8-12-2016

Minimum Ranks and Refined Inertias of Sign Pattern Matrices

Wei Gao

MINIMUM RANKS AND REFINED INERTIAS OF SIGN PATTERN MATRICES

by

WEI GAO

Under the Direction of Zhongshan Li, PhD

ABSTRACT

A sign pattern is a matrix whose entries are from the set $\{+, -, 0\}$. This thesis contains problems about refined inertias and minimum ranks of sign patterns.

The refined inertia of a square real matrix B, denoted $\mathrm{ri}(B)$, is the ordered 4-tuple $(n_+(B),\ n_-(B),\ n_z(B),\ 2n_p(B))$, where $n_+(B)$ (resp., $n_-(B)$) is the number of eigenvalues of B with positive (resp., negative) real part, $n_z(B)$ is the number of zero eigenvalues of B, and $2n_p(B)$ is the number of pure imaginary eigenvalues of B. The minimum rank (resp., rational minimum rank) of a sign pattern matrix $\mathcal{A}$ is the minimum of the ranks of the real (resp., rational) matrices whose entries have signs equal to the corresponding entries of $\mathcal{A}$.

First, we identify all minimal critical sets of inertias and refined inertias for full sign patterns of order 3. Then we characterize the star sign patterns of order $n \geq 5$ that require the set of refined inertias $\mathbb{H}_n = \{(0,n,0,0),(0,n-2,0,2),(2,n-2,0,0)\}$, which is an important set for the onset of Hopf bifurcation in dynamical systems. Finally, we establish a direct connection between condensed $m \times n$ sign patterns and zero-nonzero patterns with minimum rank r and m point-n hyperplane configurations in $\mathbb{R}^{r-1}$. Some results about the rational realizability of the minimum ranks of sign patterns or zero-nonzero patterns are obtained.

INDEX WORDS: Sign pattern, Refined inertia, Critical set of refined inertias, Star sign pattern, Minimum rank, Rational minimum rank, Point-hyperplane configuration.

MINIMUM RANKS AND REFINED INERTIAS OF SIGN PATTERN MATRICES

by

WEI GAO

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

in the College of Arts and Sciences

Georgia State University

2016

MINIMUM RANKS AND REFINED INERTIAS OF SIGN PATTERN MATRICES

by

WEI GAO

Committee Chair: Zhongshan Li

Committee: Marina Arav

Guantao Chen

Frank Hall

Hendricus van der Holst

Electronic Version Approved:

Office of Graduate Studies

College of Arts and Sciences

Georgia State University

August 2016

DEDICATION

This dissertation is dedicated to my parents.

ACKNOWLEDGEMENTS

The completion of this study could not have been possible without the participation and assistance of so many people whose names may not all be enumerated. However, I would like to express my immeasurable appreciation and deepest gratitude to the following persons:

Prof. Zhongshan Li, my research adviser, for the support, guidance, valuable comments that benefited me much in the completion and success of this study. He also gave me selfless care and help in doing research. As my mentor, he shared his knowledge both in mathematics and in life.

Prof. Guantao Chen, Chair of the department, for serving on my dissertation committee and for his time and effort in checking this manuscript. The second chapter of this thesis was initiated in Prof. Chens graduate course "Topics in Mathematics".

Prof. Frank J. Hall, Dr. Hendricus van der Holst, Dr. Marina Arav, for serving on my dissertation committee and for their time and effort in checking this manuscript, also for educating me in various courses.

My parents, Mr. Yubin Gao and Mrs. Yanling Shao, for their love, caring, understanding and for their support in both research aspect and financial aspect throughout my graduate program.

The matrix group in Georgia State University, Guangming Jing and Fei Gong, for the helpful research discussions that we have had during the past four years, many of which led to successful collaborations.

Jie Han, my boyfriend, for caring and help in research and in life.

At last, I would like to thank everyone who has helped me throughout my life at Atlanta.

TABLE OF CONTENTS

LIST OF FIGURES

PART 1

INTRODUCTION

The origin of sign pattern matrices dates back to 1947, in the book *Foundations of Economic Analysis* [42] by the Nobel Economics Prize winner Paul Anthony Samuelson. He discussed the possibility of determining unambiguously the qualitative behavior of solution values of a system of equations.

In economic analysis one may not know the exact quantitative relationships between different variables, but there may be some qualitative information such as that one quantity rises if and only if another does. For instance, it is generally agreed that the supply of a particular commodity increases as the price increases, even though the exact dependence may vary. Thus we may want to deduce qualitative information about the solution to a linear system $Ax = b$ from the knowledge of the sign patterns of the matrix A and vector b.

In Samuelson's pioneering paper [36] Lancaster put it in this way: *Economists believed for a very long time, and most economists would still hope it to be so, that a considerable body of sensible economic proposition could be expressed in a qualitative way, that is, in a form in which the algebraic sign of some effect is predicted from a knowledge of the signs, only, of the relevant structural parameters of the system.* Also, sign pattern matrices have found new applications in a number of areas such as communication complexity, neural networks, chemistry and so on.

A *sign pattern* (*matrix*) is a matrix whose entries are from the set $\{+, -, 0\}$. If no entry of $\mathcal{A}$ is zero, then $\mathcal{A}$ is said to be a *full sign pattern.* For a real matrix B, $\text{sgn}(B)$ is the sign pattern matrix obtained by replacing each positive (resp., negative) entry of B by $+$ (resp., $-$). For an $n \times n$ sign pattern matrix $\mathcal{A}$, the qualitative class of $\mathcal{A}$, denoted by $Q(\mathcal{A})$, is defined as $Q(\mathcal{A}) = \{B \in M_n(\mathbb{R}) \mid \text{sgn}(B) = \mathcal{A}\}$.

For example, if

$$\mathcal{A} = \begin{bmatrix} + & - \\ 0 & - \end{bmatrix}, \quad B = \begin{bmatrix} 7 & -2 \\ 0 & -9 \end{bmatrix},$$

then $B \in Q(\mathcal{A})$.

A *subpattern* of an $n \times n$ sign pattern $\mathcal{A}$ is an $n \times n$ sign pattern $\mathcal{B}$ obtained by replacing some (possibly empty) subset of the nonzero entries of $\mathcal{A}$ with zeros. If $\mathcal{B}$ is a subpattern of $\mathcal{A}$, then $\mathcal{A}$ is a *superpattern* of $\mathcal{B}$.

Suppose P is a property referring to a real matrix. A sign pattern $\mathcal{A}$ is said to *require* P if every real matrix in $Q(\mathcal{A})$ has property P, and $\mathcal{A}$ is said to *allow* P if some real matrix in $Q(\mathcal{A})$ has property P.

By the undirected graph G of an $n \times n$ sign pattern matrix $\mathcal{A} = (a_{ij})$, we mean a graph on n vertices $\{1, 2, \dots, n\}$ with an undirected edge between i and j if and only if $a_{ij} \neq 0$ or $a_{ji} \neq 0$.

Because of the interplay between sign pattern matrices and graph theory, the study of sign patterns is regarded as a part of combinatorial matrix theory. The 1987 dissertation of Eschenbach, directed by Johnson, studied sign patterns that require or allow certain properties and summarized the work on sign patterns up to that point. In 1995, Brualdi and Shader produced a thorough treatment *Matrices of Sign-Solvable Linear Systems* on sign pattern matrices from the sign-solvability vantage point. Since 1995 there has been a considerable number of papers on sign patterns and some generalized notions (e.g. ray patterns). [28] is a survey of important results on this topic.

A *permutation sign pattern* is a square sign pattern with entries from the set $\{0, +\}$, where the entry $+$ occurs precisely once in each row and in each column. A *signature sign pattern* is a square diagonal sign pattern all of whose diagonal entries are nonzero. Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two square sign patterns of the same order. Sign pattern $\mathcal{A}_1$ is said to be *permutationally similar* to $\mathcal{A}_2$ if there exists a permutation sign pattern $\mathcal{P}$ such that $\mathcal{A}_2 = \mathcal{P}^T \mathcal{A}_1 \mathcal{P}$. Sign pattern $\mathcal{A}_1$ is said to be *signature similar* to $\mathcal{A}_2$ if there exists a signature sign pattern $\mathcal{D}$ such that $\mathcal{A}_2 = \mathcal{D}\mathcal{A}_1\mathcal{D}$. Let $\mathcal{A}_3$ and $\mathcal{A}_4$ be two $m \times n$ sign patterns. $\mathcal{A}_3$ and

$\mathcal{A}_4$ are said to be *permutationally equivalent* if there exist permutation sign patterns $\mathcal{P}_1$ and $\mathcal{P}_2$ such that $\mathcal{A}_4 = \mathcal{P}_1\mathcal{A}_3\mathcal{P}_2$. $\mathcal{A}_3$ and $\mathcal{A}_4$ are said to be *signature equivalent* if there exist signature sign patterns $\mathcal{D}_1$ and $\mathcal{D}_2$ such that $\mathcal{A}_4 = \mathcal{D}_1\mathcal{A}_3\mathcal{D}_2$.

1.1 Refined inertias of sign pattern matrices

Let B be a real matrix of order n. The *inertia* of B is the ordered triple $\mathrm{i}(B) = (n_+(B), n_-(B), n_0(B))$, in which $n_+(B)$, $n_-(B)$ and $n_0(B)$ are the numbers of its eigenvalues (counted with multiplicities) with positive, negative and zero real parts, respectively. The concept of the refined inertia of a real matrix, which was introduced by Kim et al. in [33], splits the number of zero eigenvalues from the number of other eigenvalues on the imaginary axis in the definition of the inertia of a matrix. The *refined inertia* of B is the ordered quadruple $\mathrm{ri}(B) = (n_+(B), n_-(B), n_z(B), 2n_p(B))$ of nonnegative integers summing to n, where $n_+(B)$ (resp., $n_-(B)$) is the number of eigenvalues of B with positive (resp., negative) real part, and $n_z(B)$ (resp., $2n_p(B)$) is the number of zero (resp., nonzero pure imaginary) eigenvalues of B. For $n \times n$ sign pattern $\mathcal{A}$, the *inertia* of $\mathcal{A}$ is $\mathrm{i}(\mathcal{A}) = \{\mathrm{i}(B) \mid B \in Q(\mathcal{A})\}$, and the *refined inertia* of $\mathcal{A}$ is $\mathrm{ri}(\mathcal{A}) = \{\mathrm{ri}(B) \mid B \in Q(\mathcal{A})\}$.

1.1.1 Minimal critical sets of refined inertias of sign patterns

An $n \times n$ sign pattern $\mathcal{A}$ is called an *inertially arbitrary pattern* (IAP) if for every ordered triple (n_+, n_-, n_0) of nonnegative integers with $n_+ + n_- + n_0 = n$, there exists a real matrix $B \in Q(\mathcal{A})$ such that $\mathrm{i}(B) = (n_+, n_-, n_0)$, see, e.g., [16, 23]. Similarly, $\mathcal{A}$ is called a *refined inertially arbitrary pattern* (rIAP) if for every ordered 4-tuple $(n_+, n_-, n_z, 2n_p)$ of nonnegative integers (with $2n_p$ even) that sum to n, there exists a real matrix $B \in Q(\mathcal{A})$ such that $\mathrm{ri}(B) = (n_+, n_-, n_z, 2n_p)$, see, e.g., [14]. If every multiset of n complex numbers that is closed under conjugation is the spectrum of some $B \in Q(\mathcal{A})$, then $\mathcal{A}$ is said to be a *spectrally arbitrary pattern* (SAP).

Currently, there are some techniques for identifying inertially arbitrary patterns (see [11], for example). The Nilpotent-Jacobian method and the Nilpotent-Centralizer method are also used to prove that a sign pattern is spectrally (and hence inertially) arbitrary. When the available methods cannot be used conveniently, in order to prove that a sign pattern $\mathcal{A}$ is inertially arbitrary, for each inertia (n_+, n_-, n_0), a real matrix $B \in Q(\mathcal{A})$ has to be found to determine if $(n_+, n_-, n_0) \in \mathrm{i}(\mathcal{A})$. The following concept of a critical set eliminates the need to do this.

Let S be a proper subset of the set of all possible $(n+1)(n+2)/2$ inertias of $n \times n$ real matrices. Then S is called a *critical set of inertias* for a family $\mathcal{F}$ of sign patterns of order n if for every $\mathcal{A} \in \mathcal{F}$, $S \subseteq \mathrm{i}(\mathcal{A})$ ensures that $\mathcal{A}$ is inertially arbitrary ([33]). If no proper subset of S is a critical set of inertias for $\mathcal{F}$, then S is called a *minimal critical set of inertias* for $\mathcal{F}$. When the family $\mathcal{F}$ is the set of all $n \times n$ sign patterns, we drop the reference to $\mathcal{F}$ in these definitions.

Similarly, let S be a proper subset of the set of all possible refined inertias of $n \times n$ real matrices. Then S is called a *critical set of refined inertias* for a family $\mathcal{F}$ of sign patterns of order n if for every $\mathcal{A} \in \mathcal{F}$, $S \subseteq \mathrm{ri}(\mathcal{A})$ ensures that $\mathcal{A}$ is refined inertially arbitrary ([47]). If no proper subset of S is a critical set of refined inertias for $\mathcal{F}$, then S is called a *minimal critical set of refined inertias* for the family $\mathcal{F}$.

Minimal critical sets of refined inertias (or inertias) for sign patterns (or zero-nonzero patterns) have been investigated in several recent papers. Kim et al. in [33] give the minimal critical sets of inertias for irreducible zero-nonzero patterns of orders $n = 2, 3, 4$ and for irreducible sign patterns of orders $n = 2, 3$. Yu et al. in [47] obtain all the minimal critical sets of refined inertias and inertias for irreducible zero-nonzero patterns of orders 2 and 3. Yu in [46] identifies all the minimal critical sets of refined inertias for irreducible sign patterns of orders 2. Pereira in [40] shows that every potentially nilpotent full sign pattern is spectrally arbitrary and hence inertially arbitrary. This result implies that $\{(0, 0, n, 0)\}$ with the minimal cardinality 1 is a minimal critical set of refined inertias for $n \times n$ full sign patterns for all $n \geq 2$. In [20], Li, Zhang and I identify all the minimal critical sets of refined inertias and inertias for full sign patterns of order 3. The results and proofs are shown in Chapter 2.

1.1.2 Sign patterns that require $\mathbb{H}_n$

Substantial work has been done on the research of refined inertias. One of the most important studies focuses on three particular refined inertias for a sign pattern. This is motivated by the fact that in a dynamical system, the presence of nonzero pure imaginary eigenvalues can signal the onset of periodic solutions by Hopf bifurcation. This may occur as a parameter varies if in the linearized matrix, the eigenvalues move from all having negative real parts to a pair of pure imaginary eigenvalues that cross into the right half plane to have positive real parts. For a system of order n, this corresponds to the refined inertia going from $(0, n, 0, 0)$ to $(0, n-2, 0, 2)$ to $(2, n-2, 0, 0)$. With this motivation, Bodine et al. in [8] define $\mathbb{H}_n = \{(0, n, 0, 0),\ (0, n-2, 0, 2),\ (2, n-2, 0, 0)\}$ and consider sign patterns that require or allow $\mathbb{H}_n$. An $n \times n$ sign pattern $\mathcal{A}$ *requires* $\mathbb{H}_n$ if $\mathbb{H}_n = \mathrm{ri}(\mathcal{A})$, and $\mathcal{A}$ *allows* $\mathbb{H}_n$ if $\mathbb{H}_n \subseteq \mathrm{ri}(\mathcal{A})$.

A square sign pattern $\mathcal{A}$ is *reducible* if it is permutationally similar to a pattern of the form

$$\begin{bmatrix} \mathcal{A}_{11} & \mathcal{A}_{12} \\ 0 & \mathcal{A}_{22} \end{bmatrix},$$

where $\mathcal{A}_{11}$, $\mathcal{A}_{22}$ are square and non-vacuous. A square sign pattern is *irreducible* if it is not reducible.

For a square sign pattern whose undirected graph is a tree (possibly with loops), it is a fact that such a sign pattern is irreducible if and only if it is combinatorially symmetric, i.e., $a_{ij} \neq 0$ whenever $a_{ji} \neq 0$. We call such an irreducible sign pattern a *tree sign pattern* (t.s.p.).

Sign patterns allowing or requiring $\mathbb{H}_n$ have been the focus of several recent papers.

In [8], Bodine et al. investigate sign patterns that require or allow $\mathbb{H}_n$ for small values of n and for patterns with negative diagonal entries. Some examples are given related to these concepts. It is easy to verify that no sign pattern requires $\mathbb{H}_2$. The sign pattern

$$\begin{bmatrix} + & + \\ - & - \end{bmatrix}$$

is the only sign pattern that allows $\mathbb{H}_2$. Up to equivalence, all sign patterns that require $\mathbb{H}_3$ are given. Also, the authors conjecture that no $n \times n$ *irreducible* sign pattern that requires $\mathbb{H}_n$ exists for n sufficiently large, possibly $n \geq 8$.

Focusing on tree sign patterns, Bodine et al. [8] give a necessary and sufficient condition for a 3×3 tree sign pattern to require $\mathbb{H}_3$.

Let $\mathcal{A}$ be an $n \times n$ sign pattern. Then $\mathcal{A}$ is *potentially stable* if there is a matrix $B \in Q(\mathcal{A})$ such that $n_-(B) = n$. $\mathcal{A}$ is *sign stable* if $n_-(B) = n$ for every matrix $B \in Q(\mathcal{A})$. $\mathcal{A}$ is *sign nonsingular* if $n_z(B) = 0$ (i.e., $\det(B) \neq 0$) for all $B \in Q(\mathcal{A})$.

Theorem 1.1.1 ([8]). *A 3×3 tree sign pattern requires $\mathbb{H}_3$ if and only if it is potentially stable and sign nonsingular, but not sign stable.*

Up to equivalence, there are exactly three 3×3 star sign patterns that require $\mathbb{H}_3$:

$$\begin{bmatrix} - & + & + \\ - & 0 & 0 \\ + & 0 & - \end{bmatrix}, \begin{bmatrix} - & + & + \\ + & 0 & 0 \\ - & 0 & + \end{bmatrix}, \begin{bmatrix} + & + & + \\ - & 0 & 0 \\ - & 0 & - \end{bmatrix}.$$

In [24], Garnett et al. consider tree sign patterns of order n that require or allow the refined inertias $\mathbb{H}_n$. They point out that if an $n \times n$ sign pattern $\mathcal{A}$ requires $\mathbb{H}_n$, then $\mathcal{A}$ is potentially stable, not sign stable, sign nonsingular with $\operatorname{sgn}(\det(A)) = (-)^n$ for all $A \in Q(\mathcal{A})$ and $-\mathcal{A}$ is not potentially stable. Their result gives us some necessary conditions for a sign pattern to require $\mathbb{H}_n$.

Necessary and sufficient conditions for a 4×4 tree sign pattern to require $\mathbb{H}_4$ are also given in [24].

Theorem 1.1.2 ([24]). *A 4×4 tree sign pattern requires $\mathbb{H}_4$ if and only if it is potentially stable, sign nonsingular, not sign stable, and its negative is not potentially stable.*

Up to equivalence, there are exactly 5 star sign patterns that require $\mathbb{H}_4$.

$$\mathcal{S}_1 = \begin{bmatrix} - & + & + & + \\ - & - & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{S}_2 = \begin{bmatrix} - & + & + & + \\ + & - & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{S}_3 = \begin{bmatrix} 0 & + & + & + \\ - & 0 & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & - \end{bmatrix},$$

$$\mathcal{S}_4 = \begin{bmatrix} - & + & + & + \\ + & - & 0 & 0 \\ - & 0 & + & 0 \\ + & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{S}_5 = \begin{bmatrix} + & + & + & + \\ - & 0 & 0 & 0 \\ - & 0 & - & 0 \\ - & 0 & 0 & - \end{bmatrix}.$$

In [25], necessary and sufficient conditions for an irreducible 3×3 sign pattern to require $\mathbb{H}_3$ are given.

Theorem 1.1.3 ([25]). *Let $\mathcal{A}$ be an irreducible 3×3 sign pattern. The following are equivalent:*

- *$\mathcal{A}$ requires $\mathbb{H}_3$.*
- *$\mathcal{A}$ is potentially stable, sign nonsingular but not sign stable.*
- *$\mathcal{A}$ requires a negative determinant and allows refined inertia $(0, 1, 0, 2)$.*

These removed equivalences and complete the list of patterns given in [8].

It seems quite difficult to extend their results to all n. Thus in Chapter 3, we consider star sign patterns, a special case of tree sign patterns, that require $\mathbb{H}_n$. We obtain necessary and sufficient conditions for a star sign pattern to require $\mathbb{H}_n$ as published in [21] [22]. It is shown that up to equivalence, there are exactly five star sign patterns that require $\mathbb{H}_n$ for all $n \geq 5$, which negatively resolves the conjecture raised in [8].

1.2 Minimum Ranks of sign pattern matrices

For a sign pattern matrix $\mathcal{A}$, the *minimum rank* of $\mathcal{A}$, denoted $\mathrm{mr}(\mathcal{A})$, is defined as

$$\mathrm{mr}(\mathcal{A}) = \min\{\mathrm{rank}(A) \mid A \in Q(\mathcal{A})\}.$$

Similarly, the *maximum rank* of $\mathcal{A}$, denoted $\mathrm{MR}(\mathcal{A})$, is the maximum of the ranks of the real matrices in $Q(\mathcal{A})$. The *rational minimum rank* of a sign pattern $\mathcal{A}$, denoted $\mathrm{mr}_{\mathbb{Q}}(\mathcal{A})$, is defined to be the minimum of the ranks of the rational matrices in $Q(\mathcal{A})$. The minimum ranks of sign pattern matrices have been the focus of considerable research in recent years, see for example [3, 4, 5, 6, 7, 9, 10, 13, 18, 29, 31, 35, 37]. And they have important applications in areas such as communication complexity [1, 39, 41], machine learning [19], neural networks [15], combinatorics [17, 26, 48], and discrete geometry [38].

Consider a nonzero sign pattern $\mathcal{A}$. If $\mathcal{A}$ contains a zero row or column, then deletion of the zero row or zero column preserves the minimum rank. A sign pattern is said to be *reduced* if it does not contain any zero row or zero column. Similarly, if two nonzero rows (or two nonzero columns) of $\mathcal{A}$ are identical or are negatives of each other, then deleting such a row (or column) also preserves the minimum rank.

Following [37], we say that a sign pattern is *row condensed* if it does not contain a zero row and no two rows are identical or are negatives of each other. *Column condensed* sign patterns are defined similarly. We say that a sign pattern is a *condensed sign pattern* if it is both row condensed and column condensed. Clearly, given any nonzero sign pattern $\mathcal{A}$, we can delete zero, all but one duplicate or opposite rows and columns of $\mathcal{A}$ to get a condensed sign pattern matrix $\mathcal{A}_c$ (called the condensed sign pattern of $\mathcal{A}$) with the same minimum rank and the same rational minimum rank. Indeed, the deletion process can be reversed easily to create a matrix in the qualitative class of the original sign pattern that achieves the minimum rank. For consistency, we agree that when two rows are identical or opposite, we delete the lower one and when two columns are identical or opposite, we delete the one on the right. For a zero sign pattern $\mathcal{A}$, the condensed sign pattern $\mathcal{A}_c = \emptyset$, the empty matrix, which has minimum rank 0.

For example, for the sign pattern matrix

$$\mathcal{A} = \begin{bmatrix} 0 & - & - & - & + \\ - & + & + & + & - \\ - & - & - & + & - \\ - & - & - & + & - \\ - & 0 & - & + & - \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \mathcal{A}_c = \begin{bmatrix} 0 & - & - & - \\ - & + & + & + \\ - & - & - & + \\ - & 0 & - & + \end{bmatrix}.$$

Since every sign pattern and its condensed sign pattern have the same minimum rank and the same rational minimum rank, without loss of generality, in most of the subsequent discussions in Chapter 4, we may assume that the sign patterns involved are condensed.

Analogous to sign pattern matrices, a *zero-nonzero pattern (matrix)* is a matrix whose entries are from the set $\{0, \star\}$, where $\star$ indicates a nonzero entry. The *zero-nonzero indicator* of a number x (or a sign x in $\{+, -, 0\}$) is given by $\mathrm{zsgn}(x) = 0$ if $x = 0$ and $\mathrm{zsgn}(x) = \star$ otherwise. For a real matrix B, $\mathrm{zsgn}(B)$ gives its zero-nonzero pattern. Assuming that the ground field is $\mathbb{R}$, we define the qualitative class of a zero-nonzero pattern $\mathscr{A}$ as follows

$$Q(\mathscr{A}) = \{A \mid A \text{ is a real matrix and } \mathrm{zsgn}(A) = \mathscr{A}\}.$$

Many other notions for sign patterns, such as minimum rank, rational minimum rank, maximum rank, sign nonsingularity, condensed sign pattern, permutationally equivalence, etc., also carry over to zero-nonzero patterns. Some results on sign patterns remain valid for zero-nonzero patterns and vice versa, but there are important differences, as shown later in Chapter 4. The minimum ranks of zero-nonzero patterns have been investigated in several papers, see [32] and the references therein.

For a sign pattern or zero-nonzero pattern $\mathcal{A}$, its maximum rank, $\mathrm{MR}(\mathcal{A})$, is well understood; $\mathrm{MR}(\mathcal{A})$ is equal to the *term rank* of $\mathcal{A}$, namely, the maximum number of nonzero entries in $\mathcal{A}$ no two of which are on the same row or column [28]. It is easily shown that there is an integer matrix $A \in Q(\mathcal{A})$ that has rank $\mathrm{MR}(\mathcal{A})$. In contrast, the minimum rank of a sign pattern or zero-nonzero pattern is much more difficult to determine. There is no efficient algorithm for determining the minimum rank. As mentioned in [9], determining if a sign pattern has minimum rank at most 3 is an NP-complete problem. However, it is known ([32]) that for every integer k with $\mathrm{mr}(\mathcal{A}) \leq k \leq \mathrm{MR}(\mathcal{A})$, there is a matrix $A \in Q(\mathcal{A})$ such that $\mathrm{rank}(A) = k$.

Obviously, $\mathrm{mr}(\mathcal{A}) \leq \mathrm{mr}_{\mathbb{Q}}(\mathcal{A})$ for every sign pattern $\mathcal{A}$. When $\mathrm{mr}(\mathcal{A}) = \mathrm{mr}_{\mathbb{Q}}(\mathcal{A})$, we say that the minimum rank of $\mathcal{A}$ can be *realized rationally.* In [3, 4, 6, 43], several classes of sign patterns $\mathcal{A}$ for which $\mathrm{mr}(\mathcal{A}) = \mathrm{mr}_{\mathbb{Q}}(\mathcal{A})$ are identified, such as when $\mathcal{A}$ is full, or the minimum rank of $\mathcal{A}$ is at most 2, or $\mathrm{MR}(\mathcal{A}) - \mathrm{mr}(\mathcal{A}) \leq 2$, or the minimum rank of $\mathcal{A}$ is at least $\min\{m, n\} - 2$, where $\mathcal{A}$ is $m \times n$.

However, it has been shown in [7] and [35] that there exist sign patterns $\mathcal{A}$ for which $\mathrm{mr}(\mathcal{A}) < \mathrm{mr}_{\mathbb{Q}}(\mathcal{A})$. In particular, [35] showed the existence of a 12×12 sign pattern with $\mathrm{mr}(\mathcal{A}) = 3$ but $\mathrm{mr}_{\mathbb{Q}}(\mathcal{A}) > 3$.

Note that for every sign pattern $\mathcal{A}$ and its zero-nonzero pattern $\mathscr{A} =\mathrm{zsgn}(\mathcal{A})$, we always have $\mathrm{mr}(\mathscr{A}) \leq \mathrm{mr}(\mathcal{A})$. Conversely, given any zero-nonzero pattern $\mathscr{A}$, a real matrix $A \in Q(\mathscr{A})$ achieving the minimum rank of $\mathscr{A}$ yields a sign pattern, $\mathcal{A} = \mathrm{sgn}(A)$, such that $\mathscr{A} =\mathrm{zsgn}(\mathcal{A})$ and $\mathrm{mr}(\mathscr{A}) = \mathrm{mr}(\mathcal{A})$.

In Chapter 4, we establish a direct connection between condensed $m \times n$ sign patterns and zero-nonzero patterns with minimum rank r ($r \geq 2$) and m point-n hyperplane configurations in $\mathbb{R}^{r-1}$ [17]. We present a new and illuminating proof of the fact that for every sign pattern $\mathcal{A}$ with minimum rank 2, rational realization of the minimum rank is always possible [37]. The proof reveals many interesting properties of sign patterns with minimum rank 2, and yields some characterizations of such sign patterns. We also introduce the notions of the number of polynomial sign changes and the number of strict sign changes of a sign vector and substantially extend two known upper bounds for the minimum ranks of full sign patterns to obtain sharp upper bounds for the rational minimum ranks of general sign patterns.

Then we use the matrix factorization that guarantees the connection between condensed sign patterns and point-hyperplane configurations to prove that if the number of zero entries on each column of a sign pattern $\mathcal{A}$ with minimum rank r is at most 2 , then $\mathrm{mr}(\mathcal{A}) = \mathrm{mr}_{\mathbb{Q}}(\mathcal{A})$. We also give an upper bound for the rational minimum rank of a zero-nonzero pattern and use it to show that if the number of zero entries on each column of a zero-nonzero pattern $\mathscr{A}$ with minimum rank r is at most $r-1$, then $\mathrm{mr}(\mathscr{A}) = \mathrm{mr}_{\mathbb{Q}}(\mathcal{A})$. Furthermore, we construct the smallest known sign pattern whose minimum rank is 3 but whose rational minimum rank is greater than 3. We note that as shown in Chapter 4, rational realizability of the minimum rank for sign patterns or zero-nonzero patterns with minimum rank 3 is closely related to the central problem of rational realizability of certain point-line configurations on the plane [27].

PART 2

THE MINIMAL CRITICAL SETS OF REFINED INERTIAS FOR 3×3 FULL SIGN PATTERNS

In this chapter, we identify all the minimal critical sets of refined inertias and inertias for full sign patterns of order 3. The main results are stated below.

Theorem 2.0.1. *The only minimal critical sets of refined inertias for* 3×3 *full sign patterns are the following sets and their reversals.*

$\{(3,0,0,0),(0,3,0,0)\}, \{(3,0,0,0),(0,2,1,0)\}, \{(3,0,0,0),(0,1,0,2)\}, \{(3,0,0,0),(0,1,2,0)\},$

$\{(2,0,1,0),(0,2,1,0)\}, \{(2,0,1,0),(0,1,0,2)\}, \{(2,0,1,0),(0,1,2,0)\}, \{(1,0,0,2),(0,1,0,2)\},$

$\{(1,0,0,2),(0,1,2,0)\}, \{(1,0,2,0),(0,1,2,0)\}, \{(0,0,3,0)\}, \{(0,0,1,2)\}.$

Theorem 2.0.2. *The only minimal critical sets of inertias for* 3×3 *full sign patterns are the following sets and their reversals.*

$$\{(3,0,0),(0,3,0)\}, \{(3,0,0),(0,2,1)\}, \{(3,0,0),(0,1,2)\}, \{(2,0,1),(0,2,1)\},$$

$$\{(2,0,1),(0,1,2)\}, \{(1,0,2),(0,1,2)\}, \{(0,0,3)\}.$$

The *reversal* of an inertia (resp., refined inertia) is obtained by exchanging the first two entries in the ordered triple (resp., 4-tuple); i.e. the reversal of (n_+, n_-, n_0) (resp., $(n_+, n_-, n_z, 2n_p)$) is (n_-, n_+, n_0) (resp., $(n_-, n_+, n_z, 2n_p)$). The *reversal* of a set of inertias (resp., refined inertias) is the set of reversals of the inertias (resp., refined inertias) in that set. Clearly, for an $n \times n$ sign pattern $\mathcal{A}$, $\mathrm{i}(-\mathcal{A})$ is the reversal of $\mathrm{i}(\mathcal{A})$, and $\mathrm{ri}(-\mathcal{A})$ is the reversal of $\mathrm{ri}(\mathcal{A})$.

We will give the proofs of Theorem 2.0.1 and Theorem 2.0.2 in section 2.3 and section 2.4, respectively.

2.1 Preliminaries

Note that if $\mathcal{A}$ is a SAP, then $\mathcal{A}$ is a rIAP and an IAP. From [12], we have the following characterization of 3×3 rIAPs.

Lemma 2.1.1. *If $\mathcal{A}$ is a sign pattern of order 3, then the following statements are equivalent:*

1. *$\mathcal{A}$ is spectrally arbitrary.*
2. *$\mathcal{A}$ is inertially arbitrary.*
3. *$\mathcal{A}$ is refined inertially arbitrary.*
4. *Up to equivalence, $\mathcal{A}$ is a superpattern of one of the following sign patterns:*

$$\mathcal{D}_{3,3} = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ - & 0 & + \end{bmatrix}, \quad \mathcal{D}_{3,2} = \begin{bmatrix} - & + & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix}, \quad \mathcal{U} = \begin{bmatrix} - & + & 0 \\ - & + & + \\ 0 & - & + \end{bmatrix}, \quad \mathcal{V} = \begin{bmatrix} - & 0 & + \\ - & 0 & + \\ - & + & + \end{bmatrix}.$$

We say that two square sign patterns $\mathcal{A}$ and $\mathcal{B}$ are *equivalent* if one can be obtained from the other by any combination of negation, transposition, permutationally similarity, and signature similarity. Clearly, if two square sign patterns $\mathcal{A}$ and $\mathcal{B}$ are equivalent, then $\mathcal{A}$ is a rIAP (resp., IAP, SAP) if and only if $\mathcal{B}$ is a rIAP (resp., IAP, SAP); and $\mathrm{ri}(\mathcal{A}) = \mathrm{ri}(\mathcal{B})$ or $\mathrm{ri}(\mathcal{A}) = \mathrm{ri}(-\mathcal{B})$ (resp., $\mathrm{i}(\mathcal{A}) = \mathrm{i}(\mathcal{B})$ or $\mathrm{i}(\mathcal{A}) = \mathrm{i}(-\mathcal{B})$).

The following is an interesting example of a full sign pattern that requires a positive eigenvalue.

Lemma 2.1.2 ([12, 34])**.** *Let*

$$\mathcal{G} = \begin{bmatrix} - & + & + \\ - & + & - \\ - & - & + \end{bmatrix}.$$

Then $\mathcal{G}$ requires a positive eigenvalue.

Let $\mathcal{A} = [a_{ij}]$ be a sign pattern of order 3. We say that $\mathcal{A}$ contains a *negative 2-cycle* (resp., *positive 2-cycle*) if $a_{ij}a_{ji} < 0$ (resp., $a_{ij}a_{ji} > 0$) for some $i \neq j$. We say that $\mathcal{A}$ satisfies the *minor conditions* if $\mathcal{A}$ allows a positive and a negative principal minor of order k for each $k = 1, 2, 3$.

The following two results will be useful in our discussions later.

Lemma 2.1.3 ([12])**.** *If $\mathcal{A}$ is an irreducible sign pattern of order 3 which contains a negative 2-cycle, and $\mathcal{A}$ satisfies the minor conditions, then either $\mathcal{A}$ is equivalent to a subpattern of $\mathcal{G}$ or $\mathcal{A}$ is equivalent to a superpattern of one of the four patterns $\mathcal{D}_{3,3}$, $\mathcal{D}_{3,2}$, $\mathcal{U}$, or $\mathcal{V}$.*

Lemma 2.1.4 ([14])**.** *Let m be the maximum number of distinct refined inertias allowed by any sign pattern of order* 3*. Then $m = 13$.*

There are 13 possible distinct refined inertias:

$$(3,0,0,0), (2,1,0,0), (2,0,1,0), (1,2,0,0), (1,1,1,0), (1,0,2,0), (1,0,0,2),$$
$$(0,3,0,0), (0,2,1,0), (0,1,2,0), (0,1,0,2), (0,0,3,0), (0,0,1,2).$$

2.2 The 3×3 full sign patterns that are not rIAPs

We now identify, up to equivalence, all 3×3 full sign patterns that are not rIAPs.

Theorem 2.2.1. *Let $\mathcal{A}$ be a full sign pattern of order 3 that is not a rIAP. Then $\mathcal{A}$ is equivalent to one of the following patterns:*

$$\mathcal{A}_1 = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & + \end{bmatrix}, \quad \mathcal{A}_2 = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}, \quad \mathcal{A}_3 = \begin{bmatrix} - & + & + \\ + & - & + \\ + & + & + \end{bmatrix}, \quad \mathcal{A}_4 = \begin{bmatrix} - & + & + \\ + & - & + \\ + & + & - \end{bmatrix},$$

$$\mathcal{A}_5 = \begin{bmatrix} - & + & - \\ - & - & - \\ - & - & - \end{bmatrix}, \quad \mathcal{A}_6 = \begin{bmatrix} - & + & + \\ - & - & + \\ - & + & - \end{bmatrix}, \quad \mathcal{A}_7 = \begin{bmatrix} - & + & + \\ - & - & - \\ - & - & - \end{bmatrix}, \quad \mathcal{A}_8 = \begin{bmatrix} - & + & + \\ - & - & + \\ - & - & - \end{bmatrix},$$

$$\mathcal{G} = \begin{bmatrix} - & + & + \\ - & + & - \\ - & - & + \end{bmatrix}.$$

To prove the theorem, we need the following three lemmas.

Lemma 2.2.2. *Let $\mathcal{A}$ be a full sign pattern of order 3. If $\mathcal{A}$ contains no negative 2-cycle, then $\mathcal{A}$ is equivalent to one of $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4$ given in Theorem 2.2.1.*

Proof Let's consider the following two cases.

Case 1. The diagonal entries of $\mathcal{A}$ have different signs. Assume that $a_{11} = -, a_{22} = -, a_{33} = +$.

We now list all such sign patterns as follows:

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & + \end{bmatrix}, \quad \begin{bmatrix} - & + & + \\ + & - & - \\ + & - & + \end{bmatrix}, \quad \begin{bmatrix} - & + & - \\ + & - & + \\ - & + & + \end{bmatrix}, \quad \begin{bmatrix} - & - & + \\ - & - & + \\ + & + & + \end{bmatrix},$$

$$\begin{bmatrix} - & + & + \\ + & - & + \\ + & + & + \end{bmatrix}, \quad \begin{bmatrix} - & - & - \\ - & - & + \\ - & + & + \end{bmatrix}, \quad \begin{bmatrix} - & - & + \\ - & - & - \\ + & - & + \end{bmatrix}, \quad \begin{bmatrix} - & + & - \\ + & - & - \\ - & - & + \end{bmatrix}.$$

Note that

$$\begin{bmatrix} + & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & - \end{bmatrix} \begin{bmatrix} - & + & + \\ + & - & - \\ + & - & + \end{bmatrix} \begin{bmatrix} + & 0 & 0 \\ 0 & - & 0 \\ 0 & 0 & - \end{bmatrix} = \mathcal{A}_1.$$

Thus $\begin{bmatrix} - & + & + \\ + & - & - \\ + & - & + \end{bmatrix}$ is equivalent to $\mathcal{A}_1$.

Similarly, we see that each of the above eight sign patterns is equivalent to $\mathcal{A}_1$ or $\mathcal{A}_3$.

When one of the diagonal entries of $\mathcal{A}$ is negative and the other two are positive, a similar argument shows that $\mathcal{A}$ is also equivalent to $\mathcal{A}_1$ or $\mathcal{A}_3$.

Case 2. All diagonal entries of $\mathcal{A}$ have the same sign. Assume they are all negative.

Here are all such sign patterns:

$$\begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}, \quad \begin{bmatrix} - & + & + \\ + & - & - \\ + & - & - \end{bmatrix}, \quad \begin{bmatrix} - & + & - \\ + & - & + \\ - & + & - \end{bmatrix}, \quad \begin{bmatrix} - & - & + \\ - & - & + \\ + & + & - \end{bmatrix},$$

$$\begin{bmatrix} - & + & + \\ + & - & + \\ + & + & - \end{bmatrix}, \quad \begin{bmatrix} - & - & - \\ - & - & + \\ - & + & - \end{bmatrix}, \quad \begin{bmatrix} - & - & + \\ - & - & - \\ + & - & - \end{bmatrix}, \quad \begin{bmatrix} - & + & - \\ + & - & - \\ - & - & - \end{bmatrix}.$$

It is easy to observe that they are equivalent to $\mathcal{A}_2$ or $\mathcal{A}_4$.

When all diagonal entries of $\mathcal{A}$ are positive, a similar argument shows that $\mathcal{A}$ is also equivalent to $\mathcal{A}_2$ or $\mathcal{A}_4$.

Combining the above two cases, we see that $\mathcal{A}$ is equivalent to one of the patterns $\mathcal{A}_1$, $\mathcal{A}_2$, $\mathcal{A}_3$, $\mathcal{A}_4$. $\square$

Lemma 2.2.3. *Let $\mathcal{A}$ be a full sign pattern of order 3 such that $\mathcal{A}$ contains at least one negative 2-cycle and all diagonal entries of $\mathcal{A}$ have the same sign. Then $\mathcal{A}$ is equivalent to one of $\mathcal{A}_5, \mathcal{A}_6, \mathcal{A}_7, \mathcal{A}_8$ given in Theorem 2.2.1.*

Proof Assume that $a_{11} = a_{22} = a_{33} = -$.

Case 1. $\mathcal{A}$ contains exactly one negative 2-cycle.

By performing a permutationally similarity and a signature similarity if necessary, without loss of generality, we may assume that $a_{21} = -, a_{12} = +, a_{13} = -$, and $a_{31} = -$.

There are only two such sign patterns:

$$\begin{bmatrix} - & + & - \\ - & - & - \\ - & - & - \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} - & + & - \\ - & - & + \\ - & + & - \end{bmatrix}.$$

By multiplying the second row and the second column of the last matrix by $-$ and then switching the first two rows and columns, we see that it is equivalent to the first matrix, which is $\mathcal{A}_5$.

Case 2. $\mathcal{A}$ contains exactly two negative 2-cycles.

Without loss of generality, assume that $a_{12}a_{21} = -$ and $a_{13}a_{31} = -$. Further, by performing a suitable signature similarity if necessary, we may assume that $a_{12} = a_{13} = +$, and $a_{21} = a_{31} = -$.

There are only two such sign patterns:

$$\begin{bmatrix} - & + & + \\ - & - & + \\ - & + & - \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} - & + & + \\ - & - & - \\ - & - & - \end{bmatrix},$$

which are $\mathcal{A}_6$ and $\mathcal{A}_7$.

Case 3. $\mathcal{A}$ contains three negative 2-cycles.

Similarly as in the preceding case, we may assume that $a_{12} = a_{13} = +$, and $a_{21} = a_{31} = -$.

There are only two such sign patterns:

$$\begin{bmatrix} - & + & + \\ - & - & + \\ - & - & - \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} - & + & + \\ - & - & - \\ - & + & - \end{bmatrix},$$

both of which are easily seen to be equivalent to $\mathcal{A}_8$.

When all diagonal entries of $\mathcal{A}$ are positive, a similar argument shows that $\mathcal{A}$ is also equivalent to one of $\mathcal{A}_5$, $\mathcal{A}_6$, $\mathcal{A}_7$, and $\mathcal{A}_8$. $\square$

Lemma 2.2.4. *Let $\mathcal{A}$ be a full sign pattern of order 3 that is not a rIAP. Suppose that $\mathcal{A}$ contains at least one negative 2-cycle and the diagonal entries of $\mathcal{A}$ are not all the same. Then $\mathcal{A}$ is equivalent to $\mathcal{G}$ (as given in Theorem 2.2.1).*

Proof In view of Lemma 2.1.1 and Lemma 2.1.3, if we can show that $\mathcal{A}$ satisfies the minor conditions, then $\mathcal{A}$ is equivalent to $\mathcal{G}$.

Clearly, $\mathcal{A}$ allows a positive and a negative principal minor of order 1 because the diagonal entries of $\mathcal{A}$ are not identical.

As the three diagonal entries of $\mathcal{A}$ are nonzero and not identical, $\mathcal{A}$ contains a principal 2×2 submatrix with identical nonzero diagonal entries and $\mathcal{A}$ also contains a principal 2×2 submatrix with opposite nonzero diagonal entries. It follows that $\mathcal{A}$ allows a positive and a negative principal minor of order 2.

The fact that $\mathcal{A}$ allows a positive and a negative principal minor of order 3 follows from the well-known result (see [28]) that no 3×3 full sign pattern is sign nonsingular.

Therefore, $\mathcal{A}$ satisfies the minor conditions. By Lemma 2.1.3, $\mathcal{A}$ is equivalent to $\mathcal{G}$.

□

Proof [**Proof of Theorem 2.2.1**] Note that the matrices in Lemma 2.1.1 (4) satisfy:

(a) All matrices have a negative 2-cycle;

(b) All matrices have both positive and negative diagonal entries.

Combining the result of Lemma 2.2.4, we see that if a 3×3 sign pattern $\mathcal{A}$ is not a rIAP, then $\mathcal{A}$ must satisfy at least one of the following three conditions:

(1) $\mathcal{A}$ does not contain a negative 2-cycle.

(2) All diagonal entries of $\mathcal{A}$ have the same sign.

(3) $\mathcal{A}$ is equivalent to $\mathcal{G}$.

So the preceding three lemmas give the result of Theorem 2.2.1. □

2.3 The minimal critical sets of refined inertias for 3×3 full sign patterns

Let $\mathcal{A}$ be an $n \times n$ sign pattern that is not a rIAP. We use $R(\mathcal{A})$ to denote the set of all possible refined inertias that are not in $\mathrm{ri}(\mathcal{A})$, that is,

$$R(\mathcal{A}) = \{(n_+, n_-, n_z, 2n_p) \in \mathbb{Z}_+^4 \mid n_+ + n_- + n_z + 2n_p = n, \text{ and } (n_+, n_-, n_z, 2n_p) \notin \mathrm{ri}(\mathcal{A})\},$$

where $2n_p$ is even and $\mathbb{Z}_+$ is the set of all nonnegative integers.

Lemma 2.3.1. *Let H be a proper subset of the set of all possible refined inertias of real matrices of order n. Then H is a critical set of refined inertias for a family $\mathcal{F}$ of sign patterns of order n if and only if for every $n \times n$ sign pattern $\mathcal{A}$ in $\mathcal{F}$ that is not a rIAP, $H \cap R(\mathcal{A}) \neq \emptyset$.*

Proof Necessity. Let H be a critical set of refined inertias for a family $\mathcal{F}$ of sign patterns of order n. By the definition of critical set of refined inertias, for any $n \times n$ sign pattern $\mathcal{A}$ in $\mathcal{F}$ that is not a rIAP, $H \nsubseteq \mathrm{ri}(\mathcal{A})$. Hence, $H \cap R(\mathcal{A}) \neq \emptyset$.

Sufficiency. Assume that $H \cap R(\mathcal{A}) \neq \emptyset$ for every $n \times n$ sign pattern $\mathcal{A}$ in $\mathcal{F}$ that is not a rIAP. Then for each $n \times n$ sign pattern $\mathcal{A}$ in $\mathcal{F}$ that is not a rIAP, $H \nsubseteq \mathrm{ri}(\mathcal{A})$. Thus for every sign pattern $\widehat{\mathcal{A}}$ in $\mathcal{F}$ of order n, if $H \subseteq \mathrm{ri}(\widehat{\mathcal{A}})$, then $\widehat{\mathcal{A}}$ must be a rIAP. $\square$

Now let's consider all full sign patterns of order 3 that are not rIAPs.

Lemma 2.3.2. $R(\mathcal{A}_1) = \{(3,0,0,0),(2,0,1,0),(1,0,0,2),(1,0,2,0),(0,0,3,0),(0,0,1,2)\}$.

Proof Assume $A \in Q(\mathcal{A}_1)$ and

$$A = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & i \end{bmatrix},$$

where $a, b, c, d, e, f, g, h, i > 0$.

Then the characteristic polynomial of A is

$$p(t) = t^3 + (a+e-i)t^2 + (ae - ie - ai - cg - fh - bd)t - aei + bfg + cdh - ceg - afh + bdi.$$

We now show that $(3,0,0,0) \notin \mathrm{ri}(\mathcal{A}_1)$. Otherwise, by using the relationship between the coefficients of the characteristic polynomial and certain symmetric functions of the eigenvalues, we get the following two inequalities:

$$\begin{cases} a+e-i<0; \\ ae-ie-ai-cg-fh-bd>0. \end{cases}$$

From the first inequality, we get $a<i$. Hence, $ae-ie-ia-cg-fh-bd=e(a-i)-ai-cg-fh-bd<0$, which contradicts the second inequality.

Similarly, we can prove that $(2,0,1,0)\notin \mathrm{ri}(\mathcal{A}_1)$ and $(1,0,0,2)\notin \mathrm{ri}(\mathcal{A}_1)$.

We claim that $(1,0,2,0)\notin \mathrm{ri}(\mathcal{A}_1)$. Otherwise, we have the following equation and inequality:

$$\begin{cases} a+e-i<0; \\ ae-ie-ai-cg-fh-bd=0. \end{cases}$$

We get a contradiction by a similar argument as above.

We claim that $(0,0,3,0)\notin \mathrm{ri}(\mathcal{A}_1)$. Otherwise, we will have the following two equations:

$$\begin{cases} a+e-i=0; \\ ae-ie-ai-cg-fh-bd=0. \end{cases}$$

We get a contradiction by a similar argument as above.

We claim that $(0,0,1,2)\notin \mathrm{ri}(\mathcal{A}_1)$. Otherwise, we have the following equation and inequality:

$$\begin{cases} a+e-i=0; \\ ae-ie-ai-cg-fh-bd>0. \end{cases}$$

We get a contradiction by a similar argument as above.

By taking suitable values of a,b,c,d,e,f,g,h,i shown in the following table, we can find real matrices in $Q(\mathcal{A}_1)$ with the remaining seven refined inertias listed after Lemma 2.1.4.

Table 2.1. realization of the remaining seven refined inertias

refined inertia	a	b	c	d	e	f	g	h	i
(1, 1, 1, 0)	1	1	1	1	1	1	1	1	1
(1, 2, 0, 0)	2	1	1	1	1	1	1	1	1
(2, 1, 0, 0)	1	2	1	1	1	1	1	1	1
(0, 3, 0, 0)	2	2	2	3	16	2	$\frac{1}{10}$	3	$\frac{1}{10}$
(0, 2, 1, 0)	5	1	2	6	5	1	1	4	1
(0, 1, 2, 0)	4	1	2	5	5	1	1	4	1
(0, 1, 0, 2)	10	10	3	$\frac{80}{209}$	10	10	10	3	1

This completes the proof. □

The following lemma is straightforward.

Lemma 2.3.3. $R(-\mathcal{A}_1) = \{(0,3,0,0), (0,2,1,0), (0,1,0,2), (0,1,2,0), (0,0,3,0), (0,0,1,2)\}$.

Lemma 2.3.4. *$R(\mathcal{A}_i) \supseteq R(\mathcal{A}_1)$ and $R(-\mathcal{A}_i) \supseteq R(-\mathcal{A}_1)$ for $i = 2, 3, \ldots, 8$.*

Proof Assume $A \in Q(\mathcal{A}_3)$ and

$$A = \begin{bmatrix} -a & b & c \\ d & -e & f \\ g & h & i \end{bmatrix},$$

where $a, b, c, d, e, f, g, h, i > 0$.

Then the characteristic polynomial of A is

$$p(t) = t^3 + (a+e-i)t^2 + (ae - ei - ai - cg - fh - bd)t - aei - bfg - cdh - ceg - afh + bdi.$$

We now show that $(3,0,0,0) \notin \mathrm{ri}(\mathcal{A}_3)$. Otherwise, we have the following two inequalities:

$$\begin{cases} a + e - i < 0; \\ ae - ei - ai - cg - fh - bd > 0. \end{cases}$$

From the first inequality, we get $a < i$, which implies that $ae - ei - ai - cg - fh - bd =$

$e(a-i)-ai-cg-fh-bd<0$, contradicting the second inequality.

Similarly, we can prove that $(2,0,1,0)\notin \mathrm{ri}(\mathcal{A}_3)$ and $(1,0,0,2)\notin \mathrm{ri}(\mathcal{A}_3)$.

We claim that $(1,0,2,0)\notin \mathrm{ri}(\mathcal{A}_3)$. Otherwise, we have the following inequality and equation:

$$\begin{cases} a+e-i<0; \\ ae-ei-ai-cg-fh-bd=0, \end{cases}$$

which lead to a contradiction by a similar argument as above.

We claim that $(0,0,3,0)\notin \mathrm{ri}(\mathcal{A}_3)$. Otherwise, we have the following two equations:

$$\begin{cases} a+e-i=0; \\ ae-ei-ai-cg-fh-bd=0. \end{cases}$$

We get a contradiction by a similar argument as above.

We claim that $(0,0,1,2)\notin \mathrm{ri}(\mathcal{A}_3)$. Otherwise, we have the following equation and inequality:

$$\begin{cases} a+e-i=0; \\ ae-ei-ai-cg-fh-bd>0. \end{cases}$$

We get a contradiction by a similar argument as above.

Therefore $R(\mathcal{A}_3)\supseteq R(\mathcal{A}_1)$. Then $R(-\mathcal{A}_3)\supseteq R(-\mathcal{A}_1)$.

For $\mathcal{A}_i$ $(i=2,4,5,6,7,8)$, since all diagonal entries are negative, they don't allow the refined inertias $(3,0,0,0)$, $(2,0,1,0)$, $(1,0,2,0),(1,0,0,2)$, $(0,0,3,0)$ and $(0,0,1,2)$. Thus $R(\mathcal{A}_i)\supseteq R(\mathcal{A}_1)$. Similarly, It is easy to observe that $R(-\mathcal{A}_i)\supseteq R(-\mathcal{A}_1)$ for $i=2,4,5,6,7,8$. $\square$

Lemma 2.3.5. *$R(\mathcal{G})\supseteq R(-\mathcal{A}_1)$ and $R(-\mathcal{G})\supseteq R(\mathcal{A}_1)$.*

Proof By Lemma 2.1.2, the pattern $\mathcal{G}$ requires a positive eigenvalue. Thus, $\mathcal{G}$ does not allow the refined inertias $(0,3,0,0)$, $(0,2,1,0)$, $(0,1,0,2)$, $(0,1,2,0)$, $(0,0,3,0)$ and $(0,0,1,2)$. So $R(\mathcal{G})\supseteq R(-\mathcal{A}_1)$ and $R(-\mathcal{G})\supseteq R(\mathcal{A}_1)$. $\square$

Proof [**Proof of Theorem 2.0.1**]

Let H be a proper subset of the set of all possible refined inertias for full sign patterns of order 3. By Lemma 2.3.1 and Theorem 2.2.1, H is a critical set of refined inertias for full sign patterns of order 3 if and only if $|H \cap R(\mathcal{A}_i)| \neq \emptyset$, $|H \cap R(-\mathcal{A}_i)| \neq \emptyset$ for $i = 1, 2, \ldots, 8$, $|H \cap R(\mathcal{G})| \neq \emptyset$, and $|H \cap R(-\mathcal{G})| \neq \emptyset$. Recall that $R(\mathcal{A}_i) \supseteq R(\mathcal{A}_1)$ and $R(-\mathcal{A}_i) \supseteq R(-\mathcal{A}_1)$ for $i = 2, 3, \ldots, 8$, $R(\mathcal{G}) \supseteq R(-\mathcal{A}_1)$, and $R(-\mathcal{G}) \supseteq R(\mathcal{A}_1)$. Thus H is a critical set if and only if $|H \cap R(\mathcal{A}_1)| \neq \emptyset$ and $|H \cap R(-\mathcal{A}_1)| \neq \emptyset$.

To make H a minimal critical set of refined inertias for full sign patterns of order 3, we must have $|H \cap R(\mathcal{A}_1)| = 1$ and $|H \cap R(-\mathcal{A}_1)| = 1$. Since $\{(0, 0, 3, 0)\} \subseteq R(\mathcal{A}_1) \cap R(-\mathcal{A}_1)$ and $\{(0, 0, 1, 2)\} \subseteq R(\mathcal{A}_1) \cap R(-\mathcal{A}_1)$, each of them is a minimal critical set.

Let

$$R'(\mathcal{A}_1) = R(\mathcal{A}_1) \setminus \{(0, 0, 3, 0), (0, 0, 1, 2)\} = \{(3, 0, 0, 0), (2, 0, 1, 0), (1, 0, 0, 2), (1, 0, 2, 0)\},$$

$$R'(-\mathcal{A}_1) = R(-\mathcal{A}_1) \setminus \{(0, 0, 3, 0), (0, 0, 1, 2)\} = \{(0, 3, 0, 0), (0, 2, 1, 0), (0, 1, 0, 2), (0, 1, 2, 0)\}.$$

Then $R'(\mathcal{A}_1) \cap R'(-\mathcal{A}_1) = \emptyset$. Now we pick up exactly one refined inertia from $R'(\mathcal{A}_1)$ and one refined inertia from $R'(-\mathcal{A}_1)$ and let them form new sets as follows.

$$\{(3,0,0,0),(0,3,0,0)\}, \{(3,0,0,0),(0,2,1,0)\}, \{(3,0,0,0),(0,1,0,2)\}, \{(3,0,0,0),(0,1,2,0)\},$$
$$\{(2,0,1,0),(0,3,0,0)\}, \{(2,0,1,0),(0,2,1,0)\}, \{(2,0,1,0),(0,1,0,2)\}, \{(2,0,1,0),(0,1,2,0)\},$$
$$\{(1,0,0,2),(0,3,0,0)\}, \{(1,0,0,2),(0,2,1,0)\}, \{(1,0,0,2),(0,1,0,2)\}, \{(1,0,0,2),(0,1,2,0)\},$$
$$\{(1,0,2,0),(0,3,0,0)\}, \{(1,0,2,0),(0,2,1,0)\}, \{(1,0,2,0),(0,1,0,2)\}, \{(1,0,2,0),(0,1,2,0)\}.$$

Note that $\{(2, 0, 1, 0),\ (0, 3, 0, 0)\}$ is the reversal of $\{(3, 0, 0, 0),\ (0, 2, 1, 0)\}$, $\{(1, 0, 0, 2),\ (0, 3, 0, 0)\}$ is the reversal of $\{(3, 0, 0, 0),\ (0, 1, 0, 2)\}$, $\{(1, 0, 0, 2),\ (0, 2, 1, 0)\}$ is the reversal of $\{(2, 0, 1, 0),\ (0, 1, 0, 2)\}$, $\{(1, 0, 2, 0),\ (0, 3, 0, 0)\}$ is the reversal of $\{(3, 0, 0, 0),\ (0, 1, 2, 0)\}$, $\{(1, 0, 2, 0),\ (0, 2, 1, 0)\}$ is the reversal of $\{(2, 0, 1, 0),\ (0, 1, 2, 0)\}$, and $\{(1, 0, 2, 0),\ (0, 1, 0, 2)\}$ is the reversal of $\{(1, 0, 0, 2),\ (0, 1, 2, 0)\}$. So we drop them out.

Now we reach Theorem 2.0.1 on all minimal critical sets of refined inertias for full sign patterns of order 3. And the maximum cardinality of a minimum critical set of refined inertias for 3×3 full sign patterns is 2. $\square$

2.4 The minimal critical sets of inertias for 3×3 full sign patterns

Suppose H is a proper subset of the set of all inertias for 3×3 real matrices. Let H_r be a set of refined inertias (with the same cardinality as H) obtained by splitting each inertia in H into a refined inertia. For example, for $H = \{(0,0,3)\}$, $H_r = \{(0,0,3,0)\}$ or $\{(0,0,1,2)\}$.

Lemma 2.4.1. [47] *If every set of refined inertias H_r arising from H is a critical set of refined inertias for irreducible zero-nonzero patterns of order $n \leq 3$, then H must be a critical set of inertias. Furthermore, if any one of the critical sets of refined inertias in H_r is a minimal critical set, then H is a minimal critical set of inertias.*

Proof [**Proof of Theorem 2.0.2**] Since a 3×3 sign pattern is an IAP if and only if it is a rIAP, the result of Lemma 2.4.1 holds for 3×3 sign patterns.

Theorem 2.0.1 gives a complete list of the minimal critical sets of refined inertias for full sign patterns of order 3. Note that a 3×3 full sign pattern:

Allows $\{(3,0,0),(0,3,0)\}$ if and only if it allows $\{(3,0,0,0),(0,3,0,0)\}$.

Allows $\{(3,0,0),(0,2,1)\}$ if and only if it allows $\{(3,0,0,0),(0,2,1,0)\}$.

Allows $\{(3,0,0),(0,1,2)\}$ if and only if it allows $\{(3,0,0,0),(0,1,2,0)\}$ or $\{(3,0,0,0),(0,1,0,2)\}$.

Allows $\{(2,0,1),(0,2,1)\}$ if and only if it allows $\{(2,0,1,0),(0,2,1,0)\}$.

Allows $\{(2,0,1),(0,1,2)\}$ if and only if it allows $\{(2,0,1,0),(0,1,2,0)\}$ or $\{(2,0,1,0),(0,1,0,2)\}$.

Allows $\{(1,0,2),(0,1,2)\}$ if and only if it allows $\{(1,0,2,0),(0,1,2,0)\}$ or $\{(1,0,2,0),(0,1,0,2)\}$ or $\{(1,0,0,2),(0,1,2,0)\}$ or $\{(1,0,0,2),(0,1,0,2)\}$ (Since $\{(1,0,2,0),(0,1,0,2)\}$ is the reversal of $\{(1,0,0,2),(0,1,2,0)\}$. So $\{(1,0,2,0),(0,1,0,2)\}$ dose not appear in Theorem 2.0.1).

Allows $\{(0,0,3)\}$ if and only if it allows $\{(0,0,3,0)\}$ or $\{(0,0,1,2)\}$.

So all sets of inertias given in Theorem 2.0.2 are minimal critical sets of inertias for 3×3 full sign patterns and they are the only such sets because they correspond to all minimal critical sets of refined inertias for full sign patterns of order 3 given in Theorem 2.0.1. $\square$

PART 3

STAR SIGN PATTERNS THAT REQUIRE $\mathbb{H}_n$

In this chapter, we characterize the star sign patterns of order n ($n \geq 5$) that require $\mathbb{H}_n = \{(0, n, 0, 0), (0, n-2, 0, 2), (2, n-2, 0, 0)\}$, which is an important set for the onset of Hopf bifurcation in dynamical systems. Necessary and sufficient conditions for a star sign pattern to require $\mathbb{H}_n$ are given. We show that for $n \geq 5$, up to equivalence, there are exactly five sign patterns (identified below) that require $\mathbb{H}_n$. In [8] there is a conjecture that no $n \times n$ irreducible sign pattern that requires $\mathbb{H}_n$ exists for n sufficiently large, possibly $n \geq 8$. But our result shows that there exist $n \times n$ irreducible sign patterns that require $\mathbb{H}_n$ for $n \geq 5$, which negatively resolves the conjecture.

In this chapter, we keep using the notion of *equivalent* but slightly modify the definition. Two sign patterns are said to be *equivalent* if one can be obtained from the other by transposition, signature similarity, permutationally similarity, or any combination of these.

Here is our main result.

Theorem 3.0.2. *Let $\mathcal{A}$ be an $n \times n$ ($n \geq 5$) star sign pattern. Then $\mathcal{A}$ requires $\mathbb{H}_n$ if and only if $\mathcal{A}$ is equivalent to one of the following five patterns:*

$$\mathcal{A}_1 = \begin{bmatrix} - & + & + & \cdots & \cdots & + \\ - & 0 & & & & \\ - & & - & & & \\ \vdots & & & \ddots & & \\ - & & & & - & \\ + & & & & & - \end{bmatrix}, \mathcal{A}_2 = \begin{bmatrix} - & + & + & \cdots & \cdots & + \\ - & 0 & & & & \\ + & & - & & & \\ \vdots & & & \ddots & & \\ + & & & & - & \\ + & & & & & - \end{bmatrix},$$

$$\mathcal{A}_3 = \begin{bmatrix} 0 & + & + & \cdots & \cdots & + \\ - & 0 & & & & \\ - & & - & & & \\ \vdots & & & \ddots & & \\ - & & & & - & \\ + & & & & & - \end{bmatrix}, \mathcal{A}_4 = \begin{bmatrix} + & + & + & \cdots & \cdots & + \\ - & 0 & & & & \\ - & & - & & & \\ \vdots & & & \ddots & & \\ - & & & & - & \\ - & & & & & - \end{bmatrix},$$

$$\mathcal{A}_5 = \begin{bmatrix} - & + & + & \cdots & \cdots & + \\ + & 0 & & & & \\ + & & - & & & \\ \vdots & & & \ddots & & \\ + & & & & - & \\ - & & & & & + \end{bmatrix}.$$

3.1 Preliminaries

Up to equivalence, an $n \times n$ star sign pattern can be represented in the following form

$$\mathcal{A} = \begin{bmatrix} a_1 & + & \cdots & + \\ c_2 & a_2 & & \\ \vdots & & \ddots & \\ c_n & & & a_n \end{bmatrix}, \tag{3.1}$$

where $c_i \neq 0$ for $i = 2, 3, \ldots, n$. We let $N_+(\mathcal{A})$ (resp., $N_-(\mathcal{A})$, $N_0(\mathcal{A})$) be the row index set of positive (resp., negative, zero) entries in the main diagonal of the sign pattern $\mathcal{A}$, and let $^{\#}X$ be the number of elements of the set X.

For an $n \times n$ matrix A and a subset α of $\{1, 2, \cdots, n\}$, $A[\alpha]$ denotes the principal submatrix of A whose index set is α, and $A(\alpha)$ denotes the principal submatrix of A obtained by deleting the rows and the columns indexed by α.

The following are two necessary conditions for a sign pattern to require $\mathbb{H}_n$.

Lemma 3.1.1 ([24, p. 621]). *Let $\mathcal{A}$ be an $n \times n$ sign pattern. If $\mathcal{A}$ requires $\mathbb{H}_n$, then $\mathcal{A}$ is potentially stable, sign nonsingular with* $\operatorname{sgn}(\det(A)) = (-)^n$ *for all $A \in Q(\mathcal{A})$, $\mathcal{A}$ is not sign stable, and $-\mathcal{A}$ is not potentially stable.*

Lemma 3.1.2 ([24]). *For $n \geq 3$, let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.1). If $\mathcal{A}$ requires $\mathbb{H}_n$, then there exists a unique i such that $2 \leq i \leq n$ and $a_i = 0$.*

The following three results are known characterizations of some potentially stable star sign patterns.

Lemma 3.1.3 ([23]). *Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.1) such that $N_0(\mathcal{A}) = \emptyset$. Then $\mathcal{A}$ is potentially stable if and only if*

(1) $^{\#}\{i \mid c_i = + \text{ and } i \in N_+(\mathcal{A})\} = \lfloor \frac{^{\#}(N_+(\mathcal{A}) \setminus \{1\})}{2} \rfloor$; and

(2) when $a_1 = +$, $\{i \mid c_i = - \text{ and } i \in N_-(\mathcal{A})\} \neq \emptyset$.

Lemma 3.1.4 ([23]). *Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.1) such that $N_0(\mathcal{A}) = \{1\}$. Then $\mathcal{A}$ is potentially stable if and only if*

(1) $\{i \mid c_i = - \text{ and } i \in N_-(\mathcal{A})\} \neq \emptyset$; and

(2) $^{\#}\{i \mid c_i = + \text{ and } i \in N_+(\mathcal{A})\} = \lfloor \frac{^{\#}N_+(\mathcal{A})}{2} \rfloor$.

Lemma 3.1.5 ([23]). *Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.1), and suppose that there exists $2 \leq s \leq n$ such that $a_s = 0$. Then $\mathcal{A}$ is potentially stable if and only if $\widetilde{\mathcal{A}}$ and $\mathcal{A}(\{s\})$ are potentially stable, where $\widetilde{\mathcal{A}}$ is obtained from $\mathcal{A}$ by replacing a_s with $+$.*

The digraph $D(\mathcal{A})$ of an $n \times n$ sign pattern $\mathcal{A} = [a_{ij}]$ has the vertex set $\{1, 2, \ldots, n\}$, an arc from i to j if $a_{ij} \neq 0$, and a loop at vertex i if $a_{ii} \neq 0$.

The following is a characterization of sign stable sign patterns.

Lemma 3.1.6 ([10, 28]). *If $\mathcal{A} = [a_{ij}]$ is an $n \times n$ irreducible sign pattern, then $\mathcal{A}$ is sign stable if and only if the following five conditions hold.*

(a) $\mathcal{A}$ has nonpositive main diagonal entries.

(b) If $i \neq j$, then $a_{ij}a_{ji} \leq 0$.

(c) The digraph of $\mathcal{A}$ is a doubly directed tree.

(d) $\mathcal{A}$ does not have an identically zero determinant.

(e) There does not exist a nonempty subset β of $[1, 2, \ldots, n]$ such that each diagonal element of $\mathcal{A}[\beta]$ is zero, each row of $\mathcal{A}[\beta]$ contains at least one nonzero entry, and no row of $\mathcal{A}[\bar{\beta}, \beta]$ contains exactly one nonzero entry.

Let $\mathcal{S}_i$ $(i = 1, 2, \ldots, 5)$ be star sign patterns of order 4 defined in Section 1.1. Then we have the following result.

Theorem 3.1.7 ([24]). *The 4×4 sign patterns $\mathcal{S}_i$ $(i = 1, 2, \ldots, 5)$ require $\mathbb{H}_4$.*

3.2 Necessary conditions for a star sign pattern of order n $(n \geq 5)$ to require $\mathbb{H}_n$

Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.1). Assume that $\mathcal{A}$ requires $\mathbb{H}_n$. By Lemma 3.1.2, up to equivalence, we may assume that $a_2 = 0$ and $a_i \neq 0$ for $i = 3, 4, \ldots, n$. Furthermore, we may assume that

$$\mathcal{A} = \begin{bmatrix} a_1 & + & + & \cdots & \cdots & \cdots & \cdots & + \\ c_2 & 0 & & & & & & \\ c_3 & & - & & & & & \\ \vdots & & & \ddots & & & & \\ c_k & & & & - & & & \\ c_{k+1} & & & & & + & & \\ \vdots & & & & & & \ddots & \\ c_n & & & & & & & + \end{bmatrix}, \tag{3.2}$$

where $a_1 \in \{0, +, -\}$, $c_i \in \{+, -\}$ for $i = 2, 3, \ldots, n$, and $2 \leq k \leq n$.

If $\mathcal{A}$ has at least three positive diagonal entries, then there is some real matrix $B \in Q(\mathcal{A})$ such that B has at least three eigenvalues with positive real parts, contradicting the assumption that $\mathcal{A}$ requires $\mathbb{H}_n$. Thus the following result is clear.

Lemma 3.2.1. *Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.2) that requires $\mathbb{H}_n$. Then $\mathcal{A}$ has at most two positive diagonal entries. In particular, $n - k \leq 2$ and in case $a_1 = +$, we have $n - k \leq 1$.*

Obviously, if $\mathcal{A}$ requires $\mathbb{H}_n$, then $\mathcal{A}$ is potentially stable, and $\mathcal{A}$ is not sign stable. We now examine these necessary conditions more carefully.

3.2.1 Star sign patterns that are potentially stable

Theorem 3.2.2. *Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.2). Assume that $\mathcal{A}$ has at most two positive diagonal entries. If $\mathcal{A}$ is potentially stable, then up to equivalence, $\mathcal{A}$ must satisfy one of the following conditions:*

(1) $k = n$, $a_1 = -$, $c_2 = -$;

(2) $k = n$, $a_1 = 0$ *or* $+$, $c_2 = -$, $c_3 = -$;

(3) $k = n - 1$, $a_1 = -$, $c_2 = +$, $c_n = -$;

(4) $k = n - 1$, $a_1 = 0$ *or* $+$, $c_2 = +$, $c_3 = -$, $c_n = -$;

(5) $k = n - 2$, $a_1 = -$, $c_2 = -$, $c_{n-1} = -$, $c_n = +$;

(6) $k = n - 2$, $a_1 = 0$, $c_2 = -$, $c_3 = -$, $c_{n-1} = -$, $c_n = +$.

Proof Since $\mathcal{A}$ is a star sign pattern of order n in the form (3.2) that is potentially stable, by Lemma 3.1.5, both sign patterns

$$\widetilde{\mathcal{A}} = \begin{bmatrix} a_1 & + & + & \cdots & \cdots & \cdots & \cdots & + \\ c_2 & + & & & & & & \\ c_3 & & - & & & & & \\ \vdots & & & \ddots & & & & \\ c_k & & & & - & & & \\ c_{k+1} & & & & & + & & \\ \vdots & & & & & & \ddots & \\ c_n & & & & & & & + \end{bmatrix}, \mathcal{A}(\{2\}) = \begin{bmatrix} a_1 & + & \cdots & \cdots & \cdots & \cdots & + \\ c_3 & - & & & & & \\ \vdots & & \ddots & & & & \\ c_k & & & - & & & \\ c_{k+1} & & & & + & & \\ \vdots & & & & & \ddots & \\ c_n & & & & & & + \end{bmatrix}_{n-1}$$

are potentially stable. According to Lemma 3.2.1, it suffices to consider the following three cases.

Case 1. $k = n$.

By Lemmas 3.1.3 and 3.1.4, we have

- ${}^{\#}\{i \mid c_i = + \text{ and } i \in N_+(\widetilde{\mathcal{A}})\} = \lfloor \frac{{}^{\#}(N_+(\widetilde{\mathcal{A}}) \setminus \{1\})}{2} \rfloor = 0$, and
- ${}^{\#}\{i \mid c_i = + \text{ and } i \in N_+(\mathcal{A}(\{2\}))\} = \lfloor \frac{{}^{\#}(N_+(\mathcal{A}(\{2\})) \setminus \{1\})}{2} \rfloor = 0$.

Thus $c_2 = -$.

Furthermore, when $a_1 \neq -$, we have

- $\{i \mid c_i = - \text{ and } i \in N_-(\widetilde{\mathcal{A}})\} \neq \emptyset$, and
- $\{i \mid c_i = - \text{ and } i \in N_-(\mathcal{A}(\{2\}))\} \neq \emptyset$.

Thus $\{i \mid c_i = - \text{ and } 3 \leq i \leq n\} \neq \emptyset$. Up to equivalence, we may assume $c_3 = -$. This gives conditions (1) and (2).

Case 2. $k = n - 1$.

By Lemmas 3.1.3 and 3.1.4, we have

- ${}^{\#}\{i \mid c_i = + \text{ and } i \in N_+(\widetilde{\mathcal{A}})\} = \lfloor \frac{{}^{\#}(N_+(\widetilde{\mathcal{A}}) \setminus \{1\})}{2} \rfloor = 1$, and

- ${}^{\#}\{i \mid c_i = + \text{ and } i \in N_+(\mathcal{A}(\{2\}))\} = \lfloor \frac{{}^{\#}(N_+(\mathcal{A}(\{2\})) \setminus \{1\})}{2} \rfloor = 0.$

Thus $c_2 = +$ and $c_n = -$.

Furthermore, when $a_1 \neq -$, we have

- $\{i \mid c_i = - \text{ and } i \in N_-(\widetilde{\mathcal{A}})\} \neq \emptyset$, and

- $\{i \mid c_i = - \text{ and } i \in N_-(\mathcal{A}(\{2\}))\} \neq \emptyset$.

Thus $\{i \mid c_i = - \text{ and } 3 \leq i \leq n-1\} \neq \emptyset$. Up to equivalence, we may assume $c_3 = -$. This gives conditions (3) and (4).

Case 3. $k = n - 2$.

In this case, $a_1 \neq +$. By Lemmas 3.1.3 and 3.1.4, we have

- ${}^{\#}\{i \mid c_i = + \text{ and } i \in N_+(\widetilde{\mathcal{A}})\} = \lfloor \frac{{}^{\#}(N_+(\widetilde{\mathcal{A}}) \setminus \{1\})}{2} \rfloor = 1$, and
- ${}^{\#}\{i \mid c_i = + \text{ and } i \in N_+(\mathcal{A}(\{2\}))\} = \lfloor \frac{{}^{\#}(N_+(\mathcal{A}(\{2\})) \setminus \{1\})}{2} \rfloor = 1.$

Thus $c_2 = -$ and one of c_{n-1} and c_n is $+$. Without loss of generality, we may assume that $c_{n-1} = -$ and $c_n = +$.

Furthermore, when $a_1 = 0$, we have

- $\{i \mid c_i = - \text{ and } i \in N_-(\widetilde{\mathcal{A}})\} \neq \emptyset$, and

- $\{i \mid c_i = - \text{ and } i \in N_-(\mathcal{A}(\{2\}))\} \neq \emptyset$.

Thus $\{i \mid c_i = - \text{ and } 3 \leq i \leq n-2\} \neq \emptyset$. Up to equivalence, we may assume $c_3 = -$. This gives conditions (5) and (6). □

Example 3.2.3. *Let us consider the first star sign pattern of order* 5 *in Theorem 3.0.2*

$$\mathcal{A}_1 = \begin{bmatrix} - & + & + & + & + \\ - & 0 & 0 & 0 & 0 \\ - & 0 & - & 0 & 0 \\ - & 0 & 0 & - & 0 \\ + & 0 & 0 & 0 & - \end{bmatrix}.$$

Note that $\mathcal{A}_1$ satisfies condition (1) in Theorem 3.2.2. We have that $\mathcal{A}_1$ is potentially stable since

$$B_1 = \begin{bmatrix} -1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 0 & 0 & 0 \\ -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} \in Q(\mathcal{A}_1)$$

with $\sigma(B_1) = \{-0.43016, -1, -1, -0.78492 \pm 1.30714i\}$ *and* $n_-(B_1) = 5$.

Now consider another 5×5 *star sign pattern*

$$\mathcal{S} = \begin{bmatrix} - & + & + & + & + \\ + & 0 & 0 & 0 & 0 \\ - & 0 & - & 0 & 0 \\ - & 0 & 0 & - & 0 \\ + & 0 & 0 & 0 & - \end{bmatrix}$$

which does not satisfy any of the conditions (1)–(6) in Theorem 3.2.2. For $\widetilde{\mathcal{S}}$ *we have* ${}^{\#}\{i \mid c_i = + \text{ and } i \in N_+(\widetilde{\mathcal{S}})\} = 1$ *and* $\lfloor \frac{{}^{\#}(N_+(\widetilde{\mathcal{S}}) \setminus \{1\})}{2} \rfloor = 0$. *Thus by Lemma 3.1.3,* $\widetilde{\mathcal{S}}$ *is not potentially stable and therefore by Lemma 3.1.5,* $\mathcal{S}$ *is not potentially stable.*

3.2.2 Star sign patterns $\mathcal{A}$ that are not sign stable

Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.2). By Lemma 3.1.6, if $\mathcal{A}$ is sign stable, then $k = n$, $a_1 \neq +$, and $c_i = -$ for $i = 2, 3, \ldots, n$. So we may assume that $\mathcal{A}$ has the form

$$\mathcal{A} = \begin{bmatrix} a_1 & + & + & \cdots & + \\ - & 0 & & & \\ - & & - & & \\ \vdots & & & \ddots & \\ - & & & & - \end{bmatrix}, \tag{3.3}$$

where $a_1 \in \{0, -\}$.

On the other hand, it is easy to check that for a star sign pattern $\mathcal{A}$ of the form (3.3), all conditions in Lemma 3.1.6 hold, that is, $\mathcal{A}$ is sign stable.

So we have the following result.

Theorem 3.2.4. *Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.2). Then $\mathcal{A}$ is sign stable if and only if $k = n$, $a_1 \neq +$, and $c_i = -$ for $i = 2, 3, \ldots, n$.*

Corollary 3.2.5. *Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.2). Then $\mathcal{A}$ is not sign stable if and only if at least one of the following three conditions holds.*

(1) $k < n$;

(2) $a_1 = +$;

(3) There exists at least one $c_i = +$ for $2 \leq i \leq n$.

Theorem 3.2.6. *Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.2). Assume that $\mathcal{A}$ has at most two positive diagonal entries and $\mathcal{A}$ is potentially stable. If $\mathcal{A}$ is not sign stable, then up to equivalence, $\mathcal{A}$ must satisfy one of the following conditions:*

(1) $k = n$, $a_1 = -$, $c_2 = -$, $c_n = +$;

(2) $k = n$, $a_1 = 0$, $c_2 = -$, $c_3 = -$, $c_n = +$;

(3) $k = n$, $a_1 = +$, $c_2 = -$, $c_3 = -$;

(4) $k = n - 1$, $a_1 = -$, $c_2 = +$, $c_n = -$;

(5) $k = n - 1$, $a_1 = 0$ or $+$, $c_2 = +$, $c_3 = -$, $c_n = -$;

(6) $k = n - 2$, $a_1 = -$, $c_2 = -$, $c_{n-1} = -$, $c_n = +$;

(7) $k = n - 2$, $a_1 = 0$, $c_2 = -$, $c_3 = -$, $c_{n-1} = -$, $c_n = +$.

Proof When $k = n$ and $a_1 \neq +$, by Corollary 3.2.5, some c_i $(i = 2, 3, \ldots, n)$ is positive. Up to equivalence, we may assume that $c_n = +$. The conclusion follows from Theorem 3.2.2 and Corollary 3.2.5. $\square$

Example 3.2.7. *Let us consider the first star sign pattern of order* 5 *in Theorem 3.0.2*

$$\mathcal{A}_1 = \begin{bmatrix} - & + & + & + & + \\ - & 0 & 0 & 0 & 0 \\ - & 0 & - & 0 & 0 \\ - & 0 & 0 & - & 0 \\ + & 0 & 0 & 0 & - \end{bmatrix}$$

Note that $\mathcal{A}_1$ satisfies condition (1) in Theorem 3.2.6. We know that $\mathcal{A}_1$ is potentially stable by Example 3.2.3 and $\mathcal{A}_1$ is not sign stable since it satisfies condition (3) in Corollary 3.2.5.

Now consider another 5×5 star sign pattern

$$\mathcal{S} = \begin{bmatrix} - & + & + & + & + \\ - & 0 & 0 & 0 & 0 \\ - & 0 & - & 0 & 0 \\ - & 0 & 0 & - & 0 \\ - & 0 & 0 & 0 & - \end{bmatrix}$$

which does not satisfy any of the conditions (1)–(7) in Theorem 3.2.6. We have that $\mathcal{S}$ is sign stable since it satisfies all conditions in Theorem 3.2.4.

3.2.3 Some other necessary conditions

Proposition 3.2.8. *Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.2) that requires $\mathbb{H}_n$. Then $\mathcal{A}$ does not contain any principal submatrix of the form*

$$\mathcal{T}_1 = \begin{bmatrix} * & + & + & + \\ + & * & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & - \end{bmatrix},$$

where each $*$ indicates an arbitrary element of $\{0,+,-\}$.

Proof Consider the sign pattern

$$\mathcal{A}_1 = \begin{bmatrix} 0 & + & + & + \\ + & 0 & 0 & 0 \\ + & 0 & - & 0 \\ - & 0 & 0 & - \end{bmatrix}$$

and

$$B_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 50 & 0 & -10 & 0 \\ -10 & 0 & 0 & -1 \end{bmatrix} \in Q(\mathcal{A}_1).$$

Since the eigenvalues of B_1 are approximately $-13.5119, 0.334449, 1.08874 \pm 1.01365i$, the refined inertia of B_1 is $(3,1,0,0)$. Then $\mathcal{A}_1$ allows $(3,1,0,0)$, and so $\mathcal{T}_1$ allows $(3,1,0,0)$. If $\mathcal{A}$ contains $\mathcal{T}_1$ as a principal submatrix, by taking the $(1,1),(2,2)$ entries and the entries of $\mathcal{A}$ outside $\mathcal{T}_1$ to be sufficiently small, we see that there exists $B \in Q(\mathcal{A})$ with $n_+(B) \geq 3$. Hence, $\mathcal{A}$ does not require $\mathbb{H}_n$. □

As the following four propositions can be proved similarly, we omit minor details in the proofs.

Proposition 3.2.9. *Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.2) that requires $\mathbb{H}_n$. Then $\mathcal{A}$ does not contain any principal submatrix of the form*

$$\mathcal{T}_2 = \begin{bmatrix} + & + & + & + & + \\ - & * & 0 & 0 & 0 \\ - & 0 & - & 0 & 0 \\ - & 0 & 0 & - & 0 \\ + & 0 & 0 & 0 & - \end{bmatrix},$$

where each $*$ indicates an arbitrary element of $\{0,+,-\}$.

Proof Take

$$B_1 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ -0.01 & 0 & 0 & 0 & 0 \\ -0.1 & 0 & -2 & 0 & 0 \\ -10 & 0 & 0 & -0.3 & 0 \\ 1 & 0 & 0 & 0 & -0.01 \end{bmatrix}.$$

Then $\sigma(B_1) = \{-1.98811, 0.000151185, 0.0227976, 0.327581 \pm 2.94089i\}$ and $n_+(B_1) = 4$. The result follows. $\square$

Proposition 3.2.10. *Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.2) that requires $\mathbb{H}_n$. Then $\mathcal{A}$ does not contain any principal submatrix of the form*

$$\mathcal{T}_3 = \begin{bmatrix} * & + & + & + \\ - & - & 0 & 0 \\ + & 0 & - & 0 \\ * & 0 & 0 & + \end{bmatrix},$$

where each $*$ indicates an arbitrary element of $\{0,+,-\}$.

Proof Take

$$B_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & -0.1 & 0 & 0 \\ 8 & 0 & -1 & 0 \\ 0 & 0 & 0 & 5 \end{bmatrix}.$$

Then $\sigma(B_1) = \{-3.2452, 0.0291243, 2.11608, 5\}$ and $n_+(B_1) = 3$. The result follows. $\square$

Proposition 3.2.11. *Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.2) that requires*

$\mathbb{H}_n$. *Then $\mathcal{A}$ does not contain any principal submatrix of the form*

$$\mathcal{T}_4 = \begin{bmatrix} * & + & + & + \\ - & * & 0 & 0 \\ - & 0 & - & 0 \\ - & 0 & 0 & + \end{bmatrix},$$

where each $*$ indicates an arbitrary element of $\{0, +, -\}$.

Proof Take

$$B_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ -10 & 0 & 0 & 0 \\ -10 & 0 & -1 & 0 \\ -10 & 0 & 0 & 10 \end{bmatrix}.$$

Then $\sigma(B_1) = \{-0.518339, 9.10668, 0.205828 \pm 4.5981i\}$ and $n_+(B_1) = 3$. The result follows.

□

Proposition 3.2.12. *Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.2) that requires $\mathbb{H}_n$. Then $\mathcal{A}$ does not contain any principal submatrix of the form*

$$\mathcal{T}_5 = \begin{bmatrix} * & + & + & + \\ + & * & 0 & 0 \\ * & 0 & + & 0 \\ * & 0 & 0 & + \end{bmatrix},$$

where each $*$ indicates an arbitrary element of $\{0, +, -\}$.

Proof Note that the matrix

$$\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

has three positive eigenvalues. The result follows. □

Using these propositions we now restrict the search for star sign patterns that require $\mathbb{H}_n$ to five cases up to equivalence.

Theorem 3.2.13. *Let $\mathcal{A}$ be an $n \times n$ star sign pattern in the form (3.2). Suppose that $\mathcal{A}$ requires $\mathbb{H}_n$. Then up to equivalence, $\mathcal{A}$ satisfies one of the following conditions.*

(1) $k = n$, $a_1 = -$, $c_i = -$ for $i = 2, 3, \ldots, n-1$, $c_n = +$;

(2) $k = n$, $a_1 = -$, $c_2 = -$, $c_i = +$ for $i = 3, 4, \ldots, n$;

(3) $k = n$, $a_1 = 0$, $c_i = -$ for $i = 2, 3, \ldots, n-1$, $c_n = +$;

(4) $k = n$, $a_1 = +$, $c_i = -$ for $i = 2, 3, \ldots, n$;

(5) $k = n-1$, $a_1 = -$, $c_i = +$ for $i = 2, 3, \ldots, n-1$, $c_n = -$.

Proof By Theorem 3.2.6, we may assume that up to equivalence, $\mathcal{A}$ satisfies one of the seven conditions in Theorem 3.2.6. We now examine these seven cases using Propositions 3.2.8–3.2.11.

Case 1. $k = n$, $a_1 = -$, $c_2 = -$, $c_n = +$.

Suppose that the signs of c_i for $i = 3, 4, \ldots, n-1$ are not the same. Up to equivalence, we may assume $c_3 = +$ and $c_4 = -$. Then $\mathcal{A}[\{1, 3, 4, n\}]$ is equivalent to $\mathcal{T}_1$. By Proposition 3.2.8, $\mathcal{A}$ does not require $\mathbb{H}_n$, a contradiction. Thus, we have $c_i = -$ for $i = 3, 4, \ldots, n-1$ or $c_i = +$ for $i = 3, 4, \ldots, n-1$. This yields conditions (1) and (2).

Case 2. $k = n$, $a_1 = 0$, $c_2 = -$, $c_3 = -$, $c_n = +$.

Suppose that there exists some c_i, where $i \in \{4, 5, \ldots, n-1\}$, such that $c_i = +$. Up to equivalence, we may assume that $c_4 = +$. Then $\mathcal{A}[\{1, 3, 4, n\}]$ is equivalent to $\mathcal{T}_1$. By Proposition 3.2.8, $\mathcal{A}$ does not require $\mathbb{H}_n$, a contradiction. Thus, we have $c_i = -$ for $i = 4, 5, \ldots, n-1$. This yields condition (3).

Case 3. $k = n$, $a_1 = +$, $c_2 = -$, $c_3 = -$.

Suppose that there are at least two c_i with $i \in \{4, 5, \ldots, n\}$ such that $c_i = +$. Up to equivalence, we may assume that $c_4 = c_5 = +$. Then $\mathcal{A}[\{1, 3, 4, 5\}]$ is equivalent to $\mathcal{T}_1$. By Proposition 3.2.8, $\mathcal{A}$ does not require $\mathbb{H}_n$, a contradiction.

Suppose that there is exactly one c_i with $i \in \{4, 5, \ldots, n\}$ such that $c_i = +$. Up to equivalence, we may assume that $c_4 = +$. Then $\mathcal{A}[\{1, 2, 3, 4, 5\}]$ is equivalent to $\mathcal{T}_2$. By Proposition 3.2.9, $\mathcal{A}$ does not require $\mathbb{H}_n$, a contradiction.

Thus, we have $c_i = -$ for $i = 4, 5, \ldots, n$. This yields condition (4).

Case 4. $k = n - 1$, $a_1 = -$, $c_2 = +$, $c_n = -$.

Suppose that there exist at least two c_i with $i \in \{3, 4, \ldots, n-1\}$ such that $c_i = -$. Up to equivalence, we may assume that $c_3 = c_4 = -$. Then $\mathcal{A}[\{1, 3, 4, n\}]$ is equivalent to $\mathcal{T}_4$. By Proposition 3.2.11, $\mathcal{A}$ does not require $\mathbb{H}_n$, a contradiction.

Suppose that there is exactly one c_i with $i \in \{3, 4, \ldots, n-1\}$ such that $c_i = -$. Up to equivalence, assume that $c_3 = -$. Then $\mathcal{A}[\{1, 3, 4, n\}]$ is equivalent to $\mathcal{T}_3$. By Proposition 3.2.10, $\mathcal{A}$ does not require $\mathbb{H}_n$, a contradiction.

Thus, we have $c_i = +$ for $i = 3, 4, \ldots, n-1$. This yields condition (5).

Case 5. $k = n - 1$, $a_1 = 0$ or $+$, $c_2 = +$, $c_3 = -$, $c_n = -$.

If $c_4 = +$, then $\mathcal{A}[\{1, 3, 4, n\}]$ is equivalent to $\mathcal{T}_3$. By Proposition 3.2.10, $\mathcal{A}$ does not require $\mathbb{H}_n$, a contradiction.

If $c_4 = -$, then $\mathcal{A}[\{1, 3, 4, n\}]$ is equivalent to $\mathcal{T}_4$. By Proposition 3.2.11, $\mathcal{A}$ does not require $\mathbb{H}_n$, a contradiction.

Therefore, this case cannot arise.

Case 6. $k = n - 2$, $a_1 = -$, $c_2 = -$, $c_{n-1} = -$, $c_n = +$.

If $c_3 = +$, then $\mathcal{A}[\{1, 3, n-1, n\}]$ is equivalent to $\mathcal{T}_5$. If $c_3 = -$, then $\mathcal{A}[\{1, 2, 3, n-1\}]$ is equivalent to $\mathcal{T}_4$. So, by Propositions 3.2.12 and 3.2.11, $\mathcal{A}$ does not require $\mathbb{H}_n$, a contradiction.

Therefore, this case cannot arise.

Case 7. $k = n - 2$, $a_1 = 0$, $c_2 = -$, $c_3 = -$, $c_{n-1} = -$, $c_n = +$.

Note that $\mathcal{A}[\{1, 2, 3, n-1\}]$ is equivalent to $\mathcal{T}_4$. By Proposition 3.2.11, $\mathcal{A}$ does not require $\mathbb{H}_n$, a contradiction. Thus this case cannot arise. $\square$

It is easy to check that these five conditions determine the five star sign patterns in Theorem 3.0.2. So the necessity of Theorem 3.0.2 is established.

3.3 Star sign patterns $\mathcal{A}_1, \ldots, \mathcal{A}_5$ require $\mathbb{H}_n$

In this section, we show that the $n \times n$ ($n \geq 5$) star sign patterns $\mathcal{A}_1, \ldots, \mathcal{A}_5$ (defined in Theorem 3.0.2) require $\mathbb{H}_n$.

3.3.1 Star sign patterns $\mathcal{A}_1, \ldots, \mathcal{A}_5$ allow $\mathbb{H}_n$

Theorem 3.3.1. *For all $n \geq 5$, star sign patterns $\mathcal{A}_1$ and $\mathcal{A}_3$ allow $\mathbb{H}_n$.*

Proof For convenience, we let

$$\mathcal{S} = \begin{bmatrix} * & + & \cdots & \cdots & \cdots & + \\ - & 0 & & & & \\ - & & - & & & \\ \vdots & & & \ddots & & \\ - & & & & - & \\ + & & & & & - \end{bmatrix}, \tag{3.4}$$

be an $n \times n$ ($n \geq 4$) star sign pattern, where $* \in \{-, 0\}$. Thus $\mathcal{S} = \mathcal{A}_1$ when $*$ is $-$, and $\mathcal{S} = \mathcal{A}_3$ when $*$ is 0.

It suffices to show that $\mathcal{S}$ allows $\mathbb{H}_n$ for all $n \geq 5$.

Note that the sign pattern $\mathcal{S}$ of order 4 is equivalent to $\mathcal{S}_1$ (if $* = -$) or $\mathcal{S}_3$ (if $* = 0$) where $\mathcal{S}_1$ and $\mathcal{S}_3$ are defined in Theorem 3.1.7. Since both $\mathcal{S}_1$ and $\mathcal{S}_3$ require $\mathbb{H}_4$, for each refined inertia $(0, 4, 0, 0)$, $(0, 2, 0, 2)$ and $(2, 2, 0, 0)$, there exist suitable values of

$c, a_1, a_2, a_3, b_1, b_2$, such that the 4×4 matrix

$$B_{4\times 4} = \begin{bmatrix} c & 1 & 1 & 1 \\ -a_1 & 0 & & \\ -a_2 & & -b_1 & \\ a_3 & & & -b_2 \end{bmatrix} \in Q(\mathcal{S}_{4\times 4})$$

has the corresponding refined inertia.

Consider the $n \times n$ $(n \geq 5)$ matrix

$$B_0 = \begin{bmatrix} c & 1 & 1 & 1 & \cdots & \cdots & 1 \\ -a_1 & 0 & & & & & \\ -\frac{a_2}{n-3} & & -b_1 & & & & \\ -\frac{a_2}{n-3} & & & -b_1 & & & \\ \vdots & & & & \ddots & & \\ -\frac{a_2}{n-3} & & & & & -b_1 & \\ a_3 & & & & & & -b_2 \end{bmatrix}.$$

Then $B_0 \in Q(\mathcal{S})$, and

$$\det(\lambda I - B_0) = \begin{vmatrix} \lambda - c & -1 & -1 & -1 & \cdots & \cdots & -1 \\ a_1 & \lambda & & & & & \\ \frac{a_2}{n-3} & & \lambda + b_1 & & & & \\ \frac{a_2}{n-3} & & & \lambda + b_1 & & & \\ \vdots & & & & \ddots & & \\ \frac{a_2}{n-3} & & & & & \lambda + b_1 & \\ -a_3 & & & & & & \lambda + b_2 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda - c & -1 & -1 & -1 & \cdots & \cdots & -1 \\ a_1 & \lambda & & & & & \\ a_2 & & \lambda + b_1 & \lambda + b_1 & \cdots & \lambda + b_1 & \\ \frac{a_2}{n-3} & & & \lambda + b_1 & & & \\ \vdots & & & & \ddots & & \\ \frac{a_2}{n-3} & & & & & \lambda + b_1 & \\ -a_3 & & & & & & \lambda + b_2 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda - c & -1 & -1 & 0 & \cdots & 0 & -1 \\ a_1 & \lambda & & & & & \\ a_2 & & \lambda + b_1 & & & & \\ \frac{a_2}{n-3} & & & \lambda + b_1 & & & \\ \vdots & & & & \ddots & & \\ \frac{a_2}{n-3} & & & & & \lambda + b_1 & \\ -a_3 & & & & & & \lambda + b_2 \end{vmatrix}$$

$$= (\lambda + b_1)^{n-4} \begin{vmatrix} \lambda - c & -1 & -1 & -1 \\ a_1 & \lambda & & \\ a_2 & & \lambda + b_1 & \\ -a_3 & & & \lambda + b_2 \end{vmatrix}$$

$$= (\lambda + b_1)^{n-4} \det(\lambda I - B_{4\times 4}).$$

So the multiset of the eigenvalues of B_0 is given by $\sigma(B_0) = \{-b_1, \ldots, -b_1\} \cup \sigma(B_{4\times 4})$, in which each set is interpreted as a multiset. It follows that $n_-(B_0) = n_-(B_{4\times 4}) + (n-4)$, $n_+(B_0) = n_+(B_{4\times 4})$, $n_z(B_0) = n_z(B_{4\times 4})$, and $2n_p(B_0) = 2n_p(B_{4\times 4})$. Thus the $n \times n$ sign pattern $\mathcal{S}$ allows $\mathbb{H}_n = \{(0, n, 0, 0),\ (0, n-2, 0, 2),\ (2, n-2, 0, 0)\}$. □

Theorem 3.3.2. *For $i = 2, 4, 5$, the sign patterns $\mathcal{A}_i$ allow $\mathbb{H}_n$ for each $n \geq 5$, .*

Proof Note that the 4×4 sign patterns $\mathcal{A}_2$, $\mathcal{A}_4$ and $\mathcal{A}_5$ are equivalent to $\mathcal{S}_2$, $\mathcal{S}_5$ and $\mathcal{S}_4$ defined in Theorem 3.1.7, respectively.

Thus for each of the refined inertias $(0,4,0,0)$, $(0,2,0,2)$ and $(2,2,0,0)$, there exist suitable values of $a_1, a_2, a_3, a_4, b_1, b_2$ such that

$$B_{4\times 4} = \begin{bmatrix} a_1 & 1 & 1 & 1 \\ a_2 & 0 & & \\ a_3 & & -b_1 & \\ a_4 & & & -b_2 \end{bmatrix} \in Q(\mathcal{A}_i)$$

has this refined inertia, where $i = 2, 4, 5$.

For $n \geq 5$, consider the $n \times n$ matrix

$$B_1 = \begin{bmatrix} a_1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ a_2 & 0 & & & & & \\ \frac{a_3}{n-3} & & -b_1 & & & & \\ \frac{a_3}{n-3} & & & -b_1 & & & \\ \vdots & & & & \ddots & & \\ \frac{a_3}{n-3} & & & & & -b_1 & \\ a_4 & & & & & & -b_2 \end{bmatrix}.$$

Then $B_1 \in Q(\mathcal{A}_i)$ for $i = 2, 4, 5$, and

$$\det(\lambda I - B_1) = \begin{vmatrix} \lambda - a_1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ -a_2 & \lambda & & & & & \\ -\frac{a_3}{n-3} & & \lambda + b_1 & & & & \\ -\frac{a_3}{n-3} & & & \lambda + b_1 & & & \\ \vdots & & & & \ddots & & \\ -\frac{a_3}{n-3} & & & & & \lambda + b_1 & \\ -a_4 & & & & & & \lambda + b_2 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda - a_1 & -1 & -1 & -1 & \cdots & -1 & -1 \\ -a_2 & \lambda & & & & & \\ -a_3 & & \lambda + b_1 & \lambda + b_1 & \cdots & \lambda + b_1 & \\ -\frac{a_3}{n-3} & & & \lambda + b_1 & & & \\ \vdots & & & & \ddots & & \\ -\frac{a_3}{n-3} & & & & & \lambda + b_1 & \\ -a_4 & & & & & & \lambda + b_2 \end{vmatrix}$$

$$= \begin{vmatrix} \lambda - a_1 & -1 & -1 & 0 & \cdots & 0 & -1 \\ -a_2 & \lambda & & & & & \\ -a_3 & & \lambda + b_1 & & & & \\ -\frac{a_3}{n-3} & & & \lambda + b_1 & & & \\ \vdots & & & & \ddots & & \\ -\frac{a_3}{n-3} & & & & & \lambda + b_1 & \\ -a_4 & & & & & & \lambda + b_2 \end{vmatrix}$$

$$= (\lambda + b_1)^{n-4} \begin{vmatrix} \lambda - a_1 & -1 & -1 & -1 \\ -a_2 & \lambda & & \\ -a_3 & & \lambda + b_1 & \\ -a_4 & & & \lambda + b_2 \end{vmatrix}$$

$$= (\lambda + b_1)^{n-4} |\lambda I - B_{4\times 4}|.$$

So the multiset of the eigenvalues of B_1 is given by $\sigma(B_1) = \{-b_1, \ldots, -b_1\} \cup \sigma(B_{4\times4})$, in which each set is interpreted as a multiset. It follows that $n_-(B_1) = n_-(B_{4\times4}) + (n-4)$, $n_+(B_1) = n_+(B_{4\times4})$, $n_z(B_1) = n_z(B_{4\times4})$, and $2n_p(B_1) = 2n_p(B_{4\times4})$. Thus the $n \times n$ sign pattern $\mathcal{A}_i$ allows $\mathbb{H}_n = \{(0, n, 0, 0),\ (0, n-2, 0, 2),\ (2, n-2, 0, 0)\}$ for $i = 2, 4, 5$. $\square$

3.3.2 Star sign patterns $\mathcal{A}_1$ and $\mathcal{A}_3$ require $\mathbb{H}_n$

Throughout what follows, we let B denote a real matrix of order n of the form

$$B = \begin{bmatrix} c & 1 & 1 & \cdots & \cdots & 1 \\ -a_1 & 0 & & & & \\ -a_2 & & -b_1 & & & \\ \vdots & & & \ddots & & \\ -a_{n-2} & & & & -b_{n-3} & \\ a_{n-1} & & & & & -b_{n-2} \end{bmatrix}, \tag{3.5}$$

with $n \geq 5$, $c \leq 0$, $a_i > 0$ and $b_j > 0$ for $i = 1, 2, \ldots, n-1$ and $j = 1, 2, \ldots, n-2$. Then $B \in Q(\mathcal{S})$, where $\mathcal{S}$ is the $n \times n$ sign pattern in form (3.4).

First, we consider the case that all the b_i $(i = 1, 2, \ldots, n-2)$ are distinct and show the following result.

Theorem 3.3.3. *If all the b_i $(i = 1, 2, \ldots, n-2)$ are distinct, then* $\mathrm{ri}(B) \in \mathbb{H}_n$.

Since all the b_i $(i = 1, 2, \ldots, n-2)$ are distinct, if necessary, by doing some permutationally similarity, we may assume that B satisfies one of the following three conditions:

(1) $b_1 > b_2 > \cdots > b_{n-3} > b_{n-2}$;

(2) $b_{n-2} > b_1 > b_2 > \cdots > b_{n-3}$;

(3) There is an m with $1 \leq m \leq n-4$ such that $b_1 > b_2 > \cdots > b_{n-3}$ and $b_m > b_{n-2} > b_{m+1}$.

The following three lemmas are straightforward.

Lemma 3.3.4. *For each $i = 1, 2, \ldots, n-3$,*

$$\det(b_i I + B) = a_{i+1} b_i \prod_{\substack{j=1 \\ j \neq i}}^{n-2} (b_i - b_j),$$

and

$$\det(b_{n-2}I + B) = -a_{n-1}b_{n-2}\prod_{j=1}^{n-3}(b_{n-2} - b_j).$$

Lemma 3.3.5. *(1) If $b_1 > b_2 > \cdots > b_{n-3} > b_{n-2}$, then*

$$\operatorname{sgn}(\det(b_i I + B)) = \begin{cases} (-)^{i-1}, & \text{for } i = 1, 2, \ldots, n-3; \\ (-)^{n-2}, & \text{for } i = n-2. \end{cases}$$

(2) If $b_{n-2} > b_1 > b_2 > \cdots > b_{n-3}$, then

$$\operatorname{sgn}(\det(b_i I + B)) = \begin{cases} (-)^{i}, & \text{for } i = 1, 2, \ldots, n-3; \\ -, & \text{for } i = n-2. \end{cases}$$

(3) If there is an m with $1 \le m \le n-4$ such that $b_1 > \cdots > b_{n-3}$ and $b_m > b_{n-2} > b_{m+1}$, then

$$\operatorname{sgn}(\det(b_i I + B)) = \begin{cases} (-)^{i-1}, & \text{for } i = 1, \ldots, m; \\ (-)^{i}, & \text{for } i = m+1, \ldots, n-3; \\ (-)^{m+1}, & \text{for } i = n-2. \end{cases}$$

Lemma 3.3.6. *None of the $-b_i$ $(i = 1, 2, \ldots, n-2)$ is an eigenvalue of B.*

Lemma 3.3.7. $\operatorname{sgn}(\det(B)) = (-)^n$. *Furthermore, $n_z(B) = 0$, and $n_-(B)$ and n have the same parity.*

Proof Expanding the determinant along the second column reveals that $\operatorname{sgn}(\det(B)) = (-)^n$. Consequently, $n_z(B) = 0$. Since we also have that $\operatorname{sgn}(\det(B)) = (-)^{n_-(B)}$, $n_-(B)$ and n have the same parity. □

Lemma 3.3.8. *Let $k = n_-(B)$. Then $k \ge n-2$.*

Proof Consider the following two cases.

Case 1. $b_1 > b_2 > \cdots > b_{n-3} > b_{n-2}$, or $b_{n-2} > b_1 > b_2 > \cdots > b_{n-3}$.

Observe that by Lemma 3.3.5, the real function $p(t) = \det(tI - B)$ takes on nonzero values of opposite signs at $-b_j$ and $-b_{j+1}$, for $j = 1, \ldots, n-4$. Thus, by the Intermediate Value Theorem, $p(t)$ has at least one real zero in each open interval $(-b_j, -b_{j+1})$. It follows that the matrix B has at least one real eigenvalue in the open interval $(-b_j, -b_{j+1})$ for $j = 1, \ldots, n-4$. So B has at least $n-4$ negative eigenvalues. Let λ_j be an eigenvalue of B with $\lambda_j \in (-b_j, -b_{j+1})$ for $j = 1, \ldots, n-4$. It is easy to see that

$$\sum_{j=1}^{n-4} \lambda_j > -\sum_{i=1}^{n-4} b_i > c - \sum_{i=1}^{n-2} b_i = \operatorname{tr}(B).$$

Note that the sum of all the eigenvalues of B is equal to $\operatorname{tr}(B)$. Thus, besides λ_j ($j = 1, \ldots, n-4$), B has at least one more eigenvalue with negative real part. So $k \geq n-3$. By Lemma 3.3.7, k and n have the same parity. It follows that $k \geq n-2$.

Case 2. There is an m with $1 \leq m \leq n-4$ such that $b_1 > \cdots > b_{n-3}$ and $b_m > b_{n-2} > b_{m+1}$.

By Lemma 3.3.5, the real function $p(t) = \det(tI - B)$ takes on nonzero values of opposite signs at $-b_j$ and $-b_{j+1}$, for $j = 1, \ldots, m-1, m+1, \ldots, n-4$. Note that $\operatorname{sgn}(p(-b_{n-3})) = \operatorname{sgn}(\det(-b_{n-3}I - B)) = -$, and $\operatorname{sgn}(p(0)) = \operatorname{sgn}(\det(-B)) = +$. Thus, by the Intermediate Value Theorem, $p(t)$ has at least one real zero in each of the open intervals $(-b_j, -b_{j+1})$ for $j = 1, \ldots, m-1, m+1, \ldots, n-4$, and the open interval $(-b_{n-3}, 0)$. Thus, the matrix B has at least $n-4$ negative eigenvalues. Similarly as in Case 1, using an inequality involving the trace and the parity result, we get $k \geq n-2$. $\square$

We now complete the proof of Theorem 3.3.10.

Proof of Theorem 3.3.10 By Lemmas 3.3.7 and 3.3.8, we have $n_z(B) = 0$ and $n_-(B) = n-2$ or $n_-(B) = n$. It follows that $\operatorname{ri}(B) \in \mathbb{H}_n$. $\square$

We are now ready to establish the main result of this section.

Theorem 3.3.9. *For $n \geq 5$, the star sign patterns $\mathcal{A}_1$ and $\mathcal{A}_3$ require $\mathbb{H}_n$.*

Proof We only need to prove that the $n \times n$ sign pattern $\mathcal{S}$ in form (3.4) requires $\mathbb{H}_n$ for

$n \geq 5$. We proceed by induction on the order n.

By Theorem 3.1.7, the 4×4 sign pattern $\mathcal{S}$ requires $\mathbb{H}_4$. Suppose that the $(n-1)\times(n-1)$ sign pattern $\mathcal{S}$ requires $\mathbb{H}_{n-1}$. We prove that the $n \times n$ sign pattern $\mathcal{S}$ requires $\mathbb{H}_n$. Note that we have proved that $\mathcal{S}$ allows $\mathbb{H}_n$ in Theorem 3.3.1. Thus we only need to prove that $\mathrm{ri}(B) \in \mathbb{H}_n$ for each $B \in Q(\mathcal{S})$.

For any $B \in Q(\mathcal{S})$, by performing a diagonal similarity on B if necessary, we may assume B has the form (3.5). If all the b_i $(i = 1, 2, \ldots, n-2)$ are distinct, then $\mathrm{ri}(B) \in \mathbb{H}_n$ by Theorem 3.3.10.

Now suppose at least two of b_i $(i = 1, 2, \ldots, n-2)$ are the same. We consider the following two cases.

Case 1. $b_i = b_j$ for some i, j with $1 \leq i < j \leq n-3$.

Then

$$\det(\lambda I - B) = \begin{vmatrix} \lambda - c & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\ a_1 & \lambda & & & & & & & \\ a_2 & & \lambda + b_1 & & & & & & \\ \vdots & & & \ddots & & & & & \\ a_{i+1} & & & & \lambda + b_i & & & & \\ \vdots & & & & & \ddots & & & \\ a_{j+1} & & & & & & \lambda + b_j & & \\ \vdots & & & & & & & \ddots & \\ -a_{n-1} & & & & & & & & \lambda + b_{n-2} \end{vmatrix}.$$

By adding row $i+2$ to row $j+2$ and subtracting column $j+2$ from column $i+2$, we have

$$\det(\lambda I - B) = \begin{vmatrix} \lambda - c & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\ a_1 & \lambda & & & & & & & \\ a_2 & & \lambda + b_1 & & & & & & \\ \vdots & & & \ddots & & & & & \\ a_{i+1} & & & & \lambda + b_i & & & & \\ \vdots & & & & & \ddots & & & \\ a_{i+1} + a_{j+1} & & & & \lambda + b_i & & \lambda + b_j & & \\ \vdots & & & & & & & \ddots & \\ -a_{n-1} & & & & & & & & \lambda + b_{n-2} \end{vmatrix}$$

$$= \begin{vmatrix} \lambda - c & -1 & \cdots & -1 & 0 & -1 & \cdots & \cdots & -1 \\ a_1 & \lambda & & & & & & & \\ a_2 & & \lambda + b_1 & & & & & & \\ \vdots & & & \ddots & & & & & \\ a_{i+1} & & & & \lambda + b_i & & & & \\ \vdots & & & & & \ddots & & & \\ a_{i+1} + a_{j+1} & & & & & & \lambda + b_j & & \\ \vdots & & & & & & & \ddots & \\ -a_{n-1} & & & & & & & & \lambda + b_{n-2} \end{vmatrix}.$$

By expanding about column $i+2$, we obtain that $\det(\lambda I - B)$ is equal to the following in terms of the determinant of the $(n-1) \times (n-1)$ matrix

$$(\lambda + b_i) \begin{vmatrix} \lambda - c & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\ a_1 & \lambda & & & & & & & & \\ a_2 & & \lambda + b_1 & & & & & & & \\ \vdots & & & \ddots & & & & & & \\ a_i & & & & \lambda + b_{i-1} & & & & & \\ a_{i+2} & & & & & \lambda + b_{i+1} & & & & \\ \vdots & & & & & & \ddots & & & \\ a_{i+1} + a_{j+1} & & & & & & & \lambda + b_j & & \\ \vdots & & & & & & & & \ddots & \\ -a_{n-1} & & & & & & & & & \lambda + b_{n-2} \end{vmatrix}.$$

Take the $(n-1) \times (n-1)$ matrix

$$B_1 = \begin{bmatrix} c & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ -a_1 & 0 & & & & & & & & \\ -a_2 & & -b_1 & & & & & & & \\ \vdots & & & \ddots & & & & & & \\ -a_i & & & & -b_{i-1} & & & & & \\ -a_{i+2} & & & & & -b_{i+1} & & & & \\ \vdots & & & & & & \ddots & & & \\ -(a_{i+1} + a_{j+1}) & & & & & & & -b_j & & \\ \vdots & & & & & & & & \ddots & \\ a_{n-1} & & & & & & & & & -b_{n-2} \end{bmatrix}.$$

Then

$$\sigma(B) = \{-b_i\} \cup \sigma(B_1),$$

where the sets are interpreted as multisets. Note that $B_1 \in Q(\mathcal{S})$ has order $n-1$. By the induction hypothesis, $\mathcal{S}$ of order $n-1$ requires $\mathbb{H}_{n-1} = \{(0, n-1, 0, 0),\ (0, n-3, 0, 2),\ (2, n-3, 0, 0)\}$. Thus $\mathrm{ri}(B)$ is one of $(0, n, 0, 0)$, $(0, n-2, 0, 2)$, $(2, n-2, 0, 0)$. It follows that $\mathrm{ri}(B) \in \mathbb{H}_n$.

Case 2. $b_i = b_{n-2}$ for some i with $1 \leq i \leq n-3$.

Then

$$\det(\lambda I - B) = \begin{vmatrix} \lambda - c & -1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 \\ a_1 & \lambda & & & & & & \\ a_2 & & \lambda + b_1 & & & & & \\ \vdots & & & \ddots & & & & \\ a_{i+1} & & & & \lambda + b_i & & & \\ \vdots & & & & & \ddots & & \\ a_{n-2} & & & & & & \lambda + b_{n-3} & \\ -a_{n-1} & & & & & & & \lambda + b_{n-2} \end{vmatrix}.$$

By adding row $i+2$ to row n and subtracting column n from column $i+2$, we have

$$\det(\lambda I - B) = \begin{vmatrix} \lambda - c & -1 & -1 & \cdots & \cdots & \cdots & \cdots & -1 \\ a_1 & \lambda & & & & & & \\ a_2 & & \lambda + b_1 & & & & & \\ \vdots & & & \ddots & & & & \\ a_{i+1} & & & & \lambda + b_i & & & \\ \vdots & & & & & \ddots & & \\ a_{n-2} & & & & & & \lambda + b_{n-3} & \\ a_{i+1} - a_{n-1} & & & & \lambda + b_i & & & \lambda + b_{n-2} \end{vmatrix}$$

$$= \begin{vmatrix} \lambda - c & -1 & \cdots & -1 & 0 & -1 & \cdots & -1 \\ a_1 & \lambda & & & & & & \\ a_2 & & \lambda + b_1 & & & & & \\ \vdots & & & \ddots & & & & \\ a_{i+1} & & & & \lambda + b_i & & & \\ \vdots & & & & & \ddots & & \\ a_{n-2} & & & & & & \lambda + b_{n-3} & \\ a_{i+1} - a_{n-1} & & & & & & & \lambda + b_{n-2} \end{vmatrix}.$$

By expanding about column $i+2$, we obtain that $\det(\lambda I - B)$ is equal to the following in terms of the determinant of the $(n-1) \times (n-1)$ matrix

$$(\lambda + b_i) \begin{vmatrix} \lambda - c & -1 & -1 & \cdots & \cdots & \cdots & \cdots & \cdots & -1 \\ a_1 & \lambda & & & & & & & \\ a_2 & & \lambda + b_1 & & & & & & \\ \vdots & & & \ddots & & & & & \\ a_i & & & & \lambda + b_{i-1} & & & & \\ a_{i+2} & & & & & \lambda + b_{i+1} & & & \\ \vdots & & & & & & \ddots & & \\ a_{n-2} & & & & & & & \lambda + b_{n-3} & \\ a_{i+1} - a_{n-1} & & & & & & & & \lambda + b_{n-2} \end{vmatrix}.$$

Take the $(n-1)\times(n-1)$ matrix

$$B_2 = \begin{bmatrix} c & 1 & 1 & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ -a_1 & 0 & & & & & & & \\ -a_2 & & -b_1 & & & & & & \\ \vdots & & & \ddots & & & & & \\ -a_i & & & & -b_{i-1} & & & & \\ -a_{i+2} & & & & & -b_{i+1} & & & \\ \vdots & & & & & & \ddots & & \\ -a_{n-2} & & & & & & & -b_{n-3} & \\ a_{n-1}-a_{i+1} & & & & & & & & -b_{n-2} \end{bmatrix}.$$

Then $\sigma(B) = \{-b_i\} \cup \sigma(B_2)$.

If $a_{n-1} - a_{i+1} > 0$, then similarly as in Case 1, $\mathrm{ri}(B) \in \mathbb{H}_n$.

If $a_{n-1} - a_{i+1} = 0$, then $\sigma(B) = \{-b_i, -b_{n-2}\} \cup \sigma(B_3)$, where

$$B_3 = \begin{bmatrix} c & 1 & 1 & \cdots & \cdots & \cdots & \cdots & 1 \\ -a_1 & 0 & & & & & & \\ -a_2 & & -b_1 & & & & & \\ \vdots & & & \ddots & & & & \\ -a_i & & & & -b_{i-1} & & & \\ -a_{i+2} & & & & & -b_{i+1} & & \\ \vdots & & & & & & \ddots & \\ -a_{n-2} & & & & & & & -b_{n-3} \end{bmatrix}_{(n-2)\times(n-2)}.$$

Since $\mathrm{sgn}(B_3)$ is sign stable by Theorem 3.2.4, $\mathrm{ri}(B_3) = (0, n-2, 0, 0)$, and so $\mathrm{ri}(B) = (0, n, 0, 0) \in \mathbb{H}_n$.

If $a_{n-1} - a_{i+1} < 0$, then $\mathrm{sgn}(B_2)$ is sign stable by Theorem 3.2.4, and so $\mathrm{ri}(B_2) = (0, n-1, 0, 0)$, and $\mathrm{ri}(B) = (0, n, 0, 0) \in \mathbb{H}_n$.

The theorem now follows. $\square$

3.3.3 Star sign patterns $\mathcal{A}_2$, $\mathcal{A}_4$, and $\mathcal{A}_5$ require $\mathbb{H}_n$

Throughout what follows, we let B denote a real matrix of order $n \geq 5$ of the form

$$B = \begin{bmatrix} a_1 & 1 & 1 & \cdots & 1 & 1 \\ a_2 & 0 & & & & \\ a_3 & & -b_1 & & & \\ \vdots & & & \ddots & & \\ a_{n-1} & & & & -b_{n-3} & \\ a_n & & & & & -b_{n-2} \end{bmatrix}, \tag{3.6}$$

where $b_j > 0$ for $j = 1, 2, \ldots, n-3$, and suitable real values for $a_1, a_2, \ldots, a_n$ and b_{n-2} are taken so that $B \in Q(\mathcal{A}_i)$ for some $i \in \{2, 4, 5\}$.

First, we consider the case that all the b_j are distinct for $j = 1, 2, \ldots, n-2$ and show the following result.

Theorem 3.3.10. *If all the b_j $(j = 1, 2, \ldots, n-2)$ are distinct, then* $\mathrm{ri}(B) \in \mathbb{H}_n$.

Since all the b_j $(j = 1, 2, \ldots, n-2)$ are distinct, if necessary, by doing some permutationally similarity, we may assume that B is subjected to $b_1 > b_2 > \cdots > b_{n-2}$ in Theorem 3.3.10. To prove Theorem 3.3.10, we need the following lemmas. We also assume that $b_1 > b_2 > \cdots > b_{n-2}$ in Lemmas 3.3.11–3.3.17.

Lemma 3.3.11. *For each $j = 1, 2, \ldots, n-2$,*

$$\det(b_j I + B) = -a_{j+2} b_j \prod_{\substack{m=1 \\ m \neq j}}^{n-2} (b_j - b_m).$$

Proof Note that row $j+2$ as well as column $j+2$ of $b_j I + B$ has exactly one nonzero entry, namely the first entry, which may be used to zero out all other entries in the first row or the first column without affecting the determinant. Hence,

$$\det(b_jI+B) = \begin{vmatrix} 0 & 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & b_j & & & & & & & \\ 0 & & b_j-b_1 & & & & & & \\ \vdots & & & \ddots & & & & & \\ 0 & & & & b_j-b_{j-1} & & & & \\ a_{j+2} & & & & & 0 & & & \\ 0 & & & & & & b_j-b_{j+1} & & \\ \vdots & & & & & & & \ddots & \\ 0 & & & & & & & & b_j-b_{n-2} \end{vmatrix}$$

$$= -a_{j+2}b_j \prod_{\substack{m=1 \\ m\neq j}}^{n-2}(b_j-b_m). \qquad \square$$

In view of Lemma 3.3.11, the following two results are straightforward.

Lemma 3.3.12. *Suppose $B \in Q(\mathcal{A}_i)$ for $i \in \{2,4,5\}$. Then*

$$\operatorname{sgn}(\det(b_jI+B)) = \begin{cases} (-)^{j+1} & \textit{for } j=1,2,\ldots,n-2 \textit{ if } i=4; \\ (-)^j & \textit{for } j=1,2,\ldots,n-2 \textit{ if } i=2; \\ (-)^j & \textit{for } j=1,2,\ldots,n-3 \textit{ if } i=5. \end{cases}$$

Lemma 3.3.13. *None of the $-b_j$ $(j=1,2,\ldots,n-2)$ is an eigenvalue of B.*

Lemma 3.3.14. *Suppose $B \in Q(\mathcal{A}_i)$ for $i \in \{2,4,5\}$. Then $n_-(B) \geq n-4 \geq 1$. Furthermore, if $i \neq 5$, $n_-(B) \geq n-3$.*

Proof Observe that by Lemma 3.3.12, the real function $p(t) = \det(tI-B)$ takes on nonzero values of opposite signs at $-b_j$ and $-b_{j+1}$, for $j=1,2,\ldots,n-4$. Thus, by the Intermediate Value Theorem, $p(t)$ has at least one real zero in each open interval $(-b_j,-b_{j+1})$. It follows that the matrix B has at least one real eigenvalue in $(-b_j,-b_{j+1})$, for $j=1,2,\ldots,n-4$.

Thus $n_-(B) \geq n-4 \geq 1$. Furthermore, if $i \in \{2,4\}$, then by Lemma 3.3.12, B has at least one real eigenvalue in $(-b_j, -b_{j+1})$, for $j = 1, 2, \ldots, n-3$, so we have $n_-(B) \geq n-3$. $\square$

Lemma 3.3.15. $\operatorname{sgn}(\det(B)) = (-)^n$. *Furthermore,* $n_z(B) = 0$, *and* $n_-(B)$ *and* n *have the same parity.*

Proof Expanding the determinant along the second column reveals that $\operatorname{sgn}(\det(B)) = (-)^n$. Consequently, $n_z(B) = 0$. Since we also have that $\operatorname{sgn}(\det(B)) = (-)^{n_-(B)}$, $n_-(B)$ and n have the same parity. $\square$

For any $r \in \mathbb{R}$, define $\Delta(r)$ to be the number of eigenvalues λ of B in the closed left half-plane with $\operatorname{Re}(\lambda) \leq -r$. It is clear that

$$n_-(B) \geq \Delta(b_{n-3}) = \Delta(b_1) + \sum_{j=1}^{n-4}[\Delta(b_{j+1}) - \Delta(b_j)]. \tag{3.7}$$

Lemma 3.3.16. *For* $j = 1, 2, \ldots, n-3$, $n_-(b_jI + B)$ *and* $\Delta(b_j)$ *have the same parity.*

Proof Note that λ is an eigenvalue of B if and only if $b_j + \lambda$ is an eigenvalue of $b_jI + B$, that the non-real eigenvalues of $b_jI + B$ occur in conjugate pairs, and that $-b_1, -b_2, \ldots, -b_{n-3}$ are not eigenvalues of B by Lemma 3.3.13. We see that for $j = 1, 2, \ldots, n-3$,

- $n_-(b_jI + B) =$ the number of eigenvalues λ of B satisfying $\operatorname{Re}(\lambda) < -b_j$;
- $\Delta(b_j) =$ the number of eigenvalues λ of B satisfying $\operatorname{Re}(\lambda) \leq -b_j$;
- the number of eigenvalues λ of B satisfying $\operatorname{Re}(\lambda) = -b_j$ is even.

So $n_-(b_jI + B)$ and $\Delta(b_j)$ have the same parity. $\square$

Lemma 3.3.17. *Suppose* $B \in Q(\mathcal{A}_i)$ *for* $i \in \{2,4,5\}$. *Let* $k = n_-(B)$. *Then* $k \geq n-2$.

Proof If $i = 2$ or $i = 4$, then by Lemma 3.3.14, we have $k = n_-(B) \geq n-3$. By Lemma 3.3.15, k and n have the same parity. It follows that $k \geq n-2$, as desired. Hence, assume $i = 5$.

We claim that for every $j \leq n-3$, the parity of j and $\Delta(b_j)$ are the same. Otherwise, if there exists an even index $j \leq n-3$ such that $\Delta(b_j)$ is odd, by Lemmas 3.3.12 and 3.3.16, we have that $\det(b_jI+B) > 0$ and $n_-(b_jI+B)$ is odd, which is a contradiction; if there exists an odd index $j \leq n-3$ such that $\Delta(b_j)$ is even, by Lemmas 3.3.12 and 3.3.16, we have that $\det(b_jI+B) < 0$ and $n_-(b_jI+B)$ is even, which is a contradiction.

Thus $\Delta(b_1)$ is odd, and $\Delta(b_{j+1}) - \Delta(b_j) > 0$ is odd for $1 \leq j \leq n-4$. So by (3.1),

$$k \geq \Delta(b_{n-3}) = \Delta(b_1) + \sum_{j=1}^{n-4}[\Delta(b_{j+1}) - \Delta(b_j)] \geq n-3.$$

By Lemma 3.3.15, k and n have the same parity. It follows that $k \geq n-2$. □

We now complete the proof of Theorem 3.3.10.

Proof of Theorem 3.3.10 By Lemmas 3.3.15 and 3.3.17, we have $n_z(B) = 0$ and $n_-(B) = n-2$ or $n_-(B) = n$. It follows that $\mathrm{ri}(B) \in \mathbb{H}_n$. □

We are now ready to establish the main result of this section.

Theorem 3.3.18. *For $i = 2, 4, 5$, the $n \times n$ sign patterns $\mathcal{A}_i$ require $\mathbb{H}_n$ for each $n \geq 5$.*

Proof Fix any $i \in \{2, 4, 5\}$. We proceed by induction on the order n of $\mathcal{A}_i$.

By Lemma 3.1.7, the result holds for $n = 4$.

Suppose that the $(n-1) \times (n-1)$ sign pattern $\mathcal{A}_i$ requires $\mathbb{H}_{n-1}$ for some $n \geq 5$. We prove that the $n \times n$ sign pattern $\mathcal{A}_i$ requires $\mathbb{H}_n$. By Theorem 3.3.2, $\mathcal{A}_i$ allows $\mathbb{H}_n$. Thus we only need to prove that $\mathrm{ri}(B) \in \mathbb{H}_n$ for every $B \in Q(\mathcal{A}_i)$.

For any $B \in Q(\mathcal{A}_i)$, by performing a diagonal similarity on B if necessary, we may assume that B has the form (3.6). If all the b_j are distinct for $j = 1, 2, \ldots, n-2$, then by Theorem 3.3.10 $\mathrm{ri}(B) \in \mathbb{H}_n$.

Now suppose that two of the b_j are the same for $j = 1, 2, \ldots, n-2$. Note that in the case of $B \in Q(\mathcal{A}_3)$, b_{n-2} is different from each b_j with $j \leq n-3$ as $b_j > 0 > b_{n-2}$. By performing a permutationally similarity if necessary, without loss of generality, we may assume that $b_1 = b_2$. Then

$$\det(\lambda I - B) = \begin{vmatrix} \lambda - a_1 & -1 & -1 & -1 & -1 & \cdots & -1 \\ -a_2 & \lambda & & & & & \\ -a_3 & & \lambda + b_1 & & & & \\ -a_4 & & & \lambda + b_1 & & & \\ -a_5 & & & & \lambda + b_3 & & \\ \vdots & & & & & \ddots & \\ -a_n & & & & & & \lambda + b_{n-2} \end{vmatrix}.$$

By adding row 3 to row 4 and subtracting column 4 from column 3, we have

$$\det(\lambda I - B) = \begin{vmatrix} \lambda - a_1 & -1 & -1 & -1 & -1 & \cdots & -1 \\ -a_2 & \lambda & & & & & \\ -a_3 & & \lambda + b_1 & & & & \\ -a_3 - a_4 & & \lambda + b_1 & \lambda + b_1 & & & \\ -a_5 & & & & \lambda + b_3 & & \\ \vdots & & & & & \ddots & \\ -a_n & & & & & & \lambda + b_{n-2} \end{vmatrix}$$

$$= \begin{vmatrix} \lambda - a_1 & -1 & 0 & -1 & -1 & \cdots & -1 \\ -a_2 & \lambda & & & & & \\ -a_3 & & \lambda + b_1 & & & & \\ -a_3 - a_4 & & & \lambda + b_1 & & & \\ -a_5 & & & & \lambda + b_3 & & \\ \vdots & & & & & \ddots & \\ -a_n & & & & & & \lambda + b_{n-2} \end{vmatrix}.$$

By expanding about column 3, we obtain that $\det(\lambda I - B)$ is equal to the following in terms of the determinant of the $(n-1)\times(n-1)$ matrix

$$(\lambda+b_1)\begin{vmatrix} \lambda-a_1 & -1 & -1 & -1 & \cdots & -1 \\ -a_2 & \lambda & & & & \\ -a_3-a_4 & & \lambda+b_1 & & & \\ -a_5 & & & \lambda+b_3 & & \\ \vdots & & & & \ddots & \\ -a_n & & & & & \lambda+b_{n-2} \end{vmatrix}.$$

Take the $(n-1)\times(n-1)$ matrix

$$B_1=\begin{bmatrix} a_1 & 1 & 1 & 1 & \cdots & 1 \\ a_2 & 0 & & & & \\ (a_3+a_4) & & -b_1 & & & \\ a_5 & & & -b_3 & & \\ \vdots & & & & \ddots & \\ a_n & & & & & -b_{n-2} \end{bmatrix}.$$

Then

$$\sigma(B)=\{-b_1\}\cup\sigma(B_1),$$

where the sets are interpreted as multisets. Note that $B_1\in Q(\mathcal{A}_i)$ has order $n-1$. By the induction hypothesis, $\mathcal{A}_i$ of order $n-1$ requires $\mathbb{H}_{n-1}=\{(0,n-1,0,0),\ (0,n-3,0,2),\ (2,n-3,0,0)\}$. Thus $\mathrm{ri}(B)$ is one of $(0,n,0,0)$, $(0,n-2,0,2)$ and $(2,n-2,0,0)$. It follows that $\mathrm{ri}(B)\in\mathbb{H}_n$.

This completes the proof. $\square$

PART 4

MINIMUM RANKS OF SIGN PATTERNS, ZERO-NONZERO PATTERNS AND POINT-HYPERPLANE CONFIGURATIONS

In this chapter, we establish a direct connection between condensed $m \times n$ sign patterns, zero-nonzero patterns with minimum rank r ($r \geq 2$) and m point-n hyperplane configurations in $\mathbb{R}^{r-1}$. We present a new and illuminating proof of the fact that for every sign pattern $\mathcal{A}$ with minimum rank 2, rational realization of the minimum rank is always possible. The proof reveals many interesting properties of sign patterns with minimum rank 2, and yields some characterizations of such sign patterns. We also introduce the notions of the number of polynomial sign changes and the number of strict sign changes of a sign vector and substantially extend two known upper bounds for the minimum ranks of full sign patterns to obtain sharp upper bounds for the rational minimum ranks of general sign pattern matrices. Then we use the matrix factorization that guarantees the connection between sign patterns and point-hyperplane configurations to prove that if the number of zero entries on each column of a sign pattern $\mathcal{A}$ with minimum rank r is at most 2 , then $\mathrm{mr}(\mathcal{A}) = \mathrm{mr}_{\mathbb{Q}}(\mathcal{A})$. We also give an upper bound for the rational minimum rank of a zero-nonzero pattern and use it to show that if the number of zero entries on each column of a zero-nonzero pattern $\mathscr{A}$ with minimum rank r is at most $r-1$, then $\mathrm{mr}(\mathscr{A}) = \mathrm{mr}_{\mathbb{Q}}(\mathscr{A})$. Furthermore, we construct the smallest known sign pattern whose minimum rank is 3 but whose rational minimum rank is greater than 3.

4.1 Point-hyperplane configurations

We now establish a direct connection between $m \times n$ condensed sign patterns with minimum rank r ($r \geq 2$) and m point-n hyperplane configurations in $\mathbb{R}^{r-1}$.

To create this connection, we need the following lemma.

Lemma 4.1.1. *Let $\mathcal{A}$ be an $m \times n$ condensed sign pattern with $mr(\mathcal{A}) = r \geq 2$. Then there are suitable signature sign patterns $\mathcal{D}_1$ and $\mathcal{D}_2$, such that there is a real matrix $B \in Q(\mathcal{D}_1 \mathcal{A} \mathcal{D}_2)$ with rank$(B) = r$ such that $B = UV$, where U is $m \times r$, V is $r \times n$, and*

$$U = \begin{bmatrix} 1 & u_{12} & \cdots & u_{1r} \\ 1 & u_{22} & \cdots & u_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & u_{m2} & \cdots & u_{mr} \end{bmatrix}, \quad \text{and} \quad V = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ v_{r-1,1} & v_{r-1,2} & \cdots & v_{r-1,n} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Proof Let $B_0 \in Q(\mathcal{A})$ with rank$(B_0) = r$. Then there exist an $m \times r$ matrix U_0 and an $r \times n$ matrix V_0 such that $B_0 = U_0 V_0$, where U_0 can be taken to be any matrix whose columns form a basis for the column space of B_0 ([30]). Because $\mathcal{A}$ is a condensed sign pattern, no two rows of U_0 are linearly dependent and no two columns of V_0 are linearly dependent.

Since there are only finitely many rows of U_0 and finitely many columns of V_0, there is a suitable Givens rotation matrix $R(\theta_2; 1, 2)$ ([30]) of order r such that

(i) for each i, $1 \leq i \leq m$, if the first two components of the ith row of U_0 are not both zero, then the first component of the ith row of $U_0 R(\theta_2; 1, 2)^T$ is nonzero, and

(ii) for each j, $1 \leq j \leq n$, if the first two components of the jth column of V_0 are not both zero, then the first component of the jth column of $R(\theta_2; 1, 2) V_0$ is nonzero.

Continuing in this fashion, we can find suitable $\theta_3, \cdots, \theta_r$ such that for each k, $2 \leq k \leq r$,

(i) for each i, if the first k components of the ith row of U_0 are not all zero, then the first component of the ith row of $U_0 R(\theta_2; 1, 2)^T R(\theta_3; 1, 3)^T \cdots R(\theta_k; 1, k)^T$ is nonzero, and

(ii) for each j, if the first k components of the jth column of V_0 are not all zero, then the first component of the jth column of $R(\theta_k; 1, k) \cdots R(\theta_3; 1, 3) R(\theta_2; 1, 2) V_0$ is nonzero.

Furthermore, since U_0 has no zero row and V_0 has no zero column, θ_r may be adjusted if necessary to ensure that the first component of each row of $U_0 R(\theta_2; 1, 2)^T R(\theta_3; 1, 3)^T \cdots R(\theta_r; 1, r)^T$ is nonzero and the last component of each column of $R(\theta_r; 1, r) \cdots R(\theta_3; 1, 3)$ $R(\theta_2; 1, 2) V_0$ is nonzero.

Let $Q = R(\theta_r; 1, r) \cdots R(\theta_3; 1, 3)R(\theta_2; 1, 2)$. By replacing $B_0 = U_0V_0$ with $B = (D_1U_0Q^T)\ (QV_0D_2)$, U_0 with $U = D_1U_0Q^T$, V_0 with $V = QV_0D_2$, and $\mathcal{A}$ with a diagonally equivalent sign pattern $\text{sgn}(D_1)\mathcal{A}\ \text{sgn}(D_2)$ for some suitable nonsingular diagonal matrices D_1 and D_2 if necessary, we may assume that the first entry of each row of U is 1, and the last entry of each column of V is 1. Thus we arrive at the desired factorization $B = UV$.

□

Observe that in the preceding proof, the zero-nonzero properties of the entries are crucial, but the actual signs do not matter. Thus the lemma also holds for zero-nonzero patterns, as stated below. Of course, the signature zero-nonzero patterns are just the identity zero-nonzero patterns (all of whose diagonal entries are nonzero), so they are not needed in the following result.

Lemma 4.1.2. *Let $\mathscr{A}$ be an $m \times n$ condensed zero-nonzero pattern with $mr(\mathscr{A}) = r \geq 2$. Then there is a real matrix $B \in Q(\mathscr{A})$ with rank$(B) = r$ such that $B = UV$, where U is $m \times r$, V is $r \times n$, and*

$$U = \begin{bmatrix} 1 & u_{12} & \cdots & u_{1r} \\ 1 & u_{22} & \cdots & u_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & u_{m2} & \cdots & u_{mr} \end{bmatrix}, \quad \textit{and} \quad V = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ v_{r-1,1} & v_{r-1,2} & \cdots & v_{r-1,n} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

With the previous lemmas, we now construct an m point-n line configuration in the plane for every $m \times n$ condensed sign pattern with minimum rank 3. Let $\mathcal{A}$ be an $m \times n$ condensed sign pattern with $\text{mr}(\mathcal{A}) = 3$. Then we may assume that there is a matrix

$B \in Q(\mathcal{A})$ with $\text{rank}(B) = 3$ such that $B = UV$, where

$$U = \begin{bmatrix} 1 & u_{12} & u_{13} \\ 1 & u_{22} & u_{23} \\ \vdots & \vdots & \vdots \\ 1 & u_{m2} & u_{m3} \end{bmatrix}, \quad \text{and} \quad V = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Identify the ith row of U, $(1, u_{i2}, u_{i3})$, with the point $p_i = (u_{i2}, u_{i3})$ in $\mathbb{R}^2$, for $1 \le i \le m$. Identify the jth column of V, $(v_{1j}, v_{2j}, 1)^T$ with the straight line l_j in $\mathbb{R}^2$ given by the equation $v_{1j} + xv_{2j} + y = 0$, for $1 \le j \le n$. After taking such identifications, these resulting m points and n lines form an m point-n line configuration (unlike in [27], we use the word configuration in the sense of a collection, with no further starting assumptions) in the plane that satisfies

(i) $b_{ij} > 0$ if and only if the point p_i is above the line l_j;

(ii) $b_{ij} = 0$ if and only if the point p_i is on the line l_j;

(iii) $b_{ij} < 0$ if and only if the point p_i is below the line l_j.

Conversely, every m point-n line configuration in the Euclidean plane $\mathbb{R}^2$ gives rise to an $m \times n$ sign pattern with minimum rank at most 3. Let C be a configuration in the plane with m labeled points $p_1, p_2, \cdots, p_m$ and n labeled lines $l_1, l_2, \cdots, l_n$. By taking a suitable rotation (whose effect on the resulting sign pattern will be explained later) if necessary, we may assume that there is no vertical line in C.

Let $\mathcal{A} = [\mathbf{a}_{ij}]$ be an $m \times n$ sign pattern such that

(i) $\mathbf{a}_{ij} = +$ if and only if the point p_i is above the line l_j;

(ii) $\mathbf{a}_{ij} = 0$ if and only if the point p_i is on the line l_j;

(iii) $\mathbf{a}_{ij} = -$ if and only if the point p_i is below the line l_j.

Note that the ith row of $\mathcal{A}$ corresponds to the point p_i and the jth column of $\mathcal{A}$ corresponds to the line l_i. Then $\mathcal{A}$ is an $m \times n$ sign pattern corresponding to C and $\text{mr}(\mathcal{A}) \le 3$. Indeed, by interpreting each point $p_i = (u_{i2}, u_{i3})$ as a row $[1, u_{i2}, u_{i3}]$ of an $m \times 3$ matrix U, and interpreting each line l_j with equation $v_{1j} + v_{2j}x + y = 0$ as a column $[v_{1j}, v_{2j}, 1]^T$ of a $3 \times n$ matrix V, we obtain a real matrix $A = UV$ with $\text{rank}(A) \le 3$ and $\text{sgn}(A) = \mathcal{A}$.

For example, let C be the point-line configuration in Figure 4.1.

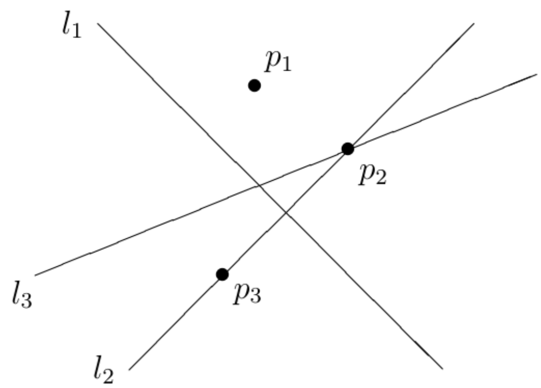

Figure 4.1. C: 3 points-3 lines configuration

Then the corresponding sign pattern matrix $\mathcal{A}$ is

$$\mathcal{A} = \begin{bmatrix} + & + & + \\ + & 0 & 0 \\ - & 0 & - \end{bmatrix}.$$

The point p_1 is above all 3 lines. So on the first row of $\mathcal{A}$ there are 3 positive signs. Similarly, we can get the second row and the third row of $\mathcal{A}$.

It is useful to think of each line in a point-line configuration on the plane to be directed (namely, oriented), so that the (i, j)-entry of the resulting sign pattern is $+$ (resp., $-, 0$) if and only if the point p_i is on the left (resp., right, inside) of the line l_j. Then it is clear that reversing the direction of a line in the configuration corresponds to negating a column of the resulting sign pattern. Further, for convenience, we could assume that each non-vertical line is pointing to the right and each vertical line is pointing upward.

However, in view of the incidence and orientation preserving dual transform D_0 (see [38]) that sends every nonzero point $a \in \mathbb{R}^2$ to the line $h_a = D_0(a) = \{x \in \mathbb{R}^2 \mid \langle a, x \rangle = 1\}$ and also sends the line h_a (which does not pass though the origin) to the point a, it is more natural to orient every line not passing through the origin in the clockwise direction, relative to the origin. Of course, the transform D_0 maps an m point-n line configuration on the plane to an n point-m line configuration, with their corresponding sign patterns being transposes of each other (assuming all the lines are oriented clockwise relative to the origin). This transform is also defined in $\mathbb{R}^d$, and the negative side of the hyperplane $h_a = D_0(a) = \{x \in \mathbb{R}^d : \langle a, x \rangle = 1\}$ is the halfspace containing the origin.

Assuming that all nonvertical lines are oriented to point to the right, and the vertical lines are oriented upward, then it is clear that any translation will preserve the resulting sign pattern matrix of a point-line configuration. Thus we can translate the configuration to a position so that none of the points in the configuration is the origin and none of the lines in the configuration passes through the origin. It is then apparent that for such a point-line configuration, any rotation of the point-line configuration through the origin preserves the resulting sign pattern, up to signature equivalence. It can be seen that more generally, if two point-line configurations can be obtained from each other through rotation and translation, then their resulting sign patterns are equivalent (through permutationally and signature equivalence). We say that two point-line configurations are *equivalent* if their resulting sign patterns are equivalent.

In order that a point-line configuration on the plane produces a condensed sign pattern, further conditions must be met. It is easy to see that an m point-n line configuration C results in a condensed sign pattern if and only if the following four conditions are satisfied by the points and lines in C.

1. No two points in C have identical or opposite relative positions (above, below, or on) relative to all the n lines in C.

2. No two lines in C have the same or opposite relative positions relative to all the m points in C.

3. No point in C is on all the n lines in C.

4. No line in C passes through all the m points in C.

Such a point-line configuration is said to be *simple*. Using the terminology of hyperplane arrangements (see [44] or [48]), the lines in C form a hyperplane arrangement in $\mathbb{R}^2$ that partitions $\mathbb{R}^2$ into relatively open connected sets (called cells) of dimensions 0 through 2, and each cell contains at most one point in C (as all the points in the same cell would yield the same sign vector relative to the hyperplanes in C). Further, since C is simple, the union of two "opposite cells" (namely, two cells whose sign vectors relative to the hyperplanes in C are negatives of each other) contain at most one point of C. Obviously, a simple point-line configuration gives rise to a sign pattern with minimum rank 1 if and only if it has exactly 1 point and 1 line.

More generally, in a similar fashion, for every reduced $m\times n$ sign pattern or zero-nonzero pattern with minimum rank $r \geq 2$, we can construct an m point-n hyperplane configuration in $\mathbb{R}^{r-1}$ using the factorization given in Lemma 4.1.1 or Lemma 4.1.2. Conversely, from an m point-n hyperplane configuration in $\mathbb{R}^{r-1} = \mathbb{R}^d$ in which no hyperplane is vertical (namely, parallel to the x_d-axis, or equivalently, the normal vector of the hyperplane is not perpendicular to the x_d-axis), we can write out a corresponding $m \times n$ sign pattern or zero-nonzero pattern whose minimum rank is at most r. We say that a point-hyperplane configuration C in $\mathbb{R}^d$ is *reduced* if no point in C is contained in all the hyperplanes in C and no hyperplane in C contains all the points in C. Of course, by saying that a point p is above a nonvertical hyperplane H in $\mathbb{R}^d$, we mean that the x_d coordinate of p is greater than the x_d coordinate of the vertical (namely, parallel to the x_d-axis) projection of p on H. We summarize this two-way correspondence between sign patterns or zero-nonzero patterns with minimum rank $r \geq 2$ and point-hyperplane configurations in $\mathbb{R}^{r-1}$ in the following two theorems.

Theorem 4.1.3. *Let $\mathcal{A}$ be a reduced $m \times n$ sign pattern with $mr(\mathcal{A}) = r \geq 2$. Then there are suitable signature sign patterns $\mathcal{D}_1$ and $\mathcal{D}_2$ such that there is a matrix $B \in Q(\mathcal{D}_1\mathcal{A}\mathcal{D}_2)$*

of rank r that has a special full-rank factorization $B = UV$ given in Lemma 4.1.1, where

$$U = \begin{bmatrix} 1 & u_{12} & \cdots & u_{1r} \\ 1 & u_{22} & \cdots & u_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & u_{m2} & \cdots & u_{mr} \end{bmatrix}, \quad \text{and} \quad V = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ v_{r-1,1} & v_{r-1,2} & \cdots & v_{r-1,n} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Identifying the ith row of U with the point $p_i = (u_{i2}, \cdots, u_{ir}) \in \mathbb{R}^{r-1}$ $(i = 1, \ldots, m)$ and identifying the jth column of V with the hyperplane H_j in $\mathbb{R}^{r-1}$ satisfying the equation $[1\ x_1\ \ldots\ x_{r-1}]v_j = 0$ $(j = 1, \ldots, n)$, we get a reduced m point-n hyperplane configuration in $\mathbb{R}^{r-1}$. Furthermore, p_i is above (resp., below, on) H_j if and only if the (i, j) entry of $\mathcal{D}_1\mathcal{A}\mathcal{D}_2$ is $+$ (resp., $-$, 0).

Conversely, given any point-hyperplane configuration C in $\mathbb{R}^{r-1}$ consisting of m points $p_1, \ldots, p_m$ and n nonvertical (i.e., the normal vector is not perpendicular to the x_{r-1}-axis) hyperplanes $H_1, \ldots, H_n$. Write $p_i = (u_{i2}, \cdots, u_{ir})$, and suppose that H_j is given by the equation $[1\ x_1\ \ldots\ x_{r-1}]v_j = 0$, where the last component of v_j is 1. Let U, V be defined as above. Then $B = UV$ is a matrix of rank at most r. Furthermore, $\mathcal{A}$=sgn$(B) = [a_{ij}]$ is an $m \times n$ sign pattern with mr$(\mathcal{A}) \le r$ such that $a_{ij} = +$ (resp., $-$, 0) if and only if p_i is above (resp., below, on) H_j. Also, $\mathcal{A}$ is reduced if and only if C is reduced.

We remark that if $C = \{p_1, \ldots, p_m; H_1, \ldots, H_n\}$ is non-reduced configuration in $\mathbb{R}^{r-1}$, then the corresponding sign pattern $\mathcal{A}$ satisfies $\text{mr}(\mathcal{A}) \le r-1$. For instance, if the hyperplane H_1 in C contains all the points in C and C does not contain other hyperplanes parallel to H_1, then the nonzero columns of $\mathcal{A}$ are determined by the configuration $C' = \{p_1, \ldots, p_m; H_1 \cap H_2, \ldots, H_1 \cap H_n\}$, a point-hyperplane configuration in the $r-2$ dimensional affine space H_1.

Theorem 4.1.4. *Let $\mathscr{A}$ be a reduced $m \times n$ zero-nonzero pattern with mr$(\mathscr{A}) = r \ge 2$. Then there is a matrix $B \in Q(\mathscr{A})$ of rank r with a special full-rank factorization $B = UV$*

(given in Lemma 4.1.2), where

$$U = \begin{bmatrix} 1 & u_{12} & \cdots & u_{1r} \\ 1 & u_{22} & \cdots & u_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & u_{m2} & \cdots & u_{mr} \end{bmatrix}, \quad \text{and} \;\; V = \begin{bmatrix} v_1 & \ldots & v_n \end{bmatrix} = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ v_{21} & v_{22} & \cdots & v_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ v_{r-1,1} & v_{r-1,2} & \cdots & v_{r-1,n} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Identifying the ith row of U with the point $p_i = (u_{i2}, \cdots, u_{ir}) \in \mathbb{R}^{r-1}$ $(i = 1, \ldots, m)$ and identifying the jth column of V, v_j, with the hyperplane H_j in $\mathbb{R}^{r-1}$ satisfying the equation $[1\ x_1\ \ldots\ x_{r-1}]v_j = 0$ $(j = 1, \ldots, n)$, we get an m point-n hyperplane configuration in $\mathbb{R}^{r-1}$. Furthermore, p_i is not on (resp., on) H_j if and only if the (i, j) entry of $\mathscr{A}$ is $\star$ (resp., 0).

Conversely, given any point-hyperplane configuration C in $\mathbb{R}^{r-1}$ consisting of m points $p_1, \ldots, p_m$ and n nonvertical (i.e., the normal vector is not perpendicular to the x_{r-1}-axis) hyperplanes $H_1, \ldots, H_n$. Write $p_i = (u_{i2}, \cdots, u_{ir})$, and suppose that H_j is given by the equation $[1\ x_1\ \ldots\ x_{r-1}]v_j = 0$, where the last component of v_j is 1. Let U, V be defined as above. Then $B = UV$ is a matrix of rank at most r. Furthermore, $\mathscr{A} = zsgn(B) = [a_{ij}]$ is an $m \times n$ zero-nonzero pattern with $mr(\mathscr{A}) \leq r$ such that $a_{ij} = \star$ (resp., 0) if and only if p_i is not on (resp., on) H_j. Also, $\mathscr{A}$ is reduced if and only if C is reduced.

We give applications of the above correspondence between sign patterns (or zero-nonzero patterns) and point-hyperplane configurations in the proofs of the following three theorems. We say that a reduced sign pattern $\mathcal{A}$ has a *direct point-hyperplane representation* if a special full-rank factorization for a matrix $B \in Q(\mathcal{D}_1 \mathcal{A} \mathcal{D}_2)$ with $\text{rank}(B) = \text{mr}(\mathcal{A})$ given in Lemma 4.1.1 can be done with $\mathcal{D}_1$ and $\mathcal{D}_2$ being the identity sign patterns. For example, for the sign patterns

$$\mathcal{A}_1 = \begin{bmatrix} + & + & + \\ - & + & + \\ - & 0 & + \end{bmatrix} \quad \text{and} \quad \mathcal{A}_2 = \begin{bmatrix} + & + & + \\ - & + & + \\ + & 0 & - \end{bmatrix},$$

it can be easily verified that both $\mathcal{A}_1$ and $\mathcal{A}_2$ have minimum rank 2, $\mathcal{A}_1$ has a direct point-hyperplane (in fact, point-point in this case) representation, but $\mathcal{A}_2$ does not. Indeed, if $\mathcal{A}_2$ has a direct point-point representation in $\mathbb{R}^1$ (regarded as the vertical number line) with points p_1, p_2, p_3 and hyperplanes (also points in this case) h_1, h_2, h_3, then the second row of $\mathcal{A}_2$ indicates that $p_2 < h_1, p_2 > h_2$, so we must have $h_2 < h_1$, while the last row of $\mathcal{A}_2$ shows that $p_3 > h_1, p_3 = h_2$, so that $h_2 > h_1$, contradicting $h_2 < h_1$.

Theorem 4.1.5. *Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be two reduced sign patterns that have direct point-hyperplane representations. Suppose that $mr(\mathcal{A}_1) = r_1 \geq 2$ and $mr(\mathcal{A}_2) = r_2 \geq 2$. Then*

$$mr\left(\begin{bmatrix} \mathcal{A}_1 & + \\ - & \mathcal{A}_2 \end{bmatrix}\right) = \max\{r_1, r_2\},$$

where the $+$ (resp., $-$) block has all the entries equal to $+$ (resp., $-$).

Proof Let $\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & + \\ - & \mathcal{A}_2 \end{bmatrix}$. Since each of $\mathcal{A}_1$ and $\mathcal{A}_2$ is a submatrix of $\mathcal{A}$, it is obvious that $\mathrm{mr}(\mathcal{A}) \geq \max\{r_1, r_2\}$. To complete the proof, we need to show the opposite inequality.

Without loss of generality, assume that $r_1 \leq r_2$ and let $d = r_2 - 1$. In a minimum rank factorization $A_1 = UV$ given in Lemma 4.1.1 of some matrix $A_1 \in Q(\mathcal{A}_1)$, we may insert $r_2 - r_1$ zero columns in U after the first column and also insert as many zero rows in V after the first row. The resulting new factorization $A_1 = U_1V_1$ can give rise to a point-hyperplane configuration C_1 in $\mathbb{R}^d$ that corresponds to $\mathcal{A}_1$, similarly as in Theorem 4.1.3. From the hypothesis and Theorem 4.1.3, we can also get a point-hyperplane configuration C_2 in $\mathbb{R}^d$ that corresponds to $\mathcal{A}_2$. The hyperplanes in C_1 divide $\mathbb{R}^d$ into connected open regions, one of which consists of all the points in $\mathbb{R}^d$ that are below all the hyperplanes in C_1. This unbounded region is called the lowest region of the arrangement of hyperplanes in C_1. Similarly, the arrangement of hyperplanes in C_2 has a highest (unbounded) region, consisting of all points in $\mathbb{R}^d$ that are above all the hyperplanes in C_2. Since translation of a configuration does not affect the resulting sign pattern, we may assume that C_1 is placed "far above" C_2, in the sense that all the points (of course, not the hyperplanes) of C_1 are in the highest region of C_2, and all the points of C_2 are in the lowest region of C_1. It is then clear that the point-hyperplane configuration $C_1 \cup C_2$ yields the sign pattern $\mathcal{A} = \begin{bmatrix} \mathcal{A}_1 & + \\ - & \mathcal{A}_2 \end{bmatrix}$. The fact that this representation is possible in $\mathbb{R}^d$ ensures that we can get a factorization of a matrix $A \in Q(\mathcal{A})$ of the form $A = U_0 V_0$, where U_0 has $d+1$ columns. It follows that $\mathrm{mr}(\mathcal{A}) \leq d+1 = r_2 = \max\{r_1, r_2\}$. This completes the proof. □

Repeated applications of the preceding theorem yield the following result.

Corollary 4.1.6. *Let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be k ($k \geq 2$) reduced sign patterns that have direct point-hyperplane representations. Suppose that $mr(\mathcal{A}_i) = r_i \geq 2$ for each $i, 1 \leq i \leq k$. Then*

$$mr\left(\begin{bmatrix} \mathcal{A}_1 & + & \dots & + \\ - & \mathcal{A}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & + \\ - & \dots & - & \mathcal{A}_k \end{bmatrix}\right) = \max\{r_1, \ldots, r_k\}.$$

A similar argument as in the proof of Theorem 4.1.5 establishes an analogous result for two reduced zero-nonzero patterns, repeated applications of which yield the following result.

Theorem 4.1.7. *Let $\mathscr{A}_1, \ldots, \mathscr{A}_k$ be k ($k \geq 2$) reduced zero-nonzero patterns such that $mr(\mathscr{A}_i) = r_i \geq 2$ for each $i, 1 \leq i \leq k$. Then*

$$mr\left(\begin{bmatrix} \mathscr{A}_1 & \star & \dots & \star \\ \star & \mathscr{A}_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \star \\ \star & \dots & \star & \mathscr{A}_k \end{bmatrix}\right) = \max\{r_1, \ldots, r_k\},$$

where all the off-diagonal blocks have all the entries equal to $\star$.

4.2 Sign patterns with minimum rank 2

The following theorem is established in [3]. We present a simpler and more insightful proof based on Lemma 4.1.1.

Theorem 4.2.1 ([3]). *Let $\mathcal{A}$ be any sign pattern with $mr(A) = 2$. Then there is a rational matrix $A \in Q(\mathcal{A})$ such that $rank(A) = 2$.*

Proof Without loss of generality, we may assume that $\mathcal{A}$ is a condensed sign pattern with a direct point-line representation. By Lemma 4.1.1, we then have a special minimum rank factorization $A = UV$ of a certain matrix $A \in Q(\mathcal{A})$, where

$$U = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix}, \quad \text{and} \quad V = \begin{bmatrix} -y_1 & -y_2 & \dots & -y_n \\ 1 & 1 & \dots & 1 \end{bmatrix}.$$

Since any two rows of U (and any two columns of V) are linearly independent, $x_1, \dots, x_m$ are distinct (and $y_1, \dots, y_n$ are distinct). By permuting the rows of U and the columns of V (and replacing $\mathcal{A}$ with a permutationally equivalent sign pattern) if necessary, we may also assume that $x_1 < x_2 < \dots < x_m$ and $-y_1 < -y_2 < \dots < -y_n$.

Note that the (i, j) entry of $A = UV$ is $x_i - y_j$ (the inner product of the ith row of U and the jth column of V), which is a continuous function of x_i and y_j. We now perturb the x_i and y_j to nearby rational values $\tilde{x}_i$ and $\tilde{y}_j$, such that $\tilde{x}_i - \tilde{y}_j = 0$ whenever $x_i - y_j = 0$ (by identifying such y_j with x_i throughout) and $\text{sgn}(\tilde{x}_i - \tilde{y}_j) = \text{sgn}(x_i - y_j)$ for all i and j. Thus we obtain a rational matrix $\tilde{B} = \tilde{U}\tilde{V}$ of rank 2 in $Q(\mathcal{A})$. $\square$

It is worth mentioning that in the above proof, the (i, j) entry of UV has a certain sign iff the jth column of V is in an appropriate half-plane (or its boundary) determined by the ith row of U in $\mathbb{R}^2$. Geometrically speaking, the proof shows that given two finite sets $\{u_1, \dots, u_m\}$ and $\{v_1, \dots, v_n\}$ of nonzero vectors in $\mathbb{R}^2$, it is possible to perturb the vectors to rational vectors so that the relative positions of the "positive" half-planes determined by the vectors remain the same, namely, the sign of $\langle v_i, v_j\rangle$ remains unchanged for all i and j.

More careful analysis of the above proof of Theorem 4.2.1 reveals many interesting properties of sign patterns with minimum rank 2 and leads to much better understanding of such sign patterns in the form of two characterizations. In particular, we now establish an upper bound for the absolute values of the entries of some integer matrix achieving the minimum rank of a sign pattern matrix with minimum rank 2.

Theorem 4.2.2. *Let $\mathcal{A} = [\mathbf{a}_{ij}]$ be an $m \times n$ sign pattern with $mr(\mathcal{A}) = 2$. Then there is an integer matrix $A = [a_{ij}] \in Q(\mathcal{A})$ such that rank $(A) = 2$ and $|a_{ij}| \leq 2\min\{m, n\} - 1$ for all i, j.*

Proof Proceeding as in the proof of Theorem 4.2.1, we may assume that $\mathcal{A}$ is a condensed sign pattern and there is a real matrix $A \in Q(\mathcal{A})$ in the form of

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix} \begin{bmatrix} -y_1 & -y_2 & \dots & -y_n \\ 1 & 1 & \dots & 1 \end{bmatrix} = [x_i - y_j]$$

where $x_1 < x_2 < \cdots < x_m$ and $-y_1 < -y_2 < \cdots < -y_n$. Clearly, in each row (or column) of A, the entries are strictly increasing when read from left to right (or from top down). We now think of the x_i and the y_j as variables subject to the restrictions that $x_1 < x_2 < \cdots < x_m$ and $-y_1 < -y_2 < \cdots < -y_n$. Then A may be regarded as a matrix depending on these variables. In order that the jth column of A agrees in sign with the jth column of $\mathcal{A}$, y_j must fall into the correct subinterval of $\mathbb{R}$ obtained by using $x_1, \dots, x_m$ as the dividing points, and in case $a_{ij} = x_i - y_j = 0$, we have $y_j = x_i$. More precisely, $\text{sgn}(x_i - y_j) = \text{sgn}(a_{ij})$ for all i and j. Assume that $x_i < y_{j_1} < y_{j_2} < x_{i+1}$. Then it is clear that the signs of the j_1th and j_2th columns of A are identical, contradicting the assumption that $\mathcal{A}$ is a condensed sign pattern. Thus there is at most one y_j strictly between two consecutive values of $x_1, x_2, \dots, x_m$. Hence, to construct an integer matrix $A \in Q(\mathcal{A})$, it suffices to leave a gap of 2 between integer values x_i and x_{i+1}, $i = 1, \dots, m-1$, to allow appropriate choice of integer value y_j so that all $x_i - y_j$ have expected signs, namely, $\text{sgn}(x_i - y_j) = \mathbf{a}_{ij}$, for all i, j. Each y_j must be equal to one of the x_i or in one of the $m + 1$ open intervals that arise when $\mathbb{R}$ is divided by the points

$x_1, x_2, \ldots, x_m$. Without loss of generality, we may set $x_1 = 2, x_2 = 4, \ldots, x_m = 2m$. We can then choose the y_j so that $1 \leq y_j \leq 2m+1$, such that $\text{sgn}(x_i - y_j) = \mathbf{a}_{ij}$, for all i, j. It follows that the resulting rank two integer matrix $A = [x_i - y_j] \in Q(\mathcal{A})$ satisfies $|a_{ij}| = |x_i - y_j| \leq 2m - 1$.

Similarly, by interchanging the roles of the x_i and y_j, it can be shown that there is a rank two integer matrix $A = [x_i - y_j] \in Q(\mathcal{A})$ satisfying $|a_{ij}| = |x_i - y_j| \leq 2n - 1$. Thus, we get the desired conclusion. $\square$

We comment that the upper bound given in the above theorem for the absolute values of the entries of an integer matrix achieving the minimum rank 2 is linear in terms of the size of the sign pattern and may appear to be quite small. However, for small values of m or n, it is easy to see that the upper bound $2\min\{m, n\} - 1$ is not sharp. For instance, clearly a $2 \times n$ sign pattern matrix with minimum rank 2 has an integer matrix realization of the minimum rank with each entry having absolute value at most 1. It can be seen that for every 3×3 non-row-condensed sign pattern $\mathcal{A}$ with minimum rank 2, there is a rank 2 integer matrix $A \in Q(\mathcal{A})$ with the absolute value of each entry at most 1, while for every 3×3 row condensed sign pattern $\mathcal{A}$ with minimum rank 2, there is a rank 2 integer matrix $A \in Q(\mathcal{A})$ such that one of its rows is the sum of the other two rows and the absolute value of each entry is at most 2. We are going to show in Theorem 4.2.8 that for $m \times n$ sign patterns with minimum rank 2, the upper bound $2\min\{m, n\} - 1$ for the absolute values of an integer matrix achieving the minimum rank can be sharpened to $2\min\{m, n\} - 3$ in general.

We are now in a position to present a characterization of sign patterns with minimum rank 2. We say that a sign vector $v = [v_1, v_2, \ldots v_n]$ is *nondecreasing* if $v_i \leq v_{i+1}$ for all i. We also say that v is *nonincreasing* if $-v$ is nondecreasing.

Theorem 4.2.3. *A sign pattern matrix $\mathcal{A}$ has minimum rank 2 iff its condensed sign pattern $\mathcal{A}_c$ satisfies the following conditions*

(i) $\mathcal{A}_c$ has at least two rows and two columns,

(ii) each row and each column of $\mathcal{A}_c$ has at most one zero entry, and

(iii) there are signature sign patterns $\mathcal{D}_1$ and $\mathcal{D}_2$ and permutation sign patterns $\mathcal{P}_1$ and $\mathcal{P}_2$ such that each row and each column of $\mathcal{P}_1\mathcal{D}_1\mathcal{A}_c\mathcal{D}_2\mathcal{P}_2$ is nondecreasing.

Proof Since $\mathrm{mr}(\mathcal{A}) = \mathrm{mr}(\mathcal{A}_c)$, without loss of generality, we may assume that $\mathcal{A}$ is condensed, namely, $\mathcal{A} = \mathcal{A}_c$.

Necessity. Suppose that $\mathrm{mr}(\mathcal{A}) = 2$. Clearly, (i) holds. Proceeding as in the proof of Theorem 4.2.1, we can find suitable signature sign patterns $\mathcal{D}_1$ and $\mathcal{D}_2$ and permutation sign patterns $\mathcal{P}_1$ and $\mathcal{P}_2$ such that for the sign pattern $\mathcal{A}_1 = \mathcal{P}_1\mathcal{D}_1\mathcal{A}\mathcal{D}_2\mathcal{P}_2$, there is a real matrix $A_1 \in Q(\mathcal{A}_1)$ of the form

$$A_1 = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix} \begin{bmatrix} -y_1 & -y_2 & \dots & -y_n \\ 1 & 1 & \dots & 1 \end{bmatrix} = [x_i - y_j]$$

where $x_1 < x_2 < \cdots < x_m$ and $-y_1 < -y_2 < \cdots < -y_n$. It follows that the entries in each row (column) of A_1 are strictly increasing, which implies (ii) and (iii).

Sufficiency. Suppose that $\mathcal{A} = \mathcal{A}_c$ is a condensed $m \times n$ sign pattern satisfying conditions (i)-(iii). Condition (i) and the fact that $\mathcal{A}$ is condensed imply that $\mathrm{mr}(\mathcal{A}) \geq 2$. It suffices to find a real matrix in the sign pattern class of $\mathcal{A}$ with rank at most 2. Since the ith row of $\mathcal{A}_1 = \mathcal{P}_1\mathcal{D}_1\mathcal{A}\mathcal{D}_2\mathcal{P}_2$ is nondecreasing and contains at most one zero entry, there is a linear function $p_i(x) = x - r_i$ such that the ith row of $\mathcal{A}_1$ is equal to $\mathrm{sgn}([p_i(1), p_i(2), \ldots, p_i(n)]) = \mathrm{sgn}([1 - r_i, 2 - r_i, \ldots, n - r_i])$. Let A_1 be the $m \times n$ matrix whose ith row is $[1 - r_i, 2 - r_i, \ldots, n - r_i]$. Then each row of A_1 is a linear combination of the two vectors $[1, 1, \ldots, 1]$ and $[1, 2, 3, \ldots, n]$. Hence, $\mathrm{rank}(A_1) \leq 2$. It then follows that $\mathrm{mr}(\mathcal{A}_1) = \mathrm{mr}(\mathcal{A}) = 2$. $\square$

From the above proof using a similar factorization as in Lemma 4.1.1, it is easy to see that a sign pattern with minimum rank 2 corresponds to a point-point configuration on the line $\mathbb{R}^1$; such a configuration can be regarded as a degenerate point-line configuration

C on the plane in which all the lines in the configuration C are parallel to each other (or equivalently, when all the points in the configuration C are collinear (but the line containing all the points may not be in C)). Thus we have the following result.

Theorem 4.2.4. *A simple point-line configuration C on the plane gives rise to a sign pattern with minimum rank 2 if and only if it satisfies the following three conditions:*

(i) *C contains at least 2 points and 2 lines;*

(ii) *Each point in C is on at most one line in C and each line in C passes through at most 1 point in C; and*

(iii) *C is equivalent to a simple point–line configuration C_1 all of whose points are on collinear.*

The Figure 4.2 is such a simple configuration yielding a sign pattern with minimum rank 2.

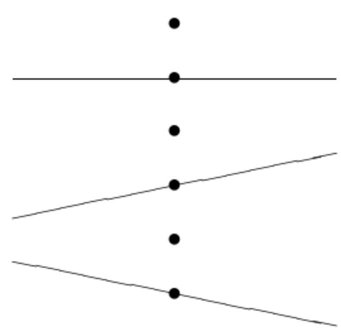

Figure 4.2. Configuration with minimum rank 2

The proof of Theorem 4.2.1 shows that if $\mathcal{A}$ is an $m \times n$ condensed sign pattern with minimum rank 2, then replacing $\mathcal{A}$ with an equivalent sign pattern if necessary, we can find an integer matrix $A \in Q(\mathcal{A})$ of the form

$$A = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_m \end{bmatrix} \begin{bmatrix} -y_1 & -y_2 & \dots & -y_n \\ 1 & 1 & \dots & 1 \end{bmatrix} = [x_i - y_j]$$

where $x_1 < x_2 < \cdots < x_m$ and $-y_1 < -y_2 < \cdots < -y_n$. As $\mathcal{A}$ is condensed, there is at most one y_j in each open interval (x_i, x_{i+1}) for $i = 1, \dots, m-1$ and there is at most one y_j value in $(-\infty, x_1) \cup (x_m, \infty)$. Otherwise, A would have two columns whose corresponding entries have the same sign (or opposite signs). In addition, the y_j can take on the x_i values. Thus, there are at most $2m$ possible y_j values. Namely, $n \leq 2m$. Similarly, we get $m \leq 2n$. This can also be seen directly from Theorem 4.2.3 since each row (column) of $\mathcal{A}$ may be assumed to be nondecreasing and contains at most one zero entry. We state this result as follows.

Corollary 4.2.5. *Let $\mathcal{A}$ be an $m \times n$ condensed sign pattern with minimum rank 2. Then $n \leq 2m$ and $m \leq 2n$.*

The proof of the above theorem can be adapted to obtain the following more efficient characterization of sign patterns with minimum rank 2.

Theorem 4.2.6. *Let $\mathcal{A}$ be an $m \times n$ condensed sign pattern with $n \geq 2$. Then $\mathcal{A}$ has minimum rank 2 iff each row of $\mathcal{A}$ has at most one zero entry, and there exist a permutation sign pattern $\mathcal{P}$ and a signature sign pattern $\mathcal{D}$ such that each row of $\mathcal{ADP}$ is nondecreasing or nonincreasing.*

Proof Observe that if every row of $\mathcal{P}_1\mathcal{D}_1\mathcal{ADP}$ is nondecreasing, then every row of $\mathcal{ADP}$ is nondecreasing or nonincreasing. Thus the necessity follows from Theorem 4.2.3.

To prove the sufficiency, assume that each row of $\mathcal{A}$ has at most one zero entry, and there exist a permutation sign pattern $\mathcal{P}$ and a signature sign pattern $\mathcal{D}$ such that each row of $\mathcal{A}_1 = \mathcal{ADP}$ is nondecreasing or nonincreasing. Since $\mathcal{A}$ is condensed with at least two rows, we have $\mathrm{mr}(\mathcal{A}) = \mathrm{mr}(\mathcal{A}_1) \geq 2$. To complete the proof, it suffices to find a matrix $A_1 \in Q(\mathcal{A}_1)$

such that $\text{rank}(A_1) \le 2$. Since the ith row of $\mathcal{A}_1$ is nondecreasing or nonincreasing and contains at most one zero entry for each i, slight modification of the argument used in the last part of the proof of the preceding theorem (by choosing a polynomial function of the form $p_i(x) = r_i - x$ when the ith row of $\mathcal{A}_1$ is nonincreasing) shows that there is a matrix A_1 in $Q(\mathcal{A}_1)$ such that $\text{rank}(A_1) \le 2$. □

By applying the above theorem to the transpose of a sign pattern $\mathcal{A}$, we get the following result.

Corollary 4.2.7. *Let $\mathcal{A}$ be an $m \times n$ condensed sign pattern with $n \ge 2$. Then $\mathcal{A}$ has minimum rank 2 iff each column of $\mathcal{A}$ has at most one zero entry, and there exist a permutation sign pattern $\mathcal{P}$ and a signature sign pattern $\mathcal{D}$ such that each column of $\mathcal{PDA}$ is nondecreasing or nonincreasing.*

For example, upon switching the second and third columns of the condensed matrix

$$\mathcal{A}_c = \begin{bmatrix} 0 & - & - & - \\ - & + & + & + \\ - & - & - & + \\ - & 0 & - & + \end{bmatrix},$$

the resulting condensed sign pattern has the property that each row is either nondecreasing or nonincreasing and there is at most one zero in each row. Thus, by Theorem 4.2.6, $\mathcal{A}_c$ (and hence $\mathcal{A}$) has minimum rank 2.

In view of Theorem 4.2.3, Theorem 4.2.6 and Corollary 4.2.7, it is apparent that a condensed sign pattern $\mathcal{A}$ satisfies the conditions in Theorem 4.2.3 iff it satisfies the row conditions in Theorem 4.2.6, iff it satisfies the column conditions in Corollary 4.2.7

Note that for the nondecreasing sign vector $v = [-, -, \ldots, -, +, \ldots, +]$ with n components in which $v_j = -$ for $1 \le j \le k$ and $v_j = +$ for $k+1 \le j \le n$, the linear function that takes on only integer values at $j = 1, 2, \ldots, n$ with $\text{sgn}(p_v(j)) = v_j$ for all j and has the least maximum absolute value over the interval $[1, n]$ is $p_v(x) = 2(x - \frac{k+k+1}{2}) = 2x - (2k+1)$; in this case, the maximum absolute value of $p_v(x)$ over $[1,n]$ is at most $2n-(2+1)=2n-3$. However, if the a nondecreasing sign vector v has exactly one zero entry $v_k=0$, then the linear function that takes on only integer values at $j=1,2,\dots,n$ with $\operatorname{sgn}(p_v(j))=v_j$ for all j and has the least maximum absolute value over the interval $[1,n]$ is $p_v(x)=x-k$; in this case, the maximum absolute value of $p_v(x)$ over $[1,n]$ is at most $n-1$. If all the components of v are $-$ (or $+$), then the constant polynomial function $p_v(x)=-1$ (or $p_v(x)=1$) is the ideal choice with the maximum absolute value of $p_v(x)$ over $[1,n]$ equal to 1. These observations can be used in the proof of the preceding theorem to establish the following improved bounds for the absolute values of the entries of some integer matrix achieving the minimum rank 2.

Theorem 4.2.8. *Let $\mathcal{A}$ be an $m\times n$ sign pattern with $mr(\mathcal{A})=2$. Suppose that the condensed sign pattern of $\mathcal{A}$ is $m'\times n'$. Then there is an integer matrix $A=[a_{ij}]\in Q(\mathcal{A})$ such that $rank(A)=2$ and $|a_{ij}|\le 2\min\{m',n'\}-3$ for all i,j. Furthermore, if each row of the condensed sign pattern of $\mathcal{A}$ contains exactly one zero entry, then there is an integer matrix $A=[a_{ij}]\in Q(\mathcal{A})$ such that $rank(A)=2$ and $|a_{ij}|\le\min\{m'-1,n'-1\}$ for all i,j.*

We now illustrate specific constructions of integer matrices with rank 2.

Example 4.2.9. *For each $n\ge 2$, the $n\times n$ integer Hankel matrix $H_n=[i+j-n-1]$ has rank 2 and has all the entries in the interval $[-(n-1),n-1]$. Note that this matrix has zero entries on the secondary diagonal. When the rows of H_n are arranged in reverse order, we get the Toeplitz matrix $PH_n=[j-i]$*

$$=\begin{bmatrix} 0 & 1 & \cdots & n-1 \\ -1 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 \\ 1-n & \cdots & -1 & 0 \end{bmatrix},\ \textit{with sgn}(PH_n)=\begin{bmatrix} 0 & + & \cdots & + \\ - & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & + \\ - & \cdots & - & 0 \end{bmatrix},$$

where P is the backward identity permutation matrix.

Example 4.2.10. *For each $n \geq 2$, the integer Toeplitz matrix $T_n = [2(j-i)-1]$ has rank 2 and has all the entries in the interval $[-(2n-1), 2n-3]$, with*

$$sgn(T_n) = \begin{bmatrix} - & + & \cdots & + \\ - & - & \ddots & \vdots \\ \vdots & \ddots & \ddots & + \\ - & \cdots & - & - \end{bmatrix}.$$

If the last row of T_n is replaced with $[-1, -1, \ldots, -1]$, we get a rank 2 matrix all of whose entries have absolute value at most $2n-3$. Note that every $m \times n$ condensed sign pattern $\mathcal{A}$ with minimum rank 2 each of whose rows is nondecreasing is equivalent (up to left multiplications by a permutation sign pattern and a signature sign pattern) to a submatrix of $sgn\left(\left[\begin{smallmatrix} PH_n \\ T_n \end{smallmatrix}\right]\right)$, where $PH_n = [j-i]_{n \times n}$ as in the preceding example.

The following is an important result (see [2, 9]) from the study of communication complexity.

Theorem 4.2.11. *If each row of a full sign pattern $\mathcal{A}$ has at most k sign changes, then $mr(\mathcal{A}) \leq k+1$.*

Obviously, the above result does not hold for a sign pattern with zero entries if the number of sign changes in a sign vector $v = [v_1, v_2, \ldots, v_n]$ is simply defined as $|\{i : v_i \neq v_{i+1}\}|$. For instance, the $n \times n$ ($n \geq 3$) upper triangular sign pattern $\mathcal{T} = [\mathbf{t}_{ij}]$ with all entries on or above the diagonal equal to $+$ has at most one sign change in each row in this sense, but its minimum rank, n, is not bounded above by $1 + 1 = 2$.

Motivated by a proof of the above theorem, where polynomials are used to construct a real vector whose entries agree in sign with a given sign vector, we introduce the following useful notion of the number of polynomial sign changes of a sign vector, which enables us to extend Theorem 4.2.11 to all sign patterns.

Let $v = [v_1, v_2, \ldots, v_n]$ be a sign vector. The *number of polynomial sign changes* of v, denoted psc(v), is defined as $\mathrm{psc}(v) = c_1(v) + c_0(v) + a(v)$, where

$$c_1(v) = |\{i \ : \ 1 \le i \le n-1, v_i v_{i+1} = -\}|, \ c_0(v) = |\{j : v_j = 0\}|$$

and $a(v)$ is the number of segments of consecutive zeros $v_{i+1} = v_{i+2} = \cdots = v_{i+k} = 0$ surrounded by nonzero entries such that $v_i v_{i+k+1}(-1)^k = -1$ (such a segment is said to be *abrupt*).

For example, for the sign vector $v = [0, -, +, 0, 0, -, 0, -, +, 0, 0, 0, -]$, we have $c_1(v) = 2, c_0(v) = 7$ and $a(v) = 2$. Note that the segments $[0, 0]$ (surrounded by $+$ and $-$) and $[0]$ (surrounded by $-$ and $-$) are the only abrupt segments since $(+)(-)(-1)^2 = -1$ and $(-)(-)(-1)^1 = -1$, while the segment $[0, 0, 0]$ is not abrupt as $(+)(-)(-1)^3 \neq -1$.

Observe that if a polynomial $p(x)$ with no multiple roots has zeros at the k numbers $i+1, \ldots, i+k$ and there is no additional zero of $p(x)$ in the interval $[i, i+k+1]$, then the sign of $p(x)$ is expected to change k times as x runs from i to $i+k+1$ (or backward), namely, $\mathrm{sgn}(p(i)p(i+k+1))(-1)^k = 1$. Thus if $(-1)^k$ does not agree with $v_i v_{i+k+1}$, then there is no such polynomial function with $\mathrm{sgn}(p(i)) = v_i$ and $\mathrm{sgn}(p(i+k+1)) = v_{i+k+1}$; in this case, it is necessary to introduce another zero of $p(x)$ in the interval $(i, i+k+1)$, which increases the degree of the polynomial by 1 and makes a contribution of 1 to the value of a in the above definition of psc(v). In other words, a segment of zeros $v_{i+1} = \cdots = v_{i+k} = 0$ surrounded by nonzero signs in a sign vector v is abrupt iff there is no polynomial $p(x)$ of degree k such that $\mathrm{sgn}(p(j)) = v_j$ for all j from i to $i+k+1$.

In fact, psc(v) is the smallest possible degree of a polynomial $p(x)$ such that $\mathrm{sgn}(p(j)) = v_j$ for all $j = 1, 2, \ldots, n$, as shown in the following lemma.

Lemma 4.2.12. *Let $v = [v_1, v_2, \ldots, v_n]$ be a sign vector and let $s = psc(v)$. Then there is a rational coefficient polynomial $p(x)$ of degree s such that $sgn(p(j)) = v_j$ for all $j = 1, 2, \ldots, n$.*

Proof Clearly, if $\mathrm{psc}(v) = 0$, then either all the entries of v are $+$ or all the entries of v are $-$, and the constant polynomial $p(x) = 1$ or $p(x) = -1$ is a desired polynomial.

If all components of v are zero, then $\mathrm{psc}(v) = n$ and $p(x) = (x-1)(x-2)\cdots(x-n)$

is a desired polynomial.

Suppose that $v \neq [0, 0, \ldots, 0]$. Note that $\mathrm{psc}(v) = \mathrm{psc}(-v)$. Replacing v by $-v$ if necessary, we may assume that the last nonzero entry of v, v_{n-r}, satisfies $v_{n-r} = (-)^r$.

We now specify the linear factors for a polynomial $p(x)$ that will satisfies our requirements. Each index i with $v_i v_{i+1} = -$ gives rise to a factor $x - (i + \frac{1}{2})$. Each index i with $v_i = 0$ gives rise to a factor $x - i$. An abrupt segment of zero entries $v_{i+1} = \cdots = v_{i+k} = 0$ (such that $v_i v_{i+k+1}(-1)^k = -1$) gives rise to a factor $x - (i+1)$. Note that the total number of linear factors described above is $s = \mathrm{psc}(v)$. Let $p(x)$ be the product of these s linear factors. Then $p(x) = (x - r_1)(x - r_2) \ldots (x - r_s)$, where $1 \leq r_1 \leq r_2 \leq \cdots \leq r_s \leq n$, r_j is a root of multiplicity 2 iff r_j is the initial index of an abrupt segment of zeros of v, and all other roots are simple. It follows that the sign of $p(x)$ changes as x passes through each simple root r_j and the sign of $p(x)$ remains fixed in any open interval between any two consecutive zeros of $p(x)$. Furthermore, $p(n+1) > 0$, $p(n) \geq 0$, $\mathrm{sgn}(p(n)) = v_n$. If $v_n = 0$ and v_{n-r} is the last nonzero entry of v, then $p(n) = \cdots = p(n-r+1) = 0$ and $\mathrm{sgn}(p(n-r)) = (-)^r = v_{n-r}$. Checking the signs of $p(n), p(n-1), \ldots, p(1)$ by keeping track of the roots of the polynomial $p(x)$, we see that the construction of $p(x)$ ensures that $\mathrm{sgn}(p(j)) = v_j$ for all $j = n, n-1, \ldots, 1$. In particular, as x runs through an abrupt segment of zero entries of v from right to left, we note that available square factor ensures that $p(j)$ has the right sign v_j as j runs through the abrupt segment all the way to the nonzero sign preceding the segment. □

We now establish a neat generalization of Theorem 4.2.11 that gives sharp upper bounds for the minimum ranks for general sign patterns. For a sign pattern $\mathcal{A}$, the *number of polynomial sign changes* of $\mathcal{A}$, denoted $\mathrm{psc}(\mathcal{A})$, is the largest number of polynomial sign changes of the rows of $\mathcal{A}$.

Theorem 4.2.13. *Let $\mathcal{A} = [\mathbf{a}_{ij}]$ be an $m \times n$ sign pattern and let s=psc($\mathcal{A}_c$). Then $mr_{\mathbb{Q}}(\mathcal{A}) \leq s + 1$.*

Proof Since $\mathcal{A}$ and $\mathcal{A}_c$ have the same rational minimum rank, we may assume that $\mathcal{A} = \mathcal{A}_c$.

Applying the preceding lemma to the ith row of $\mathcal{A}$, we see that there is a rational coefficient polynomial $p_i(x)$ of degree at most s such that $\operatorname{sgn}(p_i(j)) = \mathbf{a}_{ij}$ for all j, $1 \le j \le n$. Consider the matrix $A = [p_i(j)]$. Then A is a rational matrix in $Q(\mathcal{A})$. Note that since each $p_i(x)$ is a polynomial of degree at most s, each row of A is a linear combination of the rows of the $(s+1) \times n$ Vandermonde matrix $V = [j^{i-1}]$. It follows that $\operatorname{rank}(A) \le s+1$, and hence, $\operatorname{mr}_{\mathbb{Q}}(\mathcal{A}) \le s+1$. $\square$

Since obviously $\operatorname{mr}(\mathcal{A}) \le \operatorname{mr}_{\mathbb{Q}}(\mathcal{A})$ for every sign pattern matrix $\mathcal{A}$, the following result is immediate from the theorem above.

Corollary 4.2.14. *For every sign pattern $\mathcal{A}$, $mr(\mathcal{A}) \le psc(\mathcal{A}_c) + 1$.*

For example, for any $n \times n$ sign nonsingular upper triangular sign pattern $\mathcal{T}$, the largest of number of polynomial sign changes of the rows of $\mathcal{T}$ is $s = n-1$ and $s+1 = n$ gives a sharp upper bound for the minimum rank, n.

Example 4.2.15. *Consider the sign pattern*

$$\mathcal{B} = \begin{bmatrix} - & + & - & 0 & - & + \\ - & + & + & 0 & + & - \\ - & + & - & 0 & + & - \\ + & - & 0 & 0 & + & - \end{bmatrix}. \quad \textit{We have} \quad \mathcal{B}_c = \begin{bmatrix} - & - & - \\ - & + & + \\ - & - & + \\ + & 0 & + \end{bmatrix},$$

$psc(\mathcal{B}) = 5$, and $psc(\mathcal{B}_c) = 2$. Hence, by Corollary 4.2.14, $mr(\mathcal{B}) \le 3$, which also follows from the fact the $\mathcal{B}_c$ has 3 columns only. Note that the 3×3 submatrix of $\mathcal{B}_c$ consisting of the last three rows is sign nonsingular. Thus, $mr(\mathcal{B}) = 3$.

To take full advantage of Theorem 4.2.13, we should replace a sign pattern with its condensed sign pattern and perform multiplications on the right side by a signature sign pattern and by a permutation sign pattern to minimize the largest number of polynomial sign changes in the rows and then apply the theorem.

Theorem 4.2.16. *Let $\mathcal{A}$ be a sign pattern. Suppose that $\mathcal{A}_c$ has n columns. Then for any signature sign pattern $\mathcal{D}$ and permutation sign pattern $\mathcal{P}$ of order n, $mr_{\mathbb{Q}}(\mathcal{A}) \leq psc(\mathcal{A}_c\mathcal{D}\mathcal{P})+1$.*

Notice that a nonincreasing or nondecreasing sign vector v with at most one zero entry satisfies $\text{psc}(v) \leq 1$, as a zero entry in such a vector would not yield an abrupt segment. Hence, the following result follows from Theorem 4.2.6

Corollary 4.2.17. *Let $\mathcal{A}$ be a sign pattern with $mr(A) = 2$. Then $\mathcal{A}_c$ has at least two rows and there exist a signature sign pattern $\mathcal{D}$ and a permutation sign pattern $\mathcal{P}$ such that $psc(\mathcal{A}_c\mathcal{D}\mathcal{P}) = 1$.*

We point out that in general, for a sign pattern $\mathcal{A}$ with $r = \text{mr}(\mathcal{A}) \geq 3$, there may not exist a signature sign pattern $\mathcal{D}$ and a permutation sign pattern $\mathcal{P}$ such that $\text{psc}(\mathcal{A}_c\mathcal{D}\mathcal{P}) = r-1$. For instance, this is true for every sign pattern $\mathcal{A}$ with $\text{mr}(\mathcal{A}) = 3$ but $\text{mr}_{\mathbb{Q}}(\mathcal{A}) > 3$.

Given a condensed $m \times n$ sign pattern $\mathcal{A}$, by estimating upper bounds for certain integer coefficient polynomials related to the rows of the sign pattern over the interval $[1, n]$, we can obtain the following upper bound for the absolute values of an integer matrix $A \in Q(\mathcal{A})$ such that $\text{rank}(A) \leq \text{psc}(A) + 1$.

Theorem 4.2.18. *Let $\mathcal{A}$ be a sign pattern. Suppose that $\mathcal{A}_c$ is $m \times n$ and $k = psc(\mathcal{A}_c)$. Then there are nonnegative integers k_1 and k_2 such that $k_1 + k_2 \leq k$ and an integer matrix $A = [a_{ij}] \in Q(\mathcal{A})$ such that $rank(A) \leq k+1$ and $|a_{ij}| \leq (n-1)^{k_1}(2n-3)^{k_2}$ for all i, j.*

Proof Without loss of generality, we may assume that $\mathcal{A} = \mathcal{A}_c$. For the ith row of $\mathcal{A}$ (denoted by $\mathcal{A}_i$), as in the proof of Lemma 4.2.12, note that each factor $x-j$ or $2(x-(j+\frac{1}{2})) = 2x-(2j+1)$ (for some $j \in \{1, 2, \ldots, n\}$) satisfies $|x-j| \leq n-1$ or $|2x-(2j+1)| \leq 2n-3$. Note that there are $k_{i1} = c_0(\mathcal{A}_i) + a(\mathcal{A}_i)$ factors of the form $x-j$ and $k_{i2} = c_1(\mathcal{A}_i)$ factors of the form $2x-(j+1)$. Hence, the product $p_i(x)$ of these integer coefficient factors satisfies $|p_i(x)| \leq (n-1)^{k_{i1}}(2n-3)^{k_{i2}}$ and $\text{sgn}(p_i(j)) = a_{ij}$ for all $j = 1, 2, \ldots, n$ and $x \in [1, n]$. Therefore, for all $i = 1, \ldots, m$ and for all $x \in [1, n]$, $|p_i(x)| \leq (n-1)^{k_1}(2n-3)^{k_2} =$

$\max_{1\le i\le m}\{(n-1)^{k_{i1}}(2n-3)^{k_{i2}}\}$, where k_1 and k_2 are some nonnegative integers such that $k_1+k_2\le k$. The matrix $A=[p_i(j)]$ is then a desired integer matrix in $Q(\mathcal{A})$. □

The following is another important result (see [9]) from the study of communication complexity.

Theorem 4.2.19. *Let $\mathcal{A}$ be a full sign pattern with n columns. Let k be the smallest number of sign changes in the rows of $\mathcal{A}$. Then $mr(\mathcal{A})\le n-k$.*

For a sign vector v, the *number of strict sign changes* of v, denoted $\mathrm{ssc}(v)$, is defined as the number of sign changes in v after all the zero entries are deleted.

Let V be the $n\times k$ Vandermonde matrix $V=[i^{j-1}]$. It is clear that the submatrix of V consisting of the first k rows is nonsingular and the left null space of V has dimension $n-k$. It can be seen from a proof of the above theorem that in fact, for every sign vector $v=[v_1,v_2,\ldots,v_n]$, there is a rational vector u in the left null space of V such that $\mathrm{sgn}(u)=v$ iff $\mathrm{ssc}(u)\ge k$. Therefore, if an $m\times n$ sign pattern $\mathcal{A}$ has the property that each row has at least k strict sign changes, then there is a rational matrix $A\in Q(\mathcal{A})$ such that each row of A is in the left null space of V, and hence, $\mathrm{rank}(A)\le n-k$. Thus we arrive at the following extension of the above theorem for a general sign pattern.

Theorem 4.2.20. *Let $\mathcal{A}=\mathcal{A}_c$ be an $m\times n$ condensed sign pattern, and let $\mathcal{A}_i$ denote the ith row of $\mathcal{A}$. Let $k=\min_{1\le i\le m}\{ssc(\mathcal{A}_i)\}$ be the smallest number of strict sign changes in the rows of $\mathcal{A}$. Then $mr(\mathcal{A})\le n-k$.*

4.3 Sign patterns and zero-nonzero patterns with few zeros in each column

By scrutinizing a minimum rank factorization given in Lemma 4.1.1 for a condensed $m\times n$ sign pattern with minimum rank at least 3, we can establish the following result.

Theorem 4.3.1. *Let $\mathcal{A}$ be a condensed $m\times n$ sign pattern with $mr(\mathcal{A})=r\ge 3$. If the number of zero entries in each column of $\mathcal{A}$ is at most 2, then $mr_{\mathbb{Q}}(\mathcal{A})=mr(\mathcal{A})$.*

Proof Without loss of generality, we assume that $\mathcal{A} = [\mathbf{a}_{ij}]$ has a direct point-line representation. By Lemma 4.1.1, we then have a special minimum rank factorization $A = UV$ of a certain matrix $A = [a_{ij}] \in Q(\mathcal{A})$, where

$$U = \begin{bmatrix} 1 & u_{12} & \dots & u_{1r} \\ 1 & u_{22} & \dots & u_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & u_{m2} & \dots & u_{mr} \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} v_{11} & v_{12} & \cdots & v_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ v_{r-1,1} & v_{r-1,2} & \cdots & v_{r-1,n} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

We now treat the entries u_{ij} and v_{ij} not in the first column of U or the last row of V as variables allowed to take real values around their initial values in U or V. Thus $A = UV$ becomes a matrix whose entries are polynomial functions of the variables u_{ij} and v_{ij}.

As the number of zero entries in the jth column of $\mathcal{A}$ is at most 2, the set $Z_j = \{i : 1 \leq i \leq m \text{ and } \mathbf{a}_{ij} = 0\}$ satisfies $|Z_j| \leq 2$, for each $j = 1, \dots, n$.

For a fixed j $(1 \leq j \leq n)$, suppose that $Z_j \neq \emptyset$. We solve for certain entries of in the jth column of V in terms of other entries of V in the same column and entries of U (not in the first column). We separate the following two cases.

Case 1. $|Z_j| = 1$.

Write $Z_j = \{i\}$. Then $\mathbf{a}_{ij} = 0$ means that $v_{1j} + u_{i2}v_{2j} + \cdots + u_{i,r-1}v_{r-1,j} + u_{ir} = 0$. Solving the last equation for v_{1j}, we get

$$v_{1j} = -u_{i2}v_{2j} - \cdots - u_{i,r-1}v_{r-1,j} - u_{ir}. \tag{4.1}$$

Thus we can regard v_{1j} as a rational function (in fact, a polynomial function in this case) of the independent variables involved on the right side of the above equation.

Case 2. $|Z_j| = 2$.

Write $Z_j = \{i, k\}$. Then $\mathbf{a}_{ij} = 0$ and $\mathbf{a}_{kj} = 0$ mean that

$$\begin{cases} v_{1j} + u_{i2}v_{2j} + \cdots + u_{i,r-1}v_{r-1,j} + u_{ir} = 0 \\ v_{1j} + u_{k2}v_{2j} + \cdots + u_{k,r-1}v_{r-1,j} + u_{kr} = 0 \end{cases}$$

Note that if $u_{it} = u_{kt}$ for all $t = 1, \ldots, r-1$, then the above equations would imply $u_{ir} = u_{kr}$, so that the ith and kth rows of U are the same, which would imply that A has two equal rows, contradicting the fact that $\mathcal{A}$ is a condensed sign pattern. Thus $u_{il} \neq u_{kl}$ for some $l \in \{1, \ldots, r-1\}$. Therefore, we can solve the above system of two equations for v_{1j} and v_{lj} to obtain

$$\begin{bmatrix} v_{1j} \\ v_{lj} \end{bmatrix} = - \begin{bmatrix} 1 & u_{il} \\ 1 & u_{kl} \end{bmatrix}^{-1} \begin{bmatrix} u_{ir} + \sum_{2 \leq t \leq r-1, t \neq l} u_{it}v_{tj} \\ u_{kr} + \sum_{2 \leq t \leq r-1, t \neq l} u_{kt}v_{tj} \end{bmatrix}. \tag{4.2}$$

Thus we can regard v_{1j} and v_{lj} as rational functions of the independent variables involved on the right side of equations in (4.2), with all coefficients being rational.

The continuity of the functions given by (4.1) or (4.2) at the initial values (given in U and V) of the independent variables for each j ensures that we can let the independent variables take suitable rational values sufficiently close to their initial values in U or V, and compute the dependent variables using the rational functions given above, to yield rational perturbations $\tilde{U}$ and $\tilde{V}$ of U and V respectively, so that the resulting rational matrix $\tilde{A} = \tilde{U}\tilde{V}$ is in $Q(\mathcal{A})$ and has rank at most r. Indeed, the dependence relations given by (4.1) or (4.2) for various j ensure that $(\tilde{U}\tilde{V})_{ij} = 0$ whenever $\mathbf{a}_{ij} = (UV)_{ij} = 0$. Thus $\tilde{A} = \tilde{U}\tilde{V}$ is a rational realization of the minimum rank of $\mathcal{A}$. □

It is easy to see that the preceding proof also works for zero-nonzero patterns, so the result is valid for zero-nonzero patterns too. In fact, we are going to strengthen the preceding theorem for zero-nonzero patterns. It can be seen that for every zero-nonzero vector $\mathbf{v}$, there is a sign vector v such that $\text{zsgn}(v) = \mathbf{v}$ and $\text{psc}(v)$ is the number of zero entries of $\mathbf{v}$.

We can now establish an upper bound on the rational minimum rank of a zero-nonzero pattern.

Theorem 4.3.2. *Let $\mathscr{A}$ be a zero-nonzero pattern and let k be the maximum number of zero entries in the columns of $\mathscr{A}$. Then $mr_{\mathbb{Q}}(\mathscr{A}) \leq k+1$.*

Proof Let $\mathbf{v}_j, j = 1, \ldots, n$, denote the columns of $\mathscr{A}$. For each j, let v_j be a sign vector such that $\mathrm{zsgn}(v_j) = \mathbf{v}_j$ and $\mathrm{psc}(v_j)$ is equal to the number of zero entries in $\mathbf{v}_j$. Then $\mathrm{psc}(v_j) \leq k$ from the hypotheses. Let $\mathcal{A} = [v_1\ v_2\ \ldots\ v_n]$. Then by Theorem 4.2.13, we have $\mathrm{mr}_{\mathbb{Q}}(\mathcal{A}) \leq k+1$. But a rational matrix in $Q(\mathcal{A})$ with rank at most $k+1$ is also in $Q(\mathscr{A})$, it follows that $mr_{\mathbb{Q}}(\mathscr{A}) \leq k+1$. $\square$

As an immediate consequence, we get the following result, which can be viewed as a stronger version of Theorem 4.3.1 for zero-nonzero patterns. Note that for every zero-nonzero pattern with minimum rank at most 2, its rational minimum rank agrees with its minimum rank, due to a similar fact about sign patterns in Theorem 4.2.1.

Theorem 4.3.3. *Let $\mathscr{A}$ be a zero-nonzero pattern with $mr(\mathscr{A}) = r \geq 3$. If the number of zero entries in each column of $\mathscr{A}$ is at most $r-1$, then $mr_{\mathbb{Q}}(\mathscr{A}) = r$.*

By applying the preceding theorem to the transpose of the condensed zero-nonzero pattern of a zero-nonzero pattern, we get the following result.

Corollary 4.3.4. *Let $\mathscr{A}$ be any zero-nonzero pattern and let $r = mr(\mathscr{A})$. If the number of zero entries on each row of $\mathscr{A}_c$ is at most $r-1$, then $mr_{\mathbb{Q}}(\mathscr{A}) = mr(\mathscr{A})$.*

4.4 The smallest known sign pattern whose minimum rank is 3 but whose rational minimum rank is greater than 3

Kopparty and Rao [35], using an indirect method that provided some motivation to our direct point-hyperplane approach in section 2, showed the existence of a 12×12 sign pattern with minimum rank 3 and rational minimum rank greater than 3, based on the following configuration.

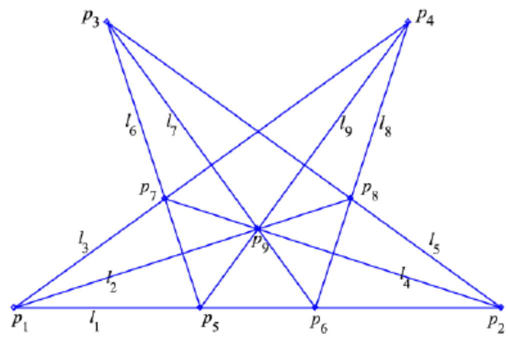

Figure 4.3. Configuration corresponding to a sign pattern with minimum rank 3 and rational minimum rank greater than 3

Example 4.4.1. *Using our approach in Section 4.1, we obtain the following* 9×9 *sign pattern* $\mathcal{A}_0$ *corresponding to the preceding point-line configuration, with* $mr(\mathcal{A}_0) \leq 3$*.*

$$\mathcal{A}_0 = \begin{bmatrix} 0 & 0 & 0 & - & - & - & - & + & + \\ 0 & - & - & 0 & 0 & + & + & - & - \\ + & + & + & + & 0 & 0 & 0 & + & + \\ + & + & 0 & + & + & + & + & 0 & 0 \\ 0 & - & - & - & - & 0 & - & + & 0 \\ 0 & - & - & - & - & + & 0 & 0 & - \\ + & + & 0 & 0 & - & 0 & - & + & + \\ + & 0 & - & + & 0 & + & + & 0 & - \\ + & 0 & - & 0 & - & + & 0 & + & 0 \end{bmatrix}.$$

Note that the submatrix $\mathcal{A}_0[\{4,5,6\},\{7,8,9\}]$ *is sign nonsingular. So* $mr(\mathcal{A}_0) = 3$*. Furthermore, since Figure 4.4 cannot be achieved by using only rational points and rational lines (lines passing through 2 rational points) [27],* $mr_{\mathbb{Q}}(\mathcal{A}_0) > 3$*. Observe that after deleting the point* p_9 *from Figure 4.4, the resulting 8 point-9 line configuration can be achieved by using only rational points and rational lines. Thus after deleting the last row of* $\mathcal{A}_0$*, the rational* *minimum rank of the new sign pattern, $\mathcal{A}_1$, is 3. Since deleting 1 row can decrease the rank of a matrix by at most 1, $mr_{\mathbb{Q}}(\mathcal{A}_0) = 4$. As indicated in [27], the 9 point-9 line configuration in Figure 4.4 is probably the smallest point-line configuration that does not have rational realization. Thus the sign pattern $\mathcal{A}_0$ is probably the smallest sign pattern whose minimum ranks over the reals and the rationals are different. Similar arguments can be used to show that the zero-nonzero pattern $\mathscr{A}_0$ =zsgn($\mathcal{A}_0$) also has the properties that $mr(\mathscr{A}_0) = 3$ and $mr_{\mathbb{Q}}(\mathscr{A}_0) = 4$.*

4.5 Open problems

For the sign pattern $\mathcal{A}_0$ in the last example, there are 4 zeros in the first column. Theorem 4.3.1 says that for a sign pattern with minimum rank 3, if the number of zeros in each column is at most 2, then the rational minimum rank of the sign pattern is 3 as well. This leaves the case of having up to 3 zeros in each column open. The lack of known not rationally realizable point-line configurations on the plane in which each line contains at most 3 points in the configuration suggests the following conjecture. Additional motivation is given by a result in [45] that states that all point-line configurations in the plane with 12 points and 12 lines such that each point is on 3 lines and each line passes through three points have rational realizations in the plane (and a similar result with 12 replaced with 11) and a conjecture in [27] on the rational realizability of 3-configurations (namely, n point-n line configurations in the plane such that each point is on 3 lines and each line passes through 3 points).

Conjecture 4.5.1. *Let $\mathcal{A}$ be any sign pattern with $mr(\mathcal{A}) = 3$. If the number of zero entries in each column of $\mathcal{A}$ is at most 3, then $mr_{\mathbb{Q}}(\mathcal{A}) = 3$.*

A related, slightly weaker conjecture is the following.

Conjecture 4.5.2. *Let $\mathscr{A}$ be any zero-nonzero pattern with $mr(\mathscr{A}) = 3$. If the number of zero entries in each column of $\mathscr{A}$ is at most 3, then $mr_{\mathbb{Q}}(\mathscr{A}) = mr(\mathscr{A})$.*

We remark that the preceding two conjectures seem to be very difficult. A positive answer to the last conjecture would immediately lead to a positive answer to the above mentioned conjecture in [27] about rational realizability of 3-configurations.

Generalizing Conjecture 4.5.1, we have the following natural question.

Problem 4.5.3. *Let $\mathcal{A}$ be any sign pattern with $mr(\mathcal{A}) = r \geq 3$. If the number of zero entries on each column of $\mathcal{A}$ is at most r, is it always true that $mr_{\mathbb{Q}}(\mathcal{A}) = mr(\mathcal{A})$?*

It is shown in [27] that all points and lines in Figure 4.4 can be represented by using numbers in $\mathbb{Q}(\sqrt{5})$. We say that the configuration in Figure 4.4 requires $\sqrt{5}$. By intricately mixing point-line configurations that require irrational algebraic numbers that are linearly independent over $\mathbb{Q}$, we suspect that one can construct a new configuration such that the corresponding sign pattern has real minimum rank 3 and large rational minimum rank. A natural question is the following.

Problem 4.5.4. *Is it true that for each integer $k \geq 4$, there exists a sign pattern $\mathcal{A}$ such that*

$$mr(\mathcal{A}) = 3 \quad \textit{and} \quad mr_{\mathbb{Q}}(\mathcal{A}) = k?$$

The approaches used in the proofs of Theorems 4.3.1 and 4.3.3 cannot be adapted to settle the case of a sign pattern matrix $\mathcal{A}$ with minimum rank r and each column having at most $r-1$ zeros. So the following weaker version of Problem 4.5.3 still remains open.

Problem 4.5.5. *Is it true that for every sign pattern $\mathcal{A}$ such that $mr(\mathcal{A}) = r \geq 4$ and each column of $\mathcal{A}$ has at most $r-1$ zeros, we have $mr_{\mathbb{Q}}(\mathcal{A}) = r$?*

REFERENCES

[1] N. Alon, P. Frankl, and V. Rödl. Geometric realization of set systems and probabilistic communication complexity. In *Proc. 26th Annual Symposium on Foundations of Computer Science*, pages 139–172, Portland, 1985. IEEE Computer Society.

[2] N. Alon and J.H. Spencer. *The Probabilistic Method*. Wiley, NY, 2 edition, 2000.

[3] M. Arav, F.J. Hall, S. Koyuncu, Z. Li, and B. Rao. Rational realizations of the minimum rank of a sign pattern matrix. *Linear Algebra and its Applications*, 409:111–125, 2005.

[4] M. Arav, F.J. Hall, Z. Li, A. Merid, and Y. Gao. Sign patterns that require almost unique rank. *Linear Algebra and its Applications*, 430:7–16, 2009.

[5] M. Arav, F.J. Hall, Z. Li, and B. Rao. Rational solutions of certain matrix equations. *Linear Algebra and its Applications*, 430:660–663, 2009.

[6] M. Arav, F.J. Hall, Z. Li, J. Sinkovic, H. van der Holst, and L. Zhang. The minimum ranks of sign patterns via sign vectors and oriented matroid duality. *Electronic Journal of Linear Algebra*, 30:360–371, 2015.

[7] A. Berman, S. Friedland, L. Hogben, U. Rothblum, and B. Shader. Minimum rank of matrices described by a graph or pattern over the rational, real and complex numbers. *Electronic Journal of Combinatorics*, 15:1–19, 2008.

[8] E. Bodine, L. Deaett, J.J. McDonald, D.D. Olesky, and P. van den Driessche. Sign patterns that require or allow particular refined inertias. *Linear Algebra and its Applications*, 437:2228–2242, 2012.

[9] R. Brualdi, S. Fallat, L. Hogben, B. Shader, and P. van den Driessche. Final report: Workshop on theory and applications of matrices described by patterns, banff international research station. Jan. 31–Feb. 5, 2010.

[10] R.A. Brualdi and B.L. Shader. *Matrices of Sign-Solvable Linear Systems*. Cambridge University Press, Cambridge, 1995.

[11] M.S. Cavers, C. Garnett, I.-J. Kim, D.D. Olesky, P. van den Driessche, and K.N. Vander Meulen. Techniques for identifying inertially arbitrary patterns. *Electronic Journal of Linear Algebra*, 26:71–89, 2013.

[12] M.S. Cavers and K.N. Vander Meulen. Spectrally and inertially arbitrary sign patterns. *Linear Algebra and its Applications*, 394:53–72, 2005.

[13] G. Chen, F.J. Hall, Z. Li, and B. Wei. On ranks of matrices associated with trees. *Graphs and Combinatorics*, 19:323–334, 2003.

[14] L. Deaett, D.D. Olesky, and P. van den Driessche. Refined inertially and spectrally arbitrary zero-nonzero patterns. *Electronic Journal of Linear Algebra*, 20:449–467, 2010.

[15] P. Delsarte and Y. Kamp. Low rank matrices with a given sign pattern. *SIAM Journal on Discrete Mathematics*, 2:51–63, 1989.

[16] J.H. Drew, C.R. Johnson, D.D. Olesky, and P. van den Driessche. Spectrally arbitrary patterns. *Linear Algebra and its Applications*, 308:121–137, 2000.

[17] W. Fang, W. Gao, Y. Gao, F. Gong, G. Jing, Z. Li, Y. Shao, and L. Zhang. Minimum ranks of sign patterns and zero-nonzero patterns and point–hyperplane configurations. submitted.

[18] J. Forster. A linear lower bound on the unbounded error probabilistic communication complexity. *Journal of Computer and System Sciences*, 65:612–625, 2002.

[19] J. Forster, N. Schmitt, H.U. Simon, and T. Suttorp. Estimating the optimal margins of embeddings in euclidean half spaces. *Machine Learning*, 51:263–281, 2003.

[20] W. Gao, Z. Li, and L. Zhang. The minimal critical sets of refined inertias for 3×3 full sign patterns. *Linear Algebra and Its Applications*, 458:183–196, 2014.

[21] W. Gao, Z. Li, and L. Zhang. Characterization of star sign patterns that require $\mathbb{H}_n$. *Linear Algebra and Its Applications*, 499:43–65, 2016.

[22] W. Gao, Z. Li, and L. Zhang. Sign patterns that require $\mathbb{H}_n$ exist for each $n \geq 4$. *Linear Algebra and Its Applications*, 489:15–23, 2016.

[23] Y. Gao and J. Li. On the potential stability of star sign pattern matrices. *Linear Algebra and its Applications*, 327:61–68, 2001.

[24] C. Garnett, D.D. Olesky, and P. van den Driessche. Refined inertias of tree sign-patterns. *Electronic Journal of Linear Algebra*, 26:620–635, 2013.

[25] C. Garnett, D.D. Olesky, and P. van den Driessche. A note on sign patterns of order 3 that require particular refined inertias. *Linear Algebra and Its Applications*, 450:293–300, 2014.

[26] R.L. Graham, M. Grötschel, and L. Lovász. *Handbook of Combinatorics*. MIT Press, 1996.

[27] B. Grunbaum. *Configurations of Points and Lines*. AMS, 2009.

[28] F.J. Hall and Z. Li. Sign pattern matrices. In *Handbook of Linear Algebra*. Chapman and Hall/CRC Press, Boca Raton, FL, second edition, 2014.

[29] F.J. Hall, Z. Li, and B. Rao. From boolean to sign pattern matrices. *Linear Algebra and its Applications*, 393:232–251, 2004.

[30] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, 2 edition, 2013.

[31] C.R. Johnson. Some outstanding problems in the theory of matrices. *Linear and Multilinear Algebra*, 12:99–108, 1982.

[32] C.R. Johnson and Y. Zhang. On the possible ranks among matrices with a given pattern. *Discrete Mathematics, Algorithms and Applications*, 2:363–377, 2010.

[33] I.-J. Kim, D.D. Olesky, B.L. Shader, P. van den Driessche, H. van der Holst, and K.N. Vander Meulen. Generating potentially nilpotent full sign patterns. *Electronic Journal of Linear Algebra*, 18:162–175, 2009.

[34] S.J. Kirkland, J.J. McDonald, and M.J. Tsatsomeros. Sign-patterns which require a positive eigenvalue. *Linear and Multilinear Algebra*, 41:199–210, 1996.

[35] S. Kopparty and K.P. Rao. The minimum rank problem: a counterexample. *Linear Algebra and its Applications*, 428:1761–1765, 2008.

[36] K. Lancaster. The scope of qualitative economics. *Rev. Econ. Studies*, 29:99–132, 1962.

[37] Z. Li, Y. Gao, M. Arav, F. Gong, W. Gao, F. Hall, and H. van der Holst. Sign patterns with minimum rank 2 and upper bounds on minimum ranks. *Linear and Multilinear Algebra*, 61:895–908, 2013.

[38] J. Matoušek. *Lectures on Discrete Geometry*. Springer, 2 edition, 2002.

[39] R. Paturi and J. Simon. Probabilistic communication complexity. *Journal of Computer and System Sciences*, 33:106–123, 1986.

[40] R.J. Pereira. Nilpotent matrices and spectrally arbitrary sign patterns. *Electronic Journal of Linear Algebra*, 16:232–236, 2007.

[41] A.A. Razborov and A.A. Sherstov. The sign rank of ac^0. *SIAM Journal of Computing*, 39:1833–1855, 2010.

[42] P.A. Samuelson. *Foundations of Economic Analysis*. Harvard University Press, Cambridge, 1947.

[43] Y. Shitov. Sign patterns of rational matrices with large rank. *European Journal of Combinatorics*, 42:107–111, 2014.

[44] Richard P. Stanley. An introduction to hyperplane arrangements. In *Lecture notes, IAS/Park City Mathematics Institute*, 2007.

[45] B. Sturmfels and N. White. All 11_3 and 12_3-configurations are rational. *Aequationes Mathematicae*, 39:254–260, 1990.

[46] B. Yu. Minimal critical sets of refined inertias for irreducible sign patterns of orders 2. *Advances in Linear Algebra & Matrix Theory*, 3:7–10, 2013.

[47] B. Yu, T. Huang, and H. Hua. Critical sets of refined inertias for irreducible zero-nonzero patterns of orders 2 and 3. *Linear Algebra and its Applications*, 437:490–498, 2012.

[48] G. M. Ziegler. *Lectures on Polytopes*. Springer-Verlag, 1994.