that $S_1 = \{y \in S \mid N_2(y) = N_2(x), xy \in E(G)\}$ and $S_2 = \{y \in S \mid N_2(y) = W_1 \cup W_1'\}$.

Claim 4.3.5. *Both $S_1 \cup N_2(x)$ and $S_2 \cup W_1 \cup W_1'$ are cliques.*

Proof: By Claim 4.3.1, $S_1 \cup N_2(x)$ is clearly a clique. Since V_2 is not a clique, there exists $w_1 \in W_1'$ and $w_2 \in N_2(x) \setminus \{W_1'\}$. For any $y_1, y_2 \in S_2$, to avoid $\langle w_1; y_1, y_2, w_2\rangle$ be a claw, $y_1y_2 \in E(G)$. Together with Claim 4.3.1, we have $S_2 \cup W_1 \cup W_1'$ as a clique. ■

Claim 4.3.6. *If $S_2 \neq \emptyset$, then $|W_1'| \geq 2$.*

Proof: Since V_2 is not a clique, $N_2(x) \setminus W_1'$ is a component of $G \setminus (S_1 \cup W_1')$. By the minimality of $|S|$, we have $|S_1 \cup W_1'| \geq |S| \geq |S_1 \cup S_2|$, which implies $|W_1'| \geq |S_2|$. If $|W_1'| = 1$, then $|S_2| = 1$, which implies $S_2 \cup W_1'$ is a vertex cut of size 2, contradicting that G is 3-connected. ■

Since V_1 and V_2 are symmetric in this case, we can get similar properties for $\langle V_1\rangle$ and can partition S into two subsets: $S_1' = \{y \in S \mid N_1(y) = N_1(x)\}$ and $S_2' = \{y \in S \mid N_1(y) = U_1 \cup U_1'\}$, where $U_1 = N_1^2(x)$ and $U_1' = N(U_1) \cap N(x)$.

Our goal is to find a strong spanning Halin subgraph H in G. First we want to find the part of H in $\langle V_2 \cup S\rangle$ by considering two cases.

Case 1: $S_2 = \emptyset$.

In this case, $N_2(y) = N_2(x)$ for any $y \in S$. Since G is 3-connected, $|W_i'| \geq 3$. Let $w_{i_1}', w_{i_2}', w_{i_3}' \in W_i'$, $V_2' = N_2(x) \setminus (\cup_{i=1}^{k}\{w_{i_1}', w_{i_2}', w_{i_3}'\}) = \{w_1', w_2', \cdots, w_{t'}'\}$, $P_i = w_{i_2}' P_i w_{i_3}'$ be hamiltonian paths in $(W_i \cup W_i') \setminus \{w_{i_2}'\}$ for $i \in [1, k]$ and $P_0 = w_1 P_0 w_{t'}$ be a hamiltonian path in $\langle V_2'\rangle$. Let $S = \{x_1, x_2, \cdots, x_t\}$ and $P' = x_3 P' x_t$ be a hamiltonian path in $S \setminus \{x_1, x_2\}$. Let $C_2 = P_0 P_1 \cdots P_k P' \cup \{x_2 w_1\}$ be a path and the vertices on the star $\{x_1\} \cup (\cup_{i=1}^{k}\{w_{i_2}'\})$, where $\{x_1\}$ is the center, be stems of T_2 with $N_C(x_1) = V(S) \cup V_2'$ and $N_C(w_{i_2}') = W_i \cup \{w_{i_1}', w_{i_3}'\}$ for all $i \in [1, k]$. Then T_2 is a HIST of $\langle V_2 \cup S\rangle$ and $V(C_2) = \{u \in T_2 \mid deg_{T_2}(u) = 1\}$.

Case 2: $S_2 \neq \emptyset$.

In this case, both $S_1 \cup N_2(x)$ and $S_2 \cup W_1 \cup W_1'$ are cliques and $N_2^2(x) = W_1$. We let $S_1 = \{x_1, x_2, \cdots, x_{t_1}\}$ and $S_2 = \{y_1, y_2, \cdots, y_{t_2}\}$, where we assume $t_1 \geq 2$ since $|S| \geq 3$. In particular, we regard y_1 is the same vertex as x_t in Case 1. We denote $W_1 = \{w_{2_1}, w_{2_2}, \cdots, w_{2_{k_2}}\}$, $N_2(x) = \{w_{1_1}, w_{1_2}, \cdots, w_{1_k}\}$ and assume $w_{1_1}, w_{1_2} \in W_1'$ by Claim 4.3.6. Let $P' = x_2 P' x_{t_1}$, $P'' = y_{t_2} P'' y_1$, $P_1 = w_{1_2} P_1 w_{1_3}$ and $P_2 = w_{2_1} P_2 w_{2_{k_2}}$ be hamiltonian paths in $S_1 \setminus \{x_1\}$, S_2,

$N_2(x) \setminus \{w_{1_1}\}$ and W_1, respectively. Let $C_2 = P'P_1P_2P''$ be a path and $\{x_1, w_{1_1}\}$ be stems of T_2, with $N_{C_2}(w_1) = (V(P') \setminus \{x_2\}) \cup V(P_1) \cup V(P_2) \cup V(P'')$ and $N_{C_2}(x_1) = \{x_2\}$. Then T_2 is a HIST of $\langle V_2 \cup S \rangle$ and $V(C_2) = \{u \in T_2 \,|\, deg_{T_2}(u) = 1\}$.

We now find the other part of H in $\langle V_1 \cup \{x_1\} \rangle$ by considering three following cases.

Case 1: We assume that $|V_1| \geq 3$ and it is not a clique.

If $|V_1| \geq 3$ and V_1 is not a clique, similarly as V_2 is not a clique, we can also find a HIST T_1 in $\langle V_1 \cup \{x_1\} \rangle$ and a path C_1 goes through all leaves of T_1. Moreover, we can also guarantee $C_1 \cup T_1$ is a planar graph and x_1 is one stem of T_1. We may denote the two endpoints of C_1 by u_1 and u_2.

Case 2: We assume that $|V_1| \geq 3$ and it is a clique.

Let $u_1, u_2, u_3 \in V_1$ such that $u_3x_1, u_1x_2, u_2x_t \in E(G)$. We denote by $C_1 = u_1C_1u_2$ as a hamiltonian path in $V_1 \setminus \{u_3\}$ and T_1 as a HIST of $\langle V_1 \cup \{x_1\} \rangle$ with stem u_1, which connects to all leaves.

Case 3: We assume that $|V_1| \leq 2$.

Note that $N_1(x) = V_1$ for all $x \in S$. Let $C_1 = u_1u_2$ and T_1 be a HIST of $\langle V_1 \cup \{x_1\} \rangle$ with stem x_1, which connects to leaves $\{u_1, u_2\}$.

Note that we can assume $u_1x_2, u_2y_1 \in E(G)$. This is clearly true if V_1 is clique or $S_2' = \emptyset$ or $S_2 = \emptyset$. If neither S_2 nor S_2' is empty, since x_2 is an arbitrary vertex of S_1 and y_1 is an arbitrary vertex of S_2, we only need to show that $x_2 \in S_1'$ and $y_1 \in S_2'$. Suppose this is not true. Then either $x_2, y_1 \in S_1'$, which implies $S = S_1'$, or $x_2, y_1 \in S_2'$, which implies $S_2 \cup (S_1 \setminus \{x_1\}) \subseteq S_2'$. Contradicting to the assumption that $S_2' \neq \emptyset$ and $|S_1'| \geq 2$.

Now we let $T = T_1 \cup T_2$, $C = C_1 \cup C_2 \cup \{u_1x_2, u_2y_1\}$ and $H = T \cup C$. Then T is a HIST of G and C is a cycle obtained by connecting all leaves of T, H is planar. Moreover, the subgraph induced on all stems of T is a star with center x. Furthermore, we have $|S - 1| \geq 2$ or $x_1w_{1_2} \in E(G)$. Thus, H is a strong spanning Halin subgraph of G. We only give the figure for $S_2 \neq \emptyset$ and $S_2' = \emptyset$ (See Figure 4.1). ■

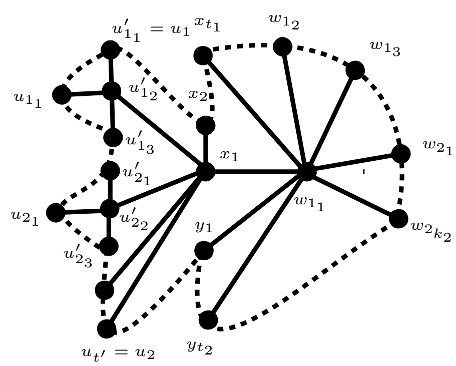

Figure 4.1. $\{claw, Z_3\}$-free graph with $|V_1| \geq 2$ and V_1 or V_2 is not a clique.

Chapter 5

LINE GRAPHS

A graph G is called the line graph of H if $V(G) = E(H)$ and two vertices in G are adjacent if and only if the corresponding edges in H are incident. In this chapter, we will show following two propositions.

Proposition 7. *If G is a 3-connected Z_3-free line graph, then G contains a strong spanning Halin subgraph.*

Proposition 8. *If G is a 3-connected $B_{1,2}$-free line graph, then G contains a strong spanning Halin subgraph.*

5.1 Characterization of line graphs

In 1970, Beineke [3] used the following lemma to give a full characterization of line graphs.

Lemma 5.1.1. [3] *The following statements are equivalent for a graph G.*

1. *G is a line graph of some graphs.*

2. *The edges of G can be partitioned into complete subgraphs in such a way that no vertex belongs to more than two of the subgraphs.*

3. *None of the nine graphs in Figure 5.1 is an induced subgraph of G.*

Assume that G is a graph with n-vertex and S is a minimum vertex cut of G. We still follow the definitions and notations in Section 4.1. Let G_1 and G_2 be the two components of $G \setminus S$, and $V_1 = V(G_1)$, $V_2 = V(G_2)$. Subject to the minimality of $|S|$, we always assume that $|V_1|$ is minimum. From Lemma 5.1.1, we can easily get the following two corollaries.

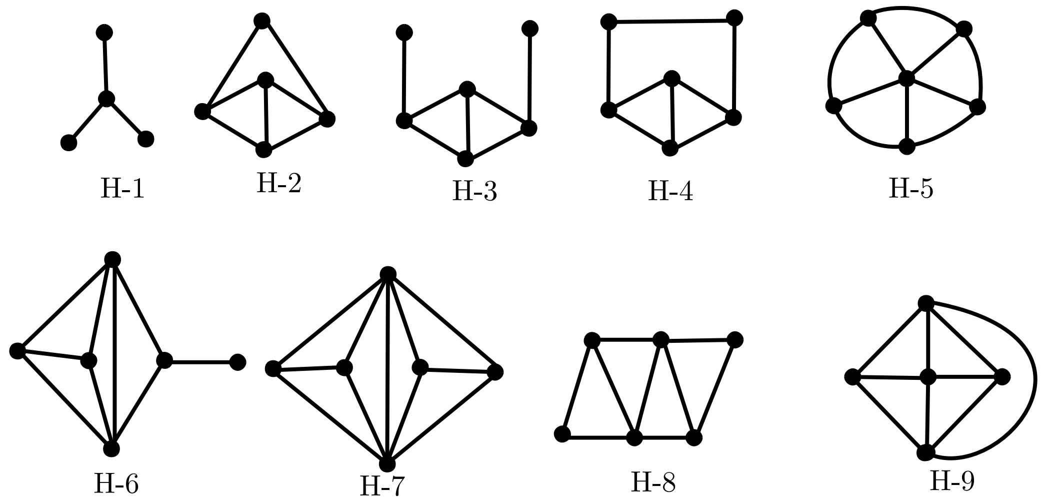

Figure 5.1. Forbidden subgraphs for line graph

Corollary 5.1.1. *For any $x, y \in S$ and $i \in [1, 2]$, if $|N_i(x) \cap N_i(y)| \geq 2$, then $xy \in E(G)$.*

Proof. Suppose to the contrary, there exist two vertices x, y in S having two common neighbors w_1, w_2 in V_2. Since $N_i(x)$ is a clique, $w_1w_2 \in E(G)$. If $N_{3-i}(x) \cap N_{3-i}(y) \neq \emptyset$, then G contains an induced subgraph isomorphic to H-2. Otherwise, let $u_1 \in N_{3-i}(x) \setminus N_{3-i}(y)$ and $u_2 \in N_{3-i}(y) \setminus N_{3-i}(x)$. Then either $\langle u_1, u_2, x, w_1, w_2, y\rangle \cong$ H-3 or $\langle u_1, u_2, x, w_1, w_2, y\rangle \cong$ H-4, showing a contradiction. ■

Corollary 5.1.2. *If there exists $x \in S$ such that $N_1(x) = V_1$, then $N_1(y) = V_1$ for all $y \in S$. Consequently, $V_1 \cup S$ is a clique.*

Proof. By Claim 4.1.2, $|N_1(x)| \geq 2$ for every $x \in S$. So this corollary holds if $|V_1| \leq 2$. Thus we suppose $|V_1| \geq 3$, by Claim 4.1.2 again, for any $y \in S$, $|N_1(x) \cap N_1(y)| = |N_1(y)| \geq 2$. By Corollary 5.1.1, $xy \in E(G)$. Let $u_1, u_2 \in N_1(y)$. If there exists $u_3 \in V_1 \setminus N_1(y)$, then $\langle u_3, x, u_1, u_2, y\rangle \cong$ H-9, the contradiction implies $N_1(y) = V_1$ for all $y \in S \setminus \{x\}$. When apply Corollary 5.1.1 again, we get $V_1 \cup S$ is a clique. ■

5.2 Proof of 3-connected Z_3-free line graphs

Proposition 9. *If G is a 3-connected Z_3-free line graph, then G contains a strong spanning Halin subgraph.*

To prove Proposition 9, we only need to consider two cases depending on whether $|V_1| \geq 2$. By Proposition 6, we only need to consider either $|V_1| \geq 2$ and both V_1 and V_2 are cliques or $|V_1| = 1$.

Case 1: $|V_1| \geq 2$.

By Proposition 6, both V_1 and V_2 are cliques.

Claim 5.2.1. *For each $x \in S$, if $|V_i| \geq 4$ and $deg_{G_i}(x) \geq 2$, then $N_i(x) = V_i$ for each $i \in [1, 2]$.*

Proof. Let $u_1, u_2 \in N_i(x)$ and $w_1 \in N_{3-i}(x)$. If $|V_i \setminus N_i(x)| \geq 2$, let $u_3, u_4 \in V_i \setminus N_i(x)$, then $\langle u_3, u_4, u_1, u_2, x, w_1 \rangle \cong$ H-6. Assume $|V_i \setminus N_i(x)| = 1$. Let $u_1, u_2, u_3 \in N_i(x)$ and $u_4 \in V_i \setminus N_i(x)$. Then $\langle u_4, u_3, u_1, u_2, x \rangle \cong$ H-9, showing a contradiction. ■

We will consider the following two cases to show that there exists a strong spanning Halin subgraph in G.

Case 1.1: $|V_2| \neq 3$.

Claim 5.2.2. *For any $x \in S$, we have $N_1(x) = V_1$ or $N_2(x) = V_2$.*

Proof. Suppose to the contrary, there exists $x \in S$ such that $N_1(x) \neq V_1$ and $N_2(x) \neq V_2$. Since $deg_{G_1}(x) \geq 2$, by Claim 5.2.1 and we assume $|V_2| \neq 3$, we have $|V_1| = 3$. Since $|V_2| \geq |V_1|$, we have $|V_2| \geq 4$. Applying Claim 5.2.1 again, $|N_2(x)| = 1$. Denote $V_1 = \{u_1, u_2, u_3\}$, $N_1(x) = \{u_1, u_2\}$ and $N_2(x) = \{w_1\}$. Then $\langle w_2, w_3, w_1; x, u_2, u_3 \rangle$ is a Z_3, giving a contradiction. ■

Claim 5.2.3. *We claim that $V_1 \cup S$ or $V_2 \cup S$ is a clique.*

Proof. Suppose there exists $x \in S$ such that $V_1 \setminus N_1(x) \neq \emptyset$. Then $N_2(x) = V_2$ by Claim 5.2.2. Since $|V_2| \geq 4$, similarly as Corollary 5.1.2, we can show that $N_2(y) = V_2$ for all $y \in S \setminus \{x\}$. Thus $V_2 \cup S$ is a clique. ■

By Claim 5.2.3, we may assume $V_1 \cup S$ is a clique. Let $x, y, z, z' \in S$, $u_1, u_2 \in V_1$, $w_1, w_2, w_3 \in V_2$ such that $u_1, w_2 \in N(y)$, $w_1 \in N(x)$, $u_2 \in N(z)$ and $w_3 \in N_2(z')$. Since V_1, $S \setminus \{x, y\}$ and $V_2 \setminus \{w_1\}$ are cliques, let $P_1 = u_1P_1u_2$, $P_2 = zP_2z'$ and $P_3 = w_3P_3w_2$ be hamiltonian paths in them, respectively. Let $C = P_1P_2P_3 \cup \{w_2y, yu_1\}$ be a cycle and $\{x, w_1\}$ be stems of T with $N_C(x) = V(P_1) \cup V(P_2) \cup \{y\}$ and $N(w_1) = V(P_3)$. Let $H = T \cup C$. Clearly, H is a strong spanning Halin subgraph of G (See Figure 5.2(1)).

Case 1.2: We assume that $|V_2| = 3$.

If there exists $x \in S$ such that $N_1(x) = V_1$ or $N_2(x) = V_2$, then similarly as Case 1.1, we can find a strong spanning Halin subgraph in G. Thus we can assume $N_1(x) \neq V_1$ and $N_2(x) \neq V_2$ for all $x \in S$. Consequently, we have $|V_1| = |V_2| = 3$. Denote $V_1 = \{u_1, u_2, u_3\}$ and $V_2 = \{w_1, w_2, w_3\}$. By Lemma 4.1.2, we know $S = \{x, y, z\}$. By symmetric, we may assume $N_1(x) = \{u_1, u_2\}$, $N_1(y) = \{u_2, u_3\}$, $N_1(z) = \{u_3, u_1\}$, $N_2(x) = \{w_1, w_2\}$, $N_2(y) = \{w_1, w_3\}$ and $N_2(z) = \{w_2, w_3\}$. Let $C = xu_1u_3zw_2w_1x$ be a cycle and all vertices on the path u_2yw_3 be stems of T with $N_C(u_2) = \{u_1, u_3, x\}$, $N_C(y) = \{w_1\}$ and $N_C(w_3) = \{w_2, z\}$. Let $H = T \cup C$. Clearly, that H is a strong spanning Halin subgraph of G (See Figure 5.2(2)).

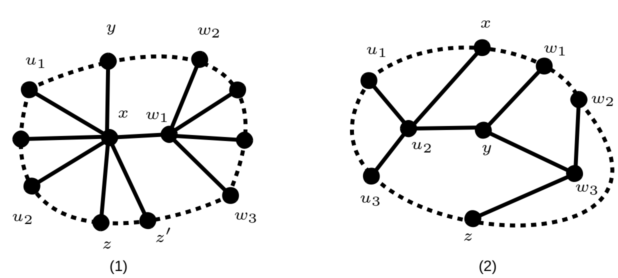

Figure 5.2. $|V_1| \geq 2$, both V_1 and V_2 are cliques.

Case 2: $|V_1| = 1$.

Lemma 5.2.1. *Let G be a 3-connected, claw-free graph, and let S be as a minimum vertex cut, V_1 and V_2 be the two components of $G \setminus S$. If $V_1 = \{u\}$ is singleton and V_2 is a clique, then there exists a strong spanning Halin subgraph in G.*

Proof. We will find a strong spanning Halin subgraph according to following two cases.

Subcase 1: $\langle S \rangle$ is connected.

Chvátal and Erdös [21], in 1972 , showed that: *If G is a k-connected graph with no independent set of $k+2$ vertices, then G contains a hamiltonian path.* Since $V_1 = \{u\}$ and $N(u) \geq S$, $\langle S \rangle$ does not contain an independent set with more than two vertices. Thus $\langle S \rangle$ contains a hamiltonian path, say $P = x_1 P x_t$, where we denote $S = \{x_1, x_2, \cdots, x_t\}$ if S is connected.

If $V_2 = \{w\}$ and $S = \{x_1, x_2, x_3\}$, let $C = ux_1wx_3u$ be a cycle and $\{x_2\}$ be the stem of T with $N_C(x_2) = \{u, x_1, w, x_3\}$. If $|S| \geq 4$, let $C = x_2 P x_t \cup \{x_t w, wx_2\}$ be a cycle and $\{u, x_1\}$ be stems of T with $N_C(u) = S \setminus \{x_2\}$ and $N_C(x_1) = \{x_2, w\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G.

If $V_2 = \{w_1, w_2\}$ and $N_2(x_1) = \{w_1, w_2\}$(or $N_2(x_t) = \{w_1, w_2\}$), we may assume $w_1 \in N_2(x_2)$ and $w_2 \in N_2(x_t)$ by Lemma 4.1.2. Let $C = x_2 P x_t \cup \{x_t w_2, w_1 w_2, w_1 x_2\}$ be a cycle and $\{u, x_1\}$ be stems of T with $N_C(u) = S$ and $N_C(x_1) = \{w_1, w_2\}$. If $V_2 = \{w_1, w_2\}$, $N_2(x_1) = \{w_1\}$ and $N_2(x_t) = \{w_2\}$, by Lemma 4.1.2 again, $N_2(x_i) = \{w_1, w_2\}$ for all $x_i \neq x_1, x_t$. Let $C = x_3 P x_t \cup \{x_t u, ux_1, x_1 w_1, w_1 x_3\}$ be a cycle and $\{w_2, x_2\}$ be stems of T with $N_C(x_2) = \{x_1, u\}$ and $N_2(w_2) = (S \setminus \{x_1, x_2\}) \cup \{w_1\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.3(1)).

If $|V_2| \geq 3$ and $S = \{x_1, x_2, x_3\}$, denote $w_1 \in N_2(x_1)$, $w_2 \in N_2(x_2)$ and $w_3 \in N_2(x_3)$ by Lemma 4.1.2. Let $P_2 = w_1 P_2 w_3$ be a hamiltonian path in $\langle V_2 \setminus \{w_2\} \rangle$. Set $C = P_2 \cup \{x_1 u, ux_3\}$ be a cycle and $\{x_2, w_2\}$ be stems of T with $N_2(x_2) = \{x_1, u, x_3\}$ and $N_2(w_2) = V(P_2)$. If $|V_2| \geq 3$ and $|S| \geq 4$, denote $w_3 \in N_2(x_t)$. Let $P_2 = w_2 P_2 w_3$ be a hamiltonian path in

$\langle V_2 \setminus \{w_1\}\rangle$. Set $C = x_2Px_tP_2$ be a cycle and all vertices on the path ux_1w_1 be stems of T with $N_C(u) = V(P) \setminus \{x_1, x_2\}$, $N_C(x_1) = \{x_2\}$ and $N_C(w_1) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.3 (2)).

Subcase 2: The induced subgraph $\langle S\rangle$ is disconnected.

Since G is claw-free and $S \subseteq N(u)$, $\langle S\rangle$ consists of exact two vertex disjoint cliques, say S_1 and S_2. We may assume $S_1 = \{x_1, x_2, \cdots, x_t\}$, $S_2 = \{y_1, y_2, \cdots, y_{t'}\}$ and $|S_1| \geq |S_2|$. Moreover, since G is 3-connected, we know $|V_2| \geq 2$.

If $V_2 = \{w_1, w_2\}$, then $S = \{x_1, x_2\}$, $S_2 = \{y_1\}$ and $N_2(x_2) = \{w_1, w_2\}$. We denote $w_1 \in N_2(x_1)$ and $w_2 \in N_2(x_2)$. Let $C = ux_1w_1y_1u$ be a cycle and $\{x_2, w_2\}$ be stems of T with $N_C(x_2) = \{u, x_1\}$ and $N_C(w_2) = \{y_1, w_1\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G.

If $V_2 = \{w_1, w_2, w_3\}$, there must exist $x_i \in S_1$ and $y_j \in S_2$ such that $N_2(x_i) \cap N_2(y_j) \neq \emptyset$. We may denote $w_2 \in N_2(x_1) \cap N_2(y_1)$ and $w_1 \in N_2(x_2)$. Set $P = x_2Px_t$ and $P' = y_2P'y_{t'}$ be hamiltonian paths in $S_1 \setminus \{x_1\}$ and $S_2 \setminus \{y_1\}$, respectively. If $S_2 = \{y_1\}$, let $C = P \cup \{ux_t, x_2w_1, w_1w_3, w_3y_1, y_1u\}$ be a cycle and $\{x_1, w_2\}$ be stems of T with $N_C(x_1) = V(P) \cup \{u\}$ and $N_C(w_2) = \{w_1, w_3, y_1\}$. If $|S_2| \geq 2$, we may assume $w_3 \in N_2(y_2)$. Let $C = PP' \cup \{ux_t, x_2w_1w_1w_3, w_3y_2, y_{t'}u\}$ be a cycle and all vertices on the path $x_1w_2y_1$ be stems of T with $N_C(x_1) = V(P) \cup \{u\}$, $N_C(w_2) = \{w_1\}$ and $N_C(y_1) = V(P') \cup \{w_3\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G.

If $|V_2| \geq 4$ and $S_2 = \{y_1\}$, let $w_1, w_2 \in N_2(x)$. Let x_1 and x_2 be two vertices in S_1 and $w_4 \in N_2(x_1)$(we may take $w_4 = w_1$ if $N_2(x_1) \subseteq \{w_1, w_2\}$) and $w_3 \neq w_1, w_2, w_4$ in $N_2(x_2)$. Denote by $P = zPx_t$ and $P_2 = w_3P_2w_2$ be two hamiltonian paths in $S \setminus \{x_1\}$ and $V_2 \setminus \{w_1, w_4\}$. Let $C = PP_2 \cup \{w_2y_1, xu, ux_t\}$ and all vertices on the path yw_4w_1 be stems of T with $N_C(x_1) \subseteq V(P) \cup \{u\}$, $N_C(w_4) \subseteq V(P_2) \setminus \{w_2\}$ and $N_C(w_1) = \{y_1, w_2\}$. Let $H = T \cup C$, then H is a strong spanning Halin subgraph (See Figure 5.3(3)).

If $|V_2| \geq 4$, $|S_2| \geq 2$ and $|S_1| \geq 3$. Let w_1, w_2, w_3, w_4 be four distinct vertices of V_2 such that $w_1x_1, w_2x_2, x_3y_1, w_4y_2 \in E(G)$. Set $P = x_2Px_t$, $P' = y_2P'y_{t'}$ and $P_2 = w_2P_2w_4$ be hamiltonian paths in $S_1 \setminus \{x_1\}$, $S_2 \setminus \{y_1\}$ and $V_2 \setminus \{w_1, w_3\}$, respectively. Let $C =$

$PP_2P' \cup \{y_{t'}u, ux_t\}$ be a cycle and all vertices on the path $x_1w_1w_3y_1$ be stems of T with $N_C(x_1) = V(P)$, $N_C(w_1) \subseteq V(P_2) \setminus \{w_4\}$, $N_C(w_3) = \{w_4\}$ and $N_C(y_1) = V(P') \cup \{u\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.3(4)).

If $|V_2| \geq 4$, $S_1 = \{x_1, x_2\}$ and $S_2 = \{y_1, y_2\}$. Similarly, denote $w_1x_1, w_2x_2, w_3y_1, w_4y_2 \in E(G)$. Since $\langle w_1, w_4, w_3; y_1, u, x_2 \rangle$ is not a Z_3, we have either $N_2(x_1) \cap N_2(x_2) \neq \emptyset$ or $N_2(y_1) \cap N_2(y_2) \neq \emptyset$ or $N_2(x_i) \cap N_2(y_j) \neq \emptyset$, where $i, j \in [1, 2]$.

Assume $N_2(x_i) \cap N_2(y_j) \neq \emptyset$, for $\{i, j\} = \{1, 2\}$. We assume, without loss of generality, $w_1 \in N_2(x_1) \cap N_2(y_1)$. Let $P_2 = w_2P_2w_4$ be a hamiltonian path in $V_2 \setminus \{w_1, w_3\}$ and all vertices on the path $x_1w_1y_1w_3$ be stems of T with $N_C(x_1) = \{u, x_2\}$, $N_C(w_1) = \{w_2\}$, $N_C(y_1) = \{y_2\}$ and $N_C(w_3) = V(P_2) \setminus \{w_2\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.3(5)).

If $N_2(x_1) \cap N_2(x_2) \neq \emptyset$(or $N_2(y_1) \cap N_2(y_2) \neq \emptyset$). Let $w_3 \in N_2(y_1) \cap N_2(y_2)$ and $P_2 = w_2P_2w_4$ be a hamiltonian path in $V_2 \setminus \{w_1, w_3\}$. Let $P = w_4y_2y_1ux_2w_2$ and $C = P_2P$ be a cycle and all vertices on the path $x_1w_1w_3$ be stems of T with $N_C(x_1) = \{u, x_2\}$, $N_C(w_1) = V(P_2) \setminus \{w_4\}$ and $N_C(w_3) = \{w_4, y_1, y_2\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.3(6)).

By Lemma 5.2.1, we only need to show that if $V_1 = \{u\}$ and V_2 is not a clique, then the structure of $\langle V_2 \rangle$ can be described and there exists a strong spanning Halin subgraph in G, we plan to show this by following two case.

Case 2.1: There exists a triangle $x_1x_2x_3x_1$ in $\langle S \rangle$.

Lemma 5.2.2. *If G is a 3-connected claw-free graph with S as a minimum vertex cut, V_1, V_2 are the exact two components of $G \setminus S$. If $V_1 = \{u\}$ and there exists a triangle $x_1x_2x_3x_1$ in S, then either V_2 is a clique or $|V_2| \leq 6$ and the structure of V_2 can be described.*

Proof. We prove Lemma 5.2.2 by the following series of claims.

Claim 5.2.4. *If $x_1x_2x_3x_1$ is a triangle in S, then $N_2(x_1) \cap N_2(x_2) \cap N_2(x_3) = \emptyset$.*

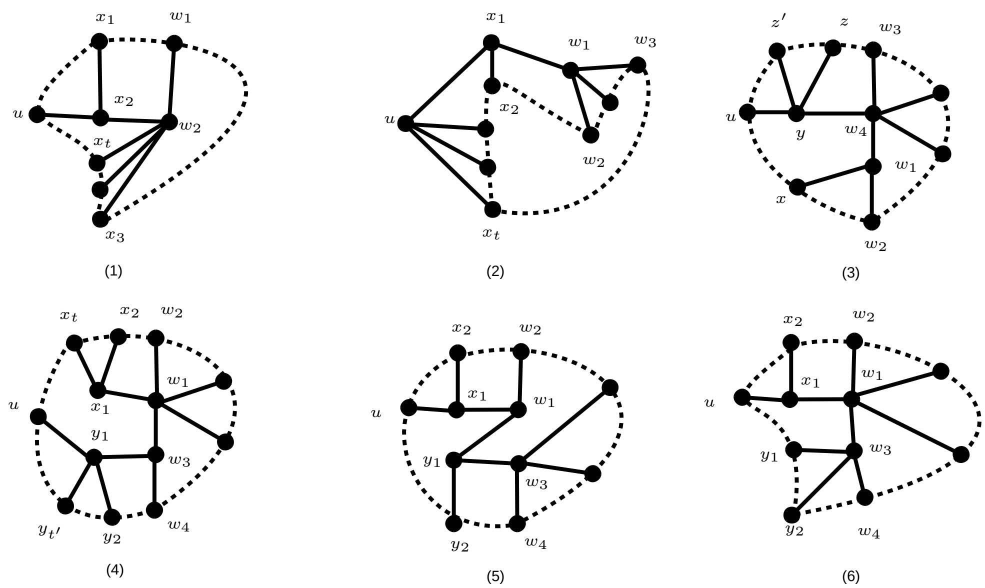

Figure 5.3. $V_1 = \{u\}$ and V_2 is a clique.

Proof. Otherwise, let $w \in N_2(x_1) \cap N_2(x_2) \cap N_2(x_3)$, then $\langle u, x_1, x_2, x_3, w\rangle \cong$ H-9, a contradiction. ■

Claim 5.2.5. *For any two distinct i and j with $1 \le i \le j \le 3$, we have $|N_2(x_i) \cap N_2(x_j)| \le 1$, for any two distinct i and j with $1 \le i \le j \le 3$.*

Proof. Otherwise, let $\{i, j, k\} = \{1, 2, 3\}$ and $w_1, w_2 \in N_2(x_i) \cap N_2(x_j)$. By Claim 5.2.4, $w_1, w_2 \notin N_2(x_k)$, we have $\langle x_k, u, x_i, x_j, w_1, w_2\rangle \cong$ H-7, showing a contradiction. ■

Claim 5.2.6. *If $N_2(x_i) \cap N_2(x_j) \neq \emptyset$, then $N_2^2(x_i) \subseteq N_2(x_j) \cup N_2(x_k)$ and $N_2^2(x_j) \subseteq N_2(x_i) \cup N_2(x_k)$, where $\{i, j, k\} = \{1, 2, 3\}$.*

Proof. Let $w_1 \in N_2(x_i) \cap N_2(x_j)$. If there exists $w \in N_2^2(x_i)$ such that $w_1 w \in E(G)$, then since $\langle x_k, u, x_i, x_j, w_1, w\rangle \ncong$ H-7, by Claim 5.2.4, $w \in N_2(x_j) \cup N_2(x_k)$. Otherwise, there must exist $w_2 \in N_2(x_i) \setminus N_2(x_j)$ such that $w_2 w \in E(G)$. To avoid $\langle u, x_k, x_j; w_1 w_2 w\rangle$ to be a Z_3 and $\langle w_2; w_1, x_k, w\rangle$ be a claw, by Claim 5.2.4 and 5.2.5, we have $w \in N_2(x_j) \cup N_2(x_k)$.

Similarly, we can show that $N_2^2(x_j) \subseteq N_2(x_i) \cup N_2(x_k)$. ■

For this case, in the following, let $R_i = N_2(x_i) \setminus (N_2(x_j) \cup N_2(x_k))$ and reserve $w_i \in R_i$ if $R_i \neq \emptyset$ and $w_{ij} \in N_2(x_i) \cap N_2(x_j)$ if $N_2(x_i) \cap N_2(x_j) \neq \emptyset$, where $\{i, j, k\} = \{1, 2, 3\}$.

Claim 5.2.7. *If $N_2(x_i) \cap N_2(x_j) = \emptyset$, denote $R_{i+3} = N_2^2(x_i) \setminus (R_1 \cup R_2 \cup R_3)$ and $w_{i+3} \in R_{i+3}$ such that $w_i w_{i+3} \in E(G)$, where $\{i, j, k\} = \{1, 2, 3\}$. The following claims hold.*

1) $V_2 = \cup_{i=1}^{6} R_i$;

2) There exist $i, j \in \{1, 2, 3\}$, such that $E(R_i, R_j) \neq \emptyset$;

3) $E(R_i, R_j) \neq \emptyset$ for all $\{i, j\} \in \{1, 2, 3\}$, $|R_1| = |R_2| = |R_3| = 1$, $R_4 = R_5 = R_6$ and V_2 is a clique.

4) G contains a strong spanning Halin subgraph.

Proof. 1) Suppose to the contrary, there exists $w \in V_2 \setminus \cup_{i=1}^{6} R_i$ such that $w_4 w \in E(G)$, then $\langle u, x_2, x_1; w_1 w_4 w \rangle$ is a Z_3, showing a contradiction.

2) If $E(R_i, R_j) = \emptyset$ for all $i, j \in \{1, 2, 3\}$, since $\langle V_2 \rangle$ is connected, we may assume $w_4 w_5 \in E(G)$. To avoid $\langle u, x_2, x_1; w_1 w_4 w_5 \rangle$ be a Z_3, $w_1 w_5 \in E(G)$. However, this will force $\langle u, x_3, x_2; w_2 w_5 w_1 \rangle$ to be a Z_3.

3) By 2), we may assume $w_1 w_2 \in E(G)$, which in turn gives us $w_2 w_4 \in E(G)$ since $\langle w_1; x_1, w_4, w_2 \rangle$ is not a claw. Since neither $\langle w_4, w_2, w_1; x_1 x_3 w_3 \rangle$ nor $\langle w_1, w_2, w_4; w_3 x_3 u \rangle$ is a Z_3, we can assume $w_2 w_3 \in E(G)$, which also implies $w_1 w_3 \in E(G)$ since $\langle w_2; x_2, w_1, w_3 \rangle$ is not a claw. If there exists $w_1' \in R_1 \setminus \{w_1\}$, then to avoid $\langle u, x_3, x_2; w_2 w_1 w_1' \rangle$ be a Z_3, $w_1' w_2 \in E(G)$. However, this will force $\langle x_2; x_1, w_1, w_1', w_2 \rangle \cong$ H-2, showing a contraction. Thus $|R_1| = 1$. Similarly, we can also prove that $|R_2| = |R_3| = 1$.

For any $w \in R_5$, since $w_2 w_3 \in E(G)$ and $\langle w_2; x_2, w_3, w \rangle$ is not a claw, $w w_3 \in E(G)$, this means $R_5 \subseteq R_6$. By the same method, we will get $R_4 = R_5 = R_6$, which illustrates V_2 is a clique. ■

4) This is clearly true by Lemma 5.2.1.

Claim 5.2.8. *If $|\{w_{12}, w_{23}, w_{13}\}| = 1$, then*

1) $V_2 = N_2(x_i) \cup N_2(x_j) \cup N_2(x_k)$, where $\{i, j, k\} = \{1, 2, 3\}$;

2) $V_2 \subseteq \{w_1, w_2, w_3, w_{12}, w_{23}, w_{13}\}$.

Proof. 1) By symmetric, we may assume w_{12} exists and w_{23}, w_{13} do not exist. By Claim 5.2.6, $N_2^2(x_1) \subseteq N_2(x_2) \cup N_2(x_3)$ and $N_2^2(x_2) \subseteq N_2(x_1) \cup N_2(x_3)$. Thus we only need to show $N_2^2(x_3) \subseteq N_2(x_1) \cup N_2(x_2)$. Suppose this is not true and there exists $w_6 \in N_2^2(x_3) \setminus (N_2(x_1) \cup N_2(x_2))$ and $w_3 \in N_2(x_3)$ such that $w_3w_6 \in E(G)$. Note that $w_{12}w_6 \notin E(G)$, otherwise $\langle x_3, u, x_2, x_1, w_{12}, w_6\rangle \cong$ H-6, which in turn gives $w_{12}w_3 \notin E(G)$, otherwise $\langle w_3; x_3, w_6, w_{12}\rangle$ is a claw. On the other hand, since $|N_2(x_1) \cup N_2(x_2)| \geq 2$, we can assume w_2 exists. To avoid $\langle w_{12}, w_2, x_2; x_3w_3w_6\rangle$ be a Z_3, we have both w_3w_2 and w_2w_6 in $E(G)$. Because if $w_3w_2 \in E(G)$, then $w_2w_6 \in E(G)$, otherwise $\langle w_3; x_3, w_6, w_2\rangle$ is a claw; if $w_2w_6 \in E(G)$, then $w_2w_3 \in E(G)$, otherwise $\langle x_1, u, x_3; w_3w_6w_2\rangle$ is Z_3. However, this will force $\langle w_3, w_6, w_2; w_{12}x_1u\rangle$ to be a Z_3, showing a contradiction.

2)*Firstly, we will show* $R_3 = \{w_3\}$ *and* $w_3w_{12} \in E(G)$. Suppose this is not true, there exist $w_3, w_6 \in R_3$. Since neither $\langle w_3, x_3, x_2, x_1, w_{12}\rangle$ nor $\langle w_6, x_3, x_2, x_1, x_{12}\rangle$ isomorphic to H-2, $w_3w_{12}, w_6w_{12} \notin E(G)$. Since $\langle V_2\rangle$ is connected and $|N_2(x_1) \cup N_2(x_2)| \geq 2$, we may assume w_2 exists and $w_2w_3 \in E(G)$. To avoid $\langle u, x_1, x_2; w_2w_3w_6\rangle$ be a Z_3, we have $w_2w_6 \in E(G)$. However, this will force $\langle x_2, x_3, w_3, w_6, w_2\rangle \cong$ H-2, a contradiction.

Secondly, we will show $R_2 = \{w_2\}$. Since $\langle V_2\rangle$ is connected, we may assume $w_2w_3 \in E(G)$. If there exists $w_5 \in R_2 \setminus \{w_2\}$, then $w_5w_3 \notin E(G)$, otherwise $\langle x_3, x_2, w_5, w_2, w_3\rangle \cong$ H-2. However, this will force $\langle w_{12}, w_5, w_2; w_3x_3u\rangle$ to be a Z_3, showing a contraction.

Thirdly, we will show $|R_1| \leq 1$. If there exists $w_1, w_4 \in R_1$, to avoid $\langle w_4, w_1, w_{12}; x_2x_3w_3\rangle$ be a Z_3 and we can assume $w_3w_1 \in E(G)$ and $w_3w_4 \notin E(G)$, otherwise, $\langle x_3; x_1, w_1, w_4, w_3\rangle \cong$ H-2. But this will in turn give $\langle w_4, w_{12}, w_1; w_3x_3u\rangle$ is a Z_3 by above three observations, we get $|R_i| \leq 1$, which implies $V_2 \subseteq \{w_1, w_2, w_3, w_{12}\}$. ■

Claim 5.2.9. *If* $|\{w_{12}, w_{23}, w_{13}\}| \geq 2$, *then*

1) $V_2 = N_2(x_1) \cup N_2(x_2) \cup N_2(x_3)$.

2) If there exists $i \in [1, 3]$ *such that* $|R_i| \geq 2$, *then* V_2 *is a clique. Otherwise* $V_2 \subseteq \{w_1, w_2, w_3, w_{12}, w_{23}, w_{13}\}$.

Proof. 1) This is clearly true by Claim 5.2.6.

2) We may assume there exist $w_3, w_4 \in R_3$. The following three statements give us $V_2 = N_2(x_3)$, which implies V_2 is a clique.

Firstly, we have $w_{13} \in V_2$. Suppose this is not true, then $w_{12}, w_{23} \in V_2$ by assumption. Since $\langle w_3, w_4, w_{23}; w_{12}x_1u\rangle$ is not a Z_3, we can assume $w_{12}w_3 \in E(G)$. However, this in turn gives $\langle x_1, w_{12}, w_{23}, w_3, x_3\rangle \cong$ H-2.

Secondly, we have $w_{12} \notin V_2$. Otherwise, since $\langle w_3, w_4, w_{13}; w_{12}x_2u\rangle$ is not a Z_3, we can assume $w_{12}w_3 \in E(G)$. But this gives us $\langle u, x_2, x_1, w_{12}, w_{13}, w_3\rangle \cong$ H-8.

Thirdly, we have $R_2 = \emptyset$ *and* $R_1 = \emptyset$. We only show that $R_2 = \emptyset$. Suppose to the contrary, there exists $w_2 \in R_2$. To avoid $\langle w_3, w_4, w_{13}; x_1x_2w_2\rangle$ be a Z_3 and $\langle w_{13}; w_3, w_2, x_1\rangle$ be a claw, we can assume $w_2w_3 \in E(G)$. Moreover, since $\langle w_{13}, w_4, w_3; w_2x_2u\rangle$ is not a Z_3, we have $w_2w_{13} \in E(G)$, which implies $\langle x_2, x_3, w_3, w_{13}, w_2\rangle \cong$ H-2, or $w_4w_2 \in E(G)$, which implies $\langle x_2, w_2, w_4, w_3, x_3\rangle \cong$ H-2.

If $|R_i| \leq 1$, then by 1), we have $V_2 \subseteq \{w_1, w_2, w_3, w_{12}, w_{23}, w_{13}\}$. ■

From Claim 5.2.7, Claim 5.2.8 and Claim 5.2.9, we can see that Lemma 5.2.2 is true.

Lemma 5.2.3. *Let G be a 3-connected Z_3-free line graph with S as a minimum vertex cut and V_1, V_2 as the exact two components of $G \setminus S$. If $V_1 = \{u\}$, $|V_2| \leq 6$ and there exists a triangle, say $x_1x_2x_3x_1$, in S, then there exists a strong spanning Halin subgraph in G.*

Proof. We still follow notations that $w_i \in N_2(x_i)\setminus(N_2(x_j)\cup N_2(x_k))$ and $w_{ij} \in N_2(x_i)\cap N_2(x_j)$, where $\{i, j, k\} = \{1, 2, 3\}$. Then, by Lemma 5.2.2, $V_2 \subseteq \{w_1, w_2, w_3, w_{12}, w_{23}, w_{13}\}$.

The following series of claims are showing this lemma is true.

Claim 5.2.10. *For any $t \in S \setminus \{x_1, x_2, x_3\}$, one of the following conclusions holds.*

1) For any $t \in S \setminus \{x_1, x_2, x_3\}$, $|N_S(t) \cap \{x_1, x_2, x_3\}| \neq 2$.

2) If $N_s(t) \cap \{x_1, x_2, x_3\} = \{x_1, x_2, x_3\}$, then $N_2(t) \subseteq \{w_1, w_2, w_3\}$.

3) If $N_s(t) \cap \{x_1, x_2, x_3\} = \{x_i\}$, then $N_2(t) = N_2(x_i)$, where $i \in [1, 3]$.

4) If $N_s(t) \cap \{x_1, x_2, x_3\} = \emptyset$, then $N_2(t) \subseteq \{w_1, w_2, w_3\}$.

5) For any $t \in S$, $N_2(t) \cap \{w_1, w_2, w_3\} \neq \emptyset$.

Proof. 1) Suppose $N_s(t) \cap \{x_1, x_2, x_3\} = \{x_1, x_2\}$, then $\langle u, x_1, x_2, x_3, t\rangle \cong$ H-9, showing a contradiction.

2) We may assume $w_{12} \in N_2(t)$, then $\langle u, x_1, x_2, t, w_{12}\rangle \cong$ H-9. Since $N_2(t) \neq \emptyset$, we have $N_2(t) \subseteq \{w_1, w_2, w_3\}$.

3) By symmetric, we may assume $N_s(t) \cap \{x_1, x_2, x_3\} = \{x_3\}$, Since neither $\langle x_3; t, x_1, w_3\rangle$ nor $\langle x_3; t, x_1, w_{23}\rangle$ nor $\langle x_3; t, x_2, w_{13}\rangle$ is a claw, we have $\{w_3, w_{13}, w_{23}\} \subseteq N_s(t)$. Moreover $tw_1, tw_2, tw_{12} \notin E(G)$, otherwise either $\langle w_1, x_1, u, x_3, t\rangle$ or $\langle w_2, x_2, u, x_3, t\rangle$ or $\langle w_{12}, x_2, u, x_3, t\rangle$ isomorphic to H-2. Thus $N_2(t) = N_2(x_3)$.

4) If $N_s(t) \cap \{x_1, x_2, x_3\} = \emptyset$, then $w_{ij} \notin N_2(t)$, otherwise $\langle u, x_i, x_j, w_{ij}, t\rangle \cong$ H-2, where $\{i, j, k\} = \{1, 2, 3\}$.

5) This is easily followed by above conclusions. ■

We may denote $S_1 = \{x_1, x_2, x_3\}$, $S_2 = \{t \in S | N_s(t) \cap \{x_1, x_2, x_3\} = \{x_1, x_2, x_3\}\}$, $S_3 = \{t \in S || N_s(t) \cap \{x_1, x_2, x_3\}| = 1\}$ and $S_4 = \{t \in S | N_s(t) \cap \{x_1, x_2, x_3\} = \emptyset\}$. Clearly $S = S_1 \cup S_2 \cup S_3 \cup S_4$.

Claim 5.2.11. *By the definition of S_1, S_2, S_3 and S_4, we have:*

1) $S_1 \cup S_2$ is a clique.

2) $S_3 \cup S_4$ is a clique.

Proof. 1) This is true since otherwise $\langle \{u\} \cup S_1 \cup S_2\rangle$ contains an induced subgraph isomorphic to H-9.

2) For any $t, t' \in S_3 \cup S_4$, we may assume $tx_1, t'x_1 \notin E(G)$. Since $\langle u; t, t', x_1\rangle$ is not a claw, we have $tt' \in E(G)$. ■

Claim 5.2.12. *For any $x_i \in S_1$, we have $|N_{S_3}(x_i)| \leq 1$, where $i \in [1, 3]$.*

Proof. We may assume there exist $t, t' \in N_{S_3}(x_1)$, then $\langle t, t', x_1\rangle$ is a triangle. Let $w' \in N_2(x_1)$, then $\langle u, t, t', x_1, w'\rangle \cong$ H-9, where $w' \in \{w_1, w_{12}, w_{13}\}$ by Claim 5.2.10. ■

Claim 5.2.13. *For any $w_i \in \{w_1, w_2, w_3\}$, we have $|N_{S_2}(w_i)| \leq 1$, $|N_{S_3}(w_i)| \leq 1$ and $|N_{S_4}(w_i)| \leq 1$.*

Proof. We only show this is true for w_1. If there exist $t, t' \in S_2$(or there exist $t, t' \in S_3$) such that $tw_1, t'w_1 \in E(G)$, then $\langle u, t, t', x_1, w_1\rangle \cong$ H-9. If there exist $t, t' \in S_4$ such that $tw_1, t'w_1 \in E(G)$, then $\langle u, t, t', w_1, x_1\rangle \cong$ H-2. ■

Claim 5.2.14. *If all w_i, w_j, w_{jk} exist, then $w_iw_{jk} \notin E(G)$, where $\{i, j, k\} = \{1, 2, 3\}$.*

Proof. Suppose to the contrary, all w_1, w_2, w_{23} exist and $w_1w_{23} \in E(G)$. Since $\langle w_{23}; w_1, w_2, x_3\rangle$ is not a claw, we have $w_1w_2 \in E(G)$, which in turn gives $\langle w_1, w_2, w_{23}, x_2, x_1\rangle \cong$ H-2, showing a contradiction. ■

Claim 5.2.15. *If there exists $t \in S_4$, then $S_4 = \{t\}$ and $N_2(t) = \{w_1, w_2, w_3\}$. Moreover, if there also exists $t' \in S_2$, then $\{t, t', w_1, w_2, w_3\}$ is a clique and $|S_2| \leq 1$.*

Proof. Suppose this is not true, we may assume $tw_3 \in E(G)$ and $tw_1 \notin E(G)$. Note that $w_1w_3, w_{12}w_3 \notin E(G)$ since neither $\langle w_3; t, x_3, w_1\rangle$ nor $\langle w_3; t, x_3, w_{12}\rangle$ is a claw. If $w_2w_3 \in E(G)$, since $\langle w_3; w_2, x_3, t\rangle$ is not a claw, we have $tw_2 \in E(G)$. Moreover, since $\langle w_2, w_3, t; ux_1w_1\rangle$ is not a Z_3, we have $w_1w_2 \in E(G)$, which implies $\langle w_2; w_1, x_2, w_3\rangle$ is a claw, showing a contradiction. If $w_2w_3 \notin E(G)$, then w_{13} or w_{23} exists since $\langle V_2\rangle$ is connected. We may assume w_{13} exists, then $\langle w_1, w_{12}, w_{13}; w_3tu\rangle$ is a Z_3, showing a contradiction. Thus we have $tw_1 \in E(G)$. Similarly, we can show that $tw_2 \in E(G)$ if w_2 exists. By Claim 5.2.10, we know $N_2(t) = \{w_1, w_2, w_3\}$, and by Claim 5.2.13, we get $S_4 = \{t\}$.

If there also exists $t' \in S_2$, we may assume $t'w_3 \in E(G)$ by symmetric. Since $\langle w_3; w_{23}, t, t'\rangle$ is not a claw, we have $tt' \in E(G)$. Moreover, since neither $\langle w_2, w_3, t', x_3, x_2\rangle$ nor $\langle w_1, w_3, t', x_3, x_1\rangle$ isomorphic to H-2, we have $t'w_2 \in E(G)$, which in turn gives us $\{t, t', w_1, w_2, w_3\}$ is a clique and $|S_2| \leq 1$ by Claim 5.2.13.

By Claim 5.2.13, we know $|S_4| \leq 1$ and $|S_2| \leq 1$. ■

Claim 5.2.16. *If $S_3 \neq \emptyset$ and there exists $t' \in S_2$, then $N_2(t') = \{w_1, w_2, w_3\}$ and $S_2 = \{t'\}$.*

Proof. We may assume there exists $t \in S_3$ such that $tx_3 \in E(G)$, then $N_2(t) = N_2(x_3)$.

Firstly, we want to show that $t'w_3 \in E(G)$. Suppose this is not true and $t'w_1 \in E(G)$, then neither w_1w_3 nor tt' in $E(G)$. Otherwise $\langle w_3, w_1, t', x_1, u, t\rangle \cong$ H-4 or $\langle w_{23}, w_1, t', x_1, u, t\rangle \cong$

H-4 or $\langle t', x_1, x_3, t, u\rangle \cong$ H-9, showing a contradiction. However, if w_{23} exists, then $w_1w_{23} \notin E(G)$ by Claim 5.2.14, which in turn gives $\langle w_{23}, w_3, t; ut'w_1\rangle$ a Z_3; if w_{12} exists, then either $\langle w_3, t, x_3; t'w_1w_{12}\rangle$ is a Z_3 or $\langle w_{12}; w_1, w_2, w_3\rangle$ is a claw; if neither w_{23} nor w_{12} exists but w_{13} exists, since $\langle V_2\rangle$ is connected and neither $\langle w_{13}; w_1, w_2, w_3\rangle$ nor $\langle x_2; w_1, w_2, w_3\rangle$ is a claw, we have either $w_1w_2 \in E(G)$ or $w_2w_3 \in E(G)$, both of them will force $w_2w_{13} \in E(G)$, contradicts to Claim 5.2.14.

Note that $tt' \notin E(G)$, otherwise $\langle u, x_3, t, t'w_3\rangle \cong$ H-9.

Secondly, we want to show that $w_1t', w_2t' \in E(G)$. Suppose this is not true, then $w_1w_2 \notin E(G)$. Otherwise $\langle t', u, x_2; w_2w_1w_{13}\rangle$ is a Z_3, which implies w_{12} and w_{13} exist since $deg_G(w_1) \geq 3$ and there dose not exist $t'' \in S_2 \cup S_3 \cup S_4$ such that $t''w_1 \in E(G)$ by Claim 5.2.10. Moreover, since neither $\langle w_3; t, t', w_1\rangle$ nor $\langle w_3; t, t', w_2\rangle$ nor $\langle w_3; t, t', w_{12}\rangle$ is a claw, we have $w_3w_1, w_3w_2 \notin E(G)$, which in turn gives $\langle w_{12}, w_1, x_1; t'w_3t\rangle$ to be a Z_3, showing a contradiction.

Thus $N_2(t') = \{w_1, w_2, w_3\}$ and we have $S_2 = \{t'\}$ by Claim 5.2.13. ■

Claim 5.2.17. *If* $S_3 = S_4 = \emptyset$ *and there exists* $t \in S_2$*, then* $\langle w_1, w_2, w_3\rangle$ *is a clique and* $N_2(t) = \{w_1, w_2, w_3\}$*. In particular,* $S_2 = \{t\}$*.*

Proof. We may assume $tw_3 \in E(G)$ by symmetric.

If both w_{23} and w_{13} exist, since neither $\langle x_1, u, t; w_3w_{23}w_2\rangle$ nor $\langle x_2, u, t; w_3w_{13}w_1\rangle$ is a Z_3 and neither $\langle x_2, w_2, w_{23}, w_3, t\rangle$ nor $\langle x_1, w_2, w_{23}, w_3, t\rangle$ isomorphic to H-2, we have w_2w_3, w_1w_3, tw_2, $tw_1 \in E(G)$. Thus $N_2(t) = \{w_1, w_2, w_3\}$ and $\langle w_1, w_2, w_3\rangle$ is a clique.

If either w_{23} or w_{13} exists, we may assume w_{23} exists. Similarly, we can show that both w_2w_3 and tw_2 in $E(G)$. Since w_{13} does not exist and w_1w_{23} does not in $E(G)$ by Claim 5.2.14, we have w_1w_2 or w_1w_3 in $E(G)$, which implies $\langle w_1, w_2, w_3\rangle$ is a clique since G is claw-free. Since $\langle x_1, x_3, t, w_3, w_1\rangle \not\cong$ H-2, we have $w_1t \in E(G)$. Thus $N_2(t) = \{w_1, w_2, w_3\}$.

If neither w_{23} nor w_{13} exists, since $\langle V_2\rangle$ is connected, we can assume $w_2w_3 \in E(G)$ by symmetric. If fact, we can show that there does not exist $t' \in S_2$ such that $t'w_1 \in E(G)$. Otherwise, since w_{12} exists, similarly as w_{23} exists, we can show that $t'w_2, t'w_3 \in E(G)$.

Then $t, t' \in N_S(w_3)$, showing a contradiction. Since $deg_G(w_1) \geq 3$, we have w_1w_2 or w_1w_3 in $E(G)$, which implies $\langle w_1, w_2, w_3\rangle$ is clique since G is claw-free. Moreover, since neither $\langle x_2, x_3, t, w_3, w_2\rangle$ nor $\langle x_1, x_3, t, w_3, w_1\rangle$ isomorphic to H-2, we have both tw_2 and tw_3 in $E(G)$. Therefore, $N_2(t) = \{w_1, w_2, w_3\}$.

By Claim 5.2.13, we know $S_2 = \{t\}$. ■

Claim 5.2.18. *If $S_2 \cup S_4 = \emptyset$ and $S_3 \neq \emptyset$, then $\langle w_1, w_2, w_3\rangle$ is a clique.*

Proof. If there exist $t, t' \in S_3$, we may assume $tx_1, t'x_3 \in E(G)$ by Claim 5.2.10. Then $N_2(t) = N_2(x_1)$ and $N_2(t') = N_2(x_3)$. If neither w_{12} nor w_{23} exists, since $\langle V_2\rangle$ is connected and $\langle w_{13}; w_1, w_2, w_3\rangle$ is not a claw, we can assume $w_2w_3 \in E(G)$ by symmetric. Moreover, to avoid $\langle w_1, t, x_1; x_2w_2w_3\rangle$ be a Z_3, $\langle w_2; x_2, w_1, w_3\rangle$ or $\langle w_3; x_3, w_1, w_2\rangle$ be a claw, we have both w_1w_2 and w_1w_3 in $E(G)$. If w_{12} or w_{23} exists, we may assume w_{12} exists. To avoid $\langle w_3, t', x_3; x_1w_{12}w_2\rangle$ be a Z_3 and $\langle t'; u, w_2, w_3\rangle$ be a claw, we have $w_2w_3 \in E(G)$. Similarly as w_{12} does not exist, to avoid $\langle w_1, t, x_1; x_2w_2w_3\rangle$ be a Z_3, we have both w_1w_2 and w_1w_3 in $E(G)$. Thus $\langle w_1, w_2, w_3\rangle$ is a clique since G is claw-free.

If $S_3 = \{t\}$, we may assume $tw_3 \in E(G)$ by symmetric. If neither w_{12} nor w_{13} exists, since $deg_G(w_1) \geq 3$, we have $w_1w_2, w_1w_3 \in E(G)$. If w_{12} does not exist and w_{13} exists, since $\langle t, w_3, w_{13}; x_1x_2w_2\rangle$ is not a Z_3, we have $w_2w_3 \in E(G)$. Moreover, if both w_{12} and w_{13} exist, since neither $\langle t, w_3, w_{13}; x_1x_2w_2\rangle$ nor $\langle t, x_3, w_3; w_2w_{12}w_1\rangle$ is a Z_3, we have w_2w_3 and w_1w_2 in $E(G)$ or w_1w_3 in $E(G)$. Since $deg_G(w_1) \geq 3$, we have w_1w_2 or w_1w_3 in $E(G)$. G is claw-free gives us that $\langle w_1, w_2, w_3\rangle$ is a clique. ■

Claim 5.2.19. *There exists a strong spanning Halin subgraph in G.*

Proof. We want to find a strong spanning Halin subgraph in G by following subcases.

Subcase 1: Suppose $S_2 \cup S_3 \cup S_4 = \{t\}$.

We may assume $tw_3 \in E(G)$ by symmetric.

Subcase 1.1: At least one of $\{w_{23}, w_{13}\}$ exists.

We may assume w_{23} exists. If w_2 exists or w_1 exists, then let $C = ux_3x_1w_{12}w_{13}w_1w_3tu$ be a cycle and all vertices on the path $x_2w_{23}w_2$ be stems of T with $N_C(x_2) = \{u, x_3, x_1\}$,

$N_C(w_{23}) = \{w_{12}, w_{13}\}$ and $N_C(w_2) = \{w_3, w_1, t\}$ (See Figure 5.4(1)). If neither w_1 nor w_2 exists, then $t \in S_2 \cup S_3$ since $|N_G(t)| \geq 3$, which implies $tx_3 \in E(G)$. Let $C = ux_2x_1w_{12}w_{13}w_3tu$ be a cycle and $\{x_3, w_{23}\}$ be stems of T with $N_C(x_3) = \{w_3, t, u, x_2, x_1\}$ and $N_C(w_{23}) = \{w_{12}, w_{13}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.4(2)).

Subcase 1.2: Neither w_{23} nor w_{13} exists.

If neither w_{23} nor w_{13} exists, then w_{12} must exist.

If $N_2(t) = N_2(x_3)$, let $C = x_3ux_1w_{12}w_1tx_3$ be a cycle and all vertices on the path $x_2w_2w_3$ be stems of T with $N_C(x_2) = \{u, x_1\}$, $N_C(w_2) = \{w_1, w_{12}\}$ and $N_C(w_3) = \{x_3, t\}$ (See Figure 5.4(3)). If $N_2(t) = \{w_1, w_2, w_3\}$, let $C = ux_3x_1w_{12}w_1w_3tu$ be a cycle and $\{x_2, w_3\}$ be stems of T with $N_C(x_2) = \{x_1, x_3, u\}$ and $N_C(w_2) = \{w_1, w_{12}, w_3, t\}$ (See Figure 5.4(4)). Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G.

Subcase 2. Assume $|S_2 \cup S_3 \cup S_4| \geq 2$.

By Claim 5.2.13 to Claim 5.2.18, we have $|N_2(S_2 \cup S_3 \cup S_4) \cap \{w_1, w_2, w_3\}| \geq 2$. And we may assume there exist $t \in S_2$ and $t' \in S_4$(since other cases are similarly and much easier), then $tw_1, t'w_3 \in E(G)$. Let $C = tt'w_3w_{13}w_{23}w_2w_{12}w_1t$ be a cycle and all vertices on the path $ux_1x_2x_3$ be stems of T with $N_C(u) = \{t, t'\}$, $N_C(x_1) = \{w_1, w_{12}\}$, $N_C(x_2) = \{w_2\}$ and $N_C(x_3) = \{w_{23}, w_{13}, w_3\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.4(5)). Note that if $S_3 \neq \emptyset$, we may assume $S_3 \subseteq \{t_1, t_2, t_3\}$ and $t_1x_1, t_2x_2, t_3x_3 \in E(G)$. Then adding x_1t_1, x_2t_2, x_3t_3 to $E(T)$, $w_1t_1, t_1w_{12}, w_{12}t_2, t_2w_2, w_{13}t_3, t_3w_3$ to $E(C)$ if they exist and deleting $w_1w_{13}, w_{12}w_2, w_{23}w_3$ from $E(C)$. Similarly, we can find a strong spanning Halin subgraph in G.

Case 2.2: There does not exist a triangle in $\langle S \rangle$.

Since G is claw-free and $ux \in E(G)$ for any $x \in S$, we have $\alpha(G) \leq 2$. Moreover, since neither H-5 nor triangle is an induced subgraph of G, there is no induced cycle with 5 vertices in $\langle S \rangle$, which implies $|S| \leq 4$. Now we want to consider following subcases.

Case 2.2.1: Assume $\langle S \rangle$ is connected.

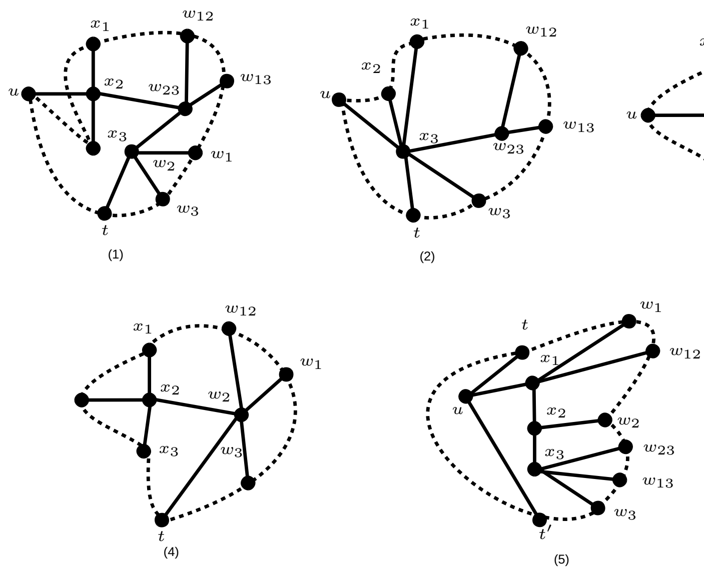

Figure 5.4. $V_1 = \{u\}$, V_2 is not a clique and $\langle S \rangle$ contains a triangle.

We may denote by $P = xyz$ is an induced path in $\langle S \rangle$, then we can easily get following claims.

Claim 5.2.20. $N_2(x) \cap N_2(z) = N_2(x) \cap N_2(y) \cap N_2(z)$.

Proof. If there exists $w \in N_2(x) \cap N_2(z) \setminus N_2(y)$, then $\langle w, x, y, z, u \rangle \cong$ H-2. ■

Claim 5.2.21. $|N_2(x) \cap N_2(y) \cap N_2(z)| \leq 1$.

Proof. Otherwise, let $w_1, w_2 \in N_2(x) \cap N_2(y) \cap N_2(z)$, then $\langle x, y, w_1, w_2, z \rangle \cong$ H-9. ■

Claim 5.2.22. *Either* $N_2(y) \subseteq N_2(x)$ *or* $N_2(y) \subseteq N_2(z)$.

Proof. For any $w \in N_2(y)$, since $\langle y; x, z, w \rangle$ is not a claw, $wx \in E(G)$ or $wy \in E(G)$. If there exist $w_1 \in N_2(y) \cap N_2(x) \setminus N_2(z)$ and $w_2 \in N_2(y) \cap N_2(z) \setminus N_2(x)$, then $\langle y; u, x, w_1, w_2, z \rangle \cong$ H-5, giving a contradiction. Thus, either $N_2(y) \subseteq N_2(x)$ or $N_2(y) \subseteq N_2(z)$. ■

From here and after, we always assume $N_2(y) \subseteq N_2(x)$, then we can show that $N_2(x) = N_2(y)$ by following claim.

Claim 5.2.23. *$N_2(x) = N_2(y)$.*

Proof. Since $N_2(y) \neq \emptyset$, we may denote $w_1 \in N_2(y) \subseteq N_2(x)$. Suppose there exists $w_2 \in N_2(x) \setminus N_2(y)$. If $w_1 \notin N_2(z)$, then $\langle w_2, w_1, x, y, u, z\rangle \cong$ H-8; if $w_1 \in N_2(z)$, since $|N_2(x) \cap N_2(y) \cap N_2(z)| = 1$ and $|N_2(y) \cup N_2(z)| \geq 2$, we may assume $w_3 \in N_2(z) \setminus N_2(y)$, however this in turn gives us $\langle w_1; w_2, w_3, y\rangle$ is a claw or $\langle w_3, w_1, w_2, x, y, u\rangle$ isomorphic to H-8. ■

Case 2.2.1.1: Suppose $\langle S\rangle$ is an induced path with four vertices, denote by P.

Claim 5.2.24. *If $\langle S\rangle$ is an induced path with four vertices, say P, then $P = txyz$ and $|V_2| \leq 6$.*

Proof. Suppose $P = xyzt$ is an induced path in $\langle S\rangle$, then there exists $w_1 \in N_2(x) = N_2(y) \setminus (N_2(z) \cap N_2(t))$ since $|N_2(x) \cup N_2(y)| \geq 2$, $|N_2(x) \cap N_2(y) \cap N_2(z)| \leq 1$ and $N_2(y) \cap N_2(t) \subseteq N_2(z)$. However, this illustrates that $\langle w_1, x, y, u, z, t\rangle \cong$ H-8. Thus $S = P = txyz$.

We want to show $|V_2| \leq 6$ by following five statements.

Firstly, There is at most one vertex, in $N_2(t) \setminus N_2(x) \cup N_2(y) \cup N_2(z)$(or $N_2(z) \setminus N_2(x) \cup N_2(y) \cup N_2(t)$). Otherwise, let $w, w' \in N_2(t) \setminus N_2(x) \cup N_2(y) \cup N_2(z)$, then $\langle w, w', t; xyz\rangle$ is a Z_3.

Secondly, $|N_2(t) \cap N_2(z)| \leq 1$. Otherwise, let $w_1, w_2 \in N_2(t) \cap N_2(z)$, then $\langle u, z, w_1, w_2, t\rangle \cong$ H-2.

Thirdly, $|N_2(t) \cap N_2(x) \cap N_2(y)| \leq 1$ and $|N_2(z) \cap N_2(x) \cap N_2(y)| \leq 1$. Otherwise, let $w_1, w_2 \in N_2(t) \cap N_2(x) \cap N_2(y)$, then $\langle t, x, y, w_1, w_2\rangle \cong$ H-9.

Fourthly, $N_2(t) \cap N_2(x) \cap N_2(y) \cap N_2(z) = \emptyset$. Otherwise, let $w \in N_2(t) \cap N_2(x) \cap N_2(y) \cap N_2(z)$, then $\langle w, t, x, u, z\rangle \cong$ H-2.

Fifthly, $|N_2(x) = N_2(y)| \leq 3$. Otherwise, we may assume there exist $w_1, w_2 \in N_2(x) \setminus (N_2(t) \cup N_2(z))$. If $N_2(x) \cap N_2(y) \cap N_2(z) \neq \emptyset$, denote $w_3 \in N_2(x) \cap N_2(y) \cap N_2(z)$, then $\langle w_1, w_2, w_3; zut\rangle$ is a Z_3. If $N_2(x) \cap N_2(y) \cap N_2(z) = \emptyset$, then there exists $w_3 \in N_2(x) \setminus (N_2(z) \cup \{w_1, w_2\})$, clearly, $dist(z, w_3) \geq 2$, we can also find a Z_3 in G. ■

By Claim 5.2.24, we may denote $S = P_4 = txyz$, $N_2(x) = N_2(y) \subseteq \{w_1, w_2, w_3\}$, $w_4 \in N_2(t) \cap N_2(z)$, $w_5 \in N_2(t) \setminus N_2(x) \cup N_2(y) \cup N_2(z)$ and $w_6 \in N_2(z) \setminus N_2(x) \cup N_2(y) \cup N_2(t)$. Note that any vertex in $\{w_1, w_2, w_3, w_4, w_5, w_6\}$ may not exist.

Claim 5.2.25. *If w_5 exists(or w_6 exists), then* $3 \geq |N_2(w_5) \cap \{w_1, w_2, w_3\}| \geq |\{w_1, w_2, w_3\}| - 1$.

Proof. If $N_2(w_5) = \{w_1, w_2, w_3\}$, then $\langle w_5, w_1, w_2, w_3, x\rangle \cong$ H-9. If there exist w_1w_5, w_2w_5 not in $E(G)$, then either $\langle w_1, w_2, x; uzw_5\rangle$ is a Z_3 or $\langle z; u, w_1, w_5\rangle$ is a claw or $\langle z; u, w_2, w_5\rangle$ is a claw, showing a contradiction. ■

Now we want to find a strong spanning Halin subgraph in G by following two subcases.

Subcase 1: Assume $|\{w_1, w_2, w_3\}| = 3$.

We may let $w_2w_6, w_3w_6, w_2w_5 \in E(G)$ by Claim 5.2.25, then we can also assume $zw_3 \in E(G)$ since $\langle w_2, w_3, w_6, z, y\rangle \ncong$ H-2, which implies $w_3w_5 \notin E(G)$ since $\langle w_3; z, w_1, w_5\rangle$ is not a claw and $\langle y, w_1, w_2, w_3, w_5\rangle \ncong$ H-9. Thus $w_1w_5 \in E(G)$, which implies $tw_1 \in E(G)$ since $\langle x, w_1, w_2, w_5, t\rangle \ncong$ H-2. Moreover, since $\langle y, w_1, w_2, w_5, w_4, z\rangle \ncong$ H-4 and $\langle w_4; t, z, w_2\rangle$ is not a claw, we have $w_3w_4 \in E(G)$. Similarly, we can show that $w_1w_4 \in E(G)$.

Let $C = utw_1w_5w_4w_6zyu$ be a cycle and all vertices on the path xw_2w_3 be stems of T with $N_C(x) = \{u, t\}$, $N_C(w_2) = \{w_1, w_5\}$ and $N_C(w_3) = \{w_4, w_6, z, y\}$(Note that even if w_4 or w_5 or w_6 does not exist, we can also find a cycle C or a tree T similarly). Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.5(1)).

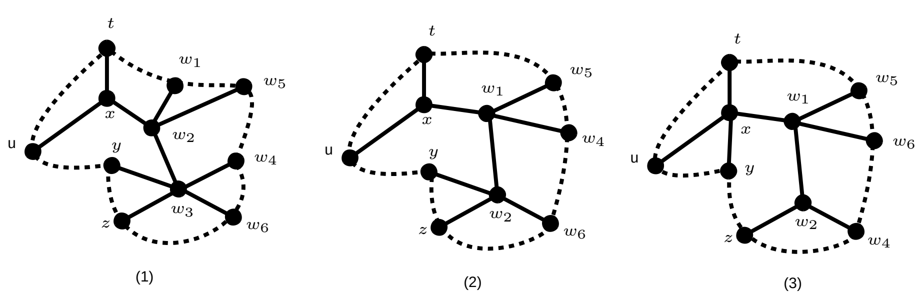

Figure 5.5. $V_1 = \{u\}$, V_2 is not a clique and $\langle S\rangle$ is an induced P_4.

Subcase 2: We assume that $|\{w_1, w_2, w_3\}| = 2$.

We may assume w_3 does not exist. By Claim 5.2.25, we can assume $w_2w_6 \in E(G)$. Since $\langle x, y, w_2; w_6w_4w_5\rangle$ is not a Z_3, we have $w_2w_4 \in E(G)$ or $w_5w_6 \in E(G)$.

If $w_2w_4 \in E(G)$, then $zw_2 \in E(G)$ since $\langle w_2, w_4, w_6, z, y\rangle \ncong$ H-2. By symmetric ,we can also show that $w_1w_4, w_1t, w_1w_5 \in E(G)$. Let $C = utw_5w_4w_6zyu$ be a cycle and all vertices on the path xw_1w_2 be stems of T with $N_C(x) = \{u, t\}$, $N_C(w_1) = \{w_5, w_4\}$ and $N_C(w_2) = \{w_6, z, y\}$(Note that even if w_4 or w_5 or w_6 does not exist, we can also find a cycle C or a tree T similarly.) Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.5(2)).

If $w_5w_6 \in E(G)$, since neither $\langle w_6; w_5, w_2, z\rangle$ is a claw nor $\langle w_5, w_2, w_6, z, u, t\rangle$ isomorphic to H-4, we have either w_2w_5 or w_2z in $E(G)$. If $w_2z \in E(G)$, to avoid $\langle u, y, z, w_2, w_6, w_4\rangle \cong$ H-4, we have $w_2w_4 \in E(G)$. Then we can find a strong spanning Halin subgraph in G as above. If $w_2w_5 \in E(G)$, to avoid $\langle w_1, x, y; zw_6w_5\rangle$ be a Z_3, we have w_1z or w_1w_6 or w_1w_5 in $E(G)$, but not both w_1w_6 and w_1w_5 in $E(G)$ since $\langle x, y, w_1, w_2, w_5, w_6\rangle \ncong$ H-7. If $w_1z \in E(G)$, since $\langle z; u, w_1, w_6\rangle$ is not a claw, we have $w_1w_6 \in E(G)$. Moreover, since $\langle w_1, z, w_6, w_4, w_5, t\rangle$ does not isomorphic to H-8, we have $w_1w_4 \in E(G)$. Let $C = utw_5w_6w_4zyu$ be a cycle and all vertices on the path xw_1w_2 be stems of T with $N_C(x) = \{y, u, t\}$, $N_C(w_1) = \{w_5, w_6\}$ and $N_C(w_2) = \{w_4, z\}$(Note that if $w_1w_6 \in E(G)$, then $w_1z \in E(G)$ since $\langle w_6; z, w_1, w_5\rangle$ is not a claw. If $w_1w_5 \in E(G)$, then $tw_1 \in E(G)$ since $\langle w_5; t, w_1, w_6\rangle$ is not a claw. Moreover, $w_1w_4 \in E(G)$ given by $\langle w_5, t, w_1, w_2, w_6, w_4\rangle \ncong$ H-5. Similarly as $zw_2 \in E(G)$, we can find a cycle C and a tree T in G.) Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.5(3))

Case 2.2.1.2: Assume $\langle S\rangle$ is an induced C_4 or P_3.

Claim 5.2.26. *If $\langle S\rangle$ is an induced cycle with four vertices or a path with three vertices, then $|V_2| \leq 6$.*

Proof. If $|S| = 4$, we denote by $S = xyztx$ be the cycle with $N_2(x) = N_2(y)$ and $N_2(z) = N_2(t)$ by Claim 5.2.23. If $|S| = 3$, we denote by $S = xyz$ with $N_2(x) = N_2(y)$.

Moreover, we reserve $w \in N_2(y) \cap N_2(z)$ if it exists. The following two statements are showing this lemma is true.

Firstly, for any $w' \in N_2(z) \setminus \{w\}$, $|N_2(w') \cap N_2(x)| \geq |N_2(x)| - 1$ *and* $|N_2(x) \setminus \{w\}| \leq 2$.

It is clearly true if $N_2(y) = \{w, w_1\}$. Thus we may assume there exist $w_1, w_2 \in N_2(y) \setminus \{w\}$ and $w' \in N_2(z) \setminus \{w\}$. Since $\langle w_1, w_2, y; uzw' \rangle$ is not a Z_3 and $\langle z, y, w_1, w_2, w' \rangle \not\cong$ H-2, we can assume $w_1 w' \in E(G)$ and $w_2 w' \notin E(G)$, which implies $|N_2(w') \cap N_2(y)| \geq |N_2(y)| - 1$ and $|N_2(y) \setminus \{w\}| \leq 2$. Similarly, we can prove that $|N_2(z) \setminus \{w\}| \leq 2$.

We may denote by $N_2(x) = N_2(y) = \{w_1, w_2, w\}$, $N_2(z) = N_2(t) = \{w, w_3, w_4\}$ and $w_1 w_4, w_2 w_3 \in E(G)$ if they exist.

Secondly, For any $w' \in V_2 \setminus N_2(x) \cup N_2(z)$, *we have* $ww' \notin E(G)$, $\{w_1, w_2, w_3, w_4\} \subseteq N_2(w')$ *and* $V_2 \subseteq \{w_1, w_2, w_3, w_4, w, w'\}$, *which implies* $|V_2| \leq 6$.

Since $\langle w; x, z, w' \rangle$ is not a claw, we get $ww' \notin E(G)$. Moreover, $N_2(w') \cap \{w_1, w_2, w_3, w_4\} \neq \emptyset$. Otherwise, let $w'' \in V_2 \setminus (N_2(x) \cup N_2(z) \cup \{w\})$ such that $w_1 w'', w'' w \in E(G)$, then $\langle z, u, y; w_1 w'' w' \rangle$ is a Z_3. Thus, we may assume $w' w_1 \in E(G)$, which implies $w' w_4 \in E(G)$ since $\langle w_1; x, w_4, w' \rangle$ is not a claw. Furthermore, there does not exist $w'' \in V_2 \setminus \{w_1, w_2, w, w_3, w_4, w'\}$ such that $w' w'' \in E(G)$. Otherwise, either $\langle z, u, y; w_1 w'' w' \rangle$ or $\langle w', w'', w_1; yzt \rangle$ is a Z_3, which in turn gives that $\langle V_2 \setminus N_2(S) \rangle$ is an independent set. On the other hand, since G is 3-connected, $\{w_1, w_2, w_3, w_4\} \subseteq N_2(w')$ for all $w' \in V_2 \setminus N_2(S)$, which implies $|V_2 \setminus N_2(S)| \leq 1$. Therefore, $V_2 \subseteq \{w_1, w_2, w_3, w_4, w, w'\}$. ■

Since any vertex in $\{w_1, w_2, w, w_3, w_4, w'\}$ may not exist and $|N_2(x) \cup N_2(y)| \geq 2$, $|N_2(z) \cup N_2(t)| \geq 2$ and $|N_2(w')| \geq 3$ if w' exists. We may assume w_2, w_3 exists. If $|S| = 4$, let $C = uxw_1w_2w'w_4wtu$ be a cycle and all vertices on the path yzw_3 be stems of T with $N_2(y) = \{u, x, w_1\}$, $N_2(z) = \{t\}$ and $N_C(w_3) = \{w_2, w', w_4, w\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.6(1)). If $|S| = 3$, let $C = uxw_1w'w_4wzu$ be a cycle and all vertices on the path yw_2w_3 be stems of T with $N_2(y) = \{u, x_1\}$, $N_2(w_2) = \{w_1\}$ and $N_C(w_3) = \{w', w_4, w, z\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.6(2)).

Case 2.2.2: Assume $\langle S \rangle$ is not connected.

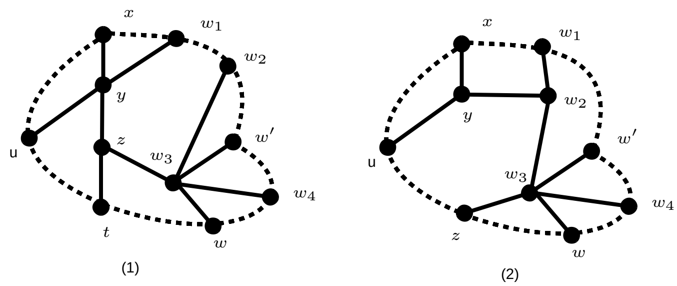

Figure 5.6. $V_1 = \{u\}$, V_2 is not a clique and $\langle S \rangle$ is an induced C_4 or P_3.

Since G is claw-free and for any $x \in S$, $ux \in E(G)$, there exist exact two components, say S_1 and S_2, in $\langle S \rangle$. Moreover, $\langle S \rangle$ does not contain an induced triangle implies $|S_1| \leq 2$ and $|S_2| \leq 2$. We let $S_1 = \{x_1, x_2\}$ and $S_2 \subseteq \{x_3, x_4\}$. The following claims are true.

Claim 5.2.27. *$N_2(x_i) \cap N_2(x_j) \cap N_2(x_k) = \emptyset$, where $i, j, k \in [1, 4]$.*

Proof. If there exists $w \in N_2(x_1) \cap N_2(x_2) \cap N_2(x_3)$, then $\langle x_3, u, x_1, x_2, w \rangle \cong$ H-2. ■

Claim 5.2.28. *$|N_2(x_i) \cap N_2(x_j)| \leq 1$, where $i, j \in [1, 4]$.*

Proof. By symmetric, we only show that $|N_2(x_1) \cap N_2(x_2)| \leq 1$ and $|N_2(x_2) \cap N_2(x_3)| \leq 1$. Suppose to the contrary, there exist $w_1, w_2 \in N_2(x_1) \cap N_2(x_2)$, then $\langle w_1, w_2, x_1, x_2, u, x_3 \rangle \cong$ H-6. If there exist $w_3, w_4 \in N_2(x_2) \cap N_2(x_3)$, then $\langle u, x_2, w_3, w_4, x_3 \rangle \cong$ H-2. ■

Claim 5.2.29. *$N_2^2(x_k) \subseteq N_2(x_i) \cup N_2(x_j)$, where $\{i, j\} = \{1, 2\}$ and $k \in \{3, 4\}$ or $\{i, j\} = \{3, 4\}$ and $k \in \{1, 2\}$.*

Proof. Suppose to the contrary, there exist $w_1 \in N_2(x_3)$ and $w_2 \in N_2^2(x_3)$. Since $\langle x_1, x_2, u; x_3 w_1 w_2 \rangle$ is not a Z_3, we may assume $w_1 x_2 \in E(G)$. However, this in turn gives us $\langle w_1; x_2, x_3, w_2 \rangle$ is a claw. ■

We may denote by $w_{ij} \in N_2(x_i) \cap N_2(x_j)$ and $w_i \in N_2(x_i) \setminus \cup_{j \neq i} N_2(x_j)$ if they exist, where $i, j \in [1, 4]$.

Claim 5.2.30. *If $S_1 = \{x_1, x_2\}$ and $S_2 = \{x_3, x_4\}$, then $V_2 \subseteq \{w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}\}$ and there exists a strong spanning Halin subgraph in G.*

Proof. We prove this claim by following three statements.

If both w_{12} *and* w_{34} *exist, then* $V_2 \subseteq \{w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}\}$. We may suppose w_1 exists, then $\langle x_3, x_4, u; x_2w_{12}w_1\rangle$ is a Z_3, showing a contradiction. Similarly, we can show that w_2, w_3, w_4 all do not exist.

If either w_{12} *or* w_{34} *exists, then* $V_2 \subseteq \{w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}\}$. We may assume w_{12} exists and w_{34} does not exist. Then neither w_1 nor w_2 exists. Since $|N_2(x_1) \cup N_2(x_2)| \geq 2$, we may assume w_{13} exists, then w_4 does not exists otherwise $\langle w_{12}, w_{13}, x_1; ux_4w_4\rangle$ is a Z_3. If w_3 exists, then w_{14} does not exist. Otherwise $\langle w_{12}, w_{14}, x_1; ux_3w_3\rangle$ is a Z_3, which in turn gives w_{24} exists since $N_2(x_4) \neq \emptyset$. To avoid $\langle w_{24}, w_{12}, x_2; ux_3w_3\rangle$ be a Z_3 and $\langle w_{12}; x_2, w_3, w_{24}\rangle$ be a claw, we have $w_{24}w_3 \in E(G)$. However, this force $\langle w_{24}; x_2, w_3, w_{24}\rangle$ to be a claw. Thus neither w_3 nor w_4 exists. Therefore, $V_2 \subseteq \{w_{12}, w_{13}, w_{14}, w_{23}, w_{24}, w_{34}\}$.

If neither w_{12} *nor* w_{34} *exists, then* $V_2 \subseteq \{w_{13}, w_{14}, w_{23}, w_{24}\}$. Firstly, if both w_{13} and w_{14} exists, then w_1 does not exist. Otherwise, $\langle w_1, w_{13}, w_{14}; x_3ux_2\rangle$ is a Z_3. Secondly, if both w_{13} and w_{24} exist, then w_1 does not exist. Since $\langle w_1, x_1, w_{13}; x_3x_4w_{24}\rangle$ is not a Z_3, we have $w_{13}w_{24} \in E(G)$, which implies $\langle w_{13}; x_1, x_3, w_{24}\rangle$ is a claw, or $w_1w_{24} \in E(G)$, which implies $\langle w_{24}; x_4, x_2, w_1\rangle$ is a claw. Thirdly, if w_1, w_2, w_4 exist, then w_{13} or w_{23} does not exist. Otherwise, since $\langle x_1, w_1, w_{13}; x_3x_4w_4\rangle$ is not a Z_3 and $\langle w_{13}; x_1, x_3, w_4\rangle$ is not a claw, we have $w_1w_4 \in E(G)$. Moreover, since $\langle w_{23}, w_2, x_2; ux_4w_4\rangle$ is not a Z_3, we have $w_2w_4 \in E(G)$ which implies $w_1w_2 \in E(G)$ since $\langle w_4; x_4, w_2, w_1\rangle$ is not a claw, or $w_{23}w_4 \in E(G)$ which implies $w_{23}w_1 \in E(G)$ since $\langle w_4; x_4, w_{23}, w_1\rangle$ is not a claw. If both w_1w_2 and w_2w_4 in $E(G)$, then $\langle w_1, w_4, w_2; x_2ux_3\rangle$ is a Z_3. If $w_{23}w_1 \in E(G)$, then $\langle w_{23}; x_2, x_3, w_1\rangle$ is a claw, showing a contradiction. Futhermore, $V_2 \neq \{w_1, w_2, w_3, w_4\}$. Otherwise, since V_2 is connected, we have $\{w_1, w_2, w_3, w_4\}$ is a clique, which implies $\langle w_1, w_4, w_2; x_2ux_3\rangle$ is a Z_3. Thus $V_2 \subseteq \{w_{13}, w_{14}, w_{23}, w_{24}\}$.

Let $C = ux_1w_{12}w_{13}w_{14}w_{34}x_4x_3u$ be a cycle and all vertices on the path $x_2w_{23}w_{24}$ be stems of T with $N_C(x_2) = \{u, x_1, w_{12}\}$, $N_C(w_{23}) = \{w_{13}, x_3\}$ and $N_C(w_{24}) = \{w_{14}, w_{34}, x_4\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.7(1)). ■

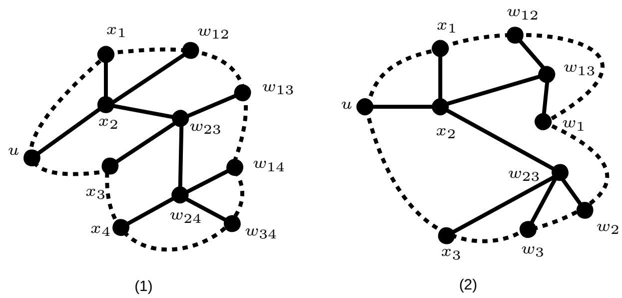

Figure 5.7. $V_1 = \{u\}$, V_2 is not a clique and $\langle S \rangle$ is not connected.

Claim 5.2.31. *If $S_1 = \{x_1, x_2\}$ and $S_2 = \{x_3\}$, then $V_2 \subseteq \{w_{12}, w_{13}, w_{23}, w_1, w_2, w_3\}$, $\langle w_1, w_2, w_3 \rangle$ is a clique and there exists a strong spanning Halin subgraph in G.*

Proof. For any $w \in V_2 \backslash N_2(S)$, $w_{12}w, w_{23}w, w_{13}w \notin E(G)$. Otherwise, $\langle w; w_{12}, x_1, x_2, u; x_3 \rangle$ isomorphic to H-3 or $\langle w_{23}; x_1, x_3, w \rangle$ is a claw or $\langle w_{13}; x_2, x_3, w \rangle$ is a claw. Let $R_1 = N_2(x_1) \setminus N_2(x_2) \cup N_2(x_3)$, $R_2 = N_2(x_2) \setminus N_2(x_1) \cup N_2(x_3)$ and $R_3 = N_2(x_3) \setminus N_2(x_1) \cup N_2(x_2)$. We may assume $V_2 \neq N_2(x_3)$ since it is not a clique. In fact $R_1 \cup R_2 \cup \{w_{12}\} \neq \emptyset$ since $N_2^2(x_3) \subseteq N_2(x_1) \cup N_2(x_2)$. This lemma is true illustrated by following two statements.

Firstly, $|R_3| \leq 1$. Suppose this is not true, we may assume there exist $w_4, w_5 \in R_3$, then w_{12} does not exist. Otherwise, $w_{12}w_4, w_{12}w_5 \notin E(G)$ since neither $\langle x_3; w_4, x_1, x_2, u \rangle$ nor $\langle x_3; w_5, x_1, x_2, u \rangle$ isomorphic to H-4, however, this will induce $\langle w_4, w_5, x_3; ux_2w_{12} \rangle$ to be a Z_3. Thus we may assume $w_6 \in R_1$, to avoid $\langle w_4, w_5, x_3; ux_1w_6 \rangle$ be a Z_3 and $\langle u; x_3, w_4, w_5, w_6; x_1 \rangle$ isomorphic to H-4, we may assume $w_6w_4 \in E(G)$ and $w_6w_5 \notin E(G)$. But this will force $\langle x_3, w_5, w_4; w_6x_1x_2 \rangle$ to be a Z_3, showing a contradiction.

Secondly, $|R_1| \leq 1$ *and* $|R_2| \leq 1$. Since $deg_G(x_3) \geq 3$ and $|R_3| \leq 1$, we may assume w_{13} exists, then $|R_2| \leq 1$. Otherwise, let $w_6, w_7 \in R_2$, then $\langle w_6, w_7, w_{13}; x_3ux_1 \rangle$ is a Z_3. Similarly, if w_{23} exists, then $|R_1| \leq 1$, which implies $|N_2(S)| \leq 6$. Since G is 3-connected and for any $w \in V_2 \setminus N_2(S)$, $w_{12}w, w_{23}w, w_{13}w \notin E(G)$, we have $R_1 \cup R_2 \subseteq N_2(w)$. However, this will force $\langle w, w_4, w_5; x_2ux_3 \rangle$ be a Z_3, where $w_4 \in R_1$, $w_5 \in R_2$. Therefore, $V_2 \subseteq \{w_{12}, w_{13}, w_{23}, w_1, w_2, w_3\}$.

Now we want to show that $\langle w_1, w_2, w_3\rangle$ is a clique.

Firstly, we can observe that $w_i w_{jk} \notin E(G)$, since $\langle u, x_1, x_2, w_{12}, w_3, x_3\rangle \cong$ H-4 and neither $\langle w_{13}; w_2, x_1, x_3\rangle$ nor $\langle w_{23}; w_1, x_2, x_3\rangle$ is a claw. If w_{12} exists, we have $w_1w_3 \in E(G)$ since $\langle w_1, w_{12}, x_1; uzw_3\rangle$ is not a Z_3. Similarly, we can show that $w_2w_3 \in E(G)$. Since $\langle w_3; w_1, w_2, x_3\rangle$ is not a claw, we have $w_1w_2 \in E(G)$. Thus $\{w_1, w_2, w_3\}$ is a clique. If w_{12} does not exist, since $deg_G(w_1) \geq 3$, $deg_G(w_2) \geq 3$, we can assume w_1w_2 in $E(G)$ or both w_1w_3 and w_2w_3 in $E(G)$. If both w_1w_3 and w_2w_3 in $E(G)$, then w_1w_2 in $E(G)$ since $\langle w_3; w_1, w_2, x_3\rangle$ is not a claw. If $w_1w_2 \in E(G)$, to avoid $\langle x_3, w_3, w_{23}; w_2w_1x_1\rangle$ be Z_3, we have $w_2w_3 \in E(G)$ or $w_1w_3 \in E(G)$, which also implies $\langle w_1, w_2, w_3\rangle$ is a clique.

Let $C = ux_1w_{12}w_1w_2w_3x_3u$ be a cycle and all vertices on the path $yw_{23}w_{13}$ be stems of T with $N_C(x_2) = \{u, x_1\}$, $N_C(w_{13}) = \{w_1, w_{12}\}$ and $N_C(w_{23}) = \{w_2, w_3, x_3\}$. Let $H = T \cup C$, then H is a strong spanning Halin subgraph in G (See Figure 5.7(2)). ■

5.3 Proof of 3-connected $B_{1,2}$-free line graphs

In this section, we always assume G is a 3-connected $B_{1,2}$-free line graph and prove Proposition 8 by following series of claims.

Claim 5.3.1. *If $|V_1| \geq 4$, then $N_1(x) = V_1$ or $N_2(x) = V_2$ for all $x \in S$.*

Proof. We may assume there exists $x \in S$, such that $N_1(x) \neq V_1$ and $N_2(x) \neq V_2$. Since $|V_2| \geq |V_1| \geq 4$, we can assume there exists $w_1 \in N_2(x)$, $w_2 \in N_2^2(x)$, $u_1, u_2 \in N_1(x)$ and $u_3 \in N_1^2(x)$, such that $w_1w_2, u_1u_3 \in E(G)$. Since $\langle u_3; u_1, u_2, x; w_1w_2\rangle$ is not a $B_{1,2}$, we have $u_3u_2 \in E(G)$, which means $N_1(x) \subseteq N_1(u_3)$. On the other hand, the fact that G does not contain H-9 as an induced subgraph gives us $N_1(x) = \{u_1, u_2\}$. Moreover, we can also assume $N_1^3(x) = \emptyset$ according to G is H-3 free, thus $|N_1^2(x)| \geq 2$ since $|V_1| \geq 4$. Let $u_3, u_4 \in N_2^2(x)$, then $\langle u_3, u_4, u_1, u_2, x, w_1\rangle \cong$ H-6, showing a contradition. ■

Claim 5.3.2. *If $|V_1| = 3$, $|V_2| \geq 4$ and $N_1(x) \neq V_1$ for all $x \in S$, then*

1) There exists $x \in S$ such that $|N_2(x)| \geq 2$;

2) $V_2 \cup S$ is a clique;

3) There exists a strong spanning Halin subgraph in G.

Proof. We can assume $\langle V_1 \rangle$ is a triangle and $|S| = 3$. Otherwise, we can find two vertices, say x and y, in S, such that $N_1(x) = N_1(y) = \{u_1, u_2\}$. Let $S' = (S \setminus \{x, y\}) \cup \{u_1, u_2\}$, $V_1' = V_1 \setminus \{u_1, u_2\}$ and $V_2' = V_2 \cup \{x, y\}$, then S' is also a minimum vertex cut with $|V_1'| < |V_1|$, contradicts to the assumption of S and V_1.

Thus we denote by $N_1(x) = \{u_1, u_2\}$, $N_1(y) = \{u_1, u_3\}$ and $N_1(z) = \{u_2, u_3\}$.

1) Suppose this is not true. Let $N_2(x) = \{w_1\}$, $N_2(y) = \{w_2\}$, $N_2(z) = \{w_3\}$ and $w_4 \in N_2^2(z) \setminus \{w_1, w_2\}$. Since $\langle x; u_2, u_3, z; w_3w_4 \rangle$ is not a $B_{1,2}$, $xz \in E(G)$. However, this will induce $\langle z; u_3, x, w_3 \rangle$ to be a claw, showing a contradiction.

Therefore, we may assume $|N_2(x)| \geq 2$ here and after.

2) Since $N_1(x) \neq V_1$ and $|V_2| \geq 4$, similarly as Claim 5.3.1, we can show that $N_2(x) = V_2$. Moreover, by the same method as Corollary 5.1.2, we will get $N_2(x) = N_2(y) = V_2$ for all $y \in S \setminus \{x\}$. Therefore, $V_2 \cup S$ is a clique.

3) We may continue to denote by $w_1 \in N_2(x)$, $w_2 \in N_2(y)$ and $w_3 \in N_2(z)$. Since $V_2 \setminus \{w_2\}$ is a clique, there exists a hamiltonian path, say $P = w_1Pw_3$, in it. Let $C = P \cup \{w_3z, zu_2, u_2u_1, u_1x, xw_1\}$ be a cycle and all vertices on the path u_3yw_2 be stems of T with $N_C(u_3) = \{u_1, u_2\}$, $N_C(y) = \{x, z\}$ and $N_C(w_2) = V(P)$. Set $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G. ■

Remark: According to Lemma 4.1.2, Claim 5.3.1 and 5.3.2, we can assume $N_1(x) = V_1$ for all $x \in S$. Moveover, if $|V_1| \geq 2$, then $V_1 \cup S$ is a clique by Corollary 5.1.1. We want to consider following cases.

Case 1: Suppose $\langle V_2 \rangle$ is a clique.

If $|V_1| = 1$, similarly as G is a 3-connected Z_3-free line graph, we can find a strong spanning Halin subgraph in G.

If $|V_1| \geq 2$, let $S = \{x_1, x_2, \cdots, x_t\}$, $w_1 \in N_2(x_1)$, $w_2 \in N_2(x_2)$ and $w_3 \in N_2(x_t)$ by Lemma 4.1.2. Since V_1, $S_1 \setminus \{x_1, x_2\}$ and V_2 are cliques, there exist hamiltonian paths, say

say $P_1 = u_1 P_1 u_2$, $P_2 = x_3 P_2 x_t$ and $P_3 = w_2 P_3 w_3$, in them, respectively. Let $C = P_1 P_2 P_3 \cup \{x_1u_1, x_1w_1, x_3u_2, x_tw_3\}$ and $\{x_2, w_2\}$ be stems of T with $N_C(x_2) = V(P_1) \cup V(P_2) \cup \{x_1\}$ and $N_2(w_2) = V(P_3)$. Set $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G. ■

Case 2: Suppose $\langle V_2 \rangle$ is not a clique.

The following two claims are showing the structure of $\langle V_2 \rangle$.

Claim 5.3.3. *There exist $x, y \in S$ such that $N_2(x) \cap N_2(y) \neq \emptyset$.*

Proof. Suppose for any $x, y \in S$, $N_2(x) \cap N_2(y) = \emptyset$. Denote by $w_1 \in N_2(x)$, $w_2 \in N_2(y)$ and $w_3 \in N_2(z)$.

Firstly, we want to show that S is a clique. This is true if $|V_1| \geq 2$, thus we assume $|V_1| = 1$. Then there is no induce path of length 3 in $\langle S \rangle$, otherwise $\langle y; x, z, w_2 \rangle$ is a claw. Therefore, $\langle S \rangle$ contains exact two cliques, say S_1 and S_2, in it. We may assume $x, y \in S_1$ and $z \in S_2$. Since $\langle w_1; x, y, u; zw \rangle$ is not a $B_{1,2}$, we have $w_1 w \in E(G)$, which implies $N_2(z) \subseteq N_2(w_1)$ and $|N_2(z)| = 1$ since G does not contain H-4 as a subgraph. Moreover, $deg_G(z) \geq 3$ gives us that there exists $t \in S_2 \setminus \{z\}$ such that $tz \in E(G)$, then $\langle y; u, t, z; ww_1 \rangle$ is a $B_{1,2}$, showing a contradiction.

Secondly, we will prove that $\langle V_2 \rangle$ is a clique. Denote by $R = V_2 \setminus N_2(S)$. If $R = \emptyset$, we may assume $w_1 w_2 \in E(G)$ since $\langle V_2 \rangle$ is connected. To avoid $\langle w_3; z, u, x; w_1 w_2 \rangle$ be a $B_{1,2}$ and claw exist, both $w_2 w_3$ and $w_1 w_3$ in $E(G)$. If there exists $w_4 \in N_2(x) \setminus \{w_1\}$, since $\langle w_4; w_1, w_2, w_3; zu \rangle$ is not a $B_{1,2}$ and $\langle V_2 \rangle$ is claw-free, we have $w_2 w_4, w_3 w_4 \in E(G)$, which implies $N_2(x) \cup N_2(y) \cup N_2(z)$ is a clique. Similarly, we can show that $V_2 = N_2(S)$ is a clique since S is a clique. If $R \neq \emptyset$, we may assume there exists $w_4 \in N_2^2(x)$ such that $w_1 w_4 \in E(G)$. Since $\langle w_2; y, u, x; w_1 w_4 \rangle$ is not a $B_{1,2}$ and $\langle w_1; x, w_4, w_2 \rangle$ is not a claw, we get $w_2 w_4 \in E(G)$, which implies $N_2^2(x) \cap R \subseteq N_2^2(y) \cap R$. Since S is a clique, by symmetric, we have $N_2^2(x) \cap R = N_2^2(S)$ for all $x \in S$. Moreover, $N_2^3(S) = \emptyset$. Otherwise, assume there exists $w_5 \in N_2^3(x)$ such that $w_4 w_5 \in E(G)$, then $\langle w_5; w_4, w_2, w_1; xu \rangle$ is a $B_{1,2}$. Furthermore, we can assume $w_1 w_2 \in E(G)$ since $\langle w; w_1, w_2, w_3 \rangle$ is not a claw for any $w \in R$. Similarly as $R = \emptyset$, we can show that $N_2(S)$ is a clique, which implies $V_2 = N_2(S) \cup N_2^2(S)$ is also a clique. ■

Let $S' = \{x \in S | \text{there exists} y \in S \text{ such that } N_2(x) \cap N_2(y) \neq \emptyset\}$ and $x \in S'$ be the vertex with $|N_2(x)|$ is maximum.

Claim 5.3.4. $V_2 = N_2(x) \cup N_2^2(x)$

Proof. Suppose this is not true. By Claim 5.3.3 and Lemma 4.1.2, we can assume there exist $w_1, w_2 \in N_2(x)$, $w_3 \in N_2^2(x)$, $w_4 \in N_2^3(x)$ and $u \in N_1(x)$ such that $w_1w_3, w_3w_4 \in E(G)$, then either $\langle u; x, w_2, w_1; w_3w_4 \rangle$ is a $B_{1,2}$ or $\langle u, x, w_1, w_2, w_3, w_4 \rangle$ isomorphic to H-2, giving a contradiction. ■

Remark: We may reserve $y \in S' \setminus \{x\}$ with $N_2(x) = N_2(y)$ if it exists. Note that if $|V_1| \geq 2$, y does not exists since G does not contain H-7 as an induced subgraph. If $V_1 = \{u\}$, there exists at most one such y, since G does not contain H-9 as an induced subgraph.

Now we want to find a strong spanning Halin subgraph in G by following subcases depending on the size of $N_2(x)$.

Case 2.1. $|N_2(x)| \geq 4$. Denote by $\{w_2, w_3, w_4\} \subseteq N_2(x) \setminus \{w_1\}$.

Claim 5.3.5. *$N_2^2(x) \subseteq N_2(z)$ for any $z \in S \setminus \{x, y\}$.*

Proof. *Firstly, we will show that $N_2^2(x) \subseteq N_2(z)$ for all $z \in S' \setminus \{x, y\}$.* For any $w \in N_2^2(x)$, if $ww_1 \in E(G)$, then we can assume $ww_3 \notin E(G)$ since G is H-9 free. This in turn gives us $wz \in E(G)$ since $\langle w_1; z, w_3, w \rangle$ is not a claw. If $ww_1 \notin E(G)$, since $V_2 = N_2(x) \cup N_2^2(x)$, we may assume $ww_2 \in E(G)$. To avoid $\langle x, w_2, w_3, w_4, w \rangle \cong$ H-9, we still have $w_3w \notin E(G)$. Since $\langle w; w_2, w_3, w_1; zu \rangle$ is not a $B_{1,2}$, we have $wz \in E(G)$.

Secondly, we want to show that $N_2^2(x) \subseteq N_2(z)$ for all $z \in S \setminus S'$. This is clearly true if $|N_2^2(x)| = 1$ or $S \setminus S' = \emptyset$. Therefore, we may assume $|N_2^2(x)| \geq 2$ and there exist $w, w' \in N_2^2(x)$ and $z \in S \setminus S'$ such that $tw \in E(G)$ and $tw' \notin E(G)$. We may also assume $ww_1 \in E(G)$, then $ww_i \notin E(G)$ for all $w_i \in N_2^2(x) \setminus \{w_1\}$. Otherwise, G contains a H-4 or H-2 as an induced subgraph. Since G is H-9 free, we can assume $w_3w' \notin E(G)$; if $w_1w' \in E(G)$, then $\langle w_3; w_1, w', w; zu\rangle$ is a $B_{1,2}$. If $w_1w' \in E(G)$, we may assume $w_2w' \in E(G)$ since $\langle w'; w_2, w_3, w_1; wz\rangle$ is not a $B_{1,2}$, then we have $ww' \in E(G)$. This forces $\langle w; w_1, w', z\rangle$ to be a claw.

We want to find a strong spanning Halin subgraph in G by following subcases for $|N_2(x)| \geq 4$.

Subcase 1: Assume $|V_1| \geq 2$.

By the remark after Claim 5.3.2, we know $V_1 \cup S$ is a clique. Then we have y does not exist since G is H-7 free; $N_2^2(x) = \{w\}$ by Claim 5.3.6; and $S \setminus \{x\} = \{z, t\}$ since $\langle S\rangle$ is a clique and G is H-9 free. We denote by $N_2(z) \cap N_2(x) = \{w_1\}$ and assume $ww_2 \in E(G)$ since $\{x, w_1\}$ is not a minimal vertex cut. Since both V_1 and $N_2(x) \setminus \{w_1\}$ are cliques, there exist hamiltonian paths, say $P_1 = u_1P_1u_2$ and $P_2 = w_2P_2w_3$, in them, respectively. Let $C = P_1P_2 \cup \{xw_3, w_2w, wt, tu_1, u_2x\}$ be a cycle and $\{z, w_1\}$ be stems of T with $N_C(z) = V_1 \cup (S \setminus \{z\}) \cup \{w_1\}$ and $N_C(w_1) = V_2 \setminus \{w_1, w\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.8(1)).

Subcase 2: Suppose that $V_1 = \{u\}$ and $|N_2^2(x)| \geq 2$.

By Claim 5.3.6 and Corollary 5.1.2, we know $|S\setminus\{x, y\}| \leq 2$. We may assume $xz \in E(G)$ since $|S| \geq 3$ and G is H-2 free and H-6 free. If $S = \{x, y, z\}$ or $S = \{x, z, t\}$, since neither $\{w_1, x\}$ nor $\{w_1, z\}$ is not a 2-cut, we can assume there exists $w \in N_2^2(x)$ such that $ww_1, ww_2 \in E(G)$. Similarly as Subcase 1, we can find a strong spanning Halin subgraph in G. If $S = \{x, y, z, t\}$, since both $N_2(x)$ and $N_2^2(x) = N_2(z)$ are cliques, there exist hamiltonian paths, say $P_1 = w_1P_1w_2$ and $P_2 = wP_2w'$, in them, respectively. Let $C = P_1P_2 \cup \{yw_2, w't, tu, uy\}$ be a cycle and $\{x, z\}$ be stems of T with $N_C(x) = V(P_1) \cup \{y\}$ and $N_C(z) = V(P_2) \cup \{t, u\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.8(2)).

Subcase 3: Assume $V_1 = \{u\}$ and $N_2^2(x) = \{w\}$.

Since G is H-9 free, $|N_2(w) \cap N_2(x)| \leq 2$. We may assume $\{w_1, w_2\} \subseteq N_2(w)$ since $V_2 = N_2(x) \cup N_2^2(x)$(Note that ww_2 may not exist).

If ww_2 does not exist, then y exists since $\{x, w_1\}$ is not a 2-cut and $S \setminus \{x, y\} \subseteq \{z, t\}$ by

Lemma 4.1.2. Similarly as Subcase 1 and 2, we can find a strong spanning Halin subgraph in G.

If ww_2 exists, we have $S\setminus\{x,y\}\subseteq\{z,t,t'\}$ by Lemma 4.1.2, we can assume $t'w_2\in E(G)$ if $|\{z,t,t'\}|=3$. Since $\langle w_3,w_4,w_1,w_2;w,t\rangle\not\cong$ H-6, we have either $tw_2\in E(G)$ or $tw_1\in E(G)$.

If $tw_2\in E(G)$ and both t and t' exist, then $tt'\in E(G)$ by Claim 5.1.1. We may assume $zt, zw_1\in E(G)$ but $zt'\notin E(G)$ since $\langle z,u,t,t',w\rangle$ is neither isomorphic to H-2 nor isomorphic to H-9 and $|N_2(x,t,t')|\geq 3$. We may also assume $w_2z\notin E(G)$, otherwise swipe w_1 as w_2. Then $zt\in E(G)$ since $\langle w;z,t,w_2\rangle$ is not a claw, which also implies $zt'\notin E(G)$. Moreover, we have $xt'\in E(G)$ since $\langle u;x,z,t'\rangle$ is not a claw and $\langle z,u,t,w,w_1,x\rangle$ does not isomorphic to H-5. Since $N_2(x)\setminus\{w_1\}$ is a clique, there exists a hamiltonian path, say $P_1=w_2P_1w_3$, in it. Let $C=P_1\cup\{w_2t',t't,tu,ux,xy,yw_3\}$ be a cycle and all vertices on the path zww_1 be stems of T with $N_C(z)=\{u,t\}$, $N_C(w)=\{t'\}$ and $N_C(w_1)=V(P_1)\cup\{x,y\}$. Let $H=T\cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See in Figure 5.8(3)).

If $tw_1\in E(G)$, by Lemma 5.1.1, $tz\in E(G)$. Since neither $\langle x;u,z,t,w_1\rangle\cong$ H-2 nor $\langle x,u,z,t,w_1,w\rangle\cong$ H-7, we can assume $xz\in E(G)$ and $xt\notin E(G)$. Moreover, to avoid $\langle u;x,t,t'\rangle$ be a claw and $\langle t',u,z,t,w_1\rangle$ isomorphic to H-2, we have either tt' in $E(G)$ or both xt' and zt' in $E(G)$. If $tt'\in E(G)$, we can find a strong spanning Halin subgraph as $tw_2\in E(G)$. If both xt' and zt' in $E(G)$, since $N_2(x)\setminus\{w_1\}$ is a clique, there exists a hamiltonian path, say $P_1=w_2P_1w_3$, in it. Let $C=P_1\cup\{ut,tw,wt',tw_2,w_3x,xu\}$ be a cycle and $\{z,w_1\}$ be stems of T with $N_C(z)=(S\setminus\{x\})\cup\{u,w\}$ and $N_C(w_1)=V(P_1)$. Let $H=T\cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.8(4)).

Case 2.2: Assume $N_2(x)=\{w_1,w_2,w_3\}$.

Claim 5.3.6. *If $N_2(x)=\{w_1,w_2,w_3\}$, then the structure of $\langle V_2\rangle$ can be described.*

Proof. For any $w\in N_2^2(x)$, $|N_2(w)\cap N_2(x)|\leq 2$ since G is H-9 free. Therefore, if $ww_1\in E(G)$, we can assume $ww_3\notin E(G)$. Since $\langle w_1;w_3,w,z\rangle$ is not a claw, $wz\in E(G)$. If $ww_1\notin E(G)$, we may assume $ww_2\in E(G)$ since $V_2=N_2(x)\cup N_2^2(x)$. The fact that

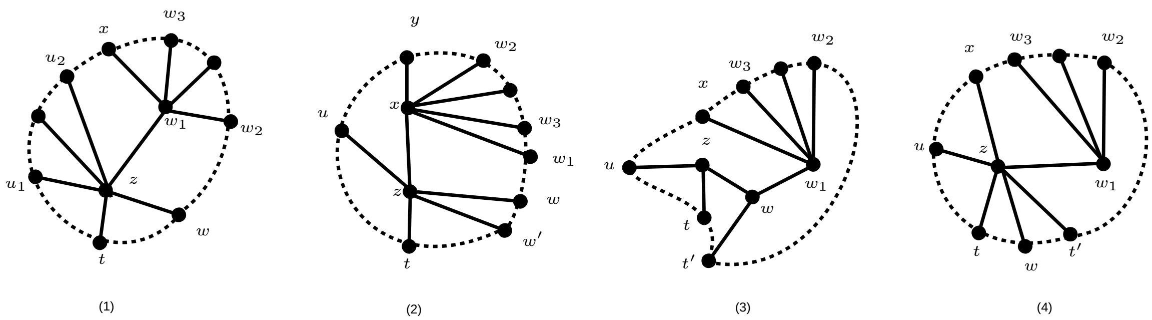

Figure 5.8. $|N_2(x)| \geq 4$.

$\langle w; w_2, w_3, w_1; zu\rangle$ is not a $B_{1,2}$ showing $ww_3 \in E(G)$. On the other hand, there exists at most one such w since if there exists another w' with the same property as w, then $\langle u, x, w_2, w_3, w, w'\rangle \cong$ H-6. This implies $|N_2(w_1) \cap N_2^2(x)| \geq |N_2^2(x)| - 1$ and also means $|N_2(z) \cap N_2^2(x)| \geq |N_2^2(x)| - 1$. Since $|N_2(x)|$ is maximum, we have $|N_2^2(x)| \leq 3$, which implies $|V_2| \leq 6$.

Here we may reserve $w \in N_2^2(x) \setminus N_2(z)$ if it exists and $w', w'' \in N_2(z) \cap N_2^2(x)$ if they exist. Since $\langle u; z, w', w_1; w_2w\rangle$ is not a $B_{1,2}$ and $\langle w_2; x, w', w\rangle$ is not a claw, $ww' \in E(G)$. Similarly, we can show that $ww'' \in E(G)$. Moreover, since neither $\langle w_2, w, w', w'', z, u\rangle$ nor $\langle w_3, w, w', w'', z, u\rangle$ isomorphic to H-3 and neither $\langle x, w_1, w_2, w_3, w'\rangle$ nor $\langle x, w_1, w_2, w_3, w''\rangle$ isomorphic to H-9, we can assume $w_3w', w_2w'' \in E(G)$ and $w_2w', w_3w'' \notin E(G)$. Note that if w does not exist, we can also assume $w_2w' \in E(G)$ since $\{x, w_1\}$ is not a 2-cut. ■

We still reserve $w \in N_2^2(x) \setminus N_2(z)$ if it exists and $w', w'' \in N_2(z) \cap N_2^2(x)$ if they exist here and after.

Claim 5.3.7. *For any $t \in S \setminus \{x, y, z\}$, if $tw \in E(G)$, then either $N_2(t) = \{w_2, w'', w\}$ or $N_2(t) = \{w_3, w', w\}$.*

Proof. If $tw \in E(G)$, since neither $\langle w; w_2, w', t\rangle$ nor $\langle w; w_3, w'', t\rangle$ is a claw and neither $\langle w'', w', z, u, t\rangle \cong$ H-2 nor $\langle w_1, w', w'', z, t\rangle \cong$ H-9 and neither $\langle w_2, w_3, x, u, t\rangle \cong$ H-2 nor $\langle w_1, w', w'', x, t\rangle \cong$ H-9, we have either $N_2(t) = \{w_2, w'', w\}$ or $N_2(t) = \{w_3, w', w\}$. ■

Claim 5.3.8. *For any $t \in S \setminus \{x, y, z\}$, $tw_1 \notin E(G)$.*

Proof. If there exists $t \in S \setminus \{x, y, z\}$ such that $tw_1 \in E(G)$, since $\langle w; w_2, w', t\rangle$ is not a claw, we can assume $tw_2 \in E(G)$. By Claim 5.3.7, we have $N_2(t) = \{w_1, w_2, w'', w\}$, which contradicts to the definition of x in S'. ■

Claim 5.3.9. *For any $t \in S\setminus\{x, y, z\}$, if $tw' \in E(G)$ (similarly as $tw'' \in E(G)$ or $tw_2 \in E(G)$ or $tw_3 \in E(G)$), then either $N_2(t) = N_2(z)$ or $N_2(t) = \{w_3, w', w\}$.*

Proof. Since neither $\langle w'; w_3, w'', t\rangle$ nor $\langle t; u, w_3, w''\rangle$ is a claw, we have either $tw_3 \in E(G)$ or $tw'' \in E(G)$. If $tw'' \in E(G)$, then $zt \in E(G)$ since $\langle w', w'', t, u, z\rangle \ncong$ H-2. To avoid $\langle w_1, w', w'', z, t\rangle \cong$ H-9, we have $N_2(t) = N_2(z)$. If $tw_3 \in E(G)$, since $\langle w_2; x, w, t\rangle$ is not a claw, we have $xt \in E(G)$. To avoid $\langle x, t, w'', w, w_2\rangle \cong$ H-2, we have $wt \in E(G)$. Therefore, $N_2(t) = \{w_3, w', w\}$. ■

Corollary 5.3.1. *For any $t \in S$, either $N_2(t) = \{w_1, w_2, w_3\}$ or $N_2(t) = \{w_1, w', w''\}$ or $N_2(t) = \{w, w'', w_2\}$ or $N_2(t) = \{w_3, w', w\}$.*

Corollary 5.3.2. *For any $t \in S$, there exists at most one $t' \in S \setminus \{t\}$, such that $N_2(t) = N_2(t')$.*

Proof. We may assume there exists $t', t'' \in S \setminus \{t\}$ such that $N_2(t) = N_2(t') = N_2(t'')$. Since $\langle V_2\rangle$ is not a clique, by Lemma 4.1.2, $|N_2(t) = N_2(t') = N_2(t'')| \geq 3$. Since G is H-2 free, $tt', tt'', t't'' \in E(G)$. This will force $\langle u, t, t', t'', w^*\rangle$ isomorphic to H-9, where $w^* \in N_2(t) = N_2(t') = N_2(t'')$. ■

Now we want to find a strong spanning Halin subgraph in G by following subcases.

Subcase 1: Assume $|V_1| \geq 2$.

By the remark after Claim 5.3.2, we have $V_1 \cup S$ is a clique. Let $P = tPz$ be a hamiltonian path in $V_1 \cup S \setminus \{x\}$ and $w_3 \in N_2(t)$. Set $C = P \cup \{zw', w'w'', w''w, ww_3, w_3t\}$ be a cycle and all vertices on the path xw_2w_1 be stems of T with $N_C(x) = V(P) \setminus \{z\}$, $N_C(w_2) = \{w_3, w\}$ and $N_C(w_1) = \{z, w', w''\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.9(1)).

Subcase 2: Suppose $V_1 = \{u\}$.

If $\langle S \rangle$ is connected, we may assume $P = x_1 x_2 \cdots x_k$ is a hamiltonian path in $\langle S \rangle$. Denote by $\{w_1, w_2, w_3\} \in N_2(x_1)$, $w_2 \in N_2(x_2)$, $w \in N_2(x_k)$ since G is H-2 free and H-9 free. Let $C = x_2 P x_k \cup \{x_k w, w w_2, w' w'', w'' w_2, w_2 x_2\}$ be a cycle and $\{u, x_1, w_3, w_1\}$ be stems of T with $N_C(u) = V(P)$, $N_C(w_3) = \{w', w\}$ and $N_C(w_1) = \{w_2, w''\}$ (See Figure 5.9(2)). If $\langle S \rangle$ is not connected, $\langle S \rangle$ has exact two cliques, denote by $S_1 = \{x_1, x_2, \cdots, x_{k_1}\}$ and $S_2 = \{y_1, y_2, \cdots, y_{k_2}\}$. If $|S_2| \neq 2$, let $w_1 \in N_2(x_1) \cap N_2(y_1)$, $w_3 \in N_2(x_1)$, $w' \in N_2(x_k)$ and $w_2 \in N_2(y_{k_2})$. Set $P_1 = x_2 P_1 x_{k_1}$ and $P_2 = y_2 P_1 y_{k_2}$ be hamiltonian paths in $S_1 \setminus \{x_1\}$ and $S_2 \setminus \{y_1\}$, respectively. Note that P_2 may not exist. Set $C = P_1 P_2 \cup \{u x_2, x_{k_1} w', w' w, w w'', w'' w_2, w_2 y_{k_2}, y_2 u\}$ be a cycle and all vertices on the path $w_3 x_1 w_1 y_1$ be stems of T with $N_C(w_3) = \{w', w\}$, $N_C(x_1) = V(P_1) \cup \{u\}$ and $N_C(w_1) = \{w_2, w''\}$ and $N_C(y_1) = V(P_2)$ (See Figure 5.9(3)). If $|S_2| = 2$, denote by $w_2 \in N_2(y_1) \cap N_2(y_2)$, $w_1 \in N_2(y_1)$, $w_3 \in N_2(x_1)$ and $w' \in N_2(x_{k_1})$. Let $P_1 = x_2 P_1 x_{k_1}$ be a hamiltonian path in $S_1 \setminus \{x_1\}$. Set $C = P_1 \cup \{u x_2, x_{k_1} w', w' w, w w'', w'' w_1, w_1 y_1, y_1 y_2, y_2 u\}$ be a cycle and all vertices on the path $x_1 w_3 w_2$ be stems of T with $N_C(x_1) = V(P_1) \cup \{u\}$, $N_C(x_1) = \{w', w_1\}$ and $N_C(w_2) = \{w'', y_1, y_2\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.9(4))

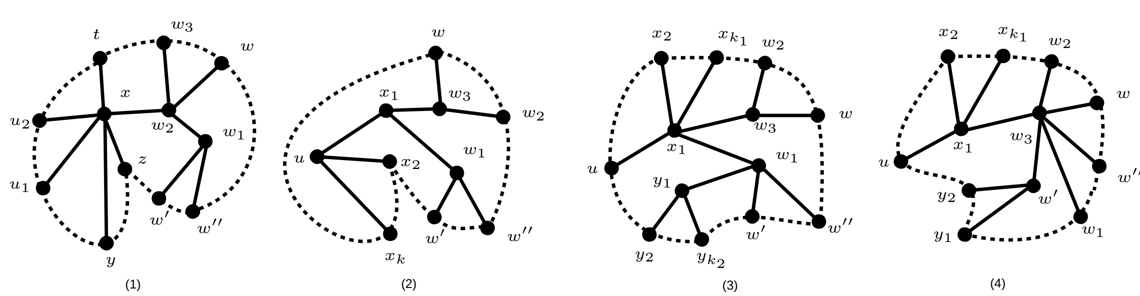

Figure 5.9. $|N_2(x)| = 3$.

Case 2.3: Suppose $N_2(x) = \{w_1, w_2\}$.

If $|V_2| = 2$, then $\langle V_2 \rangle$ is a clique since it is connected. Thus we may assume $|V_2| \geq 3$.

Claim 5.3.10. *If $|V_2| \geq 3$ and there exists $z \in S' \setminus \{x, y\}$ such that $N_2(x) \cap N_2(z) \neq \emptyset$, then there must exist at least one such z with $|N_2(z)| \geq 2$.*

Proof. Suppose this is not true. For any $z \in S'$ with $N_2(x) \cap N_2(z) \neq \emptyset$, $|N_2(z)| = 1$. Note that there exists exact one such z by Lemma 4.1.2. We may denote $N_2(x) = \{w_1, w_2\}$ and $N_2(z) = \{w_1\}$.

Firstly, we can show that for any $w \in V_2 \setminus \{w_1, w_2\}$, $ww_2 \in E(G)$ and $ww_1 \notin E(G)$. If there exists $w \in V_2 \setminus \{w_1, w_2\}$ such that $ww_1 \in E(G)$, then $w_1w_2, xz \in E(G)$ since neither $\langle w_1; z, w_2, w\rangle$ nor $\langle w_1; x, z, w\rangle$ is a claw. However, this in turn gives $\langle u, z, x, w_1, w_2, w\rangle \cong$ H-8. By Claim 5.3.4, we know $ww_2 \in E(G)$ and $ww_1 \notin E(G)$.

Secondly, $|V_2| \geq 4$. Otherwise, assume $V_2 = \{w_1, w_2, w\}$, since G is 3-connected, we have $deg_G(w) \geq 3$. This implies there exist $t, t' \in S \setminus \{x, z\}$ such that $N_2(t) = N_2(t') = \{w\}$, contradicts to Lemma 4.1.2.

Thus we may assume there exist $w_3, w_4 \in V_2 \setminus \{w_2, w_1\}$ and $t, t' \in S \setminus \{x, y, z\}$ such that $tw_3, t'w_4 \in E(G)$. To avoid $\langle w_1; w_2, w_3, w_4; t'u\rangle$ be a $B_{1,2}$, we have $t'w_3 \in E(G)$. This will force either $\langle u, t', w_4, w_3, w_1, x\rangle \cong$ H-4 or $\langle t', x, w_1, w_3, w_4\rangle \cong$ H-2, showing a contradiction. ■

Now we want to find a strong spanning Halin subgraph in G by following two subcases.

Case 2.3.1: Assume $|V_2| \geq 3$ and there exists $z \in S'$, such that $N_2(x) \cap N_2(z) \neq \emptyset$.

Denote by $N_2(x) = \{w_1, w_2\}$ and $N_2(z) = \{w_1, w_3\}$. Then,

Claim 5.3.11. *If there exists $z \in S'$ such that $N_2(x) = \{w_1, w_2\}$ and $N_2(z) = \{w_1, w_3\}$, then $V_2 \subseteq \{w_1, w_2, w_3, w\}$.*

Proof. This claim is provided by following statements.

Firstly, for any $w \in N_2^2(x)$, $ww_1 \notin E(G)$. If there exists $w \in V_2 \setminus \{w_1, w_2\}$ such that $ww_1 \in E(G)$, then $w_1w_2, xz \in E(G)$ since neither $\langle w_1; z, w_2, w\rangle$ nor $\langle w_1; x, z, w\rangle$ is a claw. However, this in turn gives us $\langle u, z, x, w_1, w_2, w\rangle \cong$ H-8 by Claim 5.3.4.

Secondly, For any $w \in N_2^2(x)$, $w_2w, w_3w \in E(G)$. If $w_2w \in E(G)$, then $w_3w \in E(G)$ because neither $\langle u; z, w_3, w_1; w_2w\rangle$ is a $B_{1,2}$ nor $\langle w_2; x, w_3, w\rangle$ is a claw. Similarly, if $w_3w \in$

$E(G)$, then $w_2w \in E(G)$. Since $V_2 = N_2(x) \cup N_2^2(x)$, we have $w_2w, w_3w \in E(G)$ for any $w \in N_2^2(x)$.

Thirdly, $V_2 \setminus \{w_1, w_2, w_3\} \subseteq \{w\}$. Otherwise, for any $w, w' \in V_2 \setminus \{w_1, w_2, w_3\}$ since $\langle w_1, w_2, w, w', w_3\rangle \not\cong$ H-2, we have $w_2w_3 \in E(G)$. This implies $V_2 \setminus \{w_1\}$ is a clique. Since $\{w_2, w_3\}$ is not a minimal vertex cut, we can assume there exists $t \in S$ such that $tw \in E(G)$. To avoid $\langle w_1; w_2, w', w; tu\rangle$ be a $B_{1,2}$, $\langle t; u, w_1, w\rangle$ be a claw and H-2 or H-4 exist in G, we have $w_2, w_3 \in N_2(t)$, which contradicts to the maximum property of $|N_2(x)|$. ■

Now we want to find a strong spanning Halin subgraph in G by following subcases.

Subcase 1: Suppose $|V_1| \geq 2$.

By remark after Claim 5.3.2, we know $V_1 \cup S$ is a clique and let $P = zPt$ be a hamiltonian path in $V_1 \cup (S \setminus \{x\})$. Since $deg_G(w) \geq 3$, there exists $t \in S$ such that $tw \in E(G)$.

If $V_2 = \{w_1, w_2, w_3, w\}$, let $C = P \cup \{zw_3, w_3w, wt\}$ be a cycle and all vertices on the path xw_2w_1 be stems of T with $N_C(x) = V(P) \setminus \{z\}$, $N_C(w_2) = \{w\}$ and $N_C(w_1) = \{z, w_3\}$. If $V_2 = \{w_1, w_2, w_3\}$, since $deg_G(w_2) \geq 2$, $deg_G(w_3) \geq 2$ and $\langle V_2\rangle$ is not a clique, there exist $t, t' \in S \setminus \{x, y, z\}$ such that $tw_2, t'w_3 \in E(G)$(Note that we may have $t = y$). Let $P = tPt'$ be a hamiltonian path in $V_1 \cup S \setminus \{x, z\}$. Set $C = P \cup \{tw_2, w_2w_1, w_1w_3, w_3t'\}$ be a cycle and $\{x, z\}$ be stems of T with $N_C(x) = V(P) \setminus \{t'\} \cup \{w_1, w_2\}$ and $N_C(z) = \{t', w_3\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.10(1) and (2)).

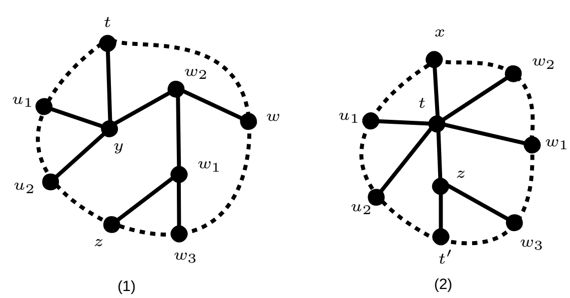

Figure 5.10. $N_2(x) = \{w_1, w_2\}$ and $|V_1| \geq 2$.

Subcase 2: Assume that $V_1 = \{u\}$.

If $V_2 = \{w_1, w_2, w_3\}$, similarly as $|V_1| \geq 2$, we can assume there exist $t, t' \in S \setminus \{x, y, z\}$ such that $tw_2, t'w_3 \in E(G)$(Note that we may have $t = y$). Since for any $x \in S$, $ux \in E(G)$ and G is claw-free and H-5 free, we have either $|\{xz, xt', zt'\}| = 1$ and $|\{xz, xt, zt\}| = 1$ or $|\{xz, xt', zt'\}| = 3$ and $|\{xz, xt, zt\}| = 3$. If $|\{xz, xt', zt'\}| = 3$ and $|\{xz, xt, zt\}| = 3$, let $C = utw_2w_1w_3t'u$ be a cycle and $\{x, z\}$ be stems of T with $N_C(x) = \{u, t, w_2\}$ and $N_C(z) = \{w_1, w_3, t'\}$. If $|\{xz, xt', zt'\}| = 1$ and $|\{xz, xt, zt\}| = 1$, then $xz \notin E(G)$. Since $\{u, w_1\}$ is not a 2 cut, we can assume $xt, xt' \in E(G)$. Moreover, since $\langle u; t, z, t' \rangle$ is not a claw, we have $tt' \in E(G)$. Let $C = tw_2w_1w_3t't$ be a cycle and all vertices on the path xuz be stems of T with $N_C(x) = \{t, w_2\}$, $N_C(u) = \{t'\}$ and $N_C(z) = \{w_1, w_3\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.11(1) and (2)).

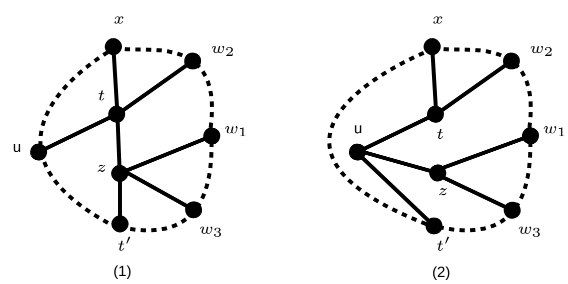

Figure 5.11. $N_2(x) = \{w_1, w_2\}$, $|V_1| = 1$ and $|V_2| = 3$.

If $V_2 = \{w_1, w_2, w_3, w\}$, we have following claims.

Claim 5.3.12. *If $V_2 = \{w_1, w_2, w_3, w\}$ and y exists, then there does not exist $z' \in S \setminus \{x, y, z\}$ such that $N_2(z) = N_2(z') = \{w_1, w_3\}$.*

Proof. Otherwise, $N_2(x) \cup N_2(y) \cup N_2(z) \cup N_2(z') = \{w_1, w_2, w_3\}$, contradicts to Lemma 4.1.2. ■

Therefore, we always assume z' does not exist and y may exist.

Claim 5.3.13. *If $V_2 = \{w_1, w_2, w_3, w\}$, for any $t \in S \setminus \{x, y, z\}$, $tw_1 \notin E(G)$.*

Proof. If $w_2w_3 \notin E(G)$, then $tw_1 \notin E(G)$ since $\langle w; t, w_2, w_3\rangle$ is not a claw. If $w_2w_3 \in E(G)$, since neither $\langle w_1; x, w_3, t\rangle$ nor $\langle w_1; w_2, z, t\rangle$ is a claw, we have $tx, tz \in E(G)$. This in turn gives $xz \in E(G)$ since $\langle t, x, u, z, w\rangle \ncong$ H-2. However, this will induce $\langle t, u, x, z, w_1\rangle \cong$ H-9. ■

Claim 5.3.14. *If $V_2 = \{w_1, w_2, w_3, w\}$ and $w_2w_3 \in E(G)$, then there exists exact one $t \in S \setminus \{x, y, z\}$, such that $tw \in E(G)$. Moreover, there exists a strong spanning Halin subgraph in G.*

Proof. Since $deg_G(w) \geq 3$, there exists $t \in S \setminus \{x, y, z\}$ such that $tw \in E(G)$. Moreover, since neither $\langle u, z, w_3, w, w_2, t\rangle \cong$ H-4 nor $\langle z, w_3, w, w_2, t\rangle \cong$ H-2 and by the maximum of $|N_2(x)|$, we have $tw_2, tw_3 \notin E(G)$. Therefore, there exists exact one such t because otherwise we can find a smaller vertex cut.

If $V_2 = \{w_1, w_2, w_3, w\}$ and $w_2w_3 \in E(G)$, let $C = utww_3zu$ be a cycle and all vertices on the path yw_2w_1 be stems of T with $N_C(y) = \{u, t\}$, $N_C(w_2) = \{w\}$ and $N_C(w_1) = \{z, w_3\}$. Let $H = T \cup C$, then H is a sstrong spanning Halin subgraph of G (See Figure 5.12(1)).■

Claim 5.3.15. *If $V_2 = \{w_1, w_2, w_3, w\}$ and $w_2w_3 \notin E(G)$, then for any $t \in S \setminus \{x, y, z\}$, either $N_2(t) = \{w, w_2\}$ or $N_2(t) = \{w, w_3\}$.*

Proof. If $tw \in E(G)$, since neither $\langle w; t, w_2, w_3\rangle$ nor $\langle t; u, w_2, w_3\rangle$ is a claw, we have either tw_2 or tw_3 in $E(G)$, but not both. If $tw_2 \in E(G)$, since $\langle w_2; t, w_1, w\rangle$ is not a claw, we have $tw \in E(G)$. Similarly, if $tw_3 \in E(G)$, then $tw \in E(G)$. Thus either $N_2(t) = \{w, w_2\}$ or $N_2(t) = \{w, w_3\}$. ■.

If all t_1, t_2 and y exist, since neither $\langle w_2, w_1, x, y, u, t_2\rangle$ nor $\langle w_3, w, t_1, t_2, u, y\rangle$ isomorphic to H-7 and neither $\langle t_2, u, x, y, w_1, w_2\rangle$ nor $\langle y, u, t_1, t_2, w_3, w\rangle$ isomorphic to H-8, we can assume $xt_1, yt_2 \in E(G)$. Moreover, since neither $\langle u, t_1, t_2, w_3, z\rangle$ nor $\langle u, x, y, w_2, t_3\rangle$ isomorphic to H-2 and $\langle t_1; x, t_2, z\rangle$ is not a claw, we can assume t_3y and t_2z in $E(G)$. Let $C = ut_3w_2w_1xt_1ww_3zu$ be a cycle and $\{y, t_2\}$ be stems of T with $N_C(y) = \{u, t_3, w_2, w_1, x\}$ and $N_C(t_2) = \{t_1, w, w_3, z\}$. If t_1, t_2 exist and y does not exist, since $\langle u, t_1, w_3, w, t_2\rangle \ncong$ H-2, we have $t_1t_2 \in E(G)$. Therefore, since $\langle z, u, t_1, t_2, w_3\rangle \ncong$ H-2, we can assume $zt_1 \in E(G)$. Let $C = ut_3w_2wt_2t_1zu$ be a cycle and all vertices on the path xw_1w_3 be stems of T with $N_C(x) = \{u, t_3\}$, $N_C(w_1) = \{w_2\}$ and $N_C(w_3) = \{w, t_1, t_2, z\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.12(2) and (3)).

If either t_1 or t_2 exists, we may assume t_2 does not exist. If y exists, since $\langle u, x, y, w_2, t_3\rangle \ncong$ H-2, we can assume $t_3y \in E(G)$. Let $C = t_3yw_2w_1zw_3wt_3$ be a cycle and all vertices on the path xut_1 be stems of T with $N_C(x) = \{w_1, w_2\}$, $N_C(u) = \{x, y, z, t_1, t_3\}$ and $N_C(t_1) = \{w_3, w\}$. If y does not exist, since $t_3w_2 \in E(G)$, we can also find a cycle C and a tree T in G similarly. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.12(4)).

If neither t_1 nor t_2 exists, then t_3 must exist since $deg_G(w) \geq 3$. If y exists, let $C = ut_3ww_3zyu$ be a cycle and all vertices on the path w_2xw_1 be stems of T with $N_C(w_2) = \{w, t_3\}$, $N_C(x) = \{u\}$ and $N_C(w_2) = \{w_3, z, y\}$. If y does not exist, since $uz \in E(G)$, we can find a cycle and a tree similarly. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.12(5)).

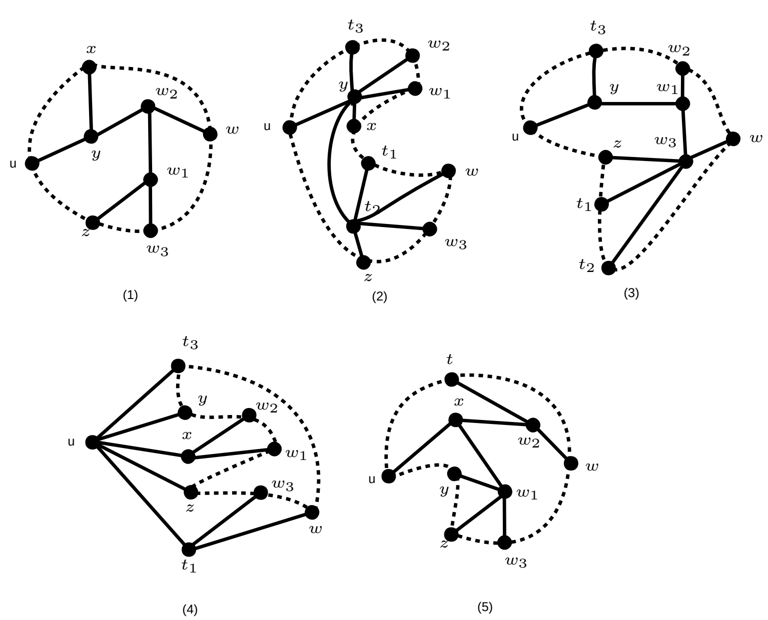

Figure 5.12. $N_2(x) = \{w_1, w_2\}$, $|V_1| = 1$ and $|V_2| \geq 4(1)$.

Case 2.3.2: Suppose $|V_2| \geq 3$ and for any $z \in S'$ with $N_2(x) \cap N_2(z) = \emptyset$.

By the definition of x, we have y must exist and $V_1 = \{u\}$. Since $V_2 = N_2(x) \cup N_2^2(x)$, we may assume there exists $w_3 \in V_2 \setminus \{w_1, w_2\}$ and $z \in S \setminus \{x, y\}$ such that $w_3w_2, w_3z \in E(G)$. Then we have following claim.

Claim 5.3.16. $|V_2| \leq 5$ *and* $S = \{x, y, z\}$.

Proof. Since $\langle u, y, x, w_2, w_3, z\rangle \cong$ H-4, we can assume $xz \in E(G)$. Moreover, $w_1w_3 \notin E(G)$, otherwise $\langle w_1, w_2, w_3, z, y\rangle \cong$ H-2. Since $deg_G(w_3) \geq 3$, we can assume there exist $w_4 \in V_2 \setminus \{w_1, w_2, w_3\}$ such that $w_3w_4 \in E(G)$ by Lemma 4.1.2 and the definition of x. To avoid $\langle w_3; z, w_2, w_4\rangle$ be a claw and $\langle w_1, w_4, w_3, z, y\rangle$ isomorphic to H-2, we have either w_2w_4 or zw_4 in $E(G)$.

If $w_2w_4 \in E(G)$, to avoid $\langle w_1; w_2, w_4, w_3; zu\rangle$ be a B_{12}, we have $w_1w_4 \in E(G)$. This implies $N_2(w_3) \subseteq N_2(w_1)$. If $zw_4 \in E(G)$, since neither $\langle u; z, w_4, w_3; w_2w_1\rangle$ is a B_{12} nor $\langle x; z, w_3, w_4, w_2\rangle \cong$ H-2 nor $\langle z; x, w_1, w_2, w_3\rangle \cong$ H-2, we have $w_1w_4 \in E(G)$, which also implies $N_2(w_3) \subseteq N_2(w_1)$. For any $w \in V_2 \setminus \{w_1, w_2, w_3, w_4\}$, if $ww_2 \in E(G)$, then $ww_3 \in E(G)$ since $\langle w_2; x, w_3, w\rangle$ is not a claw. Thus we have $N_2(w_2) \setminus \{w_1\} \subseteq N_2(w_3) \subseteq N_2(w_1)$. If $w_1w \in E(G)$, then $w_4w \in E(G)$ since $\langle w_1; x, w_4, w\rangle$ is not a claw. And if $ww_4 \in E(G)$, then $ww_1 \in E(G)$ since neither $\langle w_4; z, w_1, w\rangle$ nor $\langle w_4; w_3, w_1, w\rangle$ is a claw and $N_2(w_2) \subseteq N_2(w_1)$. Thus $N_2(w_1) \setminus \{w_2, w_3\} = N_2(w_4) \setminus \{w_2, w_3\}$.

In fact, $|V_2 \setminus \{w_1, w_2, w_3, w_4\}| \leq 1$. Otherwise, by symmetric, we may assume there exist $w_5, w_6 \in V_2 \setminus \{w_1, w_2, w_3, w_4\}$ such that $w_1w_5, w_1w_6 \in E(G)$. Then $w_5w_4, w_6w_4 \in E(G)$ since neither $\langle w_1; x, w_4, w_5\rangle$ nor $\langle w_1; x, w_4, w_6\rangle$ is a claw. This in turn gives $w_5w_3, w_6w_3 \in E(G)$ because neither $\langle w_3; w_4, w_5, w_1; yu\rangle$ nor $\langle w_3; w_4, w_6, w_1; yu\rangle$ is a B_{12}. However, this will induce $\langle w_1, w_5, w_6, w_4, w_3\rangle \cong$ H-9 since $w_1w_3 \notin E(G)$. Thus $V_2 \subseteq \{w_1, w_2, w_3, w_4, w_5\}$. Note that only w_5 may not exist.

Moreover, $S = \{x, y, z\}$. Otherwise, by the definition of x, for any $t \in S \setminus \{x, y, z\}$, $tw_1, tw_2, tw_3 \notin E(G)$. We also have $tw_4 \notin E(G)$. Otherwise $\langle w_4; t, w_1, w_3\rangle$ is a claw, which in turn gives us w_5 exists and $tw_5 \in E(G)$. If $w_2w_4 \in E(G)$, since $\langle w_5; w_4, w_2, w_3; zu\rangle$ is not a $B_{1,2}$ and $N_2(w_2) \setminus \{w_1\} \subseteq N_2(w_3)$, we can assume $w_3w_5 \in E(G)$. This forces $\langle w_5; w_1, w_3, t\rangle$

to be a claw. If $zw_4 \in E(G)$, since $\langle w_2; w_1, w_4, w_5; tu \rangle$ is not a $B_{1,2}$, we have $w_2w_5 \in E(G)$, which implies $w_3w_5 \in E(G)$. However, this will induce $\langle w_5; w_1, w_3, t \rangle$ to be a claw.

Since $deg_G(w_5) \geq 3$, we may assume $w_3w_5 \in E(G)$. If $w_2w_4 \in E(G)$, let $C = uyw_1w_5w_3zu$ be a cycle and all vertices on the path xw_2w_4 be stems of T with $N_C(x) = \{u, y\}$, $N_C(w_2) = \{y\}$ and $N_C(w_4) = \{w_1, w_5, w_3\}$. If $zw_4 \in E(G)$, let $C = uyw_1w_5w_4zu$ be a cycle and all vertices on the path xw_2w_3 be stems of T with $N_C(x) = \{u, y\}$, $N_C(w_2) = \{w_1\}$ and $N_C(w_3) = \{w_5, w_4, z\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 5.13(1) and (2)).

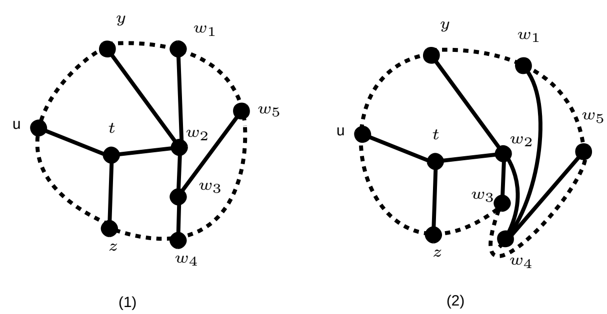

Figure 5.13. $N_2(x) = \{w_1, w_2\}$, $|V_1| = 1$ and $|V_2| \geq 4(2)$.

Chapter 6

LONG CIRCULAR INTERVAL GRAPHS

Recall that if we let Σ be a circle and $F'_1, \cdots, F'_k \subseteq \Sigma$ be homeomorphic to the interval $[0,1]$ and assume there is no three of $F'_1, \cdots, F'_k$ have union Σ and no two of $F'_1, \cdots, F'_k$ share an end-point. Set $V \subseteq \Sigma$ be finite and G be a graph with vertex set V in which, for distinct $u, v \in V$, u, v are adjacent if and only if $u, v \in F_i$ for some i. Such a graph G is called a long circular interval graph.

In this chapter, we will prove the following proposition.

Proposition 10. *If G is a 3-connected long circular interval graph, then G contains a strong spanning Halin subgraph.*

6.1 Some properties of long circular interval graphs

We still follow the definitions and notations mentioned in Section 4.1 that G is a graph with n-vertex and S is a minimum vertex cut of G. Let G_1 and G_2 be the exact two components of $G \setminus S$, and $V_1 = V(G_1)$, $V_2 = V(G_2)$. Subject to the minimality of $|S|$, we always assume that $|V_1|$ is minimum. Moreover, if we denote by $V = \{u_1, u_2, \cdots, u_n\}$ be vertex set of G and u_i, u_{i+1} are consecutive vertices along the circle Σ for any $i \in [1, n]$, where $u_{n+1} = u_1$. The following lemma is giving a partition of V.

Claim 6.1.1. *For $i \in [1, 2]$, the vertices of V_i are consecutive along Σ.*

Proof. We only show this claim is true for $i = 1$.

It is trivial if $|V_1| = 1$, thus we assume $|V_1| \geq 2$. Suppose to the contrary, there exist $1 \leq i, j \leq n$ such that $u_i, u_j \in V_1$ and $u_{i+1}, u_{j+1} \notin V_1$. Since $\langle V_1 \rangle$ is connected, there exists a path P from u_i to u_j in $\langle V_1 \rangle$. We may assume $V(u_i P u_j) \subseteq \{u_i, u_{i+1}, \cdots, u_j\}$, then $\{u_i, u_{i+1}, \cdots, u_j\} \subseteq S \cup V_1$, which implies $u_{i+1} \in S$. Moreover, since $N_2(u_{i+1}) \neq \emptyset$, there must exist $u_k \in V_2 \cap (\{u_1, u_2, \cdots, u_{i-1}\} \cup \{u_{j+1}, \cdots, u_n\})$, such that $u_{i+1}u_k \in E(G)$. by the definition, there exists an interval contains both u_{i+1} and u_k, which also has to contain either u_j or u_i. Therefore, we get either $u_ju_k \in E(G)$ or $u_iu_k \in E(G)$, showing a contradiction. ■

For simplicity, we denote by $V_1 = \{u_i, u_{i+1}, \cdots, u_j\} = \{v_{n_1}, v_{n_1-1}, \cdots, v_2, v_1\}$, $V_2 = \{u_k, u_{k+1}, \cdots, u_l\} = \{w_1, w_2, \cdots, w_{n_2}\}$, $S_1 = \{u_{j+1}, u_{j+2}, \cdots, u_{k-1}\} = \{x_1, x_2, \cdots, x_{m_1}\}$ and $S_2 = \{u_{l+1}, u_{l+2}, \cdots, u_{i-1}\} = \{y_{m_2}, y_{m_2-1}, \cdots, y_1\}$, where $u_{n+1} = u_1$. Clearly, $S = S_1 \cup S_2$. Without loss of generality, we always assume $|S_1| \geq |S_2|$, note that S_2 may be empty.

Claim 6.1.2. *1) If S_2 is empty, then S is a clique.*

2) If S_2 is not empty, then both $\langle S_1 \rangle$ and $\langle S_2 \rangle$ are connected. Moreover, both $\langle S_1 \rangle$ and $\langle S_2 \rangle$ are cliques if $E(S_1, S_2) = \emptyset$, and V_1 or V_2 is a clique if $E(S_1, S_2) \neq \emptyset$.

Proof. 1) If S_2 is empty, $S = S_1 = \{u_{j+1}, u_{j+2}, \cdots, u_{k-1}\}$. Since $N_2(u_{j+1}) \neq \emptyset$ and $u_{j+1}u_k \in E(G)$, there exists an interval contains both u_{j+1} and u_k, which also has to contain all vertices of S. Thus S is a clique.

2) Suppose $\langle S_1 \rangle$ is not connected, then there exist $u_s, u_{s+1} \in E(G)$ such that $u_su_{s+1} \notin E(G)$. Since $N_2(u_s) \neq \emptyset$, there exists an interval contains both u_s and u_l, which also has to contain all vertices of V_1. Therefore, $E(V_1, V_2) \neq \emptyset$, giving a contradiction. Similarly, we can show $\langle S_2 \rangle$ is connected.

If $E(S_1, S_2) \neq \emptyset$, we may assume $u_{k-1}u_{l+1} \in E(G)$. Then there exists an interval contains both u_{k-1} and u_{l+1}, which also has to contain all vertices in V_2 or V_1, then V_2 or V_1 is a clique.

If $E(S_1, S_2) = \emptyset$, since $N_2(u_{j+1}) \neq \emptyset$, $u_{j+1}u_k \in E(G)$. Then there exists an interval contains all vertices between u_{j+1} and u_k, which implies $\langle S_1 \rangle$ is a clique. Similarly, since $N_1(u_{l+1}) \neq \emptyset$, $u_{l+1}u_i \in E(G)$. This in turn gives that there exists an interval contains all vertices between u_{l+1} and u_i, thus $\langle S_2 \rangle$ is a clique. ■

Now we divide V_1 into following parts. Let $V_{11} = N_1(x_1) = \{v_1, v_2, \cdots, v_{s_1}\}$, $V_{12} = N_1(v_{s_1}) \backslash V_{11} = \{v_{s_1+1}, v_{s_1+12}, \cdots, v_{s_2}\}, \cdots$, and $V_{1k} = N_1(v_{s_{(k-1)}}) \backslash \cup_{i=1}^{k-1} V_{1i}$. Then $V_{11}, V_{12}, \cdots, V_{1k}$ is a partition of V_1. Similarly, let $V_{21} = N_2(x_{m_1}) = \{w_1, w_2, \cdots, w_{t_1}\}$, $V_{22} = N_2(w_{t_1}) \setminus V_{21} =$

$\{w_{t_1+1}, w_{t_1+2}, \cdots, w_{t_2}\}$, $\cdots$, and $V_{2l} = N_2(w_{t_{l-1}}) \setminus \cup_{i=1}^{l-1} V_{2i}$. Then $V_{21}, V_{22}, \cdots, V_{2l}$ is a partition of V_2. For convenience, we denote by $V_{1i} = \{v_{s_{i-1}+1}, \cdots, v_{s_i}\} = \{v_{i_1}, v_{i_2}, \cdots, v_{i_{s_i}}\}$ and $V_{2j} = \{w_{t_{j-1}+1}, \cdots, w_{t_j}\} = \{w_{j_1}, w_{j_2}, \cdots, w_{j_{t_j}}\}$ for all $i \in [1, k]$ and $j \in [1, l]$. The following claims are clearly true by the definition of long circular interval graphs and the partition of V_1 and V_2.

Claim 6.1.3. *For any $i \in [1, k]$ and $j \in [1, l]$, we have V_{1i} and V_{2j} are cliques.*

Proof. Since for any $i \in [1, k-1]$ and $j \in [1, l-1]$, we have $x_1 v_{s_1}$, $x_{m_1} w_{t_1}$, $v_{i_{s_i}} v_{i+1_{s_{i+1}}}$ $w_{j_{t_j}} w_{j+1_{t_{j+1}}}$ all in $E(G)$. ■

Claim 6.1.4. *For any $i \in [1, k-1]$ and $j \in [1, l-1]$, we have $|V_{1i}| \geq |S_1|+1$ and $|V_{2j}| \geq |S_1|$.*

Proof. If there exists $i \in [1, k-1]$ such that $|V_{1i}| \leq |S_1|$, then $V_{1i} \cup S_2$ is a vertex cut with $|V_{1i} \cup S_2| \leq |S_1 \cup S_2| = |S|$ and $|V_1'| = |V_1 \setminus \cup_{j=1}^{i} V_{1i}| < |V_1|$, this contradicts to assumptions that S is a minimum vertex cut and $|V_1|$ is minimum subject to S being minimum. Similarly, if there exists $j \in [1, l-1]$ such that $|V_{2j}| < |S_1|$, then $V_{2j} \cup S_2$ is also a vertex cut with $|V_{2j} \cup S_2| < |S_1 \cup S_2| = |S|$, contradicts to the minimality of $|S|$. ■

Claim 6.1.5. *Let $|S_1| = m_1$ and $p \in [0, m_1-1]$, then $x_{1+p} v_{1_{1+p}}$, $v_{i_{s_i-p}} v_{(i+1)_{m_1-p}}$, $x_{m_1-p} w_{m_1-p}$, $w_{j_{t_j-p}} w_{(j+1)_{m_1-p}} \in E(G)$ for any $i \in [1, k-1]$ and $j \in [1, l-1]$.*

Proof. We only proof that for any $j \in [1, l-1]$, $w_{j_{t_j-p}} w_{(j+1)_{m_1-p}} \in E(G)$. Suppose to the contrary, there exists $j \in [1, l-1]$ such that $w_{j_{t_j-p}} w_{(j+1)_{m_1-p}} \notin E(G)$. Then $\{w_{j_{t_j-p+1}}, w_{j_{t_j-p+2}}, \cdots, w_{j_{s_j}}\} \cup \{w_{(j+1)_1}, w_{(j+1)_2}, \cdots, w_{(j+1)_{m_1-p-1}}\} \cup S_2$ is a vertex cut with size $p + (m_1 - p - 1) + |S_2| = m_1 - 1 + |S_2| = |S| - 1$, showing a contradiction. ■

6.2 Proof of 3-connected long circular interval graphs

For simplicity, we denote by $S_1 = V_{20} = \{x_1, x_2, \cdots, x_{m_1}\} = \{w_{0_1}, w_{0_2}, \cdots, w_{0_{t_0}}\}$.

Our goal is to find a strong spanning Halin subgraph H in G. We divide this processor into two steps. In the first step, we find the part of H in $\langle V_2 \cup S_1 \rangle$ and in the second step, we find the other part of H in $V_1 \cup S_2 \cup \{x_t\}$.

Step 1: Finding a spanning subgraph H_2 of H in $\langle V_2 \cup S_1 \rangle$

Case 1: Assume $S_2 = \emptyset$.

We let $Q = w_{0_{t_0}} w_{1_{t_1}} \cdots w_{l-1_{t_{l-1}}}$ be a path. If for any $j \in [0, l-1]$, we have $|V_{2j}| \neq 4$. Since for any $i \in [0, l-1]$, both $V_{2j} \setminus \{w_{j_2}, w_{j_{t_j-2}}, w_{j_{t_j}}\}$ and V_{2l} are cliques, there exist hamiltonian paths, say $P_{2j} = w_{j_1} P_{2j} w_{j_{t_j-2}}$ and $P_{2l} = w_{l_1} P_{2l} w_{l_2}$, in them, respectively. Let $P_2 = w_{0_2} w_{0_{t_0-1}} w_{1_2} \cdots w_{(l-1)_{t_{l-1}-1}}$ and $C_2 = P_{20} P_{21} \cdots P_{2t} P_2$ and all vertices of Q be stems of T_2 with $N_{C_2}(w_{j_{t_j}}) = V(P_{2_j}) \cup \{w_{j_2}, w_{j_{s_j-1}}\}$ for all $j \in [0, l-2]$ and $N_{C_2}(w_{(l-1)_{t_{(l-1)}}}) = \{w | w \in V_{2l-1} \cup V_{2l}\}$. Then T_2 is a HIST of $\langle V_2 \cup S_1 \rangle$ and $V(C_2) = \{w \in V_2 \cup S_1 | deg_{T_2}(w) = 1\}$. Let $H_2 = T_2 \cup C_2$, then H_2 is planar (See Figure 6.1).

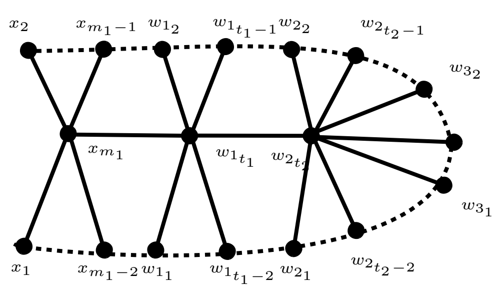

Figure 6.1. $S_2 = \emptyset$ and $|V_{2j}| \neq 4$ for all $j \in [1, l-1]$.

If there exists $j \in [0, l-1]$ such that $|V_{2j}| = 4$, first we find T_2 and C_2 in $\langle V_2 \cup S_1 \rangle$ as above, then apply Swap-Operation to the HIST (See Figure 6.2).

Swap-Operation:

1) Swapping the positions of $w_{(j+1)_1}$ and $w_{(j+1)_2}$ on the path C_2.

2) Putting w_{j_3} adjacent to w_{j_1} and $w_{(j+1)_2}$ along the path C_2.

3) Keeping all other vertices' positions are the same.

After performing Swap-Operation, we get a new T_2 and C_2, then the new T_2 is a HIST of $\langle V_2 \cup S_1 \rangle$ and $V(C_2) = \{w \in V_2 \cup S_1 | deg_{T_2}(w) = 1\}$. Let $H_2 = T_2 \cup C_2$, then H_2 is planar.

Case 2: Assume $S_2 \neq \emptyset$.

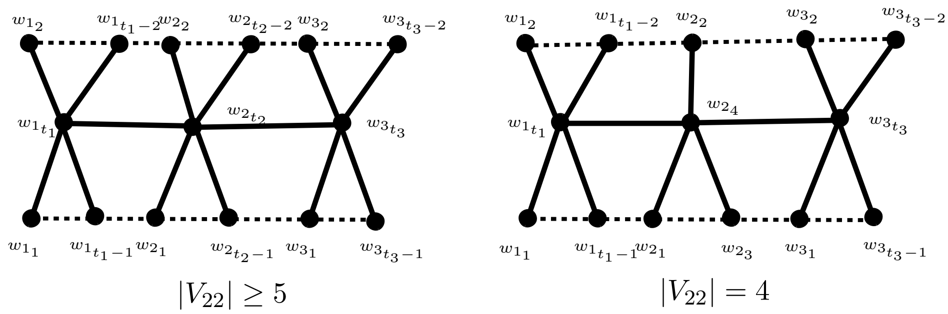

Figure 6.2. An example of swap operation with $|V_{22}| = 4$ and $|V_{22}| \geq 5$.

Since for all $j \in [0, l-1]$, both $V_{2j} \setminus \{w_{jt_j}\}$ and V_{2t} are cliques, there exist hamiltonian paths, say $P_{2j} = w_{j1} P_{2j} w_{jt_j-1}$ and $P_{2l} = w_{l_1} P_{2l} w_{l_2}$, in them, respectively. Let $C_2 = P_{20} P_{21} \cdots P_{2l}$ and all vertices of Q be stems in T_2 with $N_{C_2}(w_{jt_j}) = V(P_{2j}\}$ for all $j \in [1, l-2]$ and $N_{C_2}(w_{(l-1)_{t_{(l-1)}}}) = \{w | w \in V_{2t-1} \cup V_{2t}\}$. Then T_2 is a HIST of $\langle V_2 \cup S_1 \rangle$ and $V(C_2) = \{w \in V_2 \cup S_1 | deg_{T_2}(w) = 1\}$. Let $H_2 = T_2 \cup C_2$, then H_2 is planar (See Figure 6.3).

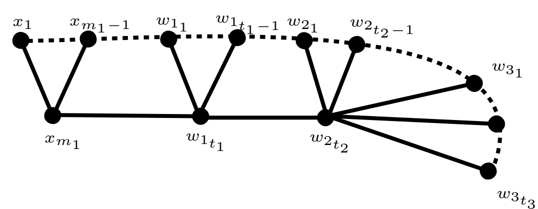

Figure 6.3. $S_2 \neq \emptyset$.

Step 2: Finding a spanning subgraph H_1 of H in $\langle V_1 \cup S_2 \cup \{x_t\} \rangle$

Case 1: Assume $S_2 = \emptyset$.

If $k \geq 2$ and $|V_{1i}| \neq 4$ for any $i \in [1, k-1]$. Since $V_{11} \setminus \{v_{1_1}, v_{1_{s_1-2}}, v_{1_{s_1}}\}$, $V_{1i} \setminus \{v_{i_2}, v_{i_{s_i-1}}, v_{i_{s_i}}\}$, for any $i \in [2, k-1]$, and V_{1k} are cliques, there exist hamiltonian paths, say $P_{11} = v_{1_2} P_{11} v_{1_{s_1-1}}$, $P_{1i} = v_{i_1} P_{1i} v_{i_{s_i-2}}$ and $P_{1k} = v_{k_1} P_{1k} v_{k_2}$, in them, respectively. Let $P_1 = v_{2_2} v_{2_{t_2-1}} v_{3_2} \cdots v_{(k-1)} v_{t_{(k-1)}-1}$ and $Q' = v_{(k-1)_{s_{k-1}}} \cdots v_{2_{s_2}} v_{1_{s_1}} v_{1_1} x_t$. Set $C_1 =$

$P_{11}P_1P_{1k}\cdots P_{13}P_{12} \cup \{v_{2_1}v_{1_{t_1-2}}\}$ and all vertices of Q' be stems of T_1 with $N_C(v_{1_1}) = V(P_{11})$, $N_C(v_{1_{s_1}}) = \{v_{1_{s_1-2}}\}$, $N_C(v_{i_{s_i}}) = V(P_{1_i}) \cup \{v_{i_2}, v_{i_{s_i-1}}\}$, for all $i \in [2, k-2]$, and $N_C(v_{(k-1)_{s_{l_{k-1}}}}) = \{v|v \in V_{1k} \cup V_{1k-1}\}$. Then T_1 is a HIST of $\langle V_1 \cup \{x_t\}\rangle$ and $V(C_1) = \{v \in V_1|deg_{T_2}(v) = 1\}$. Let $H_1 = T_1 \cup C_1$, then H_1 is planar.

If $k \geq 2$ and there exists $i \in [1, k-1]$ such that $|V_{1j}| = 4$, then we can apply Swap-Operation to $\langle V_1\rangle$ as to $\langle V_2\rangle$. Similarly, we can find a HIST T_1 and a path C_1 in $\langle V_1 \cup \{x_t\}\rangle$, then $H_1 = T_1 \cup C_1$ is planar (See Figure 6.4).

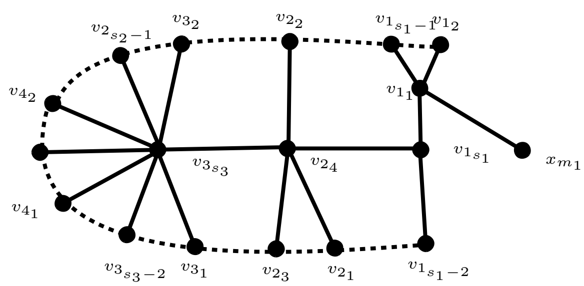

Figure 6.4. $S_2 = \emptyset$, $k = 4$ and $|V_{12}| = 4$ and $|V_{13}| \neq 4$.

If $k = 1$, which means $N_1(x_1) = V_{11} = V_1$ is a clique and $|V_1| \geq 3$. Then there exists a hamiltonian path, say $C_1 = v_{1_2}C_1v_{1n}$, in $\langle V_1 \setminus \{v_{1_1}\}\rangle$. Let $\{v_{1_1}, x_t\}$ be stems of T_1 with $N_{C_1}(v_{1_1}) = \{v|v \in V_1 \setminus \{v_{1_1}\}\}$. Then T_1 is a HIST of $\langle V_1 \cup \{x_t\}\rangle$ and $V(C_1) = \{v \in T_1|deg_{T_2}(v) = 1\}$. Let $H_1 = T_1 \cup C_1$, then H_1 is planar.

If $k = 1$ and $V_1 = \{v_{1_1}, v_{1_2}\}$(or $V_1 = \{v_{1_1}\}$). By Claim 4.1.2, $N_1(x_i) = \{v_1, v_2\}$ for all $x_i \in S_1$. Let $C_1 = v_{1_1}v_{1_2}$ and $v_{1_1}x_t, v_{1_2}x_t \in E(T_1)$. Then T_1 is a HIST of $\langle V_1 \cup \{x_t\}\rangle$ and $V(C_1) = \{v \in V_1|deg_{T_2}(v) = 1\}$. Let $H_1 = T_1 \cup C_1$, then H_1 is planar.

Case 2: Suppose that $S_2 \neq \emptyset$. Note that $v_{n_1}, v_{n_1-1} \in N_1(y)$ for all $y \in S_2$.

Case 2.1: Assume that $k \geq 2$.

Case 2.1.1: Assume that $|V_{11}| \geq 4$.

If $|V_{1k}| \geq 2$, since $V_{11} \setminus \{v_{1_1}, v_{1_{s_1}}\}$, $V_{1i} \setminus \{v_{i_{s_i}}\}$, for all $i \in [2, k]$, and S_2 are cliques, there exist hamiltonian paths, say $P_{11} = v_{1_2}P_{11}v_{1_{s_1-1}}$, $P_{1i} = v_{i_1}P_{1i}v_{i_{s_i-1}}$ and $P' = y_1P'y_{m_2}$, in them, respectively. Set $C_1 = P_{11}P_{12}\cdots P_{1s}P'$. Let $Q' = v_{k_{s_k}}v_{(k-1)_{s_{k-1}}}\cdots v_{1_{s_1}}v_{1_1}x_t$ and all vertices of Q' be stems of T_1 with $N_{C_1}(v_{1_1}) = V(P_{11}) \setminus \{v_{1_{s_1-1}}\}$, $N_C(v_{1_{s_1}}) = \{v_{1_{s_1-1}}\}$, $N_C(v_{i_{s_i}}) = V(P_{1_i})$, for all $i \in [2, k-1]$, $N_C(v_{k_{s_k}}) = \{v|v \in V_{1k} \cup S_2\}$. Then T_1 is a HIST of $\langle V_1 \cup S_2 \cup \{x_t\}\rangle$ and $V(C_1) = \{v \in V_1 \cup S_2 | deg_{T_2}(v) = 1\}$. Let $H_1 = T_1 \cup C_1$, then H_1 is planar (See Figure 6.5).

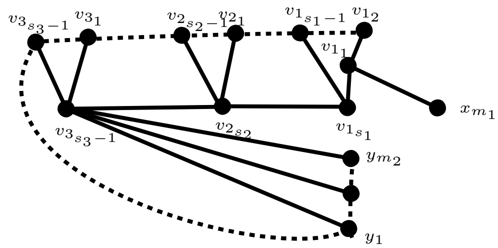

Figure 6.5. $S_2 = \emptyset$ and $k = 3$.

If $|V_{1k}| = 1$, then $v_{(k-1)_{s_{k-1}}}y, v_{k_{s_k}}y \in E(G)$ for all $y \in S_2$ and $v_{k_{s_k}}v_{(k-1)_{s_{(k-1)}-1}} \in E(G)$ since $|S_1| \geq 2$. Similarly as above, we can define P_{11}, P_{12}, $\cdots$, $P_{1(k-1)}$ and P'. Let $P_{1k} = v_{(k-1)_{s_{(k-1)}-1}}v_{k_{s_k}}y_1$ and $C_1 = P_{11}P_{12}\cdots P_{1s}P'$. Set $Q' = v_{(k-1)_{s_{k-1}}}v_{(k-2)_{s_{k-2}}}\cdots v_{1_{s_1}}v_{1_1}x_t$ be stems of T_1 with $N_{C_1}(v_{1_1}) = V(P_{11}) \setminus \{v_{1_{s_1-1}}\}$, $N_C(v_{1_{s_1}}) = \{v_{1_{s_1-1}}\}$, $N_C(v_{i_{s_i}}) = V(P_{1_i})$ for all $i \in [2, k-2]$ and $N_C(v_{(k-1)_{s_{k-1}}}) = V(P_{1k-1}) \cup S_2 \cup \{v_{k_{s_k}}\}$. Then T_1 is a HIST of $\langle V_1 \cup S_2 \cup \{x_t\}\rangle$ and $V(C_1) = \{v \in V_1 \cup S_2 | deg_{T_2}(v) = 1\}$. Let $H_1 = T_1 \cup C_1$, then H_1 is planar.

Case 2.1.2: Suppose that $|V_{11}| = 3$.

If $|V_{12}| \geq 3$, we delete the edge $v_{2_{t_2}}v_{2_1}$ from $E(T)$ and add the edge $v_{1_{t_1}}v_{2_1}$ to $E(T)$. Similarly as Case 2.1.1, we will find a HIST T_1 and a path C_1 in $\langle V_1 \cup S_2 \cup \{x_t\}\rangle$.

If $|V_{12}| = 2$, then $V_{11} = \{v_{1_1}, v_{1_2}, v_{1_3}, v_{2_1}, v_{2_2}\}$. From Claim 4.1.2, we know $v_{2_2}y, v_{2_1}y \in E(G)$ for all $y \in S_2$ and $v_{1_2}x_t \in E(G)$. In additional, $v_{1_2}v_{2_1}, v_{1_3}v_{2_2} \in E(G)$. Let $C_1 = v_{1_1}v_{1_3}v_{2_2} \cup P'$ and $Q' = v_{2_1}v_{1_2}x_t$ be paths and all vertices of Q' be stems of T_1 with $N_{C_1}(v_{1_2}) = \{v_{1_1}, v_{1_3}\}$ and $N_{C_1}(v_{2_1}) = \{v_{2_2}\} \cup \{y|y \in S_2\}$. Then T_1 is a HIST of $\langle V_1 \cup S_2 \cup \{x_t\}\rangle$ and $V(C_1) = \{v \in V_1 \cup S_2 | deg_{T_2}(v) = 1\}$. Let $H_1 = T_1 \cup C_1$, then H_1 is planar.

If $|V_{12}| = 1$, then $V_{11} = \{v_{1_1}, v_{1_2}, v_{1_3}, v_{2_1}\}$. Since $|S_1| \geq 2$ and V_1 is minimum subject to S being minimum, we have $v_{2_1}v_{1_1}, v_{2_1}v_{1_2}, v_{2_1}v_{1_3} \in E(G)$, which implies V_1 is a clique. Let $C_1 = \{v_{1_2}v_{2_1}\} \cup P'$ and $\{v_{1_3}, v_{1_1}, x_t\}$ be stems of T_1 with $N_{C_1}(v_{1_1}) = \{v_{1_2}\}$ and $N_{C_1}(v_{1_3}) = \{v_{2_1}\} \cup \{y | y \in S_2\}$. Then T_1 is a HIST of $\langle V_1 \cup S_2 \cup \{x_t\} \rangle$ and $V(C_1) = \{v \in V_1 \cup S_2 | deg_{T_2}(v) = 1\}$. Let $H_1 = T_1 \cup C_1$, then H_1 is planar.

Case 2.2: Suppose that $k = 1$, which means $N_1(x_1) = V_{11} = V_1$ is a clique. We still let $P' = y_1 P' y_m$

If $|V_1| \geq 4$, let $P_1 = v_2 P_1 v_{n_1 - 1}$ be a hamiltonian path in $\langle V_1 \setminus \{v_1, v_{n_1}\} \rangle$ and $C_1 = P_1 P'$. Set $\{v_{n_1}, v_1, x_t\}$ be stems of T_1 with $N_{C_1}(v_{n_1}) = \{v_{n_1 - 1}\} \cup V(P')$ and $N_{C_1}(v_{1_1}) = \{v | v \in V(P_1) \setminus \{v_{n_1 - 1}\}\}$. Then T_1 is a HIST of $\langle V_1 \cup S_2 \cup \{x_t\} \rangle$ and $V(C_1) = \{v \in V_1 \cup S_2 | deg_{T_2}(v) = 1\}$. Let $H_1 = T_1 \cup C_1$, then H_1 is planar.

If $|V_1| = 3$(similarly as $|V_1| = 2$). Let $C_1 = \{v_{1_1}v_{1_3}\} \cup P'$ and $\{v_{1_2}, x_t\}$ be stems of T_1 with $N_{C_1}(v_{1_2}) = \{v_{1_1}, v_{1_3}\} \cup V(P')$. Then T_1 is a HIST of $\langle V_1 \cup S_2 \cup \{x_t\} \rangle$ and $V(C_1) = \{v \in V_1 \cup S_2 | deg_{T_2}(v) = 1\}$. Let $H_1 = T_1 \cup C_1$, then H_1 is planar.

Thus, let $T = T_1 \cup T_2$, $C = C_1 \cup C_2$ and $H = T \cup C$, then H is a strong spanning Halin subgraph in G.

If $|V_1| = 1$, denote by $V_1 = \{v\}$. Since $N_2(y) \neq \emptyset$, we have $w_{n_2}y \in E(G)$ for all $y \in S_2$. Moreover $|N_2(y_{m_2})| \geq m_2$ subject to S being minimum. Let $vx_t \in E(G)$ and $C = C_2 P' \cup y_1 v x_1$, add vx_t to $E(T_2)$ and add w_{n_2}(or $w_{n_2 - 1}$ if $|V_{2l}| = 1$) to the stem set of T_2. Then $T = T_2$ is a HIST of G with $N_C(w_{n_2}) = \{y | y \in S_2\} \cup \{w_{n_2 - 1}\}$. Let $H = T \cup C$, then H is a strong spanning Halin subgraph in G.

We now let $T = T_1 \cup T_2$ and $C = C_1 \cup C_2$(note that $T_1 = \emptyset$ and $C_1 = \emptyset$ when $|V_1| = 1$). Set $H = T \cup C$, then H is a strong spanning Halin subgraph in G.

Chapter 7

ANTIPRISMATIC GRAPHS

Recall that a graph is called antiprismatic graph if for every vertex set $X \subseteq V(G)$ with $|X| = 4$, X is not a claw and there are at least two pairs of vertices in X that are adjacent. It is easy to check that every antiprismatics graph is N-free. In 1981, Duffus [28] showed that every connected $\{claw, N\}$-free graph contains a hamiltonian path, thus we have the following claim.

Claim 7.0.1. *Every connected antiprismatic graph contains a hamiltonian path.*

We will still follow definitions and notations mentioned in Section 4.1 that G is a graph with n-vertex and S is a minimum vertex cut of G. G_1 and G_2 are the exact two components of $G \setminus S$. $V_1 = V(G_1)$ and $V_2 = V(G_2)$. Subject to the minimality of $|S|$, we always assume that $|V_1|$ is minimum. In this chapter, we will show the following proposition.

Proposition 11. *If G is a 3-connected antiprismatic graph, then G contains a strong spanning Halin subgraph.*

7.1 Proof of 3-connected antiprismatic graphs

The proof of Proposition 11 will be divided into two parts depends on whether there exists a vertex x in S such that $N_1(x) = V_1$ or $N_2(x) = V_2$.

Part 1: There exists a vertex x in S such that $N_1(x) \neq V_1$ and $N_2(x) \neq V_2$.

We may reserve the notation x for this vertex and assume $v_1 \notin N_1(x)$ and $w_1 \notin N_2(x)$.

Claim 7.1.1. *If $N_1(x) \neq V_1$ and $N_2(x) \neq V_2$, then $|N_1(x)| = |V_1| - 1$ and $|N_2(x)| = |V_2| - 1$.*

Proof. Suppose to the contrary, there exists $x \in S$, $v_1, v_2 \in V_1$ and $w \in V_2$, such that $xv_1, xv_2, xw \notin E(G)$. Let $X = \{v_1, v_2, x, w\}$, we have $|E(X)| = 1$, which contradicts to the fact that G is an antiprismatic graph. ■

Thus, we denote by $N_1(x) = V_1 \setminus \{v_1\}$ and $N_2(x) = V_2 \setminus \{w_1\}$.

Claim 7.1.2. *If there exists a vertex $x \in S$ such that $N_1(x) \neq V_1$ and $N_2(x) \neq V_2$, then*

1) $|V_1| \geq 3$;

2) Both V_1 and V_2 are cliques;

3) For any $y \in S \setminus \{x\}$, if $xy \notin E(G)$, then both yv_1 and yw_1 are in $E(G)$; if $xy \in E(G)$, then at least one of $\{yv_1, yw_1\}$ is in $E(G)$.

Proof. 1) This is true following by Lemma 4.1.2.

2) We only show that V_1 is a clique. By Lemma 4.1.2, $V_1 \setminus \{v_1\}$ is a clique. For any $v \in V_1 \setminus \{v_1\}$, we have $v_1 v \in E(G)$. Otherwise, let $X = \langle x, v_1, v, w_1 \rangle$, then $|E(X)| = 1$, showing a contraction.

3) Let $X = \{x, y, v_1, w_1\}$, since $xv_1, xw_1, v_1w_1 \notin E(G)$ and $|E(X)| \geq 2$, we can easily get this conclusion. ■

In the remaining part of this subsection, we let $S_1 = \{y \in S | yv_1 \in E(G)\}$ and $S_2 = V_2 \setminus (S_1 \cup \{x\})$.

Claim 7.1.3. *If $S_2 \neq \emptyset$, then $|V_1| \geq 4$.*

Proof. Since $|V_1|$ is minimum subject to $|S|$ is minimum and $v_1 \notin N_1(S_2 \cup \{x\})$, we have $|N_1(S_2 \cup \{x\})| \geq |S_2 \cup \{x\}| + 1 \geq 2 + 1 = 3$. Thus $|V_1| \geq |N_1(S_2 \cup \{x\}) \cup \{v_1\}| \geq |N_1(S_2 \cup \{x\})| + 1 \geq 3 + 1 = 4$. ■

Claim 7.1.4. *For any $y \in S_2$, $xy, yw_1 \in E(G)$.*

Proof. Let $X = \{v_1, x, y, w_1\}$. Since $|E(X)| \geq 2$ and $v_1x, w_1x, v_1w_1, yv_1 \notin E(G)$, we have $xy, yw_1 \in E(G)$. ■

Now we want to find a strong spanning Halin subgraph in G by following cases according to whether S_1 or S_2 is empty.

Case 1: Suppose $S_2 \neq \emptyset$.

Case 1.1: Assume $S_1 = \emptyset$.

Denote by $S_2 = \{y_1, y_2, \cdots, y_t\}$. For any $v_2, v_3, v_4 \in V_1$ and $w_2, w_3 \in V_2$, since both $V_1 \setminus \{v_2\}$ and $V_2 \setminus \{w_1\}$ are cliques, there exist hamiltonian paths $P_1 = v_3P_1v_4$ and $P_2 = w_2P_2w_3$ in them, respectively.

If $\langle S_2 \rangle$ is connected, we can find a hamiltonian path $P_3 = y_2P_3y_t$ in $\langle S_2 \setminus \{y_1\} \rangle$ by Claim 7.0.1. Let $C = P_1P_3P_2 \cup \{xv_4, xw_2\}$ be a cycle and all vertices on the path $v_2y_1w_1$ be stems of T with $N_C(v_2) = V(P_1)$, $N_C(y_1) = \{x\}$ and $N_C(w_1) = V(P_2) \cup V(P_3)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.1(1)).

If $\langle S_2 \rangle$ is not connected, since $yw_1 \in E(G)$ for any $y \in S_2$, there exist exactly two cliques in S_2. We denote them by $S_{21} = \{y_1, y_2, \cdots, y_t\}$ and $S_{22} = \{z_1, z_2 \cdots, z_{t'}\}$ and always assume $|S_{21}| \geq |S_{22}|$. Let $P_3 = y_1P_3y_{t-1}$ and $P_4 = xz_1P_4z_{t'}$ be hamiltonian paths in $S_{21} \setminus \{y_t\}$ and $S_{22} \cup \{x\}$ respectively.

If $|S_{21}| \geq 2$, denote by $v_2 \in N_1(y_t)$, $v_3 \in N_1(y_{t-1})$, $w_2 \in N_2(y_1)$ and $w_3 \in N_2(z_{t'})$. Set $C = P_1P_3P_2P_4$ be a cycle and all vertices on the path $v_2y_tw_1$ be stems of T with $N_C(v_2) = V(P_1)$, $N_C(y_t) = \{x\}$ and $N_C(w_1) = V(P_2) \cup V(P_3) \cup V(P_4)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.1(2)).

If $S_{21} = \{y\}$ and $S_{22} = \{z\}$, denote by $v_3 \in N_1(y)$, $v_4 \in N_1(z)$, $w_1 \in N_2(y)$, $w_2 \in N_2(x)$ and $w_3 \in N_2(z)$. Let $P_2 = w_1P_2w_3$ be a hamiltonian path in $\langle V_2 \setminus \{w_2\} \rangle$ respectively. Set $C = P_1P_2 \cup \{v_3y, yw_1, w_3z, zv_4\}$ be a cycle and the stems of T be vertices on the path v_2xw_2 with $N_C(v_2) = V(P_1)$, $N_C(x) = \{y, z\}$ and $N_C(w_2) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.1(3)).

Case 1.2: Suppose $S_1 \neq \emptyset$.

Denote by $S_1 = \{y_1, y_2, \cdots, y_t\}$, $S_2 = \{z_1, z_2, \cdots, z_{t'}\}$, $v_2, v_4 \in N_1(z_1)$, $v_3 \in N_1(y_1)$, $w_2 \in N_2(y_t)$ and $w_3 \in N_2(z_{t'})$ by Lemma 4.1.2. Since for any $v_1, v_2, v_3, v_4 \in V_1$ and $w_1, w_2, w_3 \in V_2$, both $V_1 \setminus \{v_1, v_2\}$ and $V_2 \setminus \{w_1\}$ are cliques, there exist hamiltonian paths $P_1 = v_3P_1v_4$ and $P_2 = w_2P_4w_4$, in them, respectively.

If there exists a vertex, say z_1, in S_2 such that both $\langle S_1 \rangle$ and $\langle S_2 \setminus z_1 \rangle$ are connected, we can assume $P_3 = y_1P_3y_t$ and $P_4 = z_2P_4z_{t'}$ are hamiltonian paths in $\langle S_1 \rangle$ and $\langle S_2 \setminus z_1 \rangle$,

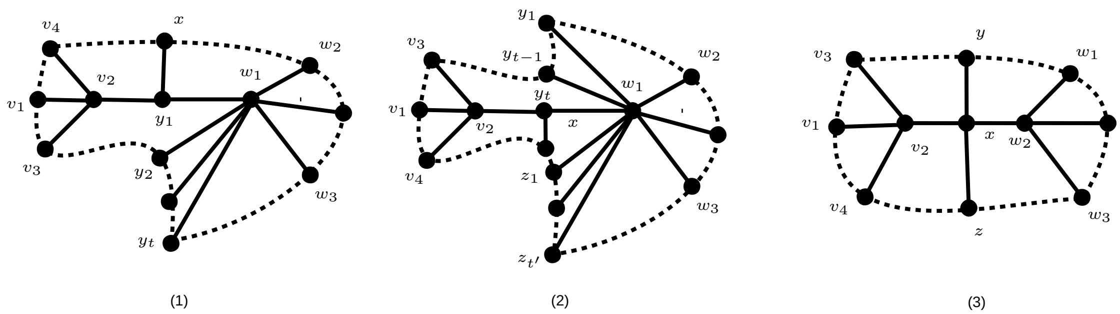

Figure 7.1. $S_1 = \emptyset$ and $S_2 \neq \emptyset$.

respectively. Let $C = P_1P_3P_2P_4 \cup \{v_4x, xz_2\}$ be a cycle and all vertices on the path $v_1v_2z_1w_1$ be stems of T with $N_C(v_1) = \{v_3\} \cup V(P_3)$, $N_C(v_2) = V(P_1) \setminus \{v_3\}$, $N_C(z_1) = \{x\}$ and $N_C(w_1) = V(P_2) \cup V(P_4)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.2(1)).

If for any $z_i \in S_2$, $\langle S_2 \setminus \{z_i\}\rangle$ is disconnected. Then there exist exactly two cliques in $\langle S_2 \setminus z_i\rangle$. We denote them by S_{21} and S_{22} and assume $P_{41} = z_1P_{41}z_j$ and $P_{42} = z_kP_{42}z_{t'}$ are hamiltonian paths in them respectively. Since $v_1z \notin E(G)$ for any $z \in S_2$, $|N_2(z)| \geq |V_2| - 1$. We may assume $w_4 \in N_2(z_j) \cap N_2(z_k)$ since $|V_2| \geq |V_1| \geq 4$. Replace P_4 by $P_{41} \cup \{z_jw_4, w_4z_k\} \cup P_{42}$ in the cycle C, similarly as $\langle S_2 \setminus z_i\rangle$ is connected, we can find a strong spanning Halin subgraph in G.

If S_1 is disconnected, we denote by S_{11} and S_{12} the exactly two cliques of S_1. If there exist $y_i \in S_{11}$ and $y_j \in S_{12}$ such that $N_1(y_i) \cap N_1(y_j) \setminus \{y_1\} \neq \emptyset$, let $v_4 \in N_1(y_i) \cap N_1(y_j)$ and $P_{31} = y_1P_{31}y_i$, $P_{32} = y_lP_{32}y_t$ be hamiltonian paths in S_{11} and S_{12}, respectively. Then replace P_3 by $P_{31} \cup \{y_kv_4, v_4y_j\} \cup P_{32}$, similarly as S_1 is connected, we can find a strong spanning Halin subgraph in G.

If S_1 is disconnected and $N_1(y_i) \cap N_1(y_j) \setminus \{y_1\} = \emptyset$ for any $y_i \in S_{11}$ and $y_j \in S_{12}$. Since $|V_1| \geq 4$, we may assume $|N_1(y_j)| \leq |V_1| - 2$ for any $y_j \in S_{12}$. Therefore, $N_2(y_j) = V_2$. Denote by $v_3 \in N_1(y_i)$, $w_2 \in N_2(y_i)$, $w_3 \in N_2(y_t)$, $w_4 \in N_2(z_j) \cap N_2(z_k)$ and $w_5 \in N_2(z_{t'} \cap N_2(y_j))$. Let $P_2 = w_2P_2w_3$, $P_{31} = y_1P_{31}y_i$, $P_{32} = y_jP_{32}y_t$, $P_{41} = xz_2P_{41}z_j$,$P_{42} = z_kP_{42}z_{t'}$ be hamiltonian paths in $\langle V_2 \setminus \{w_1, w_4, w_5\}\rangle$, $\langle S_{11}\rangle$, $\langle S_{12}\rangle$, $\langle S_{21} \cup \{x\}\rangle$ and $\langle S_{22}\rangle$, respectively. Let $C = P_1P_{31}P_2P_{32}P_{42}P_{41} \cup \{y_1w_5, w_5z_{t'}, z_kw_4, w_4z_j, z_2x, xv_4\}$ be a cycle and all vertices on the path $v_1v_2z_1w_1$ be stems of T with $N_C(v_1) = V(P_{31}) \cup \{v_3\}$, $N_C(v_2) = V(P_1) \setminus \{v_3\}$, $N_C(z_1) = \{x\}$ and $N_C(w_1) = V(P_2) \cup V(P_{32}) \cup V(P_{41}) \cup V(P_{42}) \cup \{w_4, w_5\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.2(2)).

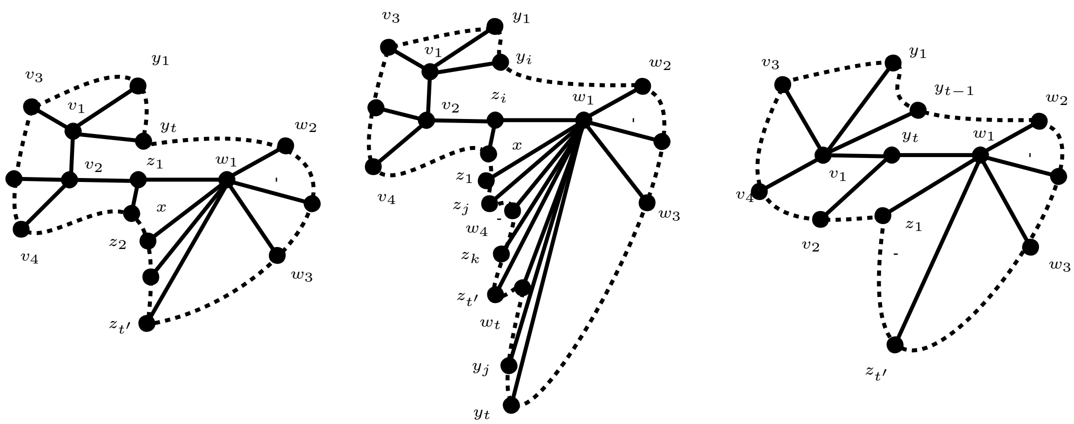

Figure 7.2. $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$.

Case 2: Suppose $S_2 = \emptyset$.

Claim 7.1.5. *For any $y \in S_1$, $xy \in E(G)$ or $yw_1 \in E(G)$.*

Proof. Let $X = \{x, y, v_1, w_1\}$. Since $xv_1, w_1v_1, xw_1 \notin E(G)$ and $v_1x \in E(G)$, we have $xy \in E(G)$ or $yw_1 \in E(G)$. ∎

Case 2.1: Assume $\langle S_1\rangle$ is connected.

Denote by $S_1 = \{y_1, y_2, \cdots, y_t\}$ and $P_3 = y_1P_3y_t$ the hamiltonian path in $\langle S_1\rangle$ according to Claim 7.0.1.

If $xy_1 \in E(G)$, denote by $v_3 \in N_1(y_t)$, $w_2 \in N_2(y_1)$, $w_3 \in N_2(x)$ and $w_4 \in N_2(y_2)$(we may have $w_4 = w_1$). Let $P_1 = v_2P_1v_3$ and $P_2 = w_3P_2w_4$ be hamiltonian paths in $V_1\setminus\{v_1\}$ and $V_2 \setminus \{w_2\}$, respectively. Set $C = P_1P_3P_2 \cup \{v_2x, xw_3\}$ be a cycle and all vertices on the path

$v_1y_1w_2$ be stems of T with $N_C(v_1) = V(P_1) \cup V(P_3)$, $N_C(y_1) = \{x\}$ and $N_C(w_2) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.3(1)).

If $xy_1 \notin E(G)$, then $y_1w_1 \in E(G)$. If $|S_1| \geq 3$, denote by $v_4 \in N_1(y_t)$, $w_2 \in N_2(x)$, $w_3 \in N_2(y_2)$. Let $P_1 = v_3P_1v_4$ and $P_2 = w_2P_2w_3$ be hamiltonian paths in $V_1 \setminus \{v_1, v_2\}$ and $V_2 \setminus \{w_1\}$, respectively. Set $C = P_1P_3P_2 \cup \{v_3x, xw_2\}$ be a cycle and all vertices on the path $v_2v_1y_1w_2$ be stems of T with $N_C(v_2) = \{v_3, x\}$, $N_C(v_1) = V(P_1) \cup (V(P_3) \setminus \{y_1, y_2\})$, $N_C(y_1) = \{y_2\}$ and $N_C(w_1) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.3(2)).

If $|S \setminus \{x\}| = 2$, we can assume $v_2 \in N_1(y_1)$ since $|N_1(y_1)| \geq 2$. Denote by $w_2 \in N_2(x)$ and $w_3 \in N_2(y_2)$. Let $C = P_1P_2 \cup \{v_3x, xw_2, w_3y_2, y_2v_1\}$ be a cycle and all vertices on the path $v_2y_1w_1$ be stems of T be with $N_C(v_2) = V(P_1) \cup \{x\}$, $N_C(y_1) = \{y_2\}$ and $N_C(w_1) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.3(3)).

Case 2.2: Assume $\langle S_1 \rangle$ is not connected.

Let $S_{11} = \{y_1, y_2, \cdots, y_t\}$ and $S_{12} = \{z_1, z_2, \cdots, z_{t'}\}$ be the exact two cliques in $\langle S_1 \rangle$ with $|S_{11}| \geq |S_{12}|$.

If $|S_{11}| \geq 2$, let $X = \{y_1, y_t, z_1, x\}$. Since $|E(X)| \geq 2$, we have $y_1x \in E(G)$ or $y_tx \in E(G)$ or $zx \in E(G)$. We may assume $y_tx \in E(G)$. Denote by $w_2 \in N_2(y_1)$, $w_1 \in N_2(y_2)$, $w_4 \in N_2(z_{t'})$ and $v_3 \in N_1(z_1)$. Since $V_1 \setminus \{v_1, v_2\}$, $V_2 \setminus \{w_2\}$, $S_{11} \setminus \{y_1\}$ and S_{12} are cliques, there exist hamiltonian paths, say $P_1 = v_4P_1v_3$, $P_2 = w_1P_2w_4$, $P_3 = y_2P_3y_t$ and $P_4 = z_1P_4z_{t'}$, in them respectively. Let $C = P_1P_3P_2P_4 \cup \{v_4x, xy_t\}$ be a cycle and all vertices on the path $v_2v_1y_1w_2$ be stems of T with $N_C(v_2) = V(P_1) \cup \{x\}$, $N_C(v_1) = V(P_4)$, $N_C(y_1) = V(P_3)$ and $N_C(w_2) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.3(4)).

If $S_{11} = \{y_1\}$ and $S_{12} = \{z_1\}$. We may assume $v_2 \in N_1(y_1)$ since $|N_1(y_1)| \geq 2$. Denote by $w_1 \in N_2(y_1)$, $w_2 \in N_2(x)$ and $w_3 \in N_2(z_1)$. Let $P_1 = v_1P_1v_3$ and $P_2 = w_2P_2w_3$ be hamiltonian paths in $V_1 \setminus \{v_2\}$ and $V_2 \setminus \{w_2\}$, respectively. Let $C = P_1P_2 \cup \{v_3x, xw_2, w_3z_1, z_1v_1\}$ be a cycle and all vertices on the path $v_2 y_1 w_1$ be stems of T with $N_C(v_2) = V(P_1) \cup \{x\}$, $N_C(y_1) = \{v_1\}$, $N_C(w_1) = \{z_1\} \cup V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.3(5)).

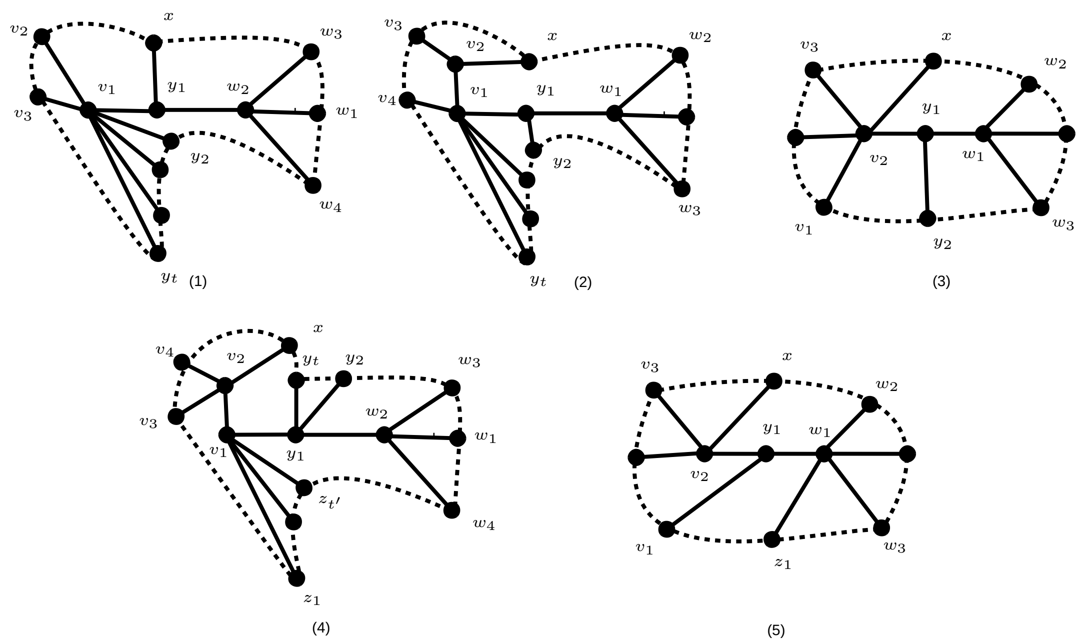

Figure 7.3. $S_1 \neq \emptyset$ and $S_2 = \emptyset$.

Part 2: For any vertex x in S, $N_1(x) = V_1$ or $N_2(x) = V_2$.

We may denote by $S_1 = \{x \in S | N_1(x) = V_1\}$ and $S_2 = S \setminus S_1$.

If $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$, then both V_1 and V_2 are cliques. Using the same method as Case 1.2 with regrading v_1 and v_2 as the same vertex, we can find a strong spanning Halin subgraph in G. Thus, we may assume $S_2 = \emptyset$. Since we can apply the same procedure for $S_1 = \emptyset$. Then V_1 is a clique. We want to show following claims before we searching a strong spanning Halin subgraph in G.

Claim 7.1.6. *For any $w \in V_2$, $|N_2(w)| \geq |V_2| - 2$. This in turn gives that there exists at most one other vertex in V_2 not adjacent to w.*

Proof. Suppose this is not true, there exist $w, w', w'' \in V_2$, such that $ww', ww'' \notin E(G)$. Let $v \in V_1$ and $X = \{v, w, w', w''\}$, then $|E(X)| \leq 1$, showing a contradiction. ■

By Claim 7.1.6, we let $V_2 = \{w_1, w_2, \cdots, w_{n_2}\}$, such that w_{2k+2} is the only possible vertex that not adjacent to w_{2k+1}.

Claim 7.1.7. *If there exists* $k \in [1, \lfloor \frac{n_2}{2} \rfloor - 1]$ *such that* $w_{2k+1}w_{2k+2} \notin E(G)$*, then for any* $x \in S_1$*, either* $xw_{2k+1} \in E(G)$ *or* $xw_{2k+2} \in E(G)$*.*

Proof. For any $x \in S_1$, if both xw_{2k+1} and xw_{2k+2} are in $E(G)$, then $\langle x; w_{2k+1}, w_{2k+2}, v \rangle$ is a claw. If neither xw_{2k+1} nor xw_{2k+2} is in $E(G)$, then $|E(\langle v, x, w_{2k+1}, w_{2k+2} \rangle)| = 1$, showing a contradiction. ■

According to the result given by Shepherd [37]: If G is a 3-connected $\{claw, N\}$-free graph then G is hamiltonian-connected. We can easily get following corollary.

Corollary 7.1.1. *For any vertex set* $Y \subseteq V_2$*, if* $|Y| \geq 5$*, then* $\langle Y \rangle$ *is hamiltonian connected.*

Proof. For any vertex set $Y \subseteq V_2$, if $|Y| \geq 5$, then $\delta(Y) \geq 3$, which implies $\langle Y \rangle$ is 3-connected, so it is hamiltonian connected. ■

We want to find a strong spanning Halin subgraph in G as follows depends on whether $\langle S_1 \rangle$ is connected.

Case 1: Assume $\langle S_1 \rangle$ is connected.

Denote by $S_1 = \{x_1, x_2, \cdots, x_t\}$. We assume $P_3 = x_3P_3x_t$ is a hamiltonian path in $\langle S_1 \rangle$ by Claim 7.0.1 and $P_1 = v_2P_1v_3$ is a hamiltonian path in $\langle V_1 \setminus \{v_1\} \rangle$ if $\langle V_1 \setminus \{v_1\} \rangle$ is not empty.

Case 1.1: Suppose $|S_1| \geq 4$.

Denote by $w_1 \in N_2(x_1)$ and $w_3 \in N_2(x_2)$. Note that we may have $w_3 = w_2$.

If $w_1w_2 \in E(G)$, we let $w_5 \in N_2(x_t)$, where $w_5 \neq w_3$ by Lemma 4.1.2 and we may have $w_5 = w_2$. By Claim 7.1.6 and Corollary 7.1.1, there exists a hamiltonian path $P_2 = w_3P_2w_5$ in $\langle V_2 \setminus \{w_1\} \rangle$. Set $C = P_1P_3P_2 \cup \{w_3x_2, x_2v_2\}$ be a cycle and all vertices on the path $v_1x_1w_1$

be stems of T with $N_C(v_1) = V(P_1) \cup V(P_3)$, $N_C(x_1) = \{x_2\}$ and $N_C(w_1) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.4(1)).

If $w_1w_2 \notin E(G)$ and $w_2x_t \in E(G)$. Let $P_2 = w_3P_2w_4$ be a hamiltonian path in $\langle V_2 \setminus \{w_1, w_5, w_2\}\rangle$ and $C = P_1P_3P_2 \cup \{v_2x_2, w_3x_2, w_4w_2, w_2x_t\}$ be a cycle. The stems of T are all vertices on the path $v_1x_1w_1w_5$ with $N_C(v_1) = V(P_1) \cup V(P_3)$, $N_C(x_1) = \{x_2\}$, $N_C(w_1) = V(P_2)$ and $N_C(w_5) = \{w_4, w_2\}$(If $|V_2| = 5$, swiping w_4 and w_5). Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.4(2)).

If $w_1w_2 \notin E(G)$ and $w_2x_t \notin E(G)$. We may denote by $w_5x_t \in E(G)$ by Lemma 4.1.2. Let $P_2 = w_2P_2w_5$ be a hamiltonian path in $\langle V_2 \setminus \{w_1, w_6, w_3\}\rangle$ and $C = P_1P_3P_2 \cup \{v_2x_2, x_2w_3, w_3w_2\}$ be a cycle. Set the stems of T be all vertices on the path $v_1x_1w_1w_6$ with $N_C(v_1) = V(P_1) \cup V(P_2)$, $N_C(x_1) = \{x_2\}$, $N_C(w_1) = V(P_2) \setminus \{w_2\}$ and $N_C(w_6) = \{w_3, w_2\}$(if $|V_2| = 5$, replacing w_6 by w_4, adding edges $\{w_1w_3, w_4w_5\}$ to $E(C)$ and deleting edge $\{w_1w_5\}$) from $E(G)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.4(3)).

Case 1.2: Suppose $|S_1| = 3$.

Let $P_1 = v_1P_1v_2$ and $P_3 = x_1x_2x_3$ be hamiltonian paths in $\langle V_1\rangle$ and $\langle S_1\rangle$, respectively. Denote by $w_2x_2 \in E(G)$.

If $w_1w_2 \in E(G)$, denote by $w_1x_1, w_3x_3 \in E(G)$. Set $P_2 = w_1P_2w_3$ be a hamiltonian path in $V_2 \setminus \{w_2\}$. Let $C = P_1P_2 \cup \{v_1x_1, x_1w_1, v_2x_3, x_3w_3\}$ be a cycle and $\{x_2, w_2\}$ be stems of T with $N_C(x_2) = V(P_1) \cup \{x_1, x_3\}$ and $N_C(w_2) = V(P-2)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.4(4)).

If $w_1w_2 \notin E(G)$, we may assume $x_1w_5, x_3w_3 \in E(G)$ by Lemma 4.1.2. Let $P_2 = w_5P_2w_1$ be a hamiltonian path in $V_2 \setminus \{w_2, w_6, w_3\}$ and $C = P_1P_2 \cup \{v_1x_1, x_1w_5, w_1w_3, w_3x_3, x_3v_2\}$ be a cycle. Set the stems of T be all vertices on the path $x_2w_2w_6$ with $N_C(x_2) = V(P_1) \cup \{x_1, x_3\}$, $N_C(w_2) = V(P_2) \setminus \{w_1\}$ and $N_C(w_6) = \{w_1, w_3\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.4(5)).

Case 2: Suppose $\langle S_1\rangle$ is disconnected.

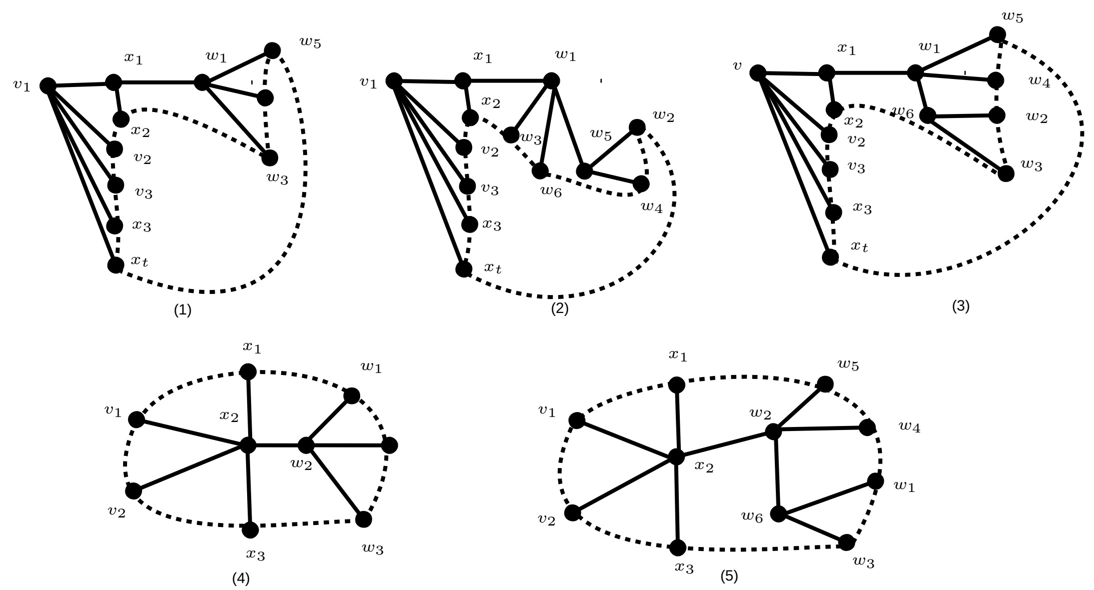

Figure 7.4. $\langle S_1 \rangle$ is connected.

Since $\langle S_1 \rangle$ is disconnected and $xv \in E(G)$ for any $x \in S_1$ and $v \in V_1$, there exist exactly two cliques in $\langle S_1 \rangle$. We denote them by $S_{11} = \{x_1, x_2, \cdots, x_t\}$, $S_{12} = \{y_1, y_2, \cdots, y_{t'}\}$ and always assume $|S_{11}| \geq |S_{12}|$. Let $P_1 = v_1 P_1 v_2$ be a hamiltonian path in $\langle V_1 \rangle$.

Case 2.1: There exist $x_i \in S_{11}$ and $y_j \in S_{12}$ such that $N_2(x_i) \cap N_2(y_j) \neq \emptyset$. We may assume $w_1 \in N_2(x_1) \cap N_2(y_1)$.

If $|S_{12}| \neq 2$ and $w_1 w_2 \in E(G)$, we denote by $w_2 \in N_2(x_t)$ and $w_3 \in N_2(y_{t'})$. Since $S_{11} \setminus \{x_1\}$, $S_2 \setminus \{y_1\}$ and $V_2 \setminus \{w_1\}$ are cliques, there exist hamiltonian paths, say $P_3 = x_2 P_3 x_t$, $P_4 = y_2 P_4 y_{t'}$ and $P_2 = w_2 P_2 w_3$, in them, respectively. Let $C = P_1 P_3 P_2 P_4$ be a cycle and the stems of T be all vertices on the path $x_1 w_1 y_1$ with $N_C(x_1) = V(P_1) \cup V(P_3)$, $N_C(y_1) = V(P_4)$ and $N_C(w_1) = V(P_2)$(If $S_{12} = \{y_1\}$, set $P_4 = y_1$). Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.5(1)).

If $|S_{12}| \neq 2$ and $w_1 w_2 \notin V_2$, we denote by $w_5 \in N_2(x_t)$ and $w_3 \in N_2(y_{t'})$. We may also let $P_2 = w_5 P_2 w_3$, $P_3 = x_2 P_3 x_t$ and $P_4 = y_2 P_4 y_{t'}$ be hamiltonian paths in $V_2 \setminus \{w_1, w_6\}$, $S_{11} \setminus \{x_1\}$ and $S_{12} \setminus \{y_1\}$, respectively. Set $C = P_1 P_3 P_4 P_2$ be a cycle and all vertices of star $\langle w_1; x_1, y_1, w_6 \rangle$ be stems of T with $N_C(x_1) = V(P_1) \cup V(P_3)$, $N_C(y_1) = V(P_4)$, $N_C(w_1) = \{w_5\}$

and $N_C(w_6) = V(P_2) \setminus \{w_5\}$(If $S_{12} = \{y_1\}$, set $P_4 = y_1$). Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.5(2)).

If $S_{12} = \{y_1, y_2\}$ and there exists $x_i \in S_{11}$ such that $N_2(x_i) \cap N_2(y_1) \cap N_2(y_2) \neq \emptyset$. Let $w_1 \in N_2(x_1) \cap N_2(y_1) \cap N_2(y_2)$ and set $\{x_1, w_1\}$ be stems of T with $w_1y_1, w_1y_2 \in E(T)$. Similarly as $|S_{12}| \neq 2$, we can find a strong spanning Halin subgraph in G.

If $S_{12} = \{y_1, y_2\}$ and for any $x_i \in S_{11}$, $N_2(x_i) \cap N_2(y_1) \cap N_2(y_2) = \emptyset$. We may assume $w_1 \in N_2(x_1) \cap N_2(y_1)$, then $w_2 \in N_2(y_2)$. Let $w_3 \in N_2(y_1)$ and $w_5 \in N_2(x_t)$ by Claim 7.1.7. Since $S_{11} \setminus \{x_1\}$ is a clique, there exists a hamiltonian path $P_3 = x_2P_3x_t$ in it. And we may denote by $P_2 = w_5P_2w_2$ a hamiltonian path in $\langle V_2 \setminus \{w_1, w_3\}\rangle$ by Claim 7.1.6 and Corollary 7.1.1. Let $C = P_1P_3P_2 \cup \{w_2y_2, y_2v_2\}$ be a cycle and all vertices on the path $x_1w_1y_1w_3$ be stems of T with $N_C(x_1) = V(P_1) \cup V(P_3)$, $N_C(w_1) = V(P_2) \setminus \{w_2, w_6\}$, $N_C(y_1) = \{y_2\}$ and $N_C(w_3) = \{w_2, w_6\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.5(3)).

Case 2.2: For any $x_i \in S_{11}$ and $y_j \in S_{12}$, $N_2(x_i) \cap N_2(y_j) = \emptyset$.

If V_2 is a clique and $|V_1| \geq 2$ or $|S| \geq 4$. Denote by $w_1 \in N_2(x_1)$, $w_2 \in N_2(x_t)$ and $w_3 \in N_2(y_1)$. Let $P_1 = v_2P_1v_3$, $P_2 = w_2P_2w_3$, $P_3 = x_2P_3x_t$ and $P_4 = y_1P_4y_{t'}$ be hamiltonian paths in $V_1 \setminus \{v_1\}$, $S_{11} \setminus \{x_1\}$, S_{12} and $V_2 \setminus \{w_1\}$, respectively. Set $C = P_1P_3P_2P_4$ be a cycle and all vertices on the path $v_1x_1w_1$ be stems of T with $N_C(v_1) = V(P_1) \cup V(P_3 \setminus \{x_t\}) \cup V(P_4)$, $N_C(x_1) = \{x_t\}$ and $N_C(w_1) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G.

If V_2 is a clique, $|V_1| = 1$ and $|S| = 3$. Denote by $w_1 \in N_2(x_1)$, $w_2 \in N_2(y_1)$ and $w_3 \in N_2(x_2)$. Let $P_2 = w_3P_4w_4$ be a hamiltonian path in $V_2 \setminus \{w_1, w_2\}$. Set $C = P_2 \cup \{v_1x_2, x_2w_3, w_4y_1, y_1v_1\}$ be a cycle and all vertices on the path $x_1w_1w_2$ be stems of T with $N_C(x_1) = \{v_1, x_2\}$, $N_C(w_1) = V_2 \setminus \{w_1, w_2, w_4\}$ and $N_C(w_2) = \{w_4\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G.

If V_2 is not a clique, there exists a pair of vertices, say $\{w_1, w_2\}$, in V_2 such that $x_iw_1, y_jw_2 \in E(G)$ for all $x_i \in S_{11}$ and $y_j \in S_{12}$ by Claim 7.1.7. We denote by $w_3 \in N_2(y_{t'})$ and $w_5 \in N_2(x_t)$. Since $\langle S_{11} \setminus \{x_1\}\rangle$ and $\langle S_{12}\rangle$ are cliques, there exist hamiltonian paths

$P_3 = x_2P_3x_t$ and $P_4 = y_1P_4y_t$ in them, respectively. We can also let $P_2 = w_5P_2w_3$ be a hamiltonian path in $\langle V_2 \setminus \{w_1, w_4, w_2\}\rangle$ by Claim 7.1.6 and Corollary 7.1.1. Let $C = P_1P_3P_2P_4$ be a cycle and all vertices on the path $x_1w_1w_4w_2$ be stems of T with $N_C(x_1) = V(P_1) \cup V(P_3)$, $N_C(w_1) = V(P_2) \setminus \{w_3, w_6\}$, $N_C(w_4) = \{w_6\}$ and $N_C(w_2) = \{w_3\} \cup V(P_4)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 7.5(4)).

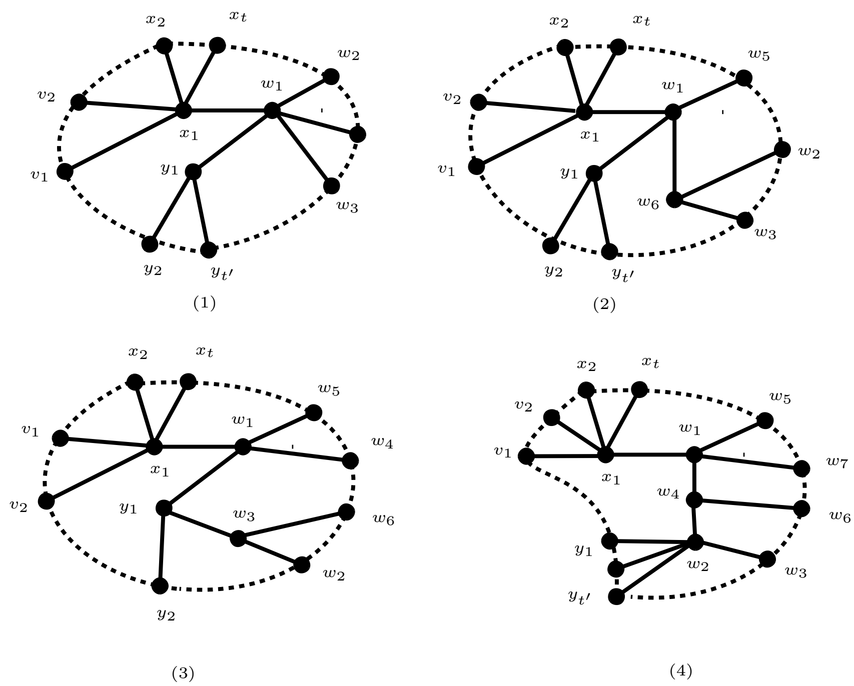

Figure 7.5. $\langle S_1 \rangle$ is not connected.

Chapter 8

1-JOIN

Recall that if we suppose W_1, W_2 is a partition of $V(G)$, and for $i \in [1, 2]$ there is a subset $A_i \subseteq W_i$ such that:

- A_i, $W_i \setminus A_i \neq \emptyset$ for $i \in [1, 2]$,
- $A_1 \cup A_2$ is a clique, and
- $E(W_1 \setminus A_1, W_2) = \emptyset$, and $E(W_1, W_2 \setminus A_2) = \emptyset$.

In these circumstances, we say that (W_1, W_2) is a 1-join.

Let $A_i = \{u_{i_j} \in W_1 | dist(u_{i_j}, B_1) = i\}$ and $B_i = \{w_{i_j} \in W_2 | dist(w_{i,j}, A_1) = i\}$, then $W_1 = A_1 \cup A_2 \cup \cdots \cup A_t$ and $W_2 = B_1 \cup B_2 \cdots B_s$, where $t = max\{dist(u, B_1) | u \in W_1\}$ and $s = max\{dist(w, A_1) | u \in W_2\}$. In particular, we assume $u_{i_1} u_{i+1_1}, w_{j_1} w_{j+1_1} \in E(G)$ for all $i \in [1, t-1]$ and $j \in [1, s-1]$. Since G is a 3-connected finite graph, the following claim is clearly true.

Claim 8.0.8. *$|A_i| \geq 3$ and $|B_j| \geq 3$ for all $i \in [1, t-1]$ and $j \in [1, s-1]$.*

In this chapter, we will show following two propositions.

Proposition 12. *If G is a 3-connected $\{claw, Z_3\}$-free graph admits 1-join, then there is a strong spanning Halin subgraph in G.*

Proposition 13. *If G is a 3-connected $\{claw, B_{1,2}\}$-free graph admits 1-join, then there exists a strong spanning Halin subgraph in G.*

8.1 Proof of 3-connected $\{claw, Z_3\}$-free graphs admit 1-joins

We want to show the following claim first.

Claim 8.1.1. *1) For any* $u_{1_i} \in A_1$, $N_{A_2}(u_{1_i})$ *is a clique.*

2) $W_1 \subseteq A_1 \cup A_2 \cup A_3$. *In particular, if there exists* $w_{1_1} \in B_1$, *such that* $|N_{B_2}(w_{1_1})| \geq 2$, *then* $W_1 = A_1 \cup A_2$.

Proof. 1) If there exist $u_{2_i}, u_{2_j} \in N_{A_2}(u_{1_i})$ such that $u_{2_i}u_{2_j} \notin E(G)$, then $\langle u_{1_i}; w_{1_1}, u_{2_i}, u_{2_j} \rangle$ is a claw, showing a contradiction.

2) Suppose to the contrary, $A_4 \neq \emptyset$, then $\langle w_{1_i}, w_{1_j}, u_{1_1}; u_{2_1}u_{3_1}u_{4_1} \rangle$ is a Z_3, showing a contradiction. Thus $W_1 \subseteq A_1 \cup A_2 \cup A_3$. If there exists $w_{2_1}, w_{2_2} \in B_2$ such that $w_{2_1}w_{1_1}, w_{2_2}w_{1_1} \in E(G)$, then $w_{2_1}w_{2_2} \in E(G)$ since G is claw-free. This in turn gives $A_3 = \emptyset$. Otherwise, $\langle w_{2_1}, w_{2_2}, w_{1_1}; u_{1_1}u_{2_1}u_{3_1} \rangle$ is a Z_3. ■

Since $E(A_1, A_3) = \emptyset$, $E(B_1, B_3) = \emptyset$, $E(A_1, B_2) = \emptyset$ and $E(A_2, B_1) = \emptyset$, we know A_2, A_1, B_1 are vertex cut. By Proposition 6, we only need to find a strong spanning Halin subgraph in G by following three cases depends on $A_3 = \emptyset$ or not.

Case 1: Assume that $A_3 \neq \emptyset$, then $W_1 = A_1 \cup A_2 \cup A_3$.

By Proposition 6 and symmetric, we may assume A_1 is the minimum vertex cut and both $V_1 = A_2 \cup A_3$ and $V_2 = W_2 = B_1 \cup B_2$ are cliques.

Claim 8.1.2. *If* $W_1 = A_1 \cup A_2 \cup A_3$, *then*

1) $|B_1| \geq |A_1| \geq 3$ *and* $|A_2| > |A_1| \geq 3$.

2) $|A_3| = |B_2| = 1$.

Proof. 1) This is true since G is 3-connected and $|V_1|$ is minimum subjects to A_1 is a minimal vertex cut.

2) Suppose this is not true. If $|A_3| \geq 2$, then $\langle u_{3_1}, u_{3_2}, u_{2_1}; u_{1_1}w_{1_1}w_{2_1} \rangle$ is a Z_3. If $|B_2| \geq 2$, then $\langle w_{2_1}, w_{2_2}, w_{1_1}; u_{1_1}u_{2_1}u_{3_1} \rangle$ is a Z_3. ■

Now we want to find a strong spanning Halin subgraph in G for this case as follows.

Since G is 3-connected, we may assume $u_{11}u_{21}, u_{12}u_{22}, u_{13}u_{23} \in E(G)$. Moreover, since $A_3 \cup A_2 \setminus \{u_{21}\}$, $A_1 \setminus \{u_{11}, u_{12}\}$ and $B_2 \cup B_1 \setminus \{w_{11}\}$ are cliques, there exist hamiltonian paths, say $P_1 = u_{22}P_1u_{23}$, $P_2 = u_{13}P_2u_{14}$ and $P_3 = w_{12}P_3w_{13}$, in them, respectively. Let

$C = P_1P_2P_3 \cup \{u_{23}u_{13}, u_{14}w_{13}, w_{12}u_{12}, u_{12}u_{22}\}$ be a cycle and all vertices on the path $u_{21}u_{11}w_{11}$ be stems with $N_C(u_{21}) = A_3 \cup A_2 \setminus \{u_{21}\}$, $N_C(u_{11}) = A_1 \setminus \{u_{11}\}$ and $N_C(w_{11}) = B_2 \cup B_1 \setminus \{w_{11}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 8.1 as an example).

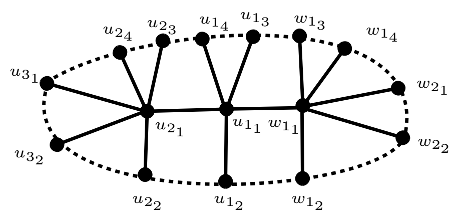

Figure 8.1. $W_1 = A_1 \cup A_2 \cup A_3$.

Case 2: Assume that $A_3 = \emptyset$ and $B_3 \neq \emptyset$.

Similarly as $A_3 \neq \emptyset$, we can show that B_1 is a minimum vertex cut and both $A_1 \cup A_2$ and $B_2 \cup B_3$ are cliques, then we can find a strong spanning Halin subgraph in G as Case 1 similarly.

Case 3: Assume that $A_3 = \emptyset$ and $B_3 = \emptyset$, then $W_1 = A_1 \cup A_2$ and $W_2 = B_1 \cup B_2$.

Since $E(A_2, B_1) = \emptyset$ and $E(A_1, B_2) = \emptyset$, by symmetric, we can assume A_1 is a minimum vertex cut.

If $|A_2| \geq 2$, Proposition 6, we can assume $B_1 \cup B_2$ is a clique. Similarly as Case 1, we can find a strong spanning Halin subgraph in G.

If $|A_2| = 1$, let $T_1, T_2, \cdots, T_m$ be maximal connected components in B_2, then we have following conclusions.

Claim 8.1.3. *1) If there exists T_i with $|T_i| \geq 3$, then for any $w \in N_{B_1}(T_i)$ and $i \in [1, m]$, $|N_{T_i}(w)| \geq |T_i| - 1$.*

2) For any $w_{21}, w_{22} \in B_2$, if $N_{B_1}(w_{21}) \cap N_{B_1}(w_{22}) \neq \emptyset$, then there exists $i \in [1, m]$ such that $w_{21}, w_{22} \in T_i$ and for any $j \in [1, i-1] \cup [i+1, m]$, $T_j \cup N_{B_1}(T_j)$ is a clique.

3) If there exists $i \in [1, m]$ such that $|T_i| \geq 3$, then $T_j \cup N_{B_1}(T_j)$ is a clique for all $j \neq i$ and

$j \in [1, m]$.

4) For any $i \in [1, m]$, if $T_i = \{w_{i_1}, w_{i_2}\}$, then $|N_{B_1}(w_{i_1})| \geq 2$ and $|N_{B_1}(w_{i_2})| \geq 2$.

Proof. 1) Suppose this is not true, there exists $w_{21}, w_{22}, w_{23} \in T_i$ and $w_{1_1} \in B_1$ such that $w_{21}w_{1_1} \in E(G)$ and $w_{22}w_{1_1}, w_{23}w_{1_1} \notin E(G)$. Then neither $w_{21}w_{22}w_{23}$ is an induced path, otherwise $\langle u_{1_1}, u_{1_2}, w_{1_1}; w_{21}w_{22}w_{23}\rangle$ is a Z_3, nor $w_{22}w_{21}w_{23}$ is an induced path, otherwise $\langle w_{21}; w_{22}, w_{23}, w_{1_1}\rangle$ is a claw, nor $w_{22}w_{21}w_{23}$ is a triangle, otherwise $\langle w_{23}w_{22}w_{21}; w_{1_1}u_{1_1}u_{2_1}\rangle$ is a Z_3, showing a contradiction. Thus $|N_{T_i}(w_{1_1}) \cap \{w_{2_1}, w_{2_2}, w_{2_3}\}| \geq 2$. Since $\langle T_i \rangle$ is connected, $|N_{T_i}(w_{1_1})| \geq |T_i| - 1$.

2) We may assume $w_{1_1} \in N_{B_1}(w_{21}) \cap N_{B_1}(w_{22})$. Since $\langle w_{1_1}; u_{1_1}, w_{21}, w_{22}\rangle$ is not a claw, we have $w_{21}w_{22} \in E(G)$, which implies there exists $i \in [1, m]$ such that $w_{2_1}, w_{2_2} \in T_i$. For any $j \in [1, i-1] \cup [i+1, m]$, if $|T_j| = 1$, then clearly $T_j \cup N_{B_1}(T_j)$ is a clique since B_1 is a clique. If there exist $w_{2_3}, w_{2_4} \in T_j$, we may let $w_{1_2} \in N_{B_1}(w_{2_3})$. Since $\langle w_{2_1}, w_{2_2}, w_{1_1}; w_{1_2}w_{2_3}w_{2_4}\rangle$ is not a Z_3, we have $w_{1_2}w_{2_4} \in E(G)$. Since $\langle T_j \rangle$ is connected, we have $T_j \cup N_{B_1}(T_j)$ is a clique.

3) This is clearly true by 1) and 2).

4) This is true since G is 3-connected. ■

Now we want to find a strong spanning Halin subgraph in G by following two subcases.

Case 3.1 There exist $w_{i_1}, w_{i_2} \in T_i$ such that $N_{B_1}(w_{i_1}) \cap N_{B_1}(w_{i_2}) \neq \emptyset$.

By Claim 8.1.3, $T_j \cup N_{B_1}(T_j)$ is a clique for all $j \in [1, i-1] \cup [i+1, m]$, and $|N_{T_i}(w)| \geq |T_i| - 1$ for any $w \in N_{B_1}(T_i)$.

Denote by $T_i = \{w_{i_1}, w_{i_2}, \cdots, w_{i_{t_i}}\}$ and $w_i^1, w_i^2, w_i^3 \in N_{B_1}(T_i)$ since G is 3-connected. We may assume $|T_1| \geq 3$ and $w_1^2 w_{1_j} \in E(G)$ for all $j \in [1, t_1 - 1]$ and $w_{1_{t_1-1}}w_{1_{t_1}} \in E(G)$. Since $T_1 \setminus \{w_{1_{t_1}}, w_{1_{t_1-1}}\}$ and T_j are cliques, there exist hamiltonian paths, say $P_{3_1} = w_{1_1}P_{3_1}w_{1_{t_1-2}}$ and $P_{3_j} = w_{j_1}P_{3_j}w_{j_{t_j}}$, in them, respectively, where $j \in [2, m]$. Let $B_1' = B_1 \setminus (\cup_{i=1}^m \{w_i^1, w_i^2, w_i^3\}) = \{w^1, w^2, \cdots, w^s\}$, then there exists a hamiltonian path, say $P_4 = w^1 P_4 w^s$, in it. There also exists a hamiltonian paths $P = u_{1_3}Pu_{1_4}$ in $A_1 \setminus \{u_{1_1}, u_{1_2}\}$ since $A_1 \setminus \{u_{1_1}, u_{1_2}\}$ is a clique. Let $C = P_{3_1} \cdots P_{3_m} P_4 P \cup (\cup_{i=1}^m \{w_i^1 w_{i_1}, w_{i_{t_i}} w_i^3, w_i^3 w_{i+1}^1\}) \cup \{w_{1_{t_1-2}}w_{1_{t_1}}, w_m^3 w^1, w_1^s u_{1_2}, u_{1_2}u_{2_1}, u_{2_1}u_{13}, u_{14}w_1^1\}$ be a cycle and the vertex set $\cup_{i=1}^m \{w_i^2\} \cup$

$\{w_{1_{t_1-1}}, u_{11}\}$ be stems of T with $N_C(u_{11}) = (A_1 \setminus \{u_{11}\}) \cup \{u_{2_1}\} \cup B_1'$, $N_C(w_1^2) = \{w_1^1, w_1^3\}$, $N_C(w_{1_{t_1-1}}) = T_1 \setminus \{w_{1_{t_1-1}}\}$ and $N_C(w_i^2) = T_i \cup \{w_i^1, w_i^3\}$, $i \in [2, m]$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 8.2(1) as an example).

Case 3.2: For any $w_{i_1}, w_{i_2} \in T_i$, $N_{B_1}(w_{i_1}) \cap N_{B_1}(w_{i_2}) = \emptyset$. Then $T_i \subseteq \{w_{i-1}, w_{i_2}\}$ and $|N_{B_1}(w_{i_1})| \geq 2$, $|N_{B_1}(w_{i_2})| \geq 2$ for all $i \in [1, m]$.

We may assume $T_i = \{w_{i_1}, w_{i_2}\}$ for all $i \in [1, l]$ and $T_j = \{w_j\}$ for all $j \in [l+1, m]$. Let $w_{i_1}^1, w_{i_1}^2 \in N_{B_1}(w_{i_1})$, $w_{i_2}^1, w_{i_2}^2 \in N_{B_1}(w_{i_2})$ for $i \in [1, l]$ and $w_i^1, w_i^2, w_i^3 \in N_{B_1}(w_i)$ for $j \in [l+1, m]$, $B_1' = B_1 \setminus ((\cup_{i=1}^{l}\{w_{i_1}^1, w_{i_1}^2, w_{i_2}^1, w_{i_2}^2\}) \cup (\cup_{j=l+1}^{m}\{w_j^1, w_j^2, w_j^3\})) = \{w^1, w^2, \cdots, w^s\}$. Since $A_1 \setminus \{u_{1_1}, u_{1_2}\}$ and B_1' are cliques, there exist hamiltonian paths, say $P = u_{1_3}Pu_{1_4}$ and $P_4 = w^1P_4w^s$, in them, respectively. Let $C = \cup_{i=1}^{l}\{w_{i_1}^1 w_{i_1}, w_{i_1}w_{i_2}, w_{i_2}w_{i_2}^2, w_{i_2}^2 w_{(i+1)_1}^1\} \cup \{w_{l_2}^2 w_{(l+1)_1}^1, w_m^3 w^1, w^s u_{1_2},\ u_{1_2}u_{2_1}, u_{2_1}u_{1_3}, u_{1_4}w_{1_1}^1\} \cup (\cup_{j=l+1}^{m}\{w_j^1 w_j, w_j w_j^3, w_j^3 w_{j+1}^1\} \cup P_4P$ be a cycle with all vertices on the star $(\cup_{i=1}^{l}\{w_{i_1}^2, w_{i_2}^1\}) \cup (\cup_{j=l+1}^{m}\{w_j^2\}) \cup \{u_{1_1}\}$ be stems of T, where u_{1_1} is the center, with $N_C(u_{1_1}) = (A_1 \setminus \{u_{1_1}\}) \cup \{u_{2_1}\} \cup B_1'$, $N_C(w_{i_1}^2) = \{w_{i_1}^1, w_{i_1}\}$, $N_C(w_{i_2}^1) = \{w_{i_2}^2, w_{i_2}\}$, $i \in [1, l]$ and $N_C(w_j^2) = \{w_j^1, w_j^3, w_j\}$, where $j \in [l+1, m]$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 8.2(2) as an example).

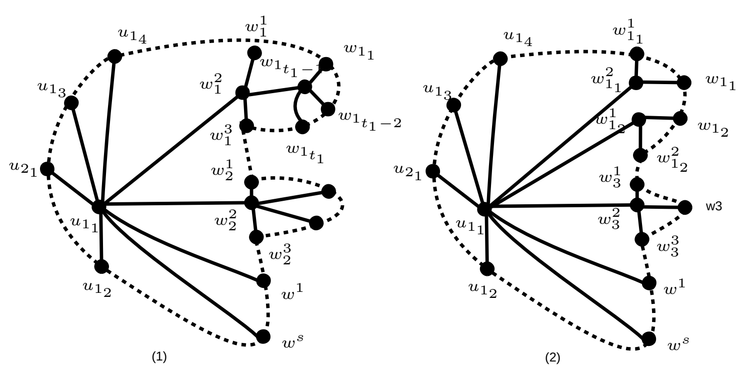

Figure 8.2. $W_1 = A_1 \cup A_2$ and $W_2 = B_1 \cup B_2$.

8.2 Proof of 3-connected $\{claw, B_{1,2}\}$-free graphs admit 1-joins

Claim 8.2.1. *Let $k = min\{i|i \in [1,t]\}$ such that $A_k \cup A_{k+1}$ is not a clique, then $k = 1$.*

Proof. We may assume this is not true and $k = 2$, then $A_1 \cup A_2$ is a clique and $A_2 \cup A_3$ is not. We can also assume $u_{3_1}u_{2_2} \notin E(G)$, then $\langle u_{3_1}; u_{2_1}, u_{2_2}, u_{1_1}; w_{1_1}w_{2_1}\rangle$ is a $B_{1,2}$, showing a contradiction. ■

Claim 8.2.2. *If $A_1 \cup A_2$ is not a clique, then*

1) $W_1 = A_1 \cup A_2$;

2) $W_2 = B_1 \cup B_2$;

3) Let $T_1, T_2, \cdots, T_m$ be maximal connected components in B_2, then $T_i \cup N_{B_1}(T_i)$ is a clique for all $i \in [1,m]$.

Proof. We may assume $u_{1_2}u_{2_1} \notin E(G)$.

1) If $A_3 \neq \emptyset$, then $\langle w_{2_1}; w_{1_1}, u_{1_2}, u_{1_1}; u_{2_1}u_{3_1}\rangle$ is a $B_{1,2}$, showing a contradiction.

2) If $B_3 \neq \emptyset$, then $\langle u_{2_1}; u_{1_1}, u_{1_2}, w_{1_1}; w_{2_1}w_{3_1}\rangle$ is a $B_{1,2}$, giving a contradiction.

3) This is clearly true if $|T_i| = 1$ since B_1 is a clique. If there exist $w_{2_1}, w_{2_2} \in T_i$ and $w_{1_1} \in N_{B_1}(T_i)$ such that $w_{1_1}w_{2_1}, w_{2_1}w_{2_2} \in E(G)$ but $w_{1_1}w_{2_2} \notin E(G)$, then $\langle u_{2_1}; u_{1_1}, u_{1_2}, w_{1_1}; w_{2_1}w_{2_2}\rangle$ is a $B_{1,2}$. Since T_i is connected, we have $T_i \cup N_{B_1}(T_i)$ is a clique. ■Let $k' = min\{j|j \in [1,s]\}$ such that $B_j \cup B_{j+1}$ is not a clique, similarly as Claim 8.2.1 and 8.2.1, we get

Corollary 8.2.1. *If $B_1 \cup B_2$ is not a clique, then*

1) $k' = 1$;

2) $W_2 = B_1 \cup B_2$;

3) $W_1 = A_1 \cup A_2$;

4) Let $R_1, R_2, \cdots, R_{m'}$ be maximal connected components in A_2, then $R_i \cup N_{A_1}(R_i)$ is a clique for all $i \in [1,m']$.

Claim 8.2.3. *If $A_1 \cup A_2$ is a clique but $B_1 \cup B_2$ is not a clique, let $T_1, T_2, \cdots, T_k$ be maximal connected components in B_2, then T_i is a clique for all $i \in [1,k]$.*

Proof. This is clearly true if $|T_i| \leq 2$, thus we can assume $|T_i| \geq 3$. Since T_i is connected, we assume $w_{2_2}w_{2_1}w_{2_3}$ is an induced path in T_i and let $w_{1_1} \in N_{B_1}(w_{2_1})$. Since $\langle w_{2_1}; w_{2_2}, w_{2_3}, w_{1_1}\rangle$ is not a claw, $w_{1_1}w_{2_2} \in E(G)$. Moreover, since $\langle w_{2_3}; w_{2_1}, w_{2_2}, w_{1_1}; u_{1_1}u_{2_1}\rangle$ is not a $B_{1,2}$, $w_{1_1}w_{2_3} \in E(G)$. But this forces $\langle w_{1_1}; u_{1_1}, w_{2_2}, w_{2_3}\rangle$ to be a claw, showing a contradiction. ■

We want to find a strong spanning Halin subgraph in G by following three cases.

Case 1: Both $A_1 \cup A_2$ and $B_1 \cup B_2$ are cliques.

If both $A_1 \cup A_2$ and $B_1 \cup B_2$ are cliques, then $A_i \cup A_{i+1}$ and $B_j \cup B_{j+1}$ are cliques for all $i \in [1, t-1]$ and $j \in [1, s-1]$. Let $P_{1_i} = u_{i_3}P_{1_i}u_{i_4}$, $P_{2_j} = w_{j_3}P_{2_j}w_{j_4}$ be hamiltonian paths in $A_i \setminus \{u_{i_1}, u_{i_2}\}$ and $B_j \setminus \{w_{j_1}, w_{j_2}\}$, where $i \in [1, t-1]$ and $j \in [1, s-1]$ and $P_{1_t} = u_{t_3}P_{1_t}u_{t_2}$, $P_{2_s} = w_{s_3}P_{2_s}w_{s_2}$ be hamiltonian paths in A_t and B_s. Let $C = P_{1_t}P_{1_{t-1}} \cdots P_{1_1}P_{2_1} \cdots P_{2_s} \cup \cup_{i=1}^{t-1}\{u_{i_4}u_{(i+1)_3}, u_{i_2}u_{(i+1)_2}\} \cup \cup_{j=1}^{s-1}\{w_{i_4}w_{(i+1)_3}, w_{i_2}w_{(i+1)_2}\} \cup \{u_{13}w_{13}, u_{12}w_{12}\}$ be a cycle and all vertices on the path $u_{(t-1)_1}u_{(t-2)_1} \cdots u_{2_1}u_{1_1}w_{1_1}w_{2_1} \cdots w_{(t-2)_1}w_{(t-1)_1}$ be stems of T with $N_C(u_{i_1}) = A_i \setminus \{u_{i_1}\}$, $i \in [1, t-2]$, $N_C(w_{j_1}) = B_j \setminus \{w_{j_1}\}$, $j \in [1, s-2]$, $N_C(u_{t-1_1}) = A_{t-1} \cup A_t \setminus \{u_{(t-1)_1}\}$ and $N_C(w_{(s-1)_1}) = B_{s-1} \cup B_s \setminus \{w_{(s-1)_1}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 8.3 as an example).

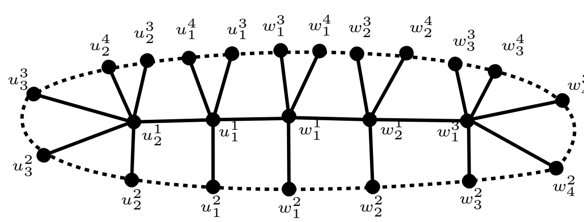

Figure 8.3. Both $A_1 \cup A_2$ and $B_1 \cup B_2$ are cliques.

Case 2: Neither $A_1 \cup A_2$ nor $B_1 \cup B_2$ is a clique.

By Claim 8.2.1 and 8.2.3, we know $W_1 = A_1 \cup A_2$ and $W_2 = B_1 \cup B_2$. Let $R_1, R_2, \cdots, R_{m'}$ be maximal connected components in A_2 and $T_1, T_2, \cdots, T_m$ be maximal connected components in B_2, then both $R_i \cup N_{A_1}(R_i)$ and $T_j \cup N_{B_1}(T_j)$ are cliques. Let $u_{i_1}, u_{i_2}, u_{i_3} \in$

$N_{A_1}(R_i)$, $w_{j_1}, w_{j_2}, w_{j_3} \in N_{B_1}(T_j)$, $A_1' = A_1 \setminus \cup_{i=1}^{m'}\{u_{i_1}, u_{i_2}, u_{i_3}\} = \{u^1, \cdots, u^t\}$ and $B_1' = B_1 \setminus \cup_{i=1}^{m}\{w_{j_1}, w_{j_2}, w_{j_3}\} = \{w^1, \cdots, w^s\}$. Since $R_i \cup \{u_{i_2}, u_{i_3}\}$ and $T_i \cup \{w_{j_2}, w_{j_3}\}$, A_1' and B_1' are cliques, there exists $P_{1_i} = u_{i_2} P_{1_i} u_{i_3}$, $P_{2_j} = w_{j_2} P_{2_j} w_{j_3}$, $P_1 = u^1 P_1 u^t$ and $P_2 = w^1 P_2 w^s$ are hamiltonian paths in them, respectively. Since $A_1 \cup B_1$ is a clique, let $C = P_{1_i} \cdots P_{1_{m'}} P_1 P_2 P_{2_m} \cdots P_{2_2} P_{2_1}$ be a cycle and all vertices on the star $\{u_{1_1}, u_{1_2}, \cdots, u_{1_{m'}}, w_{1_1}, \cdots, w_{1_m}\}$, where u_{1_1} is the center, be stems of T with $N_C(u_{i_1}) = R_i \cup \{u_{i_2}, u_{i_3}\}$, $i \in [2, m']$, $N_C(w_{j_1}) = T_j \cup \{w_{j_2}, w_{j_3}\}$, $j \in [1, m]$ and $N_C(u_{1_1}) = R_1 \cup \{u_{1_2}, u_{1_3}\} \cup A_1' \cup B_1'$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 8.4 as an example).

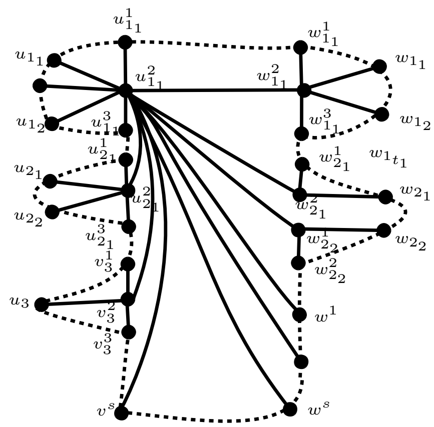

Figure 8.4. Neither $A_1 \cup A_2$ nor $B_1 \cup B_2$ is cliques.

Case 3: Either $A_1 \cup A_2$ or $B_1 \cup B_2$ is a clique. We may assume $A_1 \cup A_2$ is a clique and $B_1 \cup B_2$ is not.

By Corollary 8.2.1 and Claim 8.2.3, we know $W_1 = A_1 \cup A_2$ and $W_2 = B_1 \cup B_2$. Let $T_i = \{w_{i_1}, w_{i_2}, \cdots, w_{i_{t_i}}\}$ be maximal connected components in B_2, $w_i^1 \in N_{B_1}(w_{i_1})$, $w_i^2 \in N_{B_1}(w_{i_2})$, $w_i^3 \in N_{B_1}(w_{i_{t_i}})$ if $|T_i| \geq 3$ and $\{w_{i_1}^1, w_{i_1}^2\} \in N_{B_1}(w_{i_1})$, $\{w_{i_2}^1, w_{i_2}^2\} \in N_{B_1}(w_{i_2})$ if $T_i = \{w_{i_1}, w_{i_2}\}$. Note that we may have $w_{i_1}^2 = w_{i_2}^1$ if $N_{B_1}(w_{i_1}) \cap N_{B_1}(w_{i_2}) \neq \emptyset$. Assume $|T_i| \neq 2$ for all $i \in [1, l]$ and $|T_j| = 2$ for all $j \in [l+1, m]$. Let $B_1' = B_1 \setminus (\cup_{i=1}^{l}\{w_i^1, w_i^2, w_i^3\} \cup (\cup_{j=l+1}^{m}\{w_{j_1}^1, w_{j_1}^2, w_{j_2}^1, w_{j_2}^2\})) = \{w^1, w^2, \cdots, w^s\}$. Since $A_1 \cup A_2 \cup \{u_{1_1}\}$, $T_i \setminus \{w_{i_1}\}(i \in [1, l])$, and B_1' are cliques, there exist hamiltonian paths $P_1 = u_{12}P_1u_{13}$, $P_{2_i} = w_{i_2}P_{2_i}w_{i_{t_i}}$ and $P_3 = w^1P_3w^s$, in them, respectively. Since $A_1 \cup B_1$ is a clique, let $C = P_1P_{2_1}P_{2_2}\cdots P_{2_l}P_3$ be a cycle and all vertices $(\cup_{i=1}^{l}\{w_i^1, w_{i_1}\}) \cup (\cup_{j=l+1}^{m}\{w_{j_1}^2, w_{j_2}^1\}) \cup \{u_{1_1}\}$ be stems of T with $N_C(u_{1_1}) = A_2 \cup (A_1 \setminus \{u_{1_1}\}) \cup B_1'$, $N_C(w_i') = \{w_i^2, w_i^3\}$, $N_C(w_{i_1}) = T_i \setminus \{w_{i_1}\}$, $i \in [1, l]$, $N_C(w_{j_1}^2) = \{w_{j_1}^1, w_{i_1}\}$ and $N_C(w_{j_2}^1) = \{w_{j_2}^2, w_{j_2}\}$ for all $j \in [l+1, m]$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 8.5 as an example).

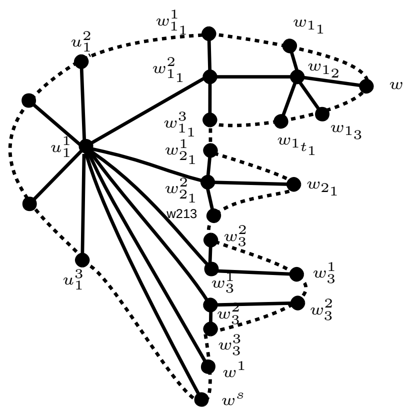

Figure 8.5. Either $A_1 \cup A_2$ or $B_1 \cup B_2$ is cliques.

Chapter 9

GENERALIZED 2-JOIN AND 2-JOIN

Recall that if we suppose that W_0, W_1, W_2 are disjoint subsets with union $V(G)$, and for $i = 1, 2$ there are subsets A_i, B_i of W_i satisfying the following:

- $W_0 \cup A_1 \cup A_2$ and $W_0 \cup B_1 \cup B_2$ are cliques, and $E(W_0, W_i \setminus (A_i \cup B_i)) = \emptyset$ for $i = 1, 2$;
- for $i = 1, 2$, $A_i \cap B_i = \emptyset$ and A_i ,B_i and $W_i \setminus (A_i \cup B_i)$ are all nonempty; and
- for all $v \in W_1$ and $w \in W_2$, either v is nonadjacent to w, or $v \in A_1$ and $w \in A_2$, or $v \in B_1$ and $w \in B_2$.

We call the triple $(W_0 \cup W_1 \cup W_2)$ a generalized 2-join, and if $W_0 = \emptyset$ we call the pair (W_1, W_2) a 2-join.

Denote by $A_1 = \{v_{1_1}, v_{1_2}, \cdots, v_{1_{t_1}}\}$, $B_1 = \{v_{2_1}, v_{2_2}, \cdots, v_{2_{t_2}}\}$, $D_1 = W_1 \setminus (A_1 \cup B_1) = D_{1_1} \cup D_{1_2} \cup \cdots \cup D_{1_{k_1}} = \{v_{3_1}, v_{3_2}, \cdots, v_{3_{t_3}}\}$, $A_2 = \{w_{1_1}, w_{1_2}, \cdots, w_{1_{s_1}}\}$, $B_2 = \{w_{2_1}, w_{2_2}, \cdots, w_{2_{s_2}}\}$, $D_2 = W_2 \setminus (A_2 \cup B_2) = D_{2_1} \cup D_{2_2} \cup \cdots \cup D_{2_{k_2}} = \{w_{3_1}, w_{3_2}, \cdots, w_{3_{s_3}}\}$ and $W_0 = \{u_1, u_2, \cdots, u_k\}$, where D_{i_j} are maximal connected components of D_i, $i \in [1, 2]$ and $j \in [1, k_1] \cup [1, k_2]$. Since G is 3-connected and $D_i \neq \emptyset$, we have $|A_i \cup B_i| \geq 3$ for $i \in [1, 2]$. Without loss of generality, we always assume $|A_1| \geq |B_1|$, thus $|A_1| \geq 2$. We have the following claim.

Claim 9.0.4. *1) For any $v \in A_i \cup B_i$, $N_{D_i}(v)$ is a clique, where $i \in [1, 2]$.*
2) If there exist $v_{1_k} \in A_i$ and $v_{2_l} \in B_i$ such that $v_{1_k} v_{2_l} \in E(G)$, then $N_{D_i}(v_{1_k}) = N_{D_i}(v_{2_l})$ for $i \in [1, 2]$.

Proof. 1) We may assume there exists $v_{1_1} \in A_1$ and $v_{3_1}, v_{3_2} \in N_{D_1}(v_{1_1})$ such that $v_{3_1} v_{3_2} \notin E(G)$, then $\langle v_{1_1}; v_{3_1}, v_{3_2}, w_{1_1} \rangle$ is a claw, showing a contradiction.

2) We may assume there exist $v_{1_k} \in A_1$ and $v_{2_l} \in B_1$. For any $v \in N_{D_1}(v_{1_k})$, since $\langle v_{1_k}; v, v_{2_l}, w_{1_1} \rangle$ is not a claw, $vv_{2_l} \in E(G)$. Similarly, for any $v \in N_{D_1}(v_{2_l})$, since $\langle v_{2_l}; v, v_{1_k}, w_{2_1} \rangle$ is not a claw, $vv_{1_k} \in E(G)$. Therefore, $N_{D_1}(v_{1_k}) = N_{D_1}(v_{2_l})$. ■

In this chapter, we will show the following four propositions.

Proposition 14. *If G is a 3-connected $\{claw, Z_3\}$-free graph admits a generalized 2-join, then G contains a strong spanning Halin subgraph.*

Proposition 15. *If G is a 3-connected $\{claw, Z_3\}$-free graph admits a 2-join, then G contains a strong spanning Halin subgraph.*

Proposition 16. *If G is a 3-connected $\{claw, B_{1,2}\}$-free graph admits a generalized 2-join, then G contains a strong spanning Halin subgraph.*

Proposition 17. *If G is a 3-connected $\{claw, B_{1,2}\}$-free graph admits a 2-join, then G contains a strong spanning Halin subgraph.*

9.1 Proof of 3-connected $\{claw, Z_3\}$-free graphs admit generalized 2-joins

Before we prove Proposition 14, we give following claim first.

Claim 9.1.1. *1) For any $v \in D_i$ and $i \in [1, 2]$, $dist(v, A_1 \cup B_1) = 1$.*
2) If $N_G(D_{i_j}) \setminus D_i \subseteq A_i$(or B_i), then $N_G(D_{i_j}) \cup D_{i_j}$ is a clique, where $i \in [1, 2]$ and $j \in [1, k_i]$.
3) If there exists $v \in A_1$(similarly as $v \in B_1$ or $v \in A_2$ or $v \in B_2$) such that $|N_{D_1}(v)| \geq 2$, then $N_{D_2}(B_2) = \emptyset$.

Proof. 1) Suppose to the contrary, there exist $v \in B_1$ and $v', v'' \in D_1$ such that $vv', v'v'' \in E(G)$ and $vv'' \notin E(G)$, then $|A_2| = 1$. Otherwise $\langle w_{1_1}, w_{1_2}, u_1; vv'v''\rangle$ is a Z_3, which implies $|B_2| \geq 2$ and $N_{D_2}(B_2) \neq \emptyset$ since G is 3-connected and $D_2 \neq \emptyset$. Since $\langle w_{1_1}, v_{1_1}, u_1; vv'v''\rangle$ is not a Z_3, if $w_{2_1}w_{3_1} \in E(G)$,, then we have $vv_{1_1} \in E(G)$ which implies $v'v_{1_1} \in E(G)$; or $v'v_{1_1} \in E(G)$ which implies $v''v_{1_1} \in E(G)$ since $\langle w_{2_1}, w_{2_2}, u_1; v_{1_1}v'v''\rangle$ is not a Z_3; or $v''v_{1_1} \in E(G)$ which implies $v'v_{1_1} \in E(G)$ since $\langle w_{2_1}, w_{2_2}, u_1; v_{1_1}v''v'\rangle$ is not a Z_3. Thus we can assume $v''v_{1_1}, v'v_{1_1} \in E(G)$ but $vv_{1_1} \notin E(G)$ since $\langle v_{1_1}; v'', v, w_{1_1}\rangle$ is not a claw. However, this will force $\langle v'', v', v_{1_1}; u_1w_{2_1}w_{3_1}\rangle$ to be a Z_3, giving a contradiction.

2) Since D_{i_j} is connected, by 1), we know $D_{i_j} \cup \{v\}$ is a clique for any $v \in N_G(D_{i_j}) \cap A_i$. Since A_i is a clique, $N_G(D_{i_j}) \cup D_{i_j}$ is a clique, where $i \in [1, 2]$ and $j \in [1, k_i]$.

3) Suppose there exist $v_{3_1}, v_{3_2} \in N_{D_1}(v_{1_1})$ and $w_{2_1}w_{3_1} \in E(G)$, then $\langle v_{3_1}, v_{3_2}, v_{1_1}; u_1 w_{2_1} w_{3_1}\rangle$ is a Z_3, showing a contradiction. ■

Now we want to find a strong spanning Halin subgraph in G by following cases.

Case 1: Assume $E(A_1, B_1) = \emptyset$ and $E(A_2, B_2) = \emptyset$.

Claim 9.1.2. *If $E(A_1, B_1) = \emptyset$ and $E(A_2, B_2) = \emptyset$, then*

1) $N_{D_i}(A_i) \cap N_{D_i}(B_i) = \emptyset$ for $i = 1, 2$.

2) If there exists $v \in A_1$such that $|N_{D_1}(v)| \geq 2$, then $N_{D_1}(B_1) = \emptyset$.

3) $N_i(D_{i_j}) \cup D_{i_j}$ is a clique.

4) $|N_i(D_{i_j})| \geq 3$ and $|W_0| \geq 3$.

Proof. 1) We only show that $N_{D_2}(A_2) \cap N_{D_2}(B_2) = \emptyset$. Suppose this is not true, there exists $w_{3_1} \in N_{D_2}(w_{1_1}) \cap N_{D_2}(w_{2_1})$, then $\langle v_{1_1}, v_{1_2}, w_{1_1}; w_{3_1} w_{2_1} v_{2_1}\rangle$ is a Z_3.

2) Suppose this is not true, there exist $v_{3_1}, v_{3_2} \in N_{D_1}(v_{1_1})$ and $v_{3_3} \in N_{D_1}(v_{2_1})$, then $\langle v_{3_1}, v_{3_2}, v_{1_1}; u_1 v_{2_1} v_{3_3}\rangle$ is a Z_3.

3) This is clearly true by 1) and Claim 9.1.1 2).

4) This is true since G is 3-connected. ■

Let $D_{1_j} = \{v_{1_j}^1, v_{1_j}^2, \cdots, v_{1_j}^{m_j}\} \subseteq D_1$, $D_{2_j} = \{w_{1_j}^1, w_{1_j}^2, \cdots, w_{1_j}^{n_j}\} \subseteq D_2$, $v_j^1, v_j^2, v_j^3 \in N_1(D_{1_j})$, $w_j^1, w_j^2, w_j^3 \in N_2(D_{2_j})$ for $j \in [1, k_1] \cup [1, k_2]$, and $A_1' = A_1 \setminus \cup_{j=1}^{k_1}\{v_j^1, v_j^2, v_j^3\} = \{v_{1_1}', v_{1_2}', \cdots, v_{1_{l_1}}'\}$, $B_1' = B_1 \setminus \cup_{j=1}^{k_1}\{v_j^1, v_j^2, v_j^3\} = \{v_{2_1}', v_{2_2}', \cdots, v_{2_{l_2}}'\}$, $A_2' = A_2 \setminus \cup_{j=1}^{k_2}\{w_j^1, w_j^2, w_j^3\} = \{w_{1_1}', w_{1_2}', \cdots, w_{1_{l_1}}'\}$, $B_2' = B_2 \setminus \cup_{j=1}^{k_2}\{w_j^1, w_j^2, w_j^3\} = \{w_{2_1}', w_{2_2}', \cdots, w_{2_{l_2}}'\}$. Since $D_{1_j} \cup \{v_j^1, v_j^3\}$, $D_{2_j} \cup \{w_j^1, w_j^3\}$, A_i', B_i' and $W_0 \setminus \{u_1, u_2\}$ are cliques, there exist hamiltonian paths, say $P_{1_j} = v_j^1 P_{1_j} v_j^3$, $P_{2_j} = w_j^1 P_{2_j} w_j^3$, $P_3 = v_{1_1}' P_3 v_{1_{l_1}}'$, $P_4 = v_{2_1}' P_4 v_{2_{l_2}}'$, $P_5 = w_{1_1}' P_5 w_{1_{l_1}}'$, $P_6 = w_{2_1}' P_6 w_{2_{l_2}}'$ and $P_7 = u_3 P_7 u_k$, in them, respectively, where $i \in [1, 2]$ and $j \in [1, k_1] \cup [1, k_2]$. Let $C = P_{1_1} \cdots P_{1_{k_1}} P_3 P_4 P_7 P_5 P_6 P_{2_{k_2}} \cdots P_{2_1}$ be a cycle and all vertices on the star $\langle u_1; v_1^2, \cdots, v_{k_1}^2, w_1^2, \cdots w_{k_2}^2\rangle$, where u_1 is the center, be stems of T with $N_C(u_1) = A_1' \cup B_1' \cup W_0 \setminus \{u_1\}$, $N_C(v_j^2) = D_{1_j} \cup \{v_j^1, v_j^3\}$ and $N_C(w_j^2) = D_{2_j} \cup \{w_j^1, w_j^3\}$, where $j \in [1, k_1] \cup [1, k_2]$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 9.1 (1) and (2) as examples).

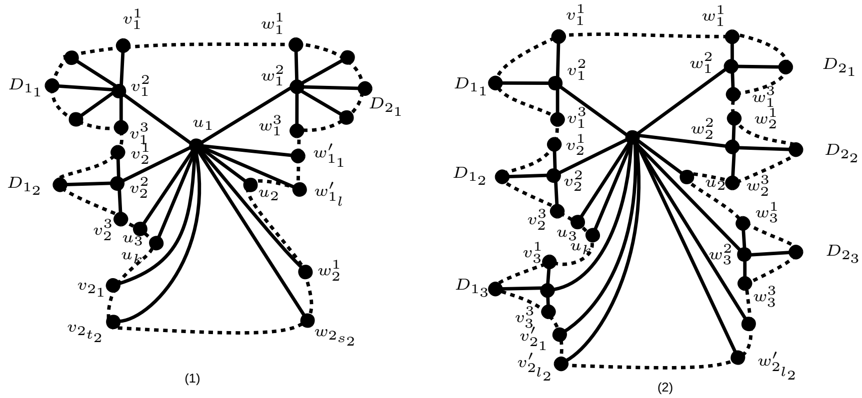

Figure 9.1. $E(A_1, B_1) = \emptyset$ and $E(A_2, B_2) = \emptyset$

Case 2: Suppose $E(A_1, B_1) \neq \emptyset$ or $E(A_2, B_2) \neq \emptyset$.

We may assume $v_{1_1}v_{2_1} \in E(G)$ and want to consider following two subcases depending on the neighborhood of $N_{D_2}(A_2)$ and $N_{D_2}(B_2)$ are empty or not.

Claim 9.1.3. *If $N_{D_2}(A_2) = \emptyset$ and $N_{D_2}(B_2) \neq \emptyset$(similarly as $N_{D_2}(B_2) = \emptyset$ and $N_{D_2}(A_2) \neq \emptyset$), then*

1) $|A_2| = 1$.

2) For any $v_{1_i} \in A_1$, $|N_{D_1}(v_{1_i})| \leq 1$.

3) For any $v_{2_i} \in B_1$, if $E(v_{2_i}, A_1) \neq \emptyset$, then $\{v_{2_i}\} \cup A_1$ is a clique.

4) We may assume $E(v_{2_j}, A_1) \neq \emptyset$ for all $j \in [1, l]$, where $l \leq t_2$, then $|N_{D_1}(A_1 \cup \{v_{2_1}, \cdots, v_{2_l}\})| \leq 1$.

5) For any $v_{2_j} \in B_1 \setminus \{v_{2_1}, \cdots, v_{2_l}\}$, $|N_{D_1}(v_{2_j})| \leq 1$. Therefore, D_1 is an independent set.

6) If $N_{D_1}(A_1) \neq \emptyset$, then D_2 is an independent set.

Proof. We may assume $w_{3_1}w_{2_1} \in E(G)$. Since $N_{D_2}(A_2) = \emptyset$, by Claim 9.0.4, we have $w_{1_i}w_{2_1} \notin E(G)$ for any $w_{1_i} \in A_2$.

1) If there exist $w_{1_1}, w_{1_2} \in A_1$, then $\langle w_{1_1}, w_{1_2}, v_{1_1}; v_{2_1}w_{2_1}w_{3_1}\rangle$ is a Z_3, showing a contradiction.

2) This is clear true by the last conclusion of Claim 9.1.1.

3) For any $v_{2_i} \in B_1$, we may assume there exists $v_{1_1}, v_{1_2} \in A_1$ such that $v_{1_1}v_{2_i} \in E(G)$

and $v_{1_2}v_{2_i} \notin E(G)$, then $\langle w_{1_1}, v_{1_2}, v_{1_1}; v_{2_1}w_{2_i}w_{3_1}\rangle$ is a Z_3, showing a contradiction. Therefore, $\{v_{2_1}\} \cup A_1$ is a clique.

4) This is clearly true by 2), 3) and Claim 9.1.1 2).

5) If there exist $v, v' \in N_{D_1}(v_{2_j})$, where $j \in [l+1, t_2]$, then $vv' \in E(G)$ by Claim 9.0.4. Moreover, $vv_{1_i}, v'v_{1_i}, vv_{2_1}, v'v_{2_1} \notin E(G)$ for all $v_{1_i} \in A_1$. Otherwise, since $\langle w_{1_1}, v_{1_k}, v_{1_i}; vv_{2_j}w_{2_1}\rangle$ is not a Z_3, we can assume $vv_{1_i} \in E(G)$ we have $w_{1_1}w_{2_1} \in E(G)$ or $vv_{1_k} \in E(G)$. If $vv_{1_k} \in E(G)$, then $\langle v_{1_k}, v_{1_i}, v; v_{2_i}w_{2_1}w_{3_1}\rangle$ is a Z_3. If $w_{1_1}w_{2_1} \in E(G)$, since $\langle v_{1_k}, v_{1_i}, w_{1_1}; w_{2_1}v_{2_j}v'\rangle$ is not a Z_3, we have $v'v_{1_i} \in E(G)$ or $v'v_{1_k} \in E(G)$, which implies $vv_{2_1}, v'v_{2_1} \in E(G)$ and $vv_{1_i}v'v_{1_i} \in E(G)$, showing a contradiction.

6) This is clearly true by Claim 9.1.1 1) and 3). ■

We denote by $D_1 = \{v_{3_1}, v_{3_2}, \cdots, v_{3_{t_3}}\}$, $v_i^1, v_i^2, v_i^3 \in N_{A_1\cup B_1}(v_{3_i})$ for all $v_{3_i} \in D_1$, $A_1' = A_1\backslash(\cup_{i=1}^{t_3}\{v_i^1, v_i^2, v_i^3\}) = \{v_{1_1}', v_{1_1}', \cdots, v_{1_{k_1}}'\}$, $B_1' = B_1\backslash(\cup_{i=1}^{t_3}\{v_i^1, v_i^2, v_i^3\}) = \{v_{2_1}', v_{2_2}', \cdots, v_{2_{k_2}}'\}$; $D_2 = \{w_{3_1}, w_{3_2}, \cdots, w_{3_{s_3}}\}$, $w_i^1, w_i^2, w_i^3 \in N_{B_2}(w_{3_i})$ for all $w_{3_i} \in D_2$, $B_2' = B_2\backslash(\cup_{i=1}^{s_3}\{w_i^1, w_i^2, w_i^3\})$ $= \{w_{2_1}', w_{2_2}', \cdots, w_{2_{k_3}}'\}$. Since A_1', B_1', B_2' and $W_0 \setminus \{u_1\}$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_1}'P_1v_{1_{k_1}}'$, $P_2 = v_{2_1}'P_2v_{2_{k_2}}'$, $P_3 = w_{2_1}'P_3w_{2_{k_3}}'$ and $P_4 = u_2P_4u_k$, in them, respectively.

We only need to find a strong spanning Halin subgraph in G for the case $N_{D_1}(A_1) \neq \emptyset$, since the other case is similar but a little bit easier.

If $|W_0| \geq 2$ (similarly as $E(A_2, B_2) \neq \emptyset$). Let $C = \cup_{i=1}^4 P_i \cup (\cup_{i=1}^{t_1}\{v_i^1v_{3_i}, v_{3_i}v_i^3\}) \cup (\cup_{i=2}^{t_1-1}\{v_i^3v_{i+1}^1\}\cup\{v_{1_{k_1}}'v_1^1, v_{t_1}^3v_{2_1}', v_{2_{k_2}}'w_{2_{k_3}}', w_{2_1}'w_{t_2}^3, w_1^1u_k, u_2w_{1_1}, w_{1_1}v_{1_1}\})\cup(\cup_{j=1}^{t_2}\{w_i^1w_{3_i}, w_{3_i}w_i^3, w_i^3 w_{i+1}^1\})$ be a cycle and all vertices on the star $\langle u_1; v_1^2, v_2^2, \cdots, v_{t_1}^2, w_1^2, \cdots, w_{t_2}^2\rangle$, where u_1 is the center, be stems of T with $N_C(u_1) = A_1' \cup A_2' \cup B_1' \cup B_2' \cup (W_0 \setminus \{u_1\})$, $N_C(v_i^2) = \{v_{3_i}, v_i^1, v_i^3\}$ and $N_C(w_i^2) = \{w_{3_i}, w_i^1, w_i^3\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 9.2 (1) as an example).

If $W_0 = \{u_1\}$ and $E(A_2, B_2) = \emptyset$, since G is 3-connected, we can assume there exist $v_{2_1}, v_{2_2} \in B_1$ such that $\{v_{2_1}, v_{2_2}\} \cup A_1$ is a clique. Let $C = \{v_{1_2}v_{3_1}, v_{3_1}v_{2_2}, v_{2_2}v_2^1, v_{t_1}^3v_{2_1}', v_{2_{k_2}}'w_{2_{k_3}}', v_{2_1}'w_{t_2}^3, w_1^1u_1, u_1w_{1_1}, w_{1_1}v_{1_{t_1}}\} \cup (\cup_{i=2}^{t_1}\{v_i^1v_{3_i}, v_{3_i}v_i^3\}) \cup (\cup_{i=1}^{t_2}\{w_i^1w_{3_i}, w_{3_i}w_i^3\})$ be a cycle and all vertices on the star $\langle v_{2_1}, v_{1_1}, v_1^2, v_{t_1}^2, w_1^2, w_{t_2}^2\rangle$, where v_{2_1} is the center, be stems of T, with $N_C(v_{2_1}) = B'_1 \cup \{u_1\} \cup B'_2$, $N_C(v_{1_1}) = (A_1 \cup \{v_{1_1}\}) \cup \{w_{1_1}, v_{3_1}\}$, $N_C(v_i^2) = \{v_i^1, v_i^3, v_{3_i}\}$ and $N_C(w_i^2) = \{w_i^1, w_i^3, w_{3_i}\}$, where $i \in [1, k_1] \cup [1, k_2]$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 9.2 (2) as an example).

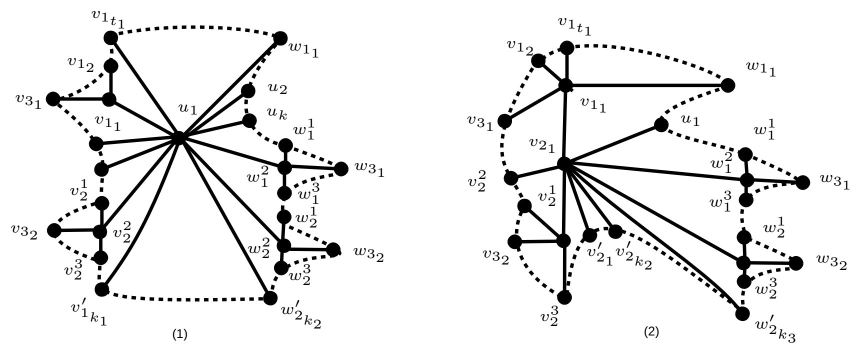

Figure 9.2. $E(A_1, B_1) \neq \emptyset$ and $N_{D_2}(A_2) = \emptyset$

Claim 9.1.4. *If $N_{D_2}(A_2) \neq \emptyset$ and $N_{D_2}(B_2) \neq \emptyset$, then*

1) For any $v_{2_i} \in B_1$(similarly as $v_{1_i} \in A_1$), if $E(v_{2_i}, A_1) \neq \emptyset$, then $|N_{A_1}(v_{2_i})| \geq |A_1| - 1$.

2) Let $A'_1 = \{v_{1_i} \in A_1 | E(v_{1_i}, B_1) \neq \emptyset\} \subseteq A_1$ and $B'_1 = \{v_{2_i} \in B_1 | E(v_{2_i}, A_1) \neq \emptyset\} \subseteq B_1$, then $|A'_1| \geq |A_1| - 1$ and $|B'_1| \geq |B_1| - 1$.

3) $|D_1| = 1$.

4) D_2 is an independent set.

5) If $N_{D_2}(A_2) \cap N_{D_2}(B_2) \neq \emptyset$ or $E(A_2, B_2) \neq \emptyset$, then $|D_2| = 1$.

Proof. Since $N_{D_2}(B_2) \neq \emptyset$, we assume $w_{2_1}w_{3_1} \in E(G)$.

1) Suppose $v_{1_1}v_{2_i} \in E(G)$. For any $v_{1_i}, v_{1_j} \in A_1 \setminus v_{1_1}$, since $\langle v_{1_i}, v_{1_j}, v_{1_1}; v_{2_i}w_{2_1}w_{3_1}\rangle$ is not a Z_3, we have $v_{1_i}v_{2_i} \in E(G)$ or $v_{1_j}v_{2_i} \in E(G)$. Thus $|N_{A_1}(v_{2_i})| \geq |A_1| - 1$.

2) This is clearly true by 1) and the symmetric of A_2 and B_2.

3) Denote by $v_{1_{t_1}} \in A_1 \setminus A'_1$, $v_{2_{t_2}} \in B_1 \setminus B'_1$ if they exist. Since $\{v_{1_{t_1}}, v_{2_{t_2}}\}$ is not 2-cut, $N_{D_1}(v_{1_{t_1}}) \cup N_{D_1}(v_{2_{t_2}}) \subseteq N_{D_1}(A'_1 \cup B'_1)$. By Claim 9.1.1 3) and the fact that $N_{D_2}(A_2) \neq \emptyset$ and $N_{D_2}(B_2) \neq \emptyset$, $|D_1| = 1$.

4) This is clearly true by 9.1.1 1) and 3).

5) Since $N_{D_1}(A_1) \neq \emptyset$, by Claim 9.1.1 3), we have $|N_{D_2}(w_{2_i})| \leq 1$ for all $w_{2_i} \in B_2$. If there exist $w_{3_i} \in D_2$ such that $w_{3_i} \in N_{D_2}(A_2) \cap N_{D_2}(B_2)$, then $N_2(w_{3_i}) \subseteq A_2 \cup B_2$ by Claim 9.1.1 1). We may assume $w_{1_1}, w_{1_2}, w_{2_1} \in N_2(w_{3_i})$. Since $\langle w_{1_1}, w_{1_2}, w_{3_i}; w_{2_1}v_{2_1}v_{3_1}\rangle$ is not a Z_3, we have $w_{1_1}w_{2_1} \in E(G)$ or $w_{1_2}w_{2_1} \in E(G)$, which implies $E(A_2, B_2) \neq \emptyset$. Similarly as 3), we can show that $|D_2| = 1$. ■

If $N_{D_2}(A_2) \cap N_{D_2}(B_2) \neq \emptyset$(similar as $E(A_2, B_2) \neq \emptyset$), we may assume $w_{3_1} \in N_{D_2}(w_{1_1}) \cap N_{D_2}(w_{2_1}) \cap N_{D_2}(w_{2_2})$ and $v_{3_1} \in N_{D_1}(v_{1_1}) \cap N_{D_1}(v_{1_2}) \cap N_{D_1}(v_{2_1})$. Since $A_1 \setminus \{v_{1_1}\}$, B_1, A_2, $B_2 \setminus \{w_{2_1}\}$ and $W_0 \setminus \{u_1\}$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_2}P_1v_{1_{t_1}}$, $P_2 = v_{2_1}P_2v_{2_{t_2}}$, $P_3 = w_{1_1}P_3w_{1_{s_1}}$, $P_4 = w_{2_2}P_4v_{2_{s_2}}$ and $P_5 = u_2P_5u_k$ in them, respectively. Let $C = P_1P_2P_3P_4P_5$ be a cycle and all vertices on the path $v_{1_1}u_1w_{2_1}$ be stems with $N_C(v_{1_1}) = \{v_{3_1}, v_{1_2}\}$, $N_C(u_1) = (A_1 \setminus \{v_{1_1}, v_{1_2}\}) \cup B_1 \cup (W_0 \setminus \{u_1\}) \cup A_2 \cup (B_2 \setminus \{w_{2_1}, w_{2_2}\})$ and $N_C(w_{2_1}) = \{w_{2_2}, w_{3_1}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 9.3 (1) as an example).

If $N_{D_2}(A_2) \cap D_2(B_2) = \emptyset$, $E(A_2, B_2) = \emptyset$ and $|W_0| \geq 2$, similarly as $N_{D_2}(A_2) = \emptyset$, we can find a strong spanning Halin subgraph in G.

If $N_{D_2}(A_2) \cap D_2(B_2) = \emptyset$, $E(A_2, B_2) = \emptyset$ and $|W_0| = 1$. Since G is 3-connected, we can assume $v_{1_1}v_{2_1}, v_{1_2}v_{2_2} \in E(G)$. Since $A_1 \setminus \{v_{1_1}\}$, $B_1 \setminus \{v_{2_1}\}$, $A_2' = A_2 \setminus (\cup_{i=1}^{s_3}\{w_i^1, w_i^2, w_i^3\})$ and $B_2' = B_2 \setminus (\cup_{i=1}^{s_3}\{w_i^1, w_i^2, w_i^3\})$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_2}P_1v_{1_{t_1}}$, $P_2 = v_{2_2}P_2v_{2_{t_2}}$, $P_3 = w_{1_1}'P_3w_{1_{k_1}}'$ and $P_4 = w_{2_1}'P_4w_{2_{k_2}}'$, in them, respectively. Let $C = \cup_{i=1}^{s_3}\{w_i^1w_{3_i}, w_{3_i}w_i^3, w_i^3w_{i+1}^1\} \cup \{w_k^3w_{1_1}', w_{1_{k_1}}'u_1, u_1w_{k+1}^1, w_{s_3}^3w_{2_1}', w_{2_{k_2}}'v_{2_{t_2}}, v_{2_2}v_{3_1}, v_{3_1}v_{1_2}, v_{1_{t_1}}w_1^1\} \cup \cup_{i=1}^{4}E(P_i)$ be a cycle and all vertices on the tree $\{v_{1_1}, v_{2_1}; w_i^1, w_i^2, \cdots, w_i^{s_3}\}$ be stems of T with $N_C(v_{1_1}) = (A_1 \setminus \{v_{1_1}\}) \cup A_2'$, $N_C(v_{2_1}) = B_1 \setminus \{v_{2_1}\} \cup \{v_{3_1}\} \cup B_2'$, $N_C(w_i^2) = \{w_i^1, w_i^3, w_{3_i}\}$, where $i \in [1, s_3]$. Let $H = T \cup C$, then H is a spanning Halin subgraph of G (See Figure 9.3 (2) as an example).

9.2 Proof of 3-connected $\{claw, Z_3\}$-free graphs admit 2-joins

In this section, we assume $W_0 = \emptyset$, which implies G contains a 2-join.

Since G is 3-connected and Z_3-free, it is easy to check that $E(A_1, B_1) \neq \emptyset$ or $E(A_2, B_2) \neq$

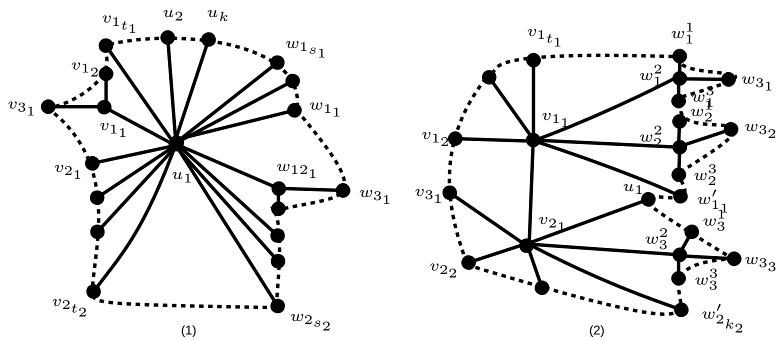

Figure 9.3. $E(A_1, B_1) \neq \emptyset$, $N_{D_2}(A_2) \neq \emptyset$ and $N_{D_2}(B_2) \neq \emptyset$

$\emptyset$. In this section, we always assume $E(A_2, B_2) \neq \emptyset$, $A_2' = \{w_{1_i} \in A_2 | E(w_{1_i}, B_2) \neq \emptyset\} = \{w_{1_1}, w_{1_2}, \cdots, w_{k_1}\} \subseteq A_2$ and $B_2' = \{w_{2_i} \in B_2 | E(w_{2_i}, A_2) \neq \emptyset\} = \{w_{2_1}, w_{2_2}, \cdots, w_{2_{k_2}}\} \subseteq A_2$, where $k_1 \leq s_1$ and $k_2 \leq s_2$. In particular, we always assume $w_{1_1}w_{2_1} \in E(G)$. Then we want to consider following two cases.

Case 1: Assume $E(A_1, B_1) = \emptyset$.

Claim 9.2.1. *If $E(A_1, B_1) = \emptyset$, then $N_{D_1}(B_1) = \emptyset$.*

Proof. Suppose this is not true, then the following three observations giving a contradiction.

Observation 1: If there exists $v_{3_j} \in N_{D_1}(B_1)$, then $|N_{A_1}(v_{3_j})| \geq |A_1| - 1$. Suppose this is not true, for any $v_{3_j} \in N_{D_1}(B_1)$, there exists $v_{1_1}, v_{1_2} \in A_1$ such that $v_{1_1}v_{3_j}, v_{1_2}v_{3_j} \notin E(G)$ and $v_{2_1}v_{3_j} \in E(G)$, then $\langle v_{1_1}, v_{1_2}, w_{1_1}; w_{2_1}v_{2_1}v_{3_j} \rangle$ is a Z_3, showing a contradiction.

Thus, we may assume there exists $v_{3_1} \in D_1$ such that $v_{3_1}v_{2_1} \in E(G)$ and let $\{v_{1_1}, v_{1_2}, \cdots, v_{1_{t_1-1}}\} \subseteq N_{A_1}(v_{3_1})$.

Observation 2: If there exists $v_{3_j} \in N_{D_1}(B_1)$ such that $|N_{A_1}(v_{3_j})| \geq 2$, then,

1) $N_{D_2}(B_2) = \emptyset$ and $|A_2 \setminus A_2'| \geq 3$;

2) $|B_2| = 1$;

3) $|B_1| = 1$ and;

4) $|N_{D_1}(B_1)| = 1$.

We may assume there exist $v_{3_1} \in N_{D_1}(B_1)$ and $v_{1_1}, v_{1_2} \in A_1$, such that $v_{1_1}v_{3_1}, v_{1_2}v_{3_1} \in E(G)$.

1) If there exists $w_{2_1} \in B_2$ and $w_{3_1} \in D_2$ such that $w_{3_1}w_{2_1} \in E(G)$, then $\langle v_{1_1}, v_{1_2}, v_{3_1}; v_{2_1}w_{2_1} w_{3_1}\rangle$ is a Z_3. Thus $N_{D_2}(B_2) = \emptyset$, which implies $N_{D_2}(A_2') = \emptyset$ by Claim 9.0.4. Since G is 3-connected and $D_2 \neq \emptyset$, we have $|A_2 \setminus A_2'| \geq 3$.

2) If there exist $w_{2_1}, w_{2_2} \in B_2$, then $\langle w_{2_1}, w_{2_2}, v_{2_1}; v_{3_1}v_{1_1}w_{1_{s_1}}\rangle$ is a Z_3, where $w_{1_{s_1}} \in A_2 \setminus A_2'$.

3) Suppose this is not true, there exist $v_{2_1}, v_{2_2} \in B_1$. We can also assume there exists $w_{3_1} \in N_{D_2}(w_{1_{s_1}})$, then $\langle v_{2_1}, v_{2_2}, w_{2_1}; w_{1_1}w_{1_{s_1}}w_{3_1}\rangle$ is a Z_3, showing a contradiction. Thus, $B_1 = \{v_{2_1}\}$.

4) If there exist $v_{3_1}, v_{3_2} \in N_{D_1}(v_{2_1})$, then $\langle v_{3_1}, v_{3_2}, v_{2_1}; w_{2_1}w_{1_1}w_{1_{s_1}}\rangle$ is a Z_3, showing a contradiction.

Observation 3: If for any $v_{3_j} \in N_{D_1}(B_1)$, $|N_{A_1}(v_{3_j})| = 1$, then

1) $|B_2| = 1$;

2) $A_2 \cup B_2$ is a clique;

3) $|N_{D_2}(A_2 \cup B_2)| = 1$ and $D_2 = N_{D_2}(A_2 \cup B_2)$;

4) $|B_1| = 1$, and;

5) $|N_{D_1}(B_1)| = 1$.

By Observation 1, we can see that $A_1 = \{v_{1_1}, v_{1_2}\}$ and always assume $v_{3_1}v_{1_1} \in E(G)$ and $v_{3_1}v_{1_2} \notin E(G)$.

1) If there exist $w_{2_1}, w_{2_2} \in B_2$, then $\langle w_{2_1}, w_{2_2}, v_{2_1}; v_{3_1}v_{1_1}v_{1_2}\rangle$ is a Z_3.

2) If there exists $w_{1_i} \in A_2$, such that $w_{1_i}w_{2_1} \notin E(G)$, then $\langle w_{1_i}, v_{1_2}, v_{1_1}; v_{3_1}v_{2_1}w_{2_1}\rangle$ is a Z_3.

3) If there exist $w_{3_1}, w_{3_2} \in N_{D_2}(A_2 \cup B_2)$, then $A_2 \cup B_2 \cup \{w_{3_1}, w_{3_2}\}$ is a clique by Claim 9.0.4. However, this will force $\langle w_{3_1}, w_{3_2}, w_{2_1}; v_{2_1}v_{3_1}v_{1_1}\rangle$ to be a Z_3. Thus $|N_{D_2}(A_2 \cup B_2)| = 1$. Since G is 3-connected, $N_{D_2}(A_2 \cup B_2)$ is not a cut vertex, we get $D_2 = N_{D_2}(A_2 \cup B_2)$.

4) We may assume there exists $v_{2_2} \in B_1$. If $v_{2_2}v_{3_1} \notin E(G)$, then $\langle w_{1_1}, v_{1_2}, v_{1_1}; v_{3_1}v_{2_1}v_{2_2}\rangle$ is a Z_3; if $v_{2_2}v_{3_1} \in E(G)$, then $\langle v_{2_1}, v_{2_2}, v_{3_1}; v_{1_1}w_{1_1}w_{3_1}\rangle$ is a Z_3, showing a contradiction. Thus

$B_1 = \{v_{2_1}\}$.

5) If there exist $v_{3_1}, v_{3_2} \in N_{D_1}(v_{2_1})$, since $\langle v_{3_2}, v_{3_1}, v_{2_1}; w_{2_1}w_{1_1}v_{1_2}\rangle$ is not a Z_3, we have $v_{3_2}v_{1_2} \in E(G)$. However, this will force $\langle v_{2_1}, v_{3_1}, v_{3_2}; v_{1_2}w_{1_1}w_{3_1}\rangle$ to be a Z_3, showing a contradiction.

From Observation 2 and 3, we can see that $|B_1| = |N_{D_1}(B_1)| = |B_2| = 1$, which means $|N_G(v_{2_1})| = 2$, contradicts to the fact that G is 3-connected. Thus $N_{D_1}(B_1) = \emptyset$. ■

Since G is 3-connected, $N_{D_1}(B_1) = \emptyset$ and $E(A_1, B_1) = \emptyset$, we have $|B_2| \geq 3$. The following claim gives us the structure of G.

Claim 9.2.2. *1) $|B_1| = 1$;*

2) Both $B_2 = B_2'$ and $A_2' \cup B_2 \cup N_{D_2}(A_2' \cup B_2)$ are cliques. In particular, $|A_2'| \geq 3$;

3) Both D_1 and $D_2 \setminus (N_{D_2}(A_2 \setminus A_2'))$ are independent sets;

4) If $D_2 \setminus N_{D_2}(A_2 \cup B_2) \neq \emptyset$, then it is an independent set.

Proof. We may denote $v_{3_1}v_{1_1} \in E(G)$ since $D_1 \neq \emptyset$.

1) If there exist $v_{2_1}, v_{2_2} \in B_1$, then $\langle v_{2_1}, v_{2_2}, w_{2_1}; w_{1_1}v_{1_1}v_{3_1}\rangle$ is a Z_3.

2) If there exists $w_{2_j} \in B_2 \setminus B_2'$, then $\langle v_{2_1}, w_{2_j}, w_{2_1}; w_{1_1}v_{1_1}v_{3_1}\rangle$ is a Z_3. Thus $\{w_{1_1}\} \cup B_2$ is a clique. Similarly, we can show that $A_2' \cup B_2$ is a clique. By Claim 9.0.4, we have $A_2' \cup B_2 \cup N_{D_2}(A_2' \cup B_2)$ is a clique. Since G is 3-connected, we have $|A_2'| \geq 3$.

3) Suppose to the contrary, there exist $v_{3_1}, v_{3_2} \in D_1$ such that $v_{3_1}v_{3_2} \in E(G)$. Since $\langle w_{2_1}, w_{2_2}, w_{1_1}; v_{1_1}v_{3_1}v_{3_2}\rangle$ is not a Z_3, we have $v_{3_2}v_{1_1} \in E(G)$. This in turn gives that $\langle v_{3_1}, v_{3_2}, v_{1_1}; w_{1_1}w_{2_1}v_{2_1}\rangle$ is a Z_3. Therefore, D_1 is an independent set. Similarly, we can show that $D_2 \setminus (N_{D_2}(A_2 \setminus A_2'))$ is also an independent set if it is not empty.

4) Let $N_{D_2}(A_2' \cup B_2) = \{w_{3_1}, w_{3_2}, \cdots, w_{3_{l_1}}\}$ and $N_{D_2}(A_2 \setminus A_2') = \{w_{3_{l_1+1}}, w_{3_{l_1+2}}, \cdots, w_{3_{l_2}}\}$. If there exists $w_{3_j} \in D_2 \setminus N_{D_2}(A_2 \cup B_2)$, then $w_{3_j}w_{3_i} \notin E(G)$ for any $i \in [l_1+1, l_2]$. Otherwise $\langle w_{2_1}, w_{2_2}, w_{1_1}; w_{1_{s_1}}w_{3_i}w_{3_j}\rangle$ is a Z_3, where $w_{3_i}w_{1_{s_1}} \in E(G)$. Thus we may let $w_{3_1}w_{3_{l_2+1}} \in E(G)$. If there exists $w_{3_j}w_{3_{l_2+1}}$ in $E(G)$, where $w_{3_j} \in D_2 \setminus (N_{D_2}(A_2 \cup B_2) \cup \{w_{3_{l_2+1}}\})$, since $\langle v_{1_1}, v_{1_2}, w_{1_1}; w_{3_1}w_{3_{l_2+1}}w_{3_j}\rangle$ is not a Z_3, we have $w_{3_1}w_{3_j} \in E(G)$. However, this will force $\langle w_{3_j}, w_{3_{l_2+1}}, w_{3_1}; w_{1_1}v_{1_1}v_{3_1}\rangle$ to be a Z_3, giving a contraction. Thus $D_2 \setminus N_{D_2}(A_2 \cup B_2)$ is an independent set. ■

Now we denote by $D_1 = \{v_{3_1}, v_{3_2}, \cdots, v_{3_{t_3}}\}$, $\{v^1_{3_i}, v^2_{3_i}, v^3_{3_i}\} \subseteq N_{A_1}(v_{3_i})$ for all $i \in [1, t_3]$, $A''_1 = A_1 \setminus \cup_{i=1}^{t_3}\{v^1_{3_i}, v^2_{3_i}, v^3_{3_i}\} = \{v'_{1_1}, v'_{1_2}, \cdots, v'_{1_{t_1}}\}$, $D'_{2_1} = \{w_{3_1}, w_{3_2}, \cdots, w_{3_{l_1}}\} = N_{D_2}(A'_2 \cup B'_2)$, $D_{2_2} = \{w_{3_{l_1+1}}, w_{3_{l_1+2}}, \cdots, w_{3_{l_2}}\} = N_{D_2}(A_2 \setminus A'_2)$, $D'_{2_3} = D_2 \setminus N_{D_2}(A_2 \cup B_2) = \{w_{3_{l_2+1}}, \cdots, w_{3_{s_2}}\}$, $\{w_{3_i}, w^2_{3_i}, w^3_{3_i}\} \subseteq N_{A_2}(w_{3_i})$ for all $i \in [l_1 + 1, s_2]$, $A''_2 = \{w_{1_i} | N_{D_2}(w_{1_i}) = \emptyset\} = \{w'_{1_1}, \cdots, w'_{1_{t_1}}\}$, $D'_{2_1} = D_{2_1} \setminus \cup_{i=l_2+1}^{s_2}\{w^1_{3_i}, w^2_{3_i}, w^3_{3_i}\}$. Since A''_1, A''_2, D'_{2_1}, $B_2 \setminus \{w_{2_1}, w_{2_2}\}$ and $(A'_2 \setminus \{w_{1_1}, w_{1_3}\}) \cup \{w_{2_1}\}$ are cliques, there exists hamiltonian paths, say $P_1 = v'_{1_1} P_1 v'_{1_{t_1}}$, $P_2 = w'_{1_1} P_2 w'_{1_{t_1}}$, $P_3 = w_{2_3} P_3 w_{2_{s_2}}$ and $P_4 = w_{2_1} P_4 w_{1_2}$, in them, respectively. Let $C = \cup_{i=1}^{t_3}\{v^1_{3_i} v_{3_i}, v_{3_i} v^3_{3_i}, v^3_{3_i} v^1_{3_{i+1}}\} \cup (\cup_{i=l+1}^{s_2}\{w^1_{3_i} w_{3_i}, w_{3_i} w^3_{3_i}, w^3_{3_i} w^1_{3_{i+1}}\}) \cup (\cup_{i=1}^4 P_i) \cup \{w_{2_2} v_{2_1}, v_{2_1} w_{2_1}, w_{1_2} v'_{3_1}\}$ be a cycle and all vertices on the tree $\langle v^2_{3_1}, w_{1_1}; v^2_{3_1}, \cdots, v^{t_3}_{3_1}, w^2_{3_{l_2}}, w^2_{3_{l_2+1}}, \cdots, w^2_{3_{s_2}}, w_{2_2}\rangle$ be stems of T with $N_C(v^2_{3_i}) = \{v^1_{3_i}, v^3_{3_i}, v_{3_i}\}$, $i \in [1, t_3]$, $N_C(w^2_{3_j}) = \{w^1_{3_j}, w^3_{3_j}, w_{3_j}\}$, $j \in [l_1+1, s_2]$, $N_C(w_{1_1}) = (A'_2 \setminus \{w_{1_1}\}) \cup D'_{2_3}$, $N_C(w_{2_2}) = (B_2 \setminus \{w_{2_2}\}) \cup \{v_{2_1}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 9.4 as an example).

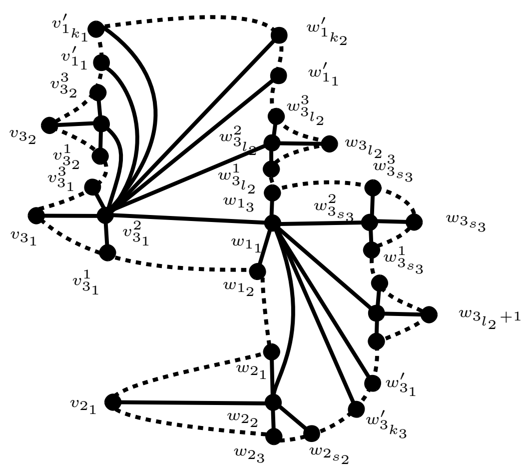

Figure 9.4. $E(A_1, B_1) = \emptyset$.

Case 2: Suppose that $E(A_1, B_1) \neq \emptyset$.

Denote by $A'_1 = \{v_{1_i} \in A_1 | E(v_{1_i}, B_1) \neq \emptyset\} = \{v_{1_1}, v_{1_2}, \cdots, v_{1_{l_1}}\} \subseteq A_1$ and $B'_1 = \{v_{2_i} \in B_1 | E(v_{2_i}, A_1) \neq \emptyset\} = \{v_{2_1}, v_{2_2}, \cdots, v_{2_{l_2}}\} \subseteq B_1$, where $l_1 \leq t_1$ and $l_2 \leq t_2$. In particular, we always assume $v_{1_1} v_{2_1} \in E(G)$.

We want to find a strong spanning Halin subgraph in G by following two subcases.

Case 2.1: Assume $N_{D_2}(B_2) \neq \emptyset$.

Claim 9.2.3. $N_{D_2}(B_2 \setminus B_2') = \emptyset$.

Proof. Suppose this is not true, there exist $w_{2_{s_2}} \in B_2 \setminus B_2'$ and $w_{3_1} \in D_2$ such that $w_{2_{s_2}} w_{3_1} \in E(G)$, then $\langle v_{1_1}, v_{1_2}, w_{1_1}; w_{2_1} w_{2_{s_2}} w_{3_1} \rangle$ is a Z_3. ■

We let $w_{2_1} w_{3_1} \in E(G)$, where $w_{2_1} \in B_2'$ and $w_{3_1} \in D_2$.

Claim 9.2.4. *1) For any $v_{2_i} \in B_1'$, we have $|N_{A_1}(v_{2_i})| \geq |A_1| - 1$, which implies $|A_1 \setminus A_1'| \leq 1$. 2) $N_{D_1}(B_1') \neq \emptyset$ and $|A_2 \setminus A_2'| \leq 1$.*

Proof. 1) For any $v_{2_i} \in B_1'$, if there exist $v_{1_1} \in A_1'$ and $v_{1_k}, v_{1_l} \in A_1 \setminus A_1'$ such that $v_{1_1} v_{2_i} \in E(G)$ and $v_{1_k} v_{2_i}, v_{1_l} v_{2_i} \notin E(G)$, then $\langle v_{1_k}, v_{1_l}, v_{1_1}; v_{2_i} w_{2_1} w_{3_1} \rangle$ is a Z_3. Thus $|N_{A_1}(v_{2_i})| \geq |A_1| - 1$, which implies $|A_1 \setminus A_1'| \leq 1$.

2) If $N_{D_1}(B_1') = \emptyset$, by Claim 9.0.4, $N_{D_1}(A_1') = \emptyset$. Since $|A_1 \setminus A_1'| \leq 1$, $D_1 \neq \emptyset$ and G is 3-connected, we can assume $v_{2_{t_2}}, v_{2_{t_2-1}} \in B_2 \setminus B_2'$, then $\langle v_{2_{t_2}}, v_{2_{t_2-1}}, v2_1; v_{1_1} w_{1_1} w_{3_1} \rangle$ is a Z_3. Similarly as 1), we can also get $|A_2 \setminus A_2'| \leq 1$. ■

Claim 9.2.5. $D_i = N_{D_i}(A_i \cup B_i)$*, where* $i \in [1, 2]$.

Proof. We only show this is true for $i = 2$. Since $N_{D_1}(B_1') \neq \emptyset$, by Claim 9.2.4, we can assume there exists $v_{3_1} \in N_{D_1}(v_{1_1}) \cap N_{D_1}(v_{2_1})$. If there exist $w_i \in A_2 \cup B_2$, $w_j \in N_{D_2}(w_i)$ and $w_k \in N_{D_2}(w_j) \setminus (A_2 \cup B_2)$, then either $\langle v_{1_1}, v_{3_1}, v_{2_1}; w_i w_j w_k \rangle$ or $\langle v_{2_1}, v_{3_1}, v_{1_1}; w_i w_j w_k \rangle$ is a Z_3, showing a contradiction. ■

Claim 9.2.6. *If there exists $i \in [1, 2]$, such that $A_i \cup B_i$ is a clique, then both D_i and $A_{3-i}' \cup B_{3-i}' \cup D_{3-i}$ are cliques.*

Proof. We only show this is true for $i = 1$. If $A_1 \cup B_1$ is a clique, then $A_1 \cup B_1 \cup D_1$ is a clique by Claim 9.0.4 and 9.2.5. Moreover, $|B_2 \setminus B_2'| \leq 1$. Otherwise, let $w_{2_j}, w_{2_k} \in B_2 \setminus B_2'$, then $\langle w_{2_j}, w_{2_k}, w_{2_1}; w_{1_1} v_{1_1} v_{3_1} \rangle$ is a Z_3. Thus $|(A_2 \setminus A_2') \cup (B_2 \setminus B_2')| \leq 2$. Furthermore, we can show that if $N_{D_2}(A_2 \setminus A_2') \neq \emptyset$, then $N_{D_2}(A_2 \setminus A_2') \subseteq N_{D_2}(A_2' \cup B_2')$. Otherwise, assume there exist $w_{3_1} \in D_2$, $w_{1_{s_1}} \in A_2 \setminus A_2'$ and $w_{2_{s_2}} \in B_2 \setminus B_2'$ such that

$w_{1_{s_1}}w_{3_1}, w_{2_{s_2}}w_{3_1} \in E(G)$. Since G is 3-connected, there exists $w_{3_2} \in D_2$ such that $w_{3_1}w_{3_2} \in E(G)$. Since $\langle w_{3_1}, w_{3_2}, w_{1_{s_1}}; v_{1_1}v_{2_1}w_{2_1}\rangle$ is not a Z_3, we have $w_{3_2}w_{2_1} \in E(G)$, this will force $\langle w_{1_{s_1}}, w_{3_1}, w_{3_2}; w_{2_1}v_{2_1}v_{3_1}\rangle$ to be a Z_3, showing a contradiction. Thus $D_2 = N_{D_2}(A_2' \cup B_2')$. Since $N_{D_1}(B_1) \neq \emptyset$, similarly as Claim 9.2.4, we can show that $|N_{A_2}(w_{2_k})| \geq |A_2| - 1$ for all $w_{2_k} \in B_2'$ or $|N_{B_2}(w_{1_k})| \geq |B_2| - 1$ for all $w_{1_k} \in A_2'$. By Claim 9.0.4, we know $A_2' \cup B_2' \cup D_2$ is also a clique. ■

Now we want to find a strong spanning Halin subgraph in G by following three subcases.

Subcase 1: Assume $|A_2| \neq 2$. Since $(A_1 \setminus \{v_{1_1}\}) \cup B_1 \cup D_1$, $(A_2 \setminus \{w_{1_1}, w_{1_{s_1}}\}) \cup D_2$ and $B_2 \setminus \{w_{2_1}\}$ are cliques, there exist hamiltonian paths, say $P_1 = v_{2_1}P_1v_{1_2}$, $P_2 = w_{1_2}P_2w_{3_1}$ and $P_3 = w_{2_2}P_3w_{2_3}$, in them, respectively. Let $C = P_1P_2P_3 \cup \{v_{1_2}w_{1_{s_1}}, w_{1_{s_1}}w_{1_2}\}$ be a cycle and all vertices on the path $v_{1_1}w_{1_1}w_{2_1}$ be stems of T with $N_C(v_{1_1}) = V(P_1)$, $N_C(w_{1_1}) = V(P_2) \cup \{w_{1_{s_1}}\}$ and $N_C(w_{2_1}) = V(P_3)$.

Subcase 2: Assume $|B_2| = 2$ and $|A_2| \neq 1$, since both $(A_1 \setminus \{v_{1_1}\}) \cup B_1 \cup D_1$ and $(A_2 \setminus \{w_{1_1}, w_{1_{s_1}}\}) \cup D_2$ are cliques, there exist hamiltonian paths, say $P_1 = v_{2_1}P_1v_{1_2}$ and $P_2 = w_{1_2}P_2w_{3_1}$ in them, respectively. Let $C = P_1P_2 \cup \{v_{1_2}w_{1_{s_1}}, w_{1_{s_1}}w_{1_2}, w_{3_1}w_{2_2}, w_{2_2}v_{2_1}\}$ be a cycle and all vertices on the path $v_{1_1}w_{1_1}w_{2_1}$ be stems with $N_C(v_{1_1}) = V(P_1)$, $N_C(w_{1_1}) = (V(A_2) \setminus \{w_{1_1}\}) \cup \{w_{1_{s_1}}\}$ and $N_C(w_{2_1}) = V(D_2) \cup \{w_{2_2}\}$.

Subcase 3: Assume $|B_2| = 2$ and $|A_2| = 1$. Since G is 3-connected, $|B_1| \geq 2$ Since $(A_1 \setminus \{v_{1_1}\}) \cup D_1$, $B_1 \setminus \{v_{2_1}\}$ and $(B_2 \setminus \{w_{2_1}\}) \cup D_2$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_2}P_1v_{3_1}$, $P_2 = v_{2_2}P_2v_{2_3}$ and $P_3 = w_{2_2}P_3w_{3_1}$, in them, respectively. Let $C = P_1P_2P_3 \cup \{v_{1_2}w_{1_1}, w_{1_1}w_{3_1}\}$ be a cycle and all vertices on the path $v_{1_1}v_{2_1}w_{2_1}$ be stems with $N_C(v_{1_1}) = V(P_1) \cup \{w_{1_1}\}$, $N_C(v_{2_1}) = V(P_2)$ and $N_C(w_{2_1}) = V(D_2) \cup \{w_{2_2}\}$.

Let $H = T \cup C$, then H is a spanning Halin subgraph of G (See Figure 9.5 as examples).

If neither $A_1 \cup B_1$ nor $A_2 \cup B_2$ is a clique, since $N_{D_2}(B_2 \setminus B_2') = \emptyset$, by Claim 9.2.3, we have $N_{D_2}(A_2) \neq \emptyset$. Moreover, we have $N_{D_1}(B_1) \neq \emptyset$ by Claim 9.2.4. Thus we only need to consider $A_1 \setminus A_1' \neq \emptyset$.

Claim 9.2.7. *If $A_1 \setminus A_1' = \{v_{1_{t_1}}\}$, we have following conclusions.*

1) $A_1' \cup B_1' \cup N_{D_1}(A_1' \cup B_1')$ is a clique;

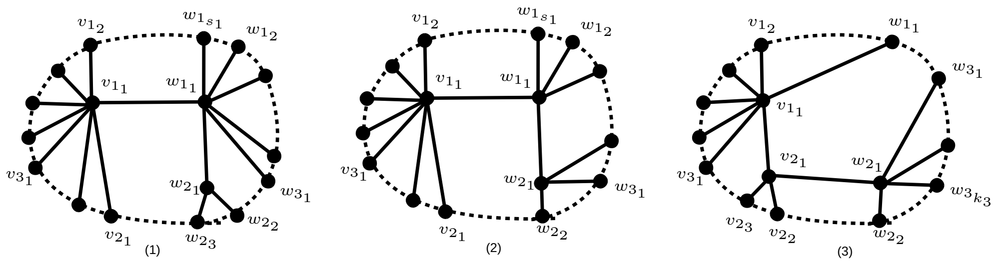

Figure 9.5. $E(A_1, B_1) \neq \emptyset$ and $N_{D_2}(B_2) \neq \emptyset$.

2) $N_{D_2}(B_2) \subseteq N_{D_2}(w_{1_i})$ for all $w_{1_i} \in A_2$. In particular, $N_{D_2}(B_2) = N_{D_2}(w_{2_j}) = \{w\}$ for all $w_{2_j} \in B_2'$;

3) $N_{D_2}(A_2) = \{w\}$, which means $D_2 = \{w\}$;

4) If $A_2 \cup B_2$ is not a clique, then $D_1 = \{v\}$.

Proof. 1) Since $v_{1_{t_1}} \in A_1 \setminus A_1'$, by Claim 9.2.4, $N_{A_1}(v_{2_j}) = A_1'$ for all $v_{2_j} \in B_1'$. This in turn gives $A_1' \cup B_1'$ is a clique. By Claim 9.0.4 and Claim 9.2.5, we get $A_1' \cup B_1' \cup D_1$ is also a clique.

2) Since $N_{D_2}(B_2 \setminus B_2') \neq \emptyset$, if there exists $w \in N_{D_2}(w_{2_j})$ for some $w_{2_j} \in B_2'$, then $ww_{1_i} \in E(G)$ for all $w_{1_i} \in N_{A_2}(w_{2_j})$. If there exists $w_{1_k} \in A_2 \setminus N_{A_2}(w_{2_j})$, then to avoid $\langle w_{1_k}, v_{1_{t_1}}, v_{1_1}; v_{2_1}, w_{2_j}, w\rangle$ be Z_3, we have $ww_{1_i} \in E(G)$. Therefore, $N_{D_2}(B_2) \subseteq N_2(w_{1_i})$ for all $w_{1_i} \in A_2$, which means $N_{D_2}(w_{2_i}) = N_{D_2}(w_{2_j})$ for all $w_{2_i}, w_{2_j} \in B_2'$. For any $w_{2_i} \in B_2$, $|N_{D_2}(w_{2_i})| = 1$. Otherwise, let $\{w, w'\} \subseteq N_{D_2}(w_{2_i})$, then $\langle w, w', w_{2_i}; v_{2_1}v_{1_1}v_{1_{t_1}}\rangle$ is a Z_3. Thus $N_{D_2}(B_2)| = |N_{D_2}(w_{2_i})| = 1$ for all $w_{2_i} \in B_2$ since $N_{D_2}(B_2 \setminus B_2') \neq \emptyset$. We may assume $N_{D_2}(B_2) = \{w\}$.

3) It is clearly true if $A_2 = A_2'$ by Claim 9.0.4. If there exists $w_{1_{s_1}} \in A_2 \setminus A_2'$ and $w' \in D_2 \setminus \{w\}$ such that $w'w_{1_{s_1}} \in E(G)$, then $ww' \in E(G)$ and $\langle w', w_{1_{s_1}}, w; w_{2_1}, v_{2_1}, v\rangle$ is a Z_3, where $v \in N_{D_1}(v_{2_1})$ since $N_{D_1}(B_1) \neq \emptyset$ by Claim 9.2.4. This implies $D_2 = \{w\}$ since $N_{D_2}(A_2 \cup B_2) = \{w\}$ and G is 3-connected.

4) If $|A_2 \setminus A_2'| = 1$, similarly as 3), we can show that $D_1 = \{v\}$. Otherwise $A_2 = A_2'$. If $A_2 \cup B_2$ is not a clique, we can assume $w_{1_1}w_{2_{s_2}} \notin E(G)$, then $|N_{D_1}(v_{1_i})| \leq 1$ for all $v_{1_i} \in A-1$,

otherwise, let $v, v' \in N_{D_1}(v_{1_i})$, then $\langle v, v', v_{1_1}; w_{1_1}w_{2_1}w_{2_{s_2}}\rangle$ is a Z_3. This in turn gives that $|N_{D_1}(A'_1 \cup B'_1)| \leq 1$. Since $N_{D_2}(A_2) \neq \emptyset$, we have $|B_1 \setminus B_1| \leq 1$. Because G is 3-connected, $(A_1 \setminus A'_1) \cup (B_1 \setminus B'_1)$ is not a vertex cut, by Claim 9.2.5, we get $D_1 = \{w\}$. ■

Similarly as $N_{D_2}(B_2) \neq \emptyset$, we can find a strong spanning Halin subgraph in G.

Case 2.2: Suppose $N_{D_2}(B_2) = \emptyset$.

We will use following claim to describe the structure of G.

Claim 9.2.8. *If $N_{D_2}(B_2) = \emptyset$, then*

1) $|A_2 \setminus A'_2| \geq 3$;

2) $|B_1| = 1$, $N_{D_1}(B_1) = \emptyset$;

3) $|B_2| = 1$;

4) $|A'_1| \geq 2$ and $|A'_2| \geq 2$;

5) For any $v_{1_j} \in A_i \setminus A'_i$, we have $|N_{D_i}(v_{1_j})| \leq 1$, where $i \in [1, 2]$;

6) For any $v' \in D_i$ and $v_{1_j} \in A_i \setminus A'_i$, we have $dist(v', v_{1_j}) \leq 2$, where $i \in [1, 2]$;

7) For any connected component $D_{i_j} \subseteq D_i$, we have $|D_{i_j}| \leq 2$, where $i \in [1, 2]$ and $j \in [1, k_1] \cup [1, k_2]$;

8) For any $v_{3_i} \in D_i$, we have $|N_{A_i}(v_{3_i})| \geq 2$, where $i \in [1, 2]$.

Proof. 1) Since $N_{D_2}(B_2) = \emptyset$, by Claim 9.0.4, $N_{D_2}(A'_2) = \emptyset$. Thus $|A_2 \setminus A'_2| \geq 3$ because of G is 3-connected and $D_2 \neq \emptyset$. We can always assume $w_{1_{s_1}}, w_{1_{s_1-1}} \in A_2 \setminus A'_2$, $w_{1_1}w_{2_1} \in E(G)$ and $w_{1_{s_1}}w_{3_1} \in E(G)$.

2) If there exist $v_{2_1}, v_{2_2} \in B_1$, then $\langle v_{2_1}, v_{2_2}, w_{2_1}; w_{1_1}w_{1_{s_1}}w_{3_1}\rangle$ is a Z_3. If there exists $v_{3_1} \in N_{D_1}(v_{2_1})$, then $\langle w_{1_{s_1}}, w_{1_{s_1-1}}, w_{1_1}; w_{2_1}v_{2_1}v_{3_1}\rangle$ is a Z_3.

3) Since $N_{D_1}(B_1) = \emptyset$, similarly as 2), we can show that $|B_2| = 1$.

4) Since G is 3-connected, $deg_G(v_{2_1}) \geq 3$ and $deg_G(w_{2_1}) \geq 3$. This in turn gives $|A'_1| \geq 2$ and $|A'_2| \geq 2$.

5) We only show this is true for $i = 1$. If there exist $v_{1_j} \in A_1 \setminus A'_1$ and $v, v' \in N_{D_1}(v_{1_j})$, then $vv' \in E(G)$ since $\langle v_{1_j}; w_{1_1}, v, v'\rangle$ is not a claw. Which in turn gives $\langle v, v', v_{1_j}; v_{1_1}v_{2_1}w_{2_1}\rangle$ is a Z_3, showing a contradiction.

6) If there exist $v' \in D_i$ and $v_{1_j} \in A_i \setminus A_i'$, such that $dist(v', v_{1_j}) = 3$. We may assume $v'v'', v''v, vv_i \in E(G)$, then $\langle w_{1_1}, w_{1_2}, v_{1_j}; vv''v' \rangle$ is a Z_3.

7) If there exist $v, v', v'' \in D_{i_j}$, we may assume $vv', v'v'' \in E(G)$, then there does not exist $v_{1_k} \in A_i \setminus A_i'$ such that $v_{1_k}v' \in E(G)$. Otherwise, either $\langle v'; v_{1_k}, v, v'' \rangle$ is a claw or $\langle v'', v, v'; v_{1_k}w_{1_j}w \rangle$ is a Z_3, where $w_{1_j} \in A_{3-i}$ and $w \in N_{D_{3-i}}(w_{1_j})$. Moreover, there does not exist $v_{1_k} \in A_i \setminus A_i'$ such that $v_{1_k}v \in E(G)$(or $v_{1_k}v'' \in E(G)$). Otherwise $\langle v'', v', v; v_{1_k}w_{1_j}w \rangle$ or $\langle w_{1_j}, w_{1_i}, v_{1_k}; vv'v'' \rangle$ is a Z_3, showing a contradiction.

8) This is true if $D_{i_j} = \{v_{3_i}\}$ since G is 3-connected. If $D_{i_j} = \{v_{3_k}, v_{3_l}\}$, since $N_{A_i}(v_{3_k}) \cap N_{A_i}(v_{3_l}) = \emptyset$ and G is 3-connected, we have $|N_{A_i}(v_{3_k})| \geq 2$ and $|N_{A_i}(v_{3_l})| \geq 2$. ■

We may denote by $D_{i_j} = \{v_{3_i}\}$ for $i \in [1, l_1]$, $D_{1_j} = \{v_{3_{j_1}}, v_{3_{j_2}}\}$ for $j \in [l_1+1, k_1]$, $D_{2_i} = \{w_{3_i}\}$ for $i \in [1, l_2]$, $D_{2_j} = \{w_{3_{j_1}}, w_{3_{j_2}}\}$ for $j \in [l_2+1, k_2]$, $N_{A_1}(v_{3_i}) = \{v_i^1, v_i^2, v_i^3\}$, $N_{A_1}(v_{3_{j_1}}) = \{v_{j_1}^1, v_{j_1}^2\}$, $N_{A_1}(v_{3_{j_2}}) = \{v_{j_2}^1, v_{j_2}^2\}$, $N_{A_2}(w_{3_i}) = \{w_i^1, w_i^2, w_i^3\}$, $N_{A_2}(w_{3_{j_1}}) = \{w_{j_1}^1, w_{j_1}^2\}$, $N_{A_2}(w_{3_{j_2}}) = \{w_{j_2}^1, w_{j_2}^2\}$, $A_1'' = (A_1 \setminus (\cup_{i=1}^{l_1}\{v_i^1, v_i^2, v_i^3\}) \cup \cup_{j=l_1+1}^{k_1}\{v_{j_1}^1, v_{j_1}^2, v_{j_2}^1, v_{j_2}^2\} = \{v'_{1_1}, \cdots v'_{1_{t_1}}\}$ and $A_2'' = (A_2 \setminus (\cup_{i=1}^{l_1}\{w_i^1, w_i^2, w_i^3\}) \cup (\cup_{j=l_1+1}^{k_1}\{w_{j_1}^1, w_{j_1}^2, w_{j_2}^1, w_{j_2}^2\}) = \{w'_{1_1}, \cdots w'_{1_{t_1}}\}$. In particular, $v'_{1_1}v_{2_1}, w'_{1_1}w_{2_1} \in E(G)$.

Since A_1'', A_2'' are cliques, there exists hamiltonian paths, say $P_1 = v'_{1_1}P_1v'_{1_{t_1}}$ and $P_2 = w'_{1_1}P_2w'_{1_{t_2}}$, in them, respectively. Let $C = \cup_{i=1}^{l_1}\{v_i^1v_{3_i}, v_{3_i}v_i^3, v_i^3v_{i+1}^1\} \cup (\cup_{j=l+1}^{k_1}\{v_{j_1}^1v_{3_{j_1}}, v_{3_{j_2}}v_{j_2}^2\} \cup \cup_{i=1}^{l_1}\{w_i^1w_{3_i}, w_{3_i}w_i^3, w_i^3w_{i+1}^1\} \cup (\cup_{j=l+1}^{k_2}\{w_{j_1}^1w_{3_{j_1}}, w_{3_{j_2}}w_{j_2}^2\} \cup P_1P_2 \cup \{v'_{1_1}v_{2_1}, v_{2_1}w_{2_1}, w_{2_1}w'_{1_1}\}$ be a cycle with all vertices on the tree $\langle v_1^2, w_1^2; v_2^2, \cdots v_{l_1+1}^2, v_{(l_1+1)_2}^1, \cdots, v_{t_{1_1}}^1, v_{t_{1_2}}^2, w_2^2, \cdots w_{l_1+1}^2, w_{(l_1+1)_2}^1, \cdots, w_{t_{2_1}}^1, w_{t_{2_2}}^2 \rangle$ be stems of T with $N_C(v_i^2) = \{v_i^1, v_i^3, v_{3_i}\}$ for $i \in [2, l_1]$, $N_C(v_{j_1}^2) = \{v_{j_1}^1, v_{3_{j_1}}\}$, $N_C(v_{j_2}^1) = \{v_{j_2}^1, v_{3_{j_2}}\}$ for $j \in [l_1+1, k_1]$, $N_C(w_i^2) = \{w_i^1, w_i^3, w_{3_i}\}$ for $i \in [2, l_2]$, $N_C(w_{j_1}^2) = \{w_{j_1}^1, w_{3_{j_1}}\}$, $N_C(w_{j_2}^1) = \{w_{j_2}^1, w_{3_{j_2}}\}$ for $j \in [l_2+1, k_2]$, $N_C(v_1^2) = \{v_1^1, v_1^3, v_{3_1}\} \cup A_1'' \cup \{v_{2_1}\}$ and $N_C(w_1^2) = \{w_1^1, w_1^3, w_{3_1}\} \cup A_2'' \cup \{w_{2_1}\}$. Let $H = T \cup C$, then H is a spanning Halin subgraph of G (See Figure 9.6 as examples).

9.3 Proof of 3-connected $\{claw, B_{1,2}\}$-free graphs admit generalized 2-joins

In this subsection, we always assume G is a 3-connected $\{claw, B_{1,2}\}$-free graph admits a generalized 2-join. Since G is connected and $D_2 \neq \emptyset$, we assume $N_{D_2}(B_2) \neq \emptyset$ because we can

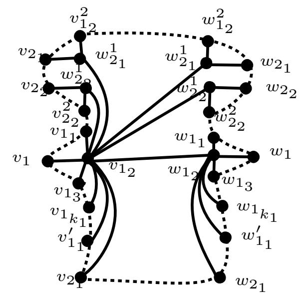

Figure 9.6. $E(A_1, B_1) \neq \emptyset$ and $N_{D_2}(B_2) = \emptyset$

similarly find a strong spanning Halin subgraph in G if $N_{D_2}(A_2) \neq \emptyset$. Let $w_{2_1}w_{3_1} \in E(G)$, then we have following claims.

Claim 9.3.1. *If $N_{D_1}(A_1) \neq \emptyset$, then both $A_1 \cup D_1$ and $B_1 \cup D_1$ are cliques.*

Proof. We may assume there exists $v_{3_1} \in D_1$ such that $v_{3_1}v_{1_1} \in E(G)$. For any $v_{1_i} \in A_1 \setminus \{v_{1_1}\}$, since $\langle v_{3_1}; v_{1_1}, v_{1_i}, u_1; w_{2_1}w_{3_1}\rangle$ is not a $B_{1,2}$, we have $v_{3_1}v_{1_i} \in E(G)$. Thus $N_{D_1}(A_1) \cup A_1$ is a clique. Moreover, for any $v_{2_j} \in B_1$ and $v_{3_k} \in N_{D_1}(A_1)$, since $\langle w_{3_1}; w_{2_1}, v_{2_j}, u_1; v_{1_1}v_{3_k}\rangle$ is not a $B_{1,2}$ and $\langle v_{1_1}; v_{3_k}, w_{1_1}, v_{2_j}\rangle$ is not a claw, we have $v_{3_k}v_{2_j} \in E(G)$. Therefore, $N_{D_1}(A_1) \subseteq N_{D_1}(v_{2_j})$ for any $v_{2_j} \in B_2$. Similarly, we can show that $N_{D_2}(B_2) \subseteq N_{D_2}(A_2)$, which implies $N_{D_2}(A_2) \neq \emptyset$ and $N_{D_1}(B_1) \subseteq N_{D_1}(A_1)$.This in turn gives $N_{D_1}(B_1) = N_{D_1}(A_1)$ and $N_{D_2}(B_2) = N_{D_2}(A_2)$. Furthermore, there does not exist $v_{3_j} \in D_1$ and $v_{2_i} \in B_1$, such that $dist(v_{3_j}, v_{2_i}) = 2$. Otherwise, assume $v_{3_j}v_{3_1} \in E(G)$, then $\langle w_{3_1}; w_{2_1}, u_1, v_{2_i}; v_{3_1}v_{3_j}\rangle$ is a $B_{1,2}$. Since G is connected, we have $D_1 = N_{D_1}(B_1)$, which in turn gives us both $A_1 \cup D_1$ and $B_1 \cup D_1$ are cliques. ■

Corollary 9.3.1. *Both $A_2 \cup D_2$ and $B_2 \cup D_2$ are cliques.*

Claim 9.3.2. *If $N_{D_1}(A_1) = \emptyset$, then*

1) $N_{D_2}(A_2) = \emptyset$ and $N_{D_i}(B_i') = \emptyset$, where $i \in [1, 2]$;

2) For any component D_{i_j} of D_i, we have $D_{i_j} \cup N_{B_i \setminus B_i'}(D_{i_j})$ is a clique, where $i \in [1, 2]$;

3) $|N_{B_i}(D_{i_j})| \geq 3$ for $i \in [1, 2]$, and if $|W_0| < 3$, then $|A_1' \cup A_2'| \geq 3 - |W_0|$.

Proof. 1) If $N_{D_1}(A_1)=\emptyset$, then $N_{D_1}(B_1)\neq\emptyset$ since $D_1\neq\emptyset$. Similarly as Claim 9.3.1, we know $N_{D_2}(A_2)=\emptyset$. By Claim 9.0.4, we know $N_{D_i}(B_i')=\emptyset$ for $i\in[1,2]$.

2) We only prove for $i=1$. This is clearly true if $|D_{1_j}|=1$. If $|D_{1_j}|\geq 2$, we may assume there exist $v_{3_2},v_{3_1}\in D_{1_j}$ and $v_{2_i}\in B_1\setminus B_1'$, such that $v_{3_2}v_{3_1},v_{3_1}v_{2_i}\in E(G)$ and $v_{3_2}v_{2_i}\notin E(G)$, then $\langle w_{3_1};w_{2_1},u_1,v_{2_i};v_{3_1}v_{3_j}\rangle$ is a $B_{1,2}$, showing a contradiction. Therefore, $D_{1_j}\cup N_{B_1}(D_{1_j})$ is a clique for any component D_{1_j} of D_1.

3) This is true since G is 3-connected. ■.

Based on Claim 9.3.1 and 9.3.2, we can find a strong spanning Halin subgraph in G as follows.

If $N_{D_1}(A_1)\neq\emptyset$, since $D_1\cup(A_1\setminus\{v_{1_1}\})$, B_1, $W_0\setminus\{u_1\}$, A_2 and $D_2\cup B_2\setminus\{w_{2_1}\}$ are cliques, there exist hamiltonian paths, say $P_1=v_{1_{t_1}}P_1v_{3_1}$, $P_2=v_{2_1}P_2v_{2_{t_2}}$, $P_3=u_2P_3u_t$, $P_4=w_{1_1}P_4w_{1_{s_1}}$ and $P_5=w_{3_1}P_5w_{2_{s_2}}$, in them, respectively. Let $C=P_1P_2P_3P_4P_5$ be a cycle and all vertices on the path $v_{1_1}u_1w_{2_1}$ be stems of T with $N_C(v_{1_1})=D_1\cup\{v_{1_2}\}$, $N_C(u_1)=(A_1\setminus\{v_{1_1},v_{1_2}\})\cup B_1\cup(W_0\setminus\{u_1\})\cup A_2\cup(B_2\setminus\{w_{2_1},w_{2_2}\})$ and $N_C(w_{2_1})=\{w_{2_2}\}\cup D_2$.

If $N_{D_1}(A_1)=\emptyset$, we denote by $D_{1_i}=\{v_{3_1},v_{3_2},\cdots,v_{3_{t_i}}\}$, $v_{2_i}^1,v_{2_i}^2,v_{2_i}^3\in N_{B_1\setminus B_1'}(D_{1_i})$, $D_{2_i}=\{w_{3_1},w_{3_2},\cdots,w_{3_{t_i}}\}$ and $w_{2_i}^1,w_{2_i}^2,w_{2_i}^3\in N_{B_2\setminus B_2'}(D_{2_i})$ for all $D_{1_i}\in D_1$ and $D_{2_i}\in D_2$. Let $B_1''=B_1\setminus\cup_{i=1}^{k_1}\{v_{2_i}^1,v_{2_i}^2,v_{2_i}^3\}=\{v_{2_1}',\cdots,v_{2_{k_2}}'\}$ and $B_2''=B_2\setminus\cup_{i=1}^{k_2}\{w_{2_i}^1,w_{2_i}^2,w_{2_i}^3\}=\{w_{2_1}',\cdots,w_{2_{s_2}}'\}$. Since A_1, A_2, B_1'', B_2'' and $D_{i_j}\cup N_{B_i\setminus B_i'}(D_{i_j})$ are cliques for all $j\in[1,k_1]\cup[1,k_2]$ and $i\in[1,2]$, there exist hamiltonian paths, say $P_1=v_{1_1}P_1v_{1_{t_1}}$, $P_2=w_{1_1}P_2w_{1_{s_1}}$, $P_3=v_{2_1}'P_3v_{2_{k_2}}'$, $P_4=w_{2_1}'P_4w_{2_{s_2}}'$, $P_{5_i}=v_{2_i}^1P_{5_i}v_{2_i}^3$ and $P_{6_i}=w_{2_i}^1P_{6_i}w_{2_i}^3$, in them, respectively.

If $|W_0|\geq 3$, let $P_7=u_3P_7u_t$ be a hamiltonian path in $W_0\setminus\{u_1,u_2\}$ and $C=P_1P_2P_3P_4P_{5_1}\cdots P_{5_{k_1}}P_{6_1}\cdots P_{6_{k_2}}\cup\{v_{1_1}u_2,u_2v_{2_1},w_{1_1}u_t,u_3w_{2_1}\}$ be a cycle. If $W_0=\{u_1,u_2\}$, we can assume $w_{1_1}w_{2_1}\in E(G)$ and let $C=P_1P_2P_3P_4P_{5_1}\cdots P_{5_{k_1}}P_{6_1}\cdots P_{6_{k_2}}\cup\{v_{1_1}u_2,u_2v_{2_1}\}$ be a cycle. Set all vertices on the star $\langle u_1;v_{2_1}^2,\cdots v_{2_{k_1}}^2,w_{2_1}^2,\cdots w_{2_{k_2}}^2\rangle$ be stems of T with $N_C(u_1)=A_1\cup A_2\cup B_1''\cup B_2''\cup\{u_2\}$, $N_C(v_{2_i}^2)=V(P_{5_i})$ and $N_C(w_{2_i}^2)=V(P_{6_i})$ for $i\in[1,k_1]\cup[1,k_2]$.

If $W_0=\{u_1\}$ and $E(A_1,B_1)\neq\emptyset$, $E(A_2,B_2)\neq\emptyset$, we can similarly find a strong spanning Halin subgraph in G as $|W_0|=2$, If $W_0=\{u_1\}$ and $E(A_2,B_2)=\emptyset$(or $E(A_1,B_1)=\emptyset$), we can assume $v_{1_1}v_{2_1}',v_{1_2}v_{2_2}'\in E(G)$. Let $C=v_{1_2}P_1v_{1_{t_1}}P_2P_3P_4P_{5_1}\cdots P_{5_{k_1}}P_{6_1}\cdots P_{6_{k_2}}\cup$

$\{w_{1_1}u_2, u_2w_{2_1}\}$ be a cycle and all vertices on the star $\langle v'_{2_1}; v_{1_1}, v^2_{2_1}, \cdots v^2_{2_{k_1}}, w^2_{2_1}, \cdots w^2_{2_{k_2}}\rangle$, where v'_{2_1} is the center, be stems of T with $N_C(v'_{2_1}) = B''_1 \cup B''_2 \cup \{u_1\}$, $N_C(v_{1_1}) = A_1 \cup A_2$, $N_C(v^2_{2_i}) = V(P_{5_i})$ and $N_C(w^2_{2_i}) = V(P_{6_i})$ for $i \in [1, k_1] \cup [1, k_2]$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 9.7 as examples).

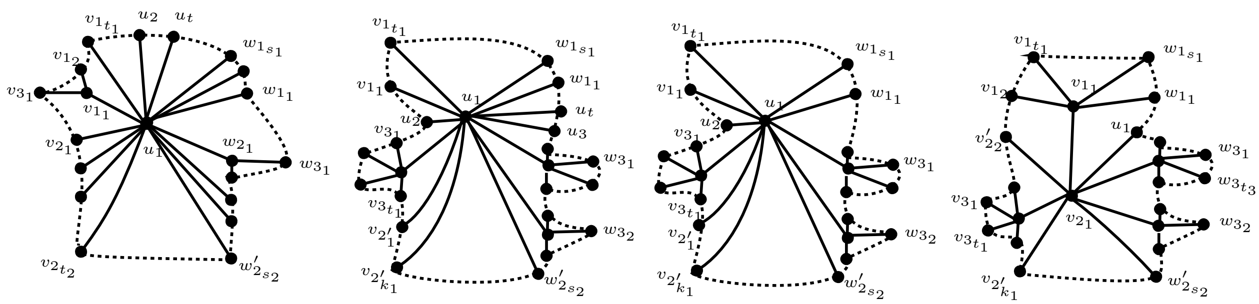

Figure 9.7. G admits a generalized 2-join

9.4 Proof of 3-connected $\{claw, B_{1,2}\}$-free graphs admit 2-joins

In this section, we assume G is a 3-connected $\{claw, B_{1,2}\}$-free graph and $W_0 = \emptyset$ which means G admits a 2-join. We will consider following two cases.

Case 1: $E(A_1, B_1) = \emptyset$ and $E(A_2, B_2) = \emptyset$.

Since G is 3-connected and $D_i \neq \emptyset$ for $i \in [1, 2]$, we can assume $N_{D_1}(A_1) \neq \emptyset$, $N_{D_2}(A_2) \neq \emptyset$ and $N_{D_2}(B_2) \neq \emptyset$. Note that if $N_{D_2}(A_2) = D_2$, then $N_{D_2}(A_2) \cap N_2(B_2) \neq \emptyset$. We have following claims.

Claim 9.4.1. *Both $N_{D_1}(A_1) \cup A_1$ and $N_{D_2}(A_2) \cup A_2$ are cliques.*

Proof. We only prove that $N_{D_1}(A_1) \cup A_1$ is a clique. If $|A_1| = 1$, then $N_{D_1}(A_1) \cup A_1$ is a clique since G is claw-free. If $|A_1| \geq 2$, we may assume there exists $v_{3_1} \in D_1$, $w' \in B_2 \cup D_2$ such that $v_{3_1}v_{1_1}, w_{3_1}w_{1_1}, w_{3_1}w' \in E(G)$ and $w_{1_1}w' \notin E(G)$. Since for any $v_{1_i} \in A_1$, $\langle v_{3_1}; v_{1_1}, v_{1_i}, w_{1_1}; w_{3_1}w'\rangle$ is not a $B_{1,2}$, we have $v_{3_1}v_{1_i} \in E(G)$, which implies $N_{A_1}(v_{3_1}) = A_1$. Therefore, $N_{D_1}(A_1) \cup A_1$ is a clique. ■

Corollary 9.4.1. *If $N_{D_1}(B_1) \neq \emptyset$, then both $N_{D_1}(B_1) \cup B_1$ and $N_{D_2}(B_2) \cup B_2$ are cliques.*

Claim 9.4.2. *If there exists $w \in D_i \setminus N_{D_i}(A_i)$ such that $dist(w, A_i) = 2$, then $N_{D_i}(A_i) \cup \{w\}$ is a clique. In particular, if we denote $D_i^j = \{w|dist(w, A_i) = j\}$, then $D_i^j \cup D_i^{j+1}$ is a clique for all possible j, where $i = 1, 2$.*

Proof. We only prove this is true for $i = 2$. This is clearly true if $|N_{D_2}(A_2)| = 1$. If $|N_{D_2}(A_2)| \geq 2$ and there exist $w \in D_2^2$, $w_{3_i}, w_{3_j} \in N_{D_2}(A_2)$ such that $w_{3_i}w \in E(G)$ and $w_{3_j}w \notin E(G)$, then $\langle w; w_{3_i}, w_{3_j}, w_{1_1}; v_{1_1}v_{3_1}\rangle$ is a $B_{1,2}$, showing a contradiction. Therefore, $D_2^1 \cup D_2^2$ is a clique. Similarly, we can show that $D_2^j \cup D_2^{j+1}$ is also a clique for all possible j. ■

We denote by $D_1^j = \{v_j^1, v_j^2, \cdots, v_j^{t_j}\}$ and $D_2^j = \{w_j^1, w_j^2, \cdots, w_j^{t_j}\}$ for all possible j.

Claim 9.4.3. *Let $i \in [1, 2]$ and k be the smallest integer such that $N_{D_i}(B_i) \cap D_i^k \neq \emptyset$, then for $i \in [1, 2]$*

1) $D_i^{k+1} = \emptyset$ and $D_i = \cup_{j=1}^k D_i^j$;

2) $D_i^k \cup N_{B_i}(D_i^k)$ is a clique;

3) If $N_{D_{3-i}}(B_{3-i}) \neq \emptyset$, then $N_{B_i}(D_i^k) = B_i$;

4) If $N_{D_{3-i}}(B_{3-i}) = \emptyset$, then $|N_{B_i}(D_i^k)| \geq 3$, $|A_i| \geq 3$, $|D_i^j| \geq 3$, $|A_{3-i}| \geq 3$ and $|D_{3-i}^j| \geq 3$ for all $j \in [1, s_i] \cup [1, s_{3-i} - 1]$, where $s_i = max\{j|dist(w, A_i) = j, w \in D_i\}$ and $s_{3-i} = max\{j|dist(w, A_{3-i}) = j, w \in D_{3-i}\}$.

Proof. We only prove this is true for $i = 2$ and denote by $D_2^0 = A_2$, $D_2^{-1} = A_1$ and $w_k^1 w_{2_1} \in E(G)$.

1) If $D_i^{k+1} \neq \emptyset$, since $\langle w_k^1; w_{2_1}, w_{k-1}^1, w_{k+1}^1\rangle$ is not a claw, we have $w_{k+1}^1 w_{2_1} \in E(G)$. However, this will force $\langle v_{2_1}; w_{2_1}, w_{k+1}^1, w_k^1; w_{k-1}^1 w_{k-2}^1\rangle$ to be a $B_{1,2}$. If there exists $w \in D_i \setminus (\cup_{j=1}^k D_i^j)$ such that $ww_{2_j} \in E(G)$, where $w_{2_j} \notin N_{B_2}(D_2^k)$, then $\langle w; w_{2_j}, v_{2_1}, w_{2_1}; w_k^1 w_{k-1}^1\rangle$ is a $B_{1,2}$, showing a contradiction.

2) For any $w_k^i \in D_2^k \setminus \{w_k^1\}$, since $\langle w_{k-2}^1; w_{k-1}^1, w_k^i, w_k^1; w_{2_1}v_{2_1}\rangle$ is not a $B_{1,2}$, $w_{2_1}w_k^i \in E(G)$. Therefore, $D_2^k \cup \{w_{2_1}\}$ is a clique. Similarly, we can show $D_2^k \cup N_{B_2}(D_2^k)$ is also a clique.

3) If $N_{D_1}(B_1) \neq \emptyset$, let $v_{3_1}v_{2_1} \in E(G)$. For any $w_{2_j} \in B_2\backslash\{w_{2_1}\}$, since $\langle v_{3_1}; v_{2_1}, w_{2_j}, w_{2_1}; w_k^1$ $w_{k-1}^1\rangle$ is not a $B_{1,2}$, we have $w_k^1 w_{2_j} \in E(G)$, which in turn gives us $N_{B_i}(D_i^k) = B_i$ is a clique.

4) This is clearly true since G is 3-connected. ■

Claim 9.4.4. *There are at most two sets in $\mathcal{S}$, $\mathcal{S} := \{A_1, A_2, B_1, B_2\} \cup (\cup_{i=1}^{s_1} D_1^i) \cup (\cup_{j=1}^{s_2} D_2^j)$, with size 1. Moreover, if there indeed exist two sets with size 1, then they are adjacent to each other.*

Proof. This is clearly true since G is 3-connected. ■

Now we want to find a strong spanning Halin subgraph in G as follows.

If $N_{D_1}(B_1) \neq \emptyset$, we may assume $N_{D_1}(B_1) \cap D_1^{k_1} \neq \emptyset$, $N_{D_2}(B_2) \cap D_2^{k_2} \neq \emptyset$ and B_1, B_2 are the only two possible sets with size 1. We denote by $D_1^0 = A_1$, $D_2^0 = A_2$. S-ince $|D_i^j| \geq 2$ and $D_i^j \setminus \{w_i^1\}$ are cliques for $i \in [1, 2]$ and $j \in [0, k_1] \cup [0, k_2]$. There exist hamiltonian paths, say $P_{i_j} = w_i^2 P_{i_j} w_i^{t_i}$, in them, respectively. Denote by $P_3 = v_{2_1} P_3 v_{2_2}$ and $P_4 = w_{2_1} P_4 w_{2_2}$ are the two hamiltonian paths in B_1 and B_2, respective-ly. Let $C = P_{1_{k_1}} P_{1_{k_1-1}} \cdots P_{1_1} P_{1_0} P_{2_0} \cdots P_{2_{k_2}} P_4 P_3$ be a cycle and all vertices on the path $v_{k_1}^1 v_{k_1-1}^1 \cdots v_1^1 v_{1_1} w_{1_1} \cdots w_{k_2}^1$ be stems of T with $N_C(v_i^1) = V(P_{1_i})$ for $i \in [0, k_1 - 1]$, $N_C(w_j^1) = V(P_{2_j})$ for $j \in [0, k_2 - 1]$, $N_C(v_{k_1}^1) = V(P_{1_{k_1}}) \cup V(P_3)$ and $N_C(w_{k_2}^1) = V(P_{2_{k_2}}) \cup V(P_4)$.

If $N_{D_1}(B_1) = \emptyset$, we denote by $\}N_{B_2}(D_2^{k_2}) = \{w_{2_1}, \cdots, w_{2_{l_2}}$ and $B_2 \setminus N_{B_2}(D_2^{k_2}) = \{w_{2_{l_2+1}}, \cdots, w_{2_{t_2}}\}$, $D_1^0 = A_1$ and $D_2^0 = A_2$. Since $|D_i^j| \geq 3$ and $D_i^j \setminus \{v_i^1, v_i^2\}$ are cliques for $i \in [1, 2]$ and $j \in [0, s_1] \cup [0, k_2]$, there exists hamiltonian paths, say $P_{i_j} = v_i^3 P_{i_j} v_i^{t_i}$, in them, respectively. Moreover, there also exist a hamiltonian path, say $P_3 = w_{2_2} P_3 w_{2_3}$ in $B_1 \cup (B_2 \setminus \{w_{2_1}\})$ since $|N_{B_2}(D_2^{k_2})| \geq 3$ and $B_1 \cup B_2$ is a clique. Let $C = P_{1_{t_1}} P_{1_{t_1-1}} \cdots P_{1_1} P_{1_0} P_{2_0} \cdots P_{2_{k_2}} P_3 \cup (\{\cup_{i=0}^{t_1-1} \{v_i^2 v_{i+1}^2\}\}) \cup (\{\cup_{i=0}^{k_2} \{w_i^2 w_{i+1}^2\}\}) \cup \{v_0^2 w_0^2, w_{2_2} w_k^2\}$ be a cycle and all vertices on the path $v_{s_1}^1 \cdots v_1^1 v_0^1 w_0^1 \cdots w_{k_2}^1 w_{2_1}$ be stems of T with $N_C(v_i^1) = V(P_{1_i})$ for $i \in [0, s_1 - 2]$, $N_C(v_{s_1-1}^1) = V(P_{1_{s_1-1}}) \cup V(P_{1_{s_1}})$ $N_C(w_j^1) = V(P_{2_j})$ for $j \in [0, k_2]$ and $N_C(w_2^1) = B_1 \cup (B_2 \setminus \{w_{2_1}\})$.

Let $H = T \cup C$, then H is a spanning Halin subgraph of G (See Figure 9.8 as examples).

Case 2: Assume $E(A_1, B_1) \neq \emptyset$ or $E(A_2, B_2) \neq \emptyset$.

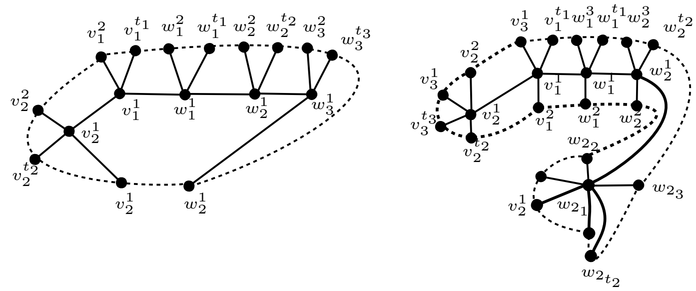

Figure 9.8. $E(A_1, B_1) = \emptyset$ and $E(A_2, B_2) = \emptyset$

Claim 9.4.5. *There exists $i \in [1, 2]$ such that $E(A_i, B_i) \neq \emptyset$ and $N_{D_i}(A_i) \cap N_{D_i}(B_i) \neq \emptyset$.*

Proof. We may assume $E(A_2, B_2) \neq \emptyset$. If $N_{D_2}(A_2) \cap N_{D_2}(B_2) = \emptyset$, we can assume there exist $w_{2_i} \in B_2 \setminus B_2'$ and $w_{3_1} \in D_2$ such that $w_{2_i}w_{3_1} \in E(G)$ by Claim 9.0.4. For any $v_{1_k} \in A_1$ and $v_{2_j} \in B_1$, since $\langle w_{3_1}; w_{2_i}, v_{2_j}, w_{2_1}; w_{1_1}v_{1_k}\rangle$ is not a $B_{1,2}$, we have $v_{1_k}v_{2_j} \in E(G)$. This implies $A_1 \cup B_1$ is a clique and $N_{D_1}(A_1) \cap N_{D_1}(B_1) \neq \emptyset$ by Claim 9.0.4 again. ■

Claim 9.4.6. *If there exists $i \in [1, 2]$ such that $E(A_i, B_i) \neq \emptyset$ and $N_{D_i}(A_i) \cap N_{D_i}(B_i) \neq \emptyset$, then*

1) Both $A_{3-i} \cup D_{3-i}$ and $B_{3-i} \cup D_{3-i}$ are cliques;

2) Both $A_i \cup D_i$ and $B_i \cup D_i$ are cliques.

Proof. We may assume $w_{1_1}w_{2_1} \in E(G)$ and $w_{3_1} \in N_{D_2}(w_{1_1}) \cap N_{D_2}(w_{2_1})$.

1) For any $v_{1_t} \in A_1$, $v_{2_s} \in B_1$ and $v_{3_j} \in N_{D_1}(v_{1_t})$, since $\langle v_{2_s}; w_{2_1}, w_{3_1}, w_{1_1}; v_{1_t}v_{3_j}\rangle$ is not a $B_{1,2}$, $v_{3_j}v_{2_s} \in E(G)$. Which implies $N_{D_1}(A_1) = N_{D_1}(B_1) = N_{D_1}(v_{1_t}) = N_{D_1}(v_{2_s})$ since v_{1_t} and v_{2_s} are arbitrary. If there exists $v' \in D_1 \setminus (N_{D_1}(A_1) \cup N_{D_1}(B_1))$ and $v \in N_{D_1}(A_1)$ such that $v'v \in E(G)$, then either $\langle v; v', v_{1_t}, v_{2_s}\rangle$ is a claw or $\langle v'; v, v_{1_1}, v_{2_1}, w_{2_1}w'\rangle$ is $B_{1,2}$, showing a contradiction. Thus $A_1 \cup D_1$ and $B_1 \cup D_1$ are cliques.

2) If $E(A_1, B_1) \neq \emptyset$, similarly as 1), we can show that both $A_2 \cup D_2$ and $B_2 \cup D_2$ are cliques. Thus we assume $E(A_1, B_1) = \emptyset$

Firstly, we have $N_{D_2}(A_2 \cup B_2) = N_{D_2}(A_2' \cup B_2')$. Otherwise, we assume there exists $w_{3_1} \in N_{D_2}(A_2 \cup B_2) \setminus N_{D_2}(A_2' \cup B_2')$ and $w_{2_{s_2}} \in B_2 \setminus B_2'$ such that $w_{3_1}w_{2_{s_2}} \in E(G)$. Then

$\langle w_{3_1}; w_{2_{s_2}}, w_{2_1}, v_{2_1}; v_{3_1}v_{1_1}\rangle$ is a $B_{1,2}$. Secondly, for any $w_{3_1} \in N_{D_2}(B_2' \cup A_2')$, we have both $\{w_{3_1}\} \cup N_{D_2}(A_2)$ and $\{w_{3_1}\} \cup N_{D_2}(B_2)$ are cliques. Otherwise, let $w_{3_1} \in N_{D_2}(w_{2_1}) \cap N_{D_2}(w_{1_1})$, then for any $w_{1_i} \in A_2 \setminus \{w_{1_1}\}$, we have $\langle w_{1_i}; w_{1_1}, w_{3_1}, w_{2_1}; v_{2_1}v_{3_1}\rangle$ is a $B_{1,2}$. Similarly, we can show that for any $w_{2_j} \in B_2 \setminus \{w_{2_1}\}$, $\langle w_{2_j}; w_{2_1}, w_{3_1}, w_{1_1}; v_{1_1}v_{3_1}\rangle$ is a $B_{1,2}$. Therefore both $A_2 \cup N_{D_2}(A_2)$ and $N_{D_2}(A_2) \cup B_2$ are cliques since w_{3_1} is arbitrary. Thirdly, $D_2 = N_{D_2}(A_2)$. Otherwise, if there exists $w' \in D_2 \setminus N_{D_2}(A_2)$ such that $w_{3_1}w' \in E(G)$, where $w_{3_1} \in N_{D_2}(A_2)$, then $\langle w'; w_{3_1}, w_{1_1}, w_{2_1}; v_{2_1}v_{3_1}\rangle$ is a $B_{1,2}$. Thus, both $A_2 \cup D_2$ and $B_2 \cup D_2$ are cliques. ■

Now we want to find a strong spanning Halin subgraph in G as follows.

If $|B_2| \geq 2$(or $|B_1| \geq 2$), since both $D_1 \cup (A_1 \setminus \{v_{1_1}\})$, $D_2 \cup (A_2 \setminus \{w_{1_1}\})$ and $B_1 \cup (B_2 \setminus \{w_{2_1}\})$ are cliques, there exist hamiltonian paths, say $P_1 = v_{3_1}P_1v_{1_2}$, $P_2 = w_{3_1}P_2w_{1_2}$ and $P_3 = w_{2_2}P_3v_{2_1}$, in them, respectively. Let $C = P_1P_2P_3$ be a cycle and all vertices on the path $v_{1_1}w_{1_1}w_{2_1}$ be stems of T with $N_C(v_{1_1}) = V(P_1)$, $N_C(w_{1_1}) = V(P_2)$ and $N_C(w_{2_1}) = V(P_3)$.

If $B_1 = \{v_{2_1}\}$ and $B_2 = \{w_{2_1}\}$, since $D_1 \cup (A_1 \setminus \{v_{1_1}\})$ and $D_2 \cup (A_2 \setminus \{w_{1_1}\})$ are cliques, there exist hamiltonian paths, say $P_1 = v_{3_1}P_1v_{1_2}$ and $P_2 = w_{3_1}P_2w_{1_2}$, in them, respectively. Let $C = P_1P_2 \cup \{w_{3_1}w_{2_1}, w_{2_1}v_{2_1}, v_{2_1}v_{3_1}\}$ be a cycle and $\{v_{1_1}, w_{1_1}\}$ be stems of T with $N_C(v_{1_1}) = V(P_1) \cup \{v_{2_1}\}$ and $N_C(w_{1_1}) = V(P_2) \cup \{w_{2_1}\}$.

Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 9.9 as examples).

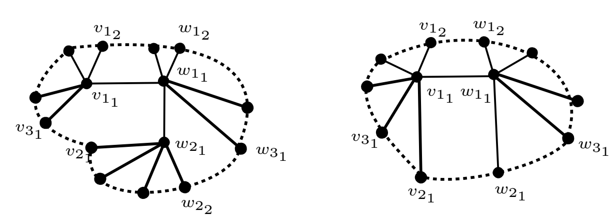

Figure 9.9. $E(A_2, B_2) \neq \emptyset$

Chapter 10

HEX JOIN

Recall that if we let (W_1, W_2) be a partition of $V(G)$, such that for $i \in [1, 2]$ there are cliques $A_i, B_i, C_i \subseteq W_i$ with the following properties:

- W_1, W_2 are both nonempty;
- for $i = 1, 2$ the sets A_i , B_i , C_i are pairwise disjoint and have union W_i;
- if $v_1 \in W_1$ and $v_2 \in W_2$, then v_1 is adjacent to v_2 unless either $v_1 \in A_1$ and $v_2 \in A_2$, or $v_1 \in B_1$ and $v_2 \in B_2$, or $v_1 \in C_1$ and $v_2 \in C_2$; and in these cases v_1, v_2 are nonadjacent.

In these circumstances we say that G is a hex-join of $\langle W_1 \rangle$ and $\langle W_2 \rangle$.

In this chapter, we will show the following proposition.

Proposition 18. *If G is a 3-connected claw-free graph and admits a hex-join, then G contains a strong spanning Halin subgraph.*

10.1 Proof of 3-connected claw-free graphs admit hex-joins

For simplicity, we denote by $D_1 = A_1$, $D_2 = B_2$, $D_3 = C_1$, $D_4 = A_2$, $D_5 = B_1$, $D_6 = C_2$ and $D_i = \{v_{i_1}, v_{i_2}, \cdots, v_{i_{t_i}}\}$ for $i \in [1, 6]$. By the definition of hex join, we know $D_i \cup D_{i+1}$ is a clique and $E(D_i, D_{i+3}) = \emptyset$, where $D_{i+6} = D_i$ for $i \in [1, 6]$. We want to consider following two cases.

Case 1: Assume that $D_i \neq \emptyset$ for all $i \in [1, 6]$.

Case 1.1: There exist at least four consecutive sets in $\{D_1, D_2, \cdots, D_6\}$ such that $|D_i| \geq 2$, where $i \in [1, 6]$.

We may assume $|D_i| \geq 2$ for $i \in \{2, 3, 4, 5\}$. Since D_1, $D_i \setminus \{v_{i_1}\}$, $i \in \{2, 3, 4, 5\}$, and D_6 are cliques, there exist hamiltonian paths, say $P_1 = v_{1_1} P_1 v_{1_{t_1}}$, $P_i = v_{i_2} P_i v_{i_{t_i}}$, $i \in [2, 3, 4, 5]$, and $P_6 = v_{6_1}P_6v_{6_{t_6}}$, in them, respectively. Let $C = P_1P_2\cdots P_6$ be a cycle and all vertices on the path $v_{2_1}v_{3_1}v_{4_1}v_{5_1}$ be stems of T with $N_C(v_{2_1}) = V(P_1)\cup V(P_2)$, $N_C(v_{3_1}) = V(P_3)$, $N_C(v_{4_1}) = V(P_4)$ and $N_C(v_{5_1}) = V(P_5)\cup V(P_6)$. Let $H = T\cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.1 (1) as an example.)

Case 1.2: There exists $i \in [1,6]$ such that $|D_i| = |D_{i+2}| = 1$ and the other four sets with at least two vertices.

We may assume $|D_1| = |D_3| = 1$. Since $D_1\cup D_3$ is not a 2-cut and $E(D_2, D_5) = \emptyset$, we can assume $v_{2_1}v_{4_1} \in E(G)$. Since $D_i \setminus \{v_{i_1}\}$, $i \in \{2,4,5\}$, and D_6 are cliques, there exist hamiltonian paths, say $P_i = v_{i_2}P_iv_{i_{t_i}}$ and $P_6 = v_{6_1}P_6v_{6_{t_6}}$, in them, respectively. Let $C = P_2P_4P_5P_6\cup\{v_{1_1}v_{2_2}, v_{2_{t_2}}v_{3_1}, v_{3_1}v_{4_2}, v_{6_{t_6}}v_{1_1}\}$ be a cycle and all vertices on the path $v_{2_1}v_{4_1}v_{5_1}$ be stems with $N_C(v_{2_1}) = V(P_2)\cup\{v_{1_1}, v_{3_1}\}$, $N_C(v_{4_1}) = V(P_4)$ and $N_C(v_{5_1}) = V(P_5)\cup V(P_6)$. Let $H = T\cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.1 (2) as an example.)

Case 1.3: There exists $i \in [1,6]$ such that $|D_i| = |D_{i+3}| = 1$ and the other four sets with at least two vertices.

We may assume $|D_1| = |D_4| = 1$. Since $D_1\cup D_4$ is not a 2-cut, then $E(D_2\cup D_3, D_5\cup D_6) \neq \emptyset$. By symmetric, we may assume $v_{3_1}v_{5_1} \in E(G)$. Since $D_i \setminus \{v_{i_1}\}$, $i \in \{2,3,5\}$, and D_6 are cliques, there exist hamiltonian paths, say $P_i = v_{i_2}P_iv_{i_{t_i}}$ and $P_6 = v_{6_1}P_6v_{6_{t_6}}$, in them, respectively. Let $C = P_2P_3P_5P_6\cup\{v_{1_1}v_{2_2}, v_{3_{t_3}}v_{4_1}, v_{4_1}v_{5_2}, v_{6_{t_6}}v_{1_1}\}$ be a cycle and all vertices on the path $v_{21}v_{3_1}v_{5_1}$ be stems of T with $N_C(v_{2_1}) = V(P_2)\cup\{v_{1_1}\}$, $N_C(v_{3_1}) = V(P_3)\cup\{v_{4_1}\}$ and $N_C(v_{5_1}) = V(P_5)\cup V(P_6)$. Let $H = T\cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.1 (3) as an example.)

Case 1.4: There exists $i \in [1,6]$ such that $|D_i| = |D_{i+1}| = |D_{i+2}| = 1$ and the other three sets with at least two vertices.

We may assume $|D_1| = |D_2| = |D_3| = 1$. Since $D_1\cup D_3$ is not a 2-cut and $E(D_2, D_5) = \emptyset$, we may assume $v_{2_1}v_{4_1} \in E(G)$. Since $D_i \setminus \{v_{i_1}\}$, $i \in \{4,5,6\}$, are cliques, there exist hamiltonian paths, say $P_i = v_{i_2}P_iv_{i_{t_i}}$ in them, respectively. Let $C = P_4P_5P_6\cup$

$\{v_{1_1}v_{2_1}, v_{2_1}v_{3_1}, v_{3_1}v_{4_2}, v_{6_{t_6}}v_{1_1}\}$ be a cycle and all vertices on the path $v_{4_1}v_{5_1}v_{6_1}$ be stems of T with $N_C(v_{4_1}) = V(P_4) \cup \{v_{2_1}, v_{3_1}\}$, $N_C(v_{5_1}) = V(P_5)$ and $N_C(v_{6_1}) = V(P_6) \cup \{v_{1_1}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.1 (4) as an example.)

Case 1.5: There exists $i \in [1, 6]$ such that $|D_i| = |D_{i+1}| = |D_{i+3}| = 1$(or $|D_i| = |D_{i+1}| = |D_{i+4}| = 1$) and the other three sets with at least two vertices.

We may assume $|D_1| = |D_2| = |D_4| = 1$. Since neither $D_1 \cup D_4$ nor $D_2 \cup D_4$ is a 2-cut and $E(D_2, D_5) = E(D_3, D_6) = \emptyset$, we may assume $v_{3_1}v_{5_1} \in E(G)$ or $\{v_{2_1}v_{6_1}, v_{1_1}v_{3_1}\} \subseteq E(G)$. If $v_{3_1}v_{5_1} \in E(G)$, we can find a strong spanning Halin subgraph in G similarly as Case 1.2. If $\{v_{2_1}v_{6_1}, v_{1_1}v_{3_1}\} \subseteq E(G)$. Since $D_i \setminus \{v_{i_1}\}$, $i \in \{5, 6\}$ and D_3 are cliques, there exist hamiltonian paths, say $P_i = v_{i_2}P_iv_{i_{t_i}}$ and $P_3 = v_{3_1}P_3v_{3_{t_3}}$, in them, respectively. Let $C = P_5P_6 \cup \{v_{1_1}v_{3_1}, v_{3_{t_3}}v_{4_1}, v_{4_1}v_{5_2}, v_{5_{t_5}}v_{6_2}, v_{6_{t_6}}v_{1_1}\}$ be a cycle and all vertices on the path $v_{2_1}v_{5_1}v_{6_1}$ be stems of T with $N_C(v_{2_1}) = V(P_3) \cup \{v_{1_1}\}$, $N_C(v_{5_1}) = V(P_5) \cup \{v_{4_1}\}$ and $N_C(v_{6_1}) = V(P_6)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.1 (5) as an example.)

Case 1.6: There exists $i \in [1, 6]$ such that $|D_i| = |D_{i+2}| = |D_{i+4}| = 1$ and the other three sets with at least two vertices.

We may assume $|D_1| = |D_3| = |D_5| = 1$. Since neither $D_1 \cup D_3$ nor $D_3 \cup D_5$ nor $D_1 \cup D_5$ is a 2-cut and $E(D_i, D_{i+3}) = \emptyset$ for $i \in \{1, 2, 3\}$, we may assume $v_{6_1}v_{2_1} \in E(G)$ and $v_{6_1}v_{4_1} \in E(G)$. Since $D_i \setminus \{v_{i_1}\}$, $i \in \{2, 4, 6\}$, are cliques, there exist hamiltonian paths, say $P_i = v_{i_2}P_iv_{i_{t_i}}$ in them, respectively. Let $C = P_2P_4P_6 \cup \{v_{1_1}v_{2_2}, v_{2_{t_2}}v_{3_1}, v_{3_1}v_{4_2}, v_{4_{t_4}}v_{5_1}, v_{5_1}v_{6_2}, v_{6_{t_6}}v_{1_1}\}$ be a cycle and all vertices on the path $v_{2_1}v_{4_1}v_{6_1}$ be stems of T with $N_C(v_{2_1}) = V(P_2) \cup \{v_{1_1}, v_{3_1}\}$, $N_C(v_{4_1}) = V(P_4) \cup \{v_{5_1}\}$ and $N_C(v_{6_1}) = V(P_6)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.1 (6) as an example.)

Case 1.7: There exists $i \in [1, 6]$ such that $|D_i| = |D_{i+1}| = |D_{i+2}| = |D_{i+3}| = 1$ and the other two sets with at least two vertices.

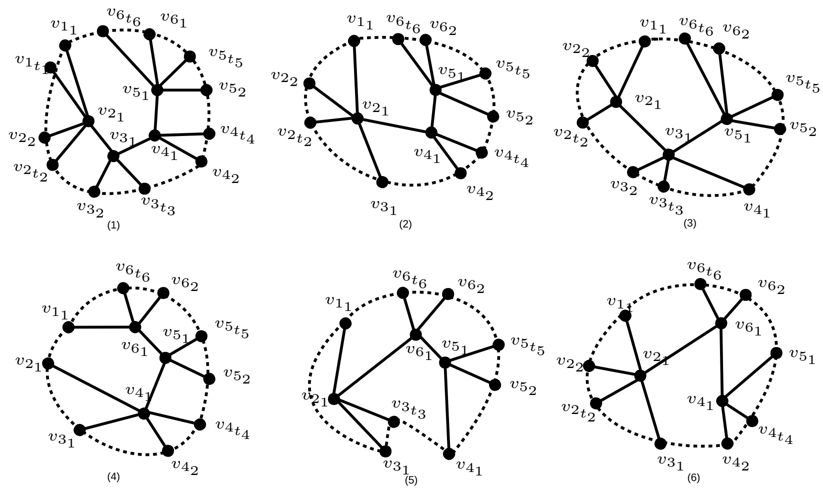

Figure 10.1. G admits hex join and there are less than three sets with size 1.

We may assume $|D_1| = |D_2| = |D_3| = |D_4| = 1$. Since $D_1 \cup D_4$ is not a 2-cut and $E(D_2, D_5) = E(D_3, D_6) = \emptyset$, we may assume $v_{2_1}v_{6_1} \in E(G)$. Moreover, since $D_2 \cup D_4$ is not a 2-cut, we have $v_{3_1}v_{5_1} \in E(G)$ or $v_{1_1}v_{3_1} \in E(G)$.

If $v_{3_1}v_{5_1} \in E(G)$, since $D_5 \setminus \{v_{5_1}\}$ and $D_6 \setminus \{v_{6_1}\}$ are cliques, there exist hamiltonian paths, say $P_5 = v_{5_2}P_5v_{5_{t_5}}$ and $P_6 = v_{6_2}P_6v_{6_{t_6}}$, in them, respectively. Let $C = P_5P_6 \cup \{v_{1_1}v_{2_1}, v_{2_1}v_{3_1}, v_{3_1}v_{4_1}, v_{4_1}v_{5_2}, v_{5_{t_5}}v_{6_2}, v_{6_{t_6}}v_{1_1}\}$ be a cycle and $\{v_{5_1}, v_{6_1}\}$ be stems of T with $N_C(v_{5_1}) = V(P_5) \cup \{v_{3_1}, v_{4_1}\}$ and $N_C(v_{6_1}) = V(P_6) \cup \{v_{1_1}, v_{2_1}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.2 (1) as an example.)

If $v_{1_1}v_{3_1} \in E(G)$, since $D_5 \setminus \{v_{5_1}\}$ and $D_6 \setminus \{v_{6_2}\}$ are cliques, there exist hamiltonian paths, say $P_5 = v_{5_2}P_5v_{5_{t_5}}$ and $P_6 = v_{6_1}P_6v_{6_{t_6}}$, in them, respectively. Let $C = P_5P_6 \cup \{v_{6_1}v_{2_1}, v_{2_1}v_{3_1}, v_{3_1}v_{4_1}, v_{4_1}v_{5_2}, v_{5_{t_5}}v_{6_{t_6}}\}$ be a cycle and all vertices on the path $v_{1_1}v_{6_2}v_{5_1}$ be stems of T with $N_C(v_{1_1}) = \{v_{2_1}, v_{3_1}\}$, $N_C(v_{6_2}) = V(P_6)$ and $N_C(v_{5_1}) = V(P_5) \cup \{v_{4_1}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.2 (2) as an example.)

Case 1.8: There exists $i \in [1, 6]$ such that $|D_i| = |D_{i+1}| = |D_{i+2}| = |D_{i+4}| = 1$ and the other two sets with at least two vertices.

We may assume $|D_1| = |D_2| = |D_3| = |D_5| = 1$. Since $D_1 \cup D_3$ is not a 2-cut and $E(D_2, D_5) = \emptyset$, we may assume $v_{2_1}v_{4_1} \in E(G)$. Moreover, since $\{v_{2_1}, v_{5_1}\}$ is not a 2-cut, we have $v_{4_1}v_{6_1} \in E(G)$ or $v_{1_1}v_{3_1} \in E(G)$.

If $v_{4_1}v_{6_1} \in E(G)$, since $D_4 \setminus \{v_{4_1}\}$ and $D_6 \setminus \{v_{6_1}\}$ are cliques, there exist hamiltonian paths, say $P_4 = v_{4_2}P_4v_{4_{t_4}}$ and $P_6 = v_{6_2}P_6v_{6_{t_6}}$, in them, respectively. Let $C = P_4P_6 \cup \{v_{1_1}v_{2_1}, v_{2_1}v_{3_1}, v_{3_1}v_{4_2}, v_{4_{t_4}}v_{5_1}, v_{5_1}v_{6_2}, v_{6_{t_6}}v_{1_1}\}$ be a cycle and $\{v_{4_1}, v_{6_1}\}$ be stems of T with $N_C(v_{4_1}) = V(P_4) \cup \{v_{2_1}, v_{3_1}, v_{5_1}\}$ and $N_C(v_{6_1}) = V(P_6) \cup \{v_{1_1}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.2 (3) as an example.)

If $v_{1_1}v_{3_1} \in E(G)$, since $D_1 \cup D_5$ is not a 2-cut, we may assume $v_{6_1}v_{2_1} \in E(G)$. Since $D_4 \setminus \{v_{4_1}\}$ and $D_6 \setminus \{v_{6_1}\}$ are cliques, there exist hamiltonian paths, say $P_4 = v_{4_2}P_4v_{4_{t_4}}$ and $P_6 = v_{6_2}P_6v_{6_{t_6}}$, in them, respectively. Let $C = P_4P_6 \cup \{v_{6_{t_6}}v_{1_1}, v_{1_1}v_{3_1}, v_{3_1}v_{4_2}, v_{4_{t_4}}v_{5_1}, v_{5_1}v_{6_2}\}$ be a cycle and all vertices on the path $v_{6_1}v_{2_1}v_{4_1}$ be stems of T with $N_C(v_{6_1}) = V(P_6) \cup \{v_{1_1}\}$, $N_C(v_{2_1}) = \{v_{3_1}\}$ and $N_C(v_{4_1}) = V(P_4) \cup \{v_{5_1}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.2 (4) as an example.)

Case 1.9: There exists $i \in [1, 6]$ such that $|D_i| = |D_{i+1}| = |D_{i+3}| = |D_{i+4}| = 1$ and the other two sets with at least two vertices.

We may assume $|D_1| = |D_2| = |D_4| = |D_5| = 1$. Since $D_2 \cup D_5$ is not a 2-cut and $E(D_i, D_{i+3}) = \emptyset$ for all $i \in \{1, 2, 3\}$, we may assume $v_{1_1}v_{3_1} \in E(G)$. Moreover, since $\{v_{1_1}, v_{4_1}\}$ is not a 2-cut, we have $v_{2_1}v_{6_2} \in E(G)$ or $v_{3_1}v_{5_1} \in E(G)$.

If $v_{2_1}v_{6_1} \in E(G)$, since $D_3 \setminus \{v_{3_1}\}$ and $D_6 \setminus \{v_{6_1}\}$ are cliques, there exist hamiltonian paths, say $P_3 = v_{3_2}P_3v_{3_{t_3}}$ and $P_6 = v_{6_2}P_6v_{6_{t_6}}$, in them, respectively. Let $C = P_3P_6 \cup \{v_{6_2}v_{2_1}, v_{2_1}v_{3_2}, v_{3_{t_3}}v_{4_1}, v_{4_1}v_{5_1}, v_{5_1}v_{6_{t_6}}\}$ be a cycle and all vertices on the path $v_{3_1}v_{1_1}v_{6_1}$ be stems of T with $N_C(v_{3_1}) = V(P_3) \cup \{v_{4_1}\}$, $N_C(v_{1_1}) = \{v_{2_1}\}$ and $N_C(v_{6_1}) = V(P_6) \cup \{v_{5_1}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.2 (5) as an example.)

If $v_{3_1}v_{5_1} \in E(G)$, since $\{v_{1_1}, v_{5_1}\}$ is not a 2-cut, we may assume $v_{6_1}v_{4_1} \in E(G)$. Similarly as $v_{2_1}v_{6_1} \in E(G)$, we can choose $\{v_{3_1}, v_{4_1}, v_{6_1}\}$ as stems of T and find a strong spanning Halin subgraph in G.

Case 1.10: There exists $i \in [1,6]$ such that $|D_i| = |D_{i+1}| = |D_{i+2}| = |D_{i+3}| = |D_{i+4}| = 1$ and the other set with at least two vertices.

We may assume $|D_1| = |D_2| = |D_3| = |D_4| = |D_5| = 1$. Since $D_1 \cup D_5$ is not a 2-cut and $E(D_i, D_{i+3}) = \emptyset$ for all $i \in \{1,2,3\}$, we may assume $v_{6_1}v_{2_1} \in E(G)$. Moreover, since $\{v_{2_1}, v_{5_1}\}$ is not a 2-cut, we have $v_{1_1}v_{3_1} \in E(G)$. Furthermore, since $\{v_{3_1}, v_{5_1}\}$ is not 2-cut,we have $v_{4_1}v_{6_1} \in E(G)$ or $v_{4_1}v_{2_1} \in E(G)$.

If $v_{4_1}v_{6_1} \in E(G)$, since $D_6 \setminus \{v_{6_1}\}$ is a clique, there exist a hamiltonian path $P_6 = v_{6_2}P_6v_{6_{t_6}}$ in it. Let $C = P_6 \cup \{v_{6_2}v_{1_1}, v_{1_1}v_{3_1}, v_{3_1}v_{4_1}, v_{4_1}v_{5_1}, v_{5_1}v_{6_{t_6}}\}$ be a cycle and $\{v_{2_1}, v_{6_1}\}$ be stems of T with $N_C(v_{2_1}) = \{v_{1_1}, v_{3_1}\}$ and $N_C(v_{6_1}) = V(P_6) \cup \{v_{4_1}, v_{5_1}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.2 (6) as an example.)

If $v_{4_1}v_{2_1} \in E(G)$, we only need to remove the edge $v_{4_1}v_{6_1}$ from $E(T)$ and add the edge $v_{2_1}v_{4_1}$ to $E(T)$, then we can find a strong spanning Halin subgraph in G similarly.

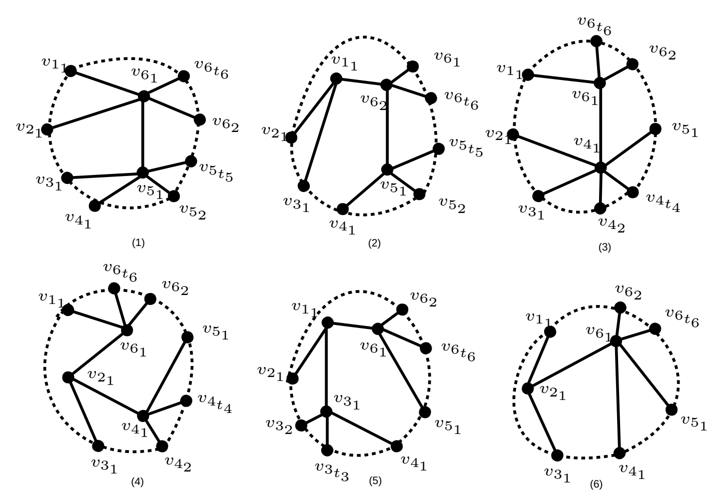

Figure 10.2. G admits hex join and three are four sets with size 1.

Case 2: There exist some $i \in [1,6]$ such that $D_i = \emptyset$.

Since neither W_1 nor W_2 is empty, there exists at least one set in W_i, $i \in [1,2]$, is nonempty. Moreover, because $E(D_i, D_{i+3}) = \emptyset$ for $i \in \{1,2,3\}$, it is easy to check that G contains a strong spanning Halin subgraph if at least three sets of $\{D_1, D_2, \cdots, D_6\}$ are empty. Thus we assume only one or two sets of $\{D_1, D_2, \cdots, D_6\}$ are empty.

Case 2.1: There exists exactly one set in $\{D_1, D_2, \cdots, D_6\}$ is empty. By symmetric, we may assume $D_6 = \emptyset$ and $|D_2| \geq |D_4|$.

Note that if $|D_2| \geq 2$, then $E(D_2, D_4) \neq \emptyset$ since there is no twins in G. If $|D_2| = 1$, then we can assume $v_{1_{t_1}}v_{3_1}, v_{3_{t_3}}v_{5_1} \in E(G)$.

Case 2.1.1: Assume that $\langle D_1 \cup D_5 \rangle$ is 2-connected. We denote by $v_{1_1}v_{5_1}, v_{1_2}v_{5_2} \in E(G)$. Since $(D_1 \setminus \{v_{1_1}\}) \cup (D_2 \setminus \{v_{2_1}\})$, D_3 and $(D_5 \setminus \{v_{5_1}\}) \cup D_4$ are cliques, there exist hamiltonian paths, $P_1 = v_{1_2}P_1v_{2_{t_2}}$, $P_3 = v_{3_1}P_3v_{3_{t_3}}$ and $P_2 = v_{4_1}P_2v_{5_2}$, in them, respectively. Let $C = P_1P_3P_2 \cup \{v_{5_2}v_{1_2}\}$ be a cycle and all vertices on the path $v_{2_1}v_{1_1}v_{5_1}$ be stems of T with $N_C(v_{2_1}) = (V(P_1) \setminus \{v_{1_2}\}) \cup V(P_3)$, $N_C(v_{1_1}) = \{v_{1_2}\}$ and $N_C(v_{5_1}) = V(P_4)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.3 (1) as an example).

Case 2.1.2: Assume that $\langle D_1 \cup D_5 \rangle$ is 1-connected, we denote by $v_{1_1}v_{5_1} \in E(G)$. If $E(D_2, D_4) \neq \emptyset$, we assume $v_{2_1}v_{4_1} \in E(D_2, D_4)$. Since $D_1 \cup (D_2 \setminus \{v_{2_1}\})$, D_3 and $D_5 \cup (D_4 \setminus \{v_{4_1}\})$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_1}P_1v_{2_{t_2}}$, $P_3 = v_{3_1}P_3v_{3_{t_3}}$ and $P_2 = v_{4_2}P_2v_{5_1}$, in them, respectively. Let $C = P_1P_3P_2$ be a cycle and $\{v_{2_1}, v_{4_1}\}$ be stems of T with $N_C(v_{2_1}) = V(P_1) \cup V(P_3)$ and $N_C(v_{4_1}) = V(P_2)$, where $v_{1_{t_1}}v_{3_1} \in E(G)$ if $D_2 = \{v_{2_1}\}$ since G is 3-connected. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.3 (2) as an example).

If $E(D_2, D_4) = \emptyset$, since there is no twins in G, we get $|D_2| = |D_4| = 1$. Moreover, because G is 3-connected, we can assume $v_{1_2}v_{3_1}, v_{3_2}v_{5_2} \in E(G)$. Since D_1, $D_3 \setminus \{v_{3_2}\}$ and $D_5 \setminus \{v_{5_2}\}$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_1}P_1v_{1_2}$, $P_3 = v_{3_1}P_3v_{3_{t_3}}$ and $P_2 = v_{5_1}P_2v_{5_{t_5}}$, in them, respectively. Let $C = P_1P_3P_2 \cup \cup\{v_{1_2}v_{3_1}, v_{3_{t_3}}v_{4_1}, v_{4_1}v_{5_{t_5}}, v_{5_1}v_{1_1}\}$ be a cycle and all vertices on the path $v_{2_1}v_{3_2}v_{5_2}$ be stems of T with $N_C(v_{2_1}) = V(P_1)$,

$N_C(v_{3_2}) = V(P_3)$ and $N_C(v_{5_2}) = V(P_{2_1}) \cup \{v_{4_1}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.3 (3) as an example).

Case 2.1.3: Suppose that $\langle D_1 \cup D_5 \rangle$ is disconnected. We may assume $E(D_2, D_4) \neq \emptyset$, otherwise G admits 1-join, we can find a strong spanning Halin subgraph in G as Chapter 7.

If $D_2 \cup D_4$ is 2-connected, we denote by $v_{2_1}v_{4_1}, v_{2_2}v_{4_2} \in E(G)$. Since $D_1 \cup (D_2 \setminus \{v_{2_1}, v_{2_2}\})$, D_3 and $D_5 \cup (D_4 \setminus \{v_{4_1}, v_{4_2}\})$ are cliques, there exist hamiltonian paths, $P_1 = v_{1_1}P_1v_{2_{t_2}}$, $P_3 = v_{3_1}P_3v_{3_{t_3}}$ and $P_2 = v_{4_3}P_2v_{5_1}$, in them, respectively. Let $C = P_1P_3P_2 \cup \{v_{5_1}v_{4_2}, v_{4_2}v_{2_2}, v_{2_2}v_{1_1}\}$ be a cycle and $\{v_{2_1}, v_{4_1}\}$ be stems of T with $N_C(v_{2_1}) = V(P_1) \cup V(P_3) \cup \{v_{2_2}\}$ and $N_C(v_{4_1}) = V(P_2) \cup \{v_{4_2}\}$, where $v_{1_1}v_{3_{t_3}} \in E(G)$ if $|D_2| = 2$ since G is 3-connected. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.3 (4) as an example).

If $D_2 \cup D_4$ is 1-connected, then $|D_2| \leq 2$, $|D_4| \leq 2$ since there is no twins in G and $|D_3| \geq 2$, $E(D_1, D_3) \neq \emptyset$, $E(D_3, D_5) \neq \emptyset$ since G is 3-connected. Thus, we may denote by $v_{2_1}v_{4_1}, v_{1_1}v_{3_1}, v_{3_2}v_{5_1} \in E(G)$. Since D_1, D_3 and D_5 are cliques, there exist hamiltonian paths, $P_1 = v_{1_1}P_1v_{1_2}$, $P_3 = v_{3_1}P_3v_{3_2}$ and $P_2 = v_{5_1}P_2v_{5_{t_5}}$, in them, respectively. We may assume $v_{1_2}v_{3_{t_3}} \in E(G)$ if $|D_2| = 1$ since G is 3-connected. Let $C = P_1P_3P_2 \cup \{v_{5_{t_5}}v_{4_2}, v_{4_2}v_{2_2}, v_{2_2}v_{1_2}\}$ be a cycle and $\{v_{2_1}, v_{4_1}\}$ be stems of T with $N_C(v_{2_1}) = V(P_1) \cup (V(P_2) \setminus \{v_{2_1}\}) \cup V(P_3)$ and $N_C(v_{2_1}) = (V(P_4) \setminus \{v_{4_1}\}) \cup V(P_5)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.3 (5) as an example).

Case 2.2: There exist exactly two sets in $\{D_1, D_2, \cdots, D_6\}$ are empty. We will find a strong spanning Halin subgraph in G by following subcases.

Case 2.2.1: The two empty sets are consecutive.

We may assume $D_5 = D_6 = \emptyset$. Since $E(D_1, D_4) = \emptyset$ and G is 3-connected, we know $|D_2 \cup D_3| \geq 3$. We may assume $|D_2| \geq |D_3|$, then $|D_2| \geq 2$. Since $D_1 \cup (D_2 \setminus \{v_{2_1}, v_{2_2}\})$ and $D_4 \cup (D_3 \setminus \{v_{3_1}, v_{3_2}\})$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_1}P_1v_{2_{t_2}}$ and $P_2 = v_{4_1}P_2v_{3_{t_3}}$, in them, respectively. Let $C = P_1P_2 \cup \{v_{1_1}v_{2_2}, v_{2_2}v_{3_2}, v_{3_2}v_{4_1}\}$ be a cycle and

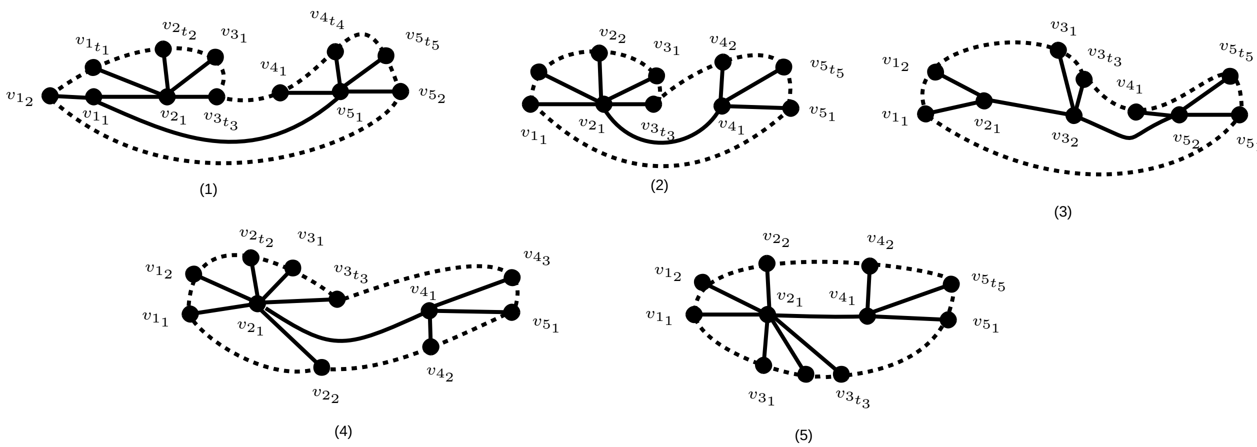

Figure 10.3. G admits hex join and there exists exactly one set is empty.

$\{v_{2_1}, v_{3_1}\}$ be stems of T with $N_C(v_{2_1}) = V(P_1) \cup \{v_{2_2}\}$ and $N_C(v_{3_1}) = V(P_2) \cup \{v_{3_2}\}$, where $v_{2_{t_2}} v_{4_{t_4}} \in E(G)$ if $|D_3| = 2$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.4 (1) as an example).

Case 2.2.2: There exists $i \in \{1, 2, 3\}$ such that $D_i = D_{i+3} = \emptyset$.

We may assume $D_3 = D_6 = \emptyset$. Since $E(D_1, D_4) = E(D_2, D_5) = \emptyset$ and G is 3-connected, we may assume $v_{1_1} v_{5_1} \in E(D_1, D_5)$ and $\{v_{2_1} v_{4_1}, v_{2_2} v_{4_2}\} \in E(D_2, D_4)$. Since $D_1 \cup (D_2 \setminus \{v_{2_1}\})$ and $D_5 \cup (D_4 \setminus \{v_{4_1}\})$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_1} P_1 v_{2_2}$ and $P_2 = v_{4_2} P_2 v_{5_1}$, in them, respectively. Let $C = P_1 P_2 \cup \{v_{5_1} v_{1_1}\}$ be a cycle and $\{v_{2_1}, v_{3_1}\}$ be stems of T with $N_C(v_{2_1}) = V(P_1)$ and $N_C(v_{4_1}) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.4 (2) as an example).

Case 2.2.3: There exists $i \in [1, 6]$ such that $D_i = D_{i+2} = \emptyset$.

By symmetric, we may assume $D_4 = D_6 = \emptyset$ and $|D_1| \geq |D_3|$. Since G contains no twins and $E(D_2, D_5) = \emptyset$, we have $D_2 = \{v_{2_1}\}$, which implies $|D_2| \geq 2$ since G is 3-connected. If $|D_5| \leq 2$, we can find a strong spanning Halin subgraph in G easily. Therefore, we assume $|D_5| \geq 3$. We may also assume $v_{5_1} v_{1_1}, v_{5_2} v_{1_2}, v_{5_3} v_{3_1} \in E(G)$ and there exists $v \in D_1 \cup (D_5 \setminus v_{5_3})$ such that $v_{3_2} v \in E(G)$ since G is 3-connected, where $v_{3_2} = v_{3_1}$ if $|D_3| = 1$.

If $E(D_1, D_3) \neq \emptyset$, we let $v_{3_2}v_{1_3} \in E(G)$ if $|D_1| \geq 3$ and $v_{3_2}v_{1_2} \in E(G)$ if $|D_1| = 2$. Since $D_1 \setminus \{v_{1_2}\}$, D_3 and $D_5 \setminus \{v_{5_2}\}$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_1}P_1v_{1_3}$, $P_2 = v_{3_2}P_2v_{3_1}$ and $P_3 = v_{5_1}P_3v_{5_3}$, in them, respectively. Let $C = P_1P_2P_3$ be a cycle and all vertices on the path $v_{2_1}v_{1_2}v_{5_2}$ be stems of T with $N_C(v_{2_1}) = (D_1 \setminus \{v_{1_1}, v_{1_2}\}) \cup D_3$, $N_C(v_{1_2}) = \{v_{1_1}\}$ and $N_C(v_{5_2}) = D_5 \setminus \{v_{5_2}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.4 (3) as an example).

If $E(D_1, D_3) = \emptyset$, then $|D_5| \geq 4$ since D_1, D_3 contains no twins. Moreover, if there exist $v_{5_i} \in D_5$, $v_{1_j} \in D_1$ and $v_{3_k} \in D_3$ such that $v_{1_j}v_{5_i}, v_{3_k}v_{5_i} \in E(G)$ but $v_{1_j}v_{3_k} \notin E(G)$, then $N_{D_5}(v_{1_j}) \cup N_{D_5}(v_{3_k}) = D_5$ since $\langle v_{5_i}; v_{1_j}, v_{3_k}, v_{5_t}\rangle$ is not a claw for any $v_{5_t} \in D_5 \setminus \{v_{5_1}\}$. Therefore, we assume $v_{3_2}v_{5_4} \in E(G)$. Since $D_1 \setminus \{v_{1_2}\}$, $D_3 \setminus \{v_{3_1}\}$ and $D_5 \setminus \{v_{5_2}, v_{5_3}\}$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_1}P_1v_{1_{t_1}}$, $P_2 = v_{3_{t_3}}P_3v_{3_2}$ and $P_3 = v_{5_1}P_3v_{5_4}$, in them, respectively. Let $C = P_1P_3P_2 \cup \{v_{1_{t_1}}v_{2_1}, v_{2_1}v_{3_{t_3}}, v_{3_2}v_{5_4}, v_{5_1}v_{1_1}\}$ be a cycle and all vertices on the path $v_{1_2}v_{5_2}v_{3_1}$ be stems of T with $N_C(v_{2_1}) = V(P_1)$, $N_C(v_{5_2}) = \{v_{5_1}\}$, $N_C(v_{5_3}) = V(P_3) \setminus \{v_{5_1}\}$ and $N_C(v_{3_1}) = V(P_2) \cup \{v_{2_1}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 10.4 (4) as an example).

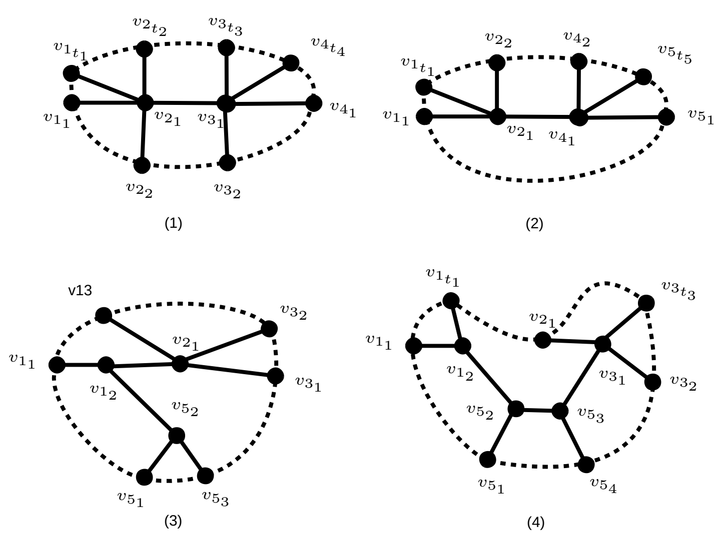

Figure 10.4. G admits hex join and there exist exactly two sets are empty.

Chapter 11

NONDOMINATING W-JOIN

For a vertex a and a set $B \subseteq V(G) \setminus \{a\}$ we say that a is complete to B or B-complete if a is adjacent to every vertex in B; and that a is anticomplete to B or B-anticomplete if a has no neighbor in B. For two disjoint subsets A and B of $V(G)$ we say that A is complete(respectively anticomplete) to B, if every vertex in A is complete(respectively anticomplete) to B. Recall that if (A, B) are a homogeneous pair, such that A is neither complete nor anticomplete to B, and at least one of A, B has at least two members. In these circumstances, we call (A, B) a W-join. A homogeneous pair (A, B) is nondominating if some vertex of $G \setminus (A \cup B)$ has no neighbor in $A \cup B$.

If G admits W-join, let (A, B) be the homogeneous pair always with $|A| \geq |B|$, D_1 be the set of vertices that A-complete and B-anticomplete, D_2 be the set of vertices that both A-complete and B-complete, D_3 vertices that A-anticomplete and B-complete, D_4 be the set of vertices that both A-anticomplete and B-anticomplete. We denote by $A = \{w_{1_1}, w_{1_2}, \cdots, w_{1_{k_1}}\}$, $B = \{w_{2_1}, w_{2_2}, \cdots, w_{2_{k_2}}\}$, $D_1 = \{v_{1_1}, v_{1_2}, \cdots, v_{1_{t_1}}\}$, $D_2 = \{v_{2_1}, v_{2_2}, \cdots, v_{2_{t_2}}\}$, $D_3 = \{v_{3_1}, v_{3_2}, \cdots, v_{3_{t_3}}\}$, $D_4 = \{v_{4_1}, v_{4_2}, \cdots, v_{4_{t_4}}\}$, $D_1' = \{v_{1_i} \in D_1 | N_{D_4}(v_{1_i}) \neq \emptyset\}$ and $D_3' = \{v_{3_i} \in D_3 | N_{D_4}(v_{3_i}) \neq \emptyset\}$. Since A is neither complete nor anticomplete to B, we always assume $w_{1_2} w_{2_1} \in E(G)$ and $w_{1_1} w_{2_1} \notin E(G)$. Thus, we have the following claim.

Claim 11.0.1. *1) Both D_1 and D_3 are cliques;*

2) $E(D_2, D_4) = \emptyset$;

3) If there exists $v_{1_i} \in D_1$ and $v_{3_j} \in D_3$ such that $v_{1_i} v_{3_j} \in E(G)$, then $N_{D_4}(v_{1_i}) = N_{D_4}(v_{3_j})$.

Proof. 1) Suppose to the contrary, there exist $v_{1_1}, v_{1_2} \in E(G)$ such that $v_{1_1} v_{1_2} \notin E(G)$, then $\langle w_{1_2}; v_{1_1}, v_{1_2}, w_{2_1} \rangle$ is a claw, showing a contradiction. Thus D_1 is a clique. Similarly, we can show that D_3 is also a clique.

2) Suppose there exist $v_{4_i} \in D_4$ and $v_{2_j} \in D_2$ such that $v_{4_i} v_{2_j} \in E(G)$, then $\langle v_{2_j}; v_{4_i}, w_{1_1}, w_{2_1} \rangle$

is a claw, giving a contradiction.

3) Suppose this is not true. If there exists $v_{4_k} \in N_{D_4}(v_{1_i}) \backslash N_{D_4}(v_{3_j})$, then $\langle v_{1_i}; v_{4_k}, v_{3_j}, w_{1_1}\rangle$ is a claw. If there exists $v_{4_t} \in N_{D_4}(v_{3_j}) \setminus N_{D_4}(v_{1_i})$, then $\langle v_{3_i}; v_{4_t}, v_{1_i}, w_{2_1}\rangle$ is a claw, showing a contradiction. ∎

In this chapter, we will show the following two propositions.

Proposition 19. *If G is a 3-connected $\{claw, Z_3\}$-free graph contains no twins and admits a nondominating W-join, then G contains a strong spanning Halin subgraph.*

Proposition 20. *If G is a 3-connected $\{claw, B_{1,2}\}$-free graph contains no twins and admits a nondominating W-join, then G contains a strong spanning Halin subgraph.*

11.1 Proof of 3-connected $\{claw, Z_3\}$-free graphs admit nondominating W-joins

Before we prove Proposition 19, we want to show following claims first.

Claim 11.1.1. *For all $v_{3_i} \in D_3$, $|N_{D_4}(v_{3_i})| \leq 1$.*

Proof. Suppose to the contrary, there exist $v_{3_i} \in D_3$ and $v_{4_1}, v_{4_2} \in D_4$ such that $v_{4_1}v_{3_i}, v_{4_2}v_{3_i} \in E(G)$, then $\langle v_{4_1}, v_{4_2}, v_{3_i}; w_{2_1}w_{1_2}w_{1_1}\rangle$ is a Z_3, giving a contradiction. ∎

Now we want to find a strong spanning Halin subgraph in G by following three cases depending on whether D_1 or D_3 is empty.

Case 1: Suppose $D_1 \neq \emptyset$ and $D_3 \neq \emptyset$.

Claim 11.1.2. *If $D_1 \neq \emptyset$ and $D_3 \neq \emptyset$, then $E(D_3, D_4) \neq \emptyset$.*

Proof. Suppose this is not true, $E(D_3, D_4) = \emptyset$. Since $E(D_2, D_4) = \emptyset$ and G is 3-connected, we have $|D_1'| \geq 3$. We always assume that $v_{1_1}v_{4_1} \in E(G)$ and know that $E(D_1', D_3) = \emptyset$ by Claim 11.1.1. For any $w_{1_i} \in A$, either $N_B(w_{1_i}) = \emptyset$ or $N_B(w_{1_i}) = B$. Otherwise, assume there exist $w_{2_j}, w_{2_k} \in B$ such that $w_{1_i}w_{2_j} \in E(G)$ and $w_{1_i}w_{2_k} \notin E(G)$, then $\langle v_{3_1}, w_{2_k}, w_{2_j}; w_{1_i}v_{1_1}v_{4_1}\rangle$ is a Z_3. This in turn gives us $|A| = 2$ and $|B| \leq 2$ since A contains no twins and $|A| \geq |B|$.

If $D_2 = \emptyset$, then $|D_3| = 1$, otherwise $\langle v_{3_1}, v_{3_2}, w_{2_1}; w_{1_2}v_{1_1}v_{4_1}\rangle$ is a Z_3. This implies $|B| = 2$ since $deg_G(w_{2_1}) \geq 3$. Moreover, D_4 is an independent set. Otherwise, we may assume $v_{4_1}v_{4_2} \in E(G)$, then either $\langle w_{2_2}, w_{2_1}, w_{1_2}; v_{1_1}v_{4_1}v_{4_2}\rangle$ or $\langle v_{4_1}, v_{4_2}, v_{1_1}; w_{1_2}w_{2_1}v_{3_1}\rangle$ is a Z_3. Since G is 3-connected, we may assume $v_{1_1}, v_{1_2}, v_{1_3} \in N_{D_4}(v_{4_1})$, then $v_{1_1}, v_{1_2}, v_{1_3}$ are twins, showing a contradiction.

If $D_2 \neq \emptyset$, we claim that $D_1 \cup D_2$ is a clique. Since for any $v_{2_i} \in D_2$ and $v_{1_j} \in D_1 \setminus \{v_{1_1}\}$, to avoid $\langle v_{1_1}, v_{1_j}, w_{1_1}; v_{2_i}w_{2_1}v_{3_1}\rangle$ be a Z_3, we have $v_{2_i}v_{1_1} \in E(G)$ or $v_{2_i}v_{1_j} \in E(G)$ or $v_{3_1}v_{2_i} \in E(G)$ or $v_{3_1}v_{1_j} \in E(G)$. In fact, we can assume $v_{2_i}v_{3_1} \in E(G)$ or $v_{2_i}v_{1_1} \in E(G)$, otherwise, either $\langle v_{1_1}, v_{1_3}, v_{1_j}; v_{2_i}w_{2_1}v_{3_1}\rangle$ is a Z_3 or $\langle v_{1_3}, v_{1_1}, v_{1_j}; v_{3_1}w_{2_1}v_{2_i}\rangle$ is a Z_3 or $\langle v_{1_j}; v_{1_1}, v_{2_i}, v_{3_i}\rangle$ is a claw, where $v_{1_3} \in D_1'$. If $v_{2_i}v_{3_1} \in E(G)$, for any $v_{1_k} \in D_1'$ and $v_{4_k} \in N_{D_4}(v_{1_k})$, since $\langle v_{3_1}, w_{2_1}, v_{2_i}; w_{1_1}v_{1_k}v_{4_k}\rangle$ is not a Z_3 and $\langle v_{1_k}; v_{3_1}, v_{4_k}, w_{1_1}\rangle$ is not a claw, we have $v_{1_k}v_{2_i} \in E(G)$. This implies $\{v_{2_i}\} \cup D_1'$ is a clique. If there exists $v_{1_t} \in D_1 \setminus D_1'$, to avoid $\langle v_{1_k}; v_{1_t}, v_{2_i}, v_{4_k}\rangle$ be a claw, we have $v_{1_t}v_{2_i} \in E(G)$, which implies $D_1 \cup D_2$ is a clique. If $v_{1_1}v_{2_i} \in E(G)$ but $v_{2_i}v_{3_1} \notin E(G)$, for any $v_{1_t} \in D_1 \setminus D_1'$, since $\langle v_{1_1}; v_{1_t}, v_{2_i}, v_{4_1}\rangle$ is not a claw, we have $v_{1_t}v_{2_i} \in E(G)$. For any $v_{1_k} \in D_1'$, since $\langle v_{1_1}; v_{1_k}, v_{4_1}, v_{2_i}\rangle$ is not a claw, we all get $v_{1_k}v_{4_1} \in E(G)$. Moreover, $\langle v_{4_1}, v_{1_k}, v_{1_1}; v_{2_i}w_{2_1}v_{3_1}\rangle$ is not a Z_3 implies $v_{1_k}v_{2_i} \in E(G)$. Thus, $D_1 \cup D_2$ is a clique.

Since G is Z_3-free, we have $|N_{D_4}(v_{1_i})| \leq 1$ for all $v_{1_i} \in D_1$ and there does not exist any triangle in D_4. However, this will force there exists a pair of twins in D_1', showing a contradiction.

From above statements, we get $E(D_3, D_4) \neq \emptyset$. ■

Therefore, from here and after, we always assume $v_{3_1}v_{4_1} \in E(G)$ for this case.

Claim 11.1.3. *If $E(D_3, D_4) \neq \emptyset$, we have following conclusions.*

1) For any $v_{4_i} \in D_4$, if $E(v_{4_i}, D_3) \neq \emptyset$, then $\{v_{4_i}\} \cup D_1$ is a clique. Which implies $D_1 = D_1'$ and D_4 is connected;

2) If $D_2 \neq \emptyset$, then $|D_4| = 1$ and if $D_2 = \emptyset$, then $|D_4| \leq 2$.

Proof. 1) Suppose there exist $v_{4_i} \in D_4$ and $v_{3_j} \in D_3$ such that $v_{4_i}v_{3_j} \in E(G)$. Since for any $v_{1_k} \in D_1$, $\langle v_{1_k}, w_{1_1}, w_{1_2}; w_{2_1} v_{3_j} v_{4_i}\rangle$ is not a Z_3 and $\langle v_{3_j}; v_{4_i}, v_{1_k}, w_{2_1}\rangle$ is not a claw, we have $v_{4_i} v_{1_k} \in E(G)$. Since G is claw-free and 3-connected, we have D_4 is connected.

2) Suppose this is not true. If $D_2 \neq \emptyset$, by 1), we may assume there exist $v_{4_2} \in D_4 \setminus \{v_{4_1}\}$ such that $v_{4_2} v_{4_1} \in E(G)$. Since $\langle w_{1_2}, v_{2_1}, w_{2_1}; v_{3_1} v_{4_1} v_{4_2}\rangle$ is not a Z_3, we have $v_{2_1} v_{3_1} \in E(G)$, but this will force $\langle w_{1_1}, w_{1_2}, v_{2_1}; v_{3_1} v_{4_1} v_{4_2}\rangle$ to be a Z_3, giving a contradiction. If $D_2 = \emptyset$, since G is 3-connected, we may assume there exist $v_{4_2}, v_{4_3} \in D_4 \setminus \{v_{4_1}\}$ such that $dist(v_{4_2}, D_1) = 1$, $dist(v_{4_3}, D_1) = 1$ and $v_{4_2} v_{3_1}, v_{4_3} v_{3_1} \notin E(G)$. Then $v_{4_1} v_{4_2} v_{4_3}$ is a triangle since $\langle v_{4_1}; v_{4_2}, v_{4_3}, v_{3_1}\rangle$ is not a claw. However, this will force $\langle v_{4_3}, v_{4_2}, v_{4_1}; v_{3_1} w_{2_1} w_{1_2}\rangle$ to be a Z_3, showing a contradiction. ■

We begin to search a strong spanning Halin subgraph in G by following subcases depending on whether D_2 is empty.

Case 1.1: Assume that $D_2 = \emptyset$.

Claim 11.1.4. *If* $|D_4| = 2$*, then*

1) For any $w_{1_i} \in A$*, either* $N_B(w_{1_i}) = \emptyset$ *or* $N_B(w_{1_i}) = B$*, which implies* $|A| = 2$ *and* $|B| \leq 2$*;*

2) $E(D_1, D_3) \neq \emptyset$*;*

3) $|D_3'| \geq 2$*.*

Proof. Since G is 3-connected, by Claim 11.1.2 and Claim 11.1.3, we may assume there exists $v_{1_1} \in D_1$ such that $v_{4_1} v_{1_1}, v_{4_2} v_{1_1} \in E(G)$.

1) If there exist $w_{1_i} \in A$ and $w_{2_j}, w_{2_k} \in B$ such that $w_{1_i} w_{2_j} \in E(G)$ and $w_{1_i} w_{2_k} \notin E(G)$, then $\langle v_{4_1}, v_{4_2}, v_{1_1}; w_{1_i} w_{2_j} w_{2_k}\rangle$ is a Z_3, showing a contradiction. Since neither A nor B contains twins and $|A| \geq |B|$, we have $|A| = 2$ and $|B| \leq 2$.

2) Suppose to the contrary, $E(D_1, D_3) = \emptyset$. Since $E(D_4, D_3) \neq \emptyset$, then either there exists a vertex in D_4 has degree 2 or D_1 contains a twins.

3) If $D_3' = \{v_{3_1}\}$, then $\{v_{3_1}, w_{1_2}\}$ is a 2-cut, giving a contrary. ■

By Claim 11.1.4, we may assume $v_{1_2} v_{4_1}, v_{1_2} v_{4_2}, v_{1_3} v_{4_2}, v_{1_1} v_{3_2} \in E(G)$. Since both $(D_1 \setminus \{v_{1_1}, v_{1_2}\}) \cup A$ and $(D_3 \setminus \{v_{3_2}\}) \cup B$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_3} P_1 w_{1_2}$, $P_2 = w_{2_1} P_2 v_{3_1}$, in them, respectively. Let $C = P_1 P_2 \cup \{v_{4_2} v_{1_3}, v_{3_1} v_{4_1}\}$ be a cycle and all vertices on the path $v_{1_2}v_{1_1}v_{3_2}$ be stems of T with $N_C(v_{1_2}) = (D_1\backslash\{v_{1_1}, v_{1_2}\})\cup\{v_{4_1}, v_{4_2}\}$, $N_C(v_{1_1}) = A$ and $N_C(v_{3_2}) = B \cup (D_3 \setminus \{v_{3_2}\})$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 11.1 (1) as an example).

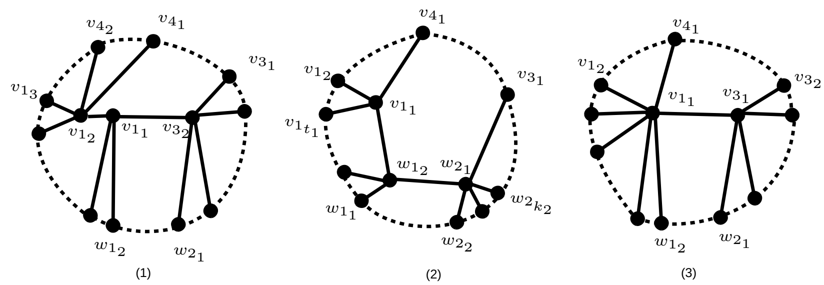

Figure 11.1. $D_1 \neq \emptyset$, $D_3 \neq \emptyset$ and $D_2 = \emptyset$.

Claim 11.1.5. *If* $|D_4| = 1$, *then*

1) $E(D_1', D_3') \neq \emptyset$;

2) $D_3 \cup D_4$ *is a clique;*

3) If $|D_3| = 1$, *then* $|B| \geq 2$ *and* $E(A \setminus \{w_{1_2}\}, B \setminus \{w_{2_1}\}) \neq \emptyset$.

Proof. 1) Since G is 3-connected and $E(D_3, D_4) \neq \emptyset$, we assume $v_{3_1}v_{4_1} \in E(G)$, then $\{v_{4_1}\} \cup D_1$ is a clique. Since v_{1_1} and v_{1_2} are not twins, we have $E(v_{1_1}, D_3') \neq \emptyset$ or $E(v_{1_2}, D_3') \neq \emptyset$ by Claim 11.0.1. Thus $E(D_1', D_3') \neq \emptyset$.

We may always assume $v_{1_1}v_{3_1} \in E(G)$, where $v_{1_1} \in D_1'$ and $v_{3_1} \in D_3'$.

2) This is clearly true if $|D_3| = 1$. If $|D_3| \geq 2$, assume there exists $v_{3_2} \in D_3 \setminus \{v_{3_1}\}$ and $v_{4_1} \in D_4$ such that $v_{3_2}v_{4_1} \notin E(G)$. Since $\langle w_{2_1}, v_{3_2}, v_{3_1}; v_{4_1}v_{1_2}w_{1_1}\rangle$ is not a Z_3, we have $v_{1_2}v_{3_1} \in E(G)$, then v_{1_1} and v_{1_2} are twins, showing a contradiction.

3) If $|B| = 1$, since A does not contain any twins, we have $|A| = 2$. If $|D_3| = 1$, then $deg_G(w_{2_1}) = 2$, showing a contradiction. Thus $|B| \geq 2$. Since neither $\{v_{3_1}, w_{1_2}\}$ nor $\{v_{3_1}, w_{2_1}\}$ is a 2-cut, we have $E(A \setminus \{w_{1_2}\}, B \setminus \{w_{2_1}\}) \neq \emptyset$. ■

If $|D_3| = 1$, let $w_{1_1}w_{2_2} \in E(G)$. Since both $(D_1 \setminus \{v_{1_1}\}) \cup (A \setminus \{w_{1_2}\})$ and $B \setminus \{w_{2_1}\}$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_2}P_1w_{1_1}$ and $P_2 = w_{2_2}P_2w_{2_{k_2}}$, in them, respectively. Let $C = P_1P_2 \cup \{w_{2_{k_2}}v_{3_1}, v_{3_1}v_{4_1}, v_{4_1}v_{1_2}\}$ be a cycle and all vertices on the path $v_{1_1}w_{1_2}w_{2_1}$ be stems of T with $N_C(v_{1_1}) = (D_1 \setminus \{v_{1_1}\}) \cup \{v_{4_1}\}$, $N_C(w_{1_2}) = A \setminus \{w_{1_2}\}$ and $N_C(w_{2_1}) = (B \setminus \{w_{2_1}\}) \cup \{v_{3_1}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 11.1 (2) as an example).

If $|D_3| \geq 2$, we also let $v_{1_1}v_{3_1} \in E(G)$. Since both $(D_1 \setminus \{v_{1_1}\}) \cup A$ and $(D_3 \setminus \{v_{3_1}\}) \cup B$ are cliques, there exist hamiltonian path, say $P_1 = v_{1_2}P_1w_{1_2}$ and $P_2 = w_{2_1}P_2v_{3_2}$, in them, respectively. Let $C = P_1P_2 \cup \{v_{3_2}v_{4_1}, v_{4_1}v_{1_2}\}$ be a cycle and $\{v_{1_1}, v_{3_1}\}$ be stems of T with $N_C(v_{1_1}) = V(P_1) \cup \{v_{4_1}\}$ and $N_C(v_{3_1}) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 11.1 (3) as an example).

Case 1.2: Assume $D_2 \neq \emptyset$, then $|D_4| = 1$.

Case 1.2.1: Assume that $D_3 \setminus D_3' \neq \emptyset$(similarly as $D_1 \setminus D_1' \neq \emptyset$).

Claim 11.1.6. *If $D_3 \setminus D_3' \neq \emptyset$, then 1) $D_1' \cup D_3'$ is a clique.*
2) $D_3' \cup D_2$ or $D_1' \cup D_2$ is a clique.

Proof. We may assume $v_{3_2} \in D_3 \setminus D_3'$.

1) For any $v_{1_i} \in D_1'$ and $v_{3_j} \in D_3'$, since $\langle w_{2_1}, v_{3_2}, v_{3_j}; v_{4_1}v_{1_i}w_{1_1}\rangle$ is not a Z_3, we have $v_{1_i}v_{3_j} \in E(G)$, which implies $D_1' \cup D_3'$ is a clique.

2) We may assume $v_{1_1}, v_{1_2} \in N_{D_1}(v_{4_1})$ by Claim 11.1.3. For any $v_{2_i} \in D_2$ and $v_{3_j} \in D_3 \setminus D_3'$, since $\langle v_{1_2}, v_{1_1}, w_{1_2}; v_{2_i}w_{2_1}v_{3_j}\rangle$ is not a Z_3, we have $v_{2_i}v_{3_j} \in E(G)$ or $v_{2_i}v_{1_2} \in E(G)$ or $v_{2_i}v_{1_1} \in E(G)$.

If $v_{2_i}v_{3_j} \in E(G)$, for any $v_{3_k} \in D_3'$, since $\langle w_{1_1}, w_{1_2}, v_{2_i}; v_{3_j}v_{3_k}v_{4_1}\rangle$ is not a Z_3, we have $v_{2_i}v_{3_k} \in E(G)$. For any $v_{2_t} \in D_2 \setminus \{v_{2_i}\}$, since $\langle v_{3_1}; v_{2_i}, v_{2_t}, v_{4_1}\rangle$ is not a claw, we get $v_{2_i}v_{2_t} \in E(G)$. Thus $D_3' \cup D_2$ is a clique.

If $v_{2_i}v_{3_j} \notin E(G)$, we may assume $v_{1_1}v_{2_i} \in E(G)$ by symmetric. For any $v_{1_j} \in D_1' \setminus \{v_{1_1}\}$, since $\langle v_{4_1}, v_{1_j}, v_{1_1}; v_{2_i}w_{2_1}v_{3_2}\rangle$ is not a Z_3, $v_{2_i}v_{1_j} \in E(G)$. For any $v_{2_t} \in D_2 \setminus \{v_{2_i}\}$, since $\langle v_{1_1}; v_{4_1}, v_{2_i}, v_{2_t}\rangle$ is not a claw, we have $v_{2_i}v_{2_t} \in E(G)$. Thus $D_1' \cup D_2$ is a clique. ■

We may always assume $D_1' \cup D_2$ is a clique.

If $|D'_3| \geq 2$, we denote by $v_{1_1}, v_{1_2}, v_{3_1}, v_{3_2} \in N_G(v_{4_1})$. Since $(D_1 \setminus \{v_{1_1}\}) \cup A$, $D_2 \setminus \{v_{3_1}\}$ and $B \cup D_3$ are cliques, there exists hamiltonian paths, say $P_1 = v_{1_2}P_1w_{1_2}$, $P_2 = v_{2_1}P_2v_{2_{t_2}}$ and $P_3 = w_{2_1}P_3v_{3_2}$, in them, respectively. Let $C = P_1P_2P_3 \cup \{v_{3_2}v_{4_1}, v_{4_1}v_{1_2}\}$ be a cycle and $\{v_{1_1}, v_{3_1}\}$ be stems of T with $N_C(v_{1_1}) = V(P_1) \cup V(P_2) \cup \{v_{4_1}\}$ and $N_C(v_{3_1}) = V(P_3)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 11.2 (1) as an example).

If $|D'_3| = 1$ and $|B| \geq 2$, we denote by $v_{1_1}, v_{1_2}, v_{3_1} \in N_G(v_{4_1})$. Since $D_1 \setminus \{v_{1_1}\}$, $(A \setminus \{w_{1_2}\}) \cup D_2$ and $(B \setminus \{w_{2_1}\}) \cup D_3$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_2}P_1v_{1_{t_1}}$, $P_2 = w_{1_1}P_2v_{2_{t_2}}$ and $P_3 = w_{2_2}P_3v_{3_1}$, in them, respectively. Let $C = P_1P_2P_3 \cup \{v_{3_1}v_{4_1}, v_{4_1}v_{1_2}\}$ be a cycle and all vertices on the path $v_{1_1}w_{1_2}w_{2_1}$ be stems of T with $N_C(v_{1_1}) = V(P_1) \cup \{v_{4_1}\}$, $N_C(w_{1_2}) = V(P_2)$ and $N_C(w_{2_1}) = V(P_3)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 11.2(2) as an example).

If $D'_3 = \{v_{3_1}\}$ and $|B| = 1$. Since $\{w_{2_1}, v_{3_1}\}$ is not a 2-cut, we have $E(D_3 \setminus D'_3, D_2) \neq \emptyset$ or $E(D_3 \setminus D'_3, D_1) \neq \emptyset$. If $E(D_3 \setminus D'_3, D_2) \neq \emptyset$, we may assume $v_{3_2}v_{2_1} \in E(G)$. Since $D_1 \setminus \{v_{1_1}\}$, $(A \setminus \{w_{1_2}\}) \cup D_2$ and D_3 are cliques, there exist hamiltonian paths, say $P_1 = v_{1_2}P_1w_{1_2}$, $P_2 = v_{2_1}P_2v_{2_{t_2}}$ and $P_3 = v_{3_2}P_3v_{3_1}$, in them, respectively. Let $C = P_1P_2P_3 \cup \{v_{3_1}v_{4_1}, v_{4_1}v_{1_2}\}$ be a cycle and all vertices on the path $v_{1_1}w_{1_2}w_{2_1}$ be stems of T with $N_C(v_{1_1}) = V(P_1) \cup \{v_{4_1}\}$, $N_C(w_{1_2}) = V(P_2)$ and $N_C(w_{2_1}) = V(P_3)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 11.2(3) as an example). If $E(D_3 \setminus D'_3, D_1) \neq \emptyset$, we may assume $v_{3_2}v_{1_{t_1}} \in E(G)$, where $v_{1_{t_1}} \in D_1 \setminus D'_1$. Since $D_1 \setminus \{v_{1_1}\}$, $A \cup D_2$ and $B \cup (D_3 \setminus \{v_{3_1}\})$ are cliques, there exists hamiltonian paths, say $P_1 = v_{1_2}P_1v_{1_{t_1}}$, $P_2 = w_{1_1}P_2v_{2_{t_2}}$ and $P_3 = w_{2_1}P_3v_{3_2}$, in them, respectively. Let $C = P_1P_2P_3$ be a cycle and all vertices on the path $v_{1_1}v_{4_1}v_{3_1}$ be stems of T with $N_C(v_{1_1}) = V(P_1) \cup V(P_2)$, $N_C(v_{4_1}) = \{v_{1_2}\}$ and $N_C(v_{3_1}) = V(P_3)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 11.2(4) as an example).

Case 1.2.2: Suppose $D_3 = D'_3$ and $D_1 = D'_1$.

If $|D_3 \cup B| \geq 3$, since $D_1 \setminus \{v_{1_1}\}$, $(A \setminus \{w_{1_2}\}) \cup D_2$ and $(B \setminus \{w_{2_1}\}) \cup D_3$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_2}P_1v_{1_{t_1}}$, $P_2 = w_{1_1}P_2v_{2_1}$ and $P_3 = w_{2_2}P_3v_{3_1}$,

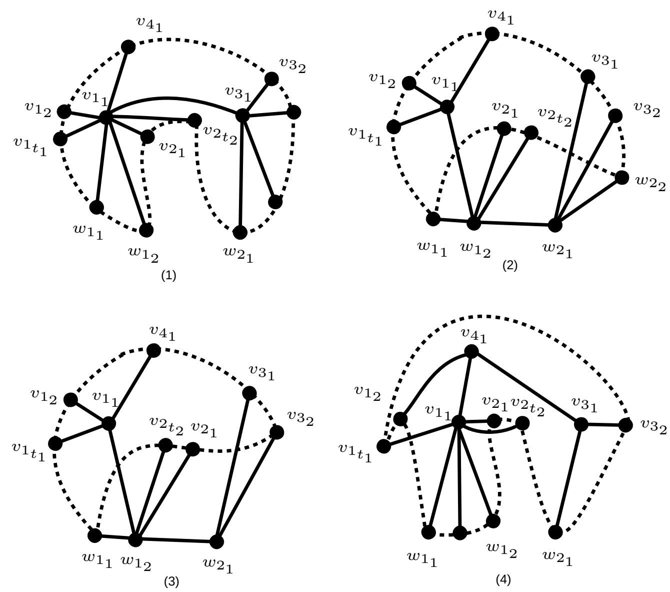

Figure 11.2. $D_3 \setminus D_3' \neq \emptyset$ and $D_2 \neq \emptyset$.

in them, respectively. Let $C = P_1P_2P_3 \cup \{v_{4_1}v_{1_2}, v_{4_1}v_{3_1}\}$ be a cycle and all vertices on the path $v_{1_1}w_{1_2}w_{2_1}$ be stems of T with $N_C(v_{1_1}) = V(P_1) \cup \{v_{4_1}\}$, $N_C(w_{1_2}) = V(P_2)$ and $N_C(w_{2_1}) = V(P_3)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 11.3(1) as an example).

If $D_3 = \{v_{3_1}\}$, $B = \{w_{2_1}\}$ and $E(D_2, D_3) \neq \emptyset$, we may assume $v_{2_1}v_{3_1} \in E(G)$. Since $D_1 \setminus \{v_{1_1}\}$ and $(A \setminus \{w_{1_2}\}) \cup D_2 \setminus \{v_{2_1}\}$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_2}P_1v_{1_{t_1}}$ and $P_2 = w_{1_1}P_2v_{2_2}$, in them, respectively. Let $C = P_1P_2 \cup \{v_{4_1}v_{1_2}, v_{4_1}v_{3_1}, v_{3_1}w_{2_1}, w_{2_1}v_{2_2}\}$ be a cycle and all vertices on the path $v_{1_1}w_{1_2}v_{2_1}$ be stems of T with $N_C(v_{1_1}) = V(P_1) \cup \{v_{4_1}\}$, $N_C(w_{1_2}) = V(P_2)$ and $N_C(w_{2_1}) = \{v_{3_1}, w_{2_1}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 11.3(2) as an example).

If $D_3 = \{v_{3_1}\}$, $B = \{w_{2_1}\}$ and $E(D_1, D_3) \neq \emptyset$, we may assume $v_{1_1}v_{3_1} \in E(G)$. Since $D_1 \setminus \{v_{1_1}\}$ and $(A \setminus \{w_{1_2}\}) \cup D_2 \setminus \{v_{2_1}\}$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_2}P_1v_{1_{t_1}}$ and $P_2 = w_{1_1}P_2v_{2_1}$, in them, respectively. Set $C = P_1P_2 \cup \{v_{4_1}v_{1_2}, v_{4_1}v_{3_1}, v_{3_1}w_{2_1}, w_{2_1}v_{2_1}\}$ be a cycle and $\{v_{1_1}, w_{1_2}\}$ be stems of T with $N_C(v_{1_1}) = V(P_1) \cup \{v_{4_1}, v_{3_1}\}$ and $N_C(w_{1_2}) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See

Figure 11.3(3) as an example).

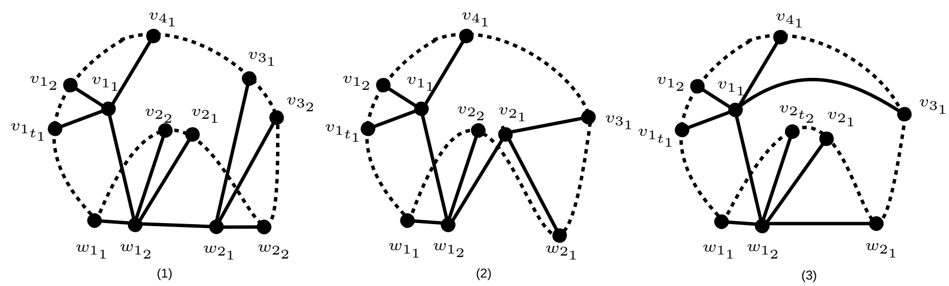

Figure 11.3. $D_1 = D_1'$, $D_3 = D_3'$ and $D_2 \neq \emptyset$.

Case 2: Suppose $D_1 = \emptyset$ and $D_3 \neq \emptyset$.

Claim 11.1.7. *If $D_1 = \emptyset$ and $D_3 \neq \emptyset$, then 1) $|D_3| \geq 3$;*

2) There does not exist $v_{3_i} \in D_3$ and $v_{4_j} \in D_4$ such that $dist(v_{3_i}, v_{4_j}) \geq 2$, which implies D_4 is an independent set;

3) $|D_2| \geq 2$.

Proof. 1) This is clearly true since $E(D_2, D_4) = \emptyset$ and G is 3-connected.

2) Suppose to the contrary, there exist $v_{3_1} \in D_3$ and $v_{4_1}, v_{4_2} \in D_4$ such that $v_{3_1}v_{4_1}, v_{4_1}v_{4_2} \in E(G)$ and $v_{3_1}v_{4_2} \notin E(G)$, then either $\langle w_{1_2}, v_{2_1}, w_{2_1}; v_{3_1}v_{4_1}v_{4_2}\rangle$ or $\langle w_{1_1}, w_{1_2}, v_{2_1}; v_{3_1}v_{4_1}v_{4_2}\rangle$ is a Z_3, giving a contradiction. By Claim 11.1.1, D_4 is an independent set.

3) Since D_4 is an independent set and G is 3-connected, for any $v_{4_i} \in D_4$, there exist at least three vertices, say $v^1_{3_i}, v^2_{3_i}, v^3_{3_i}$ in $N_{D_3}(v_{4_i})$. Since $v^1_{3_i}, v^2_{3_i}, v^3_{3_i}$ are not twins, $|D_2| \geq 2$. ■

Denote by $v^1_{3_i}, v^2_{3_i}, v^3_{3_i} \in N_{D_3}(v_{4_i})$ for every $v_{4_i} \in D_4$ and $D_3'' = D_3 \setminus \cup_{i=1}^{t_4}\{v^1_{3_i}, v^2_{3_i}, v^3_{3_i}\} = \{v_{3_1}', v_{3_2}', \cdots, v_{3_{t_3'}}'\}$. In particular, we may assume $v_{2_1}v^2_{3_1}, v_{2_2}v^1_{3_1} \in E(G)$. Since both $A \cup (D_2 \setminus \{v_{2_1}\})$ and $D_3'' \cup B$ are cliques, there exist hamiltonian paths, say $P_1 = w_{1_2}P_1v_{2_2}$ and $P_2 = w_{2_1}P_2v_{3_1}'$, in them, respectively. Let $C = P_1P_2 \cup (\cup_{i=1}^{t_4-1}\{v^1_{3_i}v_{4_i}, v_{4_i}v^3_{3_i}, v^3_{3_i}v^1_{3_{i+1}}\}) \cup \{v^1_{3_{t_4}}v_{4_{t_4}}, v_{4_{t_4}}v^3_{3_{t_4}}, v^3_{3_{t_4}}v_{3_1}', w_{2_1}w_{1_2}, v_{2_2}v^1_{3_1}\}$ be a cycle and all vertices on the path $v_{2_1}v^2_{3_1}v^2_{3_2}\cdots v^2_{3_{t_4}}$ be stems of T with $N_C(v_{2_1}) = V(P_1)$, $N_C(v^2_{3_1}) = V(P_2) \cup \{v^1_{3_1}, v^3_{3_1}, v_{4_1}\}$ and $N_C(v^2_{3_i}) =$

$\{v^1_{3_i}, v^3_{3_i}, v_{4_i}\}$ for all $i \in [2, t_4]$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 11.4(1) as an example).

Case 3: Assume $D_1 \neq \emptyset$ and $D_3 = \emptyset$.

Claim 11.1.8. *We have D_4 is an independent set or $D_1 \cup D_2$ is a clique.*

Proof. We may assume D_4 is not an independent set, i.e. there exist $v_{4_1}, v_{4_2} \in D_4$ such that $v_{4_1}v_{4_2} \in E(G)$. For any $v_{2_i} \in D_2$ and $v_{1_j} \in D_1$, if $v_{1_j} \in N_{D_1}(v_{4_1}) \setminus N_{D_1}(v_{4_2})$(or $v_{1_j} \in N_{D_1}(v_{4_2}) \setminus N_{D_1}(v_{4_1})$), since $\langle w_{2_1}, v_{2_i}, w_{1_2}; v_{1_j}v_{4_1}v_{4_2}\rangle$ is not a Z_3, we have $v_{1_j}v_{2_i} \in E(G)$; if $v_{1_j} \in N_{D_1}(v_{4_2}) \cap N_{D_1}(v_{4_1})$, since $\langle v_{4_1}, v_{4_2}, v_{1_j}; w_{1_1}v_{2_i}w_{2_1}\rangle$ is not a Z_3, we have $v_{1_j}v_{2_i} \in E(G)$. Thus $D'_1 \cup D_2$ is a clique. If $D_1 \setminus D'_1 \neq \emptyset$, we may assume there exists $v_{1_k} \in D_1 \setminus (N_{D_1}(v_{4_1}) \cup N_{D_1}(v_{4_2}))$. Since $\langle v_{1_j}; v_{4_1}, v_{2_i}, v_{1_k}\rangle$ is not a claw, we have $v_{1_k}v_{2_i} \in E(G)$. Therefore $D_1 \cup D_2$ is a clique. ■

If $D_1 \cup D_2$ is a clique and $D_3 = \emptyset$, then G admits a 1-join. Similarly as Chapter 8, we can find a strong spanning Halin subgraph in G.

If D_4 is an independent set, since G is 3-connected, we may denote by $\{v^1_{1_i}, v^2_{1_i}, v^3_{1_i}\} \in N_{D_1}(v_{4_i})$ for every $v_{4_i} \in D_4$ and $D'_1 = D_1 \setminus N_{D_1}(D_4) = \{v'_{1_1}, v'_{1_2}, \cdots, v'_{1_{t'_1}}\}$. In particular, $v^1_{1_1}v_{2_1} \in E(G)$ and $v^2_{1_1}v_{2_2} \in E(G)$ since $N_{D_1}(v_{4_1})$ does not contain twins. Since both $D'_1 \cup A$ and $(D_2 \setminus \{v_{2_1}\}) \cup B$ are cliques, there exist hamiltonian paths, say $P_1 = v'_{1_1}P_1w_{1_2}$ and $P_2 = w_{2_1}P_2v_{2_2}$, in them, respectively. Let $C = P_1P_2 \cup (\cup_{i=1}^{t_4-1}\{v^2_{1_i}v_{4_i}, v_{4_i}v^3_{1_i}, v^3_{1_i}v^2_{1_{i+1}}\}) \cup \{v^2_{1_{t_4}}v_{4_{t_4}}, v_{4_{t_4}}v^3_{1_{t_4}}, v^3_{1_{t_4}}v'_{1_1}, w_{1_2}w_{2_1}, v_{2_2}v^2_{1_1}\}$ be a cycle and all vertices on the path $v_{2_1}v^1_{1_1}v^1_{1_2}\cdots v^1_{1_{t_4}}$ be stems of T with $N_C(v_{2_1}) = V(P_2)$, $N_C(v^1_{1_1}) = V(P_1) \cup \{v_{4_1}, v^2_{1_1}, v^3_{1_1}\}$ and $N_C(v^1_{1_i}) = \{v_{4_i}, v^2_{1_i}, v^3_{1_i}\}$ for all $i \in [2, t_4]$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 11.4(2) as an example).

11.2 Proof of 3-connected $\{claw, B_{1,2}\}$-free graphs admit nondominating W-joins

Before we prove Proposition 20, we want to show the following claim first.

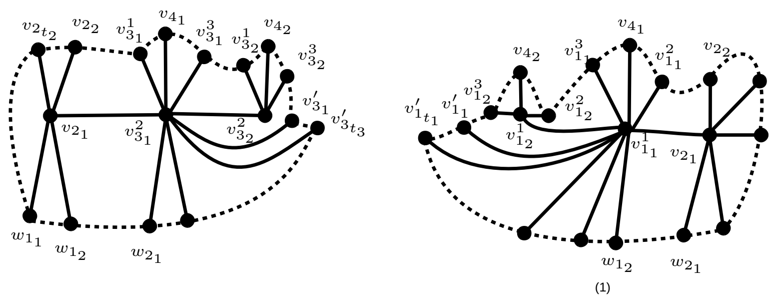

Figure 11.4. Either D_1 or D_3 is empty.

Claim 11.2.1. *1) For any component D'_{4_i} of D_4, $D'_{4_i} \cup N_{D_1}(D'_{4_i})$ is a clique;*
2) For any vertex $v_{4_i} \in D_4$, if $N_{D_3}(v_{4_i}) \neq \emptyset$, then $\{v_{4_i}\} \cup D_3$ is a clique. In particular, if $E(v_{4_i}, D_1) \neq \emptyset$, then $\{v_{4_i}\} \cup D_3$ is a clique.

Proof. We may assume there exists $v_{4_k} \in D_4$ and $v_{1_j} \in D_1$ such that $v_{4_k}v_{1_j} \in E(G)$, then there does not exist $v_{4_i} \in D_4$ such that $dist(v_{4_i}, v_{1_j}) \geq 2$. Otherwise, assume $v_{4_i}v_{4_k} \in E(G)$, then $\langle w_{2_1}; w_{1_2}, w_{1_1}, v_{1_j}; v_{4_k}v_{4_i}\rangle$ is a $B_{1,2}$, showing a contradiction. Since G is claw-free and D_1 is a clique, we have $D'_{4_i} \cup N_{D_1}(D'_{4_i})$ is a clique.

2) If there exist $v_{4_i} \in D_4$ and $v_{3_j} \in D_3$ such that $v_{4_i}v_{3_j} \in E(G)$, then for any $v_{3_k} \in D_3 \setminus \{v_{3_j}\}$, we have $v_{4_i}v_{3_k} \in E(G)$ since $\langle v_{4_i}; v_{3_j}, v_{3_k}, w_{2_1}; w_{1_2}w_{1_1}\rangle$ is not a $B_{1,2}$, which implies $\{v_{4_i}\} \cup D_3$ is a clique. In particular, if there exists $v_{1_j} \in D_1$ such that $v_{4_i}v_{1_j} \in E(G)$, then for any $v_{3_k} \in D_3$, we have $v_{4_i}v_{3_k} \in E(G)$ since $\langle v_{4_i}; v_{1_j}, w_{1_1}, w_{1_2}; w_{2_1}v_{3_k}\rangle$ is not $B_{1,2}$. Which in turn gives $\{v_{4_i}\} \cup D_3$ is a clique. ■

Now we want to find a strong spanning Halin subgraph in G depends on whether D_1 or D_3 is empty.

Case 1: Suppose $D_1 \neq \emptyset$ and $D_3 = \emptyset$.

Claim 11.2.2. *If $D_1 \neq \emptyset$ and $D_3 = \emptyset$, then D_4 is an independent set.*

Proof. Suppose this is not true. Let D'_{4_1} be a maximal component of D_4 with $|D'_{4_1}| \geq 2$, then $D'_{4_1} \cup N_{D_1}(D'_{4_1})$ is a clique by Claim 11.2.1. This implies all vertices in D'_{4_1} are twins.
■

Similarly as G is Z_3-free with $D_1 \neq \emptyset$ and $D_3 = \emptyset$ in last subsection, we can find a strong spanning Halin subgraph in G.

Case 2: Assume that $D_1 = \emptyset$ and $D_3 \neq \emptyset$.

Since G does not contain any twins, by Claim 11.2.1, $|D_4| = 1$ and $D_2 \cup D_3$ is 2-connected. We may assume $v_{3_1}v_{2_1}, v_{3_2}v_{2_2} \in E(G)$. Similarly as G is Z_3-free with $D_1 = \emptyset$ and $D_3 \neq \emptyset$ in last subsection, we can find a strong spanning Halin subgraph in G.

Case 3: Assume that $D_1 \neq \emptyset$ and $D_3 \neq \emptyset$.

Case 3.1: Assume that $E(D_1, D_4) = \emptyset$. Similarly as $D_1 = \emptyset$, we have $D_4 = \{v_{4_1}\}$, $|D_2| \geq 2$ and $D_2 \cup D_3$ is 2-connected. Moreover, we have the following claim.

Claim 11.2.3. *If $E(D_1, D_4) = \emptyset$, then*
1) $E(D_1, D_3) = \emptyset$;
2) $D_1 \cup D_2$ is a clique;
3) $|D_1| = 1$.

Proof. By Claim 11.2.1, we know $\{v_{4_1}\} \cup D_3$ is a clique.

1) If there exist $v_{1_i} \in D_1$ and $v_{3_j} \in D_3$ such that $v_{1_i}v_{3_j} \in E(G)$, then $\langle v_{3_j}; v_{4_1}, v_{1_i}, w_{2_1}\rangle$ is a claw, giving a contradiction.

2) Suppose to the contrary, there exist $v_{1_i} \in D_1$ and $v_{2_j} \in D_2$ such that $v_{1_i}v_{2_j} \notin E(G)$. Since $\langle v_{1_i}; w_{1_2}, v_{2_j}, w_{2_1}; v_{3_k}v_{4_1}\rangle$ is not a $B_{1,2}$ for any $v_{3_k} \in D_3$, we have $v_{2_j}v_{3_k} \in E(G)$. However, this will force $\langle v_{4_1}; v_{3_k}, w_{2_1}, v_{2_j}; w_{1_1}v_{1_i}\rangle$ to be a $B_{1,2}$, showing a contradiction. Therefore, $D_1 \cup D_2$ is a clique.

3) Since D_1 does not contain any twins, we have $|D_1| = 1$. ■

Since $D_2 \cup D_3$ is 2-connected, we may assume $v_{2_1}v_{3_1}, v_{2_2}v_{3_2} \in E(G)$. Let $P_1 = v_{2_2}P_1w_{1_2}$ and $P_2 = w_{2_1}P_2v_{3_3}$ be hamiltonian paths in $D_1 \cup A \cup (D_2 \setminus \{v_{2_1}\})$ and $B \cup (D_3 \setminus \{v_{3_1}\})$, respectively. Set $C = P_1P_2 \cup \{v_{3_3}v_{4_1}, v_{4_1}v_{3_2}, v_{3_2}v_{2_2}\}$ be a cycle and $\{v_{2_1}, v_{3_1}\}$ be stems of T with $N_C(v_{2_1}) = V(P_1)$ and $N_C(v_{3_1}) = V(P_2) \cup \{v_{4_1}\}$. Then $H = T \cup C$ and H is a strong spanning Halin subgraph of G (See Figure 11.5 (1) as an example).

Case 3.2: Suppose that $E(D_1, D_4) \neq \emptyset$. We may assume $v_{1_1}v_{4_1} \in E(G)$.

Claim 11.2.4. *If $E(D_1, D_4) \neq \emptyset$, then $D_4 = \{v_{4_1}\}$.*

Proof. By Claim 11.2.1, $\{v_{4_1}\} \cup D_3$ is a clique. Since G is 3-connected, if $|D_4| \geq 2$, we may assume there exists $v_{4_2} \in D_4 \setminus \{v_{4_1}\}$ such that $N_{D_1 \cup D_3}(v_{4_2}) \neq \emptyset$, then $\{v_{4_2}\} \cup D_3$ is a clique. Since $\langle v_{3_1}; v_{4_1}, v_{4_2}, w_{2_1}\rangle$ is not a claw, we have $v_{4_1}v_{4_2} \in E(G)$. Moreover, since neither $\langle w_{2_1}; v_{3_1}, v_{4_2}, w_{2_1}; v_{1_1}w_{1_1}\rangle$ is a $B_{1,2}$ nor $\langle v_{2_1}; v_{4_2}, v_{1_1}, w_{2_1}\rangle$ is a claw, we have $v_{1_1}v_{4_2} \in E(G)$, which implies v_{4_1} and v_{4_2} are twins, showing a contradiction. ■

Claim 11.2.5. *If $\{v_{4_1}\} \cup D_1$ is not a clique, then for any $w_{1_i} \in A$, either $N_B(w_{1_i}) = \emptyset$ or $N_B(w_{1_i}) = B$, which implies $|A| = 2$ and $|B| \leq 2$.*

Proof. Suppose there exist $v_{1_1}, v_{1_{t_1}} \in D_1$ such that $v_{1_1}v_{4_1} \in E(G)$ and $v_{1_{t_1}}v_{4_1} \notin E(G)$. For any $w_{1_i} \in A$, if there exists $w_{2_j}, w_{2_k} \in B$ such that $w_{1_i}w_{2_j} \in E(G)$ and $w_{1_i}w_{2_k} \notin E(G)$, then $\langle v_{4_1}; v_{1_1}, v_{1_{t_1}}, w_{1_i}; w_{2_j}w_{2_k}\rangle$ is a $B_{1,2}$. Thus either $N_B(w_{1_i}) = \emptyset$ or $N_B(w_{1_i}) = B$. Which implies $|A| = 2$ and $|B| \leq 2$ since neither A nor B contains twins. ■

If $D_2 = \emptyset$, since $deg_G(v_{4_1}) \geq 3$ and G does not contain any twins, $E(D_1', D_3') \neq \emptyset$. Moreover, if $|D_1'| = 1$, since neither $\{v_{1_1}, w_{1_2}\}$ nor $\{v_{1_1}, w_{2_1}\}$ is a 2-cut, we have $E(A \setminus \{w_{1_2}\}, B \setminus \{w_{2_1}\}) \neq \emptyset$; if $|D_3'| = 1$, since neither $\{v_{3_1}, w_{1_2}\}$ nor $\{v_{1_1}, w_{2_1}\}$ is a 2-cut, we also have $E(A \setminus \{w_{1_2}\}, B \setminus \{w_{2_1}\}) \neq \emptyset$. Thus, similarly as G is Z_3-free with $D_2 = \emptyset$ and $|D_4| = 1$, we can find a strong spanning Halin subgraph in G.

If $D_2 \neq \emptyset$ and $N_{D_1}(v_{4_1}) = D_1$, i.e $D_1' = D_1$ and $D_3' = D_3$, similarly as G is Z_3 free, we can find a strong spanning Halin subgraph in G.

If $D_2 \neq \emptyset$ and $N_{D_1}(v_{4_1}) \neq D_1$, by Claim 11.2.5, we have $|B| \leq 2$. If $|B| = 2$, we may assume $|D_1'| \geq 2$ since $deg(v_{4_1}) \geq 3$. Since $(D_1 \setminus \{v_{1_1}\}) \cup (A \setminus \{w_{1_2}\})$, $D_2 \cup (B \setminus \{w_{2_1}\})$ and D_3 are cliques, there exist hamiltonian paths, say $P_1 = v_{1_2}P_1w_{1_1}$, $P_2 = v_{2_1}P_2w_{2_2}$ and $P_3 = v_{3_1}P_3v_{3_{t_3}}$, in them, respectively. Let $C = P_1P_2P_3 \cup \{v_{1_2}v_{4_1}, v_{4_1}v_{3_1}\}$ be a cycle and all vertices on the path $v_{1_1}w_{1_2}w_{2_1}$ be stems of T with $N_C(v_{1_1}) = (D_1 \setminus \{v_{1_1}\}) \cup \{v_{4_1}\}$, $N_C(w_{1_2}) = A \setminus \{w_{1_2}\}$ and $N_C(w_{2_1}) = (B \setminus \{w_{2_1}\}) \cup D_2 \cup D_3$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 11.5 (2) as an example).

If $|B| = 1$, since $\{v_{4_1}, w_{1_2}\}$ is not a 2-cut, we can assume $v_{2_1}v_{3_1} \in E(G)$ or $v_{1_1}v_{3_1} \in E(G)$. If $v_{2_1}v_{3_1} \in E(G)$, similarly as $|B| = 2$, we can find a strong spanning Halin subgraph in G. If $v_{1_1}v_{3_1} \in E(G)$ and $D_3 = \{v_{3_1}\}$, since $(D_1 \setminus \{v_{1_1}\}) \cup (A \setminus \{w_{1_2}\})$ and D_2 are cliques, there exist hamiltonian paths, say $P_1 = v_{1_2}P_1w_{1_1}$ and $P_2 = v_{2_1}P_2v_{2_{t_2}}$, in them, respectively. Let $C = P_1P_2 \cup \{v_{4_1}v_{1_2}, v_{4_1}v_{3_1}, v_{3_1}w_{2_1}, w_{2_1}v_{2_1}\}$ be a cycle and $\{v_{1_1}, w_{1_2}\}$ be stems of T with $N_C(v_{1_1}) = (D_1 \setminus \{v_{1_1}\}) \cup \{v_{4_1}, v_{3_1}\}$ and $N_C(w_{1_2}) = (A \setminus \{w_{1_2}\}) \cup D_2 \cup \{w_{2_1}\}$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 11.5(3) as an example). If $v_{1_1}v_{3_1} \in E(G)$ and $|D_3| \geq 2$, since $D_1 \setminus \{v_{1_1}\}$, $(A \setminus \{w_{1_2}\}) \cup D_2$ and $D_3 \setminus \{v_{3_1}\}$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_2}P_1v_{1_{t_1}}$, $P_2 = w_{1_1}P_2v_{2_1}$ and $P_3 = v_{3_2}P_3v_{3_{t_3}}$, in them, respectively. Let $C = P_1P_2P_3 \cup \{v_{2_1}w_{2_1}, w_{2_1}v_{3_{t_3}}, v_{3_2}v_{4_1}, v_{4_1}v_{1_2}\}$ be a cycle and all vertices on the path $v_{3_1}v_{1_1}w_{1_2}$ be stems of T with $N_C(v_{3_1}) = (D_3 \setminus \{v_{3_1}\}) \cup \{w_{2_1}\}$, $N_C(v_{1_1}) = (D_1 \setminus \{v_{1_1}\}) \cup \{v_{4_1}\}$ and $N_C(w_{1_2}) = (A \setminus \{w_{1_2}\}) \cup D_2$. Let $H = T \cup C$, it is easy to check that H is a strong spanning Halin subgraph of G (See Figure 11.5 (4) as an example).

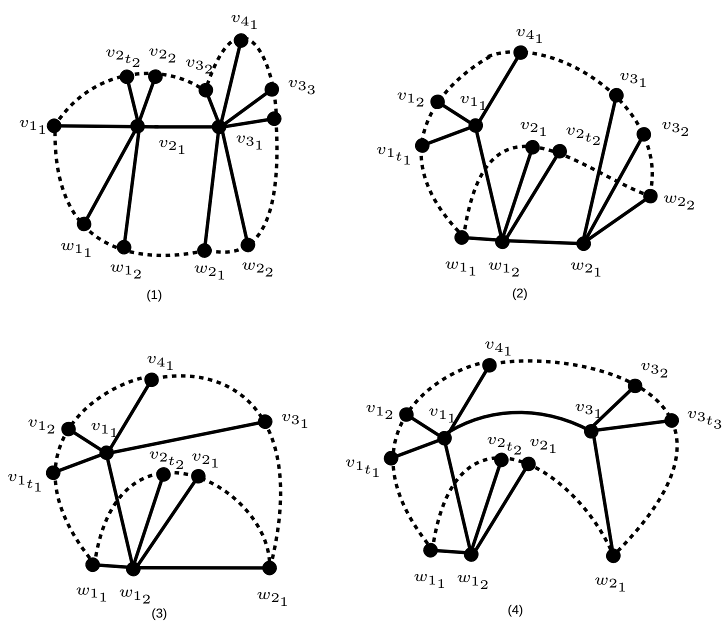

Figure 11.5. G admits a nondominating W-join.

Chapter 12

TWINS

Recall that we call u, v are *twins* if $N_{G\setminus\{u,v\}}(u) = N_{G\setminus\{u,v\}}(v)$. Let A, B be disjoint subsets of $V(G)$. By Section 4.2, we can always assume that G contains twins only in its 3-cut. We will still follow definitions and notations mentioned in Section 4.1 that G is a graph with n-vertex and S is a minimum vertex cut of G. Let G_1 and G_2 are the exact two components of $G \setminus S$, and $V_1 = V(G_1)$, $V_2 = V(G_2)$. Subject to the minimality of $|S|$, we always assume that $|V_1|$ is minimum. In particular, we denote by $S = \{x, y, z\}$ is a 3-cut of G with x and y are twins. By the definition of twins, we have $N_1(x) = N_1(y)$ and $N_2(x) = N_2(y)$. In particular, $|N_1(x)| \geq min\{2, |V_1|\}$ and $|N_2(x)| \geq min\{2, |V_2|\}$ by Lemma 4.1.2. Let $N_1^i(x) = \{v_{i_j} | dist(v_{i_j}, x) = i\} = \{v_{i_1}, \cdots, v_{i_{s_i}}\}$, $N_2^i(x) = \{w_{i_j} | dist(w_{i_j}, x) = i\} = \{w_{i_1}, \cdots, w_{i_{s_i}}\}$, $k_1 = max\{dist(v, x) | v \in V_1\}$, $k_2 = max\{dist(w, x) | w \in V_2\}$ and $v_{i_1}v_{(i+1)_1}, w_{j_1}w_{(j+1)_1} \in E(G)$ for all possible i and j.

In this chapter, we will prove following two propositions.

Proposition 21. *If G is a 3-connected $\{claw, Z_3\}$-free graph admits a pair of twins in its 3-cut. Then G contains a spanning Halin subgraph.*

Proposition 22. *If G is a 3-connected $\{claw, B_{1,2}, N\}$-free graph admits a pair of twins in its 3-cut. Then G contains a spanning Halin subgraph.*

12.1 Proof of 3-connected $\{claw, Z_3\}$-free graphs admit twins

In Section 4.3, we have been proved that if G is a 3-connected $\{claw, Z_3\}$-free graph with $|V_1| \geq 2$ and at least one of V_1 or V_2 is not a clique, then G contains a spanning Halin subgraph. Thus, in the following we always assume $|V_1| \geq 2$ and both V_1 and V_2 are cliques or $|V_1| = 1$.

If $|V_1| \geq 2$ and both V_1 and V_2 are cliques, by Claim 4.1.2, we denote by $v_1, v_3 \in N_1(y)$, $w_1, w_3 \in N_2(y)$, $v_2 \in N_1(z)$ and $w_2 \in N_2(z)$(Note that we may have $v_2 = v_3$ or $w_2 = w_3$.) If $yz \in E(G)$ and $|V_2| \geq 3$, since both $V_1 \setminus \{v_1\}$ and $V_2 \setminus \{w_1\}$ are cliques, there exist hamiltonian paths, say $P_1 = v_2P_1v_3$ and $P_2 = w_2P_2w_3$, in them, respectively. Let $C = P_1P_2 \cup \{v_3x, xw_3, w_2z, zv_2\}$ be a cycle and all vertices on the path v_1yw_1 be stems of T with $N_C(v_1) = V(P_1) \cup \{x\}$, $N_C(y) = \{z\}$ and $N_C(w_1) = V(P_2)$.

If $yz \in E(G)$ and $|V_2| = 2$. Let $C = xv_1v_2zw_2w_1x$ be a cycle and $\{y\}$ be the stem of T with $N_C(y) = \{x, v_1, v_2, z, w_2, w_1\}$

If $yz \notin E(G)$ and $|V_2| \geq 3$, we assume $w_4 \in N_2(z)$(Note that we may have $w_1 = w_4$). If $|V_1| \geq 3$, since both $V_1 \setminus \{v_1\}$ and $V_2 \setminus \{w_1, w_4\}$ are cliques, there exist hamiltonian paths, say $P_1 = v_2P_1v_3$ and $P_2 = w_2P_2w_3$, in them, respectively. Let $C = P_1P_2 \cup \{v_3x, xw_3, w_2z, zv_2\}$ be a cycle and all vertices on the path $v_1yw_1w_4$ be stems of T with $N_C(v_1) = V(P_1)$, $N_C(y) = \{x\}$, $N_C(w_1) = \{w_3\}$ and $N_C(w_4) = (V(P_2) \setminus \{w_3\}) \cup \{z\}$. If $|V_1| = 2$, Let $C = P_2 \cup \{v_1x, xw_3, w_2z, zv_2\}$ be a cycle and all vertices on the path yw_1w_4 be stems of T with $N_C(y) = \{v_1, v_2, x\}$, $N_C(w_1) = \{w_3\}$ and $N_C(w_4) = (V(P_2) \setminus \{w_3\}) \cup \{z\}$.

If $yz \notin E(G)$ and $|V_2| = 2$, then $|V_1| = 2$. Since $deg_G(z) \geq 3$, we may assume $V_2 = \{w_1, w_2\} \subseteq N_2(z)$ and $V_1 \subseteq N_1(z)$. Let $C = v_1v_2xw_1zv_1$ be a cycle and all vertices $\{y, w_2\}$ be stems of T with $N_C(y) = \{v_1, v_2, x\}$, and $N_C(w_2) = \{w_2, z\}$.

Let $H = T \cup C$, it is easy to check that H is a spanning Halin subgraph of G.

Therefore, in this section, from here and after, we always assume $V_1 = \{v\}$.

Claim 12.1.1. *1) we have $N_2^4(x) = \emptyset$, which implies $V_2 \subseteq N_2(x) \cup N_2^2(x) \cup N_2^3(x)$;*
2) If $N_2^3(x) \neq \emptyset$, then $N_2^3(x)$ is an independent set.

Proof. 1) Suppose there exists $w_{4_1} \in N_2^4(x)$, then $\langle x, y, w_{1_1}; w_{2_1}w_{3_1}w_{4_1}\rangle$ is a Z_3, giving a contradiction.

2) If there exist $w_{3_1}, w_{3_2} \in N_2^3(x)$ such that $w_{3_1}w_{3_2} \in E(G)$, then either $\langle x, y, w_{1_1}; w_{2_1}w_{3_1}w_{3_2}\rangle$ or $\langle w_{3_1}, w_{3_2}, w_{2_1}; w_{1_1}xv\rangle$ is a Z_3, showing a contradiction. ■

Let $\mathcal{C}_1, \mathcal{C}_2, \cdots, \mathcal{C}_k$ be all components of $N_2^2(x)$. Then following claims are true.

Claim 12.1.2. *Let $w_{2_s}w_{2_t}w_{2_k}$ be a path, may be not induced, in $\langle \mathcal{C}_i \rangle$. For any $w_{1_i} \in N_2(\mathcal{C}_i) \cap N_2(x)$, if $N_2(w_{1_i}) \cap \{w_{2_s}, w_{2_t}, w_{2_k}\} \neq \emptyset$, then $|N_2(w_{1_i}) \cap \{w_{2_s}, w_{2_t}, w_{2_k}\}| \geq 2$.*

Proof. If $w_{1_i}w_{2_s} \in E(G)$(or $w_{1_i}w_{2_k} \in E(G)$), since neither $\langle x, y, w_{1_i}; w_{2_s}w_{2_t}w_{2_k} \rangle$ nor $\langle w_{2_t}, w_{2_k}, w_{2_s}; w_{1_i}xv \rangle$ is a Z_3, we have $w_{1_i}w_{2_k} \in E(G)$ or $w_{1_i}w_{2_t} \in E(G)$.

If $w_{1_i}w_{2_t} \in E(G)$, since neither $\langle w_{2_t}; w_{1_i}, w_{2_k}, w_{2_s} \rangle$ is a claw nor $\langle w_{2_s}, w_{2_k}, w_{2_t}; w_{1_i}xv \rangle$ is a Z_3, we have $w_{1_i}w_{2_k} \in E(G)$ or $w_{1_i}w_{2_s} \in E(G)$.

Thus $|N_2(w_{1_i}) \cap \{w_{2_s}, w_{2_t}, w_{2_k}\}| \geq 2$. ■

Claim 12.1.3. *If $\mathcal{C}_i \cup (N_2(\mathcal{C}_i) \cap N_2(x))$ is not a clique, then for any $w_{1_i} \in N_2(\mathcal{C}_i) \cap N_2(x)$, $|N_2(w_{1_i}) \cap \mathcal{C}_i| \geq |\mathcal{C}_i| - 1$, which implies either $\mathcal{C}_i$ is a clique or $\langle \mathcal{C}_i \rangle$ is losing exact one edge.*

Proof. This is clearly true if $|\mathcal{C}_i| \leq 2$. If $|\mathcal{C}_i| \geq 3$, for any $w_{1_i} \in N_2(\mathcal{C}_i)$, by Claim 12.1.2, we know $|N_2(w_{1_i}) \cap \mathcal{C}_i| \geq 2$. We may let $u_1, u_2 \in N_2(w_{1_i}) \cap \mathcal{C}_i$, by Claim 12.1.2 again, we know $\mathcal{C}_i \setminus N_2(w_{1_i})$ is an independent set. If there exists $u_1', u_2' \in \mathcal{C}_i \setminus N_2(w_{1_i})$, we can assume there also exist $w_{1_j}, w_{1_k} \in N_2(\mathcal{C}_i)$ such that $w_{1_j}u_1', w_{1_k}u_2', u_1u_1', u_2u_2' \in E(G)$. By Claim 12.1.2 one more time, since $u_2'u_2u_1$ is a path, we have $w_{1_k}u_2 \in E(G)$. Moreover, since $u_2u_1u_1'$ is a path, we have $w_{1_k}u_1 \in E(G)$ or $w_{1_k}u_1' \in E(G)$, which implies $u_1'u_1u_2'$ or $u_1'u_2u_2'$ is a path. But this implies , $|N_2(w_{1_i}) \cap \{u_1, u_1', u_2'\}| = 1$ and $|N_2(w_{1_i}) \cap \{u_2, u_1', u_2'\}| = 1$, contradicts to Claim 12.1.2. Thus $|N_2(w_{1_i}) \cap \mathcal{C}_i| \geq |\mathcal{C}_i| - 1$ for all $w_{1_i} \in N_2(\mathcal{C}_i) \cap N_2(x)$. This in turn gives either $\mathcal{C}_i$ is a clique or $\langle \mathcal{C}_i \rangle$ is losing exactly one edge since G is claw-free. ■

Claim 12.1.4. *If there exist $w_{2_s}, w_{2_t} \in \mathcal{C}_i$ for some i such that $N_2(w_{2_s}) \cap N_2(w_{2_t}) \cap N_2(x) \neq \emptyset$, then for any $j \neq i$, $\mathcal{C}_j \cup (N_2(\mathcal{C}_j) \cap N_2(x))$ are cliques. In particular, if $\mathcal{C}_i \cup (N_2(\mathcal{C}_i) \cap N_2(x))$ is not a clique, then $|\mathcal{C}_j| = 1$.*

Proof. We may assume there exist $w_{1_i} \in N_2(w_{2_s}) \cap N_2(w_{2_t}) \cap N_2(x)$, $w_{2_j} \in \mathcal{C}_j$ and $w_{1_j} \in N_2(\mathcal{C}_j)$ such that $w_{1_j}w_{2_j} \in E(G)$. If there exists $w_{2_l} \in \mathcal{C}_j$ such that $w_{2_l}w_{2_j} \in E(G)$ and $w_{1_j}w_{2_l} \notin E(G)$, then $\langle w_{2_s}, w_{2_t}, w_{1_i}; w_{1_j}w_{2_j}w_{2_l} \rangle$ is a Z_3, showing a contradiction. Thus $\mathcal{C}_j \cup (N_2(\mathcal{C}_j) \cap N_2(x))$ is a clique for any $j \neq i$. In particular, if $\mathcal{C}_i \cup (N_2(\mathcal{C}_i) \cap N_2(x))$ is not a clique, we have $|\mathcal{C}_j| = 1$. ■

Claim 12.1.5. *For any $w \in N_2^3(x)$, if $N_2(w) \cap \mathcal{C}_i \neq \emptyset$, then*

1) $N_2(w) \subseteq \mathcal{C}_i$;

2) $|\mathcal{C}_i| \geq 2$;

3) $N_2(w) \cup (N_2(\mathcal{C}_i) \cap N_2(x))$ is a clique;

4) $\mathcal{C}_i \cup (N_2(\mathcal{C}_i) \cap N_2(x))$ is a clique;

5) $|\mathcal{C}_j| = 1$ for all $j \neq i$, which implies $N_2(N_2^3(x)) \subseteq \mathcal{C}_i$.

Proof. We may assume there exist $w_{2_i} \in \mathcal{C}_i$ and $w_{1_i} \in N_2(x)$ such that $w_{1_i}w_{2_i}, w_{2_i}w \in E(G)$.

1) If there exists $w_{2_j} \in \mathcal{C}_j$ such that $w_{2_j}w \in E(G)$ for some $j \neq i$, then $\langle x, y, w_{1_i}; w_{2_i}ww_{2_j}\rangle$ is a Z_3, showing a contradiction. Thus $N_2(w) \subseteq \mathcal{C}_i$.

2) Since G is 3-connected, $N_2^3(x)$ is an independent set and wz may be in $E(G)$, $|\mathcal{C}_i| \geq 2$.

3) Suppose this is not true, there exist $w_{2_i} \in N_2(w) \cap \mathcal{C}_i$, $w_{1_j} \in N_2(\mathcal{C}_i) \cap N_2(x)$ and $w_{2_j} \in \mathcal{C}_i$ such that $w_{1_j}w_{2_i} \notin E(G)$ and $w_{1_j}w_{2_j} \in E(G)$. We may also assume there exists $w_{2_j}w \in E(G)$, since $\langle x, y, w_{1_i}; w_{2_j}ww_{2_i}\rangle$ is not a Z_3, we have $w_{2_i}w_{2_j} \in E(G)$, which force $\langle w_{2_i}, w, w_{2_j}; w_{1_j}yv\rangle$ to be a Z_3. If $w_{2_j}w \notin E(G)$, let $Q = w_{2_j}w_{2_s}\cdots w_{2_i}w$ be the shortest path from w_{2_j} to w in $\mathcal{C}_i \cup \{w\}$, then $\langle x, y, w_{1_j}; w_{2_j}w_{2_s}w_{2_k}\rangle$ is a Z_3, where $w_{2_j}, w_{2_s}, w_{2_k}$ are the first three vertices of Q. Thus $N_2(w) \cup (N_2(\mathcal{C}_i) \cap N_2(x))$ is a clique.

4) If there exist $w_{2_k} \in \mathcal{C}_i \setminus N_2(w)$ and $w_{1_s} \in N_2(\mathcal{C}_i) \cap N_2(x)$ such that $w_{2_k}w_{1_s} \notin E(G)$. Since $\langle \mathcal{C}_i\rangle$ is connected, we may assume there exists $w_{2_i} \in N_2(w) \cap \mathcal{C}_i$ such that $w_{2_k}w_{2_i} \in E(G)$, then $\langle w_{2_i}; w_{2_k}, w_{1_s}, w\rangle$ is a claw.

5) Since $|\mathcal{C}_i| \geq 2$ and $\mathcal{C}_i \cup (N_2(\mathcal{C}_i) \cap N_2(x))$ is a clique, by Claim 12.1.4, we know $\mathcal{C}_j \cup (N_2(\mathcal{C}_j) \cap N_2(x))$ is also a clique for all $j \neq i$. If there exist two distinct vertices w_{2_k}, w_{2_l} in $\mathcal{C}_j$, we may denote by $w_{1_k} \in N_2(w_{2_k}) \cap N_2(x)$, $w_{2_i} \in N_2(w)$ and $w_{1_k} \in N_2(w)$, then $\langle w_{2_k}, w_{2_l}, w_{1_k}; w_{1_i}w_{2_i}w\rangle$ is a Z_3, showing a contradiction. Since G is 3-connected and claw-free, we have $N_2(N_2^3(x)) \subseteq \mathcal{C}_i$. ∎

Claim 12.1.6. *If there exists $w_{2_i} \in \mathcal{C}_i$ such that $zw_{2_i} \in E(G)$, then*

1) $N_2(z) \cap \mathcal{C}_j = \emptyset$ and $N_2(z) \cap (N_2(\mathcal{C}_j) \cap N_2(x)) = \emptyset$ for all $j \neq i$;

2) If $|N_2(x) \setminus N_2(w_{2_i})| \geq 2$*, then* $w_{1_i}z \in E(G)$ *for all* $w_{1_i} \in N_2(w_{2_i}) \cap N_2(x)$ *and* $xz \in E(G)$*;*

3) If $N_2^2(x) \neq \mathcal{C}_i$*, then* $|\mathcal{C}_j| = 1$ *for all* $j \neq i$*.*

Proof. 1) We may assume there exist $w_{2_i} \in \mathcal{C}_i$ and $w_{1_i} \in N_2(\mathcal{C}_i)$ such that $w_{2_i}z, w_{2_i}w_{1_i} \in E(G)$. If there exist $w_{2_j} \in \mathcal{C}_j$ such that $w_{2_j}z \in E(G)$, then $\langle z; v, w_{2_j}, w_{2_i}\rangle$ is a claw. If there exists $w_{1_j} \in N_2(\mathcal{C}_j) \cap N_2(x)$ such that $w_{1_j}z \in E(G)$, then $\langle z; v, w_{1_j}, w_{2_i}\rangle$ is a claw. Thus, we have $N_2(z) \cap \mathcal{C}_j = \emptyset$ and $N_2(z) \cap (N_2(\mathcal{C}_j) \cap N_2(x)) = \emptyset$ for all $j \neq i$.

2) We may assume there exist $w_{1_k}, w_{1_l} \in N_2(x) \setminus N_2(w_{2_i})$. For any $w_{1_i} \in N_2(w_{2_i}) \cap N_2(x)$, since $\langle w_{1_k}, w_{1_l}, w_{1_i}; w_{2_i}zv\rangle$ is not a Z_3, we have $zw_{1_i} \in E(G)$. Moreover, since $\langle w_{1_k}, w_{1_l}, x; vzw_{2_i}\rangle$ is not a Z_3, we get $xz \in E(G)$.

3) Suppose this is not true, there exist $w_{2_s}, w_{2_t} \in \mathcal{C}_j$ and $w_{1_i} \in N_2(x)$ such that $w_{1_i}w_{2_i} \in E(G)$. If there also exists $w_{1_j} \in N_2(w_{2_s}) \cap N_2(w_{2_t}) \cap N_2(x)$, then either $\langle w_{2_s}, w_{2_t}, w_{1_j}; w_{1_i}w_{2_i}z\rangle$ or $\langle w_{2_s}, w_{2_t}, w_{1_j}; w_{1_i}zv\rangle$ is a Z_3. If $N_2(w_{2_s}) \cap N_2(w_{2_t}) \cap N_2(x) = \emptyset$, we may assume there exists $w_{1_s} \in N_2(x)$ such that $w_{1_s}w_{2_s} \in E(G)$ and $w_{1_s}w_{2_t} \notin E(G)$, then $\langle z, w_{2_i}, w_{1_i}; w_{1_s}, w_{2_s}, w_{2_t}\rangle$ is a Z_3, showing a contradiction. ■

Claim 12.1.7. *If there exists* $w \in N_2^3(x)$ *such that* $wz \in E(G)$*, then*

1) $w_{2_i}z \in E(G)$ *for any* $w_{2_i} \in N_2(w) \cap \mathcal{C}_i$*;*

2) $|N_2(\mathcal{C}_i) \cap N_2(x)| \geq |N_2(x)| - 1$*. In particular,* $\mathcal{C}_i = N_2^2(x)$*;*

3) $N_2^3(x) = \{w\}$*.*

Proof. For simplicity, we assume $N_2(w) \cap \mathcal{C}_1 \neq \emptyset$. Since G is claw-free, we have $N_2(z) \cap N_2(x) = \emptyset$.

1) By Claim 12.1.5, $|\mathcal{C}_1| \geq 2$ and $\mathcal{C}_1 \cup (N_2(\mathcal{C}_1) \cap N_2(x))$ is a clique. Moreover, $|N_2(\mathcal{C}_1)| \geq 2$ since G is 3 connected. We denote by $w_{1_1}, w_{1_2} \in N_2(\mathcal{C}_1)$. Then, for any $w_{2_i} \in N_2(w) \cap \mathcal{C}_1$, since $\langle w_{1_1}, w_{1_2}, w_{2_i}; wzv\rangle$ is not a Z_3, we have $w_{2_i}z \in E(G)$.

2) By Claim 12.1.6 3), we know $|N_2(x) \setminus N_2(w_{2_i})| \leq 1$ for any $w_{2_i} \in N_2(w) \cap \mathcal{C}_1$, which implies $|N_2(\mathcal{C}_1) \cap N_2(x)| \geq |N_2(x)| - 1$. Moreover, $N_2^2(x) = \mathcal{C}_1$ since G is 3-connected.

3) Suppose this is not true, there exist $w' \in N_2^3(x) \setminus \{w\}$ and $w_{2_j} \in \mathcal{C}_1$ such that $w_{2_j}w' \in E(G)$. Then $w_{2_i}w', w_{2_j}w \notin E(G)$ since $N_2^3(x)$ is an independent set, which in

turn gives $zw_{2_j} \notin E(G)$ since $\langle z; v, w_{2_j}, w\rangle$ is not a claw. However, this will force either $\langle x, y, v; zw_{2_i}w_{2_j}\rangle$ or $\langle x, v, z; w_{2_i}w_{2_j}w'\rangle$ to be a Z_3. ■

Claim 12.1.8. *If $N_2(z) \cap N_2^3(x) = \emptyset$, $N_2(z) \cap N_2^2(x) = \emptyset$ and there exist $w_{2_i}, w_{2_j} \in N_2^2(x)$ such that $N_2(w_{2_i}) \cap N_2(w_{2_j}) \cap N_2(x) \neq \emptyset$, then $N_2(x) = N_2(z)$. In particular, $xz \in E(G)$ if $N_2^2(x) \neq \emptyset$.*

Proof. If there exists $w_{1_i} \in N_2(w_{2_i}) \cap N_2(w_{2_j}) \cap N_2(x)$ such that $zw_{1_i} \in E(G)$, then for any $w_{1_j} \in N_2(x) \setminus (N_2(w_{2_i}) \cap N_2(w_{2_j}))$, since $\langle w_{1_i}; z, w_{2_i}, w_{1_j}\rangle$ is not a claw, we have $zw_{1_j} \in E(G)$. If there exists $w_{1_j} \in N_2(x) \setminus (N_2(w_{2_i}) \cap N_2(w_{2_j}))$ such that $zw_{1_j} \in E(G)$, then for any $w_{1_i} \in N_2(w_{2_i}) \cap N_2(w_{2_j}) \cap N_2(x)$, since $\langle w_{2_i}, w_{2_j}, w_{1_i}; w_{1_j}zv\rangle$ is not a Z_3, we have $zw_{1_i} \in E(G)$. Since $N_2(z) \cap N_2^3(x) = \emptyset$ and $N_2(z) \cap N_2^2(x) = \emptyset$, thus we get $N_2(x) = N_2(z)$. In particular, if $N_2^2(x) \neq \emptyset$, then $xz \in E(G)$ because G is claw-free. ■

Now we want to find a spanning Halin subgraph in G by following cases.

Case 1. Assume that $N_2^3(x) \neq \emptyset$.

Case 1.1: Assume that $N_2^3(x) \cap N_2(z) \neq \emptyset$.

By Claim 12.1.7, we know $N_2^3(x) = \{w\}$, $N_2^2(x) = \mathcal{C}_1$, $|N_2(\mathcal{C}_1) \cap N_2(x)| \geq |N_2(x)| - 1$ and $\{w, z\} \cup (N_2(w) \cap \mathcal{C}_1)$ is a clique. Since G is 3-connected, we can assume there exist $w_{2_1}, w_{2_2}, w_{2_{s_2}} \in N_2^2(x)$ and $w_{1_1}, w_{1_2} \in N_2(x)$ such that $w_{1_1}w_{2_1}, w_{1_2}w_{2_{s_2}}, w_{2_1}w, w_{2_2}w \in E(G)$. Since both $N_2(x) \setminus \{w_{1_1}\}$ and $N_2^2(x) \setminus \{w_{2_1}\}$ are cliques, there exist hamiltonian paths, say $P_1 = w_{1_2}P_1w_{1_{s_1}}$ and $P_2 = w_{2_2}P_2w_{2_{s_2}}$, in them, respectively. Let $C = P_1P_2 \cup \{zv, vy, yw_{1_{s_1}}, w_{1_2}w_{2_{s_2}}, w_{2_2}w, wz\}$ be a cycle and all vertices on the path $xw_{1_1}w_{2_1}$ be stems of T with $N_C(x) = \{v, y\}$, $N_C(w_{1_1}) = V(P_1)$ and $N_C(w_{2_1}) = V(P_2) \cup \{w, z\}$. Let $H = T \cup C$, then H is a spanning Halin subgraph of G (See Figure 12.1 (1) as an example).

Case 1.2: Suppose that $N_2^3(x) \cap N_2(z) = \emptyset$ and $N_2^2(x) \cap N_2(z) \neq \emptyset$.

We may assume there exists $w \in N_2^3(x)$ and $N_2(w) \cap \mathcal{C}_1 \neq \emptyset$. Since G is 3-connected and $wz \notin E(G)$, we have $|N_2(w) \cap \mathcal{C}_1| \geq 3$ and $|N_2(\mathcal{C}_1) \cap N_2(x)| \geq 2$. By Claim 12.1.6 3), we get $N_2(z) \cap \mathcal{C}_1 \neq \emptyset$.

Claim 12.1.9. *There must exist $w_{2_i} \in N_2(w)$ such that $w_{2_i}z \in E(G)$ and $xz, yz \in E(G)$.*

Proof. Suppose this is not true. Since $N_2(z) \cap \mathcal{C}_1 \neq \emptyset$, there must exist $w_{2_j} \in \mathcal{C}_1 \setminus N_2(w)$ such that $w_{2_j}z \in E(G)$. Then for any $w_{2_i} \in N_2(w) \cap \mathcal{C}_1$, either $\langle x, y, v; zw_{2_j}w_{2_i}\rangle$ or $\langle x, v, z; w_{2_j}w_{2_i}w\rangle$ is a Z_3, showing a contraction. Let $w_{2_1} \in \mathcal{C}_1$ such that $ww_{2_1}, w_{2_1}z \in E(G)$. Since $\langle x, y, v; zw_{2_1}w\rangle$ is not a Z_3 and x, y are twins, we have both xz and yz in $E(G)$. ■

Denote by $N_2^3(x) = \{w_{3_1}, w_{3_2}, \cdots, w_{3_{s_3}}\}$, $\{w_{3_i}^1, w_{3_i}^2, w_{3_i}^3\} \subseteq N_2(w_{3_i}) \cap \mathcal{C}_1$, $\mathcal{C}_1 \setminus N_2(N_2^3(x)) = \{w'_{2_1}, w'_{2_2} \cdots, w'_{2_{k_2}}\}$, $\{w_{1_1}^1, w_{1_1}^2\} \subseteq N_2(\mathcal{C}_1) \cap N_2(x)$, $\mathcal{C}_j = \{w_{2_j}\}$ for all $j \neq i$, $\{w_{1_j}^1, w_{1_j}^2, w_{1_j}^3\} \subseteq N_2(w_{2_j}) \cap N_2(x)$ and $N_2(x) \setminus (\cup_{j=2}^k \{w_{1_j}^1, w_{1_j}^2, w_{1_j}^3\} \cup \{w_{1_1}^1, w_{1_1}^2\}) = \{w'_{1_1}, w'_{1_2}, \cdots, w'_{1_{t_1}}\}$.

By Claim 12.1.3, either $\mathcal{C}_1$ is a clique or $\langle \mathcal{C}_1\rangle$ is losing exactly one edge. Since both $N_2(x) \setminus (\cup_{j=2}^k \{w_{1_j}^1, w_{1_j}^2, w_{1_j}^3\} \cup \{w_{1_1}^1, w_{1_1}^2\}\})$ and $\mathcal{C}_1 \setminus N_2(N_2^3(x))$ are cliques, there exist hamiltonian paths, say $P_1 = w'_{1_1}P_1w'_{1_{t_1}}$ and $P_2 = w'_{2_1}P_2w'_{2_{k_2}}$, in them, respectively. Let $C = P_1P_2 \cup (\cup_{i=1}^{s_3-1}\{w_{3_i}^1w_{3_i}, w_{3_i}w_{3_i}^3, w_{3_i}^3w_{3_{i+1}}^1\}) \cup \{w_{3_{s_3}}^3w'_{2_1}, w'_{2_{k_2}}w_{1_1}^1, w_{1_1}^1w_{1_2}^1\} \cup (\cup_{j=2}^{k-1}\{w_{1_j}^1w_{2_j}, w_{2_j}w_{1_j}^3, w_{1_j}^3w_{1_{j+1}}^1\})$ $\cup \{w_{1_k}^3w'_{1_1}, w'_{1_{t_1}}y, yv, vz, zw_{3_1}^1\}$ be a cycle and all vertices on the tree $\{x; w_{1_1}^2, w_{1_2}^2, \cdots, w_{1_k}^2; w_{3_1}^2, \cdots, w_{3_{s_3}}^2\}$ be stems of T with $N_C(x) = V(P_1) \cup \{v, y, z\}$, $N_C(w_{1_1}^2) = V(P_1) \cup \{w_{1_1}^1\}$, $N_C(w_{1_i}^2) = \{w_{1_i}^1, w_{1_i}^3, w_{2_i}\}$ for all $i \in [2, k]$ and $N_C(w_{3_j}^2) = \{w_{3_j}^1, w_{3_j}^3, w_{3_j}\}$ for all $j \in [1, s_3]$. Let $H = T \cup C$, then H is a spanning Halin subgraph of G (See Figure 12.1 (2) as an example).

Case 1.3: Suppose that $N_2^3(x) \cap N_2(z) = \emptyset$, $N_2^2(x) \cap N_2(z) = \emptyset$ and $N_2(x) \cap N_2(z) \neq \emptyset$.

We may assume there exists $w \in N_2^3(x)$ such that $N_2(w) \cap \mathcal{C}_1 \neq \emptyset$. By Claim 12.1.5, $|\mathcal{C}_1| \geq 3$ and $N_2(N_2^3(x)) \subseteq \mathcal{C}_1$. By Claim 12.1.8, $N_2(x) = N_2(z)$ and $xz \in E(G)$. Similarly as Case 1.2, we delete the edge $xw_{1_1}^3$ form $E(T)$ and add the edge $w_{1_1}^2w_{1_1}^3$ to $E(T)$, also delete the edge $zw_{3_1}^1$ form $E(C)$ and add edges $w_{3_1}^1w_{1_1}^3, w_{1_1}^3z$ to $E(C)$, we can find a spanning Halin subgraph in G (See Figure 12.1 (1) as an example).

Case 2: Assume that $N_2^3(x) = \emptyset$ and $N_2^2(x) \neq \emptyset$.

If $\mathcal{C}_i \cup N_2(\mathcal{C}_i)$ is a clique for all possible i, we can find a spanning Halin subgraph in G similarly as Case 1.2 and 1.3. Thus we assume $\mathcal{C}_1 \cup N_2(\mathcal{C}_1)$ is not a clique, then, clearly, $|\mathcal{C}_1| \geq 2$. We denote by $\mathcal{C}_1 = \{w'_{2_1}, w'_{2_2}, \cdots, w'_{2_{k_1}}\}$. By Claim 12.1.3, we there exists

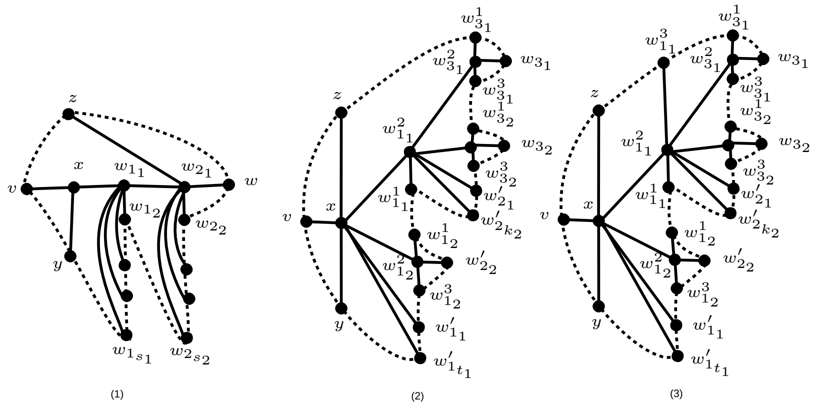

Figure 12.1. $N_2^3(x) \neq \emptyset$.

$w_{1_2} \in N_2(\mathcal{C}_1)$ such that $w_{1_2}w'_{2_2} \in E(G)$ and $w_{1_2}w'_{2_1} \notin E(G)$. Then we have following two subcases to be considered.

Case 2.1: Assume that $N_2^2(x) \cap N_2(z) \neq \emptyset$.

Claim 12.1.10. *If* $N_2^2(x) \cap N_2(z) \neq \emptyset$, *then*

1) $N_2(z) \cap \mathcal{C}_1 \neq \emptyset$;

2) If $\mathcal{C}_1 \setminus N_2(z) \neq \emptyset$, *then* $|N_2(x) \setminus N_2(\mathcal{C}_1)| \leq 1$ *and* $xz \in E(G)$;

3) $N_2^2(x) = \mathcal{C}_1$.

Proof. 1) This is clear true by Claim 12.1.6 3).

Thus we always assume $zw'_{2_1} \in E(G)$.

2) Suppose this is not true, there exists $w_{2_i} \in \mathcal{C}_1 \setminus \{w'_{2_1}\}$ such that $w'_{2_i}z \notin E(G)$ and $w'_{2_i}w'_{2_1} \in E(G)$. If there exist $w_{1_2}, w_{1_3} \in N_2(x) \setminus N_2(\mathcal{C}_1)$, then $\langle w_{1_2}, w_{1_3}, x; zw'_{2_1}w'_{2_i}\rangle$ is a Z_3. If $xz \notin E(G)$, then $\langle x, y, v; zw'_{2_1}w'_{2_i}\rangle$ is a Z_3, showing a contradiction.

3) If $|N_2(x) \setminus N_2(\mathcal{C}_1)| = 1$, then clearly $\mathcal{C}_j = \emptyset$ for all $j \neq i$. If $|N_2(x) \setminus N_2(\mathcal{C}_1)| \geq 2$, then $\mathcal{C}_1 \subseteq N_2(z)$. Since neither $\langle w'_{2_1}, w'_{2_2}, z; vxw_{1_j}\rangle$ nor $\langle w'_{2_1}, w'_{2_2}, z; xw_{1_j}w_{2_j}\rangle$ is a Z_3, where $w_{2_j} \in \mathcal{C}_j$ and $w_{1_j} \in N_2(x) \cap N_2(w_{2_j})$, we have $\mathcal{C}_j = \emptyset$. Thus $N_2^2(x) = \mathcal{C}_1$. ■

We denote by $N_2(x) = \{w_{1_1}, w_{1_2}, \cdots, w_{1_t}\}$. In particular, assume $w_{1_1}w_{1_2} \in N_2(\mathcal{C}_1)$.

If $\mathcal{C}_1 \subseteq N_2(z)$, then $\mathcal{C}_1$ is a clique. Since G is 3-connected, we assume $w_{1_1}w'_{2_1}, w'_{1_2}w'_{2_2} \in E(G)$. Since $N_2(x)\setminus\{w_{1_1}\}$ and $\mathcal{C}_1\setminus\{w'_{2_1}\}$ are cliques, there exist hamiltonian paths, say $P_1 = w_{1_2}P_1w_{1_t}$ and $P_2 = w'_{2_2}P_2w'_{2_{k_1}}$, in them, respectively. Let $C = P_1P_2\cup\{yw_{1_t}, w_{1_2}w'_{2_2}, w'_{2_{k_1}}z, zv, vy\}$ be a cycle and all vertices on the path $xw_{1_1}w'_{2_1}$ be stems of T with $N_C(x) = \{y, v\}$, $N_C(w_{1_1}) = V(P_1)$ and $N_C(w'_{2_1}) = V(P_2) \cup \{z\}$. If $|N_2(x) \setminus N_2(\mathcal{C}_1)| \leq 1$, by Claim 12.1.3, we assume $w_{1_1}w \in E(G)$ for all $w \in \mathcal{C}_1 \setminus \{w'_{2_{k_1}}\}$ and $w'_{2_1}z, w_{1_2}w'_{2_{k_1}}, w_{1_3}w'_{2_{k_1}} \in E(G)$. Since both $N_2(x) \setminus \{w_{1_1}, w_{1_2}\}$ and $\mathcal{C}_1 \setminus \{w'_{2_{k_1}}\}$ are cliques, there exist hamiltonian paths, say $P_1 = w_{1_3}P_1w_{1_t}$ and $P_2 = w'_{2_1}P_2w'_{2_{k_1-1}}$, in them, respectively. Let $C = P_1P_2 \cup \{zv, vy, yw_{1_t}, w_{1_3}w'_{2_{k_1}}, w'_{2_{k_1}}w'_{1_{k_1-1}}, w'_{2_1}z\}$ be a cycle and all vertices on the path $xw_{1_1}w_{1_2}$ be stems of T with $N_C(x) = \{z, v\}$, $N_C(w'_{1_1}) = V(P_2)$ and $N_C(w'_{1_2}) = V(P_1) \cup \{y, w_{2'_{k_1}}\}$. Let $H = T \cup C$, then H is a spanning Halin subgraph of G (See Figure 12.2 (1) and (2) as examples).

Case 2.2: Suppose $N_2^2(x) \cap N_2(z) = \emptyset$.

Since G is 3-connected, $|N_2(\mathcal{C}_i)| \geq 3$ for all possible i. We want to consider following two subcases.

Subcase 1: There exist $w_{2_s}, w_{2_t} \in \mathcal{C}_i$ such that $N_2(w_{2_s}) \cap N_2(w_{2_t}) \cap N_2(x) \neq \emptyset$.

We may assume $i = 1$, then $|N_2(w^1_{1_i}) \cap \mathcal{C}_1| \geq |\mathcal{C}_1| - 1$ for all $w_{1_i} \in N_2(\mathcal{C}_1)$ by Claim 12.1.3, $\mathcal{C}_j \cup N_2(\mathcal{C}_j)$ is a clique for all $j \neq i$, by Claim 12.1.4, and $N_2(x) = N_2(z)$ by Claim 12.1.8. Since G is 3-connected, we may assume $|\mathcal{C}_1| \geq 3$ since other cases are similar and much easier. We assume $w^1_{1_1}w^1_{2_2} \notin E(G)$. Let $\mathcal{C}_i = \{w^1_{2_i}, w^2_{2_i}, \cdots, w^{t_i}_{2_i}\}$, $\{w^1_{1_i}, w^2_{1_i}, w^3_{1_i}\} \subseteq N_2(\mathcal{C}_i)$ for all possible i, $w^1_{1_1}w^1_{2_1}, w^2_{1_1}w^3_{2_1}, w^3_{1_1}w^2_{2_1} \in E(G)$ and $D_1 = N_2(x) \setminus (\{\cup_{i=1}^k\{w^1_{1_i}, w^2_{1_i}, w^3_{1_i}\}) = \{w'_{1_1}, w'_{1_2}, \cdots, w'_{1_l}\}$. Since $\mathcal{C}_1 \setminus \{w^3_{2_1}\}$, $\mathcal{C}_j$, for all possible $j \neq i$, and D_1 are cliques, there exist hamiltonian paths, say $P_1 = w^1_{2_1}P_1w^2_{2_1}$, $P_j = w^1_{2_j}P_jw^2_{2_j}$ and $P' = w'_{1_1}P'w'_{1_l}$, in them, respectively. Let $C = P_1P_2\cdots P_kP' \cup (\cup_{i=1}^k\{w^1_{1_i}w^1_{2_i}, w^1_{2_i}w^3_{1_i}, w^3_{1_i}w^1_{1_{i+1}}\}) \cup \{w^3_{1_k}w'_{1_1}, w'_{1_l}y, yv, vz, zw^1_{1_1}\}$ be a cycle and all vertices on the tree $\{x; w^2_{1_1}, w^2_{1_2}, \cdots, w^2_{1_k}; w^3_{2_1}\}$ be stems of T with $N_C(x) = \{z, y, v\} \cup D_1$, $N_C(w^1_{1_2}) = \{w^1_{1_1}, w^3_{1_1}\}$, $N_C(w^2_{1_j}) = \{w^1_{1_j}, w^3_{1_j}\} \cup \mathcal{C}_j$ for all $j \neq i$ and $N_C(w^3_{2_1}) = \mathcal{C}_1 \setminus \{w^3_{2_1}\}$. Let $H = T \cup C$, then H is a spanning Halin subgraph of G (See Figure 12.2 (3) as an example).

Subcase 2: For any $w_{2_s}, w_{2_t} \in \mathcal{C}_i$, we have $N_2(w_{2_s}) \cap_2^1 (w_{2_s}) \cap N_2(x) = \emptyset$.

If $N_2(z) \cap N_2(x) \cap N_2(N_2^2(x)) = \emptyset$, by Claim 12.1.8. Let $|\mathcal{C}_i| \leq 2$ and we denote $\mathcal{C}_i = \{w_1^i, w_2^i\}$, $\{w_{1_1}^i, w_{1_2}^i\} \subseteq N_2(w_1^i) \cap N_2(x)$, $\{w_{2_1}^i, w_{2_2}^i\} \subseteq N_2(w_2^i) \cap N_2(x)$ for all i with $|\mathcal{C}_i| = 2$ and $\mathcal{C}_i = \{w_1^i\}$, $\{w_{1_1}^i, w_{1_2}^i, w_{1_3}^i\} \subseteq N_2(w_1^i)$ for all i with $|\mathcal{C}_i| = 1$. Denote by $|\mathcal{C}_i| = 2$ for $i \in [1, k_1]$ and $|\mathcal{C}_i| = 1$ for $i \in [k_1+1, k]$ for simplicity. We also denote by $D_1 = N_2(x) \setminus (\cup_{i=1}^{k_1}\{w_{1_1}^i, w_{1_2}^i, w_{2_1}^i, w_{2_2}^i\}) \cup (\cup_{i=k_1+1}^{k}\{w_{1_1}^i, w_{1_2}^i, w_{1_3}^i\}) = \{w'_{1_1}, w'_{1_2}, \cdots, w'_{1_l}\}$. Assume $w'_{1_1}z \in E(G)$ if $xz \in E(G)$ and $w'_{1_1}z, w'_{1_2}z \in E(G)$ if $xz \notin E(G)$ since G is 3-connected. We only find a spanning Halin subgraph in G for $xz \notin E(G)$ since the other case is similarly and easier. Since $D_1 \setminus \{w'_{1_1}, w'_{1_2}\}$ is a clique, there exists a hamiltonian path $P_1 = w'_{1_3}P_1w'_{1_l}$ in it. Let $C = P_1 \cup (\cup_{i=1}^{k_1}\{w_{1_1}^i w_1^i, w_1^i w_2^i, w_2^i w_{2_2}^i, w_{2_2}^i w_{1_1}^{i+1}\}) \cup (\cup_{k_1+1}^{k}\{w_{1_1}^i w_1^i, w_1^i w_{1_3}^i, w_{1_3}^i w_{1_1}^{i+1}\}) \cup \{w_{2_2}^{k_1} w_{1_1}^{k_1+1}, w_{1_3}^k w'_{1_1}, w'_{1_1}z, zv, vx\}$ be a cycle and all vertices on the star $\{x; w_{1_2}^1, w_{2_1}^1, w_{1_2}^2, w_{2_1}^2, \cdots, w_{1_2}^{k_1}, w_{2_1}^{k_1}, w_{1_2}^{k_1+1}, \cdots, w_{1_2}^k\}$, where x is the center, be stems of T with $N_C(x) = (D_1 \setminus \{w'_{1_1}, w'_{1_2}\} \cup \{v, x\}$, $N_C(w_{1_2}^i) = \{w_{1_1}^i, w_1^i\}$, $N_C(w_{2_1}^i) = \{w_{2_2}^i, w_1^i\}$ for all $i \in [1, k_1]$, $N_C(w_{1_2}^i) = \{w_{1_1}^i, w_{1_3}^i, w_1^i\}$ for $i \in [k_1+1, k]$ and $N_C(w'_{1_2}) = \{z, w'_{1_1}\}$ (See Figure 12.2 (4) as an example). If $N_2(z) \cap N_2(x) \cap N_2(N_2^2(x)) \neq \emptyset$, since G is claw-free, we have $xz \in E(G)$. Similarly as $N_2(z) \cap N_2(x) \cap N_2(N_2^2(x)) = \emptyset$, we delete the stem $\{w'_{1_2}\}$ from $V(T)$, delete edges $w'_{1_1}w'_{1_2}, w'_{1_2}z$ from $E(T)$ but add edges xz, xw'_{1_1} to $E(T)$; and also delete the edge $w'_{1_1}w'_{1_l}$ from $E(C)$ and add edges $w'_{1_1}w'_{1_2}, w'_{1_2}w'_{1_3}$ to $E(C)$.

Let $H = T \cup C$, then H is a spanning Halin subgraph of G

Case 3: Suppose that $N_2^2(x) = N_2^3(x)) = \emptyset$, i.e. $V_2 = N_2(x)$

Similarly as line graph, we can find a spanning Halin subgraph in G.

12.2 Proof of 3-connected $\{claw, B_{1,2}\}$-free graphs admit twins

Since, in 2015, Furuya and Tsuchiya [26] showed that: *If G is a 3-connected $\{Claw, B_{1,2}\}$-free graph but not N-free, then G contains a spanning Halin subgraph* , in this section, we assume every graph is also N-free. To prove Proposition 22, we want to consider following two cases.

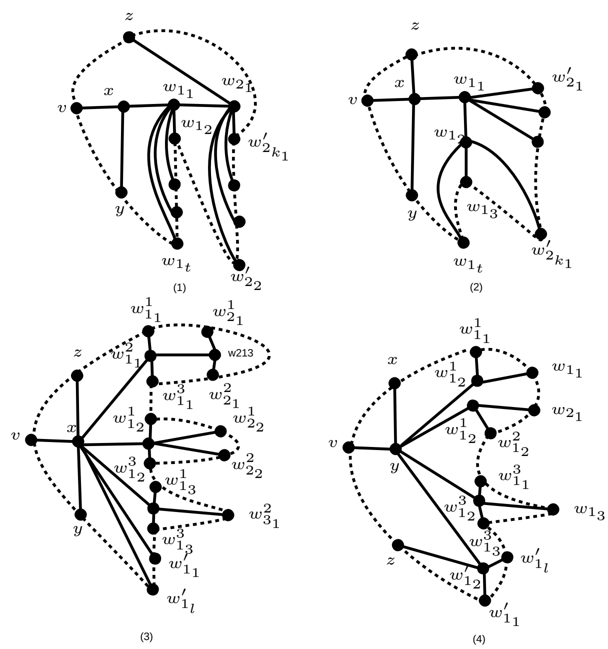

Figure 12.2. $N_2^3(x) = \emptyset$ and $N_2^2(x) \neq \emptyset$.

Case 1: For any two vertices v_{1_s}, v_{1_t}, in $N_1(x)$ and any two vertices w_{1_i}, w_{1_j} in $N_2(x)$, we have $N_1(v_{1_s}) = N_1(v_{1_t})$ and $N_2(w_{1_i}) = N_2(w_{1_j})$.

Claim 12.2.1. *If for any two vertices v_{1_s}, v_{1_t}, in $N_1(x)$ and any two vertices w_{1_i}, w_{1_j} in $N_2(x)$, we have $N_1(v_{1_s}) = N_1(v_{1_t})$ and $N_2(w_{1_i}) = N_2(w_{1_j})$, then*
1) $N_i^j(x) \cup N_i^{j+1}(x)$ is a clique for all $j \in [1, k_1 - 1] \cup [1, k_2 - 1]$ and $i \in [1, 2]$;
2) If there exists $v_j \in N_1^j(x)$(similarly as $w_{j_1} \in N_2^j(x)$) such that $v_j z \in E(G)$, then either $N_1(z) \subseteq N_1^{j-1}(x) \cup N_1^j(x)$ or $N_1(z) \subseteq N_1^j(x) \cup N_1^{j+1}(x)$. In particular, if $N_1^{j+1}(x) \cap N_1(z) = \emptyset$, then $|N_1^j(x)| \geq 3$.

Proof. we only shoe that this is true for $i = 1$. Since $N_1(v_{1_s}) = N_1(v_{1_t})$ for all $v_{1_s}, v_{1_t} \in N_1(x)$, we have $N_1(x) \cup N_1^2(x)$ is a clique. If there exists $v_{2_2} \in N_1^2(x)$ such that $v_{2_2} v_{3_1} \notin E(G)$, then $\langle v_{3_1}; v_{2_1}, v_{2_2}, v_{1_1}; x w_{1_1} \rangle$ is a $B_{1,2}$. Thus $N_1^2(x) \cup N_1^3(x)$ is a clique. Similarly, we can show that $N_1^j(x) \cup N_1^{j+1}(x)$ is a clique for all possible j.

2) This is clearly true since G is claw-free and 3-connected. ■

Let $t_i = min\{j | N_i^j(x) \cap N_i^1(z) \neq \emptyset\}$ for $i \in [1, 2]$.

Case 1.1: Suppose that $xz \notin E(G)$.

Claim 12.2.2. *Suppose $xz \notin E(G)$,*

1) If $N_1^{t_1+1}(x) \neq \emptyset$, then $N_1^{t_1+1}(x) \subseteq N_1(z)$;

2) If $N_1^{t_1+2}(x) \neq \emptyset$, then $N_1^{t_1+2}(x) \cap N_1(z) = \emptyset$;

3) $V_1 = \cup_{j=1}^{t_1+2} N_1^j(x)$.

Proof. 1) For any $v_{(t_1+1)_i} \in N_1^{t_1+1}(x)$, since $\langle v_{(t_1)_1}; v_{(t_1+1)_i}, v_{(t_1-1)_1}, z\rangle$ is not a claw, we have $v_{(t_1+1)_i}z \in E(G)$. Thus $N_1^{t_1+1}(x) \subseteq N_1(z)$.

2) This is clearly true by Claim 12.2.1 2).

3) If there exists $v' \in V_1 \setminus (\cup_{j=1}^{t_1+2} N_1^j(x))$, we may assume $v'v_{(t_1+2)_1} \in E(G)$, then $\langle w'; z, v_{(t_1)_1}, v_{(t_1+1)_1}; v_{(t_1+2)_1}v'\rangle$ is a $B_{1,2}$, showing a contradiction. ■

Claim 12.2.3. *If $xz \notin E(G)$ and $N_1^{t_1+2}(x) \neq \emptyset$, then*

1) $t_1 = t_2 = 1$;

2) $N_2(x) \subseteq N_2(z)$, $N_2^2(x) \subseteq N_2(z)$ and $N_2^3(x) = \emptyset$.

Proof. For simplicity, we denote by $y = v_{0_1} = w_{0_1}$.

1) If $t_1 \geq 2$, then $\langle v_{(t_1+2)_1}; v_{(t_1+1)_1}, z, v_{(t_1)_1}; v_{(t_1-1)_1}v_{(t_1-2)_1}\rangle$ is a $B_{1,2}$; if $t_2 \geq 2$, then $\langle v_{(t_1+2)_1}; v_{(t_1+1)_1}, v_{(t_1)_1}, z; w_{(t_2)_1}w_{(t_2-1)_1}\rangle$ is a $B_{1,2}$, giving a contradiction. Thus $t_1 = t_2 = 1$.

2) Suppose to the contrary, $w_{1_1}z \in E(G)$. For any $w_{1_i} \in N_2(x)\backslash\{w_{1_1}\}$, since $\langle v_{(t_1+2)_1}; v_{(t_1+1)_1}, v_{(t_1)_1}, z; w_{1_1}w_{1_i}\rangle$ is not a $B_{1,2}$, we have $w_{1_i}z \in E(G)$. Thus $N_2(x) \subseteq N_2(z)$. For any $w_{2_i} \in N_2(x)$, since $xz \notin E(G)$ and $\langle w_{1_1}; y, z, w_{2_i}\rangle$ is not a claw, we have $w_{1_i}z \in E(G)$. This implies $N_2^2(x) \subseteq N_2(z)$. If there exists $w_{3_i} \in N_2^3(x)$, since $N_2^3(x) \cap N_2(z) = \emptyset$, we have $\langle v_{(t_1+2)_1}; v_{(t_1+1)_1}, v_{(t_1)_1}, z; w_{2_1}w_{3_i}\rangle$ is a $B_{1,2}$. This in turn gives $N_2^3(x) = \emptyset$, showing a contradiction. ■

Claim 12.2.4. *If $xz \notin E(G)$ and $N_1^{t_1+2}(x) = \emptyset$, then*

1) $t_1 \leq 2$. In particular, if $t_1 = 2$, then $t_2 = 1$; if $t_1 = 1$, then $t_2 \leq 2$;

2) If $t_2 = 2$, then $N_2^2(x) \subseteq N_2(z)$, $N_2^3(x) \subseteq N_2(z)$ and $N_2^4(x) = \emptyset$;

3) If $t_2 = 1$, then $N_2^4(x) = \emptyset$.

Proof. For simplicity, we always assume $w_{(t_2)_1}z \in E(G)$.

1) Suppose $t_1 \geq 3$, then $\langle w_{(t_2)_1}; z, v_{(t_1+1)_1}, v_{(t_1)_1}; v_{(t_1-1)_1}v_{(t_1-2)_1}\rangle$ is a $B_{1,2}$, showing a contradiction. If $t_1 = 2$ and $t_2 \geq 2$, then $\langle w_{(t_2)_1}; z, v_{3_1}, v_{2_1}; v_{1_1}x\rangle$ is a $B_{1,2}$. If $t_1 = 1$ and $t_2 \geq 3$, then $\langle w_{(t_2)_1}; z, v_{2_1}, v_{1_1}; xw_{1_1}\rangle$ is a $B_{1,2}$, showing a contradiction.

2) If there exists $w_{2_i} \in N_2^2(x)\setminus\{w_{2_1}\}$, then $\langle x; v_{1_1}, v_{2_1}, z; w_{2_1}w_{2_i}\rangle$ is a $B_{1,2}$. Thus $N_2^2(x) \subseteq N_2(z)$. For any $w_{3_i} \in N_2^3(x)$, since $\langle w_{2_1}; w_{1_1}, z, w_{3_i}\rangle$ is not a claw, we have $w_{3_i}z \in E(G)$. Thus $N_2^3(x) \subseteq N_2(z)$. If there exists $w_{4_i} \in N_2^4(x)$, then $\langle w_{4_i}; w_{3_1}, w_{2_1}, z; v_{1_1}y\rangle$ is a $B_{1,2}$, showing a contradiction. Therefore, we get $N_2^4(x) = \emptyset$.

3) If $t_2 = 1$, by Claim 12.2.1 2), we know $N_2^3(x) \cap N_2(z) = \emptyset$. If there exists $w_{4_i} \in N_2^4(x)$, then $\langle w_{4_i}; w_{3_1}, w_{2_1}, z; v_{1_1}y\rangle$ is a $B_{1,2}$, showing a contradiction. ■

We only want to find a spanning Halin subgraph in G for following two cases since other cases are similar and much easier.

Subcase 1: Suppose $N_1^3(x) \neq \emptyset$ and $t_2 = 1$.

By Claim 12.2.1, we know $|N_1^2(x)| \geq 3$ and by Claim 12.2.4, we know $V_2 = N_2(x) \cup N_2^2(x)$. Since $(N_1^2(x) \setminus \{v_{2_1}\}) \cup N_1^3(x)$, $N_1(x) \setminus \{v_{1_1}\}$ and $(N_2(x) \setminus \{w_{1_1}\}) \cup N_2^2(x)$ are cliques, there exist hamiltonian paths, say $P_1 = v_{2_2}P_1v_{2_3}$, $P_2 = v_{1_2}P_2v_{1_3}$ and $P_3 = w_{1_2}P_3w_{2_1}$, in them, respectively. Let $C = P_1P_2P_3 \cup \{v_{1_3}x, xw_{1_2}, w_{2_1}z, zv_{2_2}\}$ be a cycle and all vertices on the path $v_{2_1}v_{1_1}yw_{1_1}$ be stems of T with $N_C(v_{2_1}) = V(P_1)$, $N_C(v_{1_1}) = V(P_2) \cup \{z\}$, $N_C(y) = \{x\}$ and $N_C(w_{1_1}) = V(P_3)$. Let $H = T \cup C$, it is easy to check that H is a spanning Halin subgraph of G (See Figure 11.5 (1) as an example).

Subcase 2: Assume that $N_1^3(x) = \emptyset$ and $t_2 = 2$.

By Claim 12.2.4, we know $V_2 = N_2(x) \cup N_2^2(x) \cup N_2^3(x)$. Since $(N_1(x) \setminus \{v_{1_1}\}) \cup N_1^2(x)$, $N_2(x) \setminus \{w_{1_1}\}$ and $(N_2^2(x) \setminus \{w_{2_1}\}) \cup N_2^3(x)$ are cliques, there exist hamiltonian paths, say $P_1 = v_{2_1}P_1v_{1_2}$, $P_2 = w_{1_2}P_2w_{1_3}$ and $P_3 = w_{2_2}P_3w_{3_1}$, in them, respectively. Let $C = P_1P_2P_3 \cup \{v_{1_2}x, xw_{1_2}, w_{3_1}z, zv_{2_1}\}$ be a cycle and all vertices on the path $v_{1_1}yw_{1_1}w_{2_1}$ be stems of T with $N_C(v_{1_1}) = V(P_1) \cup \{z\}$, $N_C(y) = \{x\}$, $N_C(w_{1_1}) = V(P_2)$ and $N_C(w_{2_1}) = V(P_3)$. Let $H = T \cup C$, it is easy to check that H is a spanning Halin subgraph of G (See Figure 11.5 (1) as an example).

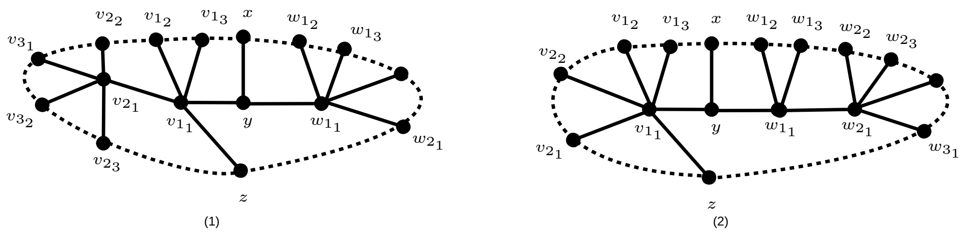

Figure 12.3. $N_1(v_{1_s}) = N_1(v_{1_t})$, $N_2(w_{1_i}) = N_2(w_{1_j})$ and $xz \notin E(G)$.

Case 1.2: Assume that $xz \in E(G)$.

Claim 12.2.5. *If $xz \in E(G)$, then $t_1 = 1$ or $t_2 = 1$.*

Proof. Suppose to the contrary, $N_1(x) \cap N_1(z) = \emptyset$ and $N_2(x) \cap N_2(z) = \emptyset$. Then $\langle y; v_{1_1}, w_{1_1}, z\rangle$ is a claw, giving a contradiction. ■

For this case, from here and after,we always assume $t_1 = 1$, which means $N_1(x) \cap N_1(z) \neq \emptyset$.

Claim 12.2.6. *If $t_2 \geq 2$, then*

1) $N_1^2(x) \cap N_1(z) = \emptyset$;

2) $N_2^{t_2+1}(x) \subseteq N_1(z)$;

3) If $N_1^2(x) \neq \emptyset$, then $t_2 = 2$ and $V_2 = N_2(x) \cup N_2^2(x)$;

4) If $N_1^2(x) = \emptyset$, then either $t_2 = 3$ and $V_2 = \cup_{j=1}^4 N_2^j(x)$ or $t_2 = 2$ and $V_2 = \cup_{j=1}^3 N_2^j(x)$.

Proof. We may always assume $w_{(t_2)_1} z \in E(G)$.

1) If there exists $v_{2_i} \in N_1^2(x) \cap N_1(z)$, then $\langle z; v_{2_i}, y, w_{(t_2)_1}\rangle$ is a claw.

2) For any $w_{(t_2+1)_i} \in N_2^{t_2+1}(x) \cap N_2(z)$, since $\langle w_{(t_2)_1}; w_{(t_2+1)_i}, w_{(t_2-1)_1}, z\rangle$ is not a claw, we have $w_{(t_2+1)_i} z \in E(G)$. Thus $N_2^{t_2+1}(x) \subseteq N_1(z)$.

We assume $w_{(t_2+1)_1} z \in E(G)$.

3) If $t_2 \geq 3$, then $\langle v_{2_1}; v_{1_1}, z, y; w_{1_1} w_{2_1}\rangle$ is a $B_{1,2}$, showing a contradiction. For any $w_{3_i} \in N_2^3(x)$, since $\langle w_{2_1}; w_{1_1}, w_{3_i}, z\rangle$ is not a claw, we have $w_{3_i} z \in E(G)$, which implies $N_2^3(x) \subseteq N_2(z)$. However, this will force $\langle w_{1_1}; w_{2_1}, w_{3_1}, z; v_{1_1} v_{2_1}\rangle$ to be a $B_{1,2}$, showing a contradiction. Thus $V_2 = N_2(x) \cup N_2^2(x)$.

4) Suppose to the contrary, $t_2 \geq 4$, then $\langle w_{1_1}; y, v_{1_1}, z; w_{4_1}w_{3_1}\rangle$ is a $B_{1,2}$. If $t_2 = 3$, then $N_2^5(x) = \emptyset$, otherwise $\langle w_{5_1}; w_{4_1}, w_{3_1}, z; yw_{1_1}\rangle$ is a $B_{1,2}$. If $t_2 = 2$, then $N_2^4(x) = \emptyset$, otherwise $\langle w_{4_1}; w_{3_1}, w_{2_1}, z; v_{1_1}; w_{1_1}\rangle$ is an induced N. ■

Claim 12.2.7. *If $t_2 = 1$, then*

1) $N_1^2(x) \cap N_1(z) = \emptyset$ or $N_2^2(x) \cap N_2(z) = \emptyset$;

2) If $N_2^2(x) \cap N_2(z) \neq \emptyset$ and $N_1^2(x) \neq \emptyset$, then $V_2 = N_2(x) \cup N_2^2(x)$;

3) If $N_2^2(x) \cap N_2(z) \neq \emptyset$ and $N_1^2(x) = \emptyset$, then $V_2 = N_2(x) \cup N_2^2(x) \cup N_2^3(x)$;

4) If $N_2^2(x) \cap N_2(z) = \emptyset$, then $|N_i^j(x)| \geq 3$ for all $j \in [1, k_1 - 1] \cup [1, k_2 - 1]$ and $i \in [1, 2]$.

Proof. 1) Suppose there exist $v_{2_i} \in N_1^2(x) \cap N_1(z)$ and $w_{2_i} \in N_2^2(x) \cap N_2(z)$, then $\langle z; v_{2_i}, w_{2_i}, y\rangle$ is a claw, showing a contradiction. Thus $N_1^2(x) \cap N_1(z) = \emptyset$ or $N_2^2(x) \cap N_2(z) = \emptyset$.

By symmetric, we may always assume $N_1^2(x) \cap N_1(z) = \emptyset$.

2) We may assume there exists $w_{2_1} \in N_2^2(x)$ such that $w_{2_1}z \notin E(G)$. If $N_2^3(x) \neq \emptyset$, then $\langle w_{3_i}; w_{2_1}, w_{1_1}, z; v_{1_1}v_{2_1}\rangle$ is a $B_{1,2}$ for any $w_{3_i} \in N_2^3(x)$, giving a contradiction.

3) Suppose $N_2^4(x) \neq \emptyset$, then for any $w_{4_i} \in N_2^4(x)$, $\langle v_{1_1}; z, w_{1_1}, w_{2_1}; w_{3_1}v_{4_1}\rangle$ is a $B_{1,2}$.

4) This is true since G is 3-connected. ■

Now we want to find a spanning Halin subgraph in G by following subcases.

Subcase 1: Assume that $t_2 \geq 2$ and $V_1 = N_1(x)$.

By Claim 12.2.6 4), we know $V_2 = \cup_{j=1}^{t_2+1} N_2^j(x)$. Since V_1, $N_2^j(x) \setminus \{w_{1_j}\}$ for $j \in [1, t_2 - 1]$ and $N_2^{t_2}(x) \cup N_2^{t_2+1}(x)$ are cliques, there exist hamiltonian paths, say $P' = v_{1_1}P'v_{1_2}$, $P_j = w_{j_2}P_jw_{j_3}$ and $P_{t_2} = w_{(t_2)_2}P_{t_2}w_{(t_2+1)_1}$, in them, respectively. Let $C = P'P_1P_2 \cdots P_{t_2} \cup \{v_{1_2}x, xw_{1_2}, w_{(t_2+1)_1}z, zv_{1_1}\}$ be a cycle and all vertices on the path $yw_{1_1} \cdots w_{(t_2)_1}$ be stems of T with $N_C(y) = V(P') \cup \{x, z\}$, $N_C(w_{i_1}) = V(P_i)$ for $i \in [1, t_2 - 1]$ and $N_C(w_{(t_2)_1}) = V(P_{t_2})$. Let $H = T \cup C$, it is easy to check that H is a spanning Halin subgraph of G (See Figure 11.5 (1) as an example).

Subcase 2: Suppose that $t_2 \geq 2$ and $V_1 = N_1(x) \cup N_1^2(x)$.

By Claim 12.2.6 3), we know $V_2 = N_2(x) \cup N_2^2(x)$. Since $(V_1 \setminus \{v_{1_1}\}) \cup V_2$ and $(N_2(x) \setminus$

$\{w_{1_1}\}) \cup N_2^2(x)$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_2}P_1v_{1_3}$ and $P_2 = w_{1_2}P_2w_{1_3}$, in them, respectively. Let $C = P_1P_2 \cup \{v_{1_2}x, xw_{1_2}, w_{2_1}z, zv_{1_3}\}$ be a cycle and all vertices on the path $v_{1_1}yw_{1_1}$ be stems of T with $N_C(v_{1_1}) = V(P_1)$, $N_C(y) = \{x, z\}$ and $N_C(w_{1_1}) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a spanning Halin subgraph of G (See Figure 11.5 (2) as an example).

Subcase 3: Suppose that $t_2 = 1$, $N_2^2(x) \cap N_2(z) \neq \emptyset$ and $V_1 = N_1(x) \cup N_1^2(x)$.

Similarly as subcase 2, we can find a spanning Halin subgraph in G.

Subcase 4: Suppose that $t_2 = 1$, $N_2^2(x) \cap N_2(z) \neq \emptyset$ and $V_1 = N_1(x)$.

We may assume $N_2^3(x) \neq \emptyset$, then by Claim 12.2.7, $V_2 \subseteq N_2(x) \cup N_2^2(x) \cup N_2^3(x)$. Since V_1, $N_2(x) \setminus \{w_{1_1}\}$ and $(N_2^2(x) \setminus \{w_{2_1}\}) \cup N_2^3(x)$ are cliques, there exist hamiltonian paths, say $P' = v_{1_1}P'v_{1_2}$, $P_1 = w_{1_2}P_1w_{1_3}$ and $P_2 = w_{2_2}P_2w_{2_3}$, in them, respectively. Let $C = P'P_1P_2 \cup \{v_{1_2}x, xw_{1_2}, w_{2_3}z, zv_{1_1}\}$ be a cycle and all vertices on the path $yw_{1_1}w_{2_1}$ be stems of T with $N_C(y) = V(P') \cup \{x, z\}$, $N_C(w_{1_1}) = V(P_1)$ and $N_C(w_{2_1}) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a spanning Halin subgraph of G (See Figure 11.5 (1) as an example).

Subcase 5: Suppose that $t_2 = 1$ and $N_2^2(x) \cap N_2(z) = \emptyset$.

Since $N_1^i(x) \setminus \{v_{i_1}, v_{i_2}\}$ for all $i \in [1, k_1 - 2]$, $(N_1^{k_1-1}(x) \setminus \{v_{(k_1-1)_1}\}) \cup N_1^{k_1}(x)$, $N_2^j(x) \setminus \{w_{i_1}, w_{i_2}\}$ for all $j \in [1, k_2 - 2]$ and $(N_1^{k_2-1}(x) \setminus \{w_{(k_2-1)_1}\}) \cup N_1^{k_2}(x)$ are cliques, there exist hamiltonian paths $P_{1_i} = v_{i_3}P_{1_i}v_{i_4}$, $P_{1_{k_1-1}} = v_{k_{1}2}P_{1_{k_1-1}}v_{k_{1}3}$, $P_{2_j} = w_{j_3}P_{2_j}w_{j_4}$, $P_{2_{k_2-1}} = w_{k_2-1_2}P_{1_{k_1-1}}w_{k_2-1_3}$, in them respectively. Let $C = P_{1_{k_1-1}} \cdots P_1P_2 \cdots P_{2_{k_2-1}} \cup \{v_{1_4}x, xw_{1_4}, v_{1_2}z, w_{1_2}z\} \cup (\cup_{i=1}^{k_1-2}(v_{i_2}v_{(i+1)_2}) \cup \cup(\cup_{j=1}^{k_2-2}(w_{j_2}w_{(j+1)_2})$ be a cycle and all vertices on the path $v_{(k_1-1)_1} \cdots v_{1_1}yw_{1_1} \cdots w_{(k_2-1)_1}$ be stems of T with $N_C(v_{i_1}) = V(P_{1_i})$ for all $i \in [1, k_1 - 1]$, $N_C(y) = \{x, z\}$ and $N_C(w_{j_1}) = V(P_{2_j})$ for all $j \in [1, k_2 - 1]$. Let $H = T \cup C$, it is easy to check that H is a spanning Halin subgraph of G (See Figure 11.5 (1) as an example).

Case 2: There exist $v_{1_i}, v_{1_j} \in N_1(x)$ or $w_{1_i}, w_{1_j} \in N_2(x)$ such that $N_1(v_{1_i}) \neq N_1(v_{1_j})$ or $N_2(w_{1_i}) \neq N_2(w_{1_j})$.

By symmetric, we always assume there exists $w_{2_1} \in N_2^2(x)$ such that $w_{1_1}w_{2_1} \notin E(G)$ and $w_{1_2}w_{2_1} \in E(G)$.

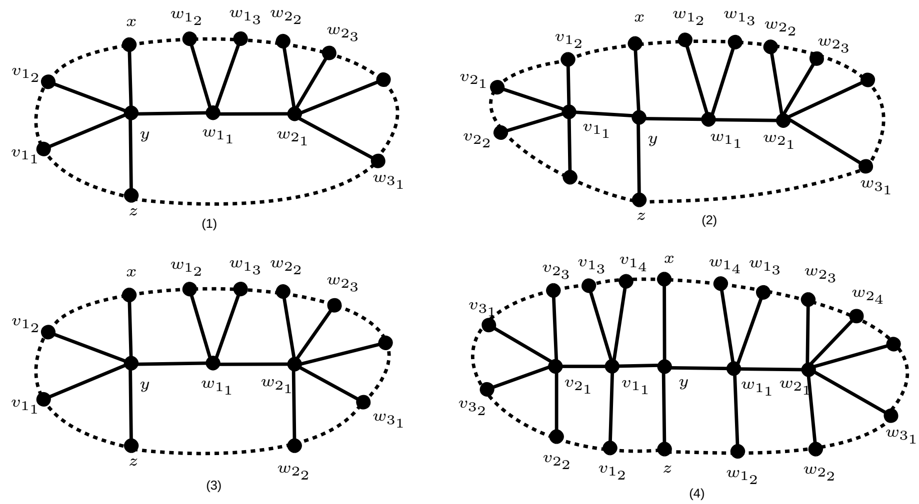

Figure 12.4. $N_1(v_{1_s}) = N_1(v_{1_t})$, $N_2(w_{1_i}) = N_2(w_{1_j})$ and $xz \in E(G)$.

Claim 12.2.8. *If there exists $w_{2_1} \in N_2^2(x)$ such that $w_{1_1}w_{2_1} \notin E(G)$ and $w_{1_2}w_{2_1} \in E(G)$, then*

1) $N_1(x) = V_1$;

2) $N_2^2(x)$ is a clique;

3) $V_2 = N_2(x) \cup N_2^2(x)$.

Proof. 1) Suppose this is not true, there exist $v_{2_1} \in V_1 \setminus N_1(x)$ and $v_{1_1} \in N_1(x)$ such that $v_{2_1}v_{1_1} \in E(G)$, then $\langle w_{2_1}; w_{1_1}, w_{1_2}, x; v_{1_1}v_{2_1}\rangle$ is $B_{1,2}$.

2) We may assume there exists $w_{2_2} \in N_2^2(x)$ such that $w_{1_2}w_{2_2} \in E(G)$ and $w_{2_1}w_{2_2} \notin E(G)$, then $\langle v_{1_1}; x, w_{1_1}, w_{1_2}; w_{2_1}; w_{2_2}\rangle$ is an induced N, giving a contradiction.

3) If there exists $w_{3_i} \in N_2^3(x)$, then $w_{2_1}w_{3_i} \notin E(G)$. Otherwise $\langle v_{1_1}; x, w_{1_2}, w_{1_1}; w_{2_1}w_{3_i}\rangle$ is a $B_{1,2}$. Thus we may assume there exist $w_{2_i} \in N_2^2(x) \setminus \{w_{2_1}\}$ such that $w_{2_i}w_{3_i} \in E(G)$, then $w_{2_i}w_{1_i} \in E(G)$ for all $w_{1_i} \in N_2(x)$. Otherwise, assume there exist $w_{1_i}, w_{1_j} \in N_2(x)$ such that $w_{1_i}w_{2_i} \in E(G)$ and $w_{1_j}w_{2_i} \notin E(G)$, then $\langle v_{1_1}; x, w_{1_j}, w_{1_i}; w_{2_i}w_{3_i}\rangle$ is a $B_{1,2}$. However, this will force $\langle w_{3_i}; w_{2_i}, w_{2_1}, w_{1_1}; xv_{1_1}\rangle$ to be a $B_{1,2}$, giving a contradiction. ■

If $N_2^2(x) = \emptyset$, then $V_1 = N_1(x)$ and $V_2 = N_2(x)$. We can find a spanning Halin subgraph in G as G is $\{claw, Z_3\}$-free. Thus we assume $N_2^2(x) \neq \emptyset$.

Case 2.1: Assume that $xz \notin E(G)$.

Claim 12.2.9. *If $xz \notin E(G)$, then $V_1 \subseteq N_1(z)$.*

Proof. This is clear true if $|V_1| = 1$ since $N_1(z) \neq \emptyset$. Thus we assume there exist $v_{1_1}, v_{1_2} \in V_1$ such that $v_{1_1}z \in E(G)$ and $v_{1_2}z \notin E(G)$. Since $\langle w_{2_1}; w_{1_1}, w_{1_2}, y; v_{1_1}z\rangle$ is not a $B_{1,2}$ and neither $\langle w_{1_1}; y, z, w_{2_1}\rangle$ nor $\langle z; v_{1_1}, w_{2_1}, w_{1_2}\rangle$ is a claw, we have either $w_{1_2}z \in E(G)$ or $w_{2_1}z \in E(G)$. If $w_{1_2}z \in E(G)$, then $\langle v_{1_2}; y, w_{1_1}, w_{1_2}; z; w_{2_1}\rangle$ is an induced N; if $w_{2_1}z \in E(G)$, then either $\langle v_{1_2}; y, w_{1_2}, w_{1_1}; w_{2_1}z\rangle$ or $\langle w_{1_2}; w_{1_1}w_{2_1}z; v_{1_1}v_{1_2}\rangle$ is a $B_{1,2}$, showing a contradiction. ■

In the following, we always assume $|N_2^2(x)| \geq 3$ since the other case is similar and much easier.

If $|V_1| \geq 2$ and $w_{2_1}z \in E(G)$, we may assume $w_{1_2}w_{2_2}, w_{1_3}w_{2_3} \in E(G)$ since G is 3-connected. Note that we may have $w_{1_1} = w_{1_3}$. Since $V_1 \setminus \{v_{1_1}\}$, $N_2(x) \setminus \{w_{1_2}\}$ and $N_2^2(x) \setminus \{w_{2_2}\}$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_2}P_1v_{1_3}$, $P_2 = w_{1_1}P_2w_{1_3}$ and $P_3 = w_{2_1}P_3w_{2_3}$, in them, respectively. Let $C = P_1P_2P_3 \cup \{v_{1_3}x, xw_{1_1}, w_{2_1}z, zv_{1_2}\}$ be a cycle and all vertices on the path $v_{1_1}yw_{1_2}w_{2_2}$ be stems of T with $N_C(v_{1_1}) = V(P_1) \cup \{z\}$, $N_C(y) = \{x\}$, $N_C(w_{1_2}) = V(P_2)$ and $N_C(w_{2_2}) = V(P_3)$. Let $H = T \cup C$, it is easy to check that H is a spanning Halin subgraph of G (See Figure 12.5(1) as an example).

If $|V_1| \geq 2$ and $w_{1_2}z \in E(G)$, we may assume $w_{1_1}w_{2_1}, w_{1_2}w_{2_2}, w_{1_3}w_{2_3} \in E(G)$ since G is 3-connected(Note that $w_{1_1} \neq w_{1_3}$). Since $V_1 \setminus \{v_{1_1}\}$, $N_2(x) \setminus \{w_{1_1}, w_{1_2}\}$ and $N_2^2(x) \setminus \{w_{2_1}\}$ are cliques, there exist hamiltonian paths, say $P_1 = v_{1_2}P_1v_{1_3}$, $P_2 = w_{1_3}P_2w_{1_4}$ and $P_3 = w_{2_2}P_3w_{2_3}$, in them, respectively. Let $C = P_1P_2P_3 \cup \{v_{1_3}x, xw_{1_4}, w_{2_2}w_{1_2}, w_{1_2}z, zv_{1_2}\}$ be a cycle and all vertices on the path $v_{1_1}yw_{1_1}w_{2_1}$ be stems of T with $N_C(v_{1_1}) = V(P_1) \cup \{z\}$, $N_C(y) = \{x\}$, $N_C(w_{1_2}) = V(P_2) \cup \{v_{1_2}\}$ and $N_C(w_{2_1}) = V(P_3)$. Let $H = T \cup C$, it is easy to check that H is a spanning Halin subgraph of G (See Figure 12.5(2) as an example).

If $|V_1| = 1$, since $deg_G(z) \geq 3$, we have $|N_2(z)| \geq 2$. If $|N_2(z) \cap N_2^2(x)| \geq 2$, we may assume $w_{2_1}z, w_{2_2}z \in E(G)$ and $w_{1_3}w_{2_3} \in E(G)$. Since $N_2(x) \setminus \{w_{1_1}\}$ and $N_2^2(x) \setminus \{w_{2_1}\}$ are cliques, there exist hamiltonian paths, say $P_1 = w_{1_2}P_1w_{1_3}$ and $P_2 = w_{2_2}P_2w_{2_3}$, in them, respectively. Let $C = P_1P_2 \cup \{v_{1_1}x, xw_{1_2}, w_{2_2}z, zv_{1_1}\}$ be a cycle and all vertices on the path $yw_{1_1}w_{2_1}$ be stems of T with $N_C(y) = \{x, v_{1_1}\}$, $N_C(w_{1_1}) = V(P_1)$ and $N_C(w_{2_1}) = V(P_2) \cup \{z\}$. Let $H = T \cup C$, it is easy to check that H is a spanning Halin subgraph of G (See Figure 12.5(3) as an example).

If $|V_1| = 1$ and $|N_2(z) \cap N_2(x)| = 1$, $|N_2(z) \cap N_2^2(x)| = 1$, we may assume $w_{1_1}z, w_{2_1}z \in E(G)$ since G is claw-free. We denote by $w_{1_2}w_{2_2}, w_{1_3}w_{2_3} \in E(G)$ since G is 3-connected(Note that we may have $w_{1_1} = w_{1_2}$). Since $N_2(x) \setminus \{w_{1_1}, w_{1_2}\}$ and $N_2^2(x) \setminus \{w_{2_2}\}$ are cliques, there exist hamiltonian paths, say $P_1 = w_{1_3}P_1w_{1_4}$ and $P_2 = w_{2_1}P_2w_{2_3}$, in them, respectively. Let $C = P_1P_2 \cup \{v_{1_1}x, xw_{1_4}, w_{2_1}z, zv_{1_1}\}$ be a cycle and all vertices on the path $yw_{1_1}w_{1_2}w_{2_2}$ be stems of T with $N_C(y) = \{x, v_{1_1}\}$, $N_C(w_{1_1}) = \{z\}$, $N_C(w_{1_2}) = V(P_1)$ and $N_C(w_{2_2}) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a spanning Halin subgraph of G (See Figure 12.5(4) as an example).

If $|V_1| = 1$ and $N_2(z) \cap N_2^2(x) = \emptyset$, which implies $|N_2(z) \cap N_2(x)| \geq 2$. We may assume $w_{1_1}w_{2_1}, w_{1_2}w_{2_2}, w_{1_3}w_{2_3} \in E(G)$ since G is 3-connected and $w_{1_4}z, w_{1_5}z \in E(G)$. Then $N_2(w_{1_4}) \cap N_2^2(x) = \emptyset$ and $N_2(w_{1_5}) \cap N_2^2(x) = \emptyset$ since G is claw-free and $xz \notin E(G)$. Because $N_2(x) \setminus \{w_{1_1}, w_{1_3}, w_{1_4}, w_{1_5}\}$ and $N_2^2(x) \setminus \{w_{2_1}\}$ are cliques, there exist hamiltonian paths, say $P_1 = w_{1_2}P_1w_{1_6}$ and $P_2 = w_{2_2}P_2w_{2_3}$, in them, respectively. Let $C = P_1P_2 \cup \{v_{1_1}x, xw_{1_6}, w_{2_3}w_{1_3}, w_{1_3}w_{1_5}, w_{1_5}z, zv_{1_1}\}$ be a cycle and all vertices on the path $yw_{1_4}w_{1_1}w_{2_1}$ be stems of T with $N_C(y) = \{x, v_{1_1}\}$, $N_C(w_{1_4}) = \{z, w_{1_5}\}$, $N_C(w_{1_1}) = V(P_1) \cup \{w_{1_3}\}$ and $N_C(w_{2_1}) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a spanning Halin subgraph of G (See Figure 12.5(5) as an example).

Case 2.2: Assume that $xz \in E(G)$. Then $yz \in E(G)$ since x and y are twins.

If $N_2(z) \cap N_2^2(x) \neq \emptyset$, we denote by $v_{1_1}z, w_{2_1}z, w_{1_2}w_{2_2}, w_{1_3}w_{2_3} \in E(G)$ since G is 3-connected(Note that we may have $w_{1_1} = w_{1_3}$). Since $V_1 \cup \{x\}$, $N_2(x) \setminus \{w_{1_2}\}$ and $N_2^2(x) \setminus \{w_{2_2}\}$ are cliques, there exist hamiltonian paths, say $P' = v_{1_1}P'x$, $P_1 = w_{1_1}P_1w_{1_3}$ and $P_2 = w_{2_1}P_2w_{2_3}$, in them, respectively. Let $C = P'P_1P_2 \cup \{w_{2_1}z, zv_{1_1}\}$ be a cycle and all vertices on the path $yw_{1_2}w_{2_2}$ be stems of T with $N_C(y) = V(P') \cup \{z\}$, $N_C(w_{1_2}) = V(P_1)$ and $N_C(w_{2_2}) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a spanning Halin subgraph

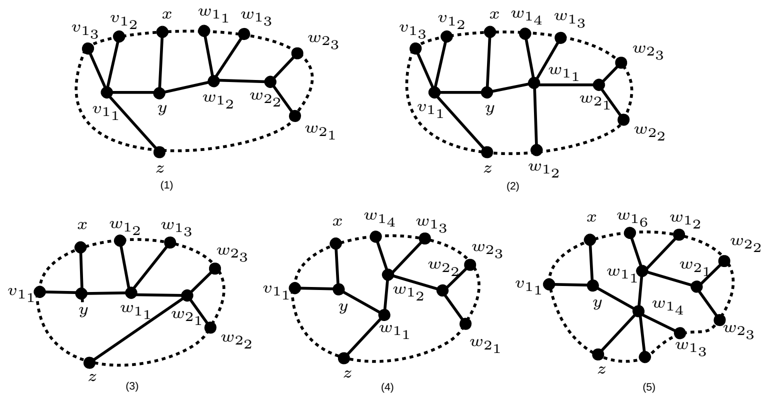

Figure 12.5. $N_2(w_{1_1}) \neq N_2(w_{1_2})$ and $xz \notin E(G)$.

of G (See Figure 12.6(1) as an example).

If $N_2(z) \cap N_2^2(x) = \emptyset$, then $N_2(z) \cap N_2(x) \neq \emptyset$ since $N_2(z) \neq \emptyset$. We denote by $v_{1_1}z, w_{1_1}z, w_{1_1}w_{2_1}, w_{1_2}w_{2_2}, w_{1_3}w_{2_3} \in E(G)$ since G is 3-connected. Since $V_1 \cup \{x\}$, $N_2(x) \setminus \{w_{1_2}, w_{1_1}\}$ and $N_2^2(x) \setminus \{w_{2_2}\}$ are cliques, there exist hamiltonian paths, say $P' = v_{1_1}P'x$, $P_1 = w_{1_4}P_1w_{1_3}$ and $P_2 = w_{2_1}P_2w_{2_3}$, in them, respectively. Let $C = P'P_1P_2 \cup \{w_{2_1}w_{1_1}, w_{1_1}z, zv_{1_1}\}$ be a cycle and all vertices on the path $yw_{1_2}w_{2_2}$ be stems of T with $N_C(y) = V(P') \cup \{z\}$, $N_C(w_{1_2}) = V(P_1) \cup \{w_{1_1}\}$ and $N_C(w_{2_2}) = V(P_2)$. Let $H = T \cup C$, it is easy to check that H is a spanning Halin subgraph of G (See Figure 12.6(2) as an example).

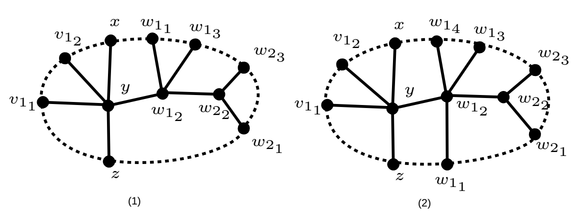

Figure 12.6. $N_2(w_{1_1}) \neq N_2(w_{1_2})$ and $xz \in E(G)$.

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