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Matchings and Tilings in Hypergraphs

Chuanyun Zang

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by

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Under the Direction of Yi Zhao, PhD

ABSTRACT

We consider two extremal problems in hypergraphs. First, given $k \geq 3$ and k-partite k-uniform hypergraphs, as a generalization of graph ($k = 2$) matchings, we determine the partite minimum codegree threshold for matchings with at most one vertex left in each part, thereby answering a problem asked by Rödl and Ruciński. We further improve the partite minimum codegree conditions to sum of all k partite codegrees, in which case the partite minimum codegree is not necessary large.

Second, as a generalization of (hyper)graph matchings, we determine the minimum vertex degree threshold asymptotically for perfect $K_{a,b,c}$-tlings in large 3-uniform hypergraphs, where $K_{a,b,c}$ is any complete 3-partite 3-uniform hypergraphs with each part of size a, b and c. This partially answers a question of Mycroft, who proved an analogous result with respect to codegree for r-uniform hypergraphs for all $r \geq 3$. Our proof uses Regularity Lemma, the absorbing method, fractional tiling, and a recent result on shadows for 3-graphs.

INDEX WORDS: Absorbing method, Regularity lemma, Hypergraph, Perfect matching, Graph tiling, Graph packing, Minimum degree.

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DEDICATION

This dissertation is dedicated to my family.

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PART 1

INTRODUCTION

Combinatorics is the study of discrete structures, and graph is perhaps the single most important discrete structure in combinatorics. A graph G is an ordered pair $(V(G), E(G))$ consisting of a set $V(G)$ of *vertices* and a set $E(G)$, disjoint from $V(G)$, of *edges*, together with an incidence function ψ_G that associates with each edge of G an unordered pair of (not necessary distinct) vertices of G. In this thesis, we only consider the graph with each edge associated to two distinct vertices and each pair of two vertices to one edge, i.e. *simple graph*. It is of great interest to find some well performed substructures (which are called subgraphs) in graphs. A subgraph is called *spanning subgraph* if it covers all vertices of the graph. A *matching* of graph G is a subgraph that consists of a collection of vertex-disjoint edges in G. A matching is called *perfect* if it is a spanning subgraph. A perfect matching is also called *1-factor* because it is a 1-regular spanning subgraph. Similarly, we can introduce *k-factor* to be a k-regular spanning subgraphs. In general, given graphs F and G, an *F-packing* or *F-tiling* of G is a subgraph of G that consists of vertex-disjoint copies of F, and an *F-factor* is a spanning F-packing.

The perfect matchings or packings are finding a partition of vertex set. Another interesting topic is finding a partition of edge set, which is usually called *graph decomposition*. One problem is the edge coloring, a partition of the edges set into as few disjoint matchings as possible (see section 1.3). When mentioning coloring, it is better to define *proper vertex coloring*, which assigns colors to the vertices of G so that no two adjacent vertices share the same color. The smallest number of colors needed is called *chromatic number* of G, denoted by $\chi(G)$. The first results in graph coloring is one of the most famous problems in graph theory – the *four color problem*.

Extremal combinatorics is a field of combinatorics which has been growing spectacularly in recent years. The word 'extremal' comes from the nature of the problems this filed deals with: if a collection of finite objects (numbers, graphs, vectors, sets. etc.) satisfies certain restrictions, how large or how small can it be? Here is one question in extremal graph theory: Given an n-vertex graph H and a g-vertex graph G, how many edges can H have so that H does not contain a copy of a fixed graph G? The celebrated Mantel's Theorem [67] says that if H has more than $n^2/4$ edges, then H contains a K_3 (triangle). This result has been generalized by Turán [89] for $G = K_r$, a complete graph on r vertices (a graph in which every pair of vertices are adjacent). In general, the *Turán number* for a graph G is the maximum number of edges in an n-vertex graph H such that H does not contain a copy of G, denoted by $ex(n, G)$. Erdös and Stone [22] extended Turán's result asymptotically to $K_r(t)$, the complete r-partite graph with t vertices in each class, see Theorem 2.4, which has been described as the "the fundamental theorem of extremal graph theory". Many work has been done on Turán type problems for graphs and hypergraphs, see surveys [30, 49]. However, for example, we still do not know the Turán number for complete bipartite graphs or Turán number for complete hypergraphs.

Another important question in extremal combinatorics: given a fixed g-vertex graph G, at least how many vertices are needed so that every 2-edge-colored complete graph on these vertices contains a monochromatic G? This is a classic two-color Ramsey problem. The smallest number of vertices needed here is called *Ramsey number* of G. For example, the Ramsey number of K_3 is 6, i.e., every 2-edge-colored K_6 contains a monochromatic K_3. Ramsey Theory is initiated and named after Frank Plumpton Ramsey who wrote a paper [73] in 1930. At about the same time, Van der Waerden [91] in 1927 proved his famous Ramsey-type result on arithmetical progressions (An arithmetic progression is a sequence of numbers that advances in steps of the same size). More detailed introduction about Ramsey theory can be found in the book [82]. Szemerédi [83] in 1975 improved Van Der Waerden's result and answered a notorious and decades-old conjecture of Erdös and Turán [26]. He showed that any positive fraction of the positive integers will contain arbitrarily long arithmetic progressions. The statement of Szemerédi Theorem is simple but the proof is much more difficult. This theorem was originally proved by Szemeredi in 1975 by a sophisticated combinatorial argument, introducing for the first time the powerful *Szemerédi regularity lemma*. There are several other deep and important proofs of this theorem, including the ergodic-theoretic proof of Furstenberg [32], the additive combinatorial proof of Gowers [33], and the hypergraph regularity proofs of Gowers [34] and Nagle, Rödl, Schacht, and Skokan [71, 79, 80, 81]. Szemerédi's Regularity Lemma has since become a powerful and now still a central tool in extremal combinatorics. The survey paper [56] provides a wide range of applications of regularity lemma. Also we briefly introduce it in Part 2.

We are more interested in finding spanning subgraphs. For matchings, there are a large number of results related to matching theory. Tutte's theorem [90] characterized the graphs that have perfect matchings and Edmonds [20] provided an efficient algorithm to find such a matching in polynomial time. However, when F is not an edge, it is NP-complete to determine whether a graph has an F-factor. Therefore it is natural to seek sufficient conditions for finding perfect a packing, like degree conditions. Given graph G and $v \in G$, define $\deg(v)$ be the number of edges in G containing v. *Minimum degree*, denoted by $\delta(G)$, is the minimum of $\deg(v)$ taken over all vertices v. In 1952 Dirac [19] proved a celebrated theorem stating that a graph on $n \geq 3$ vertices with minimum degree $n/2$ contains a Hamilton cycle (a cycle containing all vertices) and in hence a perfect matching when n is even. So problems related to minimum degree conditions are often called *Dirac-type problems*. Much work has been done on Dirac-type problems for graphs. Started from complete graphs on r vertices K_r, the celebrated Hajnal-Szemerédi theorem [35] in 1970 showed that a graph with $\delta(H) \geq (1 - 1/r)n$ contain a perfect K_r-tiling. More generally, for any fixed graph F, Komlós, Sárközy and Szemerédi [55] showed there is a constant C such that the minimum degree threshold to have a perfect F-tiling is $(1-1/\chi(F))n+C$ where $\chi(F)$ is the chromatic number of F. This confirmed a conjecture of Alon and Yuster [4] who had showed a weaker result with $o(n)$ in place of C. Finally, Kühn and Osthus [61] determined the minimum degree threshold completely up to an additive constant for any F.

As a generalization of a simple graph, for $k \geq 2$, a *k-uniform hypergraph* (in short,

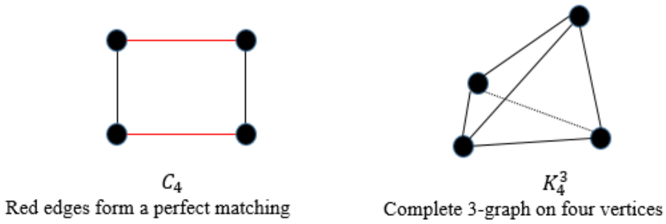

Figure 1.1. Graph and hypergraph on 4 vertices

k-graph) consists of a vertex set V and an edge set $E \subseteq \binom{V}{k}$, that is, every edge is a k-element subset of V. If $k = 2$, it is just the graph we defined earlier. One particular type of hypergraphs, a generalization of bipartite graphs, is *k-partite k-graphs*. A k-graph H is said to be k-partite if $V(H)$ can be partitioned into k parts, $V(H) = V_1 \cup \cdots \cup V_k$ such that every edge consists of exactly one vertex from each part. Similarly, given two k-graphs F and H, we can also define F-packing/tiling in H. When turning to hypergraphs with $k \geq 3$, the problem to determine the existence of a perfect packing gets harder. Even for matchings, the decision problem whether a 3-partite 3-graph contains a perfect matching is among the first 21 NP-complete problems given by Karp [46]. Therefore, we do not expect a nice characterization, and again it is natural to seek sufficient conditions for finding a perfect packing in hypergraphs.

It is worth mentioning that a relaxation of the perfect matching is to take into account of the fractional edges. A *fractional matching* of a k-graph $H = (V, E)$ is a function $\omega : E \to [0, 1]$ such that for each $v \in V$ we have $\sum_{e \ni v} \omega(e) \leq 1$. The size of ω , denoted by $\nu^*(H)$, is $\sum_{e \in E} \omega(e) = \frac{1}{k} \sum_v \sum_{e \ni v} \omega(e) \leq n/k$. It is called *perfect* if $\sum_{e \ni v} \omega(e) = 1$ for every vertex v, and hence $\nu^*(H) = n/k$. So the integer matching we introduced earlier is to take $\omega(e) = 0$ or 1.

One of the natural parameter of (hyper)graphs is *minimum degree*. Suppose H is a k-graph on n vertices. There are several definitions of the minimum degrees in H. For any set S of d vertices, where $1 \leq d \leq k - 1$, we define $\deg_H(S)$ to be the number of edges of H

which contain S. Let

$$\delta_d(H) := \delta_d = \min\{\deg_H(S) : S = \{v_1, \ldots, v_d\} \subset V(H)\}$$

be the *minimum d-degree* $\delta_d(H)$ of H. Two cases have received more attention: minimum 1-degree and minimum $(k-1)$-degree. When $d = 1$, it is referred as *minimum vertex degree*, and when $d = k-1$, it is called *minimum codegree*. Observe that $\delta_d(H) \le \binom{n-d}{k-d}$ and

$$\frac{\delta_1(H)}{\binom{n-1}{k-1}} \ge \frac{\delta_2(H)}{\binom{n-2}{k-2}} \ge \cdots \ge \frac{\delta_{k-1}(H)}{n-k+1}$$

we have that for any $c > 0$, if $\delta_d(H) \ge c\binom{n-d}{k-d}$, then $\delta_{d-1}(H) \ge \binom{n-(d-1)}{k-(d-1)}$.

Now the question change to ask the minimum d-degree threshold to force an F-factor. We briefly introduce two problems in extremal graphs in separate sections: matchings in k-graphs (section 1) and tilings in k-graphs (section 2). More results can be found in the surveys of Rödle and Ruciński [74], Köhn and Osthus [60], and Zhao [94]. In the last section, we put more information on edge-coloring.

1.1 Matchings in k-graphs

Hypergraph matchings have many practical applications such as the Santa Claus allocation problem [6]. As we stated earlier, the decision problem whether a given k-graph contains a perfect matching is NP-complete, so much attention are drawn to find sufficient conditions for a perfect matching. Suppose H is a k-graph on N vertices. The first result relating the minimum degree and the existence of a large (though, far from perfect) matching in k-graphs was given by Bollobás, Daykin and Erdös [7]. It was further extended by Daykin and Häggkvist [18] who showed that every k-graph H with $\delta_1(H) \ge (1-1/k)\binom{N-1}{k-1}$ contains a perfect matching.

Definition 1.1. *Given d, k, r and N satisfying $1 \le d \le k-1$ and $k \mid (n-r)$, define $m_d^r(k, N)$ as the smallest integer m such that every N-vertex k-graph H with $\delta_d(H) \ge m$ contains a* *matching M with $|V(M)| = N - r$. As to finding perfect matchings with $r = 0$, we suppress the subscript r, i.e., $m_d(k, N) := m_d^0(k, N)$.*

When $k = 2$, an easy greedy argument shows that $m_1(2, N) = N/2$ (see Part 2). The Dirac-type minimum d-degree thresholds for perfect matchings in general k-graphs have been studied intensively, see [2, 17, 36, 47, 51, 53, 52, 63, 66, 68, 72, 85, 87]. For $k \geq 3$, $d = k - 1$, a result of Rödle, Ruciński and Szemerédi [76] on Hamilton cycles implies that $m_{k-1}(k, N) \leq N/2 + o(N)$. Kühn and Osthus [59] sharpened this bound to $m_{k-1}(k, N) \leq N/2 + 3k^2\sqrt{N \log N}$ by using a result for the k-partite k-graphs which they had showed first. It was furthered improved to $m_{k-1}(k, N) \leq N/2 + C \log n$ by Rödle, Ruciński and Semerédi [77] in which they used the absorbing method. Rödle, Ruciński, Schacht and Semerédi in [75] found a fairly simple proof of $m_{k-1}(k, N) \leq N/2 + k/4$, and finally, in [78] they determined exactly $m_{k-1}(k, N) = N/2 - k + c$ where $c \in \{3/2, 2, 5/2, 3\}$ depends on N and k. In particular, for the decision problem of a given k-graph H under degree conditions $\delta_{k-1}(H) \geq N/k + o(N)$, Keevash, Knox and Mycroft [50] provided a polynomial-time algorithm to determine the existence of perfect matchings. Later Han [39] improved the degree condition to N/k in his polynomial-time algorithm.

For other values of d, Pikhurko [72] proved that for $d \geq k/2$, $m_d(k, N) = (1/2 + o(1))\binom{N-d}{k-d}$, which is asymptotically best possible. Treglown and Zhao [86, 87] determined the exact values of $m_d(k, N)$ when $d \geq k/2$. Independently Czygrinow and Kamat [17] determined the exact value of $m_2(4, N)$. Kühn, Osthus and Treglown [63], and independently Khan [53] determined the exact value of $m_1(3, N)$. Khan [52] also determined $m_1(4, N)$ exactly.

Based on all known results and constructions (see [94]), the following conjecture comes up.

Conjecture 1.2 ([88]). *For $k \geq 3$ and $1 \leq d \leq k - 1$,*

$$m_d(k, N) \approx \max\left\{\frac{1}{2},\ 1 - \left(\frac{k-1}{k}\right)^{k-d}\right\}\binom{N-d}{k-d}.$$

Note that the case when $d \geq k/2$ has been verified in [72]. Alon, Frankl, Huang, Rödl, Ruciński and Sudakov [2] verified Conjecture 1.2 for the case $d \geq k-4$, and very recently Treglown and Zhao [88] determined exact values of $m_2(5,N)$ and $m_3(7,N)$. More recently, Han [40] determined the exact values of $m_d(k,N)$ for $0.42k \leq d < k/2$, $m_5(12,N)$ and $m_7(17,N)$.

When $k \geq 3$ and $1 \leq d < k/2$, Hàn, Person and Schacht [36] gave a general bound as $m_d(k,N) \leq ((k-d)/k + o(1))\binom{N-d}{k-d}$. Markström and Ruciński [68] improved it to $m_d(k,N) \leq \left((k-d)/k - 1/k^{k-d} + o(1)\right)\binom{N-d}{k-d}$. Very recently, Kühn, Osthus and Townsend [62] further improved it to $m_d(k,N) \leq \left((k-d)/k - (k-d-1)/k^{k-d} + o(1)\right)\binom{N-d}{k-d}$ by using fractional matchings. The conjecture 1.2 is still far from completion.

Instead of finding a perfect matching, one question of interest is how about the minimum degree threshold for an almost perfect matching. Surprisingly, the threshold for perfect matchings in general k-graphs drops significantly if we allow even one vertex to be uncovered. When $k \nmid N$, the threshold to have a matching of size $\lfloor \frac{N}{k} \rfloor$ is shown to be between $\lfloor \frac{N}{k} \rfloor$ and $\lfloor \frac{N}{k} \rfloor + O(\log N)$ in [78], and later proved to be exactly $\lfloor \frac{N}{k} \rfloor$ in [39].

How about the minimum degree thresholds in k-partite k-graphs? Suppose H is k-partite k-graph with $V(H) = V_1 \cup \cdots \cup V_k$. A subset $S \subset V(H)$ is called *legal* if $|S \cap V_i| \leq 1$ for each $i \in [k]$. We define the *partite minimum d-degree* as the minimum of $\deg_H(S)$ taken over all legal d-vertex sets S in H, denoted by $\delta'_d(H)$. For $L \subseteq [k]$, the *partite minimum L-degree* $\delta'_L(H)$ is minimum of $\deg_H(S)$ taken over all legal $|L|$-set $S \subset \bigcup_{i \in L} V_i$.

When $k = 2$, Hall's Theorem gave a necessary and sufficient condition for the existence of perfect matching in a bipartite graph. However when $k \geq 3$, there is no such good result. Suppose H is a k-partite k-graph with each part of size n. As a simple corollary of Hall's theorem for graphs, if $\delta'_{k-1}(H) \geq n/2$ then H contains a perfect matching (It can also be derived from Dirac Theorem, or simple greedy algorithm, see Part 2). Kühn and Osthus [59] gave an analogous result when $k \geq 3$, which is that if $\delta'_{k-1}(H) \geq n/2 + \sqrt{2n \log n}$ then H has a perfect matching. Later Aharoni, Geogakopoulos and Sprüseel [1] improved this result by using conditions on only two types of partite minimum codegrees (See Part

2 for a brief proof). They showed that there is perfect matching if $\delta'_{[k]\setminus\{1\}}(H) > n/2$ and $\delta'_{[k]\setminus\{k\}}(H) \geq n/2$, and consequently, if $\delta'_{k-1}(H) > n/2$ then H has a perfect matching. Their result is best possible with k even and $n \equiv 2 (\mathrm{mod}\ 4)$. However, for other values of k and n, it is still possible to strengthen it (see survey [74]). In addition, a conjecture is stated in their paper.

Conjecture 1.3 ([1]). *If $\delta'_L(H) > n^{k-|L|}/2$ and $\delta'_{[k]\setminus L}(H) \geq n^{|L|}/2$ for some $L \subset [k-1]$, then H has a perfect matching. Or a stronger version, if $\delta'_L(H)/n^{k-|L|} + \delta'_{[k]\setminus L}(H)/n^{|L|} > 1$, then H has a perfect matching.*

Aharoni, Geogakopoulos and Sprüseel [1] gave a proof for the existence of a perfect fractional matching under the above condition. Pirkurko [72] showed an asymptotic result of the stronger version of Conjecture 1.3, see Theorem 3.9. Other than that, the conjecture is still open.

As interesting as in the general k-graph case, the minimum degree threshold for almost perfect matchings in k-partite k-graphs also drops significantly. Kühn and Osthus in [59] proved that $\delta'_{k-1}(H) \geq \lceil \frac{n}{k} \rceil$ guarantees a matching covering at least $n-(k-2)$ vertices from each part. Rödl and Ruciński asked in their survey paper [74, Problem 3.14] whether $\lceil \frac{n}{k} \rceil$ guarantees a matching in H covering at least $n-1$ vertices from each part. In Part 3, we answer this question and show that the threshold can be further weakened to $\lfloor \frac{n}{k} \rfloor$ when $n \equiv 1(\mathrm{mod\ k})$. In addition, we improve it to a new result by considering each partite minimum codegree $\delta'_{[k]\setminus\{i\}}(H)$: if there are at least three i's such that $\delta'_{[k]\setminus\{i\}}(H) > \epsilon n$ for some $\epsilon > 0$, then there is a matching covering at least $\min\{n-1, \sum_{i\in[k]} \delta'_{[k]\setminus\{i\}}(H)\}$ vertices in each vertex class.

1.2 Tilings in k-graphs

Recall that given two k-graphs F and H, an F*-tiling* (or F*-packing*) of H is a spanning subgraph which consists of a collection of vertex-disjoint copies of F in H. Given an integer n that is divisible by $|V(F)|$, we define the *tiling threshold* $t_d(n, F)$ to be the smallest integer

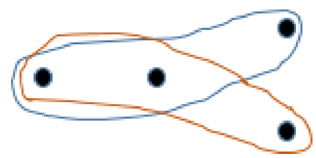

Figure 1.2. $K_4^3 - 2e$ or $\mathcal{C}_4^3$ or $K_{1,1,2}$

t such that every k-graph H on n vertices with $\delta_d(H) \geq t$ contains an F-factor.

Much work has been done on graphs ($k = 2$) as we stated earlier. When $k \geq 3$, tiling problems becomes much harder. Other than the matching problem, only a few tiling thresholds are known. Let's take a look at some codegree thresholds first. The natural starting point is 3-graphs on 4 points. Let $\mathcal{C}_4^3$ be the unique 3-graph on four vertices with two edges (this 3-graph was denoted by $K_4^3 - 2e$ in [16], and by Y in [43]). Kühn and Osthus [58] showed that $t_2(n, \mathcal{C}_4^3) = (1+o(1))n/4$. Later Czygrinow, DeBiasio and Nagle [16] determined $t_2(n, \mathcal{C}_4^3)$ exactly for large n, $t_2(n, \mathcal{C}_4^3) = n/4 + 1$ if $n \in 8\mathbb{N}$ and $t_2(n, \mathcal{C}_4^3) = n/4$ otherwise. Let K_4^3 denote the complete 3-graph on four vertices. Lo and Markström [66] proved that $t_2(n, K_4^3) = (1 + o(1))3n/4$. Simultaneously, Keevash and Mycroft [51] determined the exact value of $t_2(n, K_4^3)$ for sufficiently large n is $3n/4-2$ if $n \in 8\mathbb{N}$ or $3n/4-1$ otherwise. Let K_4^3-e denote the (unique) 3-graph on four vertices with three edges. In [65], Lo and Markström proved that $t_2(n, K_4^3 - e) = (1 + o(1))n/2$, which confirmed a conjecture of Pikhurko [72]. Exact of $t_2(n, K_4^3 - e)$ is recently proved to be $n/2 - 1$ by Han, Lo, Treglown and Zhao [41].

For the other k-graphs, Mycroft [70] determined $t_{k-1}(n, K)$ asymptotically for a wide class of k-partite k-graphs including all complete k-partite k-graphs and loose cycles. Here we state his result on complete k-partite k-graphs since there are some similarity between his result for codegree and our result in Part 4 for vertex degree. Let $K := K_{m_1,\ldots,m_k}$ be the complete k-partite k-graph with parts of size $m_1 \leq m_2 \leq \cdots \leq m_r$. We divide all complete k-partite r-graphs into *types*: K is *type* 0 if $\gcd(m_1, \ldots, m_r) > 1$ or $m_1 = \cdots = m_k = 1$; K is *type* $d \geq 1$ if $\gcd(m_1, \ldots, m_k) = 1$ and $\gcd(\{m_j - m_i : j > i\}) = d$. Mycroft [70] showed that

$$\delta_{k-1}(n,K)=\begin{cases} n/2+o(n), & \text{if } K \text{ is type } 0;\\ \frac{m_1}{|V(K)|}n+o(n), & \text{if } K \text{ is type } 1;\\ \max\{\frac{m_1}{|V(K)|}n, n/p\}+o(n), & \text{if } K \text{ is type } d\geq 2,\end{cases}$$

where p is the smallest prime factor of d. The proof of [70] makes use of Hypergraph Regularity Lemma and Blow-up Lemma of Keevash [48].

As to the vertex degree conditions, there are even fewer tiling results. Lo and Markström [66] determined $t_1(n, K_3^3(m))$ and $t_1(n, K_4^4(m))$ asymptotically, where $K_k^k(m)$ denotes the complete k-partite k-graph with m vertices in each part. Recently Han and Zhao [44] and independently Czygrinow [15] determined $t_1(n, \mathcal{C}_4^3)$ exactly for sufficiently large n. We [42] extend these results by determining $t_1(n, K)$ asymptotically for all complete 3-partite 3-graphs K, and thus partially answer a question of Mycroft [70], see Part 4.

1.3 Edge Coloring in Graphs

An *edge-coloring* is an assignment of colors to edges of a graph. A *proper edge-coloring* is an edge-coloring such that no two edges with common endpoint receive the same color. Clearly, a proper edge coloring is an edge coloring in which every color class is a matching. The smallest number of colors in a proper edge-coloring is called *edge chromatic number* of G, denoted by $\chi'(G)$. Let G be a graph with maximum degree Δ. The edge chromatic number of G is very closely related to the maximum degree Δ. Vizing's theorem shows $\chi'(G) = \Delta$ or $\Delta + 1$. In particular, if every color class is an induced matching (a matching in which no pair of two edges are joint by any edge from the host graph), it is called a *strong edge coloring*. In other words, a *strong edge-coloring* is an assignment of colors to edges of a graph such that no two edges of distance at most two receive the same color. Two edges are of distance at most two if and only if either they share an endpoint or one of their end points are adjacent. An induced matching is a set of edges such that no two edges are of distance at most two.

The strong edge chromatic number of G, usually denoted by $\chi'_s(G)$, is the minimum number of colors in a strong edge-coloring of G. For example, the strong chromatic number of Petersen graph is 5. Finding the best possible bound on $\chi'_s(G)$ in terms of Δ would be an analogue of Vizing's Theorem for strong edge-coloring. P. Erdős and J. Nešetřil [27] proposed the following conjecture in 1985.

Conjecture 1.4 ([27])**.**

$$\chi'_s(G) \leq \begin{cases} \frac{5}{4}\Delta^2 & \textit{if } \Delta \textit{ is even} \\ \frac{1}{4}(5\Delta^2 - 2\Delta + 1) & \textit{if } \Delta \textit{ is odd.} \end{cases}$$

The conjectured bounds are best possible with the constructions obtained from a blowup of C_5. When Δ is even, expanding each vertex of a 5-cycle into an independent set of cardinality $\Delta/2$ yields such a graph with $5\Delta^2/4$ edges. Similarly when Δ is odd, expanding each of two adjacent vertices into an independent set of cardinality $(\Delta+1)/2$ and each of the other three vertices of C_5 into independent set of cardinality $(\Delta-1)/2$ yields a graph with strong chromatic number $(5\Delta^2 - 2\Delta + 1)/4$. Chung, Gyărfăs, Trotter, and Tuza [11] proved that this operation gives the maximum number of edges in a $2K_2$-free graph with maximum degree Δ.

Conjecture 1.4 has been verified for graphs with maximum degree $\Delta \leq 3$. By using greedy edge coloring strategy, we can easily get $\chi'_s(G) \leq 2\Delta^2 - 2\Delta + 1$. That implies the conjecture is true for $\Delta \leq 2$. For $\Delta = 3$, it is proved by Andersen[5] and independently, by Horăk, He and Trotter [45], that $\chi'_s(G) \leq 10$ where G is a graph with maximum degree $\Delta = 3$. For $\Delta = 4$, as conjectured $\chi'_s(G) = 20$. D. Cranston [92] proved that any graph with maximum degree 4 has a strong edge-coloring using at most 22 colors. That is the best upper bound known for $\Delta = 4$. Conjecture 1.4 for $\Delta = 4$ or 5 is still open.

We [93] intend to improve the greedy algorithm to give an upper bound of the strong chromatic number in terms of Δ and use the algorithm to get an strong edge-coloring with 37 colors. See Part 5.

PART 2

TOOLS AND PRELIMINARIES

2.1 Greedy Algorithms

A greedy algorithm is an algorithm that makes the locally optimal choice at each stage with the hope of finding a global optimum. In many problems, a greedy strategy does not in general produce an optimal solution, but nonetheless a greedy algorithm may yield locally optimal solutions that approximate a global optimal solution in a reasonable time. Greedy algorithms play a useful role in the exploratory searching for matchings in k-graphs.

2.1.1 Greedy algorithms in finding maximum matchings

In this section, we state the use of greedy algorithms in finding maximum matchings. For example, when $k = 2$, we want to prove that a bipartite graph H with $V(H) = V_1 \cup V_2$, $|V_1| = |V_2| = n$ and $\delta_1'(H) \geq \delta$ contains a matching of size $\min\{n, 2\delta\}$. Suppose we find a maximum matching M and $|M| < \min\{n, 2\delta\}$. Let v, u be the leftover vertex from each part, then $N(v) \subset V(M)$ and $N(u) \subset V(M)$ otherwise contradicting the maximality of M. Since $\deg(v) + \deg(u) \geq 2\delta > |M|$, there exists one edge $e = \{xy\} \in M$ incident to both u and v, say $xu, yv \in E(H)$. Replacing e by xu and yv gives a larger matching, a contradiction.

Another example is from Kühn and Osthus [59] to find an almost perfect matching in k-partite k-graphs.

Theorem 2.1 ([59]). *For $s \geq 1$, $\ell > 0$, let H be a k-partite k-graphs with each part of size n. Denote $\delta' = \lceil n/k \rceil - \ell$ if $k \mid n$ or $n \equiv k-1 \bmod k$ and $\delta' = \lfloor n/k \rfloor - \ell$ otherwise. Suppose there are fewer than s^{k-1} legal $(k-1)$-set S with $\deg_H(S) < \delta'$. Then H has a matching which covers all but at most $(k-1)s + \ell k - 1$ vertices.*

Proof. Let $V_1, \dots, V_k$ denote the vertex classes of H. Assume the maximum matching M is of size $|M| \leq (k-1)s + \ell k$. Since each class has at least $(k-1)s + \ell k \geq (k-1)s$ vertices unmatched, for each $i = 1, \ldots, k$, one can find $(k-1)s$-sets $A_1, \ldots, A_k$ such that A_i contains exactly s unmatched vertices in V_j with $j \neq i$. Thus each A_i contains a legal $(k-1)$-set S_i with $\deg(S_i) \geq \delta'$, and all the neighbors of S_i lie entirely in $V(M)$ due to the maximality of M. Since $\sum_{i\in[k]} \deg(S_i) \geq k\delta' > n-(k-1)s - \ell k > |M|$, there exist distinct indices $i \neq j$ such that S_i and S_j have neighbors on the same edge $e \in M$, say $v_i \in S_i \cap e$ and $v_j \in S_j \cap e$. Replacing e by $\{v_i\} \cup S_i$ and $\{v_j\} \cup S_j$ gives a larger matching, a contradiction. □

At last we state the beautiful and surprisingly short proof in finding almost perfect matchings in k-partite k-graphs by Aharoni, Geogakopoulos and Sprüseel [1].

Theorem 2.2 ([1]). *Let H be a k-partite k-graphs with each part of size n. If $\delta'_{[k]\setminus\{1\}} > n/2$ and $\delta'_{[k]\setminus\{k\}} \geq n/2$, then H has a perfect matching.*

Proof. It suffices to prove the theorem for $k = 3$. To see this, let $k > 3$ and choose a perfect matching $F = g_1, g_2, \ldots, g_n$ in the complete $(k-2)$-partite $(k-2)$-graph with vertex partition $V_2, V_3, \ldots, V_{k-1}$. Let H' be the 3-partite 3-graph with vertex partition V_1, F, V_k such that for $x \in V_1, y \in V_k$, (x, g_i, y) is an edge of H' if and only if $\{x\} \cup g_i \cup \{y\}$ is an edge of H. Clearly, H' satisfies the conditions of the theorem, with $k = 3$. Assuming that the theorem is valid in this case, H' has a perfect matching, and 'de-contracting' each g_i results in a perfect matching of H.

Thus we may assume that $k = 3$. Suppose that the theorem fails. We may assume that H has a matching M that matches all but one vertex from each class; let $x_1 \in V_1$, $x_2 \in V_2$, $x_3 \in V_3$ be the unmatched vertices. Let U be the set of pairs (u, v) where $u \in V_2, v \in V_3$ such that there is an edge of M containing both u and v. Since each pair in U has more than $n/2$ neighbors in V_1, there exists a vertex $w \in V_1$ that is a neighbor of at least $n/2$ pairs in U. We consider three cases, in all of which we will be able to construct a perfect matching of H.

The first case is when $w = x_1$. Since the pair (x_2, x_3) has more than $n/2$ neighbors in V_1, there is an edge $e = (u_1, u_2, u_3) \in M$ such that $(x_1, u_2, u_3) \in E(H)$ and $(u_1, x_2, x_3) \in E(H)$. Then replacing e by (x_1, u_2, u_3) and (u_1, x_2, x_3) gives a perfect matching of H.

The next case is when w lies on an edge $f = (w, u_2, u_3) \in M$ such that $(x_1, x_2, u_3) \in E(H)$. Since the pair (u_2, x_3) has more than $n/2$ neighbors, there is an edge $g = (v_1, v_2, v_3) \in M$ such that v_1 is a neighbor of the pair (u_2, x_3) and the element $(v_2, v_3) \in U$ is in an edge with w. If $v_1 = w$ (in which case $f = g$) then replacing (w, u_2, u_3) by (x_1, x_2, u_3) and (w, u_2, x_3) gives a perfect matching of H; if $v_1 \neq w$ then replacing f and g by (x_1, x_2, v_3), (x_3, u_2, v_1) and (w, v_2, v_3) gives a perfect matching.

Finally, consider the case when w lies in an edge $f = (w, u_2, u_3) \in M$ such that $(x_1, x_2, u_3) \notin E(H)$. Since $d((u_2, u_3)) > n/2$ and $d((x_1, x_2)) \geq n/2$ there is an edge $g = (v_1, v_2, v_3) \in M$ such that $(v_1, u_2, u_3) \in E(H)$ and $(x_1, x_2, v_3) \in E(H)$. Let M' be the matching $M - f - g + (v_1, u_2, u_3) + (x_1, x_2, v_3)$. The only vertices not matched by M' are v_2, x_3 and w. Now we can repeat the argument of the first case with w playing the role of x_1. But in this case we have to be more careful: as w was a neighbor of at least $n/2$ pairs in U, and the only element of U that is not in an edge of M' is (v_2, v_3), there are still at least $n/2 - 1$ elements of U neighboring w that are each in an edge of M'. On the other hand, if $(w, v_2, x_3) \in E(H)$ we are done. Hence we can assume that the pair (v_2, x_3) has at least $(n+1)/2$ neighbors in $V_1 \setminus \{w\}$. But $n/2 - 1 + (n+1)/2 > n - 1$, thus there is an edge $e \in M'$ containing a pair neighboring w and a neighbor of (v_2, x_3). Replacing e from M' by the two corresponding edges yields a perfect matching of H. $\square$

2.2 Tools in Extremal Graph Theory

2.2.1 Some extremal graph theorems

In this section, we list some standard results from extremal graph theory mostly often used when employing the Regularity Lemma. Some of them may already be introduced in Part 1. Given a family $\mathcal{L}$ of prohibited graphs (hypergraphs), $ex(n, \mathcal{L})$ denotes the maximum number of edges (hyperedges) that an n-vertex graph (hyergraph) G can have without containing any subgraph $L \in \mathcal{L}$.

As is well known, Turán [89] proved that for every p there is a unique graph on $ex(n, K_p)$

vertices without containing K_p. The unique graph, called Turán graph, is the complete p-partite graph with n vertices whose partite sets differ in size by at most 1. The following form is weaker than Turán's original form but it is more usable.

Theorem 2.3 (Turán Theorem [89])**.**

$$ex(n, K_p) \leq \left(1 - \frac{1}{p-1}\right)\frac{n^2}{2}$$

Paul Erdös and Arthur Stone [22] extended Turán's result to $K_p(t)$, the complete p-partite graph with t vertices in each class, which is asymptotically. (For a strengthen versions, see [12, 13].)

Theorem 2.4 (Erdös-Stone Theorem [22])**.** *For any integers $p \geq 2$ and $t \geq 1$,*

$$ex(n, K_p(t)) = \left(1 - \frac{1}{p-1}\right)\binom{n}{2} + o(n^2)$$

Erdös and Simonovits [21] showed the general asymptotic result of $ex(n, \mathcal{L})$, which plays a crucial role in extremal graph theory.

Theorem 2.5 ([21])**.** *If $\mathcal{L}$ is finite and $\min_{L\in\mathcal{L}} \chi(L) = p > 1$, then*

$$ex(n, \mathcal{L}) = \left(1 - \frac{1}{p-1}\right)\binom{n}{2} + o(n^2)$$

In general, for r-graphs, we have similar results. Given $\ell_1, \ldots, \ell_r \in \mathbb{N}$, let $K^{(r)}_{\ell_1,\ldots,\ell_r}$ denote the complete r-partite r-graph whose jth part has exactly ℓ_j vertices for all $j \in [r]$.

Theorem 2.6 (Erdös Theorem [23])**.** *For r-graphs,*

$$ex(n, K^{(r)}_{\ell,\ldots,\ell}) < n^{r-\ell^{1-r}}$$

Now we state a generalization of the Erdös-Stone Theorem for hypergraphs by Erdös [24] and Brown, simonovits [8].

Theorem 2.7 ([24, 8])**.**

$$ex(n, K^{(r)}_{\ell_1,\dots,\ell_r}) = ex(n, K^{(r)}_{1,\dots,1}) + o(n^r)$$

Given $\mathcal{L}$ as a family of prohibited r-graphs, the r-graphs are called *supersaturated* if it has more edges than $ex(n, \mathcal{L})$. The basic question is how many copies of $L \in \mathcal{L}$ must occur in a r-graph on n vertices with more than $ex(n, \mathcal{L})$ edges. The following proposition is from the result of Erdós and Simonovits on supersaturation.

Proposition 2.8 ([25])**.** *Given $\mu > 0$, $l_1, \dots, l_r \in \mathbb{N}$, there exists $\mu' > 0$ such that the following holds for sufficiently large n. Let H be an r-graph on n vertices with a vertex partition $V_1 \cup \cdots \cup V_m$. Suppose $i_1, \dots, i_r \in [m]$ and H contains at least μn^r edges $e = \{v_1, \dots, v_r\}$ such that $v_1 \in V_{i_1}$, $\dots, v_r \in V_{i_r}$. Then H contains at least $\mu' n^{l_1+\cdots+l_r}$ copies of $K^{(r)}_{l_1,\dots,l_r}$ whose jth part is contained in V_{i_j} for all $j \in [r]$.*

2.2.2 Szemerédi's Regularity Lemma

Szemerédi's Regularity Lemma [84] has been proved to be an incredibly powerful and useful tool in graph theory as well as in Ramsey theory, combinatorial number theory and other areas of mathematics and theoretical computer science. The lemma essentially says that, in some sense, all large graphs can be approximated by a random-looking graphs. It helps to prove results for arbitrary graphs whenever the corresponding results are trivial for random graphs.

Given a graph H and a pair (U, W) of disjoint non-empty subsets of $V(H)$. We denote the density of (U, W) by

$$d(U, W) = \frac{e(U, W)}{|U||W|}.$$

The pair (U, W) is called (ϵ, d)*-regular* for $\epsilon > 0$ and $d \geq 0$ if

$$|d(U', W') - d| \leq \epsilon$$

for $U' \subseteq U, W' \subset W$ with $|U'| \geq \epsilon|U|, |W'| \geq \epsilon|W|$. The pair (U, W) is called *ϵ-regular* if it is (ϵ, d)-regular for some $d \geq 0$. It is immediate from the definition that in an (ϵ, d)-regular pair (U, W), if $U' \subset U$, $|U'| \geq c|U|$ and $W' \subset W$, $|W'| \geq c|W|$ for some $c \geq \epsilon$, then (U', W') is $(\epsilon/c, d)$-regular.

Lemma 2.9 (Regularity Lemma [84]). *For all $\epsilon > 0$ and $l \in \mathbb{N}$ there exist $n_0, M \in \mathbb{N}$ such that for every $n \geq n_0$ the following holds. Let G be an n-vertex graph whose vertex set is pre-partitioned into sets $V_1, V_2, \ldots, V_{l'}$, $l' \leq l$. Then there exists a partition $U_0, U_1, \ldots, U_t$ of $V(G)$, $l < t < M$, with the following properties.*

(i) For every $i, j \in [t]$ we have $|U_i| = |U_j|$ and $|U_0| < \epsilon n$

(ii) For every $i \in [t]$ and every $j \in [l']$ either $U_i \cap V_j = \emptyset$ or $U_i \subset V_j$

(iii) All but at most ϵt^2 pairs (U_i, U_j), $i, j \in [t], i \neq j$, are ϵ-regular.

The partition given in Lemma **??** is called an ϵ-regular partition of G. Given an ϵ-regular partition of G and $d \geq 0$, we refer to $V_i, i \in [t]$ as *clusters*. The *reduced graph* (or *cluster graph*) R is the graph whose vertices are clusters $U_1, \ldots, U_t$ and $\{U_i, U_j\}$ form an edge of $\mathcal{R}$ if and only if (U_i, U_j) is ϵ-regular and $d(U_i, U_j) \geq d$. Reduced graphs inherit many properties of G like the following degree result.

Proposition 2.10. *If $0 < 2\epsilon \leq d \leq c/2$ and $\delta(H) \geq c$, then $\delta(R) \geq (c - 2d)|R|$.*

Many proofs using Regularity Lemma are similar: If G has a reduced graph R then every small subgraph of R is also a subgraph of G.

For a graph R and integer $t > 0$, let $R(t)$ be the graph obtained by replacing each vertex $x \in V(R)$ with a set V_x of t vertices, and for $u \in V_x, v \in V_y$, $uv \in E(R(t))$ if and only if $xy \in E(R)$. So $R(t)$ is obtained by replacing each edge of R by a copy of the complete bipartite graph $K_{t,t}$.

A key lemma to use Regularity Lemma was stated by Komlós and Simonovites [56] as follows.

Lemma 2.11 (Key Lemma,[56]). *Given $d > \epsilon > 0$, a graph R, and a positive integer m. Let G be a graph by replacing every vertex of R by m vertices, and replacing the edges of R with ϵ-regular pairs of density at least d. Let F be s subgraph of $R(t)$ with f vertices and maximum degree $\Delta > 0$, and let $\epsilon_0 = (d-\epsilon)^\Delta/(2+\Delta)$. If $\epsilon \leq \epsilon_0$ and $t - 1 \leq \epsilon_0 m$, then $F \subset G$.*

We will state another lemma, Blow-up Lemma, which plays the same role in embedding spanning graphs as the Key Lemma played in embedding smaller graphs.

Given a graph G and two disjoint vertex sets $A, B \subset V$, we say the pair (A, B) is (ϵ, δ)-super-regular if for every $X \subset A$ and $Y \subset B$ satisfying $|X| > \epsilon|A|$ and $|Y| > \epsilon|B|$ we will have $e(X, Y) > \delta|X||Y|$, and furthermore, $\deg(a) > \delta|B|$ for all $a \in A$, and $\deg(b) > \delta|A|$ for all $b \in B$. Now we are ready to state the Blow-up Lemma.

Lemma 2.12 (Blow-up Lemma). *Given a graph R of order r and positive parameters δ, Δ, there exists an $\epsilon > 0$ such that the following holds. Let $n_1.n_2, \ldots, n_r$ be arbitrary positive integers and let us replace the vertices of R with pairwise disjoint sets $V_1, V_2, \ldots, V_r$ of sizes $n_1, n_2, \ldots, n_r$ (blowing up). We construct two graphs on the same vertex-set $V = \cup V_i$. The first graph $\boldsymbol{R}$ is obtained by replacing each edge $\{v_i, v_j\}$ of R with the complete bipartite graph between the corresponding vertex-sets V_i and V_j. A sparser graph G is constructed by replacing each edge $\{v_i, v_j\}$ with an (ϵ, δ)-duper-regular pair between v_i and V_j. If a graph H with $\Delta(H) \leq \Delta$ is embeddable into $\boldsymbol{R}$ then it is already embeddable into G.*

An example of using Regularity Lemma We prove the following result by using Regularity Lemma and Erdös-Stone Theorem. It is also discussed in Part 6.

Theorem 2.13. *Given graphs H and F with $|V(H)| = n$, $|V(F)| = f$ and $f \ll n$. Let $r = \chi(F)$. For $\gamma > 0$ if $\delta(H) \geq (1 - \frac{1}{r-1} + \gamma)n$ then every vertex of H can be covered by some copy of F.*

Proof. Let $\gamma > 0$ and $\delta(H) \geq (1 - \frac{1}{r-1} + \gamma)n$. Let v be an arbitrary vertex in $V(H)$. Let V_1 be the set of vertices that are adjacent to v, and V_2 be the set of vertices which are not adjacent to v. $|V_1| \geq (1 - \frac{1}{r-1} + \gamma)n$.

When $r = 2$, F is a bipartite graph $K_{s,t}$ with $s \leq t$. In this case, $|V_1| \geq \gamma n$. Choose $\tilde{V_1} \subset V_1$ with $|\tilde{V_1}| = \gamma n/2$. Let $V_2' = V(H) \setminus \{\tilde{V_1} \cup \{v\}\}$, then $e(\tilde{V_1}, V_2') \geq (\frac{\gamma}{2}n)^2$. By Erdös Theorem, there exists a bipartite graph $K_{t,t} \subset H[\tilde{V_1}, V_2']$. Since v is adjacent to all vertices in $\tilde{V_1}$, we can find a copy of F containing v.

When $r \geq 3$, we use Regularity Lemma. Let $0 < 2\epsilon \leq d \ll \gamma$. Apply Regularity Lemma on $V(H) \setminus \{v\}$ with pre-partition $\{V_1, V_2\}$. We get a new partition $\{U_1, U_2, \ldots, U_t\}$ and reduced graph R with $\delta(R) \geq (1 - \frac{1}{r-1} + \gamma - 2d)t \geq (1 - \frac{1}{r-1} + \gamma/2)t$ and $|R \cap V_1'| \geq (1 - \frac{1}{r-1} + \gamma - \epsilon)n/(\frac{1-\epsilon}{t}n) > (1 - \frac{1}{r-1} + \gamma/2)t$, where $V_i' \subset V(R)$ contains all clusters in V_i.

In the induced graph $R[V_1']$,

$$\begin{aligned}
\deg_{V_1'}(v) &\geq (1 - \frac{1}{r-1} + \gamma/2)t - (\frac{1}{r-1} - \gamma/2)t \\
&\geq (1 - \frac{2}{r-1} + \gamma)t \\
&\geq (1 - \frac{1}{r-2} + \gamma/2)|V_1'|
\end{aligned}$$

By Edös-Stone Theorem 2.4, $R[V_1']$ contain a K_{r-1}. Without Loss of Generality, we may assume $V(K_{r-1}) = \{U_1, \ldots, U_{r-1}\}$.

Claim 2.14. *Let $\gamma > 0$, $k \ll t$. Let X be a set of t elements and let $A_1, A_2, \ldots, A_k$ be subsets of X with size at least $(\frac{k-1}{k} + \gamma/2)t$, then $\bigcap_{i=1}^{k} A_i \neq \emptyset$.*

Proof. Let $A_i^c = X \setminus A_i$ for $i \in [k]$. It is sufficient to show that $\cup_{i=1}^{k} A_i^c \neq X$. This is obviously true since $|A_i^c| \leq (\frac{1}{k} - \gamma/2)n$ and hence $|\cup_{i=1}^{k} A_i^c| \leq \sum_{i=1}^{k} |A_i^c| < t$. □

Since $|N_R(U_i)| \geq (1 - \frac{1}{r-1} + \gamma/2)t$, one can apply Claim 2.14 on $V(R)$ with subsets $N_R(U_i)$ for $i = 1, ..., r-1$ and $k = r-1$. Therefore, there is a cluster, denoted by U', disjoint from and adjacent to all vertices of $\{U_1, \ldots, U_{r-1}\}$. Hence there exists a K_r in R with vertices $\{U_1, \ldots, U_{r-1}, U'\}$.

By Key Lemma 2.11, from $K_r \subset R$ we can get a complete r-partite graph $K \subset K_{U_1,\ldots,U_{r-1},U'}$. In $K \cup \{v\}$, one can find a copy of F containing v. □

2.2.3 Hypergraph version of Regularity Lemma

Weak Hypergraph Regularity Lemma Szemerédi's Regularity Lemma [84] has many generalizations to hypergraphs. In this thesis, we use the so-called Weak Hypergraph Regularity Lemma, which is a straightforward extension of the original Szemerédi's regularity lemma to hypergraphs (see [29, 10]).

Let $H = (V, E)$ be a k-graph and let $A_1, \dots, A_k$ be mutually disjoint non-empty subsets of V. We define $e(A_1, \dots, A_k)$ to be the number of edges with one vertex in each A_i, $i \in [k]$, and the density of H with respect to $(A_1, \dots, A_k)$ as

$$d(A_1, \dots, A_k) = \frac{e(A_1, \dots, A_k)}{|A_1| \cdots |A_k|}.$$

We say a k-tuple $(V_1, \dots, V_k)$ of mutually disjoint subsets $V_1, \dots, V_k \subseteq V$ is (ϵ, d)-*regular*, for $\epsilon > 0$ and $d \geq 0$, if

$$|d(A_1, \dots, A_k) - d| \leq \epsilon$$

for all k-tuples of subsets $A_i \subseteq V_i$, $i \in [k]$, satisfying $|A_i| \geq \epsilon |V_i|$. We say $(V_1, \dots, V_k)$ is ϵ-*regular* if it is (ϵ, d)-regular for some $d \geq 0$. It is immediate from the definition that in an (ϵ, d)-regular k-tuple $(V_1, \dots, V_k)$, if $V_i' \subset V_i$ has size $|V_i'| \geq c|V_i|$ for some $c \geq \epsilon$, then $(V_1', \dots, V_k')$ is $(\epsilon/c, d)$-regular.

Theorem 2.15 (Weak Regularity Lemma)**.** *Given $t_0 \geq 0$ and $\epsilon > 0$, there exist $T_0 = T_0(t_0, \epsilon)$ and $n_0 = n_0(t_0, \epsilon)$ so that for every k-graph $H = (V, E)$ on $n > n_0$ vertices, there exists a partition $V = V_0 \cup V_1 \cup \cdots \cup V_t$ such that*

(i) $t_0 \leq t \leq T_0$,

(ii) $|V_1| = |V_2| = \cdots = |V_t|$ *and* $|V_0| \leq \epsilon n$,

(iii) for all but at most $\epsilon\binom{t}{k}$ *k-subsets* $\{i_1, \dots, i_k\} \subset [t]$, *the k-tuple* $(V_{i_1}, \dots, V_{i_k})$ *is ϵ-regular.*

The partition given in Theorem 2.15 is called an ϵ-regular partition of H. Given an ϵ-regular partition of H and $d \geq 0$, we refer to $V_i, i \in [t]$ as *clusters* and define the *cluster* *hypergraph* $\mathcal{R} = \mathcal{R}(\epsilon, d)$ with vertex set $[t]$ and $\{i_1, \ldots, i_k\} \subset [t]$ is an edge if and only if $(V_{i_1}, \ldots, V_{i_k})$ is ϵ-regular and $d(V_{i_1}, \ldots, V_{i_k}) \geq d$.

The following corollary shows that the cluster hypergraph inherits the minimum degree of the original hypergraph. Its proof is almost the same as in [37, Proposition 16] after we replace $\frac{1}{2(k-\ell)} + \gamma$ by c – we thus omit the proof.

Corollary 2.16. *[37] Given $c, \epsilon, d > 0$ and integers $k \geq 3, t_0$ such that $0 < \epsilon < d^2/4$ and $t_0 \geq 2k/d$, there exist T_0 and n_0 such that the following holds. Let H be a k-graph on $n > n_0$ vertices such that $\delta_{k-1}(H) \geq cn$. If H has an ϵ-regular partition $V_0 \cup V_1 \cup \cdots \cup V_t$ with $t_0 \leq t \leq T_0$ and $\mathcal{R} = \mathcal{R}(\epsilon, d)$ is the cluster hypergraph, then at most $\sqrt{\epsilon} t^{k-1}$ $(k-1)$-subsets S of $[t]$ violate $\deg_{\mathcal{R}}(S) \geq (c - 2d)t$.*

We will use the Weak Hypergraph Regularity Lemma in Part 4.

Strong Hypergraph Regularity Lemma One of the main reasons for the wide applicability of Szemerédi's Regularity Lemma is the fact that it enbles one to find all small graphs as subgraphs of a regular graph. To generalize this application in finding small subgraphs in its regular partition, we need to strengthen the definition of regularity.

Before we can state the strong regularity lemma, we first define a complex. A hypergraph H consists of a vertex set $V(H)$ and an edge set $E(H)$, where every edge $e \in E(H)$ is a non-empty subset of $V(H)$. A hypergraph H is a *complex* if whenever $e \in E(H)$ and e' is a non-empty subset of e we have that $e' \in E(H)$. All the complexes considered in this section have the property that every vertex forms an edge.

For a positive integer k, a complex H is a *k-complex* if every edge of H consists of at most k vertices. The edges of size i are called i-edges of $\mathcal{H}$. Given a k-complex H, for each $i \in [k]$ we denote by H_i the underlying i-graph of H: the vertices of H_i are those of H and the edges of H_i are the i-edges of H.

Given $s \geq k$, a (k, s)-*complex* H is an s-partite k-complex, by which we mean that the vertex set of H can be partitioned into sets $V_1, \ldots, V_s$ such that every edge of H is *crossing*, namely, meets each V_i in at most one vertex.

Given $i \geq 2$, an i-partite i-graph H and an i-partite $(i-1)$-graph G on the same vertex set, we write $\mathcal{K}_i(G)$ for the family of all crossing i-sets that form a copy of the complete $(i-1)$-graph $K_i^{(i-1)}$ in G. We define the density of H with respect to G to be

$$d(H|G) := \frac{|\mathcal{K}_i(G) \cap E(H)|}{|\mathcal{K}_i(G)|} \quad \text{if} \quad |\mathcal{K}_i(G)| > 0,$$

and $d(H|G) = 0$ otherwise. More generally, if $\mathbf{Q} = (Q_1, \ldots, Q_r)$ is a collection of r subhypergraphs of G, we define $\mathcal{K}_i(\mathbf{Q}) := \bigcup_{j=1}^{r} \mathcal{K}_i(Q_j)$ and

$$d(H|\mathbf{Q}) := \frac{|\mathcal{K}_i(\mathbf{Q}) \cap E(H)|}{|\mathcal{K}_i(\mathbf{Q})|} \quad \text{if} \quad |\mathcal{K}_i(\mathbf{Q})| > 0,$$

and $d(H|\mathbf{Q}) = 0$ otherwise.

We say that H is *(d, δ, r)-regular with respect to G* if every r-tuple $\mathbf{Q}$ with $|\mathcal{K}_i(\mathbf{Q})| > \delta|\mathcal{K}_i(G)|$ satisfies $|d(H|\mathbf{Q}) - d| \leq \delta$. Instead of $(d, \delta, 1)$-regularity we simply refer to *(d, δ)-regularity.*

Given $3 \leq k \leq s$ and a (k, s)-complex H, we say that H *is $(d_k, \ldots, d_2, \delta_k, \delta, r)$-regular* if the following conditions hold:

(i) For every $i = 2, \ldots, k-1$ and every i-tuple K of vertex classes, $H_i[K]$ is (d_i, δ)-regular with respect to $H_{i-1}[K]$ unless $e(H_i[K]) = 0$, where $H_i[K]$ is the restriction of H_i to the union of all vertex classes in K.

(ii) For every k-tuple K of vertex classes, $H_k[K]$ is (d_k, δ_3, r)-regular with respect to $H_{k-1}[K]$ unless $e(H_k[K]) = 0$.

The following states that the restriction of regular complexes to a sufficiently large set of vertices is still regular, by Kühn,Mycroft and Othus [57].

Lemma 2.17 ([57], Lemma 4.1)**.** *Let k, s, r, m be positive integers and $\alpha, d_2, \ldots, d_k, \delta, \delta_k$ be positive constants such that*

$$\frac{1}{m} \leq \frac{1}{r}, \delta \leq min\{\delta_k, d_2, \ldots, d_k\} \leq \delta_k \ll \alpha \ll d_k, \frac{1}{s}$$

Let H be a $(d_k, \ldots, d_2, \delta_k, \delta, r)$-regular (k,s)-complex with vertex classes $V_1, V_2, \ldots, V_s$ with size m. For each i let $V_i' \subset V_i$ be a set of size at least αm. Then the restriction $H' = H[V_1', V_2', V_3', \ldots, V_s']$ of H to $V_1' \cup V_2' \cup V_3' \cup \cdots \cup V_s'$ is $(d_2, d_3, \sqrt{\delta_k}, \sqrt{\delta}, r)$-regular.

Statement of the Regularity Lemma In this section we state the version of the regularity lemma due to Rödl and Schacht [79] for 3-graphs, which is almost the same as the one given by Frankl and Rödl [28]. We need more notation. Suppose that V is a finite set of vertices and $\mathcal{P}^{(1)}$ is a partition of V into sets $V_1, \ldots, V_t$, which will be called *clusters*. Given any $j \in [3]$, we denote by $\mathrm{Cross}_j = \mathrm{Cross}_j(\mathcal{P}^{(1)})$ the set of all crossing j-subsets of V. For every set $A \subseteq [t]$ we write Cross_A for all the crossing subsets of V that meet V_i whenever $i \in A$. Let $\mathcal{P}_A$ be a partition of Cross_A. We refer to the partition classes of $\mathcal{P}_A$ as *cells*. Let $\mathcal{P}^{(2)}$ be the union of all $\mathcal{P}_A$ with $|A| = 2$ (so $\mathcal{P}^{(2)}$ is a partition of Cross_2). We call $\mathcal{P} = \{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$ *a family of partitions* on V.

Given $\mathcal{P} = \{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$ and $K = v_i v_j v_k$ with $v_i \in V_i$, $v_j \in V_j$ and $v_k \in V_k$, the *polyad* $P(K)$ is the 3-partite 2-graph on $V_i \cup V_j \cup V_k$ with edge set $C(v_i v_j) \cup C(v_i v_k) \cup C(v_j v_k)$, where *e.g.*, $C(v_i v_j)$ is the cell in $V_i \times V_j$ that contains $v_i v_j$. We say that $P(K)$ is (d_2, δ)-*regular* if all $C(v_i v_j), C(v_i v_k), C(v_j v_k)$ are (d_2, δ)-regular with respect to their underlying sets. We let $\hat{\mathcal{P}}^{(2)}$ be the family of all $P(K)$ for $K \in \mathrm{Cross}_3$.

Now we are ready to state the regularity lemma for 3-graphs.

Theorem 2.18 ([79])**.** *For all $\delta_3 > 0, t_0 \in \mathbb{N}$ and all functions $r : \mathbb{N} \to \mathbb{N}$ and $\delta : \mathbb{N} \to (0,1]$, there are $d_2 > 0$ such that $1/d_2 \in \mathbb{N}$ and integers T, n_0 such that the following holds for all $n \geq n_0$ that are divisible by $T!$. Let H be a 3-graph of order n. Then there exists a family of partitions $\mathcal{P} = \{\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\}$ of the vertex set V of H such that*

(i) $\mathcal{P}^{(1)} = \{V_1, \ldots, V_t\}$ is a partition of V into t clusters of equal size, where $t_0 \leq t \leq T$,

(ii) $\mathcal{P}^{(2)}$ is a partition of Cross_2 into at most T cells,

(iii) for every $K \in \mathrm{Cross}_3$, $P(K)$ is $(d_2, \delta(T))$-regular,

(iv) $\sum |\mathcal{K}_3(P)| \leq \delta_3 |V|^3$, *where the summation is over all* $P \in \hat{\mathcal{P}}^{(2)}$ *such that* H *is not* $(d, \delta_3, r(T))$*-regular with respect to* P *for any* $d > 0$.

2.3 Absorbing Method

2.3.1 Technique descriptions

The absorbing method, initiated by Rödl, Ruciński, and Szemerédi [76], has been shown to be effective handling extremal problems in graphs and hypergraphs. Roughly speaking, the absorbing method reduces the task of finding a spanning structure to finding an almost spanning structure. One example is the re-proof of Posa's conjecture by Levitt, Sárközy, and Szemerédi [64], while the original proof of Komlós, Sárközy, and Szemerédi [54] used the Regularity Lemma.

We will briefly introduce the simple and basic version of the absorbing lemma, and then use it to illustrate the algorithm of using the absorbing method. For this purpose, more detail can be found in [66].

Let H be an r-graph on n vertices. Given a vertex set $U \subset V(H)$, $H[U]$ is the subgraph of H induced by the vertices of U. Given an r-graph F of order t, $\beta > 0$, $i \in \mathbb{N}$ and two vertices $u, v \in V(H)$, we call that u, v are (F, β, i)*-reachable in* H if and only if there are at least βn^{it-1} $(it-1)$-sets W such that both $H[\{u\} \cup W]$ and $H[\{v\} \cup W]$ contain K-factors. In this case, we call W a *reachable set* for u and v. A vertex set A is (F, β, i)*-closed* in H if every two vertices in A are (F, β, i)-reachable in H. When it is clear, we use (β, i) to represent for (F, β, i).

Lemma 2.19 (Absorption Lemma for F-factors, [66]). *Let* F *be an* r*-graph of order* t. *Given* $\beta > 0$, *and* $i_0 \in \mathbb{N}$, *there exists* $\eta > 0$ *such that the following holds for all sufficiently large integers* n. *Suppose* H *is an* (F, β, i_0)*-closed* r*-graph on* n *vertices. Then there exists a vertex set* $W \subseteq V$ *and* $|W| \leq \eta n$ *such that for any vertex set* $U \subseteq V \setminus W$ *with* $|U| \leq \eta^3 n$ *and* $|U| \in t\mathbb{Z}$, *both* $H[W]$ *and* $H[W \cup U]$ *contain* F*-factors.*

Equipped with the absorption lemma, we can break down the task of finding an F-factor in large hypergraphs H into the following algorithm.

Algorithm for finding F-factors via the Absorbing Method.

1. Remove a small set $\mathcal{T}_1$ of vertex-disjoint copies of F from H such that the resultant graph $H_1 = H[V \setminus V(\mathcal{T}_1)]$ is (F, β, i_0)-closed for some integer i_0 and constant $\beta > 0$.

2. Find a vertex set $W \subset V(H_1)$ satisfying the conditions of the absorption lemma. Set $H_2 = H_1[V(H_1) \setminus W]$.

3. Show that H_2 contains an almost F-factor, i.e. a set $\mathcal{T}_2$ of vertex-disjoint copies of F such that $|V(H_2) \setminus V(\mathcal{T}_2)| < \alpha |V(H_2)|$ for small $\alpha > 0$.

4. Set $U = V(H_2) \setminus V(\mathcal{T}_2)$. Since $H_1[W \cup U]$ contains an F-factor $\mathcal{T}_3$ by the choice of W, $T_1 \cup \mathcal{T}_2 \cup \mathcal{T}_3$ is an F-factor in H.

Step 1 and Step 3 of the algorithm require most of the work. Main use of Step 1 is to get the hypergraph ready to use absorbing lemma, the closeness. However it is not always the case that $H_1 = H[V \setminus V(\mathcal{T}_1)]$ is (β, i_0)-closed, instead in Part 4 we have $V \setminus V_0$ is (β, i_0)-closed in H. In this case, we adjust the absorbing lemma, but the core idea of proof is essential, which is the crucial proof we show in the next section.

2.3.2 Crucial proof in the absorbing method

In this section, we show a classic proof of Lemma 2.19. One crucial part of the absorbing method is the probabilistic arguments. We include the well-known Chernoff's bound and Markov's bound [3] here.

Proposition 2.20 (Chernoff's bound)**.** *Let $0 < p < 1$ and let $X_1, \ldots, X_n$ be mutually independent indicator random variables with $\mathbb{P}[X_i = 1] = p$ for all i, and let $X = \sum X_i$. Then for all $a > 0$,*

$$\mathbb{P}[|X - \mathbb{E}[X]| > a] \leq 2e^{-a^2/2n}.$$

Proposition 2.21 (Markov's bound)**.** *If X is any nonnegative random variable and $a > 0$, then*

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$

We call an m-set A an *absorbing m-set* for a t-set T if $A \cap T = \emptyset$ and both $H[A]$ and $H[A \cup T]$ contain F-factors. Denote by $\mathcal{A}^m(S)$ the set of all absorbing m-sets for S. We are ready to illustrate the proof of the absorbing lemma.

Proof of Lemma 2.19. Define

$$m = i_0k^2 - i_0k, \eta = \frac{1}{m!}(\frac{\beta}{2})^k \text{ and } \alpha = \eta^3.$$

There are two steps in our proof. In the first step, we show that every t-set has sufficiently many absorbing sets; in the second step, by probabilistic argument, we build an absorbing family $\mathcal{F}'$ such that any small portion of vertices in V can be absorbed by using different members of $\mathcal{F}'$.

Claim 2.22. *For every t-set T, $|\mathcal{A}^m(T)| \geq \eta n^m$.*

Proof. Fix a t-set $T = \{v_1, v_2, \ldots, v_t\} \subset V$. We first find a t-set $S' = \{v_1, u_2, \ldots, u_t\} \subset V$ such that S' intersects T only at v_1 and spans a copy of F. Since v_1 and u is (β, i_0)-reachable for any $u \notin T$, there are at least βn^{i_0t-1} (i_0t-1)-sets S such that $H[S \cup \{v_1\}]$ contains an F-factor. Hence, by averaging argument there are at least βn^{t-1} copies of F containing v_1. Therefore there are at least

$$\beta n^t - (t-1)n^{t-2} > \tfrac{\beta}{2}n^{t-1}$$

choices for S'.

Since V is (β, i_0)-closed, there are at least βn^{i_0t-1} reachable (i_0t-1)-sets S_i for u_i and v_i where $i = 2, \ldots, t$. Next we choose a collection of pairwise disjoint sets S_i for $i = 2, \ldots, t$. Since in each step we need to avoid at most $i_0t(t-1) + t$ previously selected vertices, there are at least $\beta n^{i_0t-1}/2$ choices for each S_i. Let $A = S' \cup \left(\bigcup_i^t S_i\right)$, therefore $|A| = m$. In total, we get at least

$$\frac{\beta}{2}n^{t-1}\left(\frac{\beta}{2}n^{i_0 t-1}\right)^{t-1} = (m!)\eta n^m$$

m-sets A with multiplicity at most $m!$. Since $H[A]$ and $H[A \cup T]$ contain F-factors, A is an absorbing m-set for S and $|\mathcal{A}^m(T)| \geq \eta n^m$. $\square$

Now we will build a family $\mathcal{F}'$ by standard probabilistic arguments. Choose a family $\mathcal{F}$ of m-sets by selecting each of the $\binom{n}{m}$ possible m-sets independently with probability $p = \eta n^{1-m}/(8m)$. Then by Chernoff's bound with probability $1 - o(1)$ as $n \to \infty$, the family $\mathcal{F}$ satisfies the following properties: $\forall T \in \binom{V}{t}$,

$$|\mathcal{F}| \leq 2p\binom{n}{m} \leq \frac{\eta n}{m} \text{ and } |\mathcal{A}^m(T) \cap \mathcal{F}| \geq \frac{p|\mathcal{A}^m(T)|}{2} \geq \frac{\eta^2 n}{16m}. \tag{2.1}$$

Furthermore, the expected number of pairs of m-sets that are intersecting is at most

$$\binom{n}{m} \cdot m \cdot \binom{n}{m-1} \cdot p^2 \leq \frac{\eta^2 n}{64m}.$$

Thus, by using Markov's inequality, we derive that with probability at least 1/2

$$\mathcal{F} \text{ contains at most } \frac{\eta^2 n}{32m} \text{ intersecting pairs of } m\text{-sets.} \tag{2.2}$$

Hence, with positive probability the family $\mathcal{F}$ has all properties stated in (4.2) and (4.3). By deleting one element of each intersecting pair and removing m-sets that are not absorbing sets for any t-set $T \subset V \setminus V_0$, we get a new family $\mathcal{F}'$ and

$$|V(\mathcal{F}')| = m|\mathcal{F}'| \leq m|\mathcal{F}| \leq \eta n.$$

Note that $\mathcal{F}'$ contains pairwise disjoint m-sets. Since every m-set in $\mathcal{F}'$ is an absorbing m-set for some k-set S, $H[V(\mathcal{F}')]$ has an F-factor and therefore $|V(\mathcal{F}')| \in t\mathbb{Z}$. For any t-set T, by

(4.2) above we have

$$|\mathcal{A}^m(T) \cap \mathcal{F}'| \geq \frac{\eta^2 n}{16m} - \frac{\eta^2 n}{32m} = \frac{\eta^2 n}{32m} \tag{2.3}$$

For any set $U \subset V \setminus V(\mathcal{F}')$ of size $|U| \leq \alpha n$ and $|U| \in t\mathbb{Z}$, it can be partitioned into at most $(\alpha n/t)$ t-sets. Since

$$\frac{\alpha n}{t} \leq \frac{\eta^2 n}{32m},$$

each t-set in U can be greedily absorbed by using some unique absorbing m-set in $\mathcal{F}'$. Hence, $H[U \cup V(\mathcal{F}')]$ contains an F-factor.

Let $W = V(\mathcal{F}')$. We get the desired absorbing set W satisfying $|W| < \eta n$ such that for any vertex set U with $U \subset V \setminus W$, $|U| \in t\mathbb{Z}$ and $|U| \leq \alpha n$, both $H[W]$ and $H[U \cup W]$ contain F-factors. $\square$

When r-graph is dense enough, the absorbing method provides a powerful, global (small) absorbing structure that can absorb any (smaller) set of leftover vertices. This reduces the job of finding a spanning structure into the one of finding an almost spanning structure. Interestingly, when the minimum degree condition falls below the critical threshold for which the absorbing structure exists, a partite structure appears in the (hyper)graph (see [51, 50]). In this case, we modify Step 1 and Step 2 of the **Algorithm** (see Part 4). First we partition the vertex set of the r-graph into a few parts such that each part is closed, and then build the lattice-based absorbing structure on the partition. Our lattice-based absorbing structure works under the subcritical degree conditions and gives enough structural information in some applications.

PART 3

MATCHINGS IN K-PARTITE K-GRAPHS

3.1 Introduction

A k-graph H is said to be k-partite if $V(H)$ can be partitioned into k parts, $V(H) = V_1 \cup \cdots \cup V_k$ such that every edge consists of exactly one vertex from each class, that is, $E(H) \subset V_1 \times \cdots \times V_k$. Given such a partition, a subset $S \subset V(H)$ is called *legal* if $|S \cap V_i| \leq 1$ for each $i \in [k]$. In a k-graph H with a set S of d vertices, where $1 \leq d \leq k-1$, we define $\deg_H(S)$ to be the number of edges containing S. The *minimum d-degree* $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ taken over all d-vertex sets S in H. When $d = k-1$, it is referred as codegree. In a k-partite k-graph H, we define the *partite minimum d-degree* as the minimum of $\deg_H(S)$ taken over all legal d-vertex sets S in H, denoted by $\delta'_d(H)$.

First of all, we state our main result as follows.

Theorem 3.1 (Main Result)**.** *For any $k \geq 3$ and $\epsilon_0 > 0$, there exists n_0 such that for any $n \geq n_0$ the following holds. Let H be a k-partite k-graph with parts of size n such that $\delta_{[k]\setminus\{i\}} \geq a_i$ for all $i \in [k]$ and $(1-\epsilon_0)n \geq a_1 \geq a_2 \geq \cdots \geq a_k$, $a_3 > \epsilon_0 n$, and $\sum_{i\in[k]} a_i \geq n-k+3$. Then H contains a matching of size at least* $\min\{n-1, \sum_{i\in[k]} a_i\}$.

Let $\nu(H)$ be the size of a maximum matching in H. The following fact gives a proof of Theorem 3.1 when $\sum_{i\in[k]} a_i \leq n-k+2$.

Fact 3.2. *Fix $\epsilon > 0$ and n is sufficiently large. For $i \in [k]$, let $a_i = a_i(n)$. Let H be a k-partite k-graph with parts of size n such that $\delta_{[k]\setminus\{i\}} \geq a_i$ for all $i \in [k]$, then*

$$\nu(H) \geq \sum_{i\in[k]} a_i \text{ if } \sum_{i\in[k]} a_i \leq n-k+2.$$

Proof. Assume a maximum matching M is of size $|M| \leq \sum_{i\in[k]} a_i - 1 \leq n-k+1$. Since each class has at least $k-1$ vertices unmatched, we can find k disjoint legal $(k-1)$-sets

$A_1, \dots, A_k$ such that A_i contains exactly one unmatched vertex in V_j with $j \neq i$. Each A_i has at least a_i neighbors and all of them lie entirely in $V(M)$. Since $\sum_{i \in [k]} a_i > |M|$, there exist distinct indices $i \neq j$ such that A_i and A_j have neighbors on the same edge $e \in M$, say $v_i \in A_i \cap e$ and $v_j \in A_j \cap e$. Replacing e by $\{v_i\} \cup A_i$ and $\{v_j\} \cup A_j$ gives a larger matching, a contradiction. $\square$

To Prove Theorem 3.1, two cases are separately considered: when H is close to the extremal k-partite k-graphs (*extremal case*), and when H is far from extremal k-partite k-graphs (*nonextremal case*). In our paper, there are two types of extremal k-partite k-graphs.

Given $\epsilon > 0$, a k-partite k-graphs H is called *ϵ-close* to k-partite k-graph H' if H becomes H' after adding and deleting at most ϵn^k edges. Given k-partite k-graph H, it is called *ϵ-S-extremal* if $V(H)$ contains an independent set I such that $|I \cap V_i| \geq n - a_i - \epsilon n$ for each $i \in [k]$.

Theorem 3.3 (Non-extremal case). *Fix $k \geq 3$ and $0 < \gamma \ll \epsilon$, and let n be sufficiently large. Let H be a k-partite k-graph with parts of size n such that $\delta_{[k]\setminus\{i\}} \geq a_i$ for each $i \in [k]$ where $(1-\epsilon_0)n \geq a_1 \geq a_2 \geq \cdots \geq a_k$, $a_3 > \epsilon_0 n$, and $\sum_{i\in[k]} a_i \geq (1-\gamma)n$. Suppose H is not $5k\gamma$-S-extremal. Then H contains a matching of size at least $n-1$.*

Theorem 3.4 (Extremal case). *For any integer $k \geq 3$ and $\epsilon_0 > 0$, there exists $0 < \epsilon \ll \epsilon_0$ such that the following holds for sufficiently large integer n. Let H be a k-partite k-graph with vertex classes of size n. Suppose for each $i \in [k]$, $\delta_{[k]\setminus\{i\}}(H) \geq a_i$, where $(1-\epsilon_0)n \geq a_1 \geq a_2 \geq \cdots \geq a_k$, $a_3 > \epsilon_0 n$, and $\sum_{i\in[k]} a_i \geq n-k+3$. If H is ϵ-S-extremal, then H contains a matching of size at least $\min\{\sum_{i\in[k]} a_i, n-1\}$.*

Proof of Theorem 3.1. When $\sum_{i\in[k]} a_i \leq n-k+2$, it follows from Fact 3.2. When $\sum_{i\in[k]} a_i > n-k+2$, it follows from Theorem 3.3 and Theorem 3.4 immediately. $\square$

For the rest of this paper, in Section 3.2 we introduce the two types of absorbing lemmas in k-partite k-graphs. In Section 3.3 and Section 3.4, we give the proof of Theorem 3.3 and Theorem 3.4, respectively.

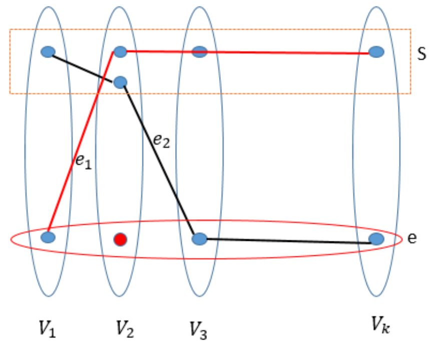

Figure 3.1. S-absorbing edge in k-partite k-graphs

Notation: Throughout this paper, we denote by H a k-partite k-graph with the vertex partition $V(H) = V_1 \cup \cdots \cup V_k$ such that $|V_1| = \cdots = |V_k| = n$. Let $V_i = V_{i \bmod k}$ if $i > k$.

3.2 Absorbing Techniques in k-partite k-graphs

In this section, we show the absorbing lemma that will be used to prove Theorem 3.3.

Let H be a k-partite k-graph. A set S is called *balanced* if it consists of the same number of vertices from each part of $V(H)$. Given balanced $2k$-set S, an edge $e \in E(H)$ disjoint from S is called *S-absorbing* if there are two disjoint edges e_1 and e_2 in $E(H)$ such that $|e_1 \cap S| = k-1$, $|e_1 \cap e| = 1$, $|e_2 \cap S| = 2$, and $|e_2 \cap e| = k-2$. Given legal k-set S, a balanced set $T \subset V(H)$ disjoint from S is called *S-perfect-absorbing* if both $H[T]$ and $H[S \cup T]$ contain a perfect matching.

Proposition 3.5. *Given $\lambda, \epsilon', \alpha > 0$, the following holds for sufficiently large n. Let H be a k-partite k-graph with parts of size n, and let $\S = \{S : S \subset V(H)\}$ such that any $S \in \S$ has at least $\lambda n^{i_0 k}$ S(-perfect)-absorbing $i_0 k$-sets in H. Then there exists a matching M' in H of size $|M'| \le \epsilon' n$ such that for every set $S \in \S$, the number of S(-perfect)-absorbing edges in M' is at least αn.*

Proof. We build the matching M' by standard probabilistic arguments. Choose a collection M of induced matching of size i_0 in H by selecting each independently with probability

$p = \epsilon'/(2n^{i_0k-1})$. For every legal k-set S, let X_S be the number of S(-perfect)-absorbing i_0k-sets in M. Then by Chernoff's bound, with probability $1-o(1)$, the family M satisfies the following properties:

$$|M| \le 2n^{i_0k}p = \epsilon' n \text{ and } |X_S \cap M| \ge \frac{p}{2}\cdot \lambda n^{i_0k} = \frac{1}{4}\lambda\epsilon' n \text{ for any legal } k\text{-set } S.$$

Furthermore, the expected number of intersecting pairs of members in M is at most $kn^{2i_0k-1}p^2 = \epsilon'^2kn/4$. By Markov's inequality, M contains at most $\epsilon'^2kn/2$ intersecting edges with probability at least $1/2$.

Let $M' \subset M$ be the obtained collection by deleting one edge of each intersecting pair and removing all edges that are not absorbing edges for any legal k-set S in M. Therefore, $|M'| \le |M| \le \epsilon' n$ and each legal k-set S has

$$|X_S \cap M'| - \frac{1}{2}\epsilon'^2kn \ge \frac{1}{4}\lambda\epsilon' n - \frac{1}{2}\epsilon'^2kn \ge \alpha n$$

S(-perfect)-absorbing edges in M'. Hence, such an absorbing matching M' exists. $\square$

Lemma 3.6 (k-partite Absorbing lemma)**.** *Given $0 < \epsilon', \alpha \ll \epsilon$, the following holds for sufficiently large n. Let H be a k-partite k-graph with parts of size n such that $\delta_{[k]\setminus\{i\}} \ge \epsilon n$ for $i \in [3]$, then there exists a matching M' in H of size $|M'| \le \epsilon' n$ such that for every balanced $2k$-set S of H, the number of S-absorbing edges in M' is at least αn.*

Proof. First we show that there are sufficiently many S-absorbing edges for each balanced $2k$-set S; then we prove the existence of a small absorbing matching by probabilistic arguments.

Given $0 < \epsilon' \ll \epsilon$ and sufficiently large n, let H be a k-partite k-graph with parts of size n such that $\delta_{[k]\setminus\{i\}} \ge \epsilon n$ for $i \in [3]$. Define $\alpha = \epsilon^3\epsilon'/16$.

Claim 3.7. *For every balanced $2k$-set S, the number of S-absorbing edges is at least $\epsilon^3 n^k/2$.*

Proof. Let $\{w, v\} := S \cap V_3$ and $\{u\} := S \cap V_2$. We only count those S-absorbing edges e for which the corresponding edge e_2 contains u and v. To this end, we will count ordered

k-sets $\{v_1, v_2, \dots, v_k\}$ such that $v_j \in V_j$, $e = \{v_1, v_2, \dots, v_k\}$ is disjoint from S, $\{w, v_2\} \in e_1$, $|e_1 \cup (S \setminus \{u, v\})| = k - 1$ and $e_2 = \{u, v, v_4, \dots, v_k, v_1\}$.

For each $j \in [4, k]$, there are exactly $n-2$ choices for v_j. Having selected $\{v_4, v_5, \dots, v_k\}$, we get the following property of v_1, v_2 and v_3: v_1 must be a neighbor of $\{u, v, v_4, \dots, v_k\}$; v_2 must be a neighbor of $S \setminus \{u, v\}$; v_3 must be a neighbor of $\{v_4, \dots, v_k, v_1, v_2\}$. Therefore, there are at least $a_j - 2$ choices of v_j for $j = 1, 2, 3$. Hence, there are at least

$$(n-2)^{k-3}(\epsilon n - 2)^3 \geq \frac{1}{2}\epsilon^3 n^k$$

S-absorbing edges. The last inequality holds since n is sufficiently large. □

Next we build the matching M' by applying Proposition 3.14. □

3.3 Nonextremal k-partite k-graphs: Proof of Theorem 3.3

In this section we derive Theorem 3.3 from the absorbing lemmas in Section 3.2 and a lemma that provides a matching covering all but a constant number of vertices when H is non-extremal.

A matching that covers all but a constant number of vertices is provided by the following lemma. We give the more general assertion with the degree condition in Theorem 3.3 fails for a small fraction of legal $(k-1)$-sets.

Lemma 3.8 (Almost perfect matching lemma)**.** *For any integer $k \geq 3$ and $0 < \epsilon \ll \alpha, \gamma$, the following holds for sufficiently large integer n. For $i \in [k]$, let $a_i = a_i(n)$ such that $\sum_{i \in [k]} a_i \geq (1-\gamma)n$. Let H be a k-partite k-graph with parts of size n which is not 2γ-extremal. Suppose for each $i \in [k]$, there are fewer than ϵn^{k-1} legal $(k-1)$-sets S such that $S \cap V_i = \emptyset$ and $\deg(S) < a_i$. Then H contains a matching that covers all but at most αn vertices in each vertex class.*

Proof. Let M be a maximum matching of size m in H. Let $V_i' = V_i \cap V(M)$ and $U_i = V_i \setminus V(M)$. Let $s := |U_1| = \cdots = |U_k|$. Suppose that $s > \alpha n$.

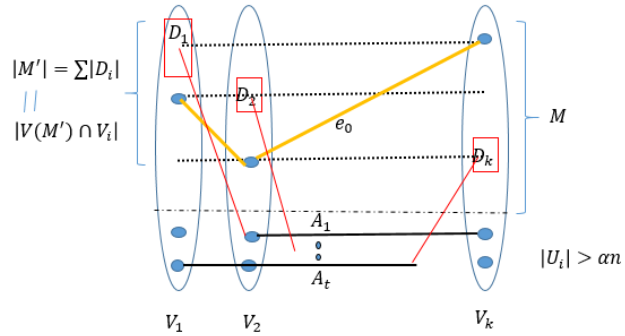

Figure 3.2. The proof of Almost Perfect Matching Lemma

For each $i \in [k]$, we greedily find a collection of $t_i = \lceil k/\epsilon \rceil$ disjoint legal $(k-1)$-sets A such that $A \cap V_i = \emptyset$ and $\deg(A) \geq a_i$. This is possible since in each step, the legal $(k-1)$-sets that cannot be picked are those either intersect the ones that have been picked, or those with low degree, whose number is at most

$$k(k-1)t_i n^{k-2} + \epsilon n^{k-1} < (\alpha n)^{k-1} < \prod_{j \in [k] \setminus \{i\}} |U_j|,$$

by $\epsilon \ll \alpha$ and that n is sufficiently large. Label these sets by $A_1, \ldots, A_t$ accordingly so that $A_j \cap V_{j'} = \emptyset$ where $j' = j \bmod k$.Therefore, $\deg(A_j) \geq a_{j'}$ and all neighbors of A_j lie entirely in $V'_{j'}$ by the maximality of M.

For $i \in [k]$, let D_i be the subset of V'_i each vertex of which has at least k sets A_j, $j \in [t]$ as neighbors, and let $D = \bigcup D_i$. We claim that $|e \cap D| \leq 1$ for each $e \in M$. Indeed, otherwise assume that $x, y \in e \cap D$ and pick A_i, A_j for some $i, j \in [t]$ such that $\{x\} \cup A_i, \{y\} \cup A_j \in E(H)$. We obtain a matching of size $m+1$ by deleting e and adding $\{x\} \cup A_i$ as well as $\{y\} \cup A_j$ in M, contradicting the maximality of M.

Next we show that $|D_i| \geq a_i - \epsilon n$ for each $i \in [k]$. Since there are no edges between A_j and V'_i for $j \neq i \bmod k$, by counting the number of edges between V'_i and $\{A_1, \ldots, A_t\}$, we get

$$t_i \cdot a_i \leq \sum_{j \equiv i \bmod k} \deg(A_j) \leq |D_i| t_i + nk.$$

Therefore, by $t_i \geq k/\epsilon$, we have

$$|D_i| \geq a_i - \frac{nk}{t_i} \geq a_i - \epsilon n.$$

Define $M' := \{e \in M : e \cap D \neq \emptyset\}$. Then for each $i \in [k]$, we have

$$|(V(M') \setminus D) \cap V_i| = \sum_{j \neq i} |D_j| \geq \sum_{j \in [k]} (a_j - \epsilon n) - a_i \geq n - a_i - 2\gamma n.$$

Since H is not 2γ-extremal, $H[V(M') \setminus D]$ contains at least one edge, denoted by e_0. Note that $e_0 \notin M$ since $e_0 \subset V(M') \setminus D$. Assume that e_0 intersects $e_1, \ldots, e_p$ in M for some $2 \leq p \leq k$. Suppose $\{v_j\} = e_j \cap D$, from the choice of e_0, we have $v_j \notin e_0$ for all $j \in [p]$. By the definition of D, we can greedily pick $A_{\ell_1}, \ldots, A_{\ell_p}$ such that $\{v_j\} \cup A_{\ell_j} \in E(H)$ for all $j \in [p]$. Let M'' be the matching obtained from replacing the edges $e_1, \ldots, e_p$ by e_0 and $\{v_j\} \cup A_{\ell_j}$ for $j \in [p]$. Thus, M'' has $m + 1$ edges, contradicting the choice of M. $\square$

Now we are ready to prove Theorem 3.3.

Proof of Theorem 3.3. Suppose H is a k-partite k-graph with parts of size n and $\delta_{[k] \setminus \{i\}} \geq a_i$ for each $i \in [k]$, $a_1 \geq a_2 \geq a_3 \geq \epsilon n$ and $\sum_{i \in [k]} a_i \geq (1 - \gamma)n$, and H is not $5k\gamma$-S-extremal. In particular, $\gamma < \frac{1}{5k}$. We first apply Lemma 3.6 on H and find the absorbing matching M' of size at most γn such that for every balanced $2k$-set $S \subset V(H)$, the number of S-absorbing edges in M' is at least αn.

Let $H' = H[V(H) \setminus V(M')]$ and $n' = |V(H') \cap V_i| \geq (1 - \gamma)n$. Note that $\sum_{i \in [k]} a'_i \geq \sum_{i \in [k]} a_i - k\gamma n \geq (1 - 2k\gamma)n'$. If H' is $4k\gamma$-extremal, i.e., $V(H')$ contains an independent set I such that $|I \cap (V_i \cap V(H'))| \geq n' - a'_i - 4k\gamma n'$ for each $i \in [k]$, then we get that H is $5k\gamma$-extremal since

$$n' - a'_i - 4k\gamma n' \geq (1 - \gamma)n - a_i - (1 - \gamma)4k\gamma n \geq n - a_i - 5k\gamma n,$$

a contradiction. Thus, H' is not $4k\gamma$-extremal. By applying Lemma 3.8 on H' with parameter

2γ, α and $\epsilon = 0$, we obtain a matching M'' in H' that covers all but at most αn vertices in each vertex class.

If there are at least three $a_i \geq \epsilon n$, then since for every balanced $2k$-set $S \subset V(H)$, the number of S-absorbing edges in M' is at least αn, we can greedily absorb the leftover (at most αn times, each time the number of the leftover vertices in each class is reduced by 1) until there is 1 leftover vertex in each class. It is possible since each class has the same number of leftover vertices and a legal k-set S always exists if the size of the leftover is greater than k. Denote by $\tilde{M}$ the matching obtained after absorbing the leftover vertices into M'. Therefore $\tilde{M} \cup M''$ is the required matching in H. $\square$

3.4 Extremal k-partite k-graphs

In this section, we prove Theorem 3.4. As inspired by [39], we apply the following weaker form of a result from Pikhurko [72]. Let H be a k-partite k-graph with parts $V_1 \cup V_2 \cup \cdots \cup V_k = V(H)$. Let $L \subseteq [k]$ and recall that

$$\delta'_L(H) = \min \left\{ \deg(S) : S \subset \bigcup_{i \in L} V_i \text{ is a legal } |L|\text{-set} \right\}.$$

Theorem 3.9. *[72, Theorem 3] For $k \geq 2, L \subseteq [k]$, let m be sufficiently large.Let H be a k-partite k-graph with parts $V_1 \cup V_2 \cup \cdots \cup V_k = V(H)$ such that $|V_i| = m$ for each $i \in [k]$. If*

$$\delta'_L(H) m^{|L|} + \delta'_{[k] \setminus L}(H) m^{k-|L|} \geq \frac{3}{2} m^k,$$

then H contains a perfect matching.

Proof of Theorem 3.4. Let $\epsilon \ll \epsilon_0$ and $\alpha = \sqrt{\epsilon}$. Suppose n is sufficiently large. Let H be a k-partite k-graph with parts $V_1 \cup V_2 \cup \cdots \cup V_k = V(H)$ such that $|V_i| = n$ for each $i \in [k]$ and $\delta_{[k]\setminus\{i\}}(H) \geq a_i$, where $(1 - \epsilon_0)n \geq a_1 \geq a_2 \geq \cdots \geq a_k$, $a_3 \geq \epsilon_0 n$, and without loss of generality,

$$n - k + 3 \leq \sum_{i \in [k]} a_i \leq n - 1.$$

Assume that H is ϵ-S-extremal, namely, there is an independent set $I \subseteq V(H)$ such that $|I \cap V_i| \geq n - a_i - \epsilon n$ for each $i \in [k]$. Note that $n - a_i - \epsilon n \geq (\epsilon_0 - \epsilon)n > 0$ by our assumption. Our goal is to find a matching in H of size at least $\sum_{i\in[k]} a_i$.

Let C be a maximum independent set of $V(H)$ satisfying $|C \cap V_i| \geq n - a_i - \epsilon n$ for each $i \in [k]$. We partition each V_i into $A_i \cup B_i \cup C_i$ for $i \in [k]$ as follows. Let $C_i = C \cap V_i$,

$$A_i = \left\{ x \in V_i \setminus C_i : \deg(x, C) \geq (1-\alpha) \prod_{j\neq i} |C_j| \right\}, \tag{3.1}$$

and $B_i = V_i \setminus (A_i \cup C_i)$. Moreover, let $A = \bigcup_{1\leq i\leq k} A_i$ and $B = \bigcup_{1\leq i\leq k} B_i$. We observe the following bounds of $|A_i|, |B_i|, |C_i|$ for each $i \in [k]$.

Claim 3.10. *$|A_i| \geq a_i - \alpha n$, $|B_i| \leq \alpha n$, and $n - a_i - \epsilon n \leq |C_i| \leq n - a_i$.*

Proof. The lower bound for $|C_i|$ follows from our hypothesis immediately. For any legal $(k-1)$-set $S \subset C \setminus V_i$, we have $N(S) \subseteq A_i \cup B_i$. By the minimum degree condition, we have

$$a_i \leq |N(S)| \leq |A_i| + |B_i| = n - |C_i| \leq a_i + \epsilon n, \tag{3.2}$$

which gives the upper bound for $|C_i|$. By the definitions of A_i and B_i, we have

$$a_i \prod_{j\neq i} |C_j| \leq |E(A_i \cup B_i, C)| \leq |B_i|(1-\alpha) \prod_{j\neq i} |C_j| + |A_i| \prod_{j\neq i} |C_j|,$$

where $E(A_i \cup B_i, C)$ is the set of edges that consist of a legal $(k-1)$-set in $\bigcup_{j\neq i} C_i$ and one vertex in $A_i \cup B_i$. Thus, we get $a_i \leq |A_i| + |B_i| - \alpha|B_i|$, which implies $\alpha|B_i| \leq |A_i| + |B_i| - a_i \leq \epsilon n$ by (3.2). So $|B_i| \leq \alpha n$ and $|A_i| \geq a_i - |B_i| \geq a_i - \alpha n$. □

Our procedure towards the desired matching consists of three steps. First, we build small disjoint matchings that cover all vertices of B. Second, we adjust the sizes of the parts such that we can finish the desired matching by Theorem 3.9. Finally, we apply Theorem 3.9 and get the final matching, leaving at most $n - \sum_{i\in[k]} a_i$ vertices in each class uncovered.

Step 1. Small disjoint matchings that cover all vertices of B.

For each $i \in [k]$, define $t_i := \max\{0, a_i - |A_i|\}$. From (3.2), we have $|B_i| \geq a_i - |A_i|$. Together with the definition of t_i and Claim 3.10, we get that for each $i \in [k]$,

$$t_i \leq |B_i| \leq \alpha n \text{ and } t_i + |A_i| \geq a_i. \tag{3.3}$$

First we build a matching M_1^i of size t_i for each $i \in [k]$. If $t_i = 0$, then $M_1^i = \emptyset$. If $t_i > 0$, since $\delta_{[k]\setminus\{i\}}(H) \geq a_i$ and C is independent, every legal $(k-1)$-set in $\bigcup_{j\neq i} C_j$ has at least $a_i - |A_i| = t_i$ neighbors in B_i. We greedily pick t_i disjoint edges each of which consists of a legal $(k-1)$-set in $\bigcup_{j\neq i} C_j$ and one vertex in B_i.

Next for each i we greedily build a matching M_2^i that covers all the vertices in $B_i \setminus V(M_1^i)$. First, for $i \neq 1$, we pick one uncovered vertex in B_i, one uncovered legal $(k-2)$-set S' in $\bigcup_{j\neq i,1} C_j$, and by codegree one uncovered vertex in V_1. Second, for $i = 1$, we pick one uncovered vertex in B_1, one uncovered legal $(k-2)$-set S' in $\bigcup_{j\neq i,2} C_j$, and by codegree one uncovered vertex in V_2. Let $M_j = \bigcup_{i=1}^k M_j^i$ for $j = 1, 2$. Now we show that the greedy process is possible. Since $a_1 \leq (1-\epsilon_0)n$ and $a_2 \geq a_3 \geq \cdots \geq a_k$, we have $a_1 \geq a_2 \geq a_3 \geq \epsilon_0 n \geq k\alpha n$, as $\epsilon \ll \epsilon_0$ and $\alpha = \sqrt{\epsilon}$. By definition, each edge in $M_1 \cup M_2$ contains at least one vertex from B. Thus the number of vertices in V_i covered by the existing matchings is at most $|M_1 \cup M_2| \leq |B| \leq k\alpha n < a_3$. So the greedy process is possible.

For each $i \in [k]$, let

$$A_i' = A_i \setminus V(M_1 \cup M_2),\ C_i' = C_i \setminus V(M_1 \cup M_2) \text{ and } V_i' = V_i \setminus V(M_1 \cup M_2).$$

Since M_1 does not contain any vertex in A, we have $|A_i'| = |A_i \setminus V(M_2^i)|$ or

$$|A_i'| \geq |A_i| - |M_2^i|. \tag{3.4}$$

Step 2. Adjust the sizes of the parts.

In this step, we will first build a small matching M_3 that adjust the sizes of the parts as follows.

Claim 3.11. *There exists a matching M_3 of size at most $k\epsilon n$ in $H[\bigcup_{i=1}^k V_i']$ so that $|C_i' \setminus V(M_3)| - \sum_{j\neq i} |A_j' \setminus V(M_3)| = r$, where $0 \le r \le |n - \sum_{i\in[k]} a_i|$.*

Proof. Let $n' = |V_i'| = |A_i'| + |C_i'|$. Note that $s := |C_i'| - \sum_{j\neq i} |A_j'| = n' - \sum_{j=1}^k |A_j'|$, which is independent of $i \in [k]$. We claim that $-k\epsilon n \le s \le |n - \sum_{i\in[k]} a_i|$. Indeed,

$$s \ge (n - |M_1 \cup M_2|) - |A| \ge n - |B| - |A| \overset{(3.2)}{\ge} n - \sum_{i\in[k]} (a_i + \epsilon n) \ge -k\epsilon n.$$

On the other hand, by (3.4) and $t_j = |M_1^j|$ for $j \in [k]$, we have

$$s \le n - \sum_{j\in[k]} (|M_1^j| + |M_2^j|) - \sum_{j\in[k]} (|A_j| - |M_2^j|) = n - \sum_{j\in[k]} (t_j + |A_j|) \overset{(3.3)}{\le} n - \sum_{j\in[k]} a_i.$$

In our process of finding M_3, we update s as $(n' - |M_3|) - \sum |A_j' \setminus V(M_3)|$. If $s \ge 0$, then set $M_3 = \emptyset$. Otherwise if $s < 0$, we do the following procedure: greedily add edges from $\bigcup (A_i' \cup C_i')$ to M_3 by picking an uncovered legal 2-set in $A_2' \cup A_3'$, an uncovered legal $(k-3)$-set in $\bigcup_{j'\in[4,k]} C_j'$, and by codegree one uncovered vertex in V_1'. Whenever one edge e is added to M_3, $|M_3|$ is increased by 1 and $\sum |A_j' \setminus V(M_3)|$ is reduced by $|e \cap A|$, which is 2 or 3, and hence s is increased by 1 or 2.

The iterations stop when $s \in \{0, 1\}$. In this case, we have added at most $-s \le k\epsilon n$ edges and thus $|M_3| \le k\epsilon n$. Note that we can always form an edge in each step. First observe that as long as we can pick the 2-set in $A_2' \cup A_3'$, we can always form the desired edge in each step because the number of covered vertices in V_1 is at most $|B| + k\epsilon n \le 2k\alpha n < a_1$. Moreover, recall that $a_2 \ge a_3 \ge \epsilon_0 n/(k-1)$, and thus by (3.4) and Claim 3.10, we have

$$|A_i'| \ge |A_i| - |M_2^2| \ge a_i - \alpha n - k\alpha n \ge k\epsilon n,$$

where $i = 1, 2$ or 3 as $\epsilon \ll \epsilon_0$. $\square$

Step 3. Find the final matching.

For each $i \in [k]$, let

$$A_i'' = A_i' \setminus V(M_3),\ C_i'' = C_i' \setminus V(M_3) \text{ and } V_i'' = V_i' \setminus V(M_3).$$

By Claim 3.10 and the definitions of M_1, M_2, M_3, for each $i \in [k]$, we have

$$|A_i''| = |A_i| - |M_1 \cup M_2 \cup M_3| \geq (a_i - \alpha n) - k\alpha n - k\epsilon n \geq a_i - 2k\alpha n.$$

Recall that $a_1 \geq a_2 \geq a_3 \geq \epsilon_0 n$, by $\epsilon \ll \epsilon_0$, we have

$$|A_1''|, |A_2''|, |A_3''| \geq a_3 - 2k\alpha n \geq \epsilon_0 n/2. \tag{3.5}$$

By Claim 3.11, we have

$$0 \leq r = |C_i''| - \sum_{j \neq i} |A_j''| \leq |n - \sum_{i \in [k]} a_i|. \tag{3.6}$$

This implies that for each $i \in [k]$,

$$|C_i''| \geq \min\{|A_1''| + |A_2''|, |A_1''| + |A_3''|, |A_2''| + |A_3''|\} \geq 2a_3 - 2k\alpha n \geq \epsilon_0 n. \tag{3.7}$$

Now we greedily match the vertices in $A_4'', \ldots, A_k''$. Indeed, for any $4 \leq j \leq k$ and any vertex $v \in A_j'' \subseteq A_j$, by (3.1), the number of legal $(k-1)$-sets S in $\prod_{l \neq j} C_l''$ such that $S \cup \{v\} \notin E(H)$ is at most

$$\alpha \prod_{l \neq j} |C_l| \leq \alpha n^{k-1} \leq \alpha (k/\epsilon_0)^{k-1} \prod_{l \neq j} |C_l''| \leq \sqrt{\alpha} \prod_{l \neq j} |C_l''|,$$

where we used (3.12) and that $\alpha \ll \epsilon_0$. So we can greedily match these vertices because the number of leftover vertices in each C_j'' is at least $\min\{|A_1''|+|A_2''|, |A_1''|+|A_3''|, |A_2''|+|A_3''|\}+r \geq \epsilon_0 n$ and thus the number of available legal $(k-1)$-sets is at least $(\epsilon_0 n)^{k-1} \geq \sqrt{\alpha} n^{k-1} >$

$\sqrt{\alpha}\prod_{l\neq j}|C_l''|$. Let M_4^0 be the resulting matching in this step.

Finally, consider the unmatched vertices of H. Let $m_i := |A_i''|$ for all $i \in [3]$. Note that the number of unmatched vertices in C_1'', C_2'' and C_3'' are $m_2 + m_3 + r$, $m_1 + m_3 + r$ and $m_1 + m_2 + r$, respectively, and the number of unmatched vertices in C_i'', $i \in [4, k]$ is $m_1 + m_2 + m_3 + r$. For $i = 1, 2$ and 3, pick arbitrary disjoint subsets C_1^2, C_1^3 of uncovered vertices in C_1'' of size m_2, m_3 and C_2^1, C_2^3 of uncovered vertices in C_2'' of size m_1, m_3; for $i \in [4, k]$, we can partition all but r vertices of C_i'' into C_i^1 of size m_1, C_i^2 of size m_2 and C_i^3 of size m_3. Therefore, we get k-partite k-graphs $H_j := H[A_j'', \bigcup_{\ell\neq j} C_\ell^j]$ for $j \in [3]$. Let us verify the assumptions of Theorem 3.9 for H_j where $j = 1, 2, 3$.

First, for any legal $(k-1)$-set $S \subset \bigcup_{\ell\neq j} C_\ell^j$, the number of its non-neighbors in $A_j \cup B_j$ is at most

$$|A_j| + |B_j| - a_j \overset{(3.2)}{\leq} \epsilon n \overset{(3.11)}{\leq} \frac{k\epsilon}{\epsilon_0} m_j \leq \alpha m_j,$$

as $\epsilon \ll \epsilon_0$. So we have

$$\delta'_{[k]\setminus\{j\}}(H_j) \geq m_j - \alpha m_j = (1-\alpha)m_j.$$

Next, for any $v \in A_j''$, by (3.1) the number of its non-neighbors in $\bigcup_{\ell\neq j} C_\ell^j$ is at most

$$\alpha \prod_{\ell\neq j} |C_\ell^j| < \alpha n^{k-1} \overset{(3.11)}{\leq} \alpha \left(\frac{k}{\epsilon_0} m_j\right)^{k-1} \leq \sqrt{\alpha} m_j^{k-1},$$

which implies that $\delta'_{\{j\}}(H_j) \geq (1-\sqrt{\alpha})m_j^{k-1}$. Thus, we have

$$\delta'_{\{j\}}(H_j) m_j + \delta'_{[k]\setminus\{j\}}(H_j) m_j^{k-1} \geq (1-\sqrt{\alpha}) m_j^{k-1} m_j + (1-\alpha) m_j m_j^{k-1} > \frac{3}{2} m_j^k,$$

since ϵ is small enough. By Theorem 3.9, we find a perfect matching M_4^j in H_j for each $j \in [3]$. Let $M_4 = M_4^0 \cup M_4^1 \cup M_4^2$, then $M_1 \cup M_2 \cup M_3 \cup M_4$ is a matching in H of size at least $n - r \geq \sum_{i\in[k]} a_i$ and we are done. $\square$

3.5 Future Work

We proved that a k-partite k-graph H with three sufficiently large codegrees has a matching of size $\min\{n-1, \sum_{i=1}^{k} a_i\}$ where $\delta_{[k]\setminus\{i\}} \geq a_i$ for all $i \in [k]$. We would further improve our result by weakening the conditions to a k-partite k-graph H with two sufficiently large codegrees. Under this condition, there are two types of extremal graphs. One is called Space Barrier, and the other is called Divisibility Barrier. Here we only state the divisibility barrier: suppose a k-partite k-graph H with $|V_1| = \cdots = |V_k| = n$. Further partition V_i for $i = 1$ or 2 into two parts such that $V_i = X_i \cup Y_i$ and $|X_i| = \lceil n/2 \rceil$, $|Y_i| = \lfloor n/2 \rfloor$ for $t = 1, 2$. The edge set consists of those edges who intersect $X_1 \cup X_2$ at either even or odd vertices. In the former case, the edge set is denoted as E_{even}; and in the later case, the edge set is denoted as E_{odd}. A k-partite k-graph H with edge set E_{even} or E_{odd} is called a *divisibility barrier*. Given k-partite k-graph H, it is called *ϵ-D-extremal* if H is ϵ-close to divisibility barriers.

Now we show some extension of our proof to a k-partite k-graph H with only two sufficiently large codegrees. The almost perfect matching lemma keeps the same as in the case with three sufficiently large codegrees. We state some work on Absorbing Lemma and the extremal case.

3.5.1 Absorbing Lemma.

The following states that the minimum vertex degree in terms of codegrees.

Fact 3.12. *Let H be a k-partite k-graph with parts of size n such that $\delta_{[k]\setminus\{i\}} \geq a_i$ for each $i \in [k]$, then any $i \in [k]$ and $v \in V_i$, we have* $\deg(v) \geq \max_{j \neq i} a_j n^{k-2}$.

Given $\beta > 0$, $i \in \mathbb{N}$, $j \in [k]$ and two vertices $u, v \in V_j$, we say that u, v are *(β, i)-reachable in H* if and only if there are at least βn^{ik-1} $(ik-1)$-sets W such that both $H[\{u\} \cup W]$ and $H[\{v\} \cup W]$ contain perfect matchings. W is called *reachable set* for u, v. If all $u, v \in V_j$ are *(β, i)-reachable*, then we say V_j is (β, i)-closed. Denote by $\tilde{N}_{\beta,i}(v)$ the set of vertices that are *(β, i)-reachable* to v.

We show that the number of one-step reachable neighbors to any vertex in each part V_i is not much less than the corresponding codegree a_i, where $i \in [k]$.

Proposition 3.13. *Suppose* $0 < 1/n \ll \alpha \ll \epsilon \ll 1/k$ *and let* H *be a* k*-partite* k*-graph such that* $\delta_{[k]\setminus\{1\}}(H), \delta_{[k]\setminus\{2\}}(H) \geq \epsilon n$*. For any* $j \in [k]$ *and* $v \in V_j$, $|\tilde{N}_{\alpha,1}(v)| \geq \delta_{[k]\setminus\{j\}}(H) - \sqrt{\alpha} n$*.*

Proof. Fix a vertex $v \in V_j$ for some $j \in [k]$, note that for any other vertex $u \in V_j$, $u \in \tilde{N}_{\alpha,1}(v)$ if and only if $|N_H(u) \cap N_H(v)| \geq \alpha n^{k-1}$. By double counting, we have

$$\sum_{S \in N_H(v)} \deg_H(S, V_j) < |\tilde{N}_{\alpha,1}(v)| \cdot |N_H(v)| + n \cdot \alpha n^{k-1}.$$

For any S in the above inequality, we know that $\deg_H(S, V_j) \geq \delta_{[k]\setminus\{j\}}(H)$. Moreover, since v is not in one of V_1 and V_2, we have that

$$|N_H(v)| \geq n^{k-2}\epsilon n \geq \sqrt{\alpha} n^{k-1},$$

as $\alpha \ll \epsilon$. Thus, $|\tilde{N}_{\alpha,1}(v)| > \delta_{[k]\setminus\{j\}}(H) - \frac{\alpha n^k}{|N_H(v)|} \geq \delta_{[k]\setminus\{j\}}(H) - \sqrt{\alpha} n$ as desired. $\square$

Throughout the rest of this subsection, without loss of generality, we may assume only $a_1, a_2 \geq \epsilon n$. The following is the key point to our proof. Here we only give a tentative outline.

Lemma 3.14 (draft)**.** *Given* $0 < \epsilon', \gamma \ll \epsilon, \epsilon^*$ *and sufficiently large* n*, there exists* $\alpha > 0$ *such that the following holds. Let* H *be a* k*-partite* k*-graph with parts of size* n *such that* $\delta_{[k]\setminus\{i\}} \geq a_i$ *for each* $i \in [k]$*. If* $\sum_{i\in[\ell]} a_i \geq (1-\gamma)n$, $a_1 \geq a_2 \geq \epsilon n$ *and* $a_j < \epsilon n$ *for* $j \geq 3$*, then one of the following holds.*

(i) $a_1 \geq a_2 \geq n/2 - k\epsilon n$*,* H *is* ϵ^**-D-extremal.*

(ii) There exists a matching M' *of size* $|M'| \leq \epsilon' n$ *such that for every legal* k*-set* S *of* H*, the number of* S*-perfect-absorbing sets in* M' *is at least* αn*.*

Proof. Given $0 < \epsilon' \ll \epsilon$ and sufficiently large n, let H be a k-partite k-graph with parts of size n such that $\delta_{[k]\setminus\{1\}}, \delta_{[k]\setminus\{2\}} \geq \epsilon n$. By Fact 3.12,

$$\delta_1'(H) \geq \min\{a_1 n^{k-2}, a_2 n^{k-2}\} \geq \epsilon n^{k-1}.$$

Claim 3.15. *If any of V_j where $j \in [2]$ is β-closed for some $\beta > 0$, then there exists a matching M' in H of size $|M'| \leq \epsilon' n$ and $\alpha > 0$ such that for every legal k-set S of H, the number of S-perfect-absorbing sets in M' is at least αn.*

Proof. If one of V_1, V_2 is β-closed for some $\beta > 0$, assume V_1 is (β, i_0)-closed, i.e., any $u, v \in V_1$ are (β, i_0)-reachable.

Fix a legal k-set $S = \{v_1, v_2, \dots, v_k\}$ such that $v_j \in V_j$, we claim there are at least $\epsilon\beta n^{i_0 k}/2$ S-perfect-absorbing $i_0 k$-sets. First of all, we find $v_1' \in V_1 \setminus \{v_1\}$ such that $\{v_1', v_2, \dots, v_k\}$ spans an edge. Since $\deg(S \setminus \{v_1\}) \geq \epsilon n$, there are at least $\epsilon n - 1$ choices of v_1'. Since V_1 is β-closed, there are at least $\beta n^{i_0 k-1}$ reachable $(i_0 k - 1)$-sets W for v_1 and v_1'. Among them, at least $\beta n^{i_0 k-1} - (k-1)n^{k-2} \geq \beta n^{i_0 k-1}/2$ reachable $(i_0 k-1)$-sets W are disjoint from S. In total, we have at least $\epsilon\beta n^{i_0 k}/2$ S-perfect-absorbing sets. Next we build the matching M' by applying Proposition 3.14. □

We have two cases.

Case 1: If $a_1 \geq n/2 + \epsilon n$, then V_2 is $(2\epsilon, 1)$-closed. Indeed, by Fact 3.12, any $v \in V_2$ has $\deg(v) \geq (1/2 + \epsilon)n^{k-1}$, therefore, for any $u, v \in V_2$, we have $|N_1(u) \cap N_1(v)| \geq 2\epsilon n^{k-1}$. By Claim 3.15, (ii) is true.

Case 2: If $a_1 < n/2 + \epsilon n$, since $a_1 + a_2 \geq n - \gamma n - (k-2)\epsilon n$, we have $a_1 \geq a_2 > n/2 - k\epsilon n$. In this case, for any $v \in V$, by Fact 3.12, $\deg(v) \geq (1/2 - k\epsilon)n^{k-1}$.

Claim 3.16. *For any $i \in [k]$, either V_i is β-closed for some $\beta > 0$ or there is a partition $V_i = X_i' \cup Y_i'$ such that X_i' and Y_i' are $(\beta', 1)$-closed for some $\beta' > 0$.*

Proof. Fix $i \in [k]$. If for any pair of vertices $x_i, y_i \in V_i$, there exists $\alpha > 0$ such that $|N(x_i) \cap N(y_i)| \geq \alpha n^{k-1}$ or at least αn vertices $z \in V_i$ such that $|N(x_i) \cap N(z)| \geq \alpha n^{k-1}$

and $|N(y_i) \cap N(z)| \geq \alpha n^{k-1}$, then V_i is $(\beta, 2)$-closed for some $\beta > 0$.

We may assume that there exists $x_i, y_i \in V_i$ such that for any $\alpha > 0$, $|N(x_i) \cap N(y_i)| < \alpha n^{k-1}$ and, at most αn vertices $z \in V_i$ such that $|N(x_i) \cap N(z)| \geq \alpha n^{k-1}$ and $|N(y_i) \cap N(z)| \geq \alpha n^{k-1}$. In this case, let $X_i = \{v \in V_i : |N(y_i) \cap N(v)| < \alpha n^{k-1}\}$ and $Y = \{v \in V_i : N(x_i) \cap N(v)| < \alpha n^{k-1}\}$. Let $Z_i = V_i \setminus (X_i \cup Y_i)$. We have the following properties of X_i, Y_i and Z_i.

(i) $x_i \in X_i$ and $y_i \in Y_i$ by definations of X_i, Y_i.

(ii) $X_i \cap Y_i = \emptyset$. Suppose $v \in X_i \cap Y_i$.

$$\begin{aligned}|N(x_i) \cup N(y_i) \cup N(v)| &= |N(v) \setminus N(x_i) \cup N(y_i)| + |N(x_i) \setminus N(y_i)| + |N(y_i)| \\ &> 3(\frac{1}{2} - \epsilon)n^{k-1} - 3\alpha n^{k-1} > n^{k-1},\end{aligned}$$

a contradiction.

(iii) $|Z_i| < \alpha n$

(iv) For any $x, x' \in X_i$, $|N(x) \Delta N(x')| < 8\alpha n^{k-1}$, and hence x, x' are 1-reachable to each other. The same holds for any pair of vertices in Y_i.

(v) For any $x \in X_i$ and $y \in Y_i$, $|N(x) \cap N(y)| < 5\alpha n^{k-1}$

For vertex $z \in Z_i$, if z_i is 1-reachable to any vertex $x \in X_i$, then add z to X_i; otherwise, there exists $x_0 \in X_i$ such that $|N(x_0) \cap N(z)| < \epsilon n^{k-1}$. In the later case, we claim that z is 1-reachable to any $y \in Y_i$, and hence we add z to Y_i. For $y \in Y_i$. assume $|N(y) \cap N(z)| < \epsilon' n$, then $|N(x_0) \cup N(y) \cup N(z)| > n^{k-1}$, a contradiction. Denote the resulted sets as X_i' and Y_i' , which will be the desired partition. □

By Proposition 3.13 with $\alpha \ll \epsilon$, for $i = 1, 2$, $|\tilde{N}_{\alpha,1}(v)| \geq \delta_{[k]\setminus\{i\}}(H) - \sqrt{\alpha} n > (1/2 - k\epsilon - \sqrt{\alpha})n$. So $|X_i'|, |Y_i| > (1/2 - \epsilon')n$ for $i = 1, 2$.

After having the partition of each part, we need to consider the edge set of H. This is the hard part, and more work need to be done. □

The almost perfect matching lemma is the same as Lemma 3.8. Hence the above would give an outline to solve the non-extremal case. For the extremal case, we need to handle two subcases: the space barrier similar to the one we did and the divisibility barrier.

3.5.2 Extremal Case

Space barrier when there are at least two large partite codegrees Most of the proof can be borrowed directly from the previous section. We first partition each part and get Claim 3.10. Step 1 follows the same approach to get small disjoint matchings that cover all vertices of B. We need some careful adjustment in Step 2.

Since we only have two sufficiently large codegrees, in the process of finding M_3 (see Claim 3.15), the iterations stop either when $s \in \{0, 1\}$ or we cannot continue. In the former case, we have added at most $-s \le k\epsilon n$ edges and thus $|M_3| \le k\epsilon n$. In the latter case, first observe that as long as we can pick the 2-set in $A_2' \cup A_j'$, we can always form the desired edge in each step because the number of covered vertices in V_1 is at most $|B| + k\epsilon n \le 2k\alpha n < a_1$. Moreover, recall that $a_2 \ge \epsilon_0 n/(k-1)$, and thus by (3.4) and Claim 3.10, we have

$$|A_2'| \ge |A_2| - |M_2^2| \ge a_2 - \alpha n - k\alpha n \ge k\epsilon n,$$

as $\epsilon \ll \epsilon_0$. So the only reason such that we cannot continue the process is that we have run out of vertices in $A_3', \dots, A_k'$, i.e., $\sum_{3\le i\le k} |A_i'| \le k\epsilon n$. By definition, we have $A_i' \subseteq (A_i \cup B_i) \setminus V(M_1 \cup M_2)$ and recall that $|M_1 \cup M_2| \le |B|$. By (3.2), we have

$$a_i \le |A_i| + |B_i| \le |A_i'| + |V(M_1 \cup M_2) \cap V_i| \le |A_i'| + k\alpha n, \tag{3.8}$$

since $|V(M_1 \cup M_2) \cap V_i| = |M_1 \cup M_2| \le |B| \le k\alpha n$. This implies that

$$\sum_{3\le i\le k} a_i \le \sum_{3\le i\le k} (|A_i'| + k\alpha n) \le k\epsilon n + (k-2)k\alpha n \le k^2\alpha n. \tag{3.9}$$

Since $a_1 \le (1-\epsilon_0)n$ and $\epsilon \ll \epsilon_0$, we get that $a_2 \ge \epsilon_0 n - k^2\alpha n \ge \epsilon_0 n/2$.

Now we do the following. We greedily add edges that consist of uncovered vertices and have type $A_1' \times A_2' \times (\prod_{j\in[3,k]} C_j')$ to M_3 . First note that when one such edge e is added to M_3, $|M_3|$ is increased by 1 and $\sum |A_j' \setminus V(M_3)|$ is reduced by $|e \cap A| = 2$, and hence s is increased by 1. So we can make $s = 0$ in at most $k\epsilon n$ steps (this includes the previous edges added to M_3).

It remains to show that the above process is possible. First, if $a_1 \geq (1/2 + \epsilon_0)n$, then the process is always possible. Indeed, by (3.8), we get

$$n - |A_1'| \leq n - (a_1 - k\alpha n) \leq (1/2 - \epsilon_0/2)n \leq a_1 - k\epsilon n.$$

This means that each legal $(k-1)$-set of type $A_2' \times (\prod_{j\in[3,k]} C_i')$ has at least $k\epsilon n$ neighbors in A_1', and thus we can pick up to $k\epsilon n$ disjoint edges of type $A_1' \times A_2' \times (\prod_{j\in[3,k]} C_i')$. So we may assume that $a_1 \leq (1/2 + \epsilon_0)n$. Together with (3.9), this implies that

$$(1/2 - 2\epsilon_0)n \leq a_1, a_2 \leq (1/2 + \epsilon_0)n. \tag{3.10}$$

Moreover, suppose that we cannot finish this process, that is, after picking at most $k\epsilon n$ such edges of type $A_1' \times A_2' \times (\prod_{j\in[3,k]} C_j')$, all other such edges intersect $V(M_3)$. Or, equivalently, all edges of type $(A_1 \cup B_1) \times (A_2 \cup B_2) \times (\prod_{j\in[3,k]} C_j)$ intersect $V(M_1 \cup M_2 \cup M_3)$. Since $|M_1 \cup M_2 \cup M_3| \leq k\alpha n + k\epsilon n \leq 2k\alpha n$, there are at most $2k\alpha n \cdot n^{k-1} = 2k\alpha n^k$ such edges. Note that since C is an independent set, there is no edge in H of type $\prod_{j\in[k]} C_j$. So all but at most $2k\alpha n^k$ edges of H contains exactly one vertex in $A_1 \cup A_2$. By (3.9) and (3.10), H is $2\epsilon_0$-D-extremal, a contradiction.

For Step 3, we only need to modify accordingly like the following equations:

$$|A_1''|, |A_2''| \geq a_2 - 2k\alpha n \geq \epsilon_0 n/k. \tag{3.11}$$

$$|C_i''| \geq \min\{|A_1''|, |A_2''|\} \geq a_2 - 2k\alpha n \geq \epsilon_0 n/k. \tag{3.12}$$

Let $m_i := |A''_i|$ for all $i \in [2]$. Note that the number of unmatched vertices in C''_1 and C''_2 are $m_2 + r$ and $m_1 + r$, respectively, and the number of unmatched vertices in C''_i, $i \in [3, k]$ is $m_1 + m_2 + r$. For $i = 1, 2$, pick arbitrary subset C_1^2 of uncovered vertices in C''_1 of size m_2 and C_2^1 of uncovered vertices in C''_2 of size m_1; for $i \in [3, k]$, we can partition all but r vertices of C''_i into C_i^1 of size m_1 and C_i^2 of size m_2. Therefore, we get k-partite k-graphs $H_j := H[A''_j, \bigcup_{\ell \neq j} C_\ell^j]$ for $j \in [2]$. Let us verify the assumptions of Theorem 3.9 for H_1 and H_2, respectively.

We can follow the same process to verify that the conditions of Theorem 3.9 are satisfied so that to find a perfect matching M_4^j in H_j for each $j \in [2]$. Let $M_4 = M_4^0 \cup M_4^1 \cup M_4^2$, then $M_1 \cup M_2 \cup M_3 \cup M_4$ is a matching in H of size at least $n - r \geq \sum_{i \in [k]} a_i$ and we are done.

Divisibility barrier when there are exactly two large partite codegrees

In this case, without loss of generality, we may assume only $a_1 \geq a_2 \geq \epsilon n$. By Lemma 3.14, $a_1, \geq a_2 \geq n/2 - k\epsilon n$ and H is divisibility barrier such that each of V_1 and V_2 is partitioned into two closed subsets of size around $n/2$, and each of V_i for $i \geq 3$ is either closed or can be partitioned into two closed subsets. This part needs real hardwork and we hope to finish it soon.

PART 4

MINIMUM VERTEX DEGREE THRESHOLD FOR TILING 3-PARTITE 3-GRAPHS

4.1 Introduction

Given $r \geq 2$, an r-uniform hypergraph (in short, *r-graph*) consists of a vertex set V and an edge set $E \subseteq \binom{V}{r}$, that is, every edge is an r-element subset of V. Given an r-graph H with a set S of d vertices, where $1 \leq d \leq r-1$, we define $\deg_H(S)$ to be the number of edges containing S (the subscript H is omitted if it is clear from the context). The *minimum d-degree* $\delta_d(H)$ of H is the minimum of $\deg_H(S)$ over all d-vertex sets S in H. The minimum 1-degree is also referred as the *minimum vertex degree*.

Given two r-graphs F and H, an *F-tiling* (also known as *F-packing*) of H is a collection of vertex-disjoint copies of F in H. An F-tiling is called a *perfect F-tiling* (or an *F-factor*) of H if it covers all the vertices of H. An obvious necessary condition for H to contain an F-factor is $|V(F)| \mid |V(H)|$. Given an integer n that is divisible by $|V(F)|$, we define the *tiling threshold* $t_d(n, F)$ to be the smallest integer t such that every r-graph H of order n with $\delta_d(H) \geq t$ contains an F-factor.

In this Part we extend these results by determining $t_1(n, K)$ asymptotically for all complete 3-partite 3-graphs K, and thus partially answer a question of Mycroft [69].

Given $a \leq b \leq c$, let $d = \gcd(b-a, c-b)$ and define

$$f(a,b,c) := \begin{cases} 1/4, & \text{if } a = 1,\ \gcd(a,b,c) = 1 \text{ and } d = 1; \\ 6 - 4\sqrt{2} \approx 0.343, & \text{if } a \geq 2,\ \gcd(a,b,c) = 1 \text{ and } d = 1; \\ 4/9, & \text{if } \gcd(a,b,c) = 1 \text{ and } d \geq 3 \text{ is odd}; \\ 1/2, & \text{otherwise.} \end{cases} \tag{4.1}$$

Theorem 4.1 (Main Result)**.**

$$t_1(n, K_{a,b,c}) = \left(\max\left\{f(a,b,c), 1-\left(\frac{b+c}{a+b+c}\right)^2, \left(\frac{a+b}{a+b+c}\right)^2\right\} + o(1)\right)\binom{n}{2}.$$

Let us compare Theorem 4.1 with the corresponding result in [70], which states that

$$t_2(n, K_{a,b,c}) = \begin{cases} n/2 + o(n) & \text{if } \gcd(a,b,c) > 1 \text{ or } a = b = c = 1; \\ an/(a+b+c) + o(n) & \text{if } \gcd(a,b,c) = 1 \text{ and } d = 1; \\ \max\{an/(a+b+c), n/p\} + o(n) & \text{otherwise.} \end{cases}$$

where p is the smallest prime factor of d. Not only is Theorem 4.1 more complicated, but also it contains a case where the coefficient of the threshold is irrational. In fact, as far as we know, all the previously known tiling thresholds had rational coefficients.

The lower bound in Theorem 4.1 follows from six constructions given in Section 2. Three of them are known as *divisibility barriers* and two are known as *space barriers*. Roughly speaking, the divisibility barriers, known as lattice-based constructions, only prevent the existence of a perfect K-tiling; in contrast, the space barriers are 'robust' because they prevent the existence of an almost perfect K-tiling. Our last construction is related to the nature of tiling, where every vertex must be contained in a copy of $K_{a,b,c}$, so we call it a *tiling barrier*. Such a barrier has never appeared before – see concluding remarks in Part 6.

Our proof of the upper bound of Theorem 4.1 consists of two parts: one is on finding an almost perfect K-tiling in H, and the other is on 'finishing up' the perfect K-tiling. Our first lemma says that H contains an almost perfect K-tiling if the minimum vertex degree of H exceeds those of the space barriers.

Lemma 4.2 (Almost Tiling Lemma)**.** *Fix integers $1 \le a \le b \le c$. For any $\gamma > 0$ and $\alpha > 0$, there exists an integer n_0 such that the following holds. Suppose H is a 3-graph of order $n > n_0$ with $\delta_1(H) \ge (\max\{1 - (\frac{b+c}{a+b+c})^2, (\frac{a+b}{a+b+c})^2\} + \gamma)\binom{n}{2}$, then there exists a $K_{a,b,c}$-tiling covering all but at most αn vertices.*

The absorbing method, initiated by Rödl, Ruciński and Szemerédi [76], has been shown to be effective in finding spanning (hyper)graphs. Our absorbing lemma says that H contains a small $K_{a,b,c}$-tiling that can *absorb* any much smaller set of vertices of H if the minimum vertex degree of H exceeds those of the divisibility barriers.

Lemma 4.3 (Absorbing Lemma)**.** *Fix integers $1 \le a \le b \le c$. For any $\gamma > 0$, there exists $\alpha > 0$ such that the following holds for sufficiently large n. Suppose H is a 3-graph on n vertices such that*

$$\delta_1(H) \ge (f(a,b,c) + \gamma)\binom{n}{2}.$$

Then there exists a vertex set W with $|W| \le \frac{1}{4}\gamma n$ such that for any vertex set $U \subset V(H) \setminus W$ with $|U| \le \alpha n$ and $|U| \in (a+b+c)\mathbb{Z}$, both $H[W]$ and $H[W \cup U]$ have $K_{a,b,c}$-factors.

The upper bound of $t_1(n, K_{a,b,c})$ in Theorem 4.1 follows from Lemmas 4.2 and 4.3 easily.

Proof of Theorem 4.1 (upper bound). Let $1 \le a \le b \le c$ be integers and $\gamma > 0$. Let $\alpha > 0$ be the constant returned by Lemma 4.3 and let $n \in (a+b+c)\mathbb{N}$ be sufficiently large. Suppose that H is a 3-graph on n vertices with $\delta_1(H) \ge (\delta + \gamma)\binom{n}{2}$, where

$$\delta = \max\left\{ f(a,b,c), 1 - \left(\frac{b+c}{a+b+c}\right)^2, \left(\frac{a+b}{a+b+c}\right)^2 \right\}.$$

We apply Lemma 4.3 to H and get a vertex set W with $|W| \le \frac{1}{4}\gamma n$ and the described absorbing property. In particular, $|W| \in (a+b+c)\mathbb{N}$. Let $H' = H[V(H) \setminus W]$. Then

$$\delta_1(H') \ge \delta_1(H) - |W|(n-1) \ge (\delta + \gamma)\binom{n}{2} - \frac{\gamma}{2}\binom{n}{2} \ge \left(\delta + \frac{\gamma}{2}\right)\binom{|V(H')|}{2}.$$

Next we apply Lemma 4.2 on H' with $\gamma/2$ in place of γ and get a $K_{a,b,c}$-tiling covering T all but a set U of at most $\alpha|V(H')| < \alpha n$ vertices of H'. Since $|V(H)|, |W|, |V(T)| \in (a+b+c)\mathbb{N}$, $|U| = |V(H)| - |W| - |V(T)| \in (a+b+c)\mathbb{N}$. By the absorbing property of W, there exists a $K_{a,b,c}$-factor on $H[W \cup U]$. Thus we get a $K_{a,b,c}$-factor of H. $\square$

Although this proof is a straightforward application of the absorbing method, there are several new ideas in the proofs of Lemmas 4.2 and 4.3. First, in order to show that almost every $(a+b+c)$-set has many absorbing sets, we use lattice-based absorbing arguments developed recently by Han [38]. Second, in order to prove Lemma 4.2, we use the concept of *fractional homomorphic tiling* given by Buß, Hàn and Schacht [9]. Third, we need a recent result of Füredi and Zhao [31] on shadows, which can be viewed as a vertex degree version of the well-known Kruskal-Katona Theorem for 3-graphs.

The rest of the paper is organized as follows. We prove the lower bound in Theorem 4.1 by six constructions in Section 2. We prove Lemma 4.3 in Section 3 and Lemma 4.2 in Section 4, respectively. Finally, we give concluding remarks in Section 5.

Notations. Throughout this paper we let $1 \le a \le b \le c$ be three integers and $k = a+b+c \ge 3$. When it is clear from the context, we write $K_{a,b,c}$ as K for short. By $x \ll y$ we mean that for any $y > 0$ there exists $x_0 > 0$ such that for any $x < x_0$ the following statement holds. We omit the floor and ceiling functions when they do not affect the proof.

4.2 Extremal Examples

In this section, we prove the lower bound in Theorem 4.1 by six constructions. Following the definition in [70], we say a 3-partite 3-graph $K_{a,b,c}$ is of *type 0* if $\gcd(a,b,c) > 1$ or $a=b=c=1$. We say $K_{a,b,c}$ is of *type* $d \ge 1$ if $\gcd(a,b,c) = 1$ and $\gcd(b-a, c-b) = d$.

Construction 4.4 (Space Barrier I). *Let V_1 and V_2 be two disjoint sets of vertices such that $|V_1| = \frac{a}{k}n - 1$ and $|V_1| + |V_2| = n$. Let G_1 be the 3-graph on $V_1 \cup V_2$ whose edge set consists of all triples e such that $|e \cap V_1| \ge 1$. Then $\delta_1(G_1) = \binom{n-1}{2} - \binom{(1-\frac{a}{k})n}{2} = (1-(1-a/k)^2)\binom{n}{2} + o(n^2)$. Since $a \le b \le c$, we have $a \le k/3$ and $0 < 1-(1-a/k)^2 \le 5/9$.*

We claim that G_1 has no perfect $K_{a,b,c}$-tiling. Indeed, consider a copy K' of $K_{a,b,c}$ in G_1. We observe that at least one color class of K' is a subset of V_1 – otherwise V_2 contains at least one vertex from each color class; since K' is complete, there is an edge in V_2, contradicting the definition of G_1. Hence a $K_{a,b,c}$-tiling in G_1 covers at most $\frac{|V_1|}{a}k < n$ vertices, so it cannot be perfect.

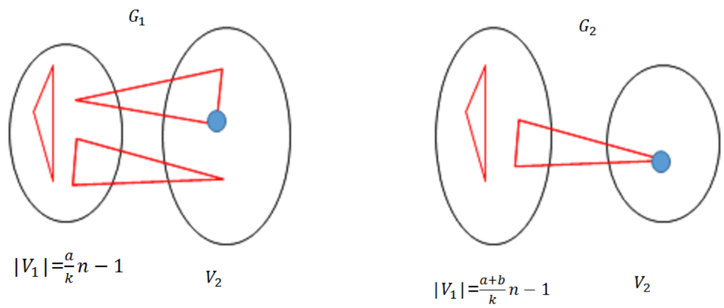

Figure 4.1. Space barriers

Construction 4.5 (Space Barrier II). *Let V_1 and V_2 be two disjoint sets of vertices such that $|V_1| = \frac{a+b}{k}n - 1$ and $|V_1| + |V_2| = n$. Let G_2 be the 3-graph on $V_1 \cup V_2$ whose edge set consists of all triples e such that $|e \cap V_1| \geq 2$. Then $\delta_1(G_2) = \binom{\frac{a+b}{k}n-1}{2} = ((a+b)^2/k^2)\binom{n}{2} + o(n^2)$. Since $a \leq b \leq c$, we have $a + b \leq 2k/3$ and $0 < (a+b)^2/k^2 \leq 4/9$.*

We claim that G_2 has no perfect $K_{a,b,c}$-tiling. Similarly as in the previous case, for any copy K' of $K_{a,b,c}$ in G_2, at least two color classes of K' are subsets of V_1. Hence a $K_{a,b,c}$-tiling in G_2 covers at most $\frac{|V_1|}{a+b}k < n$ vertices, so it cannot be perfect.

Construction 4.6 (Divisibility Barrier I). *Let V_1 and V_2 be two disjoint sets of vertices such that $|V_1| = \lfloor \frac{n}{2} \rfloor + 1$ and $|V_1| + |V_2| = n$. Let H_1 be the 3-graph on $V_1 \cup V_2$ such that $H_1[V_1]$ and $H_1[V_2]$ are two complete 3-graphs. Then $\delta_1(H_1) \geq \binom{\frac{n}{2}-2}{2} = \frac{1}{4}\binom{n}{2} + o(n^2)$.*

We claim that H_1 has no perfect $K_{a,b,c}$-tiling. Indeed, each copy of $K_{a,b,c}$ must be a subgraph of $H_1[V_1]$ or $H_1[V_2]$. Since $k \geq 3$, due to the choice of V_1 and V_2, we have $|V_1| \neq |V_2| \bmod k$ and therefore k cannot divide both $|V_1|$ and $|V_2|$. Hence H_1 has no perfect $K_{a,b,c}$-tiling.

Construction 4.7 (Divisibility Barrier II). *Suppose that $K_{a,b,c}$ is of type d for some even d. Let V_1 and V_2 be two disjoint sets of vertices such that $|V_1| + |V_2| = n$ and $|V_2| \in [\frac{n}{2} - 2, \frac{n}{2} + 2]$ is odd, and $\gcd(a, b, c) \nmid |V_2|$ if $\gcd(a, b, c) > 1$. Note that we can pick $|V_2|$ satisfying these conditions because in the interval $[\frac{n}{2} - 2, \frac{n}{2} + 2]$, there are at least two consecutive odd numbers,*

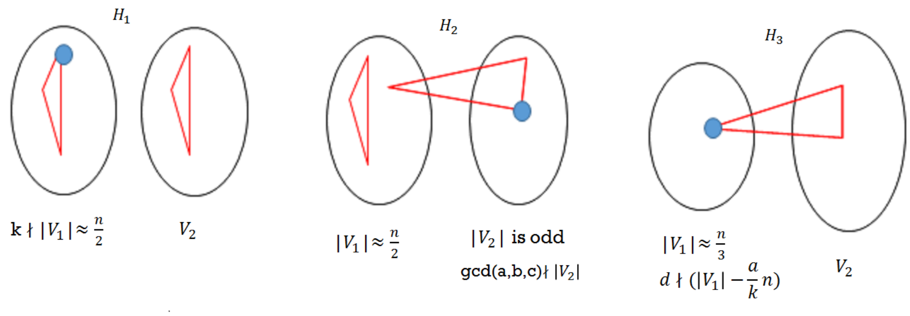

Figure 4.2. Divisibility barriers

therefore at least one of them is not divisible by $\gcd(a,b,c)$*. Let* H_2 *be the* 3*-graph on* $V_1 \cup V_2$ *whose edge set consists of all triples* e *such that* $|e \cap V_2|$ *is even (0 or 2). Then* $\delta_1(H_2) = \min\{\binom{|V_1|-1}{2} + \binom{|V_2|}{2}, |V_1|(|V_2|-1)\} = \frac{1}{2}\binom{n}{2} + o(n^2)$*.*

We claim that H_2 has no perfect $K_{a,b,c}$-tiling. Consider a copy K' of $K_{a,b,c}$ in H_2. Since every edge intersects V_2 in an even number of vertices and K' is complete, no color class of K' intersects both V_1 and V_2. Moreover, either 0 or 2 color classes of K' are subsets of V_2. Thus $|V(K') \cap V_2| \in \{0, a+b, a+c, b+c\}$. If $\gcd(a,b,c) > 1$, then $|V(K') \cap V_2|$ is divisible by $\gcd(a,b,c)$. Since $\gcd(a,b,c) \nmid |V_2|$, there is no perfect $K_{a,b,c}$-tiling. Otherwise, either $a = b = c = 1$ or $\gcd(b-a, c-b)$ is even. In either case, all of $a+b$, $a+c$ and $b+c$ are even and thus $|V(K') \cap V_2|$ is even. Since $|V_2|$ is odd, H_2 has no perfect $K_{a,b,c}$-tiling.

Construction 4.8 (Divisibility Barrier III)**.** *Suppose that* $K_{a,b,c}$ *is of type* d *for some odd* $d \geq 3$*, let* V_1 *and* V_2 *be two disjoint sets of vertices such that* $|V_1| + |V_2| = n$ *and* $|V_1| \in [\frac{n}{3} - 1, \frac{n}{3} + 1]$ *and* $d \nmid (|V_1| - \frac{n}{k}a)$*. Let* H_3 *be the* 3*-graph on* $V_1 \cup V_2$ *whose edge set consists of all triples* e *such that* $|e \cap V_1| = 1$*. Then* $\delta_1(H_3) = \min\{|V_1|(|V_2|-1), \binom{|V_2|}{2}\} = \frac{4}{9}\binom{n}{2} + o(n^2)$*.*

We claim that H_3 has no perfect $K_{a,b,c}$-tiling. Consider a copy K' of $K_{a,b,c}$ in H_3. Similarly as in the previous case, exactly one color class of K' is a subset of V_1, which implies $|V_1 \cap V(K')| \in \{a, b, c\}$. Since $\gcd(b-a, c-b) = d$, we have $a \equiv b \equiv c \bmod d$ and thus $|V_1 \cap V(K')| \equiv a \bmod d$. If H_3 contains a perfect $K_{a,b,c}$-tiling $\mathcal{K}$, then $|V_1| - \frac{n}{k}a =$

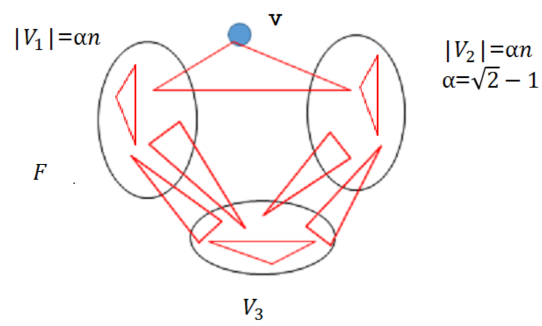

Figure 4.3. Tiling barrier

$|V(\mathcal{K}) \cap V_1| - \frac{n}{k}a \equiv 0 \bmod d$, contradicting our assumption on $|V_1|$. Hence H_3 has no perfect $K_{a,b,c}$-tiling.

Construction 4.9 (Tiling Barrier)**.** *Let $\alpha = \sqrt{2} - 1$ and suppose that V is partitioned into $\{v\} \cup V_1 \cup V_2 \cup V_3$ such that $|V_1| = |V_2| = \alpha n$ and $|V| = n$. Define a 3-graph F on V whose edge set consists of all triples vxy with $x \in V_1, y \in V_2$ and all triples e in $V_1 \cup V_2 \cup V_3$ such that $e \cap V_1 = \emptyset$ or $e \cap V_2 = \emptyset$. Therefore, $\delta_1(F) = (6 - 4\sqrt{2})\binom{n}{2} + o(n^2) \approx 0.343\binom{n}{2}$.*

It is easy to see that v is not contained in any copy of $K_{2,2,2}$, and hence not contained in any copy of $K_{a,b,c}$ with $a > 1$. Therefore, F has no perfect $K_{a,b,c}$-tiling with $a > 1$.

Proof of Theorem 4.1 (lower bound). Given positive integers $a \leq b \leq c$ and $n \in k\mathbb{N}$, where $k = a + b + c$, let $t_1 = t_1(n, K_{a,b,c})$ be the tiling threshold. By Constructions 4.4 and 4.5, we have $t_1 \geq (1 - (1 - a/k)^2)\binom{n}{2} + o(n^2)$ and $t_1 \geq ((a + b)^2/k^2)\binom{n}{2} + o(n^2)$. Furthermore, assume $K_{a,b,c}$ has type d. First, by definition, d is even if and only if $\gcd(a, b, c) > 1$ or $a = b = c = 1$ or $d \geq 2$ is even, in this case by Construction 4.7, we have $t_1 \geq \frac{1}{2}\binom{n}{2} + o(n^2)$. Second, assume that $d \geq 3$ is odd, then by Construction 4.8, we have $t_1 \geq \frac{4}{9}\binom{n}{2} + o(n^2)$. Finally assume that $d = 1$. If $a = 1$, by Construction 4.6, we have $t_1 \geq \frac{1}{4}\binom{n}{2} + o(n^2)$. If $a \geq 2$, then by Construction 4.9, we have $t_1 \geq (6 - 4\sqrt{2})\binom{n}{2} + o(n^2)$. $\square$

4.3 Proof of the Absorbing Lemma

4.3.1 Preparation

We need a simple counting result, which, for example, follows from the result of Erdős [23] on supersaturation. Given $l_1, \ldots, l_r \in \mathbb{N}$, let $K^{(r)}_{l_1,\ldots,l_r}$ denote the complete r-partite r-graph whose jth part has exactly l_j vertices for all $j \in [r]$.

Proposition 4.10. *Given $\mu > 0$, $r, m, l_1, \ldots, l_r \in \mathbb{N}$, there exists $\mu' > 0$ such that the following holds for sufficiently large n. Let H be an r-graph on n vertices with a vertex partition $V_1 \cup \cdots \cup V_m$. Suppose $i_1, \ldots, i_r \in [m]$ and H contains at least μn^r edges $e = \{v_1, \ldots, v_r\}$ such that $v_1 \in V_{i_1}, \ldots, v_r \in V_{i_r}$. Then H contains at least $\mu' n^{l_1+\cdots+l_r}$ copies of $K^{(r)}_{l_1,\ldots,l_r}$ whose jth part is contained in V_{i_j} for all $j \in [r]$.*

Given a 3-graph H, its *shadow* ∂H is the set of the pairs that are contained in at least one edge of H. We need a recent result of Füredi and Zhao [31] on the shadows of 3-graphs.

Lemma 4.11. *[31] Given a 3-graph H on n vertices, if $\delta_1(H) \geq d\binom{n}{2}$ where $d \in [\frac{1}{4}, \frac{47-5\sqrt{57}}{24}]$, then $|\partial H| \geq (4\sqrt{d} - 2d - 1)\binom{n}{2}$.*

The next lemma says that for any 3-graph, after a removal of a small portion of vertices and edges, any two vertices with a positive codegree in the remaining 3-graph has a linear codegree in H.

Lemma 4.12. *Given $\epsilon > 0$ and an n-vertex 3-graph $H = (V, E)$, there exists a vertex set $V_0' \subseteq V$ and a subhypergraph H' of H such that the following holds*

(i) $|V_0'| \leq 3\epsilon n$,

(ii) $\deg_{H'}(v) \geq \deg_H(v) - \epsilon\binom{n}{2}$ *for any* $v \in V \setminus V_0'$,

(iii) $\deg_H(S) > \epsilon^2 n$ *for any pair of vertices* $S \in \partial H'$.

Proof. If an edge $e \in E(H)$ contains a pair $S \in \binom{e}{2}$ with $\deg_H(S) \leq \epsilon^2 n$, then it is called *weak*, otherwise called *strong*. Let H' be the subhypergraph of H induced on strong edges.

Then (iii) holds. Let

$$V_0' = \left\{ v \in V : v \text{ is contained in at least } \epsilon\binom{n}{2} \text{ weak edges} \right\}.$$

Then (ii) holds. Note that the number of weak edges in H is at most $\binom{n}{2}\epsilon^2 n$. If $|V_0'| > 3\epsilon n$, then there are more than $3\epsilon n \cdot \epsilon\binom{n}{2}/3 = \binom{n}{2}\epsilon^2 n$ weak edges in H, a contradiction. Thus (i) holds. $\square$

We use the reachability arguments introduced by Lo and Markström [65, 66]. Given $\beta > 0$, $i \in \mathbb{N}$ and two vertices $u, v \in V(H)$, we call that u, v are (K, β, i)*-reachable in* H if and only if there are at least βn^{ik-1} $(ik-1)$-sets W such that both $H[\{u\} \cup W]$ and $H[\{v\} \cup W]$ contain K-factors. In this case, we call W a *reachable set* for u and v. A vertex set A is (K, β, i)*-closed in* H if every two vertices in A are (K, β, i)-reachable in H. For $x \in V(H)$, let $\tilde{N}_{\beta,i}(x)$ be the set of vertices that are (K, β, i)-reachable to x. Throughout this section, we assume that $K = K_{a,b,c}$ where $a \le b \le c$ and $k = a + b + c \ge 3$ and thus we omit K from the notations and only say (β, i)-reachable and (β, i)-closed.

We use the following two results from [66].

Proposition 4.13. *[66, Proposition 2.1] For $\beta, \epsilon > 0$ and integers $i_0' > i_0$, there exists $\beta' > 0$ such that the following holds for sufficiently large n. Given an n-vertex 3-graph H and a vertex $x \in V(H)$ with $|\tilde{N}_{\beta,i_0}(x)| \ge \epsilon n$, then $\tilde{N}_{\beta,i_0}(x) \subseteq \tilde{N}_{\beta',i_0'}(x)$. In other words, if $x, y \in V(H)$ are (β, i_0)-reachable in H and $|\tilde{N}_{\beta,i_0}(x)| \ge \epsilon n$, then x, y are (β', i_0')-reachable in H.*

The following lemma is essentially [66, Lemma 4.2] [1].

Lemma 4.14. *[66] Given $\epsilon > 0$, there exists $\eta > 0$ such that the following holds for sufficiently large n. For any n-vertex 3-graph H, two vertices $x, y \in V(H)$ are $(\eta, 1)$-reachable if the number of pairs $S \in N(x) \cap N(y)$ with $\deg(S) \ge \epsilon n$ is at least $\epsilon\binom{n}{2}$.*

[1] In fact, [66, Lemma 4.2] essentially shows that there are many copies of $K_{c,c,c+1}$ containing x and y both in the part of size $c+1$. To obtain the result we need, one can get many copies of $K_{a,b,c+1}$ containing x and y both in the part of size $c+1$, by deleting vertices from each copy.

4.3.2 Auxiliary Lemmas

We call an m-set A an *absorbing m-set* for a k-set S if $A \cap S = \emptyset$ and both $H[A]$ and $H[A \cup S]$ contain K-factors. Denote by $\mathcal{A}^m(S)$ the set of all absorbing m-sets for S.

Our proof of the Absorbing Lemma is based on the following lemma.

Lemma 4.15. *Given $0 < \eta \le 1/(2k)$, $\beta > 0$, and $i_0 \in \mathbb{N}$, there exists $\alpha > 0$ such that the following holds for all sufficiently large integers n. Suppose $H = (V, E)$ is an n-vertex 3-graph with the following two properties*

($\Diamond$) For any $v \in V$, there are at least ηn^{k-1} copies of K containing it.

($\triangle$) There exists $V_0 \subset V$ with $|V_0| \le \eta^2 n$ such that $V(H) \setminus V_0$ is (β, i_0)-closed in H.

Then there exists a vertex set W with $V_0 \subseteq W \subseteq V$ and $|W| \le \eta n$ such that for any vertex set $U \subseteq V \setminus W$ with $|U| \le \alpha n$ and $|U| \in k\mathbb{Z}$, both $H[W]$ and $H[U \cup W]$ contain K-factors.

Proof. Let
$$\eta_1 = \frac{\eta}{2}\left(\frac{\beta}{2}\right)^{k-1} \text{ and } \alpha = \frac{\eta_1^2}{32 i_0 k}.$$
There are two steps in our proof. In the first step, we build an absorbing family $\mathcal{F}_1$ that can absorb any small portion of vertices in $V \setminus V_0$. In the second step, we put the vertices in $V_0 \setminus V(\mathcal{F}_1)$ into a family $\mathcal{F}_2$ of copies of K. Then $V(\mathcal{F}_1 \cup \mathcal{F}_2)$ is the desired absorbing set.

Fix a k-set $S = \{v_1, v_2, \dots, v_k\} \subset V \setminus V_0$. Let $m = i_0 k^2 - i_0 k$. We claim that there are at least $\eta_1 n^m$ absorbing m-sets for S, namely, $|\mathcal{A}^m(S)| \ge \eta_1 n^m$. Indeed, we first find a k-set $S' = \{v_1, u_2, \dots, u_k\} \subset V \setminus V_0$ such that $S' \cap S = \{v_1\}$ and S' spans a copy of K. By ($\Diamond$), there are at least
$$\eta n^{k-1} - (k-1)n^{k-2} - \eta^2 n^{k-1} \ge \eta n^{k-1}/2$$
choices for S'. Since $V \setminus V_0$ is (β, i_0)-closed, there are at least $\beta n^{i_0 k-1}$ reachable $(i_0 k - 1)$-sets S_i for u_i and v_i for $i = 2, \dots, k$. Next we choose a collection of pairwise disjoint sets S_i for $i = 2, \dots, k$. Since in each step we need to avoid the vertices of S, S' and the previous S_i's, which are at most $m + k$ vertices, there are at least $\beta n^{i_0 k-1}/2$ choices for each S_i. Let

$A = (S' \setminus \{v_1\}) \cup \left(\bigcup_{i=2}^{k} S_i\right)$, then $|A| = m$. We claim that both $H[A]$ and $H[A \cup S]$ contain K-factors. Indeed, since for $i = 2, \ldots, k$, each $S_i \cup \{u_i\}$ spans a copy of K, $H[A]$ contains a K-factor; since S' spans a copy of K and for $i = 2, \ldots, k$, each $S_i \cup \{v_i\}$ spans a copy of K, $H[A \cup S]$ contains a K-factor. Thus A is an absorbing m-set for S. In total, we get at least

$$\frac{\eta}{2} n^{k-1} \left(\frac{\beta}{2} n^{i_0 k - 1}\right)^{k-1} = \eta_1 n^m$$

such m-sets, thus $|\mathcal{A}^m(S)| \geq \eta_1 n^m$.

Now we build the family $\mathcal{F}_1$ by standard probabilistic arguments. Choose a family $\mathcal{F}$ of m-sets in H by selecting each of the $\binom{n}{m}$ possible m-sets independently with probability $p = \eta_1 n^{1-m}/(8m)$. Then by Chernoff's bound, with probability $1 - o(1)$ as $n \to \infty$, the family $\mathcal{F}$ satisfies the following properties:

$$|\mathcal{F}| \leq 2p\binom{n}{m} \leq \frac{\eta_1 n}{4m} \text{ and } |\mathcal{A}^m(S) \cap \mathcal{F}| \geq \frac{p|\mathcal{A}^m(S)|}{2} \geq \frac{\eta_1^2 n}{16m}, \text{ for all } S \in \binom{V \setminus V_0}{k}. \tag{4.2}$$

Furthermore, the expected number of pairs of m-sets in $\mathcal{F}$ that are intersecting is at most

$$\binom{n}{m} \cdot m \cdot \binom{n}{m-1} \cdot p^2 \leq \frac{\eta_1^2 n}{64m}.$$

Thus, by using Markov's inequality, we derive that with probability at least $1/2$,

$$\mathcal{F} \text{ contains at most } \frac{\eta_1^2 n}{32m} \text{ intersecting pairs of } m\text{-sets.} \tag{4.3}$$

Hence, there exists a family $\mathcal{F}$ with the properties in (4.2) and (4.3). By deleting one member of each intersecting pair and removing m-sets that are not absorbing sets for any k-set $S \subseteq V \setminus V_0$, we get a subfamily $\mathcal{F}_1$ consisting of pairwise disjoint m-sets. Let $W_1 = V(\mathcal{F}_1)$ and thus $|W_1| = |V(\mathcal{F}_1)| = m|\mathcal{F}_1| \leq m|\mathcal{F}| \leq \eta_1 n/4$. Since every m-set in $\mathcal{F}_1$ is an absorbing m-set for some k-set S, $H[W_1]$ has a K-factor. For any k-set S, by (4.2) and (4.3) above we have

$$|\mathcal{A}^m(S) \cap \mathcal{F}_1| \geq \frac{\eta_1^2 n}{16m} - \frac{\eta_1^2 n}{32m} = \frac{\eta_1^2 n}{32m}. \tag{4.4}$$

For any set $U \subseteq V \setminus (V_0 \cup W_1)$ of size $|U| \leq \alpha n$ and $|U| \in k\mathbb{Z}$, it can be partitioned into at most $\frac{\alpha n}{k}$ k-sets. By the definition of $\mathcal{F}_1$, each k-set S has at least $\frac{\eta_1^2 n}{32m} \geq \frac{\alpha n}{k}$ absorbing sets in $\mathcal{F}_1$, thus each k-set can be greedily matched to a distinct absorbing set in $\mathcal{F}_1$. Hence, $H[U \cup W_1]$ contains a K-factor.

In the second step, by ($\diamondsuit$), we greedily build $\mathcal{F}_2$, a collection of copies of K that cover the vertices in $V_0 \setminus W_1$. Indeed, assume that we have built $i < |V_0 \setminus W_1| \leq \eta^2 n$ copies of K. Together with the vertices in W_1, at most $ki + \eta_1 n/4 \leq k\eta^2 n + \eta_1 n/4$ vertices have already been covered by $\mathcal{F}$. So for any vertex $v \in V_0$ not yet covered, we find the desired copy of K containing v by ($\diamondsuit$), because $(k\eta^2 n + \eta_1 n/4) \cdot n^{k-2} < \eta n^{k-1}$.

Let $W = V(\mathcal{F}_2) \cup W_1$, we get the desired absorbing set W with $|W| \leq k\eta^2 n + \eta_1 n/4 < \eta n$. $\square$

So it remains to show that ($\diamondsuit$) and ($\triangle$) hold in the 3-graph H. We first study the property ($\diamondsuit$). Throughout this subsection, let $d_0 = 6 - 4\sqrt{2} \approx 0.343$ and note that $(4\sqrt{d_0} - 2d_0 - 1) + d_0 = 1$ because $\sqrt{d_0} = 2 - \sqrt{2}$.

Lemma 4.16. *For any $\gamma > 0$, there exists $\eta > 0$ such that the following holds for sufficiently large n. Let H be an n-vertex 3-graph with $\delta_1(H) \geq (d_0 + \gamma)\binom{n}{2}$. Then each vertex $v \in V(H)$ is contained in at least ηn^{k-1} copies of K.*

Proof. Let $\epsilon = \gamma/12$. Let η be returned by Lemma 4.14 when $\gamma\epsilon^2/2$ plays the role of ϵ. Suppose that n is sufficiently large and H is an n-vertex 3-graph with $\delta_1(H) \geq (d_0 + \gamma)\binom{n}{2}$. We apply Lemma 4.12 on H and get V_0' and H' satisfying (i) – (iii). Let $H'' = H'[V \setminus V_0']$ and $n' = |V \setminus V_0'|$. By Lemma 4.12 (ii), we have

$$\delta_1(H'') \geq (d_0 + \gamma)\binom{n}{2} - \epsilon\binom{n}{2} - |V_0'|(n-2) > d_0\binom{n'}{2}$$

because $|V_0'| \le 3\epsilon n$. Since $\frac{1}{4} < d_0 < \frac{47-5\sqrt{57}}{24} \approx 0.385$ and $n' \ge (1-3\epsilon)n$, Lemma 4.11 implies that

$$\partial H'' \ge (4\sqrt{d_0} - 2d_0 - 1)\binom{n'}{2} \ge (4\sqrt{d_0} - 2d_0 - 1)(1-6\epsilon)\binom{n}{2}.$$

Since $\delta_1(H) \ge (d_0+\gamma)\binom{n}{2}$, for every $x \in V(H)$, we have

$$|N_H(x) \cap \partial H''| \ge \Big(d_0 + \gamma + (4\sqrt{d_0} - 2d_0 - 1) - 6\epsilon - 1\Big)\binom{n}{2} \ge \frac{\gamma}{2}\binom{n}{2},$$

by the definitions of d_0 and ϵ.

Fix $x \in V(H)$ and note that every $S \in N_H(x) \cap \partial H''$ has degree at least $\epsilon^2 n$ in H. Therefore, the number of (S, y) with $S \in N_H(x) \cap \partial H''$ and $y \in N_H(S)$ is at least $\frac{\gamma}{2}\binom{n}{2} \cdot \epsilon^2 n$. By averaging, there exists a vertex y such that

$$|N_H(y) \cap N_H(x) \cap \partial H''| \ge \gamma\epsilon^2\binom{n}{2}/2.$$

This means that x and y have at least $\gamma\epsilon^2\binom{n}{2}/2$ common neighbors with degree at least $\epsilon^2 n$. By Lemma 4.14, x and y are $(\eta, 1)$-reachable. Hence, there are at least ηn^{k-1} $(k-1)$-sets W such that $H[\{x\} \cup W]$ forms a copy of K. $\square$

Now we study the property $(\triangle)$. Following the approach in [38], given a 3-graph H, we first find a partition of $V(H)$ such that all but one part are (β, i)-closed in H and then study the reachability between different parts. The following lemma provides such a partition.

Lemma 4.17. *Given $\delta \ge 1/4$ and $\gamma > 0$, there exist constants $0 < \beta \ll \epsilon \ll \gamma$ such that the following holds for sufficiently large n. Let H be an n-vertex 3-graph with $\delta_1(H) \ge (\delta+\gamma)\binom{n}{2}$. Then there is a partition $\mathcal{P}$ of $V(H)$ into $V_0, V_1, \ldots, V_r$ such that*

- *$|V_0| \le 4\epsilon n$,*
- *$r \le \lfloor 1/(\delta + \gamma/2) \rfloor$, and*
- *$|V_i| \ge \epsilon^2 n$ and V_i is $(\beta, 2^{\lfloor 1/(\delta+\gamma/2)\rfloor - 1})$-closed in H for all $i \in [r]$.*

Proof. Let $s = \lfloor 1/(\delta+\gamma/2) \rfloor$. Then we may choose $\epsilon > 0$ such that $\binom{s+1}{2}\epsilon^2/16 < (s+1)(\delta+\gamma/2)-1$. Let η be the constant returned from applying Lemma 4.14 with $\epsilon^2/16$ in place of ϵ. Note that we may require $\eta \ll \epsilon$ because Lemma 4.14 is monotone, i.e., the conclusion holds with η replaced by any $0 < \eta' < \eta$. Furthermore, let

$$1/n \ll \beta = \beta_{s-1} \ll \cdots \ll \beta_1 \ll \beta_0 \le \eta, \alpha \ll \epsilon \ll \gamma, (s+1)(\delta+\gamma/2)-1 \ll 1/k.$$

Let $H = (V, E)$ be an n-vertex 3-graph with $\delta_1(H) \ge (\delta+\gamma)\binom{n}{2}$. We apply Lemma 4.12 on H and obtain V_0' and H' satisfying (i) – (iii).

Given $v \in V$ and $0 \le i \le s-1$, let $\tilde{N}_i(v) = \tilde{N}_{\beta_i, 2^i}(v)$ be the set of vertices in H that are (β, i)-reachable to x (note that $\tilde{N}_i(v)$ may contains the vertices of V_0'). Throughout this proof, we say 2^i-reachable (respectively, 2^i-closed) for $(\beta_i, 2^i)$-reachable (respectively, $(\beta_i, 2^i)$-closed) for short.

Fix $x \in V \setminus V_0'$, we claim that $|\tilde{N}_0(x)| \ge \frac{3}{4}\epsilon^2 n$. To see this, let

$$D = \left\{ v \in V : |N_{H'}(v) \cap N_{H'}(x)| \ge \frac{\epsilon^2}{16}\binom{n}{2} \right\}.$$

Since $\deg_H(p) > \epsilon^2 n$ for any $p \in \partial H'$, Lemma 4.14 implies that two vertices $x, v \in V$ are 1-reachable if $|N_{H'}(v) \cap N_{H'}(x)| \ge \epsilon^2\binom{n}{2}/16$. Therefore $D \subseteq \tilde{N}_0(x)$. Let t be the number of pairs (p, u) where $p \in N_{H'}(x)$ and $u \in N_{H'}(p)$. By Lemma 4.12 (iii), we have $t \ge \deg_{H'}(x) \cdot \epsilon^2 n$. Note that if $u \notin D$, the number of $p \in N_{H'}(x)$ such that $u \in N_{H'}(p)$ is $|N_{H'}(v) \cap N_{H'}(x)| < \frac{\epsilon^2}{16}\binom{n}{2}$, and thus

$$\deg_{H'}(x)\, \epsilon^2 n \le t \le n \cdot \frac{\epsilon^2}{16}\binom{n}{2} + |D| \cdot \deg_{H'}(x).$$

So we get $|D| \ge \epsilon^2 n - \epsilon^2 n\binom{n}{2}/(16 \deg_{H'}(x))$. Since $x \in V \setminus V_0'$, by Lemma 4.12 (ii), we have

$$\deg_{H'}(x) \ge (\delta+\gamma-\epsilon)\binom{n}{2} \ge (\delta+\gamma/2)\binom{n}{2} \ge \frac{1}{4}\binom{n}{2} \tag{4.5}$$

because $\delta \geq 1/4$. Consequently, $|\tilde{N}_0(x)| \geq |D| \geq \frac{3}{4}\epsilon^2 n$.

Since $|\tilde{N}_0(x)| \geq \frac{3}{4}\epsilon^2 n$, by Proposition 4.13 and the choice of β_i's, we know that $\tilde{N}_i(x) \subseteq \tilde{N}_{i+1}(x)$ for all $0 \leq i < s-1$ and all $x \in V \setminus V_0'$, and if a set $W \subseteq V \setminus V_0'$ is 2^i-closed in H for some $i \leq s-1$, then W is 2^{s-1}-closed in H.

Given a set $S \subseteq V \setminus V_0'$ of $s+1$ vertices, the Inclusion-Exclusion principle implies that

$$\sum_{x \in S} \deg_{H'}(x) - \sum_{x,y \in S} |N_{H'}(x) \cap N_{H'}(y)| \leq |\bigcup_{x \in S} N_{H'}(x)| \leq \binom{n}{2}.$$

By (4.5) and $(s+1)(\delta + \gamma/2) - 1 > \binom{s+1}{2}\epsilon^2/16$, there are two vertices $x, y \in S$ such that $|N_{H'}(x) \cap N_{H'}(y)| \geq \frac{\epsilon^2}{16}\binom{n}{2}$, so x, y are 1-reachable to each other. Consequently, if $s = 1$, then $V \setminus V_0'$ is 1-closed and we get the desired partition $\mathcal{P} = \{V_0', V \setminus V_0'\}$.

We may thus assume that $s \geq 2$ and there are two vertices in $V \setminus V_0'$ that are *not* 2^{s-1}-reachable to each other (otherwise we are done). Let r' be the largest integer such that there exist $v_1, \dots, v_{r'} \in V \setminus V_0'$ such that no pair of them are $2^{s+1-r'}$-reachable to each other. Earlier arguments show that r exists and $2 \leq r' \leq s$. Fix such $v_1, \dots, v_{r'} \in V \setminus V_0'$. By Proposition 4.13, we can assume that any two of them are not $2^{s-r'}$-reachable to each other. Then $\tilde{N}_{s-r'}(v_i)$, $i \in [r']$ satisfy the following properties.

(a) Any $v \in (V \setminus V_0') \setminus \{v_1, \dots, v_{r'}\}$ must be in $\tilde{N}_{s-r'}(v_i)$ for some $i \in [r']$ – otherwise $\{v, v_1, \dots, v_{r'}\}$ contradicts the definition of r'.

(b) $|\tilde{N}_{s-r'}(v_i) \cap \tilde{N}_{s-r'}(v_j)| < \alpha n$ for any $i \neq j$ – otherwise there are at least

$$\begin{aligned} &\frac{\alpha n}{(2^{s+1-r'}k-1)!}(\beta_{s-r'} n^{2^{s-r'}k-1} - n^{2^{s-r'}k-2})(\beta_{s-r'} n^{2^{s-r'}k-1} - 2^{s-r'}k n^{2^{s-r'}k-2}) \\ &\geq \beta_{s+1-r'} n^{2^{s+1-r'}k-1} \end{aligned}$$

reachable $(2^{s+1-r'}k-1)$-sets for v_i, v_j, contradicting the assumption that v_i, v_j are not $2^{s+1-r'}$-reachable to each other. Note that we get the lower bound of the number of the reachable sets for v_i, v_j above by fixing one element $w \in \tilde{N}_{s-r'}(v_i) \cap \tilde{N}_{s-r'}(v_j)$, one $(s-r')$-reachable set S for v_i and w (not containing v_j), and then one $(s-r')$-reachable set for v_j and w (not intersecting $\{v_i\} \cup S$). Finally, it is divided by $(2^{s+1-r'}k-1)!$ to eliminate the effect of overcounting.

For $i \in [r']$, let $V_i = (\tilde{N}_{s-r'}(v_i) \cup \{v_i\}) \setminus (V_0' \cup \bigcup_{j\in[r']\setminus\{i\}} \tilde{N}_{s-r'}(v_j))$. We observe that V_i is $2^{s-r'}$-closed for all $i \in [r']$. Indeed, if there exist $u_1, u_2 \in V_i$ that are not $2^{s-r'}$-reachable to each other, then $\{u_1, u_2, v_1, \ldots, v_{r'}\} \setminus \{v_i\}$ contradicts the definition of r'. Without loss of generality, we may assume $|V_1| \geq \cdots \geq |V_{r'}|$. Let r be the largest integer $i \in [r']$ such that $|V_i| \geq \epsilon^2 n$. Let $V_0 = V \setminus (\bigcup_{1\leq i\leq r} V_i)$. Clearly $V_0' \subseteq V_0$. By (a) and (b), we have $|V_0| \leq |V_0'| + \binom{r'}{2}\alpha n + r'\epsilon^2 n \leq 4\epsilon n$. So $\mathcal{P} = \{V_0, V_1, \ldots, V_r\}$ is the desired partition. $\square$

We need some definitions from [51]. Fix an integer $r > 0$, let H be a 3-graph and let $\mathcal{P} = \{V_0, V_1, \ldots, V_r\}$ be a partition of $V(H)$. The *index vector* $\mathbf{i}_{\mathcal{P}}(S) \in \mathbb{Z}^r$ of a subset $S \subset V(H)$ with respect to $\mathcal{P}$ is the vector whose coordinates are the sizes of the intersections of S with each part of $\mathcal{P}$ except V_0, i.e., $\mathbf{i}_{\mathcal{P}}(S)_{V_i} = |S \cap V_i|$ for $i \in [r]$. We call a vector $\mathbf{i} \in \mathbb{Z}^r$ an *s-vector* if all its coordinates are nonnegative and their sum equals s. Given $\mu > 0$, a 3-vector $\mathbf{v}$ is called a *μ-robust edge vector* if at least $\mu|V(H)|^3$ edges $e \in E(H)$ satisfy $\mathbf{i}_{\mathcal{P}}(e) = \mathbf{v}$. A k-vector $\mathbf{v}$ is called a *μ-robust K-vector* if at least $\mu|V(H)|^k$ copies K' of K in H satisfy $\mathbf{i}_{\mathcal{P}}(V(K')) = \mathbf{v}$. Let $I^{\mu}_{\mathcal{P}}(H)$ be the set of all μ-robust edge vectors and let $I^{\mu}_{\mathcal{P},K}(H)$ be the set of all μ-robust K-vectors. Let $L^{\mu}_{\mathcal{P},K}(H)$ be the lattice generated by the vectors of $I^{\mu}_{\mathcal{P},K}(H)$, in other words, $L^{\mu}_{\mathcal{P},K}(H)$ consists of all linear combinations of the vectors of $I^{\mu}_{\mathcal{P},K}(H)$.

Given a partition $\mathcal{P}$, $0 < \mu < 1$ and a μ-robust edge vector $\mathbf{i}$, Proposition 4.10 implies that there exists $\mu' > 0$ such that the edges with index vector $\mathbf{i}$ form at least $\mu' n^k$ copies of K with certain index vectors. For example, when $r = 2$ and $(1, 2) \in I^{\mu}_{\mathcal{P}}(H)$, we have $(a, b+c), (b, a+c)$ and $(c, a+b) \in I^{\mu'}_{\mathcal{P},K}(H)$ for some $\mu' > 0$. For $j \in [r]$, let $\mathbf{u}_j \in \mathbb{Z}^r$ be the jth *unit vector*, namely, $\mathbf{u}_j$ has 1 on the jth coordinate and 0 on other coordinates.

Given a partition $\mathcal{P} = \{V_0, V_1, \ldots, V_r\}$ of $V(H)$ provided by Lemma 4.17, the following lemma shows that $V(H) \setminus V_0$ is closed if $\mathbf{u}_j - \mathbf{u}_l \in L^{\mu'}_{\mathcal{P},K}(H)$ for all $1 \leq j < l \leq r$.

Lemma 4.18. *Given integers r, i_0 and constants $\epsilon, \beta, \mu' > 0$, there exists $\beta' > 0$ and an integer $i_0' > 0$ such that the following holds for sufficiently large n. Let H be a 3-graph with a partition $\mathcal{P} = \{V_0, V_1, \dots, V_r\}$ such that for each $j \in [r]$, $|V_j| \geq \epsilon^2 n$ and V_j is (β, i_0)-closed in H for some $\beta > 0$ and integer i_0. If $\mathbf{u}_j - \mathbf{u}_l \in L^{\mu'}_{\mathcal{P},K}(H)$ for all $1 \leq j < l \leq r$ and some μ', then $V(H) \setminus V_0$ is (β', i_0')-closed in H.*

Proof. We call a set I of k-vectors in $\mathbb{Z}^r$ *base* if all $\mathbf{u}_j - \mathbf{u}_l$, $1 \leq j < l \leq r$, can be written as linear combinations of the vectors in I, namely, there exist $a^{j,l}_{\mathbf{v}} \in \mathbb{Z}$ such that

$$\mathbf{u}_j - \mathbf{u}_l = \sum_{\mathbf{v} \in I} a^{j,l}_{\mathbf{v}} \mathbf{v}. \tag{4.6}$$

For example, the set of all k-vectors in $\mathbb{Z}^r$ is a base. We denote by $C(r, k, I)$ the largest $|a^{j,l}_{\mathbf{v}}|$ over all $\mathbf{v} \in I$ and $1 \leq j < l \leq r$ and let $C(r, k) = \max C(r, k, I)$ among all bases I. Given integers r, i_0 and constants $\epsilon, \beta, \mu' > 0$, let n be sufficiently large and in particular, $1/n \ll 1/C(r, k)$.

We claim that for any $1 \leq j < l \leq r$, any $x_j \in V_j$ and any $x_l \in V_l$ are $(\beta_{j,l}, i_{j,l})$-reachable for some $\beta_{j,l} > 0$ and some $i_{j,l} \geq i_0$. Once this is done, since $|\tilde{N}_{\beta,i_0}(v)| \geq |V_j| - 1 \geq \epsilon^2 n/2$ for any $j \in [r]$ and $v \in V_j$, we can apply Proposition 4.13 with $\epsilon^2/2$ in place of ϵ and $i_0' = \max\{i_{jl}\}$ and derive that any $x_j \in V_j$ and any $x_l \in V_l$ are $(\tilde{\beta}, i_0')$-reachable for some $\tilde{\beta} > 0$. For the same reason, any two vertices in V_j, $j \in [r]$, are (β'', i_0')-reachable for some $\beta'' > 0$. We thus conclude that any two vertices of $V(H) \setminus V_0$ are (β', i_0')-reachable with $\beta' = \min\{\tilde{\beta}, \beta''\}$.

Below we prove the claim for $j = 1$ and $l = 2$. By our assumption, there are nonnegative integers $p_{\mathbf{v}}, q_{\mathbf{v}}$, $\mathbf{v} \in I^{\mu'}_{\mathcal{P},K}(H)$, such that

$$\mathbf{u}_1 - \mathbf{u}_2 = \sum_{\mathbf{v} \in I^{\mu'}_{\mathcal{P},K}(H)} (p_{\mathbf{v}} - q_{\mathbf{v}})\mathbf{v} \quad \textit{i.e.,} \quad \sum_{\mathbf{v} \in I^{\mu'}_{\mathcal{P},K}(H)} q_{\mathbf{v}}\mathbf{v} + \mathbf{u}_1 = \sum_{\mathbf{v} \in I^{\mu'}_{\mathcal{P},K}(H)} p_{\mathbf{v}}\mathbf{v} + \mathbf{u}_2. \tag{4.7}$$

By comparing the sums of all the coordinates from two sides of either equation in (4.7), we obtain that

$$\sum_{\mathbf{v}\in I^{\mu'}_{\mathcal{P},K}(H)} p_{\mathbf{v}} = \sum_{\mathbf{v}\in I^{\mu'}_{\mathcal{P},K}(H)} q_{\mathbf{v}}$$

Denote this constant by C and thus $C \le |I^{\mu'}_{\mathcal{P},K}(H)|C(r,k) \le \binom{k+r-1}{r-1}C(r,k) < \mu' n/(4k)$, as n is sufficiently large. For each $\mathbf{v} \in I^{\mu'}_{\mathcal{P},K}(H)$, we select $p_{\mathbf{v}} + q_{\mathbf{v}}$ vertex-disjoint copies of K with index vector $\mathbf{v}$ that do not contain x_1 or x_2, and form two disjoint families $\mathcal{K}^p$ and $\mathcal{K}^q$, where $\mathcal{K}^p$ consists of $p_{\mathbf{v}}$ copies of K with index vector $\mathbf{v}$ for all $\mathbf{v} \in I^{\mu'}_{\mathcal{P},K}(H)$, and $\mathcal{K}^q$ consists of $q_{\mathbf{v}}$ vertex-disjoint copies of K with index vector $\mathbf{v}$ for all $\mathbf{v} \in I^{\mu'}_{\mathcal{P},K}(H)$. Note that $|V(\mathcal{K}^p)| = |V(\mathcal{K}^q)| = kC$. When we select any copy of K, we need to avoid at most $2kC$ vertices, which are incident to at most $2kCn^{k-1} \le \mu' n^k/2$ copies of K. Therefore, the number of choices of these copies is at least $(\mu' n^k/2)^{2C}$.

By (4.7), we have $\mathbf{i}_{\mathcal{P}}(V(\mathcal{K}^q)) + \mathbf{u}_1 = \mathbf{i}_{\mathcal{P}}(V(\mathcal{K}^p)) + \mathbf{u}_2$. Fix two vertices $x_1' \in V(\mathcal{K}^p) \cap V_1$ and $x_2' \in V(\mathcal{K}^q) \cap V_2$. Let $V(\mathcal{K}^p) \setminus \{x_1'\} = \{y_1, \dots, y_{kC-1}\}$ and $V(\mathcal{K}^q) \setminus \{x_2'\} = \{y_1', \dots, y_{kC-1}'\}$ such that for $i \in [kC-1]$, y_i and y_i' are from the same part of $\mathcal{P}$ and thus are (β, i_0)-reachable to each other. We next select a reachable (i_0k-1)-set S_i for each y_i, y_i', $i \in [kC-1]$ such that all these (i_0k-1)-sets are vertex disjoint and also disjoint from $V(\mathcal{K}^p \cup \mathcal{K}^q) \cup \{x_1, x_2\}$. Since in each step we need to avoid at most $(kC-1)(i_0k-1) + 2kC + 2$ vertices, there are at least $\frac{\beta}{2} n^{i_0k-1}$ choices for each S_i. Finally, since x_1 and x_1' and respectively, x_2 and x_2' are (β, i_0)-reachable, we pick two vertex-disjoint reachable (i_0k-1)-sets S_{x_1}, S_{x_2} for them such that they are also disjoint from all $S_1, \dots, S_{kC-1}$ and $V(\mathcal{K}^p \cup \mathcal{K}^q) \cup \{x_1, x_2\}$. Since in each step we need to avoid at most $(kC-1)(i_0k-1) + 2kC + 2 + i_0k - 1$ vertices, there are also at least $\frac{\beta}{2} n^{i_0k-1}$ choices for each of S_{x_1}, S_{x_2}. We claim that $A := \bigcup_{i\in[kC-1]} S_i \cup S_{x_1} \cup S_{x_2} \cup V(\mathcal{K}^p \cup \mathcal{K}^q)$ is a reachable $(i_0k^2C + kC + i_0k - 1)$-set for x_1 and x_2. Indeed, $H[A \cup \{x_1\}]$ contains a K-factor because all of $H[S_i \cup \{y_i\}]$ for $i \in [kC-1]$, $H[S_{x_1} \cup \{x_1, x_1'\}]$ and $\mathcal{K}^q$ contain K-factors; $H[A \cup \{x_2\}]$ contains a K-factor because all of $H[S_i \cup \{y_i'\}]$ for $i \in [kC-1]$, $H[S_{x_2} \cup \{x_2, x_2'\}]$ and $\mathcal{K}^p$ contain K-factors. There are at least

$$\frac{\left(\frac{\mu'}{2} n^k\right)^{2C} \left(\frac{\beta}{2} n^{i_0k-1}\right)^{kC+1}}{(i_0k^2C + kC + i_0k - 1)!} = \frac{\left(\frac{\mu'}{2}\right)^{2C} \left(\frac{\beta}{2}\right)^{kC+1}}{(i_0k^2C + kC + i_0k - 1)!} n^{i_0k^2C + kC + i_0k - 1}$$

such reachable sets. We thus obtain the desired $\beta_{1,2} = \left(\frac{\mu'}{2}\right)^{2C} \left(\frac{\beta}{2}\right)^{kC+1} / (i_0 k^2 C + kC + i_0 k - 1)!$ and $i_{1,2} = i_0 kC + C + i_0$. □

4.3.3 Proof of Lemma 4.3

The following simple fact will be used later for finding linear combinations of robust K-vectors.

Fact 4.19. *Let $a, b, c \in \mathbb{Z}$. If* $\gcd(a, b, c) = 1$ *and* $\gcd(b - a, c - b)$ *is odd, then* $\gcd(a + b, a + c, b + c) = 1$.

Proof. Let $l = \gcd(a + b, a + c, b + c)$. Then $l \mid (b - a)$ and $l \mid (c - b)$ and consequently $l \mid \gcd(b - a, c - b)$. Thus l is odd. On the other hand, $l \mid 2(a + b + c)$. Since l is odd, it follows that $l \mid (a + b + c)$. Consequently, $l \mid a, l \mid b$ and $l \mid c$, which implies $l \mid \gcd(a, b, c) = 1$, namely, $l = 1$. □

Proof of Lemma 4.3. Fix $\delta \geq f(a, b, c)$ and $\gamma > 0$. Let $\eta = \eta(\gamma)$ be the constant returned by Lemma 4.16. In addition, assume that $\eta \leq \min\{1/(2k), \gamma/4, \mu_1'/2\}$, where μ_1' is the constant returned by Proposition 4.10 with inputs $\mu = 1/8$, $l_1 = b$, and $l_2 = c$. Let $i_0 = 2^{\lfloor 1/(\delta+\gamma/2) \rfloor - 1}$. Let $\beta \ll \epsilon \ll \gamma$ be the constants returned by Lemma 4.17, and assume that $\epsilon \leq \eta^2/4$. We pick $0 < \mu \ll \epsilon$ and let μ' the constant returned by Proposition 4.10 with μ, $l_1 = a$, $l_2 = b$, and $l_3 = c$. We apply Lemma 4.18 with β, i_0 and μ' and get β' and i_0'. Finally, we apply Lemma 4.15 with β', η and i_0', and get $\alpha > 0$.

Let n be sufficiently large and let H be a 3-graph on n vertices such that $\delta_1(H) = (\delta + \gamma)\binom{n}{2}$. It suffices to verify the assumptions ($\triangle$) and ($\diamondsuit$) in Lemma 4.15 – Lemma 4.15 thus provides the desired vertex set W (here $|W| \leq \eta n \leq \gamma n/4$).

If $\delta_1(H) \geq (6 - 4\sqrt{2} + \gamma)\binom{n}{2}$, then ($\diamondsuit$) holds by Lemma 4.16. Otherwise by the definition of $f(a, b, c)$, we know that $a = 1$ and $\delta_1(H) \geq (\frac{1}{4} + \gamma)\binom{n}{2}$, then by Proposition 4.10, there are at least $\mu_1'(n-1)^{b+c} \geq \eta n^{k-1}$ copies of $K_{b,c}^{(2)}$ in the link graph[2] of each vertex of H (thus

[2]Given a 3-graph H and a vertex $v \in V(H)$, the link graph is defined as the graph with the vertex set $V(H) \setminus \{v\}$ and the edge set $\{S \setminus \{v\} : v \in S, S \in E(H)\}$.

($\diamondsuit$) holds).

In the rest of the proof we verify ($\triangle$) in cases depending on the type of K. We first apply Lemma 4.17 to H and obtain a partition $\mathcal{P} = \{V_0, V_1, \dots, V_r\}$ of $V(H)$ such that $|V_0| \le 4\epsilon n \le \eta^2 n$, $r \le \lfloor 1/(\delta + \gamma/2) \rfloor$, $|V_i| \ge \epsilon^2 n$ and V_i is (β, i_0)-closed in H for all $i \in [r]$. In particular, $r = 1$ when $d = \gcd(b-a, c-b)$ is even (and $\delta \ge \frac{1}{2}$); $r \le 2$ if $d \ge 3$ is odd (and $\delta \ge \frac{4}{9}$); $r \le 3$ if $d = 1$ (and $\delta \ge \frac{1}{4}$).

We are done if $r = 1$. When $r \ge 2$, we consider μ-robust edge vectors in H with respect to the partition $\mathcal{P}$. By Lemma 4.18, it suffices to verify the assumption in Lemma 4.18, that is, $(1,-1) \in L^{\mu'}_{\mathcal{P},K}(H)$ when $r = 2$ and respectively, $(1,-1,0)$, $(1,0,-1)$, $(0,1,-1) \in L^{\mu'}_{\mathcal{P},K}(H)$ when $r = 3$. For convenience, write

$$\mathbf{t}_1 = (a, b+c), \mathbf{t}_2 = (b, a+c), \mathbf{t}_3 = (c, a+b), \mathbf{t}_4 = (a+b+c, 0)$$

and

$$\mathbf{t}'_i = (a+b+c, a+b+c) - \mathbf{t}_i \text{ for } 1 \le i \le 4.$$

Claim 4.20. *For any partition $\mathcal{P}' = \{V_0, V', V''\}$ of $V(H)$ with $|V_0| \le 4\epsilon n$ and $|V''|, |V'| \ge \epsilon^2 n$, we have $(1,2)$ or $(2,1) \in I^{3\mu}_{\mathcal{P}'}(H)$.*

Proof. Without loss of generality, assume that $|V'| \le n/2$. Fix $v \in V'$. We observe that v is contained in at least ϵn^2 crossing edges (those with index vector $(1,2)$ or $(2,1)$) – otherwise $\delta_1(H) \le \binom{n/2}{2} + \epsilon n^2 + |V_0| n < (\frac{1}{4} + \gamma)\binom{n}{2}$, contradicting our assumption on $\delta_1(H)$. Hence v is in at least $\epsilon n^2/2$ edges with index vector $(1,2)$ or $\epsilon n^2/2$ edges with index vector $(2,1)$. Without loss of generality, assume that at least half of the vertices in V' are in at least $\epsilon n^2/2$ edges with index vector $(1,2)$. Thus the number of edges with index vector $(1,2)$ is at least $\frac{1}{2}\epsilon^2 n \cdot \epsilon n^2/2 \ge 3\mu n^3$ as $\mu \ll \epsilon$. This means that $(1,2) \in I^{3\mu}_{\mathcal{P}'}(H)$. $\square$

Case 1: K is of type $d \ge 3$ with d odd.

In this case, $\delta_1(H) \ge (\frac{4}{9} + \gamma)\binom{n}{2}$. Thus $r = 2$ and $\mathcal{P} = \{V_0, V_1, V_2\}$. By Claim 4.20, without loss of generality, assume that $(1,2) \in I^{\mu}_{\mathcal{P}}(H)$. If $I^{\mu}_{\mathcal{P}}(H) = \{(1,2)\}$, then assume that $|V_2| = pn$ for some $0 < p < 1$. The number of edges with index vector $(1,2)$ is at most

$$|V_1|\binom{|V_2|}{2} < (1-p)p^2n^3/2 \le \frac{4}{9}\cdot\frac{n^3}{6},$$

where equality holds when $p = 2/3$. Thus, $e(H) \le \frac{4}{9}\frac{n^3}{6} + 3\mu n^3 + |V_0|n^2 < \frac{4}{9}\binom{n}{3} + 5\epsilon n^3$ (where $3\mu n^3$ bounds the number of edges with other index vectors), contradicting our assumption on $\delta_1(H)$. Therefore, $|I^\mu_{\mathcal{P}}(H)| \ge 2$ and there are 3 possibilities: $I^\mu_{\mathcal{P}}(H) \supseteq \{(1,2),(3,0)\}$, $I^\mu_{\mathcal{P}}(H) \supseteq \{(1,2),(0,3)\}$ and $I^\mu_{\mathcal{P}}(H) \supseteq \{(1,2),(2,1)\}$. By Proposition 4.10,

$$I^{\mu'}_{\mathcal{P},K}(H) \supseteq \{\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}_4\} \text{ or } I^{\mu'}_{\mathcal{P},K}(H) \supseteq \{\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}'_4\} \text{ or } I^{\mu'}_{\mathcal{P},K}(H) \supseteq \{\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}'_1,\mathbf{t}'_2,\mathbf{t}'_3\},$$

respectively. If $\{\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}_4\} \subseteq I^{\mu'}_{\mathcal{P},K}(H)$,

$$\mathbf{t}_4-\mathbf{t}_1 = (b+c,-(b+c)),\ \mathbf{t}_4-\mathbf{t}_2 = (a+c,-(a+c)),\ \mathbf{t}_4-\mathbf{t}_3 = (a+b,-(a+b)) \in L^{\mu'}_{\mathcal{P},K}(H).$$

Since K is of type $d \ge 3$ and d is odd, Fact 4.19 implies that $\gcd(b+c,a+c,a+b) = 1$ and hence $(1,-1) = x(\mathbf{t}_4-\mathbf{t}_1)+y(\mathbf{t}_4-\mathbf{t}_2)+z(\mathbf{t}_4-\mathbf{t}_3) \in L^{\mu'}_{\mathcal{P},K}(H)$ for some integers x,y,z. Otherwise $\{\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}'_4\} \subseteq I^{\mu'}_{\mathcal{P},K}(H)$ or $\{\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3,\mathbf{t}'_1,\mathbf{t}'_2,\mathbf{t}'_3\} \subseteq I^{\mu'}_{\mathcal{P},K}(H)$, it is easy to see that in either case

$$(a,-a),(b,-b),(c,-c) \in L^{\mu'}_{\mathcal{P},K}(H).$$

Since $\gcd(a,b,c) = 1$, we conclude that $(1,-1) \in L^{\mu'}_{\mathcal{P},K}(H)$.

Case 2: K is of type 1 and $r = 2$.

By Claim 4.20, without loss of generality, assume that $(1,2) \in I^\mu_{\mathcal{P}}(H)$. By Proposition 4.10, we have

$$\mathbf{t}_1,\mathbf{t}_2,\mathbf{t}_3 \in I^{\mu'}_{\mathcal{P},K}(H),$$

and thus

$$\mathbf{t}_2-\mathbf{t}_1 = (b-a,a-b), \mathbf{t}_3-\mathbf{t}_2 = (c-b,b-c) \in L^{\mu'}_{\mathcal{P},K}(H).$$

Since $K_{a,b,c}$ is of type 1, namely, $\gcd(b-a,c-b) = 1$, we conclude that $(1,-1) \in L^{\mu'}_{\mathcal{P},K}(H)$.

Case 3: K is of type 1 and $r = 3$.

If $(1,2,0) \in I^{\mu}_{\mathcal{P}}(H)$, then the arguments in Case 2 show that $(1,-1,0) \in L^{\mu'}_{\mathcal{P},K}(H)$. If we also have $(0,1,2) \in I^{\mu}_{\mathcal{P}}(H)$, then $(0,1,-1) \in L^{\mu'}_{\mathcal{P},K}(H)$. Consequently $(1,0,-1) \in L^{\mu'}_{\mathcal{P},K}(H)$, and we are done. In general, let T be the set of all vectors with three coordinates $0,1,2$ (in any order). If

$$I^{\mu}_{\mathcal{P}}(H) \text{ contains any two members of } T \text{ not having 0 on the same coordinate,} \tag{4.8}$$

then the arguments above show that $(1,-1,0),(0,1,-1),(1,0,-1) \in L^{\mu'}_{\mathcal{P},K}(H)$.

We claim that (4.8) holds if $(1,1,1) \notin I^{\mu}_{\mathcal{P}}(H)$. In fact, we prove a stronger statement: *for each $i \in [3]$, there is a member of T in $I^{\mu}_{\mathcal{P}}(H)$ whose ith coordinate is positive.* Fix $i \in [3]$. By applying Claim 4.20 to $\mathcal{P}' = \{V_0, V_i, V_{i+1} \cup V_{i+2}\}$ (the addition is modulo 3), we may assume that at least $3\mu n^3$ edges have index vector $(1,2)$ with respect to $\mathcal{P}'$. Since $(1,1,1) \notin I^{\mu}_{\mathcal{P}}(H)$, at most μn^3 of these edges have index vector $(1,1,1)$ with respect to $\mathcal{P}$. Thus, there exists $j \neq i$ such that at least μn^3 of these edges intersect V_j with two vertices. This proves the desired statement.

What remains is the case when $(1,1,1) \in I^{\mu}_{\mathcal{P}}(H)$. In this case, by Proposition 4.10,

$$(a,b,c),(b,a,c),(a,c,b),(b,c,a),(c,a,b),(c,b,a) \in I^{\mu'}_{\mathcal{P},K}(H).$$

This implies $(y,-y,0),(0,y,-y),(y,0,-y) \in L^{\mu'}_{\mathcal{P},K}(H)$ for all $y \in \{b-a, c-b\}$. Since $\gcd(b-a, c-b) = 1$, we derive that $(1,-1,0),(0,1,-1),(1,0,-1) \in L^{\mu'}_{\mathcal{P},K}(H)$. $\square$

4.4 Proof of the Almost Tiling Lemma

4.4.1 The Weak Regularity Lemma and cluster hypergraphs

We first introduce the *Weak Regularity Lemma* for 3-graphs, a straightforward extension of Szemerédi's regularity lemma for graphs [84].

Let $H = (V,E)$ be a 3-graph and let V_1, V_2, V_3 be mutually disjoint non-empty subsets of V. We denote the number of edges with one vertex in each V_i, $i \in [3]$ by $e(V_1, V_2, V_3)$, and the density of H with respect to (V_1, V_2, V_3) by

$$d(V_1, V_2, V_3) = \frac{e(V_1, V_2, V_3)}{|V_1||V_2||V_3|}.$$

The triple (V_1, V_2, V_3) of mutually disjoint subsets $V_1, V_2, V_3 \subseteq V$ is called (ϵ, d)-*regular* for $\epsilon > 0$ and $d \geq 0$ if

$$|d(A_1, A_2, A_3) - d| \leq \epsilon$$

for all triples of subsets $A_i \subseteq V_i$, $i \in [3]$, satisfying $|A_i| \geq \epsilon |V_i|$. The triple (V_1, V_2, V_3) is called ϵ-*regular* if it is (ϵ, d)-regular for some $d \geq 0$. By definition, if $A_i \subseteq V_i$, $i \in [3]$, has size $|A_i| \geq p|V_i|$ for some $p \geq \epsilon$, then (A_1, A_2, A_3) is $(\epsilon/p, d)$-regular.

Let $H = (V, E)$ be an n-vertex 3-graph, a partition of V into $V_0, V_1, \ldots, V_t$ is called an (ϵ, t)-*regular partition* if

(i) $|V_1| = |V_2| = \cdots = |V_t|$ and $|V_0| \leq \epsilon n$,

(ii) for all but at most $\epsilon\binom{t}{3}$ sets $\{i, j, l\} \in \binom{[t]}{3}$, the triple (V_i, V_j, V_l) is ϵ-regular.

We call $V_1, \ldots, V_t$ *clusters*. Given an (ϵ, t)-regular partition $\mathcal{P} = \{V_0, V_1, V_2, \ldots, V_t\}$ and $d > 0$, the *cluster hypergraph* $\mathcal{R} = \mathcal{R}(\epsilon, d, \mathcal{P})$ is defined as the 3-graph whose vertices are clusters $V_1, \ldots, V_t$ and $\{V_i, V_j, V_l\}$ forms an edge of $\mathcal{R}$ if and only if (V_i, V_j, V_l) is ϵ-regular and $d(V_i, V_j, V_l) \geq d$.

The following is a simple corollary of the Weak Regularity Lemma; it shows that the cluster hypergraph inherits the minimum degree of the original hypergraph. Since its proof is the same as that of [9, Proposition 15], we omit the proof.

Proposition 4.21. *[9] For $0 < \epsilon < d \ll \delta$ and $t_0 \geq 0$ there exist T and n_2 such that the following holds. Suppose H is a 3-graph on $n > n_2$ vertices with $\delta_1(H) \geq \delta\binom{n}{2}$. Then there exists an (ϵ, t)-regular partition $\mathcal{P}$ with $t_0 < t < T$ such that the cluster hypergraph $\mathcal{R} = \mathcal{R}(\epsilon, d, \mathcal{P})$ satisfies $\delta_1(\mathcal{R}) \geq (\delta - \epsilon - d)\binom{t}{2}$.*

Next we show that every regular triple can be almost perfectly tiled by copies of $K_{a,b,c}$ provided the sizes of its three parts is somewhat balanced.

Proposition 4.22. *Let $a \le b \le c$ be integers, $0 < 2\epsilon \le d$, and m be sufficiently large. Suppose (V_1, V_2, V_3) is (ϵ, d)-regular, $|V_1| \le |V_2| \le |V_3| = m$, and*

$$\frac{|V_1|}{a} \ge \frac{|V_2|}{b} \ge \frac{|V_3|}{c}. \tag{4.9}$$

Then there is a $K_{a,b,c}$-tiling on $V_1 \cup V_2 \cup V_3$ covering all but at most $\frac{c}{a}\epsilon(|V_1| + |V_2| + |V_3|)$ vertices.

Proof. We will greedily pick vertex-disjoint $K_1, K_2, \ldots, K_s$ until $|V_i \setminus \bigcup_{\ell=1}^{s} V(K_\ell)| < \epsilon m$ for some $i \in [3]$, where each K_ℓ is a copy of $K_{a,b,c}$ or $K_{k,k,k}$ by the algorithm described below. This gives rise to a $K_{a,b,c}$-tiling because each copy of $K_{k,k,k}$ consists of three vertex-disjoint copies of $K_{a,b,c}$. Our algorithm is as follows. For $i \in [3]$, let $U_i^0 = V_i$. For $j \in [s]$, let

$$\left\{U_1^j, U_2^j, U_3^j\right\} = \left\{V_i \setminus \bigcup_{\ell=1}^{j} V(K_\ell) : i \in [3]\right\} \text{ such that } |U_1^j| \le |U_2^j| \le |U_3^j|,$$

and $U^j = U_1^j \cup U_2^j \cup U_3^j$. In other words, U_1^j, U_2^j, U_3^j are the subsets of V_1, V_2, V_3 obtained from removing the vertices of $K_1, \ldots, K_j$ and arranged in the ascending order of size. Suppose that we have already found $K_1, \ldots, K_j$ and $|U_1^j| \ge \epsilon m$. We let K_{j+1} be a copy of $K_{k,k,k}$ from U^j if

$$|U_3^j| - |U_1^j| \le c - a; \tag{4.10}$$

otherwise we let K_{j+1} be a copy of $K_{a,b,c}$ with a vertices from U_1^j, b vertices from U_2^j, and c vertices from U_3^j. In either case this is possible because $|U_i^j| \ge \epsilon m$ for $i \in [3]$; by the regularity, we have $d(U_1^j, U_2^j, U_3^j) \ge d - \epsilon \ge \epsilon$ and

$$e(U_1^j, U_2^j, U_3^j) \ge \epsilon |U_1^j||U_2^j||U_3^j| \ge \epsilon^4 m^3.$$

By Proposition 4.10, we can find a copy of $K_{k,k,k}$ or $K_{a,b,c}$ from U^j. The algorithm terminates when $|U_1^s| < \epsilon m$. We need to show that $|U^s| \leq \frac{c}{a}\epsilon(|V_1| + |V_2| + |V_3|)$. By (4.9), $|V_1| + |V_2| + |V_3| \geq \frac{k}{c}m$ and thus $\frac{c}{a}\epsilon(|V_1| + |V_2| + |V_3|) \geq \frac{k}{a}\epsilon m$. So it suffices to show that $|U^s| \leq \frac{k}{a}\epsilon m$.

First, assume that (4.10) holds for some $0 \leq j < s$. In this case $K_{j+1} \cong K_{k,k,k}$ and

$$|U_3^{j+1}| - |U_1^{j+1}| = |U_3^j| - |U_1^j| \leq c - a.$$

Therefore $K_\ell \cong K_{k,k,k}$ for all $\ell > j$ and consequently $|U_3^s| - |U_1^s| \leq c - a$. Since $|U_1^s| < \epsilon m$, it follows that $|U^s| < 3\epsilon m + 2(c - a)$. If $a = c$, then $|U^s| \leq 3\epsilon m = \frac{k}{a}\epsilon m$ and we are done. Otherwise $\frac{k}{a} \geq 3 + \frac{1}{a}$. Since m is large enough, it follows that $|U^s| < (3 + \frac{1}{a})\epsilon m \leq \frac{k}{a}\epsilon m$, as desired.

Second, assume that (4.10) fails for all $0 \leq j < s$. We claim that for all $0 \leq j \leq s$,

$$\frac{|U_1^j|}{a} \geq \frac{|U_2^j|}{b} \geq \frac{|U_3^j|}{c}. \tag{4.11}$$

This suffices because $|U_1^s| < \epsilon m$ and (4.11) with $j = s$ together imply that $|U^s| \leq (1 + \frac{b}{a} + \frac{c}{a})|U_1^s| < \frac{k}{a}\epsilon m$.

Let us prove (4.11) by induction. The $j = 0$ case follows from (4.9) and the assumption $|V_1| \leq |V_2| \leq |V_3|$. Suppose that (4.11) holds for some $j \geq 0$. By our algorithm, K_{j+1} is a copy of $K_{a,b,c}$ with a vertices from U_1^j, b vertices from U_2^j, and c vertices from U_3^j. Let $\tilde{U}_i^j = U_i^j \setminus V(K_{j+1})$ for $i \in [3]$ and thus $|\tilde{U}_1^j|/a = |U_1^j|/a - 1$, $|\tilde{U}_2^j|/b = |U_2^j|/b - 1$ and $|\tilde{U}_3^j|/c = |U_3^j|/c - 1$. By the inductive hypothesis,

$$\frac{|\tilde{U}_1^j|}{a} \geq \frac{|\tilde{U}_2^j|}{b} \geq \frac{|\tilde{U}_3^j|}{c}. \tag{4.12}$$

Since $|U_i^j| \geq \epsilon m \geq b + c$ for all $i \in [3]$,

$$b^2 - a^2 \leq (b - a)|U_1^j| \leq b|U_2^j| - a|U_1^j|$$

and

$$c^2 - b^2 \le (c-b)|U_2^j| \le c|U_3^j| - b|U_2^j|$$

which implies that

$$\frac{|\tilde{U}_2^j|}{a} \ge \frac{|\tilde{U}_1^j|}{b}, \quad \text{and} \quad \frac{|\tilde{U}_3^j|}{b} \ge \frac{|\tilde{U}_2^j|}{c}. \tag{4.13}$$

Now we separate cases according to the order of $|\tilde{U}_1^j|$, $|\tilde{U}_2^j|$ and $|\tilde{U}_3^j|$. Since $|\tilde{U}_3^j| - |\tilde{U}_1^j| = |U_3^j| - |U_1^j| - (c-a) > 0$, we only have three cases.

Case 1. $|\tilde{U}_1^j| \le |\tilde{U}_2^j| \le |\tilde{U}_3^j|$. Then (4.11) for $j+1$ follows from (4.12) immediately.

Case 2. $|\tilde{U}_2^j| \le |\tilde{U}_1^j| \le |\tilde{U}_3^j|$. Together with (4.12) and (4.13), we derive that

$$\frac{|\tilde{U}_2^j|}{a} \ge \frac{|\tilde{U}_1^j|}{b} \ge \frac{|\tilde{U}_2^j|}{b} \ge \frac{|\tilde{U}_3^j|}{c}.$$

Case 3. $|\tilde{U}_1^j| \le |\tilde{U}_3^j| \le |\tilde{U}_2^j|$. Together with (4.12) and (4.13), we derive that

$$\frac{|\tilde{U}_1^j|}{a} \ge \frac{|\tilde{U}_2^j|}{b} \ge \frac{|\tilde{U}_3^j|}{b} \ge \frac{|\tilde{U}_2^j|}{c}.$$

This implies that (4.11) holds for $j+1$ and we are done. □

When $a = b = c$, the proof of Lemma 4.2 is a standard application of the regularity method. This was given implicitly in [53] and stated as [66, Lemma 4.4] without a proof. For completeness, we include the proof here.

Proof of Lemma 4.2 when $a = b = c$. Let $0 < 4\epsilon = d \ll \min\{\gamma, \alpha\}$ and $t_0 = 1/\epsilon$. Suppose T and n_2 are the parameters returned by Proposition 4.21 with $\delta = 5/9 + \gamma$. Let H be a 3-graph on n vertices with $\delta_1(H) \ge (\frac{5}{9} + \gamma)\binom{n}{2}$ for some sufficiently large $n \ge n_2$. We apply Proposition 4.21 and obtain an (ϵ, t)-regular partition $\mathcal{P}$ with $t_0 < t < T$ and a cluster hypergraph $\mathcal{R} = \mathcal{R}(\epsilon, d, \mathcal{P})$ satisfying $\delta_1(\mathcal{R}) \ge (\frac{5}{9} + \gamma - \epsilon - d)\binom{t}{2}$. Suppose that $t \equiv r \bmod 3$ for some $r \in \{0, 1, 2\}$. Let $\mathcal{R}'$ be the induced subgraph of $\mathcal{R}$ on clusters $V_{r+1}, \ldots, V_t$ each of size $n/t \geq n/T$. Then $\delta_1(\mathcal{R}') \geq \delta_1(\mathcal{R}) - 2t \geq \left(\frac{5}{9} + \frac{\gamma}{2}\right)\binom{t}{2}$. We apply [36, Theorem 6] [3] to $\mathcal{R}'$ and get a perfect matching M. For each edge $e = \{V_i, V_j, V_l\} \in M$, Proposition 4.22 provides a $K_{a,b,c}$-tiling that covers all but at most $\epsilon(|V_i| + |V_j| + |V_l|)$ vertices of $V_i \cup V_j \cup V_l$. The union of these $K_{a,b,c}$-tilings covers all but at most

$$|V_0| + \epsilon(|V_1| + |V_2| + \cdots + |V_t|) + 2|V_1| \leq 2\epsilon n + 2n/t \leq 4\epsilon n \leq \alpha n$$

vertices of $V(H)$, as desired. □

We assume that $a < c$ in the next two subsections.

4.4.2 Fractional homomorphic tilings

To obtain a large $K_{a,b,c}$-tiling in H when $a < c$, we follow the idea of Buß, Hàn and Schacht [9] considering a fractional homomorphism from $K_{a,b,c}$ to the cluster hypergraph $\mathcal{R}$. Let us first define a fractional hom($K_{a,b,c}$)-tiling (hom(K)-tiling for short).

Definition 4.23. *Given a 3-graph $H = (V, E)$, a function $h : V \times E \to [0, 1]$ is called a fractional hom(K)-tiling of H if*

(1) $h(v, e) = 0$ if $v \notin e$,

(2) $h(v) = \sum_{e \in E} h(v, e) \leq 1$,

(3) for every $e \in E$ there exists a labeling $e = uvw$ such that $h(u, e) \leq h(v, e) \leq h(w, e)$ and

$$\frac{h(u, e)}{a} \geq \frac{h(v, e)}{b} \geq \frac{h(w, e)}{c}.$$

Given $e = uvw \in E$, we simply write $h(u, v, w) = (h(u, e), h(v, e), h(w, e))$ and given a constant λ, we write $\lambda(x_1, x_2, x_3) = (\lambda x_1, \lambda x_2, \lambda x_3)$. We denote by $h_{\min}$ the smallest non-

[3] We may alternatively use the exact result in [53, 63].

zero value of $h(v,e)$ *and by* $w(h)$ *the (total) weight of* h*:*

$$w(h) = \sum_{(v,e)\in V\times E} h(v,e).$$

For example, suppose that the vertex classes of K are X, Y, Z with $|X| = a$, $|Y| = b$ and $|Z| = c$. We obtain a fractional hom(K)-tiling h by letting $h(x,y,z) = (\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab})$ for every $xyz \in E(K)$ with $x \in X, y \in Y, z \in Z$. Then $w(h) = k$ (the largest possible) and $h_{\min} = \frac{1}{bc}$. We later refer to $(\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab})$ as the *standard weight* of an edge of K and refer to the function mentioned above as the *standard weight function* on K.

The following proposition shows that a fractional hom(K)-tiling in the cluster hypergraph can be "converted" to an integer K-tiling in the original hypergraph.

Proposition 4.24. *Let* $1 \le a \le b \le c$ *be integers. Suppose* $\epsilon, \phi > 0$, $d \ge 2\epsilon/\phi$, $t \in \mathbb{Z}$, *and* n *is sufficiently large. Let* H *be a* 3*-graph on* n *vertices with an* (ϵ, t)*-regular partition* $\mathcal{P}$ *and a cluster hypergraph* $\mathcal{R} = \mathcal{R}(\epsilon, d, \mathcal{P})$*. Suppose that there is a fractional hom*(K)*-tiling* h *of* $\mathcal{R}$ *with* $h_{\min} \ge \phi$*. Then there exists a* K*-tiling of* H *that covers at least* $(1 - 2c\epsilon/\phi)\, w(h)n/t$ *vertices.*

Proof. Let $\mathcal{R}'$ be the subhypergraph of $\mathcal{R}$ consisting of the hyperedges $e = uvw \in E(\mathcal{R}')$ with $h(u,e)$, $h(v,e)$, $h(w,e) \ge h_{\min} \ge \phi$. For each $u \in V(\mathcal{R}')$, let V_u be the corresponding cluster of H. Since $\mathcal{P}$ is an (ϵ, t)-regular partition, all the clusters have size ℓ for some $\ell \ge (1-\epsilon)n/t$. In each V_u we find disjoint subsets V_u^e of size $h(u,e)\ell$ for all $e \in E(\mathcal{R}')$ with $u \in e$ – this is possible because $\sum_{e\in E(\mathcal{R}')} h(u,e) \le 1$. Note that every edge $e = uvw \in E(\mathcal{R}')$ corresponds to an (ϵ, d')-regular triple (V_u, V_v, V_w) for some $d' \ge d$. Hence for every $e = uvw \in E(\mathcal{R}')$, (V_u^e, V_v^e, V_w^e) is $(\epsilon/\phi, d')$-regular with at least $\phi\ell \ge (1-\epsilon)\phi n/t$ vertices in each part. Because of Definition 4.23 (3) and assumption $d \ge 2\epsilon/\phi$, we can apply Proposition 4.22 and obtain a K-tiling covering at least

$$\left(1 - \frac{c}{a}\cdot\frac{\epsilon}{\phi}\right) h(e)\ell \ge (1 - c\epsilon/\phi)h(e)(1-\epsilon)\frac{n}{t} \ge (1 - 2c\epsilon/\phi)\, h(e)\frac{n}{t}$$

vertices of $V_u \cup V_v \cup V_w$, where $h(e) = h(u,e) + h(v,e) + h(w,e)$. Repeating this to all hyperedges of $\mathcal{R}'$, we obtain a K-tiling that covers at least

$$\sum_{uvw \in E(\mathcal{R}')} (1 - 2c\epsilon/\phi)\, h(e)\frac{n}{t} = (1 - 2c\epsilon/\phi)\, w(h)\frac{n}{t}$$

vertices of H. $\square$

Suppose H is a 3-graph satisfying the assumptions of Lemma 4.2 and $\mathcal{R}$ is the reduced graph found by Proposition 4.21. By Proposition 4.24, if the reduced graph $\mathcal{R}$ has a large K-tiling, so does H. The core of the proof of Lemma 4.2 says that if a maximum K-tiling in $\mathcal{R}$ is not large enough, then we improve it fractionally, which also gives a large K-tiling in H by Proposition 4.24. The following two propositions show how we improve the tiling fractionally.

Given a copy K_1 of K and two vertices $u, u' \notin V(K_1)$, let $\mathcal{L}_1(K_1, u, u')$ denote the family of all 3-graphs on $\{u, u'\} \cup V(K_1)$ whose edge set contains $E(K_1)$ and at least $a+1$ triples $uu'v$ with $v \in V(K_1)$.

Proposition 4.25. *Let $1 \le a \le b \le c$ be integers with $a < c$ and $k = a+b+c$. Let K_1 be a copy of $K_{a,b,c}$ and let $u, u' \notin V(K_1)$ be two vertices. For any 3-graph $L \in \mathcal{L}_1(K_1, u, u')$, there is a fractional hom(K)-tiling h of L with $w(h) \ge k + \frac{1}{abc}$ and $h_{\min} \ge \frac{1}{bc^2}$.*

Proof. Let the vertex classes of K_1 be X, Y, Z with $|X| = a$, $|Y| = b$, and $|Z| = c$. Since $\deg(uu') \ge a + 1 = |X| + 1$, we have $N(uu', Y \cup Z) \ne \emptyset$.

If there exists $z \in N(uu', Z)$, then we pick $x \in X$ and $y \in Y$ and assign weights $h(z, u, u') = (\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab})$,

$$h(x,y,z) = \left(\frac{1}{bc} - \frac{a}{bc^2}, \frac{1}{ac} - \frac{1}{c^2}, \frac{1}{ab} - \frac{1}{bc}\right) = \frac{c-a}{abc^2}\,(a,b,c)\,,$$

and assign the standard weight to all other edges of K_1. Then h is a fractional hom(K)-tiling of L with $w(h) = k + \frac{1}{ab} + \frac{1}{ac} - \frac{a}{bc^2} - \frac{1}{c^2} \ge k + \frac{1}{abc}$ and $h_{\min} = \frac{c-a}{bc^2} \ge \frac{1}{bc^2}$.

Otherwise $N(uu',Z)=\emptyset$, then there exists $y\in N(uu',Y)$. First assume $a<b$. We assign $h(y,u,u')=(\frac{1}{bc},\frac{1}{ac},\frac{1}{ab})$,

$$h(x,y,z)=\left(\frac{1}{bc}-\frac{a}{b^2c},\frac{1}{ac}-\frac{1}{bc},\frac{1}{ab}-\frac{1}{b^2}\right)=\frac{b-a}{ab^2c}\,(a,b,c)$$

for some $x\in X$ and $z\in Z$, and the standard weight to all other edges. Then h is a fractional hom(K)-tiling with $w(h)=k+\frac{1}{ab}+\frac{1}{ac}-\frac{a}{b^2c}-\frac{1}{b^2}\geq k+\frac{1}{abc}$ and $h_{\min}=\frac{b-a}{b^2c}\geq\frac{1}{bc^2}$. Second, we assume $a=b$. By the degree condition, we have $N(uu',X)\neq\emptyset$. Pick $x\in N(uu',X)$ and $z\in Z$. By assigning $h(x,u,u')=h(y,u,u')=h(x,y,z)=(\frac{1}{2ac},\frac{1}{2ac},\frac{1}{2a^2})$ and the standard weight to all other edges, we get a fractional hom(K)-tiling with $w(h)=k+\frac{1}{a^2}+\frac{1}{ac}-\frac{1}{2a^2}\geq k+\frac{1}{abc}$ and $h_{\min}=\frac{1}{2ac}\geq\frac{1}{bc^2}$ as $c\geq 2$. □

Given two vertex-disjoint copies K_1,K_2 of K and a vertex $u\notin V(K_1)\cup V(K_2)$, let $\mathcal{L}_2(K_1,K_2,u)$ denote the family of all 3-graphs on $\{u\}\cup V(K_1)\cup V(K_2)$ whose edge set contains $E(K_1)\cup E(K_2)$ and at least $\max\{a^2+2a(b+c),(a+b)^2\}+1$ triples uvw with $v\in V(K_1)$ and $w\in V(K_2)$.

The following proposition shows that any 3-graph $L\in\mathcal{L}_2(K_1,K_2,u)$ has a hom(K)-tiling with weight greater than $2k$. In its proof we assign weights to an edge as follows. Suppose $0<\lambda\leq 1$, then $(\frac{a}{c}\lambda,\frac{b}{c}\lambda,\lambda)$ satisfies (3) in Definition 4.23. Furthermore, given $\mu_1,\mu_2\geq 0$ such that $\frac{a}{c}\lambda+\mu_1\leq\frac{b}{c}\lambda\leq\lambda-\mu_2$, then $(\frac{a}{c}\lambda+\mu_1,\frac{b}{c}\lambda,\lambda-\mu_2)$ satisfies (3) in Definition 4.23 as well.

Proposition 4.26. *Let $1\leq a\leq b\leq c$ be integers with $a<c$ and $k=a+b+c$. Let K_1,K_2 be two vertex-disjoint copies of K and let $u\notin V(K_1)$ be a vertex. For any 3-graph $L\in\mathcal{L}_2(K_1,K_2,u)$, there exists a fractional hom(K)-tiling of L with $w(h)\geq 2k+\frac{1}{abc^2}$ and $h_{\min}\geq\frac{1}{bc^2}$.*

Proof. For $i=1,2$, let the vertex classes of K_i be X_i, Y_i, Z_i with $|X_i|=a$, $|Y_i|=b$, and $|Z_i|=c$. Let L_u be the link graph of u in L, that is, L_u is a bipartite graph with edges between $X_1\cup Y_1\cup Z_1$ and $X_2\cup Y_2\cup Z_2$. Then L_u satisfies the following properties.

(i) Since $\deg_L(u) \geq a^2 + 2a(b+c) + 1$, L_u must have an edge not incident to $X_1 \cup X_2$.

(ii) Since $\deg_L(u) \geq (a+b)^2 + 1$, L_u must have an edge incident to $Z_1 \cup Z_2$.

Write $\lambda = \frac{1}{abc}$. Our proof is now divided into cases based on the values of a, b and c.

Case 1. $b < c$.

First we assume that there is $z_1z_2 \in L_u$ for $z_1 \in Z_1$ and $z_2 \in Z_2$. Let $x_i \in X_i$, $y_i \in Y_i$ for $i = 1, 2$. In this case let $h(u, z_1, z_2) = (\lambda, \lambda, \lambda)$ and $h(x_1, y_1, z_1) = h(x_2, y_2, z_2) = (\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab} - \lambda)$. Other edges of K_1 or K_2 receive the standard weight $(\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab})$. (For the rest of the proof, any edge of K_1 or K_2 not specified receives the standard weight.) Therefore we get a fractional hom(K)-tiling of L with $w(h) = 2k + \lambda$ and $h_{\min} = \lambda$. We thus assume $L_u[Z_1, Z_2] = \emptyset$ and proceed in two subcases.

Case 1.1. $a < b < c$. We first assume that there exists $z_1y_2 \in L_u$ for $z_1 \in Z_1$ and $y_2 \in Y_2$. Let $x_i \in X_i$ for $i = 1, 2$, $y_1 \in Y_1$ and $z_2 \in Z_2$. In this case we let $h(y_2, z_1, u) = (\frac{a}{c}\lambda, \frac{b}{c}\lambda, \lambda)$, $h(x_1, y_1, z_1) = (\frac{1}{bc}, \frac{1}{ac}, \frac{1}{ab} - \frac{b}{c}\lambda)$, and $h(x_2, y_2, z_2) = (\frac{1}{bc}, \frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{ab} - \frac{a}{b}\lambda)$. So we get a fractional hom(K)-tiling of L with $w(h) = 2k + (1 - \frac{a}{b})\lambda \geq 2k + \frac{1}{b}\lambda$ and $h_{\min} = \frac{a}{c}\lambda$.

We thus assume that $L_u[Z_1, Y_2] = \emptyset$ and by symmetry, $L_u[Y_1, Z_2] = \emptyset$. By (i), it follows that $L_u[Y_1, Y_2] \neq \emptyset$. By (ii), without loss of generality, assume that $L_u[Z_1, X_2] \neq \emptyset$. Suppose $y_1y_2, z_1x_2 \in L_u$ with $y_1 \in Y_1$, $y_2 \in Y_2$, $z_1 \in Z_1$ and $x_2 \in X_2$. Let $x_1 \in x_1$ and $z_2 \in Z_2$. We let $h(y_1, y_2, u) = (\frac{b}{c}\lambda, \frac{b}{c}\lambda, \lambda)$, $h(x_2, u, z_1) = (\frac{a}{c}\lambda, \frac{b}{c}\lambda, \lambda)$, $h(x_1, y_1, z_1) = \left(\frac{1}{bc}, \frac{1}{ac} - \frac{b}{c}\lambda, \frac{1}{ab} - \lambda\right)$, and $h(x_2, y_2, z_2) = \left(\frac{1}{bc} - \frac{a}{c}\lambda, \frac{1}{ac} - \frac{b}{c}\lambda, \frac{1}{ab} - \lambda\right)$. So we get a fractional hom(K)-tiling of L with $w(h) = 2k + \frac{b}{c}\lambda$ and $h_{\min} = \frac{a}{c}\lambda$.

Case 1.2. $a = b < c$. We first assume that both $L_u[Z_1, X_2] \neq \emptyset$ and $L_u[Z_1, Y_2] \neq \emptyset$. Suppose $z_1x_2, z_1'y_2 \in L_u$ with $z_1, z_1' \in Z_1$, $x_2 \in X_2$ and $y_2 \in Y_2$ (we may have $z_1 = z_1'$). We assign the weights $h(z_1, x_2, u) = h(z_1', y_2, u) = (\frac{a}{c}\lambda, \frac{a}{c}\lambda, \lambda)$. If $a \geq 2$, then pick $x_1, x_1' \in X_1$, $y_1, y_1' \in Y_1$ and $z_2 \in Z_2$. We assign $h(x_1, y_1, z_1) = h(x_1', y_1', z_1') = \left(\frac{1}{ac}, \frac{1}{ac}, \frac{1}{a^2} - \frac{a}{c}\lambda\right)$ and $h(x_2, y_2, z_2) = \left(\frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{a^2} - \lambda\right)$. Otherwise $a = 1$ and $c \geq 2$. Pick $x_1 \in X_1$, $y_1 \in Y_1$

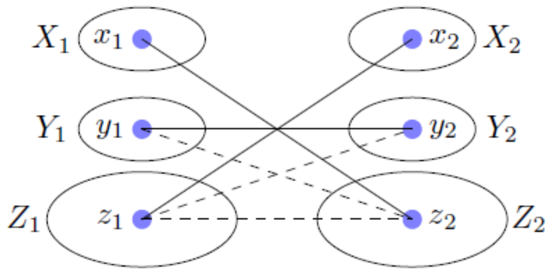

Figure 4.4. L_u in the last subcase of Case 1.2.

and $z_2 \in Z_2$. We assign

$$h(x_1, y_1, z_1) = h(x_1', y_1', z_1') = \begin{cases} \left(\frac{1}{ac}, \frac{1}{ac}, \frac{1}{a^2} - \frac{2a}{c}\lambda\right) & \text{if } z_1 = z_1', \\ \left(\frac{1}{ac}, \frac{1}{ac}, \frac{1}{a^2} - \frac{a}{c}\lambda\right) & \text{if } z_1 \neq z_1'. \end{cases}$$

(note that if $z_1 = z_1'$ then $x_1y_1z_1$ and $x_1'y_1'z_1'$ are the same edge) and $h(x_2, y_2, z_2) = \left(\frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{a^2} - \lambda\right)$. In all cases we get a fractional hom(K)-tiling of L with $w(h) = 2k + \lambda$ and $h_{\min} = \frac{a}{c}\lambda$.

We may thus assume that at least one of $L_u[Z_1, X_2]$ and $L_u[Z_1, Y_2]$ is empty, and by symmetry, at least one of $L_u[X_1, Z_2]$ and $L_u[Y_1, Z_2]$ is empty. Since $a = b$, X_i and Y_i $(i = 1, 2)$ play the same role. Without loss of generality, assume that $L_u[Z_1, Y_2] = L_u[Y_1, Z_2] = \emptyset$. Furthermore, we observe that $L_u[Z_1, X_2] \neq \emptyset$ and $L_u[X_1, Z_2] \neq \emptyset$ – otherwise, as $L_u[Z_1, Z_2] = \emptyset$, it follows that $\deg_L(u) \leq 4a^2 + ac < a^2 + 2a(b + c)$, a contradiction.

Suppose $z_1x_2, x_1z_2 \in L_u$, where $z_1 \in Z_1$, $x_2 \in X_2$, $x_1 \in X_1$, $z_2 \in Z_2$. By (i), there exists $y_1y_2 \in L_u$, where $y_1 \in Y_1$, $y_2 \in Y_2$ (see Figure 1). We assign the weights $h(u, x_2, z_1) = h(u, x_1, z_2) = h(y_1, y_2, u) = (\frac{a}{c}\lambda, \frac{a}{c}\lambda, \lambda)$ and $h(x_1, y_1, z_1) = h(x_2, y_2, z_2) = (\frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{a^2} - \lambda)$. This gives a fractional hom(K)-tiling of L with $w(h) = 2k + \lambda + \frac{2a}{c}\lambda$ and $h_{\min} = \frac{a}{c}\lambda$. Note that $h(u) = \frac{2a}{c}\lambda + \lambda = \frac{2a+c}{a^2c^2} \leq 1$ because $a \geq 1$ and $c \geq 2$. Thus this weight assignment is possible.

Case 2. $a < b = c$.

Since $b = c$, Y_i and Z_i $(i = 1, 2)$ play the same role. Thus by (i), without loss of generality, assume that there exists $z_1z_2 \in L_u$ for $z_1 \in Z_1$ and $z_2 \in Z_2$. Furthermore, generalizing (ii), we know that there must be an edge incident to $Y_1 \cup Y_2$ and without loss of generality, say that edge is incident to Y_1. We now proceed with three cases.

Case 2.1. There exists $y_1x_2 \in L_u$ where $y_1 \in Y_1$ and $x_2 \in X_2$. Pick $x_1 \in X_1$, $y_2 \in Y_2$ and $z_i \in Z_i$ for $i = 1, 2$. We assign $h(u, z_1, z_2) = (\lambda, \lambda, \lambda)$, $h(x_2, y_1, u) = (\frac{a}{c}\lambda, \lambda, \lambda)$, $h(x_1, y_1, z_1) = \left(\frac{1}{c^2}, \frac{1}{ac} - \lambda, \frac{1}{ac} - \lambda\right)$, and $h(x_2, y_2, z_2) = \left(\frac{1}{c^2} - \frac{a}{c}\lambda, \frac{1}{ac} - \lambda, \frac{1}{ac} - \lambda\right)$. Thus, we get a fractional hom(K)-tiling of L with $w(h) = 2k + \lambda$ and $h_{\min} = \frac{a}{c}\lambda$.

Case 2.2. There exists $y_1y_2 \in L_u$ where $y_1 \in Y_1$ and $y_2 \in Y_2$. Pick $x_i \in X_i$ and $z_i \in Z_i$ for $i = 1, 2$. We assign the weights $h(u, z_1, z_2) = h(u, y_1, y_2) = (\lambda, \lambda, \lambda)$ and $h(x_1, y_1, z_1) = h(x_2, y_2, z_2) = \left(\frac{1}{c^2}, \frac{1}{ac} - \lambda, \frac{1}{ac} - \lambda\right)$ and get a fractional hom(K)-tiling of L with $w(h) = 2k + 2\lambda$ and $h_{\min} = \lambda$.

Case 2.3. There exists $y_1z_2' \in L_u$ where $y_1 \in Y_1$ and $z_2' \in Z_2$ (it is possible to have $z_2 = z_2'$). Pick $z_1 \in Z_1$. We assign the weights $h(z_2, u, z_1) = h(z_2', u, y_1) = (\frac{a}{c}\lambda, \lambda, \lambda)$. Pick $x_1 \in X_1$, $x_2 \in X_2$ and distinct $y_2, y_2' \in Y_2$, which is possible as $b > a \geq 1$. Let $h(x_1, y_1, z_1) = \left(\frac{1}{c^2}, \frac{1}{ac} - \lambda, \frac{1}{ac} - \lambda\right)$ and $h(x_2, y_2, z_2) = h(x_2, y_2', z_2') = \left(\frac{1}{c^2}, \frac{1}{ac} - \frac{a}{c}\lambda, \frac{1}{ac} - \frac{a}{c}\lambda\right)$. Thus, we get a fractional hom(K)-tiling of L with $w(h) = 2k + 2\lambda - \frac{2a}{c}\lambda \geq 2k + \frac{2}{c}\lambda$ and $h_{\min} = \frac{a}{c}\lambda$.

In all cases we obtain a fractional hom(K)-tiling with $w(h) \geq 2k + \frac{\lambda}{c} = 2k + \frac{1}{abc^2}$ and $h_{\min} \geq \frac{a}{c}\lambda = \frac{1}{bc^2}$. □

4.4.3 Proof of Lemma 4.2 when $a < c$

Let H be a 3-graph on n vertices. Given $0 \leq \beta \leq 1$, a K-tiling of H is called *β-deficient* if it covers all but at most βn vertices of $V(H)$.

Proposition 4.27. *Given $0 < d \leq 3/5$ and $\beta, \rho > 0$, there exists an n_0 such that the following holds. If every 3-graph H on $n > n_0$ vertices with $\delta_1(H) \geq d\binom{n}{2}$ has a β-deficient K-tiling, then every 3-graph H' on $n' > \max\{n_0, 5\}$ vertices with $\delta_1(H') \geq (d - \rho)\binom{n'}{2}$ has a $(\beta + 2k\rho)$-deficient K-tiling.*

Proof. Let H' be a 3-graph on n' vertices with $\delta_1(H') \geq (d-\rho)\binom{n'}{2}$. By adding a set A of $2\rho n'$ new vertices and all the triples of $V(H') \cup A$ that intersect A as edges, we obtain a 3-graph H on $n = n' + 2\rho n'$ vertices. Thus

$$\delta_1(H) = \delta_1(H') + 2\rho n'(n'-1) + \binom{2\rho n'}{2} \geq (d-\rho)\binom{n'}{2} + 4\rho\binom{n'}{2} + \binom{2\rho n'}{2}.$$

Note that $3\rho\binom{n'}{2} \geq 2d\rho n'^2$ because $d \leq 3/5$ and $n' \geq 5$. Thus, $\delta_1(H) \geq d\binom{n'}{2} + 2d\rho n'^2 + d\binom{2\rho n'}{2} = d\binom{n}{2}$. By assumption, H has a β-deficient K-tiling. After removing at most $2\rho n'$ copies of K that intersect A, we obtain a $(\beta + 2k\rho)$-deficient K-tiling of H'. $\square$

Proof of Lemma 4.2 when $a < c$. Since $a < c$, we have $k \geq 4$. Let $\delta = \max\{1 - (\frac{b+c}{k})^2, (\frac{a+b}{k})^2\}$. Since $a \leq b \leq c$, it follows that $\delta \leq \max\{5/9, 4/9\} = 5/9$. Without loss of generality, assume that $0 < \gamma \leq \min\{3/5 - \delta, 2\delta, a/(3k)\}$. Assume for a contradiction that there is an α such that for all n_0 there is some 3-graph H on $n > n_0$ vertices with $\delta_1(H) \geq (\delta+\gamma)\binom{n}{2}$ but which does not contain an α-deficient K-tiling. Let α_0 be the supremum of all such α.

Let $\epsilon \ll \gamma\alpha_0$. By the definition of α_0, there is an integer n_0 such that

$$\text{all 3-graphs } H \text{ on } n > n_0 \text{ vertices with } \delta_1(H) \geq (\delta+\gamma)\binom{n}{2} \text{ have an } (\alpha_0+\epsilon)\text{-deficient } K\text{-tiling.} \tag{4.14}$$

We may also assume that n_0 is sufficiently large so that we can apply Proposition 4.10 with $r = 3$, $m = 1$, $l_1 = a$, $l_2 = b$, $l_3 = c$ on 3-graphs of order at least $\alpha_0 n_0/2$. Our goal is to show that there exists an n_1 such that all 3-graphs H on $n > n_1$ vertices with $\delta_1(H) \geq (\delta+\gamma)\binom{n}{2}$ have an $(\alpha_0 - \epsilon)$-deficient K-tiling, thus contradicting the definition of α_0.

Let n_2 and T by returned from Proposition 4.21 with inputs ϵ, $d = 2bc^2\epsilon$, $t_0 = \max\{n_0, k/\epsilon\}$. Let $n_1 = \max\{n_0, n_2\}$ and let H be a 3-graph on $n > n_1$ vertices with $\delta_1(H) \geq (\delta+\gamma)\binom{n}{2}$. We assume that H does *not* contain an $(\alpha_0 - \epsilon)$-deficient K-tiling – otherwise we are done. After applying Proposition 4.21 to H with the constants chosen above, we get an (ϵ, t)-regular partition $\mathcal{P}$ with $t_0 < t < T$ and a cluster hypergraph $\mathcal{R} = \mathcal{R}(\epsilon, d, \mathcal{P})$

on $t > t_0$ vertices with $\delta_1(\mathcal{R}) \geq (\delta+\gamma-(2bc^2+1)\epsilon)\binom{t}{2}$. By (4.14) and assumption $\delta+\gamma \leq 3/5$, we can apply Proposition 4.27 and obtain an $(\alpha_0 + \epsilon + 2k(2bc^2+1)\epsilon)$-deficient K-tiling of $\mathcal{R}$. Let $\mathcal{M} = \{K_1, K_2, \ldots, K_m\}$ be a largest K-tiling in $\mathcal{R}$ and let U be the set of uncovered vertices.

Claim 4.28. *Let h be a fractional hom(K)-tiling of $\mathcal{R}$ with $h_{\min} \geq \frac{1}{bc^2}$. Then $w(h) < (1-\alpha_0+\sqrt{\epsilon}/2)t \leq mk + \sqrt{\epsilon}t$.*

Proof. We know that $|U| \leq (\alpha_0 + \epsilon + 2k(2bc^2+1)\epsilon)t \leq (\alpha_0 + 5kbc^2\epsilon)t$. As $\epsilon \ll 1$, it follows that

$$mk + \sqrt{\epsilon}t \geq (1-\alpha_0-5kbc^2\epsilon)t + \sqrt{\epsilon}t \geq (1-\alpha_0+\sqrt{\epsilon}/2)t.$$

So it suffices to show that $w(h) < (1-\alpha_0+\sqrt{\epsilon}/2)t$. Suppose this is not the case. By Proposition 4.24, there is a K-tiling of H that covers at least

$$\left(1-2bc^3\epsilon\right) w(h)\frac{n}{t} \geq \left(1-2bc^3\epsilon\right)(1-\alpha_0+\sqrt{\epsilon}/2)t\frac{n}{t} \geq (1-\alpha_0+\epsilon)\, n$$

vertices (as $\epsilon \ll 1$). Therefore it is an $(\alpha_0-\epsilon)$-deficient K-tiling, contradicting our assumption on H. □

In the rest of the proof we will derive a contradiction to Claim 4.28. Immediately Claim 4.28 implies that

$$|U| \geq \frac{\alpha_0}{2}t \tag{4.15}$$

otherwise $\mathcal{M}$ gives a fractional hom(K)-tiling h with $w(h) = mk \geq (1-\alpha_0/2)t \geq (1-\alpha_0+\sqrt{\epsilon}/2)t$, as $\epsilon \ll \alpha_0$.

Let $E_3 = \{e \in E(\mathcal{R}) : e \subseteq U\}$ and $E_2 = \{e \in E(\mathcal{R}) : |e \cap U| = 2\}$.

Claim 4.29. $|E_3| \leq \gamma\binom{|U|}{3}/2$ *and* $|E_2| \leq \delta\binom{|U|}{2}mk$.

Proof. By (4.15) and Proposition 4.10, we have $|E_3| \leq \gamma\binom{|U|}{3}/2$, as otherwise there exists a copy of K in U, contradicting the maximality of $\mathcal{M}$.

Suppose, to the contrary, that $|E_2| > \delta\binom{|U|}{2}mk$. Recall that $\mathcal{L}_1(K_1, u, u')$ is the family of all 3-graphs on $\{u, u'\} \cup V(K_1)$ whose edge set contains $E(K_1)$ and at least $a+1$ triples $uu'v$ with $v \in V(K_1)$, where K_1 is a copy of K and $u, u' \notin V(K_1)$. Let $\mathcal{A}$ be the set of all triples iuu', $i \in [m]$, $u \neq u' \in U$ such that uu' is adjacent to at least $a+1$ vertices in K_i, in other words, $\mathcal{R}[V(K_i) \cup \{u, u'\}] \in \mathcal{L}_1(K_i, u, u')$. Let $\mathcal{A}_0$ be a largest matching in $\mathcal{A}$. By the maximality of $\mathcal{A}_0$, for any $i \in [m] \setminus V(\mathcal{A}_0)$ and any $u \neq u' \in U \setminus V(\mathcal{A}_0)$, at least $k-a$ vertices of K_i are not adjacent to uu'. Counting the number of non-edges $e \notin E(\mathcal{R})$ with $|e \cap U| = 2$, we have

$$(k-a)(m - |\mathcal{A}_0|)\binom{|U| - 2|\mathcal{A}_0|}{2} \leq \binom{|U|}{2}mk - |E_2| < (1-\delta)\binom{|U|}{2}mk.$$

Since $(1-\delta)k \leq (\frac{b+c}{k})^2 k = \frac{(k-a)^2}{k}$, it follows that

$$(m - |\mathcal{A}_0|)\binom{|U| - 2|\mathcal{A}_0|}{2} \leq \frac{k-a}{k}m\binom{|U|}{2}. \tag{4.16}$$

We claim that $|\mathcal{A}_0| \geq \gamma\alpha_0 m$. Indeed, (4.15) implies that $|U| \geq \alpha_0 t/2 \geq \alpha_0 mk/2 \geq 2\alpha_0 m$ (as $k \geq 4$). If $|\mathcal{A}_0| < \gamma\alpha_0 m$, then $m - |\mathcal{A}_0| \geq (1-\gamma\alpha_0)m$ and $|U| - 2|\mathcal{A}_0| \geq |U| - 2\gamma\alpha_0 m \geq (1-\gamma)|U|$. Thus (4.16) implies that

$$\frac{k-a}{k}m\binom{|U|}{2} \geq (1-\gamma\alpha_0)m\binom{(1-\gamma)|U|}{2} > (1-\gamma\alpha_0)(1-2\gamma)m\binom{|U|}{2} > (1-3\gamma)m\binom{|U|}{2}$$

contradicting $\gamma \leq \frac{a}{3k}$. Now let $\mathcal{A}' \subseteq \mathcal{A}_0$ be of size $\gamma\alpha_0 m$. By Proposition 4.25, for each member of $\mathcal{A}'$, there is a fractional hom(K)-tiling h' of $\mathcal{R}[V(K_i) \cup \{u, u'\}]$ with $w(h') \geq k + \frac{1}{abc}$ and $h'_{\min} \geq \frac{1}{bc^2}$. This gives rise to a fractional hom(K)-tiling h of $\mathcal{R}$ with $h_{\min} \geq \frac{1}{bc^2}$ and $w(h) \geq mk + \gamma\alpha_0 m/(abc)$.

To complete the proof, we need a lower bound for m. Recall that $\delta_1(\mathcal{R}) \geq (1 - \left(\frac{b+c}{k}\right)^2 + \gamma - (2bc^2+1)\epsilon)\binom{t}{2}$. Thus if $|U| > \frac{b+c}{k}t$, then $\binom{|U|}{2} \geq (\frac{b+c}{k})^2\binom{t}{2} - t$ and

$$\delta_1(\mathcal{R}[U]) \geq \delta_1(\mathcal{R}) - \binom{t}{2} + \binom{|U|}{2} > (\gamma - (2bc^2+1)\epsilon)\binom{t}{2} - t \geq \frac{\gamma}{2}\binom{t}{2},$$

where the last inequality holds because $t \geq t_0 \geq 1/\epsilon$. This implies that $|E_3| > \frac{1}{3}|U|\gamma\binom{t}{2}/2 > \gamma\binom{|U|}{3}/2$, contradicting the first part of Claim 4.29. Therefore $|U| \leq \frac{b+c}{k}t$ and $|V(\mathcal{M})| = mk \geq \frac{a}{k}t$, which gives $m \geq \frac{a}{k^2}t$. The fractional hom(K)-tiling h of $\mathcal{R}$ thus satisfies

$$w(h) \geq mk + \frac{\gamma\alpha_0 m}{abc} \geq mk + \frac{\gamma\alpha_0 t}{k^2bc} > mk + \sqrt{\epsilon}t,$$

as $\epsilon \ll \gamma\alpha_0$, contradicting Claim 4.28. $\square$

Let $\mathcal{T}$ be the set of all triples uij, $u \in U$, $i \neq j \in [m]$ such that there are at least $\delta k^2 + 1$ edges uvw of $\mathcal{R}$ with $v \in V(K_i)$ and $w \in V(K_j)$. Recall that given two vertex-disjoint copies K_1, K_2 of K and a vertex $u \notin V(K_1) \cup V(K_2)$, $\mathcal{L}_2(K_1, K_2, u)$ denotes the family of all 3-graphs on $\{u\} \cup V(K_1) \cup V(K_2)$ whose edge set contains $E(K_1) \cup E(K_2)$ and at least $\max\{a^2 + 2a(b+c), (a+b)^2\} + 1$ triples uvw with $v \in V(K_1)$ and $w \in V(K_2)$. Note that $\delta k^2 + 1 = \max\{a^2 + 2a(b+c), (a+b)^2\} + 1$, and thus each triple uij in $\mathcal{T}$ corresponds to a member of $\mathcal{L}_2(K_i, K_j, u)$. Let $\mathcal{T}_0$ be a largest matching in $\mathcal{T}$.

Claim 4.30. $|\mathcal{T}_0| \geq \frac{\gamma\alpha_0}{6k}t$.

Proof. We first derive a lower bound for $|\mathcal{T}|$ by considering $\sum_{u \in U} \deg_{\mathcal{R}}(u)$. Note that the edges of $\mathcal{R}$ intersecting U may contain one, two or three vertices in U. Moreover, we can partition the edges uxy of $\mathcal{R}$ with $u \in U$ and $x \in V(\mathcal{K}_i), y \in V(\mathcal{K}_j)$ (so exactly one vertex in U) into three classes: such edges with $i = j$, which is bounded above by $\binom{k}{2}m|U|$; the edges with $i \neq j$ and $uij \notin \mathcal{T}$, bounded above by $\delta k^2 |U|\binom{m}{2}$; and the edges with $i \neq j$ and $uij \in \mathcal{T}$, bounded above by $k^2|\mathcal{T}|$. Thus, we get

$$|U|\delta_1(\mathcal{R}) \leq \sum_{u \in U} \deg_{\mathcal{R}}(u) \leq 3|E_3| + 2|E_2| + \binom{k}{2}m|U| + \delta k^2 |U|\binom{m}{2} + k^2|\mathcal{T}|.$$

By Claim 4.29, it follows that

$$\begin{aligned}|U|\delta_1(\mathcal{R}) &\le |U|\left(\frac{\gamma}{2}\binom{|U|}{2} + \delta|U|mk + \binom{k}{2}m + \delta k^2\binom{m}{2}\right) + k^2|\mathcal{T}| \\ &\le |U|\left(\delta\binom{|U|}{2} + \delta|U|mk + \delta k^2\binom{m}{2} + \binom{k}{2}m\right) + k^2|\mathcal{T}| \quad \text{as } \gamma \le 2\delta \\ &\le |U|\left(\delta\binom{t}{2} + \frac{kt}{2}\right) + k^2|\mathcal{T}|.\end{aligned}$$

On the other hand, $\delta_1(\mathcal{R}) \ge (\delta + \gamma - (2bc^2+1)\epsilon)\binom{t}{2}$. Using $t \ge k/\epsilon$ and $\epsilon \ll \gamma$, we derive that $k^2|\mathcal{T}| \ge |U| \cdot \frac{\gamma}{2}\binom{t}{2}$ or $|\mathcal{T}| \ge \frac{\gamma}{2k^2}\binom{t}{2}|U|$.

By the maximality of $\mathcal{T}_0$, all triples of $\mathcal{T}$ are covered by $V(\mathcal{T}_0)$. Given some $uij \in \mathcal{T}_0$, the number of triples of $\mathcal{T}$ containing u is at most $\binom{m}{2}$ and the number of triples containing at least one of i, j is at most $2(m-1)|U|$. Also, since $mk - k + |U| = t - k$, we get that $(mk-k)|U| \le (t-k)^2/4$. Thus, we get

$$|\mathcal{T}| \le 2|\mathcal{T}_0|(m-1)|U| + \binom{m}{2}|\mathcal{T}_0| \le \frac{|\mathcal{T}_0|}{k}\binom{t}{2} + \frac{|\mathcal{T}_0|}{k^2}\binom{t}{2}$$

because $(m-1)|U| = (mk-k)|U|/k \le (t-k)^2/(4k) \le \binom{t}{2}/(2k)$. Together with $|\mathcal{T}| \ge \frac{\gamma}{2k^2}\binom{t}{2}|U|$, we derive that $|\mathcal{T}_0| \ge \frac{\gamma|U|}{2k+2} \ge \frac{\gamma\alpha_0 t}{6k}$ using (4.15). □

For every $uij \in \mathcal{T}_0$, Proposition 4.26 provides a fractional hom(K)-tiling h of $\mathcal{R}[\{u\} \cup V(K_i) \cup V(K_j)]$ with $w(h) \ge 2k + \frac{1}{abc^2}$ and $h_{\min} \ge \frac{1}{bc^2}$. Furthermore, for every $K_i \in \mathcal{M}$ with $i \notin V(\mathcal{T}_0)$, we assign the standard weight on K_i. Hence, the union of all these fractional hom(K)-tilings gives a fractional hom(K)-tiling of $\mathcal{R}$ with $h_{\min} \ge \frac{1}{bc^2}$ and

$$w(h) \ge \left(2k + \frac{1}{abc^2}\right)|\mathcal{T}_0| + k(m - 2|\mathcal{T}_0|) = mk + \frac{1}{abc^2}|\mathcal{T}_0| \ge mk + \sqrt{\epsilon}t,$$

as $\epsilon \ll \gamma\alpha_0$, contradicting Claim 4.28. This completes the proof of Lemma 4.2. □

4.5 Summary

In this part, we investigate the minimum vertex degree conditions for tiling complete 3-partite 3-graphs K. Our result is best possible, up to the error term γn^2. We remark that in some cases (*e.g.*, $K = K_{1,1,t}$ for $t \geq 2$) it seems possible to remove the error term and obtain exact results – this was done for $K_{1,1,2}$ in [15, 44]. In general, in order to obtain an exact result, we need to have a stability version of the almost tiling lemma or a stability version of the absorbing lemma, together with an analysis of the 3-graphs that look like extremal examples. (See Part 6.)

PART 5

STRONG EDGE COLORING

5.1 Introduction

Recall that *the strong edge chromatic number* of G, usually denoted by $\chi'_s(G)$, is the minimum number of colors in a strong edge-coloring of G. In this part, inspired by papers of Anderson [5] and Cranston [14], we want to get an upper bound for strong edge chromatic number of graphs with maximum degree Δ. The *girth* of a graph is the length of the shortest cycle. We prove that a Δ-regular graph G with girth at leat 5 has a strong edge-coloring that uses $2\Delta^2 - 3\Delta + 2$ colors. By applying this algorithm to graphs with maximum degree 5, we obtain a strong edge-coloring using 37 colors.

Our main results are as follows.

Theorem 5.1. *If G is a graph with maximum degree Δ and girth at least 5, then G has a strong edge-coloring that uses $2\Delta^2 - 3\Delta + 2$ colors.*

Theorem 5.2. *If G is a graph with maximum degree $\Delta = 5$, then G has a strong edge-coloring that uses 37 colors.*

For the rest of this part, we will prove Theorem 5.1 in Section 5.2, and give an outline of proof for Theorem 5.2.

5.2 Graphs with Maximum Degree Δ: Proof of Theorem 5.1

We refer to the color classes as the integers started from 1. A greedy coloring strategy is to use the least color class that is not forbidden from use on an edge at the time the edge is colored, i.e., when coloring an edge $e = xy$, we need to forbid the colors that are already used by the edges incident to x or y, as well as the colors by the edges having an end-vertex adjacent to x or y. Define the neighborhood of e as the set of edges that are incident to e, or has an end-vertex adjacent to some end-vertex of e, denoted by $N(e)$. Then $|N(e)| \leq 2\Delta(\Delta-1)$. Let $F(e)$ be the set of colors occurring on edges of $N(e)$; edges of $N(e)$ that are still uncolored do not contribute to $F(e)$, therefore $|F(e)| \leq |N(e)| \leq 2\Delta(\Delta-1)$.

Our aim is to find an order of the edges in which we can color the edges of G one by one. Let v be an arbitrary vertex of G. For $i = 0, 1, 2, ..$, let D_i be the set of vertices of distance i from v and we call D_i *distance class* i. So $D_0 = \{v\}$. For any edge e of G, its distance denoted as $d_v(e)$ is the smallest distance among end-vertices of e. We say an edge order is *compatible* with vertex v if e_1 precedes e_2 in the order only when $d_v(e_1) \geq d_v(e_2)$. Intuitively, we color all the edges in distance class $i+1$ before we color any edge in distance class i.

The following is an observation for graphs with maximum degree Δ.

Lemma 5.3. *If G is a graph with maximum degree Δ, then G has a strong edge-coloring that uses $2\Delta^2 - 3\Delta + 1$ colors except that it leaves those edges incident to a single vertex.*

Proof. Let v be a vertex of G. Greedily color the edges in an order that is compatible with v. If e is an edge not incident to v, then $d_v(e) \geq 1$, and an end-vertex x of e with $x \in D_{d(e)}$ will be adjacent to a vertex u in $D_{d(e)-1}$. When we color e, none of the Δ edges incident to u has yet been colored, so at most $2\Delta^2 - 3\Delta$ edges of $N(e)$ have been colored, i.e $|F(e)| \leq 2\Delta^2 - 3\Delta$. Hence, we get a strong edge-coloring that uses $2\Delta^2 - 3\Delta + 1$ colors except that it leaves those edges incident to v. $\square$

Now we are ready to prove Theorem 5.1 in two cases: G is not regular, and G is Δ-regular with girth at least 5

Case 1: G is not regular.

In this case, we get a even stronger result as follows.

Lemma 5.4. *Any graph with maximum degree Δ that has a vertex with degree at most $\Delta - 1$ has a strong edge-coloring that uses $2\Delta^2 - 3\Delta + 1$ colors.*

Proof. Let v be the vertex with degree at most $\Delta - 1$. Greedily color the edges in an order that is compatible with v, by Lemma 5.3 we get a partial strong edge-coloring using

$2\Delta^2 - 3\Delta + 1$ except leaving those edges incident to v. Let e_i be the edge incident to v, $|N(e_i)| \leq 2\Delta^2 - 3\Delta$, where $i = 1, 2, \ldots, \Delta - 1$. We can color those edges incident to v in the order $e_1, e_2, e_3, ..., e_{\Delta-1}$, and $|F(e_1)| \leq 2\Delta^2 - 3\Delta - \Delta + 2$, $|F(e_2)| \leq 2\Delta^2 - 3\Delta - \Delta + 3$, $\ldots$, $|F(e_{\Delta-1})| \leq 2\Delta^2 - 3\Delta$, so there are colors available for each edge incident to v. □

Case 2: G is Δ-regular with girth at least 5.

In this case, we need to prove the following lemma.

Lemma 5.5. *Any Δ-regular graph with girth at least 5 has a strong edge-coloring that uses $2\Delta^2 - 3\Delta + 2$ colors.*

Before proving Lemma 5.5, we first do some observations. Let v be a vertex of G. We want to greedily color the edges in an order that is compatible with v. By Lemma 5.3, we get a partial strong edge-coloring that uses $2\Delta^2 - 3\Delta + 1$ colors except that it leaves those edges incident to v. To finish the proof, we need to consider the local structure of those uncolored edges incident to v. By adding one more color class, we release Δ colors available to be greedily assigned to those edges incident to v.

Let D_1, D_2 be the vertex distance classes of v with distance 1 and 2, respectively. Since the girth is at leat 5, there are no induced edges within D_1, and any two distinct vertices in D_1 don't have common neighbor in D_2. Let $E[D_1, D_2]$ be the set of edges which have one end in D_1 and the other end in D_2.

Proposition 5.6. *By recoloring, we can assign the same color (say color α) to Δ edges of $E[D_1, D_2]$.*

Proof. Let $D_1 = w_1, w_2, ..., w_\Delta$. For $i = 1, 2, 3, \ldots, \Delta$, w_i has $\Delta - 1$ neighbors in D_2, and denote the set of these neighbors as W_i. Since there is no triangle in G, $w_i \cup W_i$ induces a $K_{1,\Delta-1}$. Now we give some observations as follows:

a. $W_i \cap W_j = \emptyset$, for any $i \neq j$.

b. no induced edges within D_1 .

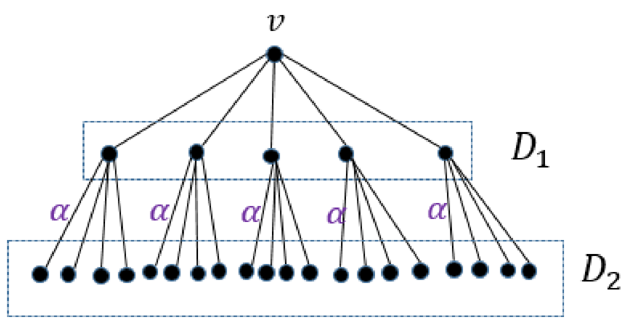

Figure 5.1. Δ edges of $E[D_1, D_2]$ receive the same color α

c. no induced edges within each $W_i, i = 1, 2, 3, \ldots, \Delta$.

d. $|N(u) \cap W_j| \leq 1$ for any $u \in W_i$ where $i \neq j$ and $i, j = 1, 2, 3, \ldots, \Delta$.

Let $N_2(u) := N(u) \cap D_2$ for any $u \in D_2$. Our goal is to find an induced matching in $E[D_1, D_2]$ with size Δ and assign new color α to them. It is sufficient to find an independent set V_0 of size Δ consisting of exactly one vertex from each W_i where $i = 1, 2, 3, \ldots, \Delta$.

Case 1. $|N_2(u)| \leq 1$ for any u in D_2. If $|N_2(u)| = 0$ for any u in D_2, i.e., there is no edge in D_2, then we can choose Δ edges in $E[D_1, D_2]$ by choosing one vertex from each $W_i, i = 1, 2, 3, \ldots, \Delta$. If there exists $u \in D_2$ such that $|N_2(u)| = 1$, note that the set of edges in D_2 is an induced matching. We choose a vertex with one neighbor in D_2, say it is from W_1, denoted as v_1. Suppose the only neighbor of v_1 in D_2 is from W_2, then we can choose one vertex from W_2 which is not adjacent to v_1, denoted as v_2. This is possible since $W_1 \cap W_2 = \emptyset$. Consider the only neighbor of v_2 in D_2, if it is not in W_1, we may assume $N_2(v_2) \subset W_3$, then we choose one vertex from W_3 which is different from this neighbor, denoted as v_3; otherwise we can arbitrarily choose one vertex from W_3 with Δ choices. Continue this process, and each step we have at least $\Delta - 1$ choices. So we get a vertex subset $V_0 \subset D_2$ of size Δ such that $E[D_1, V_0]$ is an induced matching.

Case 2. There exists a vertex u in D_2 such that $|N_2(u)| \geq 2$. Let $v_1, v_2 \in N_2(u)$, and suppose $u \in W_\Delta, v_1 \in W_1, v_2 \in W_2$. It is obvious that v_1, v_2 are nonadjacent otherwise there is a triangle. Let $V_1 = \{v_1\}$, we will choose vertices sequentially as follows:

If we already have $V_{k-1} = \{v_1, v_2, \ldots, v_{k-1}\}$, then choose $v_k \in W_k \setminus N_2(V_{k-1})$ and let $V_k = V_{k-1} \cup \{v_k\}$. This process is possible since $|W_k| = \Delta - 1$ and $|W_k \cap N_2(V_{k-1})| \leq k - 1$ because of observation (d), we get $|W_k \cap N_2(V_{k-1})| \geq (\Delta - 1) - (k - 1) \geq 1$ when $k \leq \Delta - 1$. When $k = \Delta$, since $N_2(v_1, v_2) \cap W_\Delta = \{u\}$, we have $|W_\Delta \setminus N_2(V_{\Delta-1})| \leq (\Delta - 1) - 1 = \Delta - 2 < |W_\Delta|$. So we choose $v_\Delta \in W_\Delta$ and let $V_0 = V_{k-1} \cup v_\Delta$, and $E[D_1, V_0]$ is an induced matching. □

Proof of Lemma 5.5. First by Lemma 5.3, we get a partial strong edge-coloring π with $2\Delta^2 - 3\Delta + 1$ colors except that it leaves those edges incident to some vertex v. Now consider the local structure within 2 distance classes from v, by Proposition 5.6, we can assign a new color α to Δ edges in $E[D_1, D_2]$ and release those colors used by these Δ edges in π. By greedily assign these released color to those Δ edges incident to v, we obtain a strong edge-coloring that uses $2\Delta^2 - 3\Delta + 2$ colors. □

5.3 Graphs with $\Delta = 5$

Lemma 5.3 with $\Delta = 5$ provides a partial strong edge-coloring with 36 colors. So we only need to consider the local structure within distance 2 from a single vertex v. When the girth of the graph is at least 5, Theorem 5.2 can be obtained from Lemma 5.5 since $2\Delta^2 - 3\Delta + 2 = 37$ with $\Delta = 5$. When there exists a vertex with degree less than 5, Theorem 5.2 is true by Lemma 5.4. Therefore the remaining cases are the 5-regular graphs with the girth at most 4.

G is 5-regular with girth 3.

We have the following lemma.

Lemma 5.7. *If G is a 5-regular graph with girth 3, then G has a strong edge-coloring that uses 37 colors.*

Proof. Start from a triangle $\{v_1, v_2, v_3\}$ with edges $c_1 = v_1v_2, c_2 = v_2v_3, c_3 = v_3v_1$, First by Lemma 5.3, we get a partial strong edge-coloring using 36 colors with the edges incident to

v_1 uncolored. Now release the colors used by the edges incident to v_2 and edges incident to v_3, and we have 12 uncolored edges. Assign colors to all the edges incident to the triangle first and then the edges on the triangle. Since any edge e incident to the triangle, we have $|N(e)| = 39$, and $|F(e)| \leq 39 - 3 < 37$, we can greedily color it. For $i = 1, 2, 3, |N(c_i)| = 35, |F(c_i)| \leq 35 < 37$, we can also greedily color c_1, c_2, c_3. $\square$

G is 5-regular with girth 4.

Lemma 5.8. *If G is a 5-regular graph with girth 4, then G has a strong edge-coloring that uses 37 colors.*

Proof. Let G be a 5-regular graph with girth 4. Let v be a vertex on a 4-cycle of G. Color the edges in an order compatible with v, by Lemma 5.3, we get a partial strong edge-coloring with 36 colors.

Let $e_i = vw_i$ and $W_i = N_(w_i) \cap D_2$ where $i = 1, 2, 3, 4, 5$. Suppose w_1 and w_2 have a common neighbor in D_2. Because girth is 4, we have D_1 is independent. Observation:

(a) Since $|N(e_i)| = 40 - |E(D_1 \setminus \{w_i\}, W_i)|$, if $|E(D_1 \setminus \{w_i\}, W_i)| \geq 4$ then we can greedily color e_i where $i = 1, \ldots, 5$.

(b) $|D_2| \geq 7$. Otherwise, we can greedily color e_i where $i = 1, \ldots, 5$.

Since $|F(e_i)| \leq 39 - 4 = 35$ for $i = 1, 2$ and $|F(e_i)| \leq 40 - 4 = 36$ when $i = 3, 4, 5$, we'll have a similar argument with the proof of Lemma 5.5 to show that we can reassign a new color to 3 edges in $E(D_1, D_2)$, otherwise the neighborhood of e_i where $i = 1, 2, 3, 4, 5$ is small enough for us to greedily color it. If we have at least three W_i that contains vertices with no neighbor in $D_2 \cup W_i \setminus \{w_i\}$, then we can choose an induced matching in $E(D_1, D_2)$ of size at least three. Otherwise we have the following cases.

Case 1. There are two W_i, say W_4, W_5 that contains vertices with no neighbor in $D_2 \cup W_i \setminus \{w_i\}$. Choose $v_4 \in W_4, v_5 \in W_5$ such that $N(v_i) \cap (D_2 \cup W_i \setminus \{w_i\}) = \emptyset$ for $i = 4, 5$. So we only need to choose one vertex $v_i \in W_i$ for some $i = 1, 2, 3$ such that w_4v_4, w_5v_5, w_iv_i form an induced matching. If such vertex does not exist, then any $v \in \cup_{i=1}^{3} W_i$, v is adjacent to w_1 or w_2. Because w_1 and w_2 have a common neighbor in D_2, we have $| \cup_{i=3}^{5} W_i| \leq 6$.

Therefore, $|E(D_1 \setminus \{w_i\}, W_i)| \geq 4$ for each $i = 1, 2, 3$.

Case 2. There is one W_i, say W_5 that contains vertices with no neighbor in $D_2 \cup W_i \setminus \{w_i\}$, and choose one such vertex as v_5. By observation (b), we have at least one vertex $v \in D_2 \subset W_5$, say $v \in W_4$, then v_5w_5 and vw_4 is an induced matching. Following the same argument in Case 1, we either find an induced matching of size 3 or $|E(D_1 \setminus \{w_i\}, W_i)| \geq 4$ for each $i \in [5]$.

Case 3. There is no W_i that contains vertices with no neighbor in $D_2 \cup W_i \setminus \{w_i\}$. Let v be the common neighbor of w_1 and w_2 in D_2. If v is adjacent to all w_i, then $|E(D_1 \setminus \{w_i\}, W_i)| \geq 4$ for each $i = 1, \ldots, 5$. We may assume v is not adjacent to w_5. Since w_i has at most 3 neighbors in W_5 for $i = 1, 2$, we have either v is adjacent to at least one vertex in W_5 or at least one of w_1, w_2 is adjacent to any vertex in W_5. In either case, we can find an induced matching of size 3 in $E(D_1, D_2)$ otherwise $|E(D_1 \setminus \{w_i\}, W_i)| \geq 4$ for each $i \in [5]$. □

5.4 Summary

In this part, we provide an algorithm to find a strong edge-coloring using $2\Delta^2 - 3\Delta + 2$ colors in graphs with maximum degree Δ and girth at least 5. With the help of this algorithm, we get a strong edge-coloring with 37 colors for graphs with maximum degree 5. As for our knowledge, this is the best upper bound known for $\Delta = 5$. However, by Conjecture 1.4, every graph with maximum degree 5 has a strong edge-coring using 29. We are still far from finishing this journey in finding strong chromatic numbers.

PART 6

CONCLUSION REMARKS

In Part 2, we provide the partite minimum codegree condition for (almost) perfect matchings in k-partite k-graphs. So far we show that a k-partite k-graph with each part of size n, three sufficiently large partite minimum codegrees and sum of all partite codegrees at least $n-1$ has a matching of size at least $n-1$. If we only have two sufficiently large partite minimum codegrees, we encounter the divisibility barriers and the clarity of these divisibility barriers would be the key to tackle this problem.

In Part 3, we investigate the minimum vertex degree conditions for tiling complete 3-partite 3-graphs K. Our result is best possible, up to the error term γn^2. We remark that in some cases (*e.g.*, $K = K_{1,1,t}$ for $t \geq 2$) it seems possible to remove the error term and obtain exact results – this was done for $K_{1,1,2}$ in [15, 44]. In general, in order to obtain an exact result, we need to have a stability version of the almost tiling lemma or a stability version of the absorbing lemma, together with an analysis of the 3-graphs that look like extremal examples. In many cases, when analyzing extremal examples, we need to know $\mathrm{ex}_1(n, K)$, the vertex-degree Turán number for K, which is a challenging question in general. (The *generalized Turán number* of $\mathrm{ex}_d(n, F)$ of an r-graph F is the smallest integer t such that every r-graph H of order n with $\delta_d(H) \geq t+1$ contains a copy of F.)

When proving the lower bound of Theorem 4.1, we introduced the covering barrier. In general, given an r-graph F, let $c_d(n, F)$ denote the minimum integer c such that every r-graph H of order n with $\delta_d(H) \geq c$ has the property that every vertex of H is covered by some copy of F. When F is a graph, it is not hard to see that $c_1(n, F) = (1 - 1/(\chi(F)-1) + o(1))n$: the lower bound follows from the $(\chi(F)-1)$-partite Turán graph, and the upper bound can be derived after applying the Regularity Lemma to $V(H) \setminus \{v\}$ for an arbitrary vertex v (see

[?] for details). Given an r-graph F, trivially

$$\mathrm{ex}_d(n, F) < c_d(n, F) \le t_d(n, F). \tag{6.1}$$

We know that $c_1(n, F) = \mathrm{ex}_1(n, F) + o(n)$ for all 2-graphs F. Construction 4.9 and Lemma 4.16 together show that $c_1(n, K_{a,b,c}) = (6 - 4\sqrt{2} + o(1))\binom{n}{2}$ if $2 \le a \le b \le c$, while Theorem 4.1 shows that $t_1(n, K_{a,b,c}) = (6 - 4\sqrt{2} + o(1))\binom{n}{2}$ for certain a, b, c (for example, $K_{2,3,6}$). This shows that the upper bound for $c_d(n, F)$ in (6.1) could be asymptotically tight as well. For small 3-graphs F, determining $c_2(n, F)$ seems easier than determining $\mathrm{ex}_2(n, F)$ or $t_2(n, F)$ (known as two difficult problems) – see [?] for recent progress.

Let us give the following constructions of space barriers for complete r-partite r-graph tilings for arbitrary r.

Construction 6.1. *Fix positive integers $i < r$ and $a_1 \le \cdots \le a_r$. Let $s = a_1 + \cdots + a_r$ and H_i be an n-vertex r-graph with $V(H_i) = A_i \cup B_i$ and $|A_i| = (a_1 + \cdots + a_i)n/s - 1$ such that $E(H_i)$ consists of all r-tuples containing at least i vertices of A_i.*

To see why H_i does not contain a $K_{a_1,\dots,a_r}$-factor, we observe that for each copy of $K_{a_1,\dots,a_r}$, at least i color classes of it are subsets of A_i, and thus at least $a_1 + \cdots + a_i$ vertices of it are in A_i. Since $|A_i| < (a_1 + \cdots + a_i)n/s$, there is no $K_{a_1,\dots,a_r}$-factor of H_i. Thus the minimum d-degree threshold for (almost) perfect $K_{a_1,\dots,a_r}$-tiling is at least $\max_{i\in[r-1]} \delta_d(H_i)$. Note that $\delta_d(H_{r-d+1}) = 0$ since any d-set in B_{r-d+1} has degree zero. Thus, $\max_{i\in[r-1]} \delta_d(H_i) = \max_{i\in[r-d]} \delta_d(H_i)$. This means that there are $r - d$ space barriers, *e.g.*, there is only one construction for the $(r-1)$-degree case, and there are two constructions for the vertex degree threshold in 3-graphs.

Since our main idea of proving Lemma 4.2 (see also [43]) is to analyze the bipartite link graph of any uncovered vertex on two existing copies of K in the partial tiling, new ideas are needed to attack the general vertex degree tiling problem. On the other hand, this also means that it is possible to extend our result to tiling r-partite r-graphs under minimum $(r-2)$-degree, provided a corresponding absorbing lemma.

Another direction to strengthen the result of this paper is to study the minimum vertex degree conditions for non-complete 3-partite 3-graphs. Clearly if F is a spanning subgraph of $K_{a,b,c}$ then $t_1(n, F) \leq t_1(n, K_{a,b,c})$. Note that there may be more than one choice of $K_{a,b,c}$ that contains F as a spanning subgraph. It seems not clear whether $t_1(n, F) = \min t_1(n, K_{a,b,c})$, where the minimum is over all $K_{a,b,c}$ that contain F as a spanning subgraph.

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