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Essays in Production Networks, Capital Structure and Asset Prices

Carlos Nunez

ESSAYS ON PRODUCTION NETWORKS, CAPITAL STRUCTURE AND ASSET PRICES

BY

CARLOS NUNEZ

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ACCEPTANCE

This dissertation was prepared under the direction of the CARLOS NUNEZ Dissertation Committee. It has been approved and accepted by all members of that committee, and it has been accepted in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Business Administration in the J. Mack Robinson College of Business of Georgia State University.

Richard Phillips, Dean

DISSERTATION COMMITTEE

Dr. Ajay Subramanian (Chair)

Dr. Stephen Shore

Dr. Liang Peng

Dr. Alejandro Del Valle

Dr. Baozhong Yang (External - Department of Finance)

ABSTRACT

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BY

CARLOS NUNEZ

JULY 14^{TH}, 2023

Committee Chair: Dr. Ajay Subramanian

Major Academic Unit: Maurice R. Greenberg School of Risk Science

This dissertation is comprised of two separate essays: (1) Production Networks and Capital Structure, and (2) Product Variety and Asset Prices.

In the first essay, we develop a dynamic structural model to investigate the relationships among characteristics of an economy's production network—the network of input-output linkages that influence firms' production decisions—and firms' financial decisions. We analytically characterize the equilibrium of the economy and derive novel implications for the impact of the production network on firms' capital structures and default risks. An increase in network concentration leads to lower leverage ratios and default probabilities for larger industries, but higher leverage ratios and default probabilities for smaller industries. Network sparsity has a positive effect on firms' leverage ratios. We then calibrate the model to match relevant identifying moments in the data and obtain quantitative implications for the effects of network characteristics on firms' financial structures and default risk. As proportions of their baseline values, a 20% increase (decrease) in the network concentration alters (i) the mean leverage ratios of the representative firms in the largest, median and smallest industries by -12.4%,66.3% and 254% (14.4%,-38.8% and -69.2%), respectively; and (ii) the mean default probabilities of the representative firms in the largest, median and smallest industries by -17.3%,49.7% and 164.8% (25.3%,-30.6% and -62.8%),

respectively. An increase in network sparsity increases the mean leverage ratios of firms in all industries with a 20% increase in network sparsity leading to a 54.3% increase in average industry leverage. We examine how resilient the U.S. production network is to contagion by analyzing how unexpected shocks to industry parameters impact the default probabilities firms in other industries. A 200% increase in the firm-level idiosyncratic volatility in the largest industry leads to a relative increase of 19.2% in firms' default probability in the most impacted industry and generates an average relative increase of 13.3% in the default probability for firms in the remaining industries. A similar percentage increase in the default threshold parameter of the largest industry causes a relative increase of 151% for firms in the most impacted industry and an average increase of 106.7% for firms in the other industries.

In the second essay, we show how product variety affects asset prices in a general-equilibrium model. We analytically characterize the unique equilibrium and estimate the model to match asset pricing and product market moments. The equity premium and risk-free rate can be reconciled for risk aversion levels around 4 and plausible annual discount factors. Our model generates new implications for how product market characteristics influence asset prices. We find that while competition leads to product substitution within industries, product complementarity is observed between industries. The market risk premium decreases with the both the average intra-industry and inter-industry product substitutabilities. We show empirical support for the novel cross-sectional prediction that industry excess returns increase with intra-industry product substitutabilities.

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I dedicate this dissertation to the loving memory of my aunt ***Margery Trujillo****, who passed away in 2021. I wish that she could have witnessed this achievement – a sizeable part of which is thanks to her. She lives forever in my memories, as I fondly remember our precious time together.*

Production Networks and Capital Structure

Abstract

We develop a dynamic structural model to investigate the relationships among characteristics of an economy's production network—the network of input-output linkages that influence firms' production decisions—and firms' financial decisions. We analytically characterize the equilibrium of the economy and derive novel implications for the impact of the production network on firms' capital structures and default risks. An increase in network concentration leads to lower leverage ratios and default probabilities for larger industries, but higher leverage ratios and default probabilities for smaller industries. Network sparsity has a positive effect on firms' leverage ratios. We then calibrate the model to match relevant identifying moments in the data and obtain quantitative implications for the effects of network characteristics on firms' financial structures and default risk. As proportions of their baseline values, a 20% increase (decrease) in the network concentration alters (i) the mean leverage ratios of the representative firms in the largest, median and smallest industries by $-12.4\%, 66.3\%$ and 254% (14.4%, -38.8% and -69.2%), respectively; and (ii) the mean default probabilities of the representative firms in the largest, median and smallest industries by $-17.3\%, 49.7\%$ and 164.8% (25.3%, -30.6% and -62.8%), respectively. An increase in network sparsity increases the mean leverage ratios of firms in all industries with a 20% increase in network sparsity leading to a 54.3% increase in average industry leverage. We examine how resilient the U.S. production network is to contagion by analyzing how unexpected shocks to industry parameters impact the default probabilities firms in other industries. A 200% increase in the firm-level idiosyncratic volatility in the largest industry leads to a relative increase of 19.2% in firms' default probability in the most impacted industry and generates an average relative increase of 13.3% in the default probability for firms in the remaining industries. A similar percentage increase in the default threshold parameter of the largest industry causes a relative increase of 151% for firms in the most impacted industry and an average increase of 106.7% for firms in the other industries.

JEL Classification: G32, G33, D21, D22, L11, L14

Keywords: Networks, Capital Structure, Production, Default Risk, Contagion

1 Introduction

Networks are a fundamental feature of the modern production economy. Firms do not produce in isolation, but as part of a system of input-output linkages. Relatively independent streams of the literature explore how (i) characteristics of the economy's production network influence firms' production decisions and asset prices; and (ii) how the nature of links between financial institutions—specifically banks—influence systemic risk and the potential for contagion in bank failures. The first stream of the literature abstracts away from firms' financial structures, while the second stream focuses solely on financial firms with *exogenous* financial structures comprising primarily of debt. Hence, the research questions of how input-output links among *non-financial* firms *endogenously* influence their capital structures and default risks as well as the degree of resilience of the production network to contagion among firm defaults are largely unexplored.

We develop a dynamic structural model to show how characteristics of the economy's production network influence non-financial firms' production and financing decisions as well as the potential for contagion among firm defaults. We analytically characterize the equilibrium of the economy in which firms' production and financing decisions are endogenously determined by the input-output linkages among industries, as well as industry-wide and firm-specific productivity shocks. We derive novel implications for the impacts of key characteristics of the production network—the concentration and sparsity—-on firms' capital structures and default risks. Interestingly, network concentration has *differential* impacts on firms' leverage ratios and default probabilities in large and small industries. A higher network concentration *lowers* leverage ratios and default probabilities for firms in *larger* industries, but *raises* leverage ratios and default probabilities for firms in *smaller* industries. A higher network sparsity increases firms' leverage ratios in all industries. We then calibrate the model to match relevant moments in the data and exploit the calibrated model to obtain quantitative implications for the effects of network characteristics on firms' financial structures and default risk. Network concentration and sparsity both have quantitatively substantial effects on firms' leverage ratios and default probabilities. We examine the resilience of the U.S. production network to contagion by analyzing how unexpected shocks to key parameters in each industry affect firm default probabilities in other industries. Shocks to the firm-specific productivity shock volatility and default threshold of the largest industry have quantitatively substantial impacts on firm default probabilities in other industries, but shocks to the bankruptcy cost have only marginal effects. The effects of contagion are highly nonlinear and decline significantly with the size of the industry being impacted.

We model a discrete-time open economy over an infinite time horizon. There are two types of agents: entrepreneurs who operate firms, and investors who provide capital to firms via equity and debt. The economy is organized into a finite number of industries or sectors each with *ex ante* identical firms. Firms in each industry produce a distinct industry good by combining capital as well as goods produced by all industries as intermediate inputs. Agents derive utility from the consumption of the basket of imperfectly substitutable industry goods at the end of each period. Industry goods are, therefore, used for consumption and as intermediate inputs for the production of other industry goods.

Firms' financial and production decisions are affected by two types of shocks. At the beginning of a period, firms in each industry experience an industry-wide productivity shock. After the shock is realized, firms rent capital from investors via equity and debt. For tractability, we consider single-period debt that is due at the end of the period. We can, however, reformulate the model with longer-term debt, but with firms being able to frictionlessly alter their debt levels in each period. In either scenario, a firm's debt level is determined by the debt payment due at the end of each period. After they rent capital, firms in each industry experience idiosyncratic productivity shocks that are independently and identically distributed across firms *conditional* on the industry-level productivity shock. Firms then make their production decisions by purchasing industry goods in spot markets at endogenously determined prices. Finally, firms' earnings for the period are realized from which they make debt and dividend payments.

The key friction that drives capital structure choices is that insiders (management and employees) can extract private benefits from a firm's free cash flow. Debt is a hard claim that reduces a firm's free cash flow—total earnings net of debt payments—thereby restricting the extraction of private benefits by insiders (e.g., Jensen (1986), Stulz (1990), Hart and Moore (1995), Williams (1995), and Zwiebel (1996)). Firms' financing decisions in each period reflect the tradeoff between the benefits of debt in limiting private benefit extraction by insiders and financial distress costs that are incurred when firms' earnings are insufficient to make debt payments. We model distress costs via a decline in an insolvent firm's productivity relative to the level when it is solvent. For tractability, similar to studies such as Elenev et al. (2021), a firm is insolvent when its maximum earnings fall below a proportion of the required debt payment. An insolvent firm returns to its normal, solvent state at the beginning of the next period, that is, insolvency does not persist over time.

We analytically derive the unique equilibrium in which firms' financing decisions at the beginning of each period, the capital rental rate, the spot prices of industry goods, and firms' production decisions at the end of the period are all endogenously determined. In the equilibrium, firm insolvency propagates through the production network. Specifically, the lower productivity of insolvent firms in an industry affects the supply of the industry good, thereby influencing industry good prices as well as all firms' output decisions and agents' consumption decisions. Firms' financing decisions reflect the impact of industry good prices on their outputs and earnings. In this manner, the production network influences firms' financial structures. We show that the equilibrium is determined by a complex fixed-point problem for the vector of default probabilities of firms in different industries. We analytically show that the fixed-point problem has a unique solution, thereby determining the equilibrium.

Firms' production and financing decisions are determined by two key characteristics of the production network—the network concentration and sparsity—that serve as sufficient statistics for how the network structure influences firms' capital structures and default probabilities. A high network concentration corresponds to a scenario in which a relatively small number of industries dominate economic output. We show that we can express the network concentration in terms of a parameter that closely approximates a power law in the data, thereby greatly facilitating a "comparative static" analysis. Interestingly, the network concentration has *sharply contrasting* effects on large and small industries, where industry size is determined by the fraction of the economy's total gross output that is generated by the industry. An increase in the network concentration *decreases* firms' leverage ratios and default probabilities in each period for firms in large industries, but *increases* leverage ratios and default probabilities in small industries. The intuition is that, *ceteris paribus*, the insolvency of firms in larger industries has a relatively greater negative impact on the economy via industry good prices that determine all firms' output decisions and earnings. Firms' financing decisions incorporate this negative externality via its impact on industry good prices and firm profits. Hence, firms in larger industries are more conservative in their financing decisions relative to firms in smaller industries. A higher network concentration implies a redistribution of weights from larger industries to smaller ones. Consequently, larger industries shrink in size, which increases the risk of insolvency for firms in these industries with the opposite happening for smaller industries. As a result, a higher network concentration leads to lower leverage ratios for firms in larger industries, but higher leverage ratios for firms in smaller industries. Network sparsity measures the extent to which industries are self-reliant. The less that industries depend on the inputs from other industries for their production, the more sparse the network is. Controlling for network concentration, an increase in sparsity, therefore, leads to an increase in leverage ratios for firms in all industries.

We also analytically explore the effects of contagion by showing how unanticipated shocks to key parameters in each industry—the firm-level idiosyncratic volatility, default threshold and bankruptcy cost—influence firms in other industries. Specifically, as the shocks are unanticipated, we assume that firms' leverage levels are equal to their equilibrium values, and then examine how the shocks affect the default probabilities of firms in other industries. Interestingly, the default probability of firms in any industry increases with an unexpected increase in the firm-level idiosyncratic volatility in another industry if it is below a threshold, but decreases if it is above the threshold. The intuition for the *non-monotonic* impact of the idiosyncratic volatility is that volatility has two offsetting effects. A higher volatility increases the risk of insolvency by making negative scenarios more common, but it can also increase the upside by augmenting the likelihood of positive shocks. Given that shocks have a positively skewed log-normal distribution, the negative effects dominate when the volatility is below a threshold, but the positive effects prevail when the volatility is above the threshold. The default probability of firms in any industry unambiguously increases with an unexpected increase in the default threshold or the proportional bankruptcy cost in another industry. Indeed, in either scenario, firm insolvency in the affected industry has a more negative impact on firm profits in other industries, thereby increasing firms' default probabilities.

We then take our structural model to the data to conduct a *quantitative* analysis of how characteristics of the production network influence firm leverage ratios, default probabilities and the likelihood of contagion. We combine firm-level financial statements data from Compustat and industry-level data on network/productivity factors from the Bureau of Economic Analysis (BEA) as well as data on default probabilities from Standard and Poors (S&P). We group firms into 55 major industries, following the classification used by the BEA in which industries are aggregated according to the first three NAICS numbers. We exclude financial and regulated industries as our model focuses on non-financial productive firms that are not subject to significant regulation. We exploit the analytical tractability of the model to directly calibrate the industry-level parameters by using relevant identifying moments in the data for each industry that have closed-form expressions in the model. We, thereby, circumvent the need to numerically solve the model to estimate a large number of parameters across the 55 industries.

We use our calibrated model as the baseline for our quantitative analysis. A 20% relative increase (decrease) in the network concentration parameter changes the average leverage ratios of the representative firms in the largest, median and smallest industries by $-12.4\%, 66.3\%$ and 254% ($14.4\%, -38.8\%$ and -69.2%), respectively; and the default probabilities by $-17.3\%, 49.7\%$ and 164.8% ($25.3\%, -30.6\%$ and -62.8%) , respectively. A 20% increase in network sparsity leads to a relative increase of 54.3% in the industry average leverage ratio, while a 20% decrease in network sparsity leads to a decrease of 35.2% in the average industry average leverage. Hence, the effects of concentration and sparsity that we derive analytically in the model are also quantitatively significant.

We examine the resiliency of the U.S. production network to contagion by investigating the propagation of firm defaults through the network as a result of unexpected shocks to industry parameters. Specifically, mirroring our earlier theoretical analysis of contagion, we apply unanticipated shocks to key parameters—the firm-level idiosyncratic volatility, the default threshold, and bankruptcy cost—in each industry, and then determine their effects on firms' default probabilities in other industries. A 200% increase in the idiosyncratic volatility of firms in the largest industry leads to a 19.2% increase in the default probability of firms in the most affected industry and an average increase of 13.3% in firms' default probabilities in the remaining industries. A 200% unexpected increase in the default threshold parameter in the largest industry can generate up to a 151% relative increase in the probability of default of firms in the most impacted industry.

This shock also causes the average default probability of firms in other industries to rise by 106.7%. The shock in the default threshold parameter is more significant than the industry-level volatility, since the former has an unambiguously positive effect on the probability of default. Finally, shocks to the bankruptcy cost parameters are relatively small as they only affect firm profits in bankruptcy and firm default probabilities in the data are relatively small across industries. The overall takeaways from our quantitative analysis is that the contagious effects of unexpected shocks are highly nonlinear and depend critically on industry size, the nature of the shocks and their magnitudes.

The paper is organized as follows. In the next section we review of relevant literature and highlight what we believe to be our main contributions. In Section 3 we present the model. We derive the equilibrium and our main analytical results in Section 4. We describe the calibration of the model in Section 4.6. We present the results of our quantitative analysis in Section 5. Section 6 concludes. We provide supplementary analysis, including the proofs of all results, in the Appendix.

2 Literature Review

One stream of the literature theoretically analyzes how the characteristics of financial networks—the links among financial institutions, specifically banks—influence systemic risk and the potential for contagion in bank failures. Eisenberg & Noe (2001) develop an equilibrium model in which banks are linked to other banks via the interbank lending market, and the default of one institution can trigger additional defaults in the network. Acemoglu et al. (2015) analyze how the network structure impacts the propagation of shocks in financial institutions. In this vein, Amini & Minca (2016) *and* Amini et al. (2016) propose a random graphs approach, where contagion spreads throughout the banking system depending on the distribution of in-degrees and out-degrees and the number of defaults that each bank can tolerate from its counterparties. A different strand within this stream of the literature focuses on how fire sales affect contagion via common holdings by financial institutions. In these papers, the authors focus on how deleveraging in response to an initial shock can cause a feedback loop where the pressured sale of assets leads to fire sales, market losses and further deleveraging (see Cont & Shaanning (2017)*,* Bichuch & Feinstein (2019)*,* Braouzec & Wagalath (2019), Banerjee & Feinstein (2021) and Detering et al. (2022) for details).

Although the literature on financial networks and contagion in financial institutions (particularly banks) is substantial, it is relatively more sparse in the case of non-financial firms that represent the main focus of our paper. One of the key issues being addressed by several studies, such as Acemoglu et al. (2012)*,* Acemoglu & Tahbaz-Salehi (2020) *and* Herzkovic et al. (2020), is how macroeconomic fluctuations arise from production networks and the resulting impacts on firms' production decisions and asset returns. These studies develop models where firms react to exogenous shocks by modifying inputs demanded from other firms. This change in the amount of inputs also leads to fluctuations in production, which propagate through the production network. In terms of the modeling approach, Herzkovic (2018) is the closest antecedent to our paper. He proposes an endogenous model of production, in which network sparsity and concentration affect asset prices and macroeconomic quantities such as GDP and consumption.

Another less closely related, but important strand of the literature endogenizes the formation of production networks. Oberfield (2018) examines a dynamic model of network formation, wherein producers strategically select one input from a set of suppliers that evolves randomly. His findings reveal that this endogenous decision-making process leads to the emergence of star suppliers who serve as major intermediaries. Other studies, such as Acemoglu and Azar (2020), develop a tractable model where multiple products can be produced by combining labor and an endogenous subset of other products as inputs. They highlight that the introduction of new products significantly expands the set of technological possibilities for existing industries and these cost reductions then propagate to other industries through lower input prices. Finally, Taschereau-Dumouchel (2020) investigates an economy where firms' interconnections and decisions to operate or shut down affect the production network. The findings suggest that the endogenous reorganization of the network plays a crucial role in mitigating the impact of shocks on aggregate fluctuations. In contrast to this literature, we take the network as given in our framework as in Herzkovic (2018).

We contribute to the literature by developing a dynamic equilibrium model to show how the production network influences firms' financial and production decisions. Our framework is parsimonious and tractable, thereby allowing us to analytically derive testable implications for how characteristics of the production network—its concentration and sparsity—influence firms' capital structures and default probabilities. We are also able to exploit the analytical tractability of the model to directly calibrate its parameters to the data, thereby facilitating a robust quantitative analysis of the effects of network characteristics and the degree of resilience of the network to contagion.

3 Model

We model a discrete-time *open economy* with an infinite time horizon and dates $t \in \{0, 1, 2, ...\}$. The economy is populated by investors and entrepreneurs, who establish and operate firms. We alternatively refer to entrepreneurs as "firms" wherever there is no danger of confusion. The firms are organized into $N > 0$ industries with firms in each industry producing a distinct industry good $I \in \{1, 2, ..., N\}$. There is a mass, M_I, of firms in industry I. Industry goods can be directly consumed or used as intermediate inputs for the production of other industry goods. Goods are non-durable and cannot be stored.

Each individual firm in an industry manufactures the industry good in each period using physical capital and other industry goods as inputs. Firms rent capital from investors at the beginning of each period via equity and debt contracts and purchase industry goods in spot markets. Capital cannot be consumed; it is used solely as an input in firms' production processes. The aggregate capital stock, which is held by investors, is K_t at date t. We take K_t, which can evolve stochastically over time, as a parameter of the model in each period. We can also interpret the capital stock, K_t,as the "effective" or "installed" capital that is employed in production.

Figure 1: Model Timeline

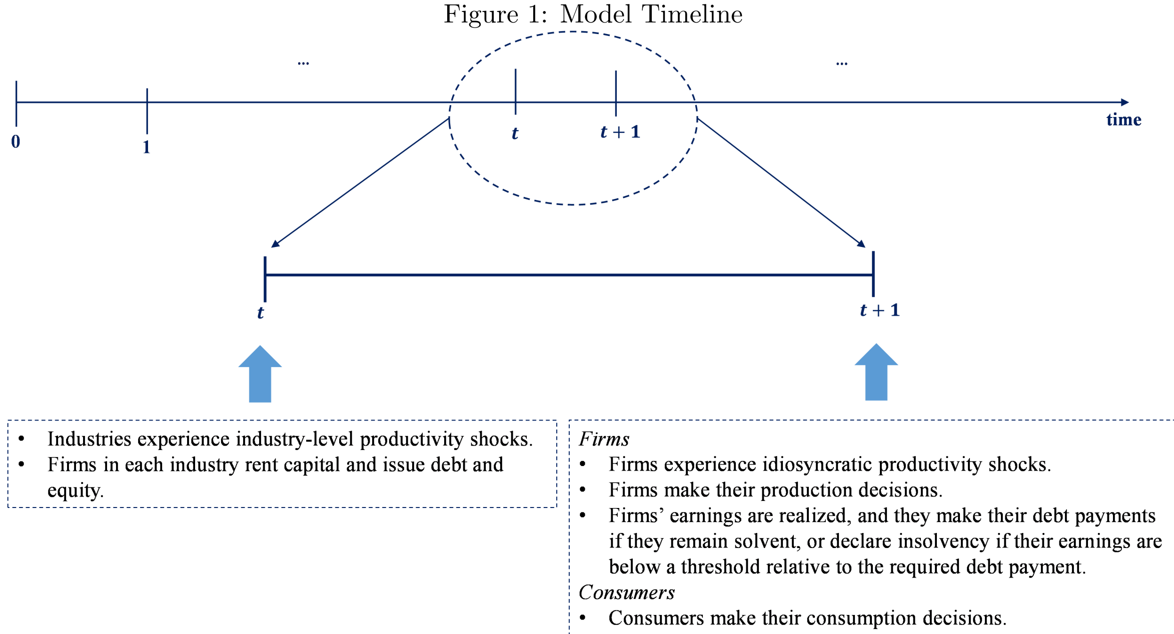

Figure 1 shows the timeline of events in each period. A time period in the model is defined as $[t, t+1]$,where $t \in \{0, 1, 2...\}$. At the beginning of each period, firms in each industry experience an industry-wide productivity shock. Firms observe the industry-wide productivities and rent capital from investors via debt and equity. At time $t + 1$, firms experience idiosyncratic productivity shocks. Firms then make their productions decision using the capital that they have rented and industry goods that they purchase in spot markets. Consumers choose their baskets of consumption goods. Firms' earnings are realized. Firms remain solvent and make their required debt payments if their realized earnings are above a threshold relative to their debt. If the realized earnings are below the threshold, they declare insolvency and their earnings flow to debtholders. We now describe the main ingredients of the model in detail.

3.1 Consumers

Agents in the economy—investors and entrepreneurs—have identical preferences over the consumption of the N industry goods in each realized state of the economy at date t that are described by the utility function,

$$u_t = \prod_{I \in [N]} c_{I,t}^{\alpha_I}, \tag{1}$$

where $c_{I,t}$ is the amount consumed of good I at date t by the representative agent, and $\{\alpha_I > 0; I \in [N]\}$ are preference weights, such that $\sum_{I \in [N]} \alpha_I = 1$. We hereafter employ the shorthand notation $[N]$ to refer to the set $\{1, 2, ..., N\}$. In each realized state at date t, the representative agent maximizes her utility given by (1) subject to her budget constraint,

$$\sum_{I \in [N]} P_{I,t} c_{I,t} \leq y_t. \tag{2}$$

In the above, y_t is the agent's allocation to consumption in the realized state at date t and $\{P_{I,t}; I \in [N]\}$ are the spot prices of the industry goods that are determined endogenously in equilibrium. Given that $\sum_{I \in [N]} \alpha_I = 1$, each agent's maximum utility in each state is proportional to her payoff, y_t. Note that different agents have different payoffs depending on their roles in the economy. We develop an industry equilibrium framework in which the focus is on firms' production and financing decisions. Consequently, as will become clear below, our analysis only requires the specification of agent preferences over the basket of consumption goods in each realized state at date t as described by (1), but not preferences vis-a-vis the risk and timing of consumption.

3.2 Firms

At the beginning of each period, $[t, t+1]$, each industry I experiences an industry-wide productivity shock, $Z_{I,t}$, that affects the productivity of all firms in the industry. The subscript, I, explicitly indicates that *all* firms in the industry experience the industry shock, $Z_{I,t}$. Throughout, we use uppercase and lowercase letters as subscripts to denote industry-level and firm-level variables, respectively. The industry shocks are publicly observable and could be correlated across industries. After observing the industry shock, $Z_{I,t}$, each firm i in industry I rents capital from investors via equity and debt. The capital rental rate in period t, which is the same for all firms in the economy, is $R_{K,t}$, and is endogenously determined in equilibrium. For simplicity, a firm's debt level in each period is determined by the required debt payment at the end of each period. This is the case if debt is single-period with the principal and interest payment being due at the end of the period, or if debt is longer-term, but firms can frictionlessly restructure—either increase or decrease—their debt levels in each period. We remain agnostic about debt maturity as it is not pertinent to our analysis and its main implications.

After it rents capital, each firm i experiences an idiosyncratic productivity shock, $z_{i,t+1}$, where the subscript, $t+1$, explicitly indicates that the productivity shock is realized at the end of the period *after* the firm rents its capital. Conditional on the industry shock, firm-specific productivity shocks are independently and identically distributed across firms in the industry. After realizing its firm-specific productivity shock, the firm makes its production decision by using its rented capital and industry goods as intermediate inputs that it purchases in spot markets.

Productivity shocks for firm $i \in I$ have the following lognormal forms:

$$
\begin{aligned}
z_{i,t+1} &= e^{(1-\beta_I)\left(-\frac{1}{2}\eta_I^2+\eta_I\varepsilon_{i,t+1}\right)}, \\
Z_{I,t} &= e^{(1-\beta_I)(\mu_I t+\sigma_I\varepsilon_t)},
\end{aligned}
\tag{3}
$$

where $\sigma_I > 0$ is a parameter that determines the industry shock volatility, $\eta_I > 0$ is a parameter that determines the firm-level idiosyncratic volatility, and μ_I determines the drift of the industry shock. The parameter, β_I,is the elasticity of the basket of industry goods in the production function of firms in industry I and is defined in (4) below when we specify firms' production functions. We introduce it as a normalizing variable in (3) without loss of generality to simplify the subsequent analytical expressions. The industry-level shocks $\varepsilon_t \ \forall t$ and idiosyncratic shocks $\varepsilon_{i,t} \forall i,t$ are independent and identically distributed (i.i.d.) standard normal random variables.

We model financial distress as in Elenev et al. (2021). If a firm i's realized idiosyncratic productivity, $z_{i,t+1}$, is high enough that its maximum earnings are above a threshold relative to its required debt payment, then its total productivity is $Z_{I,t}z_{i,t+1}$. However, if $z_{i,t+1}$ is below a threshold such that the firm's maximum earnings are below the threshold relative to its debt, then the firm's productivity declines to $\chi_I Z_{I,t} z_{i,t+1}$ where $\chi_I \in (0,1)$ is an industry-wide parameter that determines the firm's cost of insolvency.[1] In other words, the firm's productivity is lower than it would otherwise be if it were not in distress, thereby lowering its earnings. The precise specification of the insolvency condition necessitates a description of the firm's production functions when it is solvent and insolvent, respectively.

[1] Elenev et al. (2021) assume that a firm is insolvent if its earnings fall below the required debt payment. We adopt a more general formulation in which a firm is insolvent if its earnings fall below a proportion of its debt level, where the proportion is a calibratable parameter.

An individual firm i's production function in period $[t, t+1]$ is given by

$$x_{i,t+1} = A_I Z_{I,t} z_{i,t+1} k_{i,t}^{\gamma_I} \left[\prod_{J \in [N]} x_{iJ,t+1}^{w_{IJ}} \right]^{\beta_I} \text{ when it is solvent,}$$

$$x_{i,t+1}^{(b)} = A_I \chi_I Z_{I,t} z_{i,t+1} k_{i,t}^{\gamma_I} \left[\prod_{J \in [N]} x_{iJ,t+1}^{w_{IJ}} \right]^{\beta_I} \text{ when it is insolvent.} \tag{4}$$

In the above, the superscript in $x_{i,t+1}^{(b)}$ explicitly indicates that this is the firm's production when it is bankrupt or insolvent. In (4), $\beta_I \in (0, 1)$ is the elasticity of the basket of industry goods for industry I in the production function; $A_I := \beta_I^{-\frac{\beta_I}{1-\beta_I}}$ is a normalizing constant that is introduced to simplify the subsequent analytical expressions; $k_{i,t}$ is the amount of capital that the firm rents at date t; and $x_{iJ,t+1}$ is the amount of industry good J that the firm chooses. The additional time subscripts on capital and the industry goods explicitly indicate that capital is rented *at the beginning* of the period, while industry goods are purchased in spot markets *at the end* of the period after the firm realizes its idiosyncratic productivity shock. In (4), $\gamma_I > 0$ is the weight of capital, and $w_{IJ} \in (0, 1)$ is the weight that the input from industry $J's$ good has in industry I's production function. Here $\sum_{J \in [N]} w_{IJ} = 1$ for all $I \in [N]$. The matrix, $[w_{IJ}]$, represents the production network of the economy.

A firm i's maximum gross profit or earnings given its capital, $k_{i,t}$, are given by

$$\pi_{i,t+1}(k_{i,t}) = \sup_{\{\{x_{iJ,t+1}\}_{J \in [N]}\}} P_{I,t} x_{i,t+1} - \sum_{J \in [N]} P_{J,t} x_{iJ,t+1} \text{ when it is solvent,}$$

$$\pi_{i,t+1}^{(b)}(k_{i,t}) = \sup_{\{\{x_{iJ,t+1}\}_{J \in [N]}\}} P_{I,t} x_{i,t+1}^{(b)} - \sum_{J \in [N]} P_{J,t} x_{iJ,t+1} \text{ when it is insolvent.} \tag{5}$$

The argument in the firm's earnings explicitly indicates that it depends on the capital it chooses at the beginning of the period. In the above, the firm's outputs when it is solvent and insolvent, respectively, $x_{i,t+1}$ and $x_{i,t+1}^{(b)}$, are given by (4). As the notation above indicates, the vector of spot prices of industry goods, $\{P_{I,t}; I \in [N]\}$, depend only on the industry-wide productivity shocks that are realized at the beginning of the period.

We now precisely specify the condition for firm solvency. Let $d_{i,t}$ be the debt level chosen by the firm at the beginning of the period, that is, $d_{i,t}$ is the debt payment that is due at the end of the period. We assume that

$$\text{If } \pi_{i,t+1}(k_{i,t}) \geq \rho_I d_{i,t}; \text{the firm is solvent,}$$
$$\text{If } \pi_{i,t+1}(k_{i,t}) < \rho_I d_{i,t}, \text{the firm is insolvent,} \tag{6}$$

where $\rho_I > 0$ is an industry-level parameter. In the above, $\pi_{i,t+1}(k_{i,t})$, is the maximum earnings the firm would generate if its productivity were not affected by solvency, and is given by the first equation in (5). For tractability, we assume that each firm is in financial distress (if at all) only in the current period, that is, insolvency does not persist over time.

3.3 Firm Financial Decisions

We now describe each firm's financial decisions or, equivalently, its capital structure choice at the beginning of each period $[t, t+1]$. Firms' decisions at any date maximize their expected discounted stream of future earnings, where the discount rate is determined by the firm's cost of capital, $r_I, I \in [N]$, that is constant through time and is the same for all firms in an industry. As in several prior studies, the friction that drives capital structure choices is that a firm's insiders (its management and employees) can extract private benefits from a firm's free cash flow. Debt is a hard claim that reduces a firm's free cash flow—total earnings net of capital rental and debt payments—thereby restricting the extraction of private benefits by insiders (e.g., Jensen (1986), Stulz (1990), Hart and Moore (1995), Williams (1995), and Zwiebel (1996)). Firms' financing decisions in each period reflect the tradeoff between the benefits of debt in limiting private benefit extraction by insiders and costs of insolvency.

By the above discussion, for a given choice of capital, $k_{i,t}$, and debt, $d_{i,t}$, the end-of-period payout to the firm's shareholders and debtholders is

$$\begin{aligned} \pi^{\text{net}}_{i,t+1}(k_{i,t}, d_{i,t}) &= \overbrace{[(1-\tau_I)\left[\pi_{i,t+1}(k_{i,t}) - (1+r_I)R_{K,t}k_{i,t}\right] + \tau_I d_{i,t}]\, 1_{\pi_{i,t+1}(k_{i,t}) \geq \rho_I d_{i,t}}}^{\text{solvent}} \\ &\quad + \overbrace{(1-\tau_I)\left[\pi^{(b)}_{i,t+1}(k_{i,t}) - (1+r_I)R_{K,t}k_{i,t}\right] 1_{\pi_{i,t+1}(k_{i,t}) < \rho_I d_{i,t}}}^{\text{insolvent}} \end{aligned} \tag{7}$$

In the above, $\tau_I \in (0,1)$ is the proportion of the firm's free cash flow that the firm's insiders are able to extract as their private benefits. We allow for the private benefit parameter to vary across industries, which captures the fact that the nature of industries could determine the extent to which insiders can extract private benefits. The first term on the R.H.S. of (7) indicates the benefit of debt in limiting private benefit extraction. Recall that $R_{K,t}k_{i,t}$ is the cost of renting capital, $k_{i,t}$, where the capital rental rate, $R_{K,t}$, is endogenously determined in the model by the market-clearing condition for capital.

Firms rent capital in each period, there are no capital adjustment costs, the debt level is determined by the payment due at the end of the period, and firm insolvency does not persist over time. Hence, each firm's full optimization problem is reduced to a period-by-period optimization problem. Each firm in industry, I, therefore chooses its capital structure at the beginning of each period to maximize its expected discounted end-of-period net payoff as described by the following:

$$\sup_{k_{i,t},d_{i,t}} \mathbb{E}_t \left[\frac{1}{1+r_I} \pi^{\text{net}}_{i,t+1}(k_{i,t}, d_{i,t}) \right]. \tag{8}$$

3.4 Equilibrium Conditions

In each period, $[t, t+1]$, the following conditions must hold in the industry equilibrium.

1. Each firm i in an industry chooses its capital structure to solve (8) after observing the industry shock, $Z_{I,t}$.

2. At the end of the period, after each firm's idiosyncratic productivity shock, $z_{i,t+1}$, is realized, the firm purchases industry goods in competitive spot markets to solve (5).

3. In each realized state at the end of the period, each agent in the economy purchases industry goods to maximize her utility given by (1) subject to her budget constraint, (2).

4. Markets for industry goods clear, that is, the total production of each industry good equals its total consumption plus its use as an intermediate good in the production of industry goods.

5. The market for capital clears in each period, that is, the aggregate demand for capital by firms in the economy equals the aggregate supply of capital, K_t.

4 The Equilibrium

We now derive the unique equilibrium of the model in which firms' financing and production decisions are endogenously determined. We first analyze firms' production decisions at the end of each period given their financing decisions and then their capital structure choices at the beginning of the period. Recall that firms in each industry experience the same industry-wide productivity shock, make their capital and financing decisions, experience idiosyncratic productivity shocks, and then make their production decision. As firms are ex ante identical when they choose their capital structures, it follows that, in equilibrium, all firms in an industry choose the same capital structure in each period. The capital structures of firms, therefore, depend only on the vector of industry productivity shocks, $\{Z_{I,t}; I \in [N]\}$ that are realized before firms make their capital and financing decisions as we formalize in the following preliminary proposition.

Proposition 1. *[Symmetric Equilibrium] All firms in an industry I choose the same capital structure, $\left(k^*_{I,t}, d^*_{I,t}\right)$, that depends only on the vector of industry productivity shocks, $\{Z_{I,t}; I \in [N]\}$.*

Our focus in this study is on how the network structure impacts *inter-industry* differences in firm leverage rather than on intra-industry leverage heterogeneity. We can, therefore, interpret our theory as one of *industry leverage*, which can be viewed as the average firm leverage in an industry. An extensive body of empirical evidence shows that industry leverage is among the most important determinants of firm leverage (Frank and Goyal (2009)). By Proposition 1, we can hereafter focus on the *representative firm* in each industry $I \in [N]$ without loss of generality.

Remark 1. We can, in fact, extend our model to incorporate intra-industry firm heterogeneity without altering our main implications, but at the expense of additional notational complexity. Specifically, instead of firms' productivities being identical at the beginning of each period, suppose that the productivity of firm i in industry I at the beginning of period t is $\zeta_i Z_{I,t}$, where ζ_i is a strictly positive parameter that is drawn from a distribution $F_I(.)$. As the notation indicates, the distribution $F_I(.)$ can vary across industries. We can then show that firm i's optimal capital structure is given by

$$k^*_{i,t} = g_I(\zeta_i) k^*_{I,t}; d^*_{i,t} = h_I(\zeta_i) d^*_{I,t}, \tag{9}$$

where $g_I(.), h_I(.)$ are functions determined by industry-level parameters, and $\left(k^*_{I,t}, d^*_{I,t}\right)$ is the optimal capital structure of firms in the main model without intra-industry heterogeneity in capital structures that we characterize in Theorem 1 below. We can show that our main results regarding the impacts of network characteristics on leverage, default risk and contagion extend to this more general setting.

4.1 Aggregate Production

In equilibrium, industry goods markets must clear. Hence, the total amount produced by industry I in period $[t, t+1]$ must be equal to the sum of the consumption of the industry good and the amount used as an intermediate input by firms in industry I and other industries.

$$X^*_{I,t} = C^*_{I,t} + \sum_{J \in [N]} X^*_{JI,t}. \tag{10}$$

By the law of large numbers, the total production and consumption depend only on the industry-wide productivity shocks realized at the beginning of the period as the notation above indicates.

Let $\mathbf{\Pi}_t^* \equiv \left(\Pi_{1,t}^*, ..., \Pi_{N,t}^*\right)$ denote the vector of industry gross profits, that is, $\Pi_{I,t}^*$ is the sum of the profits generated by all firms in industry I in period $[t, t+1]$. The industry gross profits are known at the beginning of period $[t, t+1]$ after the industry-wide productivity shocks are realized. The following proposition shows the familiar result that each industry's gross profit is a constant proportion of the total gross profit generated by all industries, which follows from the Cobb-Douglas preferences of consumers (see (1)) and the production functions of firms (see (4)).

Proposition 2. *[Industry Gross Profits] The vector of industry gross profits in period* $[t, t+1]$*is*

$$\mathbf{\Pi}_t^* = \boldsymbol{\theta} \sum_{I \in [N]} \Pi_{I,t}^*, where \tag{11}$$

$$\boldsymbol{\theta} := (\boldsymbol{I} - diag(\boldsymbol{\beta}))[\boldsymbol{I} - \boldsymbol{W}' diag(\boldsymbol{\beta})]^{-1} \boldsymbol{\alpha}, and \tag{12}$$

$$\sum_{I \in [N]} \theta_I = 1.$$

In the above, $\mathbf{\Pi}_t^* := (\Pi_{1,t}^*, ..., \Pi_{N,t}^*)'$, $\boldsymbol{\alpha} := (\alpha_1, ..., \alpha_N)'$, $\boldsymbol{\beta} := (\beta_1, ..., \beta_N)'$, $\boldsymbol{W}$ is the matrix with IJ-th entry $[w_{IJ}]$ and $\boldsymbol{\theta} := (\theta_1, ..., \theta_N)'$. The $\text{diag}(\boldsymbol{x})$ operator denotes the diagonalization function, which takes a vector $\boldsymbol{x}$ and transforms it into a diagonal square matrix with entry $[i, i]$ equal to x_i. This result shows shows that each industry I's share of the total gross profit of the economy is $\theta_I \in [0, 1]$. The share, θ_I, depends on the vector of elasticities of intermediate inputs, $\boldsymbol{\beta}$, the input-output network marix $\boldsymbol{W}$, and the vector of elasticities, $\boldsymbol{\alpha}$, that determine consumer preferences. The profit proportions vector, $\boldsymbol{\theta}$, corresponds to the Katz centrality measure[2]. This measure quantifies the indirect effects that each industry has on the rest as follows.

$$\boldsymbol{\theta} = \overbrace{(\boldsymbol{I} - \text{diag}(\boldsymbol{\beta}))\boldsymbol{\alpha}}^{\text{Preference component}} + \overbrace{[\underbrace{\boldsymbol{W}'\text{diag}(\boldsymbol{\beta})}_{1^{\text{st}} \text{ order effect}} + \underbrace{(\boldsymbol{W}'\text{diag}(\boldsymbol{\beta}))^2}_{2^{\text{nd}} \text{ order effect}} + \underbrace{(\boldsymbol{W}'\text{diag}(\boldsymbol{\beta}))^3}_{3^{\text{rd}} \text{ order effect}} + \underbrace{...}_{\text{Higher order effects}}]\boldsymbol{\alpha}}^{\text{Network component}}$$

The above captures the intuitive implications that industry $I \in [N]$ generates a higher proportion of total profits if (i) consumers have a greater preference for the industry good (i.e., larger α_I); (ii) a relatively greater number of other industries in the economy use the good as an intermediate input (which is captured by the input-output matrix, $\boldsymbol{W}$); or (iii) the industry's production function puts more weight on inputs from other industries (i.e., higher β_I).

As firm-level shocks are independently and identically distributed across firms, and all firms in an industry

[2] See *Katz (1953)*.

choose the same capital structure, it follows from the law of large numbers that the proportion of firms that default in an industry coincides with the default probability of the representative firm in the industry. Further, the spot prices of industry goods at the end of the period and the gross profits of all industries depend only on the vector of industry-wide productivity shocks, $\{Z_{I,t}; I \in [N]\}$ that are realized at date t. Define the aggregate price index, which is a function of the vector of spot prices of industry goods, $\{P_{1,t,...,}P_{N,t}\}$,as follows.

$$\overline{P}_t = \prod_{I \in [N]} P_{I,t}^{\alpha_I} > 0 \tag{13}$$

Note that, by our earlier discussion, the industry good prices depend only on the industry-wide productivity shocks that are realized at date t. As with other aggregate variables, we incorporate this explicitly in our notation for clarity.

The following proposition analytically characterizes the total gross profit in the economy in period $[t, t+1]$.

Proposition 3. *[Total Economic Profit] Let $\{(k_{I,t}, d_{I,t}; I \in [N]\}$ be the vector of capital and debt level choices by the representative firms in the different industries. The total gross profit in the economy is given by*

$$\begin{aligned}\sum_{I \in [N]} \mathbf{\Pi}^*_{I,t} = \exp\Big\{&\delta - \boldsymbol{\theta}' \log(\boldsymbol{\theta}) + \boldsymbol{\theta}' \log(\mathbf{1} - \boldsymbol{\beta}) + \boldsymbol{\theta}' \log(\boldsymbol{M}) + \log(\overline{P}_t) + \\ &\boldsymbol{\theta}' \boldsymbol{\mu} t + \boldsymbol{\theta}' \boldsymbol{\sigma} \varepsilon_t + \boldsymbol{\theta}' \log(\boldsymbol{\xi}_t) + \boldsymbol{\theta}' [\boldsymbol{I} - diag(\boldsymbol{\beta})]^{-1} diag(\boldsymbol{\gamma}) \log(\boldsymbol{k}_t)\Big\},\end{aligned} \tag{14}$$

$$\delta := \boldsymbol{\theta}' [\boldsymbol{I} - diag(\boldsymbol{\beta})]^{-1} diag(\boldsymbol{\beta}) \boldsymbol{\mathcal{W}}_s, \tag{15}$$

$$\xi_{I,t} := 1 - \big(1 - \chi_I^{\frac{1}{1-\beta_I}}\big) \Phi \left(\psi_{I,t}(k_{I,t}, d_{I,t}) - \frac{1}{2} \eta_I\right), \tag{16}$$

$$\psi_{I,t}(k_{I,t}, d_{I,t}) := \frac{1}{\eta_I} \log \left(\frac{\rho_I d_{I,t}}{M_I^{-1} \theta_I \mathbf{1}' \mathbf{\Pi}^*_{t+1}} \xi_{I,t}\right). \tag{17}$$

In the above, $\boldsymbol{\mathcal{W}}_s$ is a vector with I-th entry equal to $\sum_{J \in [N]} w_{IJ} \log(w_{IJ})$, $\boldsymbol{\xi}_t := (\xi_{1,t}, ..., \xi_{N,t})'$, $\boldsymbol{\gamma} := (\gamma_1, ..., \gamma_N)'$ and $\boldsymbol{k}_t := (k_{1,t}, ..., k_{N,t})'$. The vector, $\boldsymbol{M} := (M_1, ..., M_N)'$, is the vector of the masses of firms in the various industries.

Note that, by the discussion preceding the proposition, the total economic profit depends only on the industry-wide productivity shocks that are realized at date t. The expression in (14) can be unpacked as follows. The term, $\boldsymbol{\mathcal{W}}_s$, in (15) is a measure of the sparsity of the network (see Herzkovic (2018)), which captures how *self-reliant* industries are. The more that industries depend on their own industry good as an input for production and less on other industry goods, the larger the value for the sparsity measure. To further highlight the intuition for the sparsity measure, consider its values in the extreme cases of complete self-reliance and equal dependence on all industries for any industry $I \in [N]$:

- *Complete self-reliance*: In this scenario, the vector of weights for industry I is given by

$$w_{IJ} = \begin{cases} 1 & \text{if } I = J \\ 0 & \text{if } I \neq J \end{cases}$$

  In this case, the sparsity measure takes the highest possible value of 0 (i.e., $\sum_{J\in[N]} w_{IJ} \log(w_{IJ}) = (N-1)\log(0^0) + \log(1^1) = 0$).

- *Equal dependence*: The other extreme corresponds to the case in which $w_{IJ} = \frac{1}{N}$ for all $J \in [N]$. The sparsity measure now takes on the lowest possible value of $-\log(N)$ (i.e., $\sum_{J\in[N]} w_{IJ} \log(w_{IJ}) = \sum_{J\in[N]} \frac{1}{N} \log\left(\frac{1}{N}\right) = -\log(N)$).

Hence, we conclude that $-\log(N)\mathbf{1} \leq \boldsymbol{\mathcal{W}}_s \leq \mathbf{0}$. We make the important observation from (14) and (15) that sparsity affects total profits positively. The intuition is that, as sparsity increases, firms in each industry are less affected by negative shocks that impact other industries.

Continuing with the interpretation of the terms in (14), the expression, $\boldsymbol{\theta}' \log(\boldsymbol{\theta})$, can be interpreted as a *concentration* measure (or the negative of entropy). Following the same extreme scenarios that we used for the case of sparsity, we find that $-\log(N) \leq \boldsymbol{\theta} \log(\boldsymbol{\theta}) \leq 0$. This implies that the total gross profit depends negatively on concentration. Intuitively, a more concentrated network leads to increased dependence on shocks that affect more central industries in the economy.

The variable $\xi_{I,t}$ in (16) determines the impact on gross profits of firm insolvency that lowers firm productivity as indicated by (4). As expected, $\xi_{I,t}$ depends positively on the recovery rate parameter, χ_I, in (4) that determines firm productivity in insolvency and negatively on the probability of default which is given by the term $\Phi\left(\psi_{I,t}(k_{I,t}, d_{I,t}) - \frac{1}{2}\eta_I\right)$. The term $\psi_{I,t}(k_{I,t}, d_{I,t})$, which influences the probability of default, is the default threshold that depends on the capital structure choice of firms in industry I. Intuitively, the probability of default increases with the default threshold parameter, ρ_I and the debt level, $d_{I,t}$,and declines with the industry's profit as a proportion of the total economic profit, $\theta_I \sum_{J\in[N]} \Pi^*_{J,t}$.

An important feature of the model is that $\xi_{I,t}$ and $\psi_{I,t}(k_{I,t}, d_{I,t})$ are *jointly determined* by (16) and (17). This captures the externality imposed on other industries by insolvent firms in industry I. Insolvency lowers firm productivity, which lowers the production of the industry good that is employed as an intermediate input by firms in other industries. This, in turn, affects all industry good prices and, thereby, the profits of firms in all industries. As we discuss later, the coupled set of equations, (16) and (17), generates a *fixed-point* problem for the vector of equilibrium default probabilities of firms in all industries that plays a central role in determining firms' capital structure decisions in equilibrium. For future reference, the probability that the representative firm in industry $I \in [N]$ is insolvent at time $t+1$ is given by

$$\varphi_{I,t} = \Pr(\pi_{i,t+1}(k_{I,t}) < \rho_I d_{I,t}) = \Phi\left(\psi_{I,t}(k_{I,t}, d_{I,t}) + \frac{1}{2}\eta_I\right). \tag{18}$$

4.2 Firm Capital Structures

We now characterize the equilibrium capital structures and default probabilities of firms. It is useful to verbally describe the characterization before presenting the formal results. Firms' capital structure choices trade off the private benefits of debt against costs of insolvency that manifest in the lowering of firm productivities upon insolvency. The insolvency of a firm affects the production of its good that is used as an intermediate input for other firms in its own industry and other industries as well as for consumption by agents. As we discussed above, the distortions in the production of goods due to firm insolvency in each industry affects equilibrium spot market prices for the goods. The spot market prices, in turn, influence firm profits and, therefore, their capital structure decisions.

We make the following assumption on parameter values to ensure the existence of equilibrium.

$$\sum_{I\in[N]} \frac{\theta_I \gamma_I}{1-\beta_I} < 1 \tag{19}$$

The above inequality implies decreasing returns to scale for the overall economy. Note that we don't require each individual industry to exhibit decreasing returns to scale in production, but for all the industries to collectively experience decreasing returns.

We need to define the following intermediate functions that play key roles in the characterizations of firms' capital structures and default probabilities. Let $\varphi_I \in [0,1]$ for $I \in [N]$ be a parameter. Define the following functions of φ_I:

$$\Xi_I(\varphi_I) := 1 - \left(1 - \chi_I^{\frac{1}{1-\beta_I}}\right)\Phi\left(\Phi^{-1}(\varphi_I) - \eta_I\right), \tag{20}$$

$$\Lambda_I(\varphi_I) := \left(1 + \frac{\eta_I \Xi_I(\varphi_I)}{\left(1 - \chi_I^{\frac{1}{1-\beta_I}}\right)\phi\left[\Phi^{-1}\left(\frac{1-\Xi_I(\varphi_I)}{1-\chi_I^{\frac{1}{1-\beta_I}}}\right)\right]}\right)^{-1}, \tag{21}$$

$$\Psi_I(\varphi_I) := \frac{\phi\left(\Phi^{-1}(\varphi_I)\right)\left(1 - \Lambda_I(\varphi_I)\right)}{\eta_I}, \tag{22}$$

$$\Omega_I(\varphi_I) := \left[\frac{(1-\varphi_I - \Psi_I(\varphi_I))\left(1-\boldsymbol{\theta}'\boldsymbol{\Lambda}\left(\boldsymbol{\varphi}\right)\right)}{\theta_I \Lambda_I(\varphi_I)} - \Psi_I(\varphi_I) \right]^{-1}, \tag{23}$$

where $\boldsymbol{\Lambda}\left(\boldsymbol{\varphi}\right) \equiv (\Lambda_1(\varphi_1), ..., \Lambda_N(\varphi_N))$.

Theorem 1. *[The Equilibrium] There is a unique equilibrium of the economy that can be characterized as follows*

1. The default probability of firms in industry I during period $[t, t+1]$ is $\varphi_I^* \in (0, \bar{\varphi}_I)$, where φ_I^* is the unique solution of the following fixed point problem for each $I \in [N]$:

$$\varphi_I^* = \Phi\left(\frac{1}{\eta_I}\left[\log\left(\frac{(1-\tau_I)\rho_I}{\tau_I}\right) + \log(\Omega_I\left(\varphi_I^*\right)) + \log(\Xi_I\left(\varphi_I^*\right)) \right] + \frac{1}{2}\eta_I \right). \tag{24}$$

2. The upper bound, $\bar{\varphi}_I$, on possible values of the equilibrium default probability, φ_I^*, is given by

$$\bar{\varphi}_I := \min\left\{ \varphi_I \middle| \varphi_I - \left[1 - \nu_I\left(1 + \frac{\theta_I \zeta_I}{1-\boldsymbol{\theta}'\boldsymbol{\zeta}}\right)\right] = 0 \right\}. \tag{25}$$

3. The capital and debt level choices of firms in industry $I \in [N]$ are given by

$$k_{I,t}^* = \frac{\theta_I^2 \frac{\gamma_I}{1-\beta_I}\left(1 + \Psi_I\left(\varphi_I^*\right)\Omega_I\left(\varphi_I^*\right)\right)}{M_I \sum_{J\in[N]} \theta_J^2 \frac{\gamma_J}{1-\beta_J}\left(1+\Psi_J\left(\varphi_J^*\right)\Omega_J\left(\varphi_J^*\right)\right)} K_t, \tag{26}$$

$$d_{I,t}^* = \frac{(1-\tau_I)\theta_I}{M_I \tau_I}\Omega_I\left(\varphi_I^*\right) \sum_{I\in[N]} \boldsymbol{\Pi}_{I,t}^*. \tag{27}$$

4. The gross profit of all firms in the economy in period $[t, t+1]$ is

$$\sum_{I\in[N]} \boldsymbol{\Pi}_{I,t}^* = \exp\Bigg\{ \delta - \boldsymbol{\theta}'\log(\boldsymbol{\theta}) + \boldsymbol{\theta}'\log(\mathbf{1}-\boldsymbol{\beta}) + \log(\overline{P}_t) + \boldsymbol{\theta}'\boldsymbol{\mu}t + \boldsymbol{\theta}'\boldsymbol{\sigma}\varepsilon_t + \boldsymbol{\theta}'\log(\boldsymbol{\Xi}(\boldsymbol{\varphi}^*)) + \sum_{I\in[N]} \frac{\gamma_I}{1-\beta_I}\theta_I \log\left(\frac{\gamma_I \theta_I^2 K_t (1+\Psi_I\left(\varphi_I^*\right)\Omega_I\left(\varphi_I^*\right))}{(1-\beta_I)\sum_{J\in[N]}\theta_J^2 \frac{\gamma_J}{1-\beta_J}(1+\Psi_J\left(\varphi_J^*\right)\Omega_J\left(\varphi_J^*\right))} \right) \Bigg\}, \tag{28}$$

5. The leverage ratio of firms in industry I is

$$Lev_{I,t} = \frac{d_{I,t}^*}{k_{I,t}^*} = \frac{(1-\tau_I)\Omega_I\left(\varphi_I^*\right)\left[\sum_{J\in[N]}\theta_J^2 \frac{\gamma_J}{1-\beta_J}(1+\Psi_J\left(\varphi_J^*\right)\Omega_J\left(\varphi_J^*\right))\right]}{\tau_I \theta_I \frac{\gamma_I}{1-\beta_I}(1+\Psi_I\left(\varphi_I^*\right)\Omega_I\left(\varphi_I^*\right))} \frac{\sum_{I\in[N]}\boldsymbol{\Pi}_{I,t}^*}{K_t} \tag{29}$$

6. The rental rate of capital is given by

$$R_{K,t} = \frac{\left[\sum_{I\in[N]}\theta_I^2 \frac{\gamma_I}{1-\beta_I}(1+\Psi_I\left(\varphi_I^*\right)\Omega_I\left(\varphi_I^*\right))\right]\sum_{I\in[N]}\boldsymbol{\Pi}_{I,t}^*}{(1+r_I)(1-\boldsymbol{\theta}'\boldsymbol{\Lambda}\left(\boldsymbol{\varphi}^*\right))K_t}. \tag{30}$$

As noted earlier, firm insolvency in each industry affects the production of the industry good that, in turn, influences the production decisions of all other firms and industry good prices. The industry good prices, in turn, affect the earnings of an individual firm in each industry. Each firm chooses its debt level rationally anticipating its probability of insolvency. In equilibrium, however, the law of large numbers implies that the probability of insolvency of an individual firm equals the proportion of insolvent firms in the industry that is reflected in industry good prices. Hence, as described by Theorem 1, firms' capital structure choices and default probabilities are determined by the solution to the system of fixed-point equations, (24), for the equilibrium default probabilities that, in turn, depend on the functions defined in (23) – (25). Note in particular that the function, Ω_I, defined in (27) captures the interaction among the different industries via the term $\boldsymbol{\theta}'\boldsymbol{\Lambda}(\boldsymbol{\varphi})$.

4.3 Network Characteristics, Capital Structures and Firm Default Probabilities

4.3.1 Concentration and Sparsity

We now analytically examine how the structure of the production network affects firms' capital structures and probabilities of default. The network affects leverage via two channels: the *industry weights* vector $\boldsymbol{\theta}$ defined in (12) and the *weighted sparsity*, δ defined in (15). As discussed earlier, the $\boldsymbol{\theta}$ vector represents industry profits as proportions of the total profit of the economy. The parameter, δ, represents the weighted average sparsity of the input-output matrix $\boldsymbol{W}$, which captures how much of an industry's inputs come from the same industry or from its peers in the economy.

As shown in Theorem 1, the industry weights vector, $\boldsymbol{\theta}$, directly affects the probabilities of default and leverage ratios of firms. However, the weighted average sparsity only affects leverage via the total gross profit in the economy, $\sum_{I\in[N]} \boldsymbol{\Pi}^*_{I,t}$ given by (14). As each industry's gross profit scales with the total gross profit, each firm's optimal debt level also scales proportionally with the total gross profit given that it reflects the tradeoff between the private benefits of debt and insolvency costs both of which are proportional to profits. Consequently, *controlling for the industry weights vector,* $\boldsymbol{\theta}$, the default probability, which depends on the ratio of the debt level to firm profit, is unaffected by network sparsity. The only channel through which the network structure affects default probabilities is via the vector of industry weights, $\boldsymbol{\theta}$.

4.3.2 Power Law for Industry Weights

To simplify the analysis and facilitate more concrete implications, we assume that the vector of industry weights, $\boldsymbol{\theta}$, *sorted in decreasing order*, can be calibrated to a power law function that depends on a single *scalar parameter*, $\theta \in [0,1]$, as follows.

$$\theta_I = \frac{\theta}{(1-\theta)(1-(1-\theta)^N)}(1-\theta)^I. \tag{31}$$

We can verify that $\sum_{I\in[N]} \theta_I = 1$, as expected. The scalar θ can be interpreted as a measure of industry concentration. The higher the value of θ, the higher the concentration. This can be gleaned from measures such as the Herfindahl–Hirschman index - HHI, that can be expressed as a function of θ:

$$\text{HHI}(\boldsymbol{\theta}) = \sum_{I\in[N]} \theta_I^2 = \frac{\theta}{(1-\theta)(1-(1-\theta)^N)} \sum_{I\in[N]} (1-\theta)^{2I} = \frac{\theta}{2-\theta} \cdot \frac{1+(1-\theta)^N}{1-(1-\theta)^N}.$$

We can show that $\frac{\partial HHI}{\partial \theta} > 0$. We can also generate the extremes of lowest and highest concentration with the above functional form. Specifically,

$$\boldsymbol{\theta}_{\min} = \lim_{\theta\to 0} \boldsymbol{\theta} = \left(\frac{1}{N}, \frac{1}{N}, ..., \frac{1}{N}\right)' \Rightarrow \text{HHI}(\boldsymbol{\theta}_{\min}) = 0,$$

and

$$\boldsymbol{\theta}_{\max} = \lim_{\theta\to 1} \boldsymbol{\theta} = (1,0,...,0)' \Rightarrow \text{HHI}(\boldsymbol{\theta}_{\max}) = 1.$$

The following graph shows the relationship between the actual and estimated patterns for the weight vector, $\boldsymbol{\theta}$, with the 55 industries in our dataset sorted from largest to smallest on the horizontal axis. It shows that a power law is a good approximation for the observed variation of $\boldsymbol{\theta}$ across industries.

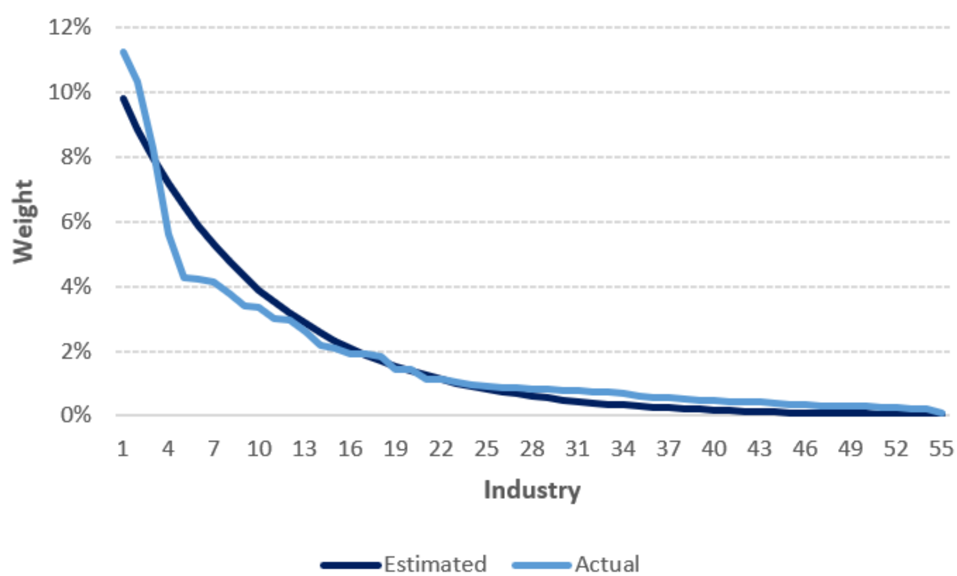

Figure 2: Comparison between actual and estimated weights $\boldsymbol{\theta}$

4.3.3 Impacts of Network Structure

The following proposition shows how the network structure impacts the leverage levels and default probabilities.

Proposition 4. *[Network Characteristics, Leverage Levels, and Default Probabilities] Let $I \in [N]$ be ordered in decreasing order by the vector of industry weights, $\boldsymbol{\theta}$.*

1. *An increase in concentration (i.e., increase in θ) affects the industry weights as follows.*

$$\frac{\partial \log(\theta_N)}{\partial \log(\theta)} \leq \frac{\partial \log(\theta_{N-1})}{\partial \log(\theta)} \leq ... \leq \frac{\partial \log(\theta_2)}{\partial \log(\theta)} \leq \frac{\partial \log(\theta_1)}{\partial \log(\theta)} \tag{32}$$

2. *There exists a strictly positive vector $\boldsymbol{\rho^* := \left(\rho_1^*, \rho_2^*, ..., \rho_N^*\right)} > 0$, such that if $\rho_I < \rho_I^*$ for all $I \in [N]$, then we have the following results:*

$$\frac{\partial \log(\varphi_1^*)}{\partial \log(\theta)} \leq 0 \leq \frac{\partial \log(\varphi_N^*)}{\partial \log(\theta)}. \tag{33}$$

3. *Similarly for leverage, if $\rho_I < \rho_I^*$ for all $I \in [N]$, then we have the following results:*

$$\frac{\partial \log(Lev_1)}{\partial \log(\theta)} \leq 0 \leq \frac{\partial \log(Lev_N)}{\partial \log(\theta)}. \tag{34}$$

4. *Controlling for the industry weights,$\boldsymbol{\theta}$, the sensitivity of leverage with respect to sparsity satisfies*

$$\frac{\partial \log(Lev_I(\theta, \delta))}{\partial \delta} > 0, \forall I \in [N]. \tag{35}$$

The above proposition shows that network concentration has *differential* impacts on the leverage levels and default probabilities of firms in larger and smaller industries. More precisely, provided the default threshold in each industry, $I \in [N]$, is below a trigger level, ρ_I^*, the probabilities of default and leverage levels increase with θ for firms in smaller industries (i.e., higher values of I) and decrease with θ for firms in larger industries (i.e., lower values of I). In the calibrated model of Section 4.3.4, we find that the default thresholds in all the industries are, indeed, below the trigger levels, $\{\rho_I^*; I \in [N]\}$. Since θ is a measure of concentration (with higher values implying greater concentration), we can conclude that a higher concentration, *ceteris paribus*, leads to a lower (higher) probability of default and leverage for firms in larger (smaller) industries.

The intuition for these results hinges on the observation that firms' leverage choices rationally incorporate the impact of firm insolvency on their profits via the production network. Firms in larger industries impose greater externalities on firms in other industries and the total output of the economy due to their potential insolvency relative to firms in smaller industries. An increase in network concentration further increases the influence of larger industries, and decreases the influence of smaller industries. Consequently, firms in larger industries become more conservative in their leverage choices, while firms in smaller industries become less conservative. These effects hold provided the industry default thresholds are below respective thresholds. If the default thresholds are sufficiently high, then the nonlinear effects of firm insolvency make it difficult to pin down the comparative statics of leverage levels and default probabilities with network concentration. However, as we mentioned above, in the calibrated model, we find that the default thresholds are, indeed, below the trigger levels so that the conclusions of Proposition 4 hold.

As we discussed in Section 4.3.1, controlling for the vector of industry weights, $\boldsymbol{\theta}$, and hence the network concentration, an increase in sparsity (i.e., increase in δ) increases firm leverage levels, without affecting the probability of default. As mentioned in Section 4.1, sparsity captures how self-reliant industries are with regard to their production inputs. The more sparse the network, the less that each industry depends on the others for their productive activity. Hence, an increase in sparsity allows industries to increase their leverage, without generating additional insolvency risk. Given that industries share the total gross profit in the economy, only the weighted average measure for sparsity affects leverage, and not individual industry sparsities. This implies that the sensitivity of leverage with respect to the aggregate sparsity measure is similar for all industries in the economy.

4.3.4 Contagion Analysis

In this section, we examine the impact of *unancipated* shocks to key parameters in each industry—the firm-level idiosyncratic volatility, η_I,the default threshold, ρ_I, and the recovery proportion, χ_I, upon firm insolvency—on the probabilities of default of firms in the other industries. As the shocks are unanticipated, we assume that firms have already chosen their optimal leverage levels according to (29) and that the unexpected shock occurs in period $t+1$.

Proposition 5. *[Effects of Unanticipated Shocks]*

1. There exists a vector of threshold values of the firm-level idiosyncratic volatility, $\boldsymbol{\eta}^* \equiv (\eta_1^*, ..., \eta_N^*)$, *such that the probability of default for industry I with respect to an unexpected increase in firm-level idiosyncratic* *volatility of industry J satisfies the following.*

$$\frac{\partial \varphi_I^*}{\partial \eta_J} = \begin{cases} \geq 0 & \eta_J \leq \eta_J^* \\ < 0 & \eta_J > \eta_J^* \end{cases}. \tag{36}$$

2. The default probability for industry I with respect to an unexpected increase in the default threshold of industry J satisfies

$$\frac{\partial \varphi_I^*}{\partial \rho_J} \geq 0, \tag{37}$$

3. The default probability for industry I with respect to an unexpected increase in the recovery proportion of industry J satisfies

$$\frac{\partial \varphi_I^*}{\partial \chi_J} \leq 0\cdot \tag{38}$$

Part 1 of the proposition shows that the default probability of firms in an industry increases with an unexpected increase in the firm-level idiosyncratic volatility in another industry if it is below a threshold, but decreases if it is above the threshold. The intuition for the non-monotonic behavior of the idiosyncratic volatility is that volatility has two offseting effects. A higher volatility increases the risk of insolvency by making negative scenarios more common, but it can also increase the upside by augmenting the likelihood of positive shocks. Given that shocks have a positively skewed log-normal distribution, the negative effects dominate when the volatility is below a threshold, but the positive effects prevail when the volatility is above the threshold. In the calibrated model, we find that the idiosyncratic volatilities are below the respective thresholds so that $\frac{\partial \varphi_I^*}{\partial \eta_J} \geq 0$, for all $I, J \in [N]$. This implies that unexpected shocks to the idiosyncratic volatility can generate contagion effects, by increasing the probability of default in other industries of the economy.

Part 2 shows that the default probability of firms in an industry increases with an unexpected increase in the default threshold in another industry. The intuition is that, an unexpected increase in the default threshold for an industry increases the proportion of insolvent firms that, in turn, lowers the total profit of the economy and the profits of firms in other industries. Hence, the default probabilities of firms increase.

Part 3 shows that the default probability of firms in an industry decreases with an unanticipated increase in the recovery proportion upon insolvency of firms in another industry. The intuition is analogous to that of Part 2. An increase in the recovery proportion lowers the negative impact of firm insolvency on the profits of firms in other industries, thereby lowering their default probabilities.Calibration

We now calibrate our structural model to the data to obtain baseline values of the parameters to facilitate our subsequent quantitative analysis.

4.4 Data

We obtain data from two primary sources—Compustat and the Bureau of Economic Analysis (BEA)—from 1997 to 2021. The time period is determined by the availabaility of relevant data at the BEA. The Fundamentals Annual data from Compustat contains information on balance sheet variables for a large sample of firms. BEA provides annual information on the input-output tables that we use to determine the industry network weights. We also obtain additional data on firm productivities from the Bureau of Labor Statistics (BLS), and firm default probabilities from *S&P Global Ratings (2021)*.

To arrive at the final Compustat dataset, we select only companies incorporated or legally registered in the United States and with the US dollar as the native currency. We exclude firms that have missing or nonsensical values (i.e., negative values for asset size, debt, sales and interest payments). The final dataset includes $89,715$ observations of $11,655$ unique firms. The following table presents summary statistics for the Compustat data.

Table 1: Summary Statistics

Variables	Obs	Mean	Std. Dev	Min	P25	P50	P75	Max
Assets ($bn)	89,715	3,28	15,46	0,00	0,04	0,29	1,44	551,67
Sales ($bn)	89,715	2.75	13.45	0.00	0.04	0.27	1.29	569.96
Debt Expense ($bn)	89,715	0.19	1.60	0.00	0.00	0.01	0.06	105.09
Gross Profit ($bn)	89,715	0.89	4.33	0.00	0.01	0.09	0.41	220.39
Debt Expense-to-Assets	89,715	0.43	10.94	0.00	0.02	0.04	0.10	1,280.00
Debt Expense-to-Gross Profit	89,715	2.60	55.21	0.00	0.05	0.15	0.39	6,182.75
Asset Turnover	89,715	1.42	44.15	0.00	0.57	0.99	1.55	13,203.00

We group the firms into 55 major industries aggregated according to the first three NAICS codes following the classication used by the BEA. We exclude FIRE (i.e., Finance, Insurance, Real Estate, Rental, and Leasing), Utilities and Government as our model is one of non-financial firms that are not regulated by the Government. Figure 3 illustrates the network structure implied by the input-output matrix for firms grouped into 12 major industries at the two-digit NAICS level purely for illustrative convenience. The nodes represent each industry and the edges correspond to the linkages with other industries. The closed loops (i.e., edges that go from one industry back to itself) indicate that firms in the industry utilize goods produced within the industry. We, however, carry out our analysis for firms grouped into 55 industries at the three-digit NAICS level.

Figure 3: Network Structure: 12 Major Industries (2-digit NAICS)

We obtain the matrix of input-output weights using the "commodities by industries" database provided by the BEA. Figure 4 shows the average weight matrix for the entire historical period for a subset of industries. We present the complete matrix for all 55 industries in Appendix B.

	Wholesale trade	Other transportation equipment	Computer and electronic products	Administrative and support services	Chemical products	Motion picture and sound recording industries	Air transportation	peline transportatic	Machinery	Miscellaneous professional, scientific, and technical services	Data processing, internet publishing, and other information services	...	Hospitals
Wholesale trade	23.12%	1.81%	1.57%	0.50%	2.25%	0.29%	0.74%	0.47%	2.71%	0.46%	0.51%	...	0.95%
Other transportation equipment	0.03%	90.41%	0.13%	0.03%	0.05%	0.02%	0.70%	0.03%	0.35%	0.04%	0.03%	...	0.02%
Computer and electronic products	0.39%	4.43%	57.35%	0.52%	0.84%	0.91%	0.21%	0.28%	2.18%	0.60%	1.04%	...	0.28%
Administrative and support services	1.39%	1.02%	0.73%	31.74%	0.78%	1.12%	1.71%	2.44%	0.98%	1.64%	2.06%	...	1.64%
Chemical products	0.26%	0.86%	0.84%	0.45%	32.62%	0.24%	0.29%	0.24%	1.19%	0.45%	0.31%	...	1.81%
Motion picture and sound recording industries	0.11%	0.06%	0.06%	0.15%	0.07%	87.90%	0.09%	0.37%	0.08%	0.47%	0.20%	...	0.10%
Air transportation	0.37%	0.25%	0.20%	0.72%	0.25%	0.40%	81.31%	0.19%	0.37%	0.44%	0.64%	...	0.23%
Pipeline transportation	0.08%	0.09%	0.07%	0.09%	0.44%	0.06%	0.54%	85.10%	0.13%	0.06%	0.07%	...	0.13%
Machinery	0.17%	1.71%	0.37%	0.32%	0.71%	0.21%	0.28%	0.43%	59.86%	0.23%	0.37%	...	0.23%
Miscellaneous professional, scientific, and technical services	1.64%	0.88%	0.85%	1.73%	1.11%	2.80%	0.81%	2.03%	1.24%	24.15%	2.20%	...	1.43%
Data processing, internet publishing, and other information services	0.36%	0.41%	0.31%	1.50%	0.29%	0.30%	0.34%	0.31%	0.52%	0.76%	75.27%	...	0.48%
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋱	
Hospitals	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	...	99.99%

Figure 4: Average Network Weight Matrix for Subset of Industries

The average network matrix (i.e., W) has the property that all rows must add up to 1 (i.e., $\sum_{j \in [N]} w_{IJ} = 1$). The colors indicate the relative magnitude of the weight values, ranging from 0.03% (i.e., white) to 98.71%

(i.e., dark blue).

4.5 Model Parameters

The model has nine parameters for each industry that can be represented by the following parameter vectors: $\boldsymbol{\alpha}$, the elasticities of consumption of the industry goods in the representative consumer's utility function; $\boldsymbol{\beta}$, the elasticities of the intermediate good inputs in the production function; $\boldsymbol{\gamma}$, the elasticities of capital in the production function; $\boldsymbol{\eta}$, the volatilities of firm-level idiosyncratic productivity shocks; $\boldsymbol{\mu}$, the mean growth rates of the industry-level productivity shocks; $\boldsymbol{\sigma}$, the volatilities of the industry-level productivity shocks; $\boldsymbol{\rho}$; the insolvency threshold parameters; $\boldsymbol{\chi}$, the proportional recovery rates in insolvency; and τ; the agency cost/private benefit parameters. As the parameters are all at the industry level, and there are 55 industries, we have a total of 495 parameters. We calibrate the parameters for each industry using industry-level statistics in the data that facilitate their direct identification, thereby significantly alleviating the computational complexity of estimating the model. The following table summarizes the parameters of the model.

Table 2: Model Parameters

Parameter	Description
$\boldsymbol{\alpha}$	Preference elasticities
$\boldsymbol{\beta}$	Intermediate input elasticities
$\boldsymbol{\eta}$	Volatilities of firm-level idiosyncratic productivity shocks
$\boldsymbol{\mu}$	Mean growth rates of industry-level productivity shocks
$\boldsymbol{\sigma}$	Volatilities of industry-level productivity shocks
$\boldsymbol{\rho}$	Default threshold parameters
$\boldsymbol{\chi}$	Proportional insolvency recovery rates
γ	Capital elasticities
$\boldsymbol{\tau}$	Private benefit parameters

4.6 Calibration of Model Parameters

The calibration of the model parameters requires the following intermediate result.

Proposition 1. *The revenue of firm $i \in I$ is linked to the average revenue of firms in industry $I \in [N]$ via the following expression:*

$$P_{I,t}x^*_{i,t+1} = z_{i,t+1}^{\frac{1}{1-\beta_I}} P_{I,t}\bar{x}^*_{I,t} = e^{-\frac{1}{2}\eta_I^2+\eta_I\varepsilon_{i,t+1}} P_{I,t}\bar{x}^*_{I,t}, \tag{39}$$

where $P_{I,t+1}\bar{x}^*_{I,t+1} := \frac{1}{m_1}\sum_{i\in I} P_{I,t+1}x^*_{i,t+1}$.

We now describe the calibration of each model parameter. Since the target moments for our calibration are at the industry level, we aggregate firm-level variables according to their 3-digit NAICS codes. The following table summarizes the variables used in the model and their sources.

Table 3: Model variables and their sources

Variable	Source	Description
$\boldsymbol{X}_t$	Compustat (variable: sale)	Revenue
$\boldsymbol{\Pi}_t$	Compustat (variable: sale - cogs)	Gross profits
$\boldsymbol{K}_t$	Compustat (variable: at)	Capital levels
$\boldsymbol{D}_t$	Compustat (variable: dlc + xint)	Debt levels
$[N]$	Compustat (variable: naics)	Industry classification
$\boldsymbol{W}$	BEA	Input-output network
$\boldsymbol{\alpha}$	BEA	Preference elasticies
$\boldsymbol{\chi}$	Jankowitsch et al. (2014)	Recovery rates given default
$\boldsymbol{Z}_t$	BLS	Multifactor productivities
$\boldsymbol{\varphi}$	Standard & Poor's	Probabilities of default

- $\boldsymbol{\alpha}$ - We calibrate the consumption elasticities to match the means of relative consumption expenditures from the BEA

$$\alpha_I = \mathbb{E}\left[\frac{\text{Consumption Expenditures}_{I,t}}{\sum_{J\in[N]} \text{Consumption Expenditures}_{J,t}}\right], \forall I \in [N]. \tag{40}$$

  We approximate the expectation on the R.H.S. above by the time-series average of the corresponding relative consumption expenditures of each industry $I \in [N]$. We obtain consumption expenditure data by industry from the BEA.

- $\boldsymbol{\beta}$ - We calibrate the elasticities of the intermediate good inputs in the production function as follows. By the first order conditions for (5), a firm's profit is a constant proportion of its revenue , that is, $\pi_{i,t+1}(k_{I,t}) = (1-\beta_I)P_{I,t}x^*_{i,t+1}$.[3] As firms in an industry are ex ante identical in the model, the total profit of all firms in industry $I \in [N]$ equals $(1-\beta_I)$ times the total revenue. Accordingly, we calibrate the elasticity vector to match the expectation of industry profit as a proportion of industry sales,

$$\beta_I = 1 - \mathbb{E}\left[\frac{\Pi^*_{I,t}}{P_{I,t}X^*_{I,t}}\right], \forall I \in [N]. \tag{41}$$

  We approximate the expectation on the R.H.S. above by the time-series average of the industry profits as a proportion of industry revenues. The time series for firm-level gross profits $(\pi^*_{i,t})$ and sales $(P_{I,t}x^*_{i,t})$ are provided by Compustat. We calculate gross profit for each firm by subtracting the cost of goods sold from sales.

[3]See the Appendix 6 for further details.

- **η** - In the model, the volatility of the firm-level idiosyncratic productivity shock for each industry equals the volatility of the growth in firm sales. By (39), the sales of each firm is equal to the industry average sales multiplied by the idiosyncratic shock. Accordingly, we can calibate η_I using the following relation:

$$\eta_I = \frac{1}{\sqrt{2}}\sqrt{\mathbb{V}\left[\log\left(\frac{P_{I,t+1}x^*_{i,t+2}}{P_{I,t}x^*_{i,t+1}}\right)\right] + \mathbb{V}\left[\log\left(\frac{P_{I,t+1}\bar{x}^*_{I,t+1}}{P_{I,t}\bar{x}^*_{I,t}}\right)\right]}, \forall I \in [N]. \tag{42}$$

In the R.H.S. above, $P_{I,t}x^*_{i,t+1}$ represents the sales of the representative firm i in industry I in period t, while $P_{I,t+1}\bar{x}^*_{I,t}$ is the average sales of all firms in the industry in period t. We approximate the variance on the R.H.S. above using time-series data on individual and industry-average firm sales. Recall that, as firm-level productivity shocks are i.i.d., it follows by the law of large numbers that the average sales of all firms in an industry depends only on the industry-level productivity shocks that are realized at the beginning of the period. In contrast, the revenue of an individual firm depends on its firm-specific productivity shock that is realized at the end of the period. The notation explicitly incorporates the distinction as in Sections 3 and 4.

- **μ** and **σ** - The mean growth rate and volatility of the industry-level productivity shock can be calibrated by using (3) and noticing that the logarithmic return for the industry-level productivity factor has the form $\log\left(\frac{Z_{I,t}}{Z_{I,t-1}}\right) = (1-\beta_I)[\mu_I + \sigma_I(\varepsilon_t - \varepsilon_{t-1})]$. We then proceed to find the expected value and variance of the previous equation and solve for μ_I and σ_I. The results are as follows:

$$\mu_I = \frac{1}{1-\beta_I}\mathbb{E}\left[\log\left(\frac{Z_{I,t}}{Z_{I,t-1}}\right)\right], \forall I \in [N]. \tag{43}$$

$$\sigma_I = \frac{1}{\sqrt{2}(1-\beta_I)}\sqrt{\mathbb{V}\left[\log\left(\frac{Z_{I,t}}{Z_{I,t-1}}\right)\right]}, \forall I \in [N]. \tag{44}$$

We use time-series data on multi-factor productivities provided by the Bureau of Labor Statistics (BLS) to represent $Z_{I,t}$ and approximate the expectation and variance on the R.H.S. of the equations above using the corresponding time-series mean and variance, respectively.

- **ρ** - The default threshold parameter is obtained by using (18), solving for ρ_I and calculating expectations.

$$\log(\rho_I) = \eta_I\mathbb{E}[\Phi^{-1}(\varphi^*_I)] - \frac{1}{2}\eta_I^2 - \mathbb{E}\log\left(\frac{D^*_{I,t}}{\Pi^*_{I,t}}\right) - \log(\Xi_I(\varphi^*_I)), \forall I \in [N]. \tag{45}$$

The above equation expresses the default threshold in terms of the default probability, φ^*_I, of firms in industry I and the ratio of debt expenses to profit. We use data on default probabilities provided by S&P Global Ratings to set the values of the default probabilities. We employ the ratio of total industry debt to industry profit as the proxy for the term, $\frac{D^*_{I,t}}{\Pi^*_{I,t}}$, and approximate the expectations by the time-series averages of the quantities. We obtain data on the firm-level debt expense ($d^*_{i,t}$) from Compustat by adding short term debt and interest payments.

- $\boldsymbol{\chi}$: We obtain the proportional salvage values upon insolvency from Jankowitsch et al. (2014).

- $\boldsymbol{\gamma}$: We calibrate the capital intensities vector so that the following system of fixed-point equations that determine the expected leverage ratio is satisfied. We solve the system numerically to obtain the capital intensity parameters.

$$\mathbb{E}\left[\frac{D^*_{I,t}}{K^*_{I,t}}\right] = \frac{(1-\tau_I)\Omega_I\left(\varphi^*_I\right)\sum_{J\in[N]}\theta^2_J\frac{\gamma_J}{1-\beta_J}(1+\Psi_J\left(\varphi^*_J\right)\Omega_J\left(\varphi^*_J\right))}{\tau_I\theta_I\frac{\gamma_I}{1-\beta_I}(1+\Psi_I\left(\varphi^*_I\right)\Omega_I\left(\varphi^*_I\right))}\mathbb{E}\left[\frac{\sum_{I\in[N]}\Pi^*_{I,t}}{K_t}\right], \forall I\in[N]. \quad (46)$$

We obtain data on firm-level capital (i.e., assets) ($k_{i,t}$) from Compustat.

- $\boldsymbol{\tau}$: The agency cost/private benefits parameter vector can be obtained by solving for τ_I in the following expression for the expected debt expense-to-gross profit ratio

$$\mathbb{E}\left[\frac{D^*_{I,t}}{\Pi^*_{I,t}}\right] = \frac{1-\tau_I}{\tau_I}\Omega_I\left(\varphi^*_I\right), \forall I\in[N]. \quad (47)$$

4.7 Baseline Parameter Values

Table 4 in Appendix C shows the baseline parameter values from our calibration of the model. From the table we can see that there is a lot of variation across industries. We can summarize the results as follows:

- $\boldsymbol{\alpha}$: Industries have a very varied behavior with respect to the preference weight parameter. The concentration is very high, since only six industries represent over 50% of the purchasing decisions made by consumers. There are several industries that have a negligible weight, since their goods are primarily used as intermediate goods rather than for final consumption.

- $\boldsymbol{\beta}$: The input elasticities are more evenly distributed across industries, with values ranging from 37.69% for the Publishing industries to 87.77% for the Wholesale trade industry. Most industries use input proportions lying between 60% and high 90%.

- $\boldsymbol{\eta}$: The firm-level idiosyncratic volatilities vary from 12.06% for General Merchandise Stores up to 53.33% for Air Transportation.

- $\boldsymbol{\mu}$: The mean growth rates of industry-level productivities vary widely across industries. While some industries have a large and positive mean growth rate (e,g., Computer and Electronic Products and

Wholesale trade), some industries exhibit negative mean growth rates (e.g., Other Transportation and Support Activities and Hospitals).

- $\boldsymbol{\sigma}$: The volatilities of industry-level productivity shocks also vary widely across industries, capturing the fact that productivity risks are quite heterogeneous as is the case with mean growth rates.

- $\boldsymbol{\chi}$: We obtain the default recovery parameters from Jankowitsch et al. (2014). Since the authors have a less granular grouping of industries, we match industries in their grouping to industries in our sample and assign the corresponding default recovery parameters from their study.

- $\boldsymbol{\gamma}$: The capital elasticities also exhibit a wide range of values, with most being significantly lower than $\mathbf{1} - \boldsymbol{\beta}$. This confirms that most industries exhibit decreasing returns to scale, since $\beta_I + \gamma_I < 1$. The constraint in (19) is preserved, since $\sum_{I \in [N]} \frac{\theta_I \gamma_I}{1-\beta_I} \approx 0.89 < 1$ for our data.

- $\boldsymbol{\tau}$: the agency cost/private benefits parameters also vary widely across industries ranging from 0.07% for Pipeline Transportation to 18.73% for Apparel and Leather and Allied Products.

5 Quantitative Analysis

We now analyze the calibrated model to obtain quantitative implications for the impact of network characteristics.

5.1 Network Characteristics, Firm Leverage Ratios, and Default Probabilities

We examine how changes in the network concentration parameter, θ, and sparsity parameter, δ, infuence the mean firm leverage ratios and default probabilities across industries. Figures 5, 6 and 7 show the variations of mean firm leverage with respect to the concentration parameter, θ, and sparsity, δ, across different industry samples. Industry size is determined by the parameter vector $\boldsymbol{\theta}$, which are the proportions of gross profits that each industry generates.

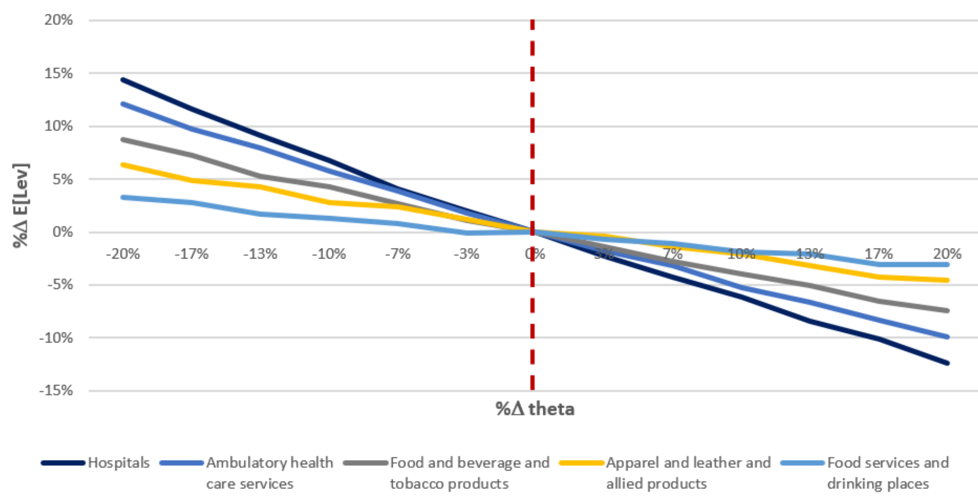

Figure 5: Variation of mean firm leverage with network concentration for large industries

Figure 5 shows that, for the largest industries, a 20% increase (decrease) in the network concentration, θ, results in a 12.4% decrease (14.4% increase) in mean firm leverage. Figure 6 shows that, for the smallest industries, similar increases (decreases) in network concentration lead to changes in mean firm leverage of 254% and -69.2%, respectively. The results are consistent with Proposition 4 showing differential impacts of network concentration on firm leverage levels in large and small industries. The figures demonstrate that the effects of network concentration are quantitatively more significant for firm leverage levels in smaller industries.

Figure 7 shows that, as predicted by Proposition 4, mean firm leverage increases with network sparsity. Mean firm leverage increases by 54.3% for a 20% increase in network sparsity but decreases by 35.2% for a similar percentage decrease in network sparsity.

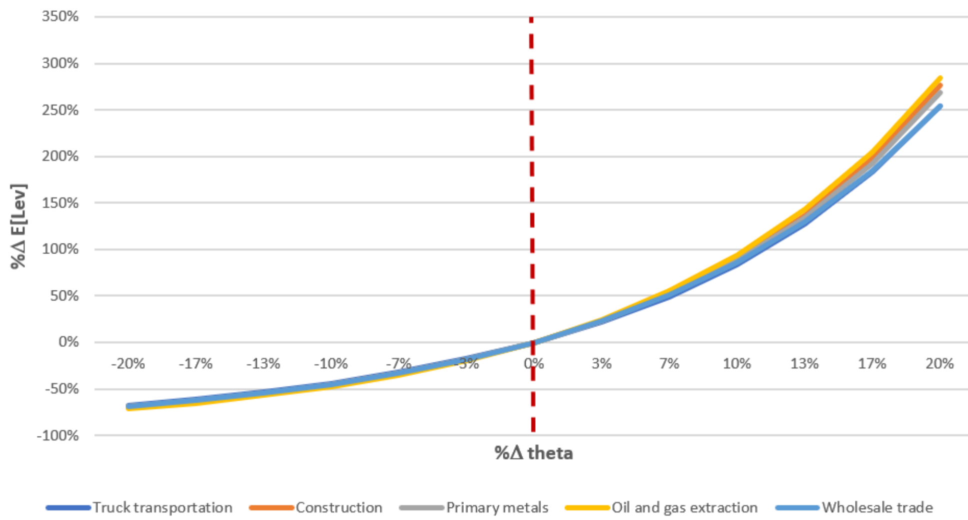

Figure 6: Variation of mean firm leverage with network concentration for small industries

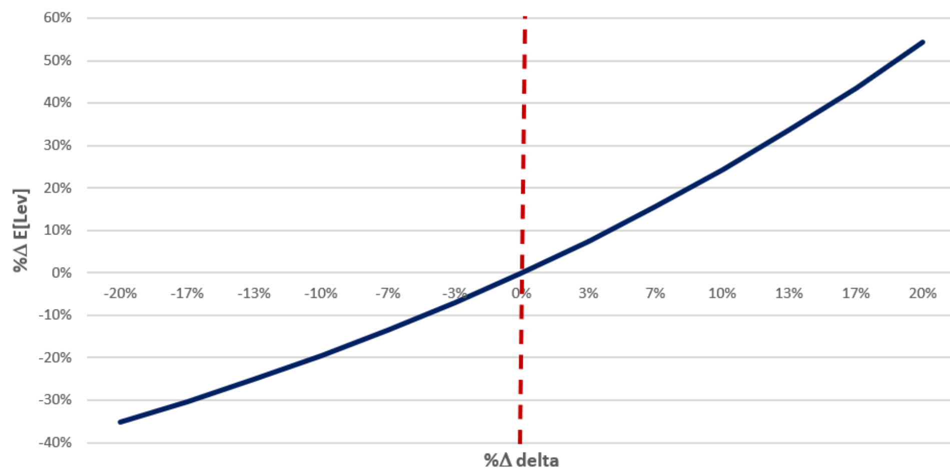

Figure 7: Variation of mean firm leverage with sparsity

Figures 8 and 9 show how firm default probabilities vary with network concentration.

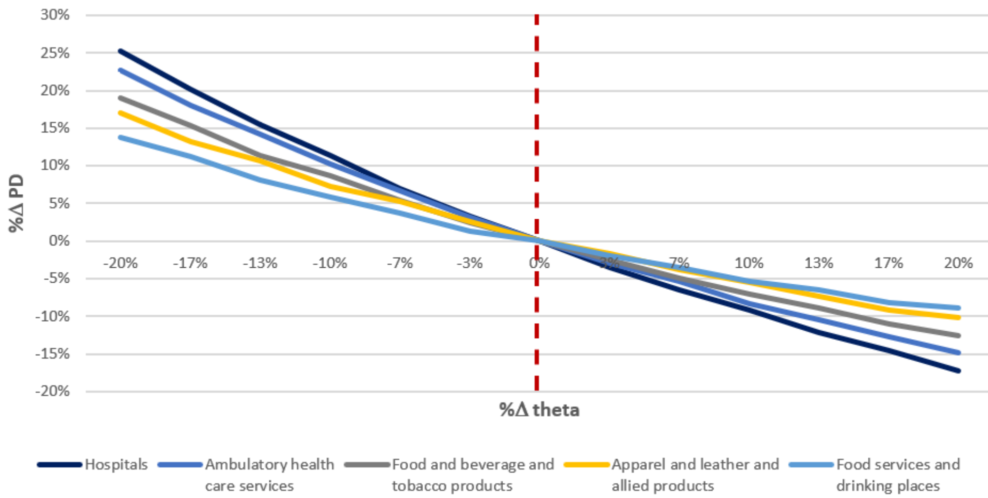

Figure 8: Variation of firm default probabilities with network concentration for large industries

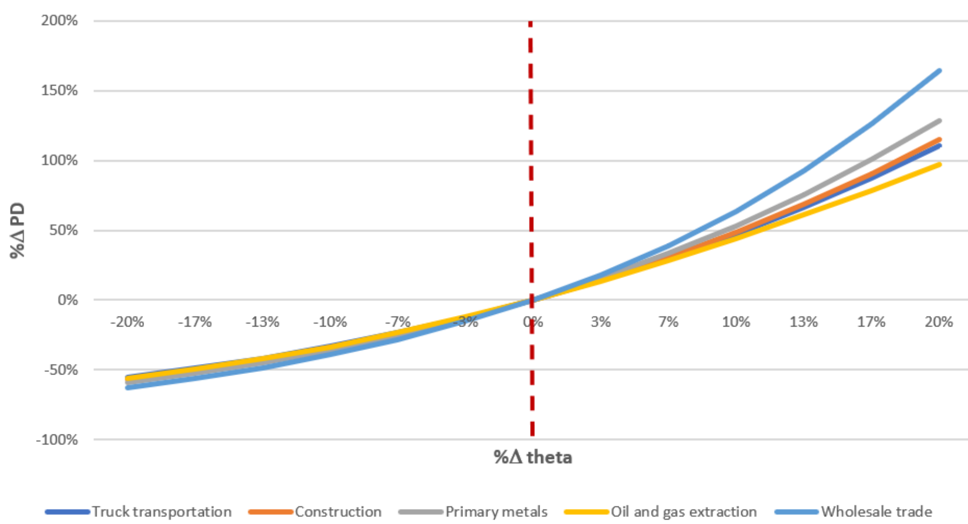

Figure 9: Variation of firm default probabilities with network concentration for small industries

These graphs confirm the results in Proposition 4 that the default probabilities of firms in larger industries decline with network concentration, while those of firms in smaller industries increase with network concentration. A 20% increase (decrease) in θ leads to a -17.3% (25.3%) relative change in the probability of default for the largest industry. The median and smallest industries, on the other hand, exhibit a sharp increase (decrease) in the probabilities of default of 49.7% (30.6%) and 164.8% (62.8%), respectively.

5.2 Contagion Analysis

We now examine the potential for contagion in the network as a consequence of unexpected shocks to key parameters. In this analysis, we assume that firms in all industries choose their equilibrium debt levels described in Theorem 1. We then apply unanticipated shocks to the firm-level idiosyncratic volatility (η_I), the default threshold parameter (ρ_I) and the recovery rate parameter (χ_I) in each industry $I \in [N]$ and examine the effects on firm default risk in the remaining industries given firms' equilibrium debt levels. According to the analytical results outlined in section 4.3.4, a positive shock to η_I and ρ_I increases the probability of default of firms in all industries, while a positive shock to χ_I has the opposite effect. Intuitively, a higher firm-level idiosyncratic volatility increases the likelihood of firm insolvency, which increases the negative impact on firms in other industries. The same is true for the default threshold parameter, with higher values indicating that firms need additional profits to pay off their debt expenses. In the case of the recovery rate parameter, the greater the portion of the production that is retained after financial distress, the lower the probability of the firm defaulting.

Figure 10 shows how large discrete shocks to the top four largest industries impact the rest of the industries in the economy. Each graph represents a box plot of the percentage change in the probability of default for all industries $J \neq I$ in response to a significant unexpected shock in the firm-level idiosyncratic volatility for industry I. We apply three shocks of size $50\%, 100\%$ and 200% for η_I. For example, the first graph on the top left corner shows the distribution of the impact of unexpected shocks to the firm-level idiosyncratic volatility of the largest industry on the probabilities of default of firms in the remaining industries.

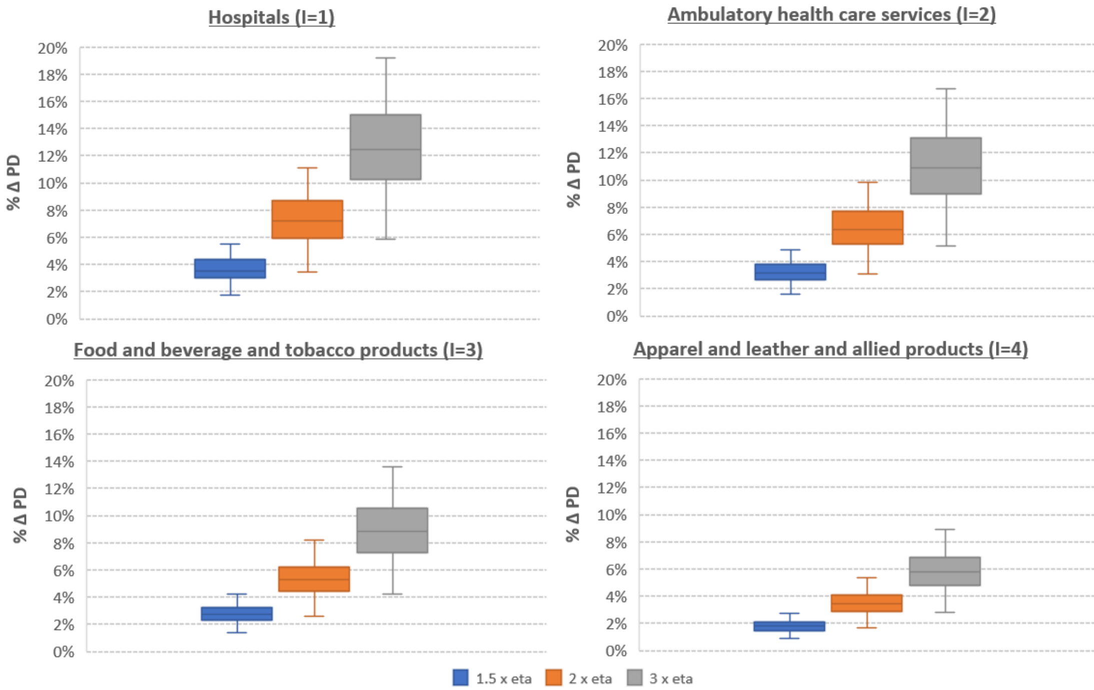

Figure 10: Effects of shocks to firm-level idiosyncratic volatilities in top four industries

As expected, the potential for contagion is higher for the largest industries in the economy, and the effect decreases rapidly with industry size. The median (maximum) relative increases in the probabilities of default are 12.5% (19.2%), 12.9% (16.7%), 8.8% (13.6%) and 5.8% (8.9%), given the largest shock in the firm-level idiosyncratic volatility in each of the top four industries, respectively.

Figure 11 shows the weighted average increase (i.e., weighted by the industry proportions vector, $\boldsymbol{\theta}$) in the overall probability of default in the economy, given the unexpected shock in each industry. Consistent with the previous graphs, industries are sorted from largest to smallest on the horizontal axis.

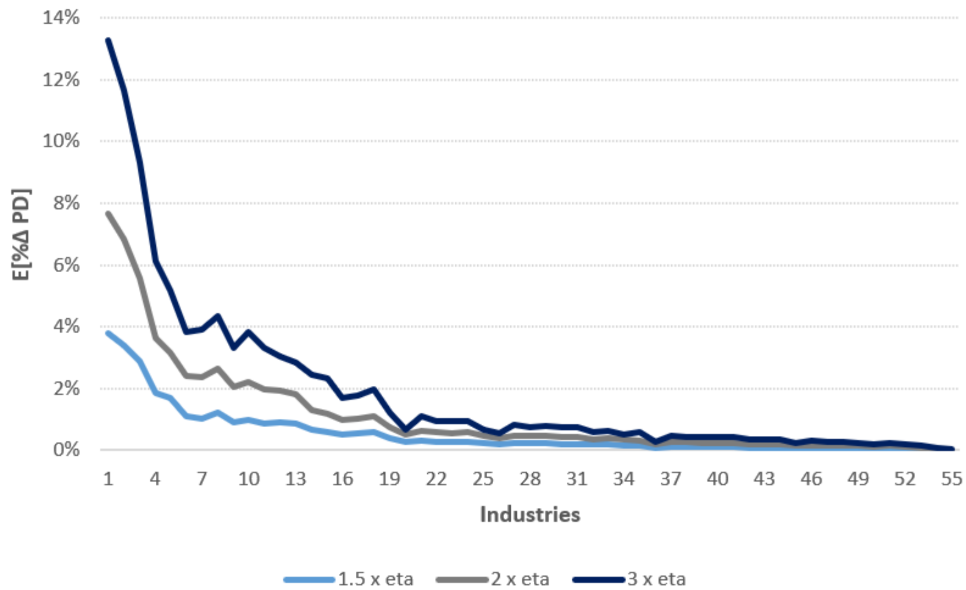

Figure 11: Effects of shocks to firm-level idiosyncratic volatilities on weighted average default probability across industries

Figure 11 shows that unexpected shocks to the firm-level idiosyncratic volatilities of larger industries have quantitatively significant and nonlinear effects on the probabilities of default of firms in other industries. A 200% increase in the firm-level idiosyncratic volatility of the largest industry can increase the average probability of default in the remaining industries by 13.3%. The effect dies out quickly with a reduction in industry size, and it never exceeds 5% starting with the sixth largest industry.

A similar analysis can be performed for the default threshold parameter. Figure 12 plots the impact of different shocks to the four largest industries on the default probabilities of firms in the remaining industries.

The potential for contagion increases exponentially with shock size, but decreases fast with impacted industry size. The median (maximum) relative increases in the the probabilities of default are 99.6% (151%), 80.3% (122.6%), 74% (113.3%) and 48.5% (74.7%), given the largest shock for the idiosyncratic volatility in each of the top four industries, respectively.

Figure 13 shows how the shock in each industry increases the weighted (by concentration) average default probability of firms in the remaining industries .

The contagious effect of an unanticipated shock to the default threshold parameter is even larger than that of the firm-level idiosyncratic volatility. A 200% increase in ρ_I for the largest industry more than doubles the average of the probability of default of firms in the remaining industries. The key difference between η_I and ρ_I is that, while an increase in the former can also lower default probabilities in certain cases (i.e., such as when the probability of default is already relatively high and the greater volatility increases the

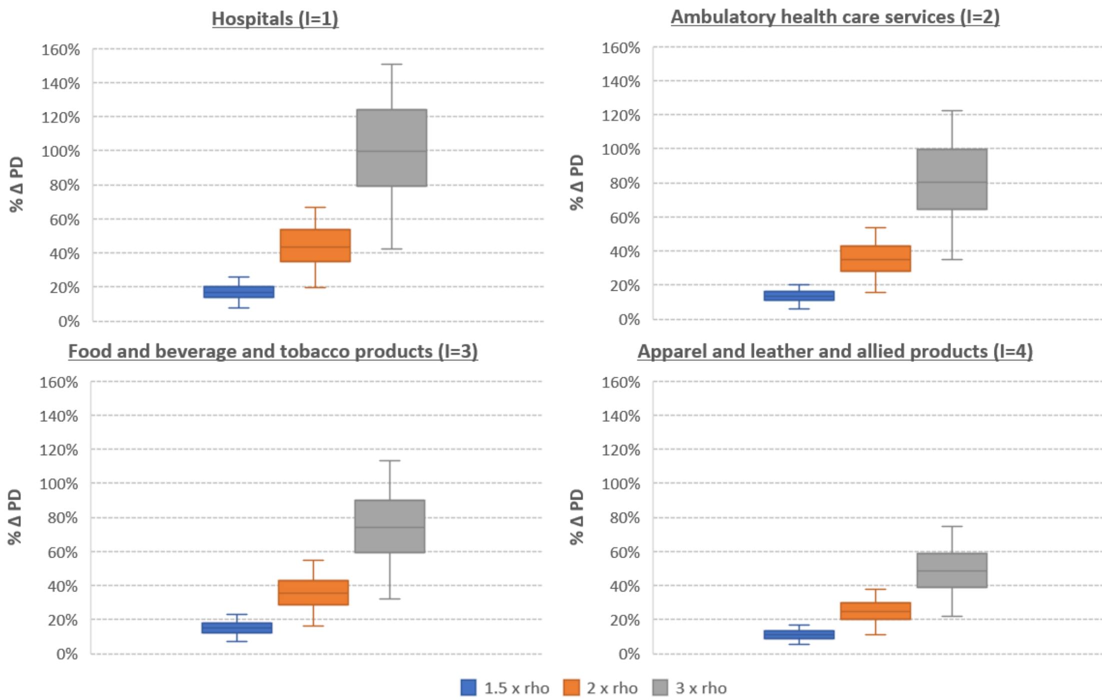

Figure 12: Effects of shocks to the default threshold in top four industries

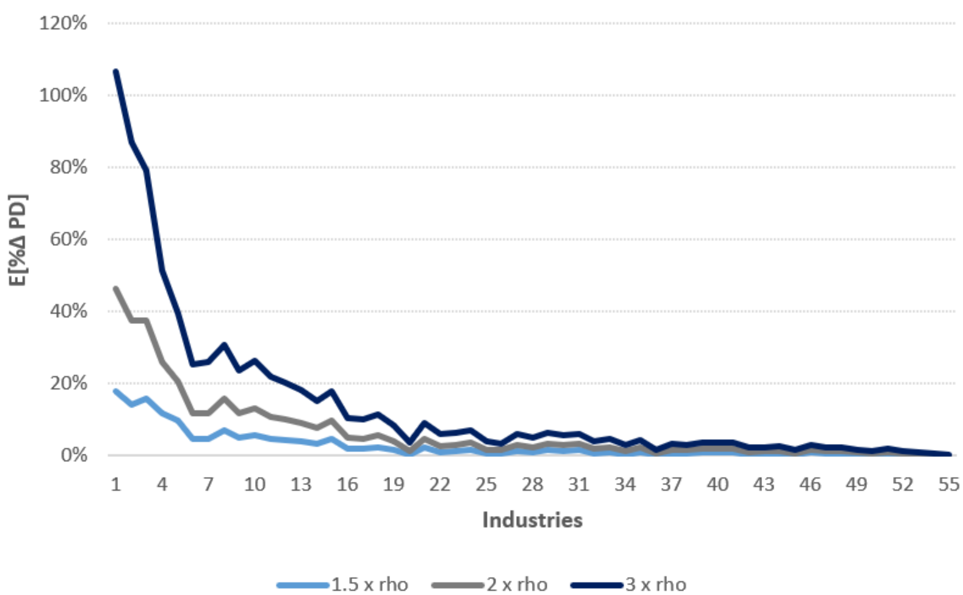

Figure 13: Effects of shocks to default thresholds on weighted average default probability across industries

likelihood of positive scenarios), the latter unambiguously increases firm default probabilities.

For the insolvency recovery parameter, we perform an analysis with different shock levels (i.e., 25%, 50% and 100%). The impacts of these shocks to χ_I are marginal, since the level of the actual probability of default in all industries is relatively low. Intuitively, the recovery rate only has an effect in the scenario in which firms default, but this scenario is relatively unlikely. Hence, shocks to the recovery rates have limited potential for contagion. Figure 14 shows how even the largest shock (i.e., 100%) only generates a modest relative decrease in the average probability of default of firms in the remaining industries of only -0.5%.

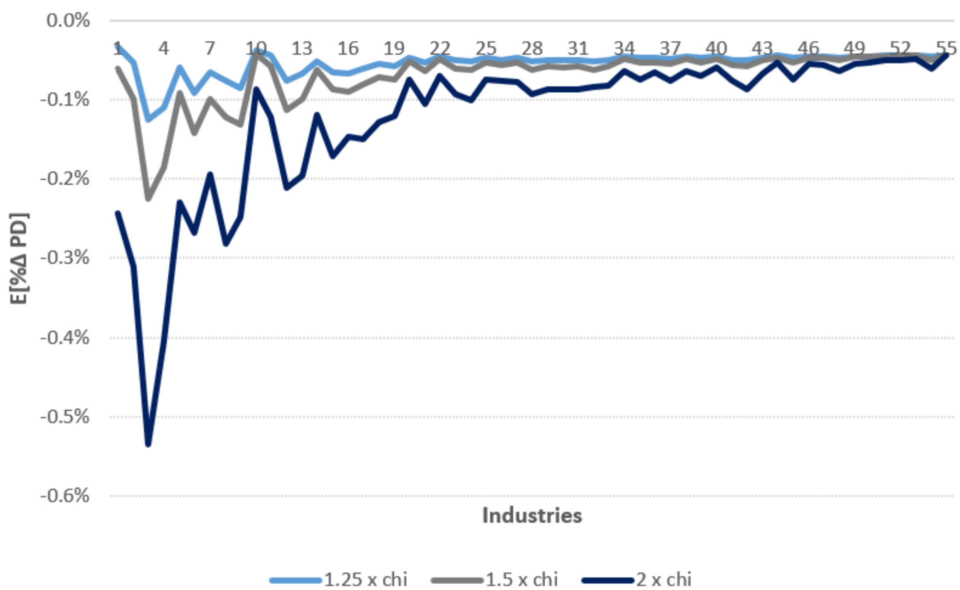

Figure 14: Effects of shocks to insolvency recovery rates on weighted average default probabilities across industries

5.3 Summary

To summarize, we have the following quantitative implications:

- Concentration: An increase in concentration redistributes weight from smaller industries to larger ones. This redistribution leads to larger industries having a higher weight, which decreases their expected leverage and probability of default. On the other hand, smaller industries have a smaller weight, which increases their expected leverage and consequently, their probability of default. The relationship between concentration and leverage/ probability of default shows a higher curvature for both the largest and smallest industries in the economy. A 20% relative increase (decrease) in the concentration parameter leads to relative changes in leverage of $-12.4\%, 66.3\%$ and 254% ($14.4\%, -38.8\%$ and -69.2%) for the largest, median and smallest industries. In the case of the probability of default we have that a 20% relative increase (decrease) in the concentration parameter translates to $-17.3\%, 49.7\%$ and 164.8% ($25.3\%, -30.6\%$ and -62.8%) relative changes in the largest, median and smallest industries, respectively.

- Sparsity: An increase in sparsity leads to higher expected leverage, without affecting the probability of default. This means that, all else equal, industries that depend less (i.e., more sparse) on other industries in the economy, can afford to have higher expected leverage levels, without affecting their risk profile. Intuitively, more sparse networks imply that industries are more self-reliant, which means that they depend less on the risk factors that affect other industries. The elasticity of expected leverage to network sparsity is identical for all industries and exhibits a positive curvature, with a 20% increase in δ leading to a relative change of 54.3% in leverage, vis-a-vis a change of -35.2% for a drop of equal magnitude.

- Contagion and Spillover Effects: As expected, shocks to larger industries have a higher impact on the economy. A 200% unexpected increase in the firm-level idiosyncratic volatility η_I in the largest industry can generate up to a 19.2% relative increase in the probability of default for firms in the most impacted industry. This same shock generates an average relative increase of 13.3% in the probabilities of default of all of the remaining industries in the economy. The potential for contagion is highly nonlinear and dies off swiftly with decreases in industry size. A 200% unexpected increase in the default threshold parameter ρ_I in the largest industry can generate up to a 151% relative increase in the probability of default for the most impacted industry. This shock also causes the average of the probabilities of default in other industries in the economy to rise by 106.7%. The shock in the default threshold parameter is more significant than the industry-level volatility, since the former has an unambiguously positive effect on the probability of default. Finally, shocks to the recovery rate parameter χ_I are marginal, since the current level for the probabilities of default is relatively low and χ_I only affects the profit in the distress scenario.

6 Conclusions

We demonstrate how an economy's production network influences non-financial firms' production and financing decisions in a dynamic structural framework. We analytically characterize the equilibrium of the economy and derive novel implications for the impacts of key characteristics of the production network—the concentration and sparsity—-on firms' capital structures and default risks. Network concentration has sharply contrasting effects on firms' leverage ratios and default probabilities in large and small industries. A higher network concentration lowers leverage ratios and default probabilities for larger industries, but increases them for smaller industries. An increase in network sparsity increases firms' leverage ratios in all industries. We carry out a quantitative analysis of the effects of network characteristics by calibrating the model to match identifying moments in the data. Network concentration and sparsity both have quantitatively substantial effects on firms' average leverage ratios and default probabilities. Shocks to the firm-level idiosyncratic volatility and default threshold of the largest industry have quantitatively significant impacts on firm default probabilities in other industries, thereby suggesting the potential for significant contagion in firm defaults. However, shocks to firm salvage values upon insolvency have only marginal effects. The effects of contagion are highly nonlinear and depend on the size of the industry being impacted as well as the magnitudes of the shocks.

In future research, it would be interesting to incorporate investment and capital accumulation with possible capital adjustment costs, which could generate implications for how production networks influence economic growth via firms' investment and financing decisions. It would also be important to allow for intra-industry firm and product heterogeneity as well as production networks within an industry. We could allow for the production network itself to evolve over time, perhaps stochastically.

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Appendix A

Proof of Proposition 1

We begin by deriving the expressions for firm profits that are determined by firms' production decisions at the end of the period after their idiosyncratic productivity shocks are realized. Let a firm i in industry I choose capital and debt levels that are given by $k_{i,t}, d_{i,t}$, respectively. The first order conditions for the maximization of the profit function in (5) are

$$x^*_{iJ,t+1}(k_{i,t}) = \beta_I \frac{P_{I,t}}{P_{J,t}} w_{IJ} x^*_{i,t+1}(k_{i,t}),$$

and

$$x^{(b)*}_{iJ,t+1}(k_{i,t}) = \beta_I \frac{P_{I,t}}{P_{J,t}} w_{IJ} x^{(b)*}_{i,t+1}(k_{i,t}).$$

In the above, we explicitly indicate the dependence of the output levels on the capital level, $k_{i,t}$, for clarity. Plugging $x^*_{iJ,t+1}(k_{i,t})$ and $x^{(b)*}_{iJ,t+1}(k_{i,t})$ into the profit functions, we get

$$\pi^*_{i,t+1}(k_{i,t}) = (1-\beta_I) P_{I,t} x^*_{i,t+1}(k_{i,t}),$$

$$\pi^{(b)*}_{i,t+1}(k_{i,t}) = (1-\beta_I) P_{I,t} x^{(b)*}_{i,t+1}(k_{i,t}).$$

where

$$x^*_{i,t+1}(k_{i,t}) = Z_{I,t}^{\frac{1}{1-\beta_I}} z_{i,t+1}^{\frac{1}{1-\beta_I}} P_{I,t}^{\frac{\beta_I}{1-\beta_I}} k_{i,t}^{\frac{\gamma_I}{1-\beta_I}} \left[\prod_{J\in[N]} \left(\frac{w_{IJ}}{P_{J,t}}\right)^{w_{IJ}} \right]^{\frac{\beta_I}{1-\beta_I}},$$

and

$$x^{(b)*}_{i,t+1}(k_{i,t}) = \chi_I^{\frac{1}{1-\beta_I}} Z_{I,t}^{\frac{1}{1-\beta_I}} z_{i,t+1}^{\frac{1}{1-\beta_I}} P_{I,t}^{\frac{\beta_I}{1-\beta_I}} k_{i,t}^{\frac{\gamma_I}{1-\beta_I}} \left[\prod_{J\in[N]} \left(\frac{w_{IJ}}{P_{J,t}}\right)^{w_{IJ}} \right]^{\frac{\beta_I}{1-\beta_I}}.$$

Let

$$\tilde{x}^*_{i,t+1}(k_{i,t}, d_{i,t}) =: x^*_{i,t+1}(k_{i,t}) \mathbb{1}_{\pi^*_{i,t+1}(k_{i,t}) \geq \rho_I d_{i,t}} + x^{(b)*}_{i,t+1} \mathbb{1}_{\pi^*_{i,t+1}(k_{i,t}) < \rho_I d_{i,t}}$$

be the final production for an individual firm i in industry I. We have the following:

$$\mathbb{E}_t[\tilde{x}^*_{i,t+1}(k_{i,t}, d_{i,t})] = \left(1 - \left(1 - \chi_I^{\frac{1}{1-\beta_I}}\right) \mathbb{E}_t\left[\mathbb{1}_{\pi^*_{i,t+1}(k_{i,t}) < \rho_I d_{i,t}} z_{i,t+1}^{\frac{1}{1-\beta_I}} \right]\right) \bar{x}^*_{I,t}(k_{i,t}) \tag{48}$$

where

$$\bar{x}^*_{I,t}(k_{i,t}) := Z_{I,t}^{\frac{1}{1-\beta_I}} P_{I,t}^{\frac{\beta_I}{1-\beta_I}} k_{i,t}^{\frac{\gamma_I}{1-\beta_I}} \left[\prod_{J\in[N]} \left(\frac{w_{IJ}}{P_{J,t}}\right)^{w_{IJ}} \right]^{\frac{\beta_I}{1-\beta_I}} \tag{49}$$

We note that, as its subscripts indicate, the function, $\bar{x}^*_{I,t}(k_{i,t})$,is the same for all firms in the industry, and is determined at date t after the industry-wide productivity shocks are realized.

We can rewrite the expectation in (48) as follows.

$$
\begin{aligned}
\mathbb{E}_t\left[\mathbb{1}_{\pi^*_{i,t+1}(k_{i,t})<\rho_I d_{i,t}} z_{i,t+1}^{\frac{1}{1-\beta_I}}\right] &= \mathbb{E}_t\left[\mathbb{1}_{z_{i,t+1}^{\frac{1}{1-\beta_I}} < \frac{\rho_I d_{i,t}}{(1-\beta)P_{I,t}\bar{x}^*_{i,t}(k_{i,t})}} z_{i,t+1}^{\frac{1}{1-\beta_I}}\right] \\
&= \mathbb{E}_t\left[\mathbb{1}_{\varepsilon_{i,t+1}<\psi_{I,t}(k_{i,t},d_{i,t})+\frac{1}{2}\eta_I} e^{-\frac{1}{2}\eta_I^2+\eta_I\varepsilon_{i,t+1}}\right] \\
&= \int_{-\infty}^{\psi_{I,t}(k_{i,t},d_{i,t})+\frac{1}{2}\eta_I} e^{-\frac{1}{2}\eta_I^2+\eta_I z}\phi(z)dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\psi_{I,t}(k_{i,t},d_{i,t})+\frac{1}{2}\eta_I} e^{-\frac{1}{2}(z-\eta_I)^2}dz \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\psi_{I,t}(k_{i,t},d_{i,t})-\frac{1}{2}\eta_I} e^{-\frac{1}{2}z'^2}dz' \\
&= \Phi\left(\psi_{I,t}(k_{i,t},d_{i,t}) - \frac{1}{2}\eta_I\right),
\end{aligned}
$$

where,

$$
\psi_{I,t}(k_{i,t},d_{i,t}) := \frac{1}{\eta_I}\log\left(\frac{\rho_I d_{i,t}}{(1-\beta_I)P_{I,t}\bar{x}^*_{I,t}(k_{i,t})}\right).
$$

The subscript on the function $\psi_{I,t}(.)$ indicates that it is the same for all firms in industry I.

Plugging the above into (48), we conclude that

$$
\mathbb{E}_t[\tilde{x}^*_{i,t+1}(k_{i,t},d_{i,t})] = \xi_{I,t}\bar{x}^*_{I,t}(k_{i,t}),
$$

where

$$
\xi_{I,t}(k_{i,t},d_{i,t}) := 1-(1-\chi_I^{\frac{1}{1-\beta_I}})\Phi\left(\psi_{I,t}(k_{i,t},d_{i,t})-\frac{1}{2}\eta_I\right).
$$

The expected end-of-period payout to the firm's shareholders and debtholders in (7) can be expressed as follows.

$$
\begin{aligned}
\mathbb{E}_t\left[\pi^{*\text{net}}_{i,t+1}(k_{i,t},d_{i,t})\right] &= (1-\tau_I)\mathbb{E}_t\left[\pi^*_{i,t+1}(k_{i,t})1_{\pi^*_{i,t+1}(k_{i,t})\geq\rho_I d_{i,t}} + \pi^{(b)*}_{i,t+1}(k_{i,t})1_{\pi^*_{i,t+1}(k_{i,t})<\rho_I d_{i,t}} + \frac{\tau_I}{1-\tau_I}d_{i,t}1_{\pi^*_{i,t+1}(k_{i,t})\geq\rho_I d_{i,t}} - (1+r_I)\right. \\
&= (1-\tau_I)\mathbb{E}_t\left[(1-\beta_I)P_{I,t}\left(x^*_{i,t+1}(k_{i,t})1_{\pi^*_{i,t+1}(k_{i,t})\geq\rho_I d_{i,t}} + x^{(b)*}_{i,t+1}(k_{i,t})1_{\pi^*_{i,t+1}(k_{i,t})<\rho_I d_{i,t}}\right) + \frac{\tau_I}{1-\tau_I}d_{i,t}1_{\pi^*_{i,t+1}(k_{i,t})\geq\rho}\right. \\
&= (1-\tau_I)\mathbb{E}_t\left[(1-\beta_I)P_{I,t}\tilde{x}^*_{i,t+1}(k_{i,t},d_{i,t}) + \frac{\tau_I}{1-\tau_I}d_{i,t}\left(1-1_{\pi^*_{i,t+1}(k_{i,t})<\rho_I d_{i,t}}\right) - (1+r_I)R_{K,t}k_{i,t}\right] \\
&= (1-\tau_I)\left[(1-\beta_I)P_{I,t}\xi_{I,t}\bar{x}^*_{I,t}(k_{i,t}) + \frac{\tau_I}{1-\tau_I}d_{i,t}\left(1-\mathbb{E}_t\left[1_{\pi^*_{i,t+1}(k_{i,t})<\rho_I d_{i,t}}\right]\right) - (1+r_I)R_{K,t}k_{i,t}\right] \\
&= (1-\tau_I)\left[(1-\beta_I)P_{I,t}\xi_{I,t}\bar{x}^*_{I,t}(k_{i,t}) + \frac{\tau_I}{1-\tau_I}d_{i,t}\left(1-\mathbb{P}_t\left[\pi^*_{i,t+1}(k_{i,t})<\rho_I d_{i,t}\right]\right) - (1+r_I)R_{K,t}k_{i,t}\right] \\
&= (1-\tau_I)\left[(1-\beta_I)P_{I,t}\xi_{I,t}\bar{x}^*_{I,t}(k_{i,t}) + \frac{\tau_I}{1-\tau_I}d_{i,t}(1-\varphi_{I,t}) - (1+r_I)R_{K,t}k_{i,t}\right], \qquad (50)
\end{aligned}
$$

where

$$
\varphi_{I,t}(k_{i,t},d_{i,t}) = \mathbb{P}_t(\pi^*_{i,t+1}(k_{i,t})<\rho_I d_{i,t}) = \Phi\left(\psi_{I,t}(k_{i,t},d_{i,t})+\frac{1}{2}\eta_I\right).
$$

We note from the R.H.S. of the final equality in (50) that the function, $\mathbb{E}_t\left[\pi^{*\text{net}}_{i,t+1}(k_{i,t},d_{i,t})\right]$, is the same for all firms in industry I.

The optimal capital structure choice of firm i in industry I solves the following.

$$(k^*_{i,t}, d^*_{i,t}) = \arg\sup_{k_{l,t},d_{l,t}} \mathbb{E}_t\left[\frac{1}{1+r_I}\pi^{*\text{net}}_{i,t+1}(k_{i,t}, d_{i,t})\right]. \tag{51}$$

As the function, $\mathbb{E}_t\left[\frac{1}{1+r_I}\pi^{*\text{net}}_{i,t+1}(k_{i,t}, d_{i,t})\right]$ is the same for all firms in industry I, each firm in industry I faces exactly the same optimal capital structure choice problem and, therefore, chooses the same capital structure.

Without loss of generality, therefore, we can consider the *representative firm* in industry I that chooses a capital structure, $(k_{I,t}, d_{I,t})$. For future reference, we collect together the expressions derived above as functions of the capital and debt levels chosen by the representative firm for convenience.

$$\bar{x}^*_{I,t}(k_{I,t}) := Z_{I,t}^{\frac{1}{1-\beta_I}} P_{I,t}^{\frac{\beta_I}{1-\beta_I}} k_{I,t}^{\frac{\gamma_I}{1-\beta_I}} \left[\prod_{J\in[N]}\left(\frac{w_{IJ}}{P_{J,t}}\right)^{w_{IJ}}\right]^{\frac{\beta_I}{1-\beta_I}} \tag{52}$$

$$\psi_{I,t}(k_{I,t}, d_{I,t}) := \frac{1}{\eta_I}\log\left(\frac{\rho_I d_{i,t}}{(1-\beta_I)P_{I,t}\bar{x}^*_{I,t}(k_{I,t})}\right), \tag{53}$$

$$\varphi_{I,t}(k_{I,t}, d_{I,t}) = \Phi\left(\psi_{I,t}(k_{I,t}, d_{I,t}) + \frac{1}{2}\eta_I\right). \tag{54}$$

$$\xi_{I,t}(k_{I,t}, d_{I,t}) := 1 - (1-\chi_I^{\frac{1}{1-\beta_I}})\Phi\left(\psi_{I,t}(k_{I,t}, d_{I,t}) - \frac{1}{2}\eta_I\right), \tag{55}$$

$$\begin{aligned}(k^*_{I,t}, d^*_{I,t}) &= \arg\sup_{k_{I,t},d_{I,t}} \mathbb{E}_t\pi^{\text{net}*}_{I,t+1}(k_{I,t}, d_{I,t}) \\ &= \arg\sup \frac{1-\tau_I}{1+r_I}\left[(1-\beta_I)P_{I,t}\xi_{I,t}\bar{x}^*_{I,t}(k_{I,t}) + \frac{\tau_I}{1-\tau_I}d_{I,t}(1-\varphi_{I,t}(k_{i,t}, d_{i,t})) - (1+r_I)R_{K,t}k_{I,t}\right]\end{aligned} \tag{56}$$

$$\varphi_{I,t}(k_{I,t}, d_{I,t}) = \mathbb{P}_t(\pi^*_{I,t+1}(k_{I,t}) < \rho_I d_{I,t}) = \Phi\left(\psi_{I,t}(k_{I,t}, d_{I,t}) + \frac{1}{2}\eta_I\right). \tag{57}$$

Proof of Proposition 2

As firm-specific productivity shocks are i.i.d., it follows by the law of large numbers that the total production of firms in industry I is given by the mass of firms, M_I, times the expected production of the representative firm.

$$X^*_{I,t} := M_I\mathbb{E}_t[\tilde{x}^*_{I,t+1}] = M_I\xi_{I,t}\bar{x}^*_{I,t}.$$

In the above, we have suppressed the arguments of the functions—the capital and debt levels of the representative firm—to ease the notation. This leads to the following expression for the total profit for industry I:

$$\Pi^*_{I,t} := (1-\beta_I)P_{I,t}X^*_{I,t}. \tag{58}$$

Since $x^*_{iJ,t+1}$ and $x^{(b)*}_{iJ,t+1}$ are linear in $x^*_{i,t+1}$ and $x^{(b)*}_{i,t+1}$, we can also express the total inputs bought by industry I from industry J as

$$X^*_{IJ,t} := M_I\beta_I\frac{P_{I,t}}{P_{J,t}}w_{IJ}\xi_{I,t}\bar{x}^*_{I,t} = \beta_I w_{IJ}\frac{P_{I,t}}{P_{J,t}}X^*_{I,t}$$

Switching to the consumer's problem, the Lagrangian function is

$$\mathcal{L}_t = \max_{\{c_{I,t}\}_{I\in[N]}} \prod_{I\in[N]} c_{I,t}^{\alpha_I} + \lambda_t \left(y_t - \sum_{I\in[N]} P_{I,t} c_{I,t} \right).$$

The first order condition with respect to $c_{J,t}$ leads to the following result:

$$\frac{\partial \mathcal{L}_t}{\partial c_{J,t}} = 0 \Rightarrow c_{J,t} = \frac{\alpha_J \prod_{I\in[N]} c_{I,t}^{\alpha_I}}{\lambda_t P_{J,t}}.$$

Take $c_{J,t}$ and $c_{K,t}$ for all $J, K \in [N]$. The previous expression implies the following:

$$P_{J,t} c_{J,t} = P_{K,t} c_{K,t} \frac{\alpha_J}{\alpha_K}.$$

Replacing this condition in the budget constraint and noticing that $\sum_{j\in[n]} \alpha_j = 1$, we obtain the optimal consumption of good I:

$$P_{I,t} c_{I,t}^* = \alpha_I y_t.$$

The total consumption of good I in the economy is given by the following expression:

$$P_{I,t} C_{I,t}^* = \alpha_I \sum_{J\in[N]} \Pi_{J,t}^* = \alpha_I \sum_{J\in[N]} (1-\beta_J) P_{J,t} X_{J,t}^*$$

Replacing the optimal values $C_{I,t}^*$ and $X_{IJ,t}^*$ in the market-clearing condition in (10), we obtain the following:

$$P_{I,t} X_{I,t}^* = \alpha_I \sum_{J\in[N]} (1-\beta_J) P_{J,t} X_{J,t}^* + \sum_{J\in[N]} \beta_J w_{JI} P_{J,t} X_{J,t}^*$$

We proceed to multiply by $(1-\beta_I)$ to obtain the following expression:

$$\frac{\beta_I}{1-\beta_I} \Pi_{I,t}^* = \alpha_I \beta_I \sum_{J\in[N]} \Pi_{J,t}^* + \beta_I \sum_{J\in[N]} \frac{\beta_J}{1-\beta_J} w_{JI} \Pi_{J,t}^*$$

This can be expressed in vector form as:

$$\begin{aligned} \mathrm{diag}(\boldsymbol{\beta})[\boldsymbol{I} - \mathrm{diag}(\boldsymbol{\beta})]^{-1} \boldsymbol{\Pi}_t^* =& \mathrm{diag}(\boldsymbol{\beta}) \boldsymbol{\alpha} \sum_{J\in[N]} \Pi_{J,t}^* + \mathrm{diag}(\boldsymbol{\beta}) \boldsymbol{W}' \mathrm{diag}(\boldsymbol{\beta}) [\boldsymbol{I} - \mathrm{diag}(\boldsymbol{\beta})]^{-1} \boldsymbol{\Pi}_t^* \\ =& [\boldsymbol{I} - \mathrm{diag}(\boldsymbol{\beta}) \boldsymbol{W}']^{-1} \mathrm{diag}(\boldsymbol{\beta}) \boldsymbol{\alpha} \sum_{J\in[N]} \Pi_{J,t}^* \\ =& \mathrm{diag}(\boldsymbol{\beta}) [\boldsymbol{I} - \boldsymbol{W}' \mathrm{diag}(\boldsymbol{\beta})]^{-1} \boldsymbol{\alpha} \sum_{J\in[N]} \Pi_{J,t}^*. \end{aligned}$$

This leads to the final result

$$\boldsymbol{\Pi}_t^* = (\boldsymbol{I} - \mathrm{diag}(\boldsymbol{\beta}))[\boldsymbol{I} - \boldsymbol{W}' \mathrm{diag}(\boldsymbol{\beta})]^{-1} \boldsymbol{\alpha} \sum_{J\in[N]} \Pi_{J,t}^* = \boldsymbol{\theta} \sum_{J\in[N]} \Pi_{J,t}^*,$$

where $\boldsymbol{\Pi}_t^* := (\Pi_{1,t}^*, ..., \Pi_{N,t}^*)'$, $\boldsymbol{\beta} := (\beta_1, ..., \beta_N)'$, $\boldsymbol{\alpha} := (\alpha_1, ..., \alpha_N)'$, $\boldsymbol{W}$ is the matrix with IJ-th entry $[w_{IJ}]$

and $\boldsymbol{\theta} := (\theta_1, ..., \theta_N)'$. We can show that $\sum_{I\in[N]} \theta_I = 1$ as follows

$$\begin{aligned}\sum_{I\in[N]} \theta_I = \mathbf{1}'\boldsymbol{\theta} =& \mathbf{1}'(\boldsymbol{I} - \text{diag}(\boldsymbol{\beta}))[\boldsymbol{I} - \boldsymbol{W}'\text{diag}(\boldsymbol{\beta})]^{-1}\boldsymbol{\alpha} \\ =& \mathbf{1}'(\boldsymbol{I} - \text{diag}(\boldsymbol{\beta}))\sum_{k=0}^{\infty}(\boldsymbol{W}'\text{diag}(\boldsymbol{\beta}))^k\boldsymbol{\alpha} \\ =& (\mathbf{1} - \boldsymbol{\beta})'\sum_{k=0}^{\infty}(\boldsymbol{W}'\text{diag}(\boldsymbol{\beta}))^k\boldsymbol{\alpha} \\ =& \mathbf{1}'\boldsymbol{\alpha} = 1\end{aligned}$$

Proof of Proposition 3

We start by multiplying both sides of (52) by $P_{I,t}$:

$$P_{I,t}\bar{x}^*_{I,t} = P_{I,t}^{\frac{1}{1-\beta_I}} Z_{I,t}^{\frac{1}{1-\beta_I}} k_{I,t}^{\frac{\gamma_I}{1-\beta_I}} \left[\prod_{J\in[N]}\left(\frac{w_{IJ}}{P_{J,t}}\right)^{w_{IJ}}\right]^{\frac{\beta_I}{1-\beta_I}}.$$

We apply the natural logarithm function to both sides of the previous equation and multiply by $1-\beta_I$

$$\begin{aligned}(1-\beta_I)\log(P_{I,t}\bar{x}^*_{I,t}) = \log(P_{I,t}) + (1-\beta_I)(\mu_I t + \sigma_I\varepsilon_t) + \\ \gamma_I\log(k_{I,t}) + \beta_I\left(\sum_{J\in[N]} w_{IJ}\log(w_{IJ}) - \sum_{J\in[N]} w_{IJ}\log(P_{J,t})\right).\end{aligned}$$

This expression can be represented in vector form as follows:

$$\begin{aligned}(\boldsymbol{I} - \text{diag}(\boldsymbol{\beta}))\log(\boldsymbol{P}_t \odot \bar{\boldsymbol{x}}^*_t) = \log(\boldsymbol{P}_t) + (\boldsymbol{I} - \text{diag}(\boldsymbol{\beta}))(\boldsymbol{\mu}t + \boldsymbol{\sigma}\varepsilon_t) + \\ \text{diag}(\boldsymbol{\gamma})\log(\boldsymbol{k}_t) + \text{diag}(\boldsymbol{\beta})(\boldsymbol{\mathcal{W}}_s - \boldsymbol{W}\log(\boldsymbol{P}_t)),\end{aligned}$$

with $\bar{\boldsymbol{x}}^*_t := (\bar{x}^*_{1,t}, ..., \bar{x}^*_{N,t})'$, $\boldsymbol{\mu} := (\mu_1, ..., \mu_N)'$,$\boldsymbol{\sigma} := (\sigma_1, ..., \sigma_N)'$, $\boldsymbol{k}_t := (k_{1,t}, ..., k_{N,t})'$,$\boldsymbol{\gamma}_t := (\gamma_1, ..., \gamma_N)'$ and $\boldsymbol{\mathcal{W}}_s$ is a vector with I-th entry equal to $\sum_{J\in[N]} w_{IJ}\log(w_{IJ})$. Solving for prices, we obtain

$$\begin{aligned}\log(\boldsymbol{P}_t) = [\boldsymbol{I} - \text{diag}(\boldsymbol{\beta})\boldsymbol{W}]^{-1}(\boldsymbol{I} - \text{diag}(\boldsymbol{\beta}))\Big\{\log(\boldsymbol{P}_t \odot \bar{\boldsymbol{x}}^*_t) - \\ (\boldsymbol{\mu}t + \boldsymbol{\sigma}\varepsilon_t) - [\boldsymbol{I} - \text{diag}(\boldsymbol{\beta})]^{-1}\text{diag}(\boldsymbol{\gamma})\log(\boldsymbol{k}_t) - [\boldsymbol{I} - \text{diag}(\boldsymbol{\beta})]^{-1}\text{diag}(\boldsymbol{\beta})\boldsymbol{\mathcal{W}_s}\Big\}.\end{aligned}$$

We proceed to pre-multiply the price vector equation by α' and notice that $\boldsymbol{\alpha}'[\boldsymbol{I} - \text{diag}(\boldsymbol{\beta})\boldsymbol{W}]^{-1}(\boldsymbol{I} - \text{diag}(\boldsymbol{\beta})) = \boldsymbol{\theta}'$ and $\boldsymbol{\theta}'\mathbf{1} = 1$, which leads to the following:

$$\begin{aligned}\boldsymbol{\alpha}'\log(\boldsymbol{P}_t) = \boldsymbol{\theta}'\log(\boldsymbol{P}_t \odot \bar{\boldsymbol{x}}^*_t) - (\boldsymbol{\theta}'\boldsymbol{\mu}t + \boldsymbol{\theta}'\boldsymbol{\sigma}\varepsilon_t) - \\ \boldsymbol{\theta}'[\boldsymbol{I} - \text{diag}(\boldsymbol{\beta})]^{-1}\text{diag}(\boldsymbol{\gamma})\log(\boldsymbol{k}_t) - \boldsymbol{\theta}'[\boldsymbol{I} - \text{diag}(\boldsymbol{\beta})]^{-1}\text{diag}(\boldsymbol{\beta})\boldsymbol{\mathcal{W}_s}.\end{aligned}$$

Using the price normalization in equation 13 we have

$$\begin{aligned}\boldsymbol{\theta}'\log(\boldsymbol{P}_t \odot \bar{\boldsymbol{x}}^*_t) = \boldsymbol{\theta}'[\boldsymbol{I} - \text{diag}(\boldsymbol{\beta})]^{-1}\text{diag}(\boldsymbol{\beta})\boldsymbol{\mathcal{N}} + \log(\overline{P}_t) + \boldsymbol{\theta}'\boldsymbol{\mu}t + \boldsymbol{\theta}'\boldsymbol{\sigma}\varepsilon_t + \\ \boldsymbol{\theta}'[\boldsymbol{I} - \text{diag}(\boldsymbol{\beta})]^{-1}\text{diag}(\boldsymbol{\gamma})\log(\boldsymbol{k}_t).\end{aligned}$$

We sum $\boldsymbol{\theta}' \log(\mathbf{1} - \boldsymbol{\beta}) + \boldsymbol{\theta}' \log(\boldsymbol{\xi}_{t+1}) + \boldsymbol{\theta}' \log(\boldsymbol{M})$ at both sides of the previous equation to obtain the final result

$$\sum_{I\in[N]} \Pi^*_{I,t} = \exp\Big\{\boldsymbol{\theta}'[\boldsymbol{I} - \mathrm{diag}(\boldsymbol{\beta})]^{-1}\mathrm{diag}(\boldsymbol{\beta})\boldsymbol{\mathcal{W}}_s - \boldsymbol{\theta}' \log(\boldsymbol{\theta}) + \boldsymbol{\theta}' \log(\mathbf{1} - \boldsymbol{\beta}) + \boldsymbol{\theta}' \log(\boldsymbol{M}) + \log(\overline{P}_t) + \theta' \boldsymbol{\mu} t + \boldsymbol{\theta}'\boldsymbol{\sigma}\varepsilon_t + \boldsymbol{\theta}' \log(\boldsymbol{\xi}_t) + \boldsymbol{\theta}'[\boldsymbol{I} - \mathrm{diag}(\boldsymbol{\beta})]^{-1}\mathrm{diag}(\boldsymbol{\gamma}) \log(\boldsymbol{k}_t)\Big\}.$$

Proof of Theorem 1

We split the proof into several steps for clarity.
Step 1:
Using the expected discounted net profits function in (56), and the total profit for industry I in (58) and (11), we have:

$$\begin{aligned}
\mathbb{E}_t\left[\frac{1}{1+r_I}\pi^{*\text{net}}_{I,t+1}(k_{I,t}, d_{I,t})\right] &= \frac{1-\tau_I}{1+r_I}\left[(1-\beta_I)P_{I,t}\xi_{I,t}\bar{x}^*_{I,t}(k_{I,t}) + \frac{\tau_I}{1-\tau_I}d_{I,t}(1-\varphi_{I,t}) - (1+r_I)R_{K,t}k_{I,t}\right] \\
&= \frac{1}{M_I}\frac{1-\tau_I}{1+r_I}\left[\Pi^*_{I,t} + \frac{\tau_I}{1-\tau_I}D_{I,t}(1-\varphi_{I,t}) - (1+r_I)R_{K,t}K_{I,t}\right] \\
&= \frac{1}{M_I}\frac{1-\tau_I}{1+r_I}\left[\theta_I \sum_{J\in[N]} \Pi^*_{J,t} + \frac{\tau_I}{1-\tau_I}D_{I,t}(1-\varphi_{I,t}) - (1+r_I)R_{K,t}K_{I,t}\right].
\end{aligned}$$

In the above, we simplify subsequent expressions by making the change of variables, $K_{I,t} := M_I k_{I,t}$ and $D_{I,t} := M_I d_{I,t}$. Ignoring the constant terms, which do not affect the optimal capital structure decision, the optimization problem (51) reduces to

$$\sup_{k_{I,t},d_{I,t}} \mathbb{E}_t\left[\frac{1}{1+r_f}\pi^{\text{net}}_{i,t+1}(k_{I,t}, d_{I,t})\right] = \sup_{K_{I,t},D_{I,t}}\left\{\theta_I \sum_{J\in[N]} \Pi^*_{J,t} + \frac{\tau_I}{1-\tau_I}D_{I,t}(1-\varphi_{I,t}) - (1+r_I)R_{K,t}K_{I,t}\right\}.$$

Let $\hat{K}_{I,t} := \log(K_{I,t})$ and $\hat{D}_{I,t} := \log\left(D_{I,t}\right)$. The optimization problem can then be re-expressed as follows.

$$\sup_{\hat{K}_{I,t},\hat{D}_{I,t}}\left\{\theta_I \sum_{J\in[N]} \Pi^*_{J,t} + \frac{\tau_I}{1-\tau_I}e^{\hat{D}_{I,t}}(1-\varphi_{I,t}) - (1+r_I)R_{K,t}e^{\hat{K}_{I,t}}\right\},$$

where

$$\sum_{J\in[N]} \Pi^*_{J,t} = \exp\Big\{\delta - \boldsymbol{\theta}' \log(\boldsymbol{\theta}) + \boldsymbol{\theta}' \log(\mathbf{1} - \boldsymbol{\beta}) + \log(\overline{P}_t) + \boldsymbol{\theta}'\boldsymbol{\mu} t + \boldsymbol{\theta}'\boldsymbol{\sigma}\varepsilon_t + \boldsymbol{\theta}' \log(\boldsymbol{\xi}_t) + \boldsymbol{\theta}'[\boldsymbol{I} - \mathrm{diag}(\boldsymbol{\beta})]^{-1}\mathrm{diag}(\boldsymbol{\gamma})\hat{\boldsymbol{K}}_t\Big\},$$

$$\psi_{I,t}(\hat{K}_{I,t}, \hat{D}_{I,t}) = \frac{1}{\eta_I}\left[\hat{D}_{I,t} + \log\left(\frac{\rho_I}{\theta_I}\right) - \log\left(\sum_{J\in[N]} \Pi^*_{J,t}\right) + \log(\xi_{I,t})\right]$$

Step 2:
The first order conditions with respect to $\hat{K}_{I,t}$ and $\hat{D}_{I,t}$ can be expressed as

$$e^{\hat{K}_{I,t}} = \frac{1}{r_K(1+r_I)}\theta_I \sum_{J\in[N]} \Pi^*_{J,t}\left(\frac{\partial\boldsymbol{\theta}' \log(\boldsymbol{\xi}_t)}{\partial\hat{K}_{I,t}} + \frac{\gamma_I}{1-\beta_I}\theta_I\right) - \frac{1}{R_{K,t}(1+r_I)}\frac{\tau_I}{1-\tau_I}e^{\hat{D}_{I,t}}\frac{\partial\varphi_{I,t}}{\partial\hat{K}_{I,t}} \tag{59}$$

$$e^{\hat{D}_{I,t}} = -\frac{\frac{(1-\tau_I)}{\tau_I}\theta_I \sum_{J\in[N]} \Pi^*_{J,t+1} \frac{\partial \boldsymbol{\theta}' \log(\boldsymbol{\xi}_t)}{\partial \hat{D}_{I,t}}}{1 - \varphi_{I,t} - \frac{\partial \varphi_{I,t}}{\partial \hat{D}_{I,t}}}. \tag{60}$$

The partial derivative term, $\frac{\partial \boldsymbol{\theta}' \log(\boldsymbol{\xi}_t)}{\partial \hat{K}_{I,t}}$, can be obtained from (16) as follows:

$$\frac{\partial \log(\xi_{J,t})}{\partial \hat{K}_{I,t}} = \frac{(1-\chi_J)\phi\left(\psi_{J,t}(\hat{K}_{J,t}, \hat{D}_{J,t}) - \frac{1}{2}\eta_J\right)}{\eta_J \xi_{J,t}} \left(\frac{\partial \boldsymbol{\theta}' \log(\boldsymbol{\xi}_t)}{\partial \hat{K}_{I,t}} - \frac{\partial \log(\xi_{J,t})}{\partial \hat{K}_{I,t}} + \frac{\gamma_I}{1-\beta_I}\theta_I \right).$$

This leads to:

$$\frac{\partial \log(\xi_{J,t})}{\partial \hat{K}_{I,t}} = \zeta_{J,t} \left(\frac{\partial \boldsymbol{\theta}' \log(\boldsymbol{\xi}_t)}{\partial \hat{K}_{I,t}} + \frac{\gamma_I}{1-\beta_I}\theta_I \right), \tag{61}$$

where

$$\zeta_{J,t} := \left(1 + \frac{\eta_J \xi_{J,t}}{(1-\chi_J^{\frac{1}{1-\beta_J}})\phi\left(\psi_{J,t}(\hat{K}_{J,t}, \hat{D}_{J,t}) - \frac{1}{2}\eta_J\right)} \right)^{-1}.$$

Next, we observe that

$$\frac{\partial \boldsymbol{\theta}' \log(\boldsymbol{\xi}_t)}{\partial \hat{K}_{I,t}} = \boldsymbol{\theta}'\boldsymbol{\zeta}_t \frac{\partial \boldsymbol{\theta}' \log(\boldsymbol{\xi}_t)}{\partial \hat{K}_{I,t}} + \boldsymbol{\theta}'\boldsymbol{\zeta}_t \frac{\gamma_I}{1-\beta_I}\theta_I,$$

which implies that

$$\frac{\partial \boldsymbol{\theta}' \log(\boldsymbol{\xi}_t)}{\partial \hat{K}_{I,t}} = \frac{\boldsymbol{\theta}'\boldsymbol{\zeta}_t \frac{\gamma_I}{1-\beta_I}\theta_I}{1-\boldsymbol{\theta}'\boldsymbol{\zeta}_t}. \tag{62}$$

Similarly, we obtain

$$\frac{\partial \log(\xi_{J,t})}{\partial \hat{D}_{I,t}} = \zeta_{J,t} \left(\frac{\partial \boldsymbol{\theta}' \log(\boldsymbol{\xi}_t)}{\partial \hat{D}_{I,t}} - \mathbb{1}_{\{I=J\}} \right), \tag{63}$$

which implies that

$$\frac{\partial \boldsymbol{\theta}' \log(\boldsymbol{\xi}_t)}{\partial \hat{D}_{I,t}} = -\frac{\theta_I \zeta_{I,t}}{1-\boldsymbol{\theta}'\boldsymbol{\zeta}_t}. \tag{64}$$

Plugging (62) and (64) into (61) and (63), respectively, and setting J to I, we obtain the following:

$$\frac{\partial \log(\xi_{I,t})}{\partial \hat{K}_{I,t}} = \frac{\gamma_I}{1-\beta_I}\theta_I \zeta_{I,t} \left(\frac{\boldsymbol{\theta}'\boldsymbol{\zeta}_t}{1-\boldsymbol{\theta}'\boldsymbol{\zeta}_t} + 1 \right), \tag{65}$$

and

$$\frac{\partial \log(\xi_{I,t})}{\partial \hat{D}_{I,t}} = -\zeta_{I,t} \left(\frac{\theta_I \zeta_{I,t}}{1-\boldsymbol{\theta}'\boldsymbol{\zeta}_t} + 1 \right). \tag{66}$$

The remaining partial derivative terms $\frac{\partial \varphi_{I,t}}{\partial \hat{K}_{I,t}}$ and $\frac{\partial \varphi_{I,t}}{\partial \hat{D}_{I,t}}$ can be obtained as follows:

$$\frac{\partial \varphi_{I,t}}{\partial \hat{K}_{I,t}} = \frac{\phi\left(\Phi^{-1}(\varphi_{I,t})\right)}{\eta_I} \left(-\frac{\partial \boldsymbol{\theta}' \log(\boldsymbol{\xi}_t)}{\partial \hat{K}_{I,t}} + \frac{\partial \log(\xi_{I,t})}{\partial \hat{K}_{I,t}} - \frac{\gamma_I}{1-\beta_I}\theta_I \right) = -\nu_{I,t}\frac{\gamma_I}{1-\beta_I}\theta_I \left(\frac{\boldsymbol{\theta}'\boldsymbol{\zeta}_t}{1-\boldsymbol{\theta}'\boldsymbol{\zeta}_t} + 1 \right),$$

$$\frac{\partial \varphi_{I,t}}{\partial \hat{D}_{I,t}} = \frac{\phi\left(\Phi^{-1}(\varphi_{I,t})\right)}{\eta_I} \left(-\frac{\partial \boldsymbol{\theta}' \log(\boldsymbol{\xi}_t)}{\partial \hat{D}_{I,t}} + \frac{\partial \log(\xi_{I,t})}{\partial \hat{D}_{I,t}} + 1 \right) = \nu_{I,t} \left(\frac{\theta_I \zeta_{I,t}}{1-\boldsymbol{\theta}'\boldsymbol{\zeta}_t} + 1 \right),$$

where

$$\nu_{I,t} := \frac{\phi\left(\Phi^{-1}(\varphi_{I,t})\right)(1-\zeta_{I,t})}{\eta_I}.$$

Step 3:

The above analysis leads to the following results for the equilibrium total capital and debt levels of the industry expressed in terms of the vector of default probabilities,

$$\boldsymbol{\varphi_t} \equiv (\varphi_{1,t}, ..., \varphi_{N,t}) \tag{67}$$

$$K^*_{I,t} = \frac{\theta_I^2 \gamma_I}{(1-\beta_I)R_{K,t}(1+r_I)} \left(\frac{1+\Psi_I(\varphi_{I,t})\,\Omega_I(\varphi_{I,t})}{1-\boldsymbol{\theta}'\Lambda(\boldsymbol{\varphi_t})} \right) \sum_{J\in[N]} \Pi^*_{J,t}, \tag{68}$$

$$D^*_{I,t} = \frac{(1-\tau_I)\theta_I}{\tau_I} \Omega_I(\varphi_{I,t}) \sum_{J\in[N]} \Pi^*_{J,t}, \tag{69}$$

where

$$\Omega_I(\varphi_{I,t}) := \left[\frac{(1-\varphi_{I,t}-\Psi_I(\varphi_{I,t}))(1-\boldsymbol{\theta}'\boldsymbol{\Lambda}(\boldsymbol{\varphi_t}))}{\theta_I \Lambda_I(\varphi_{I,t})} - \Psi_I(\varphi_{I,t}) \right]^{-1},$$

$$\Xi_I(\varphi_{I,t}) := 1 - (1-\chi_I^{\frac{1}{1-\beta_I}})\Phi(\Phi^{-1}(\varphi_{I,t}) - \eta_I),$$

$$\Lambda_I(\varphi_{I,t}) := \left(1 + \frac{\eta_I \Xi_I(\varphi_{I,t})}{(1-\chi_I^{\frac{1}{1-\beta_I}})\phi\left(\Phi^{-1}\left(\frac{1-\Xi_I(\varphi_{I,t})}{1-\chi_I^{\frac{1}{1-\beta_I}}}\right)\right)} \right)^{-1},$$

$$\Psi_I(\varphi_{I,t}) := \frac{\phi\left(\Phi^{-1}(\varphi_{I,t})\right)(1-\Lambda_I(\varphi_{I,t}))}{\eta_I}.$$

We then have

$$\log(K^*_{I,t}) = \log\left(\frac{\gamma_I \theta_I^2}{(1-\beta_I)(1+r_I)R_{K,t}} \left(\frac{1+\Psi_I(\varphi_{I,t})\,\Omega_I(\varphi_{I,t})}{1-\boldsymbol{\theta}'\boldsymbol{\Lambda}(\boldsymbol{\varphi_t})} \right) \right) + \log\left(\sum_{J\in[N]} \Pi^*_{J,t} \right) =$$

$$\log\left(\frac{\gamma_I \theta_I^2}{(1-\beta_I)(1+r_I)R_{K,t}} \left(\frac{1+\Psi_I(\varphi_{I,t})\,\Omega_I(\varphi_{I,t})}{1-\boldsymbol{\theta}'\boldsymbol{\Lambda}(\boldsymbol{\varphi_t})} \right) \right) + \delta -$$

$$\boldsymbol{\theta}'\log(\boldsymbol{\theta}) + \boldsymbol{\theta}'\log(\boldsymbol{1}-\boldsymbol{\beta}) + \log(\overline{P}_t) + \boldsymbol{\theta}'\boldsymbol{\mu}t + \boldsymbol{\theta}'\boldsymbol{\sigma}\varepsilon_t + \boldsymbol{\theta}'\log(\boldsymbol{\Xi}(\boldsymbol{\varphi_t})) + \boldsymbol{\theta}'[\boldsymbol{I} - \text{diag}(\boldsymbol{\beta})]^{-1}\text{diag}(\boldsymbol{\gamma})\log(\boldsymbol{K}^*_t),$$

This leads to the following expression for the total production:

$$\sum_{J\in[N]} \Pi^*_{J,t} = \exp\left\{ \frac{1}{1-\sum_{I\in[N]} \frac{\theta_I \gamma_I}{1-\beta_I}} \left[\delta - \boldsymbol{\theta}'\log(\boldsymbol{\theta}) + \boldsymbol{\theta}'\log(\boldsymbol{1}-\boldsymbol{\beta}) + \log(\overline{P}_t) + \right.\right.$$

$$\left.\left. \boldsymbol{\theta}'\boldsymbol{\mu}t + \boldsymbol{\theta}'\boldsymbol{\sigma}\varepsilon_t + \boldsymbol{\theta}'\log(\boldsymbol{\Xi}(\boldsymbol{\varphi_t})) + \sum_{I\in[N]} \frac{\gamma_I}{1-\beta_I}\theta_I \log\left(\frac{\gamma_I \theta_I^2}{(1-\beta_I)(1+r_I)R_{K,t}} \left(\frac{1+\Psi_I(\varphi_{I,t})\,\Omega_I(\varphi_{I,t})}{1-\boldsymbol{\theta}'\boldsymbol{\Lambda}(\boldsymbol{\varphi_t})} \right) \right) \right] \right\}. \tag{70}$$

Step 4:

The next step is to derive the equilibrium rental rate of capital $R_{K,t}$ using the market clearing condition for total capital. We start by summing up capital in (68) to obtain

$$K_t = \frac{\sum_{I\in[N]} \theta_I^2 \frac{\gamma_I}{1-\beta_I} \left(1+\Psi_I\left(\varphi_{I,t}\right)\Omega_I\left(\varphi_{I,t}\right)\right)}{R_{K,t}(1+r_I)} \sum_{J\in[N]} \Pi^*_{J,t}.$$

This allows us to write $R_{K,t}$ as follows:

$$R_{K,t} = \frac{\sum_{I\in[N]} \theta_I^2 \frac{\gamma_I}{1-\beta_I} \left(1+\Psi_I\left(\varphi_{I,t}\right)\Omega_I\left(\varphi_{I,t}\right)\right)}{(1+r_I)\left(1-\boldsymbol{\theta}'\boldsymbol{\Lambda}\left(\boldsymbol{\varphi_t}\right)\right)} \frac{\sum_{J\in[N]} \Pi^*_{J,t}}{K_t}.$$

Plugging (30) into 70 we obtain the following.

$$\sum_{J\in[N]} \Pi^*_{J,t} = \exp\Bigg\{\delta - \boldsymbol{\theta}'\log(\boldsymbol{\theta}) + \boldsymbol{\theta}'\log(\mathbf{1}-\boldsymbol{\beta}) + \log(\overline{P}_t) +$$

$$\boldsymbol{\theta}'\boldsymbol{\mu}t + \boldsymbol{\theta}'\boldsymbol{\sigma}\varepsilon_t + \boldsymbol{\theta}'\log(\boldsymbol{\Xi}\left(\boldsymbol{\varphi_t}\right)) + \sum_{I\in[N]} \frac{\gamma_I}{1-\beta_I}\theta_I \log\left(\frac{\gamma_I\theta_I^2 K_t\left(1+\Psi_I\left(\varphi_{I,t}\right)\Omega_I\left(\varphi_{I,t}\right)\right)}{(1-\beta_I)\sum_{J\in[N]}\theta_J^2\frac{\gamma_J}{1-\beta_J}\left(1+\Psi_J\left(\varphi_{J,t}\right)\Omega_J\left(\varphi_{J,t}\right)\right)}\right)\Bigg\}. \tag{71}$$

We proceed to plug in the previous value for $R_{K,t}$ in (68) to obtain:

$$K^*_{I,t} = \frac{\theta_I^2 \frac{\gamma_I}{1-\beta_I}\left(1+\Psi_I\left(\varphi_{I,t}\right)\Omega_I\left(\varphi_{I,t}\right)\right)}{\sum_{J\in[N]}\theta_J^2\frac{\gamma_J}{1-\beta_J}\left(1+\Psi_J\left(\varphi_{J,t}\right)\Omega_J\left(\varphi_{J,t}\right)\right)} K_t.$$

Step 5:

In the final step, we observe that the capital structures of firms are expressed in terms of the vector of firm default probabilities, (67). However, the default probabilities are given by (54), which depend on firms' capital structure choices. Hence, we have a fixed-point problem for firm default probabilities whose solution determines the equilibrium capital structures and default probabilities of firms. We now show that the fixed-point problem has a unique solution that, in turn, determines the unique equilibrium.

We start by defining

$$\bar{\varphi}_{I,t} := \min\left\{\varphi_{I,t}\middle|\varphi_{I,t} - \left[1-\Psi_I\left(\varphi_{I,t}\right)\left(1+\frac{\theta_I\Lambda_I\left(\varphi_{I,t}\right)}{1-\boldsymbol{\theta}'\Lambda\left(\boldsymbol{\varphi_t}\right)}\right)\right]=0\right\}.$$

We want to show that $\bar{\varphi}_{I,t}\in(0,1)$. Let

$$F(\tilde{\varphi}_{I,t}) = \Phi(\tilde{\varphi}_{I,t}) - \left[1-\Psi_I\left(\varphi_{I,t}\right)\left(1+\frac{\theta_I\Lambda_I\left(\varphi_{I,t}\right)}{1-\boldsymbol{\theta}'\Lambda\left(\boldsymbol{\varphi_t}\right)}\right)\right],$$

where $\tilde{\varphi}_{I,t} = \Phi^{-1}(\varphi_{I,t})$. We have the following intermediate result:

$$\left(\Psi_I\left(\varphi_{I,t}\right)|\tilde{\varphi_{I,t}}\to-\infty\right) = \left(\Psi_I\left(\varphi_{I,t}\right)|\tilde{\varphi}_{I,t}\to\infty\right)\to 0$$

This implies the following:

$$F(\tilde{\varphi}_{I,t}\to-\infty)\to-1,$$

and

$$F(\tilde{\varphi}_{I,t}\to\infty)\to 0.$$

We will show that there exists some $\delta_I>0$, such that $F(\delta_I)>0$. Since the function $F(\tilde{\varphi}_{I,t})$ is continuous, there exists some value $\bar{\varphi}_{I,t}\in(0,1)$ such that $F(\Phi^{-1}(\bar{\varphi}_{I,t}))=0$. First, we have the following inequality.

$$F(\tilde{\varphi}_{I,t}) = \Phi(\tilde{\varphi}_{I,t}) - \left[1-\Psi_I\left(\varphi_{I,t}\right)\left(1+\frac{\theta_I\Lambda_I\left(\varphi_{I,t}\right)}{1-\boldsymbol{\theta}'\Lambda\left(\boldsymbol{\varphi_t}\right)}\right)\right] > \Phi(\tilde{\varphi}_{I,t}) + \Psi_I\left(\varphi_{I,t}\right) - 1 =$$

$$\frac{(1-\Phi(\tilde{\varphi}_{I,t}))(1-\Lambda_I\left(\varphi_{I,t}\right))}{\eta_I}\left[\frac{\phi(\tilde{\varphi}_{I,t})}{1-\Phi(\tilde{\varphi}_{I,t})} - \frac{\eta_I}{1-\Lambda_I\left(\varphi_{I,t}\right)}\right].$$

Next, we note that

$$\Lambda_I\left(\varphi_{I,t}\right) < \left(1 + \frac{\sqrt{2\pi}\eta_I \chi_I^{\frac{1}{1-\beta_I}}}{1-\chi_I^{\frac{1}{1-\beta_I}}}\right)^{-1} = \bar{\Lambda}_I$$

This leads to the inequality

$$F(\tilde{\varphi}_{I,t}) > \frac{(1-\Phi(\tilde{\varphi}_{I,t}))(1-\bar{\Lambda}_I)}{\eta_I}\left[\frac{\phi\left(\tilde{\varphi}_{I,t}\right)}{1-\Phi(\tilde{\varphi}_{I,t})} - \frac{\eta_I}{1-\bar{\Lambda}_I}\right].$$

The function $\frac{\phi(\tilde{\varphi}_{I,t})}{1-\Phi(\tilde{\varphi}_{I,t})}$, which is known as the inverse Mills ratio, is an increasing function of $\tilde{\varphi}_{I,t}$. This means that there exists some value $-\infty < \delta_I < \infty$ such that

$$\frac{\phi\left(\delta_I\right)}{1-\Phi(\delta_I)} > \frac{\eta_I}{1-\bar{\Lambda}_I}$$

Finally,

$$F(\delta_I) > \frac{(1-\Phi(\delta_I))(1-\bar{\Lambda}_I)}{\eta_I}\left[\frac{\phi\left(\delta_I\right)}{1-\Phi(\delta_I)} - \frac{\eta_I}{1-\bar{\Lambda}_I}\right] > 0.$$

Since $F(\tilde{\varphi}_{I,t} \to -\infty) = 0$, $F(\delta_I) > 0$ and the function $F(\cdot)$ is continuous, there must exist some value $\bar{\varphi}_{I,t} \in (0,1)$ such that $F(\Phi^{-1}(\tilde{\varphi}_{I,t})) = 0$.

The final step is to show that there exists a unique value $\varphi^*_{I,t} \in (0, \bar{\varphi}_I)$ for all $I \in [N]$ such that the equilibrium probability of default is the solution to the fixed point equation

$$\varphi^*_{I,t} = \Phi\left(\frac{1}{\eta_I}\left[\log\left(\frac{(1-\tau_I)\rho_I}{\tau_I}\right) + \log(\Omega_I\left(\varphi^*_{I,t}\right)) + \log(\Xi_I\left(\varphi^*_{I,t}\right))\right] + \frac{1}{2}\eta_I\right)$$

We define the following function for all industries $I \in [N]$:

$$\Psi_I(\tilde{\varphi}_{I,t}) := \Phi(\tilde{\varphi}_{I,t}) - \Phi\left(\frac{1}{\eta_I}\left[\log\left(\frac{(1-\tau_I)\rho_I}{\tau_I}\right) + \log\left(\Omega_I\left(\tilde{\varphi}_{I,t}\right)\right) + \log\left(\Xi_I\left(\tilde{\varphi}_{I,t}\right)\right)\right] + \frac{1}{2}\eta_I\right).$$

First, we have that $\Psi_I(\tilde{\varphi}_{I,t} \to -\infty) = 0$ and $\lim_{\tilde{\varphi}_{I,t}\uparrow\Phi(\bar{\varphi}_I)} \Psi_I(\tilde{\varphi}_{I,t}) = -(1-\Phi(\bar{\varphi}_{I,t})) < 0$. In the second expression it is necessary to use the limit from the left, since the function has a discontinuity at $\tilde{\varphi}_{I,t} = \Phi(\bar{\varphi}_{I,t})$.

Next, we will show that there exists some $-\infty < \delta_I < \Phi(\bar{\varphi}_I)$ such that $\Psi_I(\delta_I) > 0$. We begin with the following series of inequalities:

$$\Psi_I(\delta_I) > \Phi(\delta_I) - \Phi\left(\frac{1}{\eta_I}\left[\log\left(\frac{(1-\tau_I)\rho_I}{\tau_I}\right) + \log\left(\Omega_I\left(\delta_I\right)\right)\right] + \frac{1}{2}\eta_I\right) =$$

$$\Phi(\delta_I) - \Phi\left(\frac{1}{\eta_I}\left[\log\left(\frac{\theta_I(1-\tau_I)\rho_I}{\tau_I}\right) - \log\left(\frac{(1-\Phi(\delta_I)-\Psi_I\left(\delta_I\right))(1-\boldsymbol{\theta}'\boldsymbol{\Lambda}\left(\boldsymbol{\delta}\right))}{\Lambda_I\left(\delta_I\right)} - \theta_I\Psi_I\left(\delta_I\right)\right)\right] + \frac{1}{2}\eta_I\right).$$

We need to show that

$$\Phi(\delta_I) > \Phi\left(\frac{1}{\eta_I}\left[\log\left(\frac{\theta_I(1-\tau_I)\rho_I}{\tau_I}\right) - \log\left(\frac{(1-\Phi(\delta_I)-\Psi_I\left(\delta_I\right))(1-\boldsymbol{\theta}'\boldsymbol{\Lambda}\left(\boldsymbol{\delta}\right))}{\Lambda_I\left(\delta_I\right)} - \theta_I\Psi_I\left(\delta_I\right)\right)\right] + \frac{1}{2}\eta_I\right).$$

This is equivalent to

$$\exp\left\{\eta_I\delta_I + \log\left(\frac{(1-\Phi(\delta_I)-\Psi_I\left(\delta_I\right))(1-\boldsymbol{\theta}'\boldsymbol{\Lambda}\left(\boldsymbol{\delta}\right))}{\Lambda_I\left(\delta_I\right)} - \theta_I\Psi_I\left(\delta_I\right)\right)\right\} > \frac{\theta_I(1-\tau_I)\rho_I}{\tau_I}e^{\frac{1}{2}\eta_I^2}.$$

We have the following inequalities

$$\exp\left\{\eta_I\delta_I + \log\left(\frac{(1-\Phi(\delta_I)-\Psi_I(\delta_I))(1-\boldsymbol{\theta}'\boldsymbol{\Lambda}(\boldsymbol{\delta}))}{\Lambda_I(\delta_I)} - \theta_I\Psi_I(\delta_I)\right)\right\} > \frac{(1-\Phi(\delta_I)-\Psi_I(\delta_I))(1-\boldsymbol{\theta}'\boldsymbol{\Lambda}(\boldsymbol{\delta}))}{\Lambda_I(\delta_I)} - \theta_I\Psi_I(\delta_I) >$$
$$\frac{(1-\Phi(\delta_I)-\frac{\phi(\delta_I)}{\eta_I})(1-\bar{\Lambda}_I)}{\Lambda_I(\delta_I)} - \frac{\theta_I\phi(\delta_I)}{\eta_I} > (1-\bar{\Lambda}_I)\left[\left(\frac{1-\Phi(\delta_I)}{\phi(\delta_I)} - \frac{1}{\eta_I}\right)\frac{\phi(\delta_I)}{\Lambda_I(\delta_I)} - \frac{\theta_I}{\eta_I\sqrt{2\pi}(1-\bar{\Lambda}_I)}\right].$$

The term $\frac{\phi(\delta_I)}{\Lambda_I(\delta_I)}$ can be bounded from below as follows

$$\frac{\phi(\delta_I)}{\Lambda_I(\delta_I)} = \phi(\delta_I)\left(1 + \frac{\eta_I\left[1-\left(1-\chi_I^{\frac{1}{1-\beta_I}}\right)\Phi\left(\delta_I-\eta_I\right)\right]}{\left(1-\chi_I^{\frac{1}{1-\beta_I}}\right)\phi\left(\delta_I-\eta_I\right)}\right) >$$
$$\frac{\eta_I\left[1-\left(1-\chi_I^{\frac{1}{1-\beta_I}}\right)\Phi\left(\delta_I-\eta_I\right)\right]}{\left(1-\chi_I^{\frac{1}{1-\beta_I}}\right)} > \frac{\eta_I\chi_I^{\frac{1}{1-\beta_I}}}{\left(1-\chi_I^{\frac{1}{1-\beta_I}}\right)}.$$

Using the previous bound, we then have the following

$$(1-\bar{\Lambda}_I)\left[\left(\frac{1-\Phi(\delta_I)}{\phi(\delta_I)} - \frac{1}{\eta_I}\right)\frac{\phi(\delta_I)}{\Lambda_I(\delta_I)} - \frac{\theta_I}{\eta_I\sqrt{2\pi}(1-\bar{\Lambda}_I)}\right] >$$
$$\frac{\eta_I(1-\bar{\Lambda}_I)\chi_I^{\frac{1}{1-\beta_I}}}{1-\chi_I^{\frac{1}{1-\beta_I}}}\left[\frac{1-\Phi(\delta_I)}{\phi(\delta_I)} - \frac{1}{\eta_I} - \frac{\theta_I\left(1-\chi_I^{\frac{1}{1-\beta_I}}\right)}{\eta_I^2\chi_I^{\frac{1}{1-\beta_I}}\sqrt{2\pi}(1-\bar{\Lambda}_I)}\right].$$

The function $\frac{1-\Phi(\delta_I)}{\phi(\delta_I)}$, i.e., the Mills ratio, is monotonically decreasing in δ_I. Hence, it is possible to choose an arbitrarily small $-\infty < \delta_I < \Phi(\bar{\varphi}_I)$ such that

$$\frac{\eta_I(1-\bar{\Lambda}_I)\chi_I^{\frac{1}{1-\beta_I}}}{1-\chi_I^{\frac{1}{1-\beta_I}}}\left[\frac{1-\Phi(\delta_I)}{\phi(\delta_I)} - \frac{1}{\eta_I} - \frac{\theta_I\left(1-\chi_I^{\frac{1}{1-\beta_I}}\right)}{\eta_I^2\chi_I^{\frac{1}{1-\beta_I}}\sqrt{2\pi}(1-\bar{\Lambda}_I)}\right] > \frac{\theta_I(1-\tau_I)\rho_I}{\tau_I}e^{\frac{1}{2}\eta_I^2}.$$

Since $\Psi_I(\cdot)$ is continuous, $\Psi(\tilde{\varphi}_I \to -\infty) = 0 > -(1-\Phi(\bar{\varphi}_I)) = \lim_{\tilde{\varphi}_I \uparrow \Phi(\bar{\varphi}_I)} \Psi(\tilde{\varphi}_I)$ and there exists some $-\infty < \delta_I < \Phi(\bar{\varphi}_I)$ such that $\Psi(\delta_I) > 0$, we can conclude that there exists a unique solution $\tilde{\varphi}^*_{I,t} \in (-\infty, \Phi(\bar{\varphi}_I))$ to the fixed point equation (i.e., $\Psi_I(\tilde{\varphi}^*_{I,t}) = 0$).

Step 6:

In Step 5, we have shown the existence of a unique solution to the fixed point problem in the vector of firm default probabilities. $\left(\varphi^*_{1,t}, ..., \varphi^*_{N,t}\right)$. Plugging these into (68) and (69), we obtain the equilibrium capital structures of all firms. Q.E.D.

Proof of Proposition 1

The only difference between individual firms and the industry average is the idiosyncratic shock. From 6 we have the following relationship

$$P_{I,t+1}x^*_{i,t+1} = z_{i,+1}^{\frac{1}{1-\beta_I}}P_{I,t+1}\bar{x}^*_{I,t+1} = e^{-\frac{1}{2}\eta_I^2+\eta_I\varepsilon_{i,t+1}}P_{I,t+1}\bar{x}^*_{I,t+1}.$$

We proceed to find the logarithmic difference

$$\log\left(\frac{P_{I,t+1}x^*_{i,t+1}}{P_{I,t}x^*_{i,t}}\right) - \log\left(\frac{P_{I,t+1}\bar{x}^*_{I,t+1}}{P_{I,t}\bar{x}^*_{I,t}}\right) = \eta_I(\varepsilon_{i,t+1} - \varepsilon_{i,t}).$$

The next step is to find the variance:

$$\mathbb{V}\left[\log\left(\frac{P_{I,t+1}x^*_{i,t+1}}{P_{I,t}x^*_{i,t}}\right)\right]+\mathbb{V}\left[\log\left(\frac{P_{I,t+1}\bar{x}^*_{I,t+1}}{P_{I,t}\bar{x}^*_{I,t}}\right)\right]=2\eta_I^2,$$

which leads to the final result

$$\eta_I=\frac{1}{\sqrt{2}}\sqrt{\mathbb{V}\left[\log\left(\frac{P_{I,t+1}x^*_{i,t+1}}{P_{I,t}x^*_{i,t}}\right)\right]+\mathbb{V}\left[\log\left(\frac{P_{I,t+1}\bar{x}^*_{I,t+1}}{P_{I,t}\bar{x}^*_{I,t}}\right)\right]}$$

Proof of Proposition 4

We can rewrite the probability of default in equation 24 as

$$\Phi^{-1}(\varphi_I^*)=\frac{1}{\eta_I}\left[\log\left(\frac{(1-\tau_I)\rho_I}{\tau_I}\right)+\log(\Omega_I\left(\varphi_I^*\right))+\log(\Xi_I\left(\varphi_I^*\right))\right]+\frac{1}{2}\eta_I$$

We begin by deriving the results for the probability of default.

$$\begin{aligned}\frac{\partial\varphi_I^*}{\partial\theta}&=\frac{1}{\eta_I}\left[\frac{\phi(\Phi^{-1}(\varphi_I^*))}{\Omega_I\left(\varphi_I^*\right)}\frac{\partial\Omega_I\left(\varphi_I^*\right)}{\partial\theta}+\frac{\phi(\Phi^{-1}(\varphi_I^*))}{\Xi_I\left(\varphi_I^*\right)}\frac{\partial\Xi_I\left(\varphi_I^*\right)}{\partial\theta}\right]\\&=\frac{1}{\eta_I}\left[\frac{\phi(\Phi^{-1}(\varphi_I^*))}{\Omega_I\left(\varphi_I^*\right)}\frac{\partial\Omega_I\left(\varphi_I^*\right)}{\partial\theta}-\frac{1}{\Xi_I\left(\varphi_I^*\right)}(1-\chi_I^{\frac{1}{1-\beta_I}})\phi(\Phi^{-1}(\varphi_I^*)-\eta_I)\frac{\partial\varphi_I^*}{\partial\theta}\right]\\&=\frac{\phi(\Phi^{-1}(\varphi_I^*))}{\eta_I\Omega_I\left(\varphi_I^*\right)}\frac{\partial\Omega_I\left(\varphi_I^*\right)}{\partial\theta}-\frac{\Lambda_I\left(\varphi_I^*\right)}{1-\Lambda_I\left(\varphi_I^*\right)}\frac{\partial\varphi_I^*}{\partial\theta}\\&=\frac{\Psi_I\left(\varphi_I^*\right)}{\Omega_I\left(\varphi_I^*\right)}\frac{\partial\Omega_I\left(\varphi_I^*\right)}{\partial\theta}\end{aligned}$$

We can rewrite equations 23, 22 and 21 as

$$\log\left(\Psi_I\left(\varphi_I^*\right)+\frac{1}{\Omega_I\left(\varphi_I^*\right)}\right)=\log(1-\varphi_I^*-\Psi_I\left(\varphi_I^*\right))-\log(\theta_I)+\log\left(1-\sum_{J\in[N]}\theta_J\Lambda_J\left(\varphi_J^*\right)\right)-\log(\Lambda_I\left(\varphi_I^*\right)),$$

$$\begin{aligned}\log(\Psi_I\left(\varphi_I^*\right))&=\log(\phi(\Phi^{-1}(\varphi_I^*))+\log(1-\Lambda_I\left(\varphi_I^*\right))-\log(\eta_I)\\&=-\frac{1}{2}\log(2\pi)-\frac{1}{2}\Phi^{-1}(\varphi_I^*)^2+\log(1-\Lambda_I\left(\varphi_I^*\right))-\log(\eta_I),\end{aligned}$$

$$\begin{aligned}\log\left(\frac{1}{\Lambda_I\left(\varphi_I^*\right)}-1\right)&=\log(\eta_I)+\log(\Xi_I\left(\varphi_I^*\right))-\log(1-\chi_I^{\frac{1}{1-\beta_I}})-\log(\phi(\Phi^{-1}(\varphi_I^*)-\eta_I))\\&=\log(\eta_I)+\log(\Xi_I\left(\varphi_I^*\right))-\log(1-\chi_I^{\frac{1}{1-\beta_I}})+\frac{1}{2}\log(2\pi)+\frac{1}{2}(\Phi^{-1}(\varphi_I^*)-\eta_I)^2,\end{aligned}$$

where,

$$\theta_I:=\frac{\theta}{(1-\theta)(1-(1-\theta)^N)}(1-\theta)^I.$$

Now we will proceed to find the partial derivatives:

$$\frac{\partial \Omega_I(\varphi_I^*)}{\partial \theta} = \Omega_I(\varphi_I^*)\left(\Omega_I(\varphi_I^*)\Psi_I(\varphi_I^*)+1\right)\cdot$$
$$\left[\frac{1}{1-\varphi_I^*-\Psi_I(\varphi_I^*)}\frac{\partial \varphi_I^*}{\partial \theta} + \frac{1+\Omega_I(\varphi_I^*)(1-\varphi_I^*)}{(1-\varphi_I^*-\Psi_I(\varphi_I^*))(\Omega_I(\varphi_I^*)\Psi_I(\varphi_I^*)+1)}\frac{\partial \Psi_I(\varphi_I^*)}{\partial \theta} + \frac{1}{\Lambda_I(\varphi_I^*)}\frac{\partial \Lambda_I(\varphi_I^*)}{\partial \theta} + \right.$$
$$\left.\frac{1}{\theta_I}\frac{\partial \theta_I}{\partial \theta} + \frac{1}{1-\sum_{J\in[N]}\theta_J\Lambda_J(\varphi_J^*)}\left(\sum_{J\in[N]}\frac{\partial \theta_J}{\partial \theta}\Lambda_J(\varphi_J^*) + \sum_{J\in[N]}\theta_J\frac{\partial \Lambda_J(\varphi_J^*)}{\partial \theta}\right)\right],$$

$$\frac{\partial \Psi_I(\varphi_I^*)}{\partial \theta} = -\frac{1-\Lambda_I(\varphi_I^*)}{\eta_I}\Phi^{-1}(\varphi_I^*)\frac{\partial \varphi_I^*}{\partial \theta} - \frac{\Psi_I(\varphi_I^*)}{1-\Lambda_I(\varphi_I^*)}\frac{\partial \Lambda_I(\varphi_I^*)}{\partial \theta},$$

$$\begin{aligned}
\frac{\partial \Lambda_I(\varphi_I^*)}{\partial \theta} &= \frac{\Lambda_I(\varphi_I^*)(1-\Lambda_I(\varphi_I^*))}{\phi(\Phi^{-1}(\varphi_I^*))}\left[\frac{1}{\Xi_I(\varphi_I^*)}(1-\chi_I^{\frac{1}{1-\beta_I}})\phi(\Phi^{-1}(\varphi_I^*)-\eta_I) - (\Phi^{-1}(\varphi_I^*)-\eta_I)\right]\frac{\partial \varphi_I^*}{\partial \theta} \\
&= \frac{\Lambda_I(\varphi_I^*)(1-\Lambda_I(\varphi_I^*))}{\phi(\Phi^{-1}(\varphi_I^*))}\left[\frac{\eta_I}{1-\Lambda_I(\varphi_I^*)} - \Phi^{-1}(\varphi_I^*)\right]\frac{\partial \varphi_I^*}{\partial \theta} \\
&= \frac{\Lambda_I(\varphi_I^*)(1+\kappa_I(\varphi_I^*))}{\Psi_I(\varphi_I^*)}\frac{\partial \varphi_I^*}{\partial \theta},
\end{aligned}$$

and

$$\frac{\partial \log(\theta_I)}{\partial \log(\theta)} = 1 - \frac{\theta}{1-\theta}\left[I-1+N\frac{(1-\theta)^N}{1-(1-\theta)^N}\right],$$

with

$$\kappa_I(\varphi_I^*) := -\Lambda_I(\varphi_I^*) - \frac{(1-\Lambda_I(\varphi_I^*))^2}{\eta_I}\Phi^{-1}(\varphi_I^*).$$

Replacing $\frac{\partial \Lambda_I(\varphi_I^*)}{\partial \theta}$ into $\frac{\partial \Psi_I(\varphi_I^*)}{\partial \theta}$ we have

$$\begin{aligned}
\frac{\partial \Psi_I(\varphi_I^*)}{\partial \theta} &= -\frac{1-\Lambda_I(\varphi_I^*)}{\eta_I}\Phi^{-1}(\varphi_I^*)\frac{\partial \varphi_I^*}{\partial \theta} - \Lambda_I(\varphi_I^*)\left[1-\frac{1-\Lambda_I(\varphi_I^*)}{\eta_I}\Phi^{-1}(\varphi_I^*)\right]\frac{\partial \varphi_I^*}{\partial \theta} \\
&= -\left[\frac{(1-\Lambda_I(\varphi_I^*))^2}{\eta_I}\Phi^{-1}(\varphi_I^*) + \Lambda_I(\varphi_I^*)\right]\frac{\partial \varphi_I^*}{\partial \theta} \\
&= \kappa_I(\varphi_I^*)\frac{\partial \varphi_I^*}{\partial \theta},
\end{aligned}$$

Replacing these results in the expression for $\frac{\partial \Omega_I(\varphi_I^*)}{\partial \theta}$ we obtain

$$\frac{\partial \Omega_I(\varphi_I^*)}{\partial \theta} = \Omega_I(\varphi_I^*)\left(\Omega_I(\varphi_I^*)\Psi_I(\varphi_I^*)+1\right)\left(A_I + B_I(\varphi_I^*)\frac{\partial \varphi_I^*}{\partial \theta} + \sum_{J\in[N]} C_J(\varphi_J^*)\frac{\partial \varphi_J^*}{\partial \theta}\right),$$

where

$$A_I := \frac{1}{\theta}\frac{\partial \log(\theta_I)}{\partial \log(\theta)} + \sum_{J\in[N]}\frac{1}{\theta}\frac{\partial \log(\theta_J)}{\partial \log(\theta)}\left[\frac{\theta_J\Lambda_J(\varphi_J^*)}{1-\sum_{J\in[N]}\theta_J\Lambda_J(\varphi_J^*)}\right]$$

$$B_I(\varphi_I^*) := \frac{1}{\Psi_I(\varphi_I^*)}\left[\frac{1+\kappa_I(\varphi_I^*)}{1-\frac{\Psi_I(\varphi_I^*)}{1-\varphi_I^*}} + \frac{\Psi_I(\varphi_I^*)\Omega_I(\varphi_I^*)\kappa_I(\varphi_I^*)}{1+\Omega_I(\varphi_I^*)\Psi_I(\varphi_I^*)}\right],$$

and

$$C_I(\varphi_I^*) := \frac{1}{\Psi_I(\varphi_I^*)}\left[\frac{\theta_I\Lambda_I(\varphi_I^*)}{1-\sum_{J\in[N]}\theta_J\Lambda_J(\varphi_J^*)}\right](1+\kappa_I(\varphi_I^*)).$$

Notice that the sign of $\frac{\partial \theta_I}{\partial \theta}$ depends on I. We have the following relationship

$$\frac{\partial \log(\theta_I)}{\partial \log(\theta)} \geq \frac{\partial \log(\theta_{I+1})}{\partial \log(\theta)}, \forall I \in \{1, 2, ..., N-1\}$$

For the extremes we have:

$$\frac{\partial \log(\theta_1)}{\partial \log(\theta)} = 1 - \frac{\theta}{1-\theta}\left[N\frac{(1-\theta)^N}{1-(1-\theta)^N}\right] = 1 - \frac{N(1-\theta)^{N-1}}{\sum_{k=0}^{N-1}(1-\theta)^k} \geq 0$$

$$\frac{\partial \log(\theta_N)}{\partial \log(\theta)} = 1 - \frac{\theta}{1-\theta}\left[N-1+N\frac{(1-\theta)^N}{1-(1-\theta)^N}\right] = \frac{1}{1-\theta}\left[1 - \frac{N}{\sum_{k=0}^{N-1}(1-\theta)^k}\right] \leq 0$$

This implies the following series of inequalities:

$$A_1 \geq A_2 \geq ... \geq A_{N-1} \geq A_N. \tag{72}$$

Assuming that $B_I \neq \frac{1}{\Psi_I(\varphi_I^*)(\Omega_I(\varphi_I^*)\Psi_I(\varphi_I^*)+1)}$ and putting these results together we have

$$\begin{aligned}\frac{\partial \varphi_I^*}{\partial \theta} &= \Psi_I(\varphi_I^*)(\Omega_I(\varphi_I^*)\Psi_I(\varphi_I^*)+1)\left(A_I + B_I(\varphi_I^*)\frac{\partial \varphi_I^*}{\partial \theta} + \sum_{J\in[N]} C_J(\varphi_J^*)\frac{\partial \varphi_J^*}{\partial \theta}\right) \\ &= -\left(B_I(\varphi_I^*) - \frac{1}{\Psi_I(\varphi_I^*)(\Omega_I(\varphi_I^*)\Psi_I(\varphi_I^*)+1)}\right)^{-1}\left(A_I + \sum_{J\in[N]} C_J(\varphi_J^*)\frac{\partial \varphi_J^*}{\partial \theta}\right).\end{aligned}$$

Multiplying by $C_I(\varphi_I^*)$ and summing through $I \in [N]$ we have

$$\sum_{I\in[N]} C_I(\varphi_I^*)\frac{\partial \varphi_I^*}{\partial \theta} = -\frac{\sum_{I\in[N]}\left(B_I(\varphi_I^*) - \frac{1}{\Psi_I(\varphi_I^*)(\Omega_I(\varphi_I^*)\Psi_I(\varphi_I^*)+1)}\right)^{-1} A_I C_I(\varphi_I^*)}{1+\sum_{I\in[N]}\left(B_I(\varphi_I^*) - \frac{1}{\Psi_I(\varphi_I^*)(\Omega_I(\varphi_I^*)\Psi_I(\varphi_I^*)+1)}\right)^{-1} C_I(\varphi_I^*)}.$$

Replacing this into the partial derivative for φ_I and transforming the expression into an elasticity, we have

$$\begin{aligned}\frac{\partial \log(\varphi_I^*)}{\partial \log(\theta)} &= -\frac{\theta}{\varphi_I^*}\left(B_I(\varphi_I^*) - \frac{1}{\Psi_I(\varphi_I^*)(\Omega_I(\varphi_I^*)\Psi_I(\varphi_I^*)+1)}\right)^{-1}\left(A_I - \frac{\sum_{J\in[N]}\frac{A_J C_J(\varphi_J^*)}{B_J(\varphi_J^*) - \frac{1}{\Psi_J(\varphi_J^*)(\Omega_J(\varphi_J^*)\Psi_J(\varphi_J^*)+1)}}}{1+\sum_{J\in[N]}\frac{C_J(\varphi_J^*)}{B_J(\varphi_J^*) - \frac{1}{\Psi_J(\varphi_J^*)(\Omega_J(\varphi_J^*)\Psi_J(\varphi_J^*)+1)}}}\right) \\ &= -\frac{\theta}{\varphi_I^*}\left(B_I(\varphi_I^*) - \frac{1}{\Psi_I(\varphi_I^*)(\Omega_I(\varphi_I^*)\Psi_I(\varphi_I^*)+1)}\right)^{-1}\left(1+\sum_{J\in[N]}\frac{C_J(\varphi_J^*)}{B_J(\varphi_J^*) - \frac{1}{\Psi_J(\varphi_J^*)(\Omega_J(\varphi_J^*)\Psi_J(\varphi_J^*)+1)}}\right)^{-1} \cdot \\ &\quad \left(A_I + \sum_{J\in[N]}\frac{(A_I - A_J)C_J(\varphi_J^*)}{B_J(\varphi_J^*) - \frac{1}{\Psi_J(\varphi_J^*)(\Omega_J(\varphi_J^*)\Psi_J(\varphi_J^*)+1)}}\right).\end{aligned}$$

Hence,

$$\begin{aligned}\sum_{J\in[N]}\frac{(A_I - A_J)C_J(\varphi_J^*)}{B_J(\varphi_J^*) - \frac{1}{\Psi_J(\varphi_J^*)(\Omega_J(\varphi_J^*)\Psi_J(\varphi_J^*)+1)}} &= \sum_{J\in[N]}\frac{\frac{1}{\theta}\left(\frac{\partial \log(\theta_I)}{\partial \log(\theta)} - \frac{\partial \log(\theta_J)}{\partial \log(\theta)}\right)C_J(\varphi_J^*)}{B_J(\varphi_J^*) - \frac{1}{\Psi_J(\varphi_J^*)(\Omega_J(\varphi_J^*)\Psi_J(\varphi_J^*)+1)}} \\ &= \frac{1}{1-\theta}\sum_{J\in[N]}\frac{\big(J-I\big)C_J(\varphi_J^*)}{B_J(\varphi_J^*) - \frac{1}{\Psi_J(\varphi_J^*)(\Omega_J(\varphi_J^*)\Psi_J(\varphi_J^*)+1)}}.\end{aligned}$$

Finally, we have

$$\frac{\partial \log(\varphi_I^*)}{\partial \log(\theta)} = -\frac{\theta}{\varphi_I^*}\left(B_I\left(\varphi_I^*\right) - \frac{1}{\Psi_I\left(\varphi_I^*\right)\left(\Omega_I\left(\varphi_I^*\right)\Psi_I\left(\varphi_I^*\right)+1\right)}\right)^{-1}\left(1+\sum_{J\in[N]}\frac{C_J\left(\varphi_J^*\right)}{B_J\left(\varphi_J^*\right) - \frac{1}{\Psi_J\left(\varphi_J^*\right)\left(\Omega_J\left(\varphi_J^*\right)\Psi_J\left(\varphi_J^*\right)+1\right)}}\right)^{-1}\cdot$$
$$\left(A_I + \frac{1}{1-\theta}\sum_{J\in[N]}(J-I)\frac{C_J\left(\varphi_J^*\right)}{B_J\left(\varphi_J^*\right) - \frac{1}{\Psi_J\left(\varphi_J^*\right)\left(\Omega_J\left(\varphi_J^*\right)\Psi_J\left(\varphi_J^*\right)+1\right)}}\right).$$

The sign of $\frac{\partial \log(\varphi_I^*)}{\partial \log(\theta)}$ depends on the level of $\boldsymbol{\rho}$ and the size of the industry given by its position I. Let $\rho_I^* > 0$

$$\rho_I^* = \left\{\rho_I \middle| \kappa_I\left(\varphi_I^*\right) = -\left[1+\frac{\Omega_I\left(\varphi_I^*\right)\Psi_I\left(\varphi_I^*\right)}{1+\Omega_I\left(\varphi_I^*\right)\Psi_I\left(\varphi_I^*\right)}\left(1-\frac{\Psi_I\left(\varphi_I^*\right)}{1-\varphi_I^*}\right)\right]^{-1}\right\}.$$

If $\rho_I < \rho_I^*$ for all $I \in [N]$, then $B_I\left(\varphi_I^*\right) > \frac{1}{\Psi_I\left(\varphi_I^*\right)\left(\Omega_I\left(\varphi_I^*\right)\Psi_I\left(\varphi_I^*\right)+1\right)}$ and $C_I\left(\varphi_I^*\right) > 0$, for all $I \in [N]$. This implies that

$$\frac{C_I\left(\varphi_I^*\right)}{B_I\left(\varphi_I^*\right) - \frac{1}{\Psi_I\left(\varphi_I^*\right)\left(\Omega_I\left(\varphi_I^*\right)\Psi_I\left(\varphi_I^*\right)+1\right)}} \geq 0, \forall I \in [N],$$

and,

$$\frac{\partial \log(\varphi_I^*)}{\partial \log(\theta)} = -\overbrace{\frac{\theta}{\varphi_I^*}}^{>0}\left(\overbrace{B_I\left(\varphi_I^*\right) - \frac{1}{\Psi_I\left(\varphi_I^*\right)\left(\Omega_I\left(\varphi_I^*\right)\Psi_I\left(\varphi_I^*\right)+1\right)}}^{>0}\right)^{-1}\left(\overbrace{1+\sum_{J\in[N]}\frac{C_J\left(\varphi_J^*\right)}{B_J\left(\varphi_J^*\right) - \frac{1}{\Psi_J\left(\varphi_J^*\right)\left(\Omega_J\left(\varphi_J^*\right)\Psi_J\left(\varphi_J^*\right)+1\right)}}}^{>0}\right)^{-1}\cdot$$
$$\left(A_I + \frac{1}{1-\theta}\sum_{J\in[N]}(J-I)\overbrace{\frac{C_J\left(\varphi_J^*\right)}{B_J\left(\varphi_J^*\right) - \frac{1}{\Psi_J\left(\varphi_J^*\right)\left(\Omega_J\left(\varphi_J^*\right)\Psi_J\left(\varphi_J^*\right)+1\right)}}}^{\geq 0}\right).$$

Hence, the sign of $\frac{\partial \log(\varphi_I^*)}{\partial \log(\theta)}$ ultimately depends on the term A_I. We have $A_1 \geq 0 \geq A_N$, which implies

$$\frac{\partial \log(\varphi_1^*)}{\partial \log(\theta)} \leq 0 \leq \frac{\partial \log(\varphi_N^*)}{\partial \log(\theta)}.$$

We can rewrite leverage in equation 29, as follows

$$\text{Lev}_I(\theta,\delta) = \frac{(1-\tau_I)\Omega_I\left(\varphi_I^*\right)\theta_I}{\tau_I\lambda_I\left(\varphi_I^*\right)}\frac{\Pi_{t+1}(\theta,\delta)}{K_t}, \tag{73}$$

where

$$\Pi_{t+1}(\theta,\delta) = \exp\Bigg\{\delta - \boldsymbol{\theta}'\log(\boldsymbol{\theta}) + \boldsymbol{\theta}'\log(\mathbf{1}-\boldsymbol{\beta}) + \log(\overline{P}) + \boldsymbol{\theta}'\boldsymbol{\mu}t + \boldsymbol{\theta}'\boldsymbol{\sigma}\varepsilon_t + \boldsymbol{\theta}'\log(\boldsymbol{\Xi}\left(\boldsymbol{\varphi}^*\right)) + \sum_{I\in[N]}\frac{\gamma_I}{1-\beta_I}\theta_I\log(\lambda_I\left(\varphi_I^*\right)) + \sum_{I\in[N]}\frac{\gamma_I}{1-\beta_I}\theta_I\log(K_t)\Bigg\}, \tag{74}$$

and

$$\lambda_I\left(\varphi_I^*\right) := \frac{\frac{\gamma_I}{1-\beta_I}\theta_I^2(1+\Omega_I\left(\varphi_I^*\right)\Psi_I\left(\varphi_I^*\right))}{\sum_{J\in[N]}\frac{\gamma_J}{1-\beta_J}\theta_J^2(1+\Omega_J\left(\varphi_J^*\right)\Psi_J\left(\varphi_J^*\right))}.$$

Finally, we will proceed to determine $\frac{\partial \mathrm{Lev}_I(\theta,\delta)}{\partial \log(\theta)}$. We can rewrite $\mathrm{Lev}_I(\theta,\delta)$ as follows:

$$\log(\mathrm{Lev}_I(\theta,\delta)) = \log(\theta_I) + \log(1-\tau_I) + \log(\Omega_I(\varphi_I^*)) + \log(\Pi_{t+1}(\theta,\delta)) - \log(\tau_I) - \log(\lambda_I(\varphi_I^*)) - \log(K_t).$$

To simplify the analysis, we will find $\frac{\partial \log(\mathrm{Lev}_I(\theta,\delta))}{\partial \log(\theta)} - \frac{\partial \log(\mathrm{Lev}(\theta,\delta))}{\partial \log(\theta)}$

$$\begin{aligned}\frac{\partial \log(\mathrm{Lev}(\theta,\delta))}{\partial \log(\theta)} - \frac{\partial \log(\mathrm{Lev}_J(\theta,\delta))}{\partial \log(\theta)} &= \frac{\partial \log(\theta_I)}{\partial \log(\theta)} - \frac{\partial \log(\theta_J)}{\partial \log(\theta)} + \\ &\theta\Big(\frac{1}{\Omega_I(\varphi_I^*)}\frac{\partial \Omega_I(\varphi_I^*)}{\partial\theta} - \frac{1}{\Omega_J(\varphi_J^*)}\frac{\partial \Omega_J(\varphi_J^*)}{\partial\theta} - \frac{1}{\lambda_I(\varphi_I^*)}\frac{\partial \lambda_I(\varphi_I^*)}{\partial\theta} + \frac{1}{\lambda_J(\varphi_J^*)}\frac{\partial \lambda_J(\varphi_J^*)}{\partial\theta}\Big) \\ &= (J-I)\frac{\theta}{1-\theta} + \\ &\theta\Big(\frac{1}{\Psi_I(\varphi_I^*)}\frac{\partial \varphi_I^*}{\partial\theta} - \frac{1}{\Psi_J(\varphi_J^*)}\frac{\partial \varphi_J^*}{\partial\theta} - \frac{1}{\lambda_I(\varphi_I^*)}\frac{\partial \lambda_I(\varphi_I^*)}{\partial\theta} + \frac{1}{\lambda_J(\varphi_J^*)}\frac{\partial \lambda_J(\varphi_J^*)}{\partial\theta}\Big).\end{aligned}$$

The next step is to determine $\frac{\partial \lambda_I(\varphi_I^*)}{\partial\theta}$

$$\begin{aligned}\frac{1}{\lambda_I(\varphi_I^*)}\frac{\partial \lambda_I(\varphi_I^*)}{\partial\theta} &= \frac{2}{\theta_I}\frac{\partial\theta_I}{\partial\theta} + \frac{1}{1+\Omega_I(\varphi_I^*)\Psi_I(\varphi_I^*)}\Big(\Psi_I(\varphi_I^*)\frac{\partial\Omega_I(\varphi_I^*)}{\partial\theta} + \Omega_I(\varphi_I^*)\frac{\partial\Psi_I(\varphi_I^*)}{\partial\theta}\Big) - \\ &\frac{\sum_{J\in[N]}\frac{\gamma_J}{1-\beta_J}\left[2\theta_J(1+\Omega_J(\varphi_J^*)\Psi_J(\varphi_J^*))\frac{\partial\theta_J}{\partial\theta} + \theta_J^2\left(\Psi_J(\varphi_J^*)\frac{\partial\Omega_J(\varphi_J^*)}{\partial\theta} + \Omega_J(\varphi_J^*)\frac{\partial\Psi_J(\varphi_J^*)}{\partial\theta}\right)\right]}{\sum_{J\in[N]}\frac{\gamma_J}{1-\beta_J}\theta_J^2(1+\Omega_J(\varphi_J^*)\Psi_J(\varphi_J^*))} \\ &= \frac{2}{\theta_I}\frac{\partial\theta_I}{\partial\theta} + \frac{\Omega_I(\varphi_I^*)(1+\kappa_I(\varphi_I^*))}{1+\Omega_I(\varphi_I^*)\Psi_I(\varphi_I^*)}\frac{\partial\varphi_I^*}{\partial\theta} - \\ &\frac{\sum_{J\in[N]}\frac{\gamma_J}{1-\beta_J}\theta_J^2(1+\Omega_J(\varphi_J^*)\Psi_J(\varphi_J^*))\left[\frac{2}{\theta_J}\frac{\partial\theta_J}{\partial\theta} + \frac{\Omega_J(\varphi_J^*)(1+\kappa_J(\varphi_J^*))}{1+\Omega_J(\varphi_J^*)\Psi_J(\varphi_J^*)}\frac{\partial\varphi_J^*}{\partial\theta}\right]}{\sum_{J\in[N]}\frac{\gamma_J}{1-\beta_J}\theta_J^2(1+\Omega_J(\varphi_J^*)\Psi_J(\varphi_J^*))}.\end{aligned}$$

This allows us to write $\frac{\partial \log(\mathrm{Lev}_I(\theta,\delta))}{\partial \log(\theta)} - \frac{\partial \log(\mathrm{Lev}_J(\theta,\delta))}{\partial \log(\theta)}$ as

$$\begin{aligned}\frac{\partial \log(\mathrm{Lev}_I(\theta,\delta))}{\partial \log(\theta)} - \frac{\partial \log(\mathrm{Lev}_J(\theta,\delta))}{\partial \log(\theta)} &= (I-J)\frac{\theta}{1-\theta} + \frac{1-\Omega_I(\varphi_I^*)\Psi_I(\varphi_I^*)\kappa_I(\varphi_I^*)}{\Psi_I(\varphi_I^*)(1+\Omega_I(\varphi_I^*)\Psi_I(\varphi_I^*))}\varphi_I^*\frac{\partial\log(\varphi_I^*)}{\partial\log(\theta)} - \\ &\frac{1-\Omega_J(\varphi_J^*)\Psi_J(\varphi_J^*)\kappa_J(\varphi_J^*)}{\Psi_J(\varphi_J^*)(1+\Omega_J(\varphi_J^*)\Psi_J(\varphi_J^*))}\varphi_J^*\frac{\partial\log(\varphi_J^*)}{\partial\log(\theta)}.\end{aligned}$$

If $\rho_I < \rho_I^*$ for all $I \in [N]$, then we have

$$\frac{\partial \log(\mathrm{Lev}_1(\theta,\delta))}{\partial \log(\theta)} \le 0 \le \frac{\partial \log(\mathrm{Lev}_N(\theta,\delta))}{\partial \log(\theta)}.$$

Finally, controlling for the concentration level, we have the following sensitivity for leverage with respect to the sparsity term:

$$\frac{\partial \log(\mathrm{Lev}_I(\theta,\delta))}{\partial\delta} = \frac{\partial \log(\Pi_{t+1}(\theta,\delta))}{\partial\delta} = 1 > 0.$$

Proof of Proposition 5

Given that leverage is held constant, equation 18 can be rewritten depending on the type of unanticipated shock as:

$$\varphi_I^{(\eta)} = \Phi\left(\frac{1}{\eta_I^{(\eta)}}\left[\eta_I\Phi^{-1}(\varphi_I) - \frac{1}{2}\eta_I^2 - \log\left(\frac{\Xi_I(\varphi_I)}{\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)}\right) + \sum_{J\in[N]}\theta_J\log\left(\frac{\Xi_J(\varphi_J)}{\Xi_J^{(\eta)}\left(\varphi_J^{(\eta)}\right)}\right)\right] + \frac{1}{2}\eta_I^{(\eta)}\right), \quad (75)$$

$$\varphi_I^{(\rho)} = \Phi\left(\frac{1}{\eta_I}\left[\eta_I\Phi^{-1}(\varphi_I) + \log\left(\frac{\rho_I^{(\rho)}}{\rho_I}\right) - \log\left(\frac{\Xi_I(\varphi_I)}{\Xi_I\left(\varphi_I^{(\rho)}\right)}\right) + \sum_{J\in[N]}\theta_J\log\left(\frac{\Xi_J(\varphi_J)}{\Xi_J\left(\varphi_J^{(\rho)}\right)}\right)\right]\right), \quad (76)$$

$$\varphi_I^{(\chi)} = \Phi\left(\frac{1}{\eta_I}\left[\eta_I\Phi^{-1}\left(\varphi_I\right) + \log\left(\frac{\Xi_I^{(\chi)}\left(\varphi_I^{(\chi)}\right)}{\Xi_I\left(\varphi_I\right)}\right) - \sum_{J\in[N]}\theta_J\log\left(\frac{\Xi_J^{(\chi)}\left(\varphi_J^{(\chi)}\right)}{\Xi_J\left(\varphi_J\right)}\right)\right]\right), \tag{77}$$

where

$$\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right) = 1 - \left(1 - \chi_I^{\frac{1}{1-\beta_I}}\right)\Phi\left(\Phi^{-1}\left(\varphi_I^{(\eta)}\right) - \eta_I^{(\eta)}\right), \tag{78}$$

and

$$\Xi_I^{(\chi)}\left(\varphi_I^{(\chi)}\right) = 1 - \left(1 - \chi_I^{(\chi)\frac{1}{1-\beta_I}}\right)\Phi\left(\Phi^{-1}\left(\varphi_I^{(\chi)}\right) - \eta_I\right). \tag{79}$$

In the previous equations the superscript denotes the parameter that suffered the unanticipated shock. From these expressions we conclude

$$\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right) = 1 - \left(1 - \chi_I^{\frac{1}{1-\beta_I}}\right)\cdot \Phi\left(\frac{1}{\eta_I^{(\eta)}}\left[\eta_I\Phi^{-1}\left(\varphi_I\right) - \frac{1}{2}\left(\eta_I^2 + \eta_I^{(\eta)2}\right) - \log\left(\frac{\Xi_I\left(\varphi_I\right)}{\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)}\right) + \sum_{J\in[N]}\theta_J\log\left(\frac{\Xi_J\left(\varphi_J\right)}{\Xi_J^{(\eta)}\left(\varphi_J^{(\eta)}\right)}\right)\right]\right), \tag{80}$$

$$\Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right) = 1 - \left(1 - \chi_I^{\frac{1}{1-\beta_I}}\right)\cdot \Phi\left(\frac{1}{\eta_I}\left[\eta_I\Phi^{-1}\left(\varphi_I\right) - \eta_I^2 + \log\left(\frac{\rho_I^{(\rho)}}{\rho_I}\right) - \log\left(\frac{\Xi_I\left(\varphi_I\right)}{\Xi_I\left(\varphi_I^{(\rho)}\right)}\right) + \sum_{J\in[N]}\theta_J\log\left(\frac{\Xi_J\left(\varphi_J\right)}{\Xi_J\left(\varphi_J^{(\rho)}\right)}\right)\right]\right), \tag{81}$$

and

$$\Xi_I^{(\chi)}\left(\varphi_I^{(\chi)}\right) = 1 - \left(1 - \chi_I^{(\chi)\frac{1}{1-\beta_I}}\right)\Phi\left(\frac{1}{\eta_I}\left[\eta_I\Phi^{-1}\left(\varphi_I\right) - \eta_I^2 + \log\left(\frac{\Xi_I^{(\chi)}\left(\varphi_I^{(\chi)}\right)}{\Xi_I\left(\varphi_I\right)}\right) - \sum_{J\in[N]}\theta_J\log\left(\frac{\Xi_J^{(\chi)}\left(\varphi_J^{(\chi)}\right)}{\Xi_J\left(\varphi_J\right)}\right)\right]\right). \tag{82}$$

Equations 80, 81 and 82 constitute each a system of fixed point equations, in which the functions $\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)$, $\Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)$ and $\Xi_I^{(\chi)}\left(\varphi_I^{(\chi)}\right)$ denote the mean fraction of production that is retained after the unanticipated shock in the parameters η_I, ρ_I and χ_I considering possible defaults. The parameters $\eta_I^{(\eta)}$, $\rho_I^{(\rho)}$ and $\chi_I^{(\chi)}$ corresponds to the shocked values for the idiosyncratic volatilty, and default threshold and recovery parameters for industry I. We can interpret the ratios $\frac{\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)}{\Xi_I(\varphi_I)}$, $\frac{\Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)}{\Xi_I(\varphi_I)}$ and $\frac{\Xi_I^{(\chi)}\left(\varphi_I^{(\chi)}\right)}{\Xi_I(\varphi_I)}$ as the relative recovery after default, with smaller ratios indicating a greater loss. If there is no surprise in the recovery, then we can see that $\varphi_I^{(\eta)} = \varphi_I^{(\rho)} = \varphi_I^{(\chi)} = \varphi_I$.

Before proving the results, we can conclude from the functional forms for $\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)$, $\Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)$ and $\Xi_I^{(\chi)}\left(\varphi_I^{(\chi)}\right)$ that

$$\frac{\partial \Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)}{\partial \varphi_I^{(\eta)}} < 0,$$

$$\frac{\partial \Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)}{\partial \varphi_I^{(\rho)}} < 0,$$

$$\frac{\partial \Xi_I^{(\chi)}\left(\varphi_I^{(\chi)}\right)}{\partial \varphi_I^{(\chi)}} < 0.$$

We will begin by deriving the elasticity results for unanticipated shocks to η_I. We start by finding the partial derivative of $\log\left(\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)\right)$ in equation 80 with respect to $\eta_J^{(\eta)}$:

$$\begin{aligned}\frac{\partial \log\left(\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)\right)}{\partial \eta_J^{(\eta)}} = {} & \frac{1}{\eta_I^{(\eta)}\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)}\left(1-\chi_I^{\frac{1}{1-\beta_I}}\right)\phi\left(\Phi^{-1}\left(\frac{1-\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)}{1-\chi_I^{\frac{1}{1-\beta_I}}}\right)\right)\cdot \\ & \Bigg(-\frac{\partial \log\left(\Xi_J^{(\eta)}\left(\varphi_J^{(\eta)}\right)\right)}{\partial \eta_J^{(\eta)}} + \sum_{K\in[N]}\theta_K\frac{\partial \log\left(\Xi_K^{(\eta)}\left(\varphi_K^{(\eta)}\right)\right)}{\partial \eta_J^{(\eta)}} + \frac{\mathbb{1}_{\{I=J\}}}{\eta_J^{(\eta)}}\cdot \\ & \left[\eta_J\Phi^{-1}(\varphi_I) - \frac{1}{2}\eta_J^2 - \log\left(\frac{\Xi_J\left(\varphi_J\right)}{\Xi_J^{(\eta)}\left(\varphi_J^{(\eta)}\right)}\right) + \sum_{K\in[N]}\theta_K\log\left(\frac{\Xi_K\left(\varphi_K\right)}{\Xi_K^{(\eta)}\left(\varphi_K^{(\eta)}\right)}\right) + \frac{1}{2}\eta_J^{(\eta)2}\right]\Bigg).\end{aligned}$$

The next step is to solve for $\frac{\partial \log\left(\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)\right)}{\partial \eta_J^{(\eta)}}$

$$\begin{aligned}\frac{\partial \log\left(\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)\right)}{\partial \eta_J^{(\eta)}} = {} & \Lambda_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)\sum_{K\in[N]}\theta_K\frac{\partial \log\left(\Xi_K^{(\eta)}\left(\varphi_K^{(\eta)}\right)\right)}{\partial \eta_J^{(\eta)}} + \frac{\Lambda_J^{(\eta)}\left(\varphi_J^{(\eta)}\right)\mathbb{1}_{\{I=J\}}}{\eta_J^{(\eta)}}\cdot \\ & \left[\eta_J\Phi^{-1}(\varphi_I) - \frac{1}{2}\eta_J^2 - \log\left(\frac{\Xi_J\left(\varphi_J\right)}{\Xi_J^{(\eta)}\left(\varphi_J^{(\eta)}\right)}\right) + \sum_{K\in[N]}\theta_K\log\left(\frac{\Xi_K\left(\varphi_K\right)}{\Xi_K^{(\eta)}\left(\varphi_K^{(\eta)}\right)}\right) + \frac{1}{2}\eta_J^{(\eta)2}\right],\end{aligned}$$

where

$$\Lambda_I^{(\eta)}\left(\varphi_I^{(\eta)}\right) := \left(1 + \frac{\eta_I^{(\eta)}\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)}{\left(1-\chi_I^{\frac{1}{1-\beta_I}}\right)\phi\left(\Phi^{-1}\left(\frac{1-\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)}{1-\chi_I^{\frac{1}{1-\beta_I}}}\right)\right)}\right)^{-1}.$$

The next step is to multiply by θ_I both sides of the equation, sum through $I\in[N]$ and solve for $\sum_{I\in[N]}\theta_I\frac{\partial \log\left(\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)\right)}{\partial \eta_J^{(\eta)}}$

$$\sum_{I\in[N]} \theta_I \frac{\partial \log\left(\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)\right)}{\partial \eta_J^{(s)}} = -\frac{\theta_J \frac{\Lambda_J^{(\eta)}\left(\varphi_J^{(\eta)}\right)}{\eta_J^{(\eta)}}}{1-\sum_{I\in[N]}\theta_I \Lambda_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)} \cdot$$

$$\left[-\eta_J \Phi^{-1}(\varphi_I) + \frac{1}{2}\eta_J^2 + \log\left(\frac{\Xi_J\left(\varphi_J\right)}{\Xi_J^{(\eta)}\left(\varphi_J^{(\eta)}\right)}\right) - \sum_{K\in[N]} \theta_K \log\left(\frac{\Xi_K\left(\varphi_K\right)}{\Xi_K^{(\eta)}\left(\varphi_K^{(\eta)}\right)}\right) - \frac{1}{2}\eta_J^{(\eta)2}\right].$$

Replacing this result into the expression for $\frac{\partial \log\left(\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)\right)}{\partial \eta_J^{(\eta)}}$ we obtain the expression

$$\frac{\partial \log\left(\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)\right)}{\partial \eta_J^{(\eta)}} = -\frac{\Lambda_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)}{\eta_J^{(\eta)}}\left(\frac{\theta_J \Lambda_J^{(\eta)}\left(\varphi_J^{(\eta)}\right)}{1-\sum_{I\in[N]}\theta_I \Lambda_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)} + \mathbb{1}_{\{I=J\}}\right)\cdot$$

$$\left[-\eta_J \Phi^{-1}(\varphi_I) + \frac{1}{2}\eta_J^2 + \log\left(\frac{\Xi_J\left(\varphi_J\right)}{\Xi_J^{(\eta)}\left(\varphi_J^{(\eta)}\right)}\right) - \sum_{K\in[N]} \theta_K \log\left(\frac{\Xi_K\left(\varphi_K\right)}{\Xi_K^{(\eta)}\left(\varphi_K^{(\eta)}\right)}\right) - \frac{1}{2}\eta_J^{(\eta)2}\right]$$

This can be restated as

$$\frac{\partial \log\left(\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)\right)}{\partial \log\left(\eta_J^{(\eta)}\right)} = -\Lambda_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)\left(\frac{\theta_J \Lambda_J^{(\eta)}\left(\varphi_J^{(\eta)}\right)}{1-\sum_{I\in[N]}\theta_I \Lambda_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)} + \mathbb{1}_{\{I=J\}}\right)\cdot$$

$$\left[-\eta_J \Phi^{-1}(\varphi_I) - \frac{1}{2}\left(\eta_J^{(\eta)2} - \eta_J^2\right) + \log\left(\frac{\Xi_J\left(\varphi_J\right)}{\Xi_J^{(\eta)}\left(\varphi_J^{(\eta)}\right)}\right) - \sum_{K\in[N]} \theta_K \log\left(\frac{\Xi_K\left(\varphi_K\right)}{\Xi_K^{(\eta)}\left(\varphi_K^{(\eta)}\right)}\right)\right].$$

Let

$$\eta_J^* = \left\{\eta_J^{(\eta)}\middle| -\eta_J \Phi^{-1}(\varphi_I) - \frac{1}{2}\left(\eta_J^{(\eta)2} - \eta_J^2\right) + \log\left(\frac{\Xi_J\left(\varphi_J\right)}{\Xi_J^{(\eta)}\left(\varphi_J^{(\eta)}\right)}\right) - \sum_{K\in[N]} \theta_K \log\left(\frac{\Xi_K\left(\varphi_K\right)}{\Xi_K^{(\eta)}\left(\varphi_K^{(\eta)}\right)}\right) = 0\right\}.$$

The sign of the sensitivity depends on the last term inside the square brackets. Hence, we obtain two different conditions

$$\frac{\partial \log\left(\Xi_I^{(\eta)}\left(\varphi_I^{(\eta)}\right)\right)}{\partial \log\left(\eta_J^{(\eta)}\right)} = \begin{cases} \leq 0 & \eta_J^{(\eta)} \leq \eta_J^* \\ > 0 & \eta_J^{(\eta)} > \eta_J^* \end{cases}.$$

This leads to the following result:

$$\frac{\partial \log\left(\varphi_I^{(\eta)}\right)}{\partial \log\left(\eta_J^{(\eta)}\right)} = \begin{cases} \geq 0 & \eta_J^{(\eta)} \leq \eta_J^* \\ < 0 & \eta_J^{(\eta)} > \eta_J^* \end{cases}.$$

Next we derive the results for unanticipated shocks to ρ_I. The partial derivative of $\log\left(\xi_I^{(\rho)}\right)$ in equation 81 with respect to $\rho_J^{(\rho)}$ corresponds to

$$\frac{\partial \log\left(\Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)\right)}{\partial \rho_J^{(\rho)}} = \frac{1}{\eta_I \Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)}\left(1-\chi_I^{\frac{1}{1-\beta_I}}\right)\phi\left(\Phi^{-1}\left(\frac{1-\Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)}{1-\chi_I^{\frac{1}{1-\beta_I}}}\right)\right)\cdot$$

$$\left(-\frac{1}{\rho_I^{(\rho)}}\mathbb{1}_{\{I=J\}} - \frac{\partial \log\left(\Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)\right)}{\partial \rho_J^{(\rho)}} + \sum_{K\in[N]}\theta_K \frac{\partial \log\left(\Xi_K^{(\rho)}\left(\varphi_K^{(\rho)}\right)\right)}{\partial \eta_J^{(\rho)}}\right).$$

The following step is to solve for $\frac{\partial \log\left(\Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)\right)}{\partial \rho_J^{(\rho)}}$

$$\frac{\partial \log\left(\Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)\right)}{\partial \rho_J^{(\rho)}} = \Lambda_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)\Big(-\frac{1}{\rho_I^{(\rho)}}\mathbb{1}_{\{I=J\}} + \sum_{K\in[N]}\theta_K \frac{\partial \log\left(\Xi_K^{(\rho)}\left(\varphi_K^{(\rho)}\right)\right)}{\partial \eta_J^{(\rho)}}\Big),$$

where

$$\Lambda_I^{(\rho)}\left(\varphi_I^{(\rho)}\right) := \left(1 + \frac{\eta_I \Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)}{\left(1-\chi_I^{\frac{1}{1-\beta_I}}\right)\phi\left(\Phi^{-1}\left(\frac{1-\Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)}{1-\chi_I^{\frac{1}{1-\beta_I}}}\right)\right)}\right)^{-1}.$$

Analogous to the previous derivation, we need to multiply by θ_I both sides of the equation, sum through $I \in [N]$ and solve for $\sum_{I\in[N]}\theta_I \frac{\partial \log\left(\Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)\right)}{\partial \rho_J^{(\rho)}}$

$$\sum_{I\in[N]}\theta_I \frac{\partial \log\left(\Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)\right)}{\partial \rho_J^{(\rho)}} = -\frac{1}{1-\sum_{I\in[N]}\theta_I \Lambda_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)} \frac{\theta_J \Lambda_J^{(\rho)}\left(\varphi_J^{(\rho)}\right)}{\rho_J^{(\rho)}}.$$

Replacing this result into the expression for $\frac{\partial \log\left(\Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)\right)}{\partial \rho_J^{(\rho)}}$ we obtain

$$\frac{\partial \log\left(\Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)\right)}{\partial \rho_J^{(\rho)}} = -\frac{\Lambda_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)}{\rho_J^{(\rho)}}\left(\frac{\theta_J \Lambda_J^{(\rho)}\left(\varphi_J^{(\rho)}\right)}{1-\sum_{I\in[N]}\theta_I \Lambda_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)} + \mathbb{1}_{\{I=J\}}\right).$$

This is equivalent to

$$\frac{\partial \log\left(\Xi_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)\right)}{\partial \log\left(\rho_J^{(\rho)}\right)} = -\Lambda_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)\left(\frac{\theta_J \Lambda_J^{(\rho)}\left(\varphi_J^{(\rho)}\right)}{1-\sum_{I\in[N]}\theta_I \Lambda_I^{(\rho)}\left(\varphi_I^{(\rho)}\right)} + \mathbb{1}_{\{I=J\}}\right).$$

We conclude that

$$\frac{\partial \log\left(\Xi_I^{(\chi)}\left(\varphi_I^{(\chi)}\right)\right)}{\partial \log\left(\chi_J^{(\chi)}\right)} \leq 0.$$

Which is equivalent to

$$\frac{\partial \log\left(\varphi_I^{(x)}\right)}{\partial \log\left(\chi_J^{(x)}\right)} \geq 0.$$

Finally, we will derive the results for unanticipated shocks to χ_I. The partial derivative of $\log\left(\Xi_I^{(x)}\left(\varphi_I^{(x)}\right)\right)$ in equation 82 with respect to $\chi_J^{(x)}$ is given by

$$\frac{\partial \log\left(\Xi_I^{(x)}\left(\varphi_I^{(x)}\right)\right)}{\partial \chi_J^{(x)}} = \frac{1}{\Xi_I^{(x)}\left(\varphi_I^{(x)}\right)} \frac{1}{1-\beta_I} \chi_I^{(x)\frac{\beta_I}{1-\beta_I}} \left(\frac{1-\Xi_I^{(x)}\left(\varphi_I^{(x)}\right)}{1-\chi_I^{(x)\frac{1}{1-\beta_I}}}\right) \mathbb{1}_{\{I=J\}} - \frac{1}{\Xi_I^{(x)}\left(\varphi_I^{(x)}\right)} \left(1-\chi_I^{(x)\frac{1}{1-\beta_I}}\right) \phi\left(\Phi^{-1}\left(\frac{1-\Xi_I^{(x)}\left(\varphi_I^{(x)}\right)}{1-\chi_I^{(x)\frac{1}{1-\beta_I}}}\right)\right) \frac{1}{\eta_I} \cdot \left(\frac{\partial \log\left(\Xi_I^{(x)}\left(\varphi_I^{(x)}\right)\right)}{\partial \chi_J^{(x)}} - \sum_{K\in[N]} \theta_K \frac{\partial \log\left(\Xi_K^{(x)}\left(\varphi_K^{(x)}\right)\right)}{\partial \chi_J^{(x)}}\right).$$

The following step is to solve for $\frac{\partial \log\left(\Xi_I^{(x)}\left(\varphi_I^{(x)}\right)\right)}{\partial \chi_J^{(x)}}$

$$\frac{\partial \log\left(\Xi_I^{(x)}\left(\varphi_I^{(x)}\right)\right)}{\partial \chi_J^{(x)}} = \Lambda_I^{(x)}\left(\varphi_I^{(x)}\right) \cdot \left[\frac{\eta_I}{1-\beta_I} \frac{\chi_I^{(x)\frac{\beta_I}{1-\beta_I}}}{1-\chi_I^{(x)\frac{1}{1-\beta_I}}} \frac{\frac{1-\Xi_I^{(x)}\left(\varphi_I^{(x)}\right)}{1-\chi_I^{(x)\frac{1}{1-\beta_I}}}}{\phi\left(\Phi^{-1}\left(\frac{1-\Xi_I^{(x)}\left(\varphi_I^{(x)}\right)}{1-\chi_I^{(x)\frac{1}{1-\beta_I}}}\right)\right)} \mathbb{1}_{\{I=J\}} + \sum_{K\in[N]} \theta_K \frac{\partial \log\left(\Xi_K^{(x)}\left(\varphi_K^{(x)}\right)\right)}{\partial \chi_J^{(x)}}\right],$$

where

$$\Lambda_I^{(x)}\left(\varphi_I^{(x)}\right) := \left(1 + \frac{\eta_I \Xi_I^{(x)}\left(\varphi_I^{(x)}\right)}{\left(1-\chi_I^{(x)\frac{1}{1-\beta_I}}\right)\phi\left(\Phi^{-1}\left(\frac{1-\Xi_I^{(x)}\left(\varphi_I^{(x)}\right)}{1-\chi_I^{(x)\frac{1}{1-\beta_I}}}\right)\right)}\right)^{-1}.$$

Next, we proceed to multiply by θ_I both sides of the equation, sum through $I \in [N]$ and solve for $\sum_{I\in[N]} \theta_I \frac{\partial \log\left(\Xi_I^{(x)}\left(\varphi_I^{(x)}\right)\right)}{\partial \chi_J^{(x)}}$

$$\sum_{I\in[N]} \theta_I \frac{\partial \log\left(\Xi_I^{(x)}\left(\varphi_I^{(x)}\right)\right)}{\partial \chi_J^{(x)}} = \frac{\theta_J \Lambda_J^{(x)}\left(\varphi_J^{(x)}\right)}{1-\sum_{I\in[N]} \theta_I \Lambda_I^{(x)}\left(\varphi_I^{(x)}\right)} \frac{\eta_J}{1-\beta_J} \frac{\chi_J^{(x)\frac{\beta_J}{1-\beta_J}}}{1-\chi_J^{(x)\frac{1}{1-\beta_J}}} \frac{\frac{1-\Xi_J^{(x)}\left(\varphi_J^{(x)}\right)}{1-\chi_J^{(x)\frac{1}{1-\beta_J}}}}{\phi\left(\Phi^{-1}\left(\frac{1-\Xi_J^{(x)}\left(\varphi_J^{(x)}\right)}{1-\chi_J^{(x)\frac{1}{1-\beta_J}}}\right)\right)}.$$

Replacing this result into the expression for $\frac{\partial \log\left(\Xi_I^{(x)}\left(\varphi_I^{(x)}\right)\right)}{\partial \chi_J^{(x)}}$ we obtain

$$\frac{\partial \log\left(\Xi_I^{(\chi)}\left(\varphi_I^{(\chi)}\right)\right)}{\partial \chi_J^{(\chi)}} = \Lambda_I^{(\chi)}\left(\varphi_I^{(\chi)}\right)\frac{\eta_J}{1-\beta_J}\frac{\chi_J^{(\chi)\frac{\beta_J}{1-\beta_J}}}{1-\chi_J^{(\chi)\frac{1}{1-\beta_J}}}\frac{\frac{1-\Xi_J^{(\chi)}\left(\varphi_J^{(\chi)}\right)}{1-\chi_J^{(\chi)\frac{1}{1-\beta_J}}}}{\phi\left(\Phi^{-1}\left(\frac{1-\Xi_J^{(\chi)}\left(\varphi_J^{(\chi)}\right)}{1-\chi_J^{(\chi)\frac{1}{1-\beta_J}}}\right)\right)}\cdot \left(\frac{\theta_J\Lambda_J^{(\chi)}\left(\varphi_J^{(\chi)}\right)}{1-\sum_{I\in[N]}\theta_I\Lambda_I^{(\chi)}\left(\varphi_I^{(\chi)}\right)}+\mathbb{1}_{\{I=J\}}\right).$$

This is equivalent to

$$\frac{\partial \log\left(\Xi_I^{(\chi)}\left(\varphi_I^{(\chi)}\right)\right)}{\partial \log\left(\chi_J^{(\chi)}\right)} = \Lambda_I^{(\chi)}\left(\varphi_I^{(\chi)}\right)\frac{\eta_J}{1-\beta_J}\frac{\chi_J^{(\chi)\frac{1}{1-\beta_J}}}{1-\chi_J^{(\chi)\frac{1}{1-\beta_J}}}\frac{\frac{1-\Xi_J^{(\chi)}\left(\varphi_J^{(\chi)}\right)}{1-\chi_J^{(\chi)\frac{1}{1-\beta_J}}}}{\phi\left(\Phi^{-1}\left(\frac{1-\Xi_J^{(\chi)}\left(\varphi_J^{(\chi)}\right)}{1-\chi_J^{(\chi)\frac{1}{1-\beta_J}}}\right)\right)}\cdot \left(\frac{\theta_J\Lambda_J^{(\chi)}\left(\varphi_J^{(\chi)}\right)}{1-\sum_{I\in[N]}\theta_I\Lambda_I^{(\chi)}\left(\varphi_I^{(\chi)}\right)}+\mathbb{1}_{\{I=J\}}\right).$$

We conclude that

$$\frac{\partial \log\left(\Xi_I^{(\chi)}\left(\varphi_I^{(\chi)}\right)\right)}{\partial \log\left(\chi_J^{(\chi)}\right)} \geq 0.$$

Which is equivalent to

$$\frac{\partial \log\left(\varphi_I^{(\chi)}\right)}{\partial \log\left(\chi_J^{(\chi)}\right)} \leq 0.$$

Appendix B

Complete average input - output matrix for all 55 industries

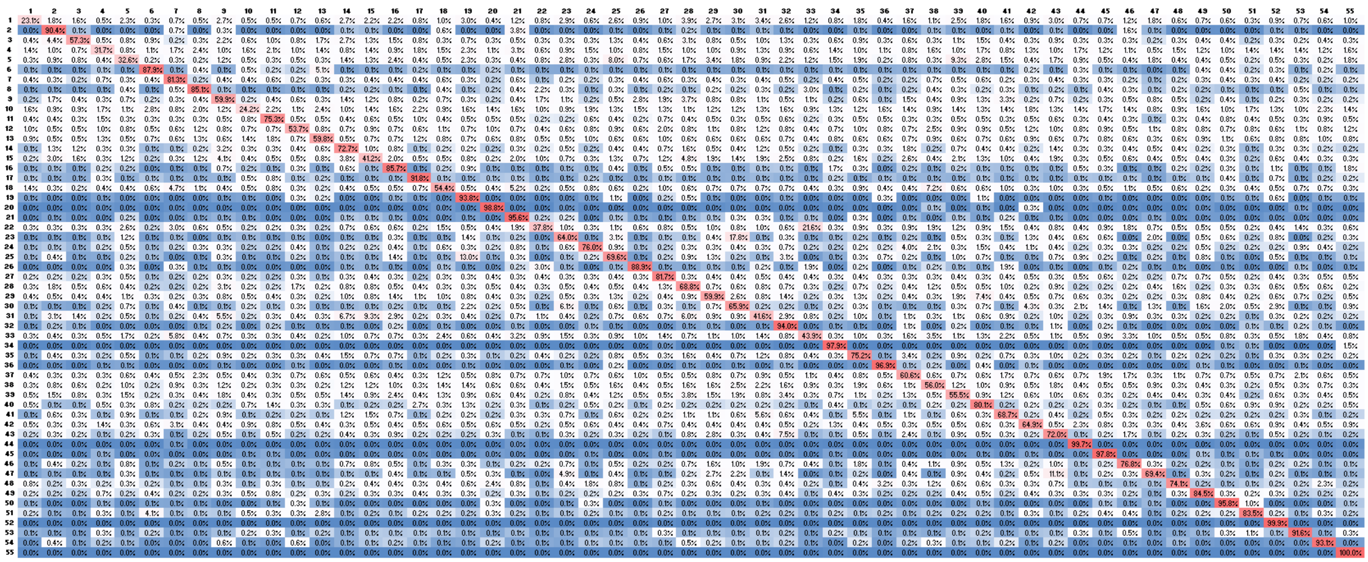

Figure 15: Average input-output matrix

The colors represent a heat map from highest proportion (in red) to lower proportion of inputs (in blue). As we can see, several industries have a concentration of less than 50%,which means that over half of their good is produced with inputs from other industries.

Appendix C

Table 4: Calibrated parameters

Industry	α (%)	β (%)	η (%)	μ (%)	σ (%)	χ (%)	ρ	γ (%)	τ (%)
Wholesale trade	0.00	87.78	23.88	9.40	14.91	33.40	3.99	89.48	0.10
Other transportation equipment	0.42	77.41	21.07	-1.60	10.11	38.93	2.73	18.29	0.84
Computer and electronic products	1.91	58.24	26.00	14.22	8.77	38.93	5.02	12.28	1.93
Administrative and support services	0.64	66.03	24.05	1.16	3.80	38.65	4.49	22.35	0.87
Chemical products	5.94	45.59	27.17	-1.14	3.25	38.93	3.50	5.28	3.44
Motion picture and sound recording industries	0.55	58.83	38.83	-0.52	9.83	34.70	2.85	31.21	0.56
Air transportation	1.40	79.62	53.34	5.96	37.27	38.17	0.87	11.26	0.22
Pipeline transportation	0.00	80.09	35.59	5.71	36.24	38.17	0.97	95.27	0.07
Machinery	0.27	65.56	23.25	-0.35	6.02	38.93	1.80	24.03	0.41
Miscellaneous professional, scientific, and technical services	0.74	71.11	26.66	0.78	4.86	38.65	3.63	37.87	0.51
Data processing, internet publishing, and other information services	0.35	45.60	27.62	2.20	15.45	34.70	4.03	32.23	0.93
Other services, except government	6.73	66.59	19.69	-0.89	4.07	38.65	3.67	5.90	5.85
Broadcasting and telecommunications	3.57	48.70	23.89	1.47	4.55	34.70	2.81	10.85	2.96
Electrical equipment, appliances, and components	0.98	70.15	25.18	0.69	7.28	38.93	3.00	14.13	0.92
Fabricated metal products	0.42	69.97	17.57	-0.64	6.05	38.93	3.70	31.77	0.76
Miscellaneous manufacturing	3.07	45.24	22.25	1.39	3.73	38.93	5.30	5.71	5.13
Publishing industries, except internet (includes software)	1.56	37.69	28.73	2.52	5.27	34.70	5.25	11.93	2.81
Other transportation and support activities	0.20	81.65	23.92	-6.04	13.64	38.17	4.44	36.36	0.50
Apparel and leather and allied products	4.81	55.21	16.76	0.01	5.51	38.93	9.56	1.68	18.73
Food and beverage stores	0.00	73.40	14.99	3.92	4.76	33.40	9.40	10.47	2.55
Water transportation	0.18	71.61	19.73	2.69	11.21	38.17	2.37	25.61	1.09
Oil and gas extraction	0.00	62.78	37.51	5.45	16.38	44.37	2.73	147.34	0.14
Farms	1.55	61.53	30.81	3.84	8.83	38.50	3.93	9.62	2.08
Other retail	0.04	69.69	18.33	3.44	4.18	33.40	8.92	24.72	1.35
Textile mills and textile product mills	0.89	70.68	16.48	1.12	6.34	38.93	2.63	11.85	1.36
Support activities for mining	0.00	73.29	33.84	8.88	26.42	44.37	3.14	50.08	0.34
Waste management and remediation services	0.25	63.90	25.99	-0.50	6.70	38.65	2.36	29.63	0.65
Motor vehicles, bodies and trailers, and parts	4.03	80.19	22.51	2.11	18.27	38.93	0.55	8.64	0.41

Table 4: Continued

Industry	α (%)	β (%)	η (%)	μ (%)	σ (%)	χ (%)	ρ	γ (%)	τ (%)
Paper products	0.57	69.63	16.81	0.33	5.96	38.93	2.40	20.97	0.83
Food and beverage and tobacco products	11.54	61.29	19.12	-0.28	3.28	38.93	3.05	1.89	8.83
Primary metals	0.04	83.75	26.34	3.52	8.95	38.93	2.50	61.53	0.15
Furniture and related products	1.36	65.43	15.04	-0.04	4.96	38.93	8.41	4.74	8.11
Petroleum and coal products	3.99	84.17	30.30	6.22	26.76	38.93	3.35	9.51	0.94
Ambulatory health care services	10.41	75.81	24.54	-1.68	5.40	38.65	3.93	1.07	8.65
Nonmetallic mineral products	0.27	73.46	19.28	1.37	6.44	38.93	3.03	28.89	0.73
General merchandise stores	0.00	73.42	12.06	3.92	4.77	33.40	8.24	19.42	2.13
Construction	0.00	83.41	28.54	-4.00	4.38	38.50	1.61	48.80	0.13
Truck transportation	0.15	86.48	19.40	1.82	8.65	38.17	2.45	29.88	0.34
Plastics and rubber products	0.78	72.07	18.43	1.78	4.36	38.93	3.07	17.25	1.05
Printing and related support activities	0.11	68.34	18.13	2.09	4.37	38.93	4.29	14.57	1.38
Mining, except oil and gas	0.01	67.43	28.63	0.97	10.34	44.37	2.24	59.28	0.34
Food services and drinking places	7.32	73.21	17.11	-1.10	5.06	34.70	4.83	2.83	11.91
Wood products	0.11	77.40	24.90	-1.02	7.34	38.93	2.35	31.24	0.37
Nursing and residential care facilities	2.49	85.95	19.54	-3.97	12.58	38.65	1.34	4.49	1.33
Amusements, gambling, and recreation industries	1.99	58.52	24.24	0.85	4.62	34.70	1.74	10.47	1.95
Rail transportation	0.01	63.57	15.04	3.12	6.20	38.17	3.92	77.41	1.10
Forestry, fishing, and related activities	0.12	57.55	47.56	-0.31	7.14	38.50	0.92	67.71	0.09
Warehousing and storage	0.01	70.89	14.58	5.26	10.39	38.17	2.56	52.98	0.51
Accommodation	1.45	68.20	27.46	0.11	3.52	34.70	1.65	15.83	1.19
Educational services	3.61	52.49	25.43	-0.86	5.14	38.65	4.59	4.82	3.69
Performing arts, spectator sports, museums, and related activities	0.73	69.81	38.27	-1.38	8.76	34.70	2.23	25.03	0.57
Social assistance	2.02	77.45	21.03	0.16	17.51	38.65	1.96	4.72	1.59
Transit and ground passenger transportation	0.42	80.74	36.97	1.20	10.76	38.17	2.05	19.43	0.36
Motor vehicle and parts dealers	0.00	76.36	19.45	4.41	5.36	33.40	1.41	19.57	0.25
Hospitals	10.02	82.54	22.31	-4.43	5.90	38.65	1.76	1.33	4.92

Product Variety and Asset Pricing*

Abstract

We show how product variety affects asset prices in a general-equilibrium model. We analytically characterize the unique equilibrium and estimate the model to match asset pricing and product market moments. The equity premium and risk-free rate can be reconciled for risk aversion levels around 4 and plausible annual discount factors. Our model generates new implications for how product market characteristics influence asset prices. We find that while competition leads to product substitution within industries, product complementarity is observed between industries. The market risk premium decreases with the both the average intra-industry and inter-industry product substitutabilities. We show empirical support for the novel cross-sectional prediction that industry excess returns increase with intra-industry product substitutabilities.

Key Words: product variety, imperfect substitutability, equity premium puzzle

JEL Classification: G11, G12, E21, E23

*This is a joint work with Dr. Ajay Subramanian, Dr. Baozhong Yang and Dr. Florin Bidian.

1 Introduction

Traditional asset pricing models assume that there is a single consumption good (see Duffie (2001), Ljungvist and Sargent (2004), Cochrane (2005) for surveys). In reality, however, there are multiple consumption goods that are only imperfectly substitutable. The incorporation of product variety could lead to a more volatile stochastic discount factor (SDF) that reflects not only the risk in consumption *levels*, but also the *composition risk* of the basket of consumption goods. Moreover, if the set of available consumption goods itself varies endogenously with aggregate shocks to the economy, then the covariation of the SDF with the market return could also be altered significantly. Hence, the incorporation of endogenous product heterogeneity and the imperfect substitutability of consumption goods could significantly impact asset prices.

We develop a parsimonious, analytically tractable general-equilibrium asset pricing model that incorporates agents' preferences for product variety. The number of products produced by the economy and their heterogeneity depend on aggregate economic shocks. We obtain closed-form analytical characterizations of the unique equilibrium and the corresponding SDF. We estimate the model and demonstrate that product variety, indeed, substantially influences asset prices. The observed market risk premium and risk-free rate can be reconciled for moderate relative risk aversion levels around 4 and empirically plausible annual discount factors. We employ our structurally estimated model to derive novel implications for how product market characteristics influence asset prices. The market risk premium *increases* with the average *intra-industry* product substitutability Our theory also generates the novel cross-sectional prediction that industries with higher product substitutabilities have higher excess returns which we verify empirically. In summary, composition risk arising from the imperfect substitutability of multiple consumption goods is, indeed, an important determinant of asset prices and can explain the observed equity premium and risk-free rate for empirically reasonable risk aversion levels and discount factors.

We begin with our baseline model of a discrete-time, single-sector economy with monopolistically competitive firms producing distinct, imperfectly substitutable products. There is a continuum of identical agents with "constant elasticity of substitution" (CES) preferences for the goods. Agents rent capital to firms in each period. At the beginning of each period, the aggregate state of the economy is realized and observed by all market participants. To operate during the period, each firm must supply a fixed amount of capital that depends on the aggregate state (Comin and Gertler (2006) and Jaimovich and Floetotto (2008)). Firms then experience idiosyncratic productivity shocks with a distribution that depends on the aggregate state and make their production decisions for the period.

We derive the unique competitive equilibrium that satisfies free entry and market clearing conditions in each period. The mass of active firms and the aggregate price index—a moment of the distribution of product prices that depends on the product substitutability—are endogenously determined in equilibrium and vary with the aggregate state. With complete markets, the *real SDF* (that values real returns) is the marginal rate of substitution of the representative agent at the aggregate real consumption process analogous to the traditional consumption-based capital asset pricing model (CCAPM). However, in contrast with the CCAPM, where there is a single consumption good (or, alternatively, all goods are perfect substitutes) so that aggregate consumption is the sum of individual good consumptions, the aggregate real consumption in our model is determined by the consumptions of individual goods aggregated using the CES aggregator.

The rationale for the equity premium and risk-free rate puzzles in the standard CCAPM is that the aggregate *real consumption* assumed by the CCAPM—the aggregate *nominal consumption* deflated by the *consumer price index* (CPI)—is not volatile enough and does not covary sufficiently with the market return. In our model, however, the aggregate real consumption is given by the aggregate nominal consumption weighted by the *aggregate price index*, which incorporates the *endogenous* heterogeneity of products as well as the nontrivial elasticity of substitution between them. This potentially increases the volatility of the utility an agent derives from a consumption bundle due to the composition risk of imperfectly substitutable goods. We show that the aggregate price index and, therefore, the SDF can be expressed in terms of the growth in the variety of products in the economy, and the productivity growth of firms. In particular, the volatilities of the aggregate price index and the SDF rise with an increase in product variety and firms' productivity, and their covariations with the market return also become more negative. We show that, if product variety grows over time and is procyclical, the volatility of the SDF and its negative covariation with the market could be high for even moderate levels of the relative risk aversion of the representative agent. Several studies predict that product variety grows over time and, moreover, the growth in product variety is procyclical (Schumpeter (1939), Schmookler (1966), Shleifer (1986)). More recent studies show empirical support for procyclical product variety growth (Axarloglou (2003), Broda and Weinstein (2006)). Therefore, the increase in the volatility of the SDF with product variety, and the fact that the SDF covaries more negatively with the market, together imply a higher equity premium and lower risk-free rate than what can be explained by the traditional CCAPM. In other words, as we alluded to earlier, the market risk premium and risk-free rate reflect not only the risk in consumption levels, but also the composition risk of consumption goods.

We conduct an empirical analysis of the extent to which the incorporation of product heterogeneity is able to reconcile the equity premium and risk-free rate puzzles using the generalized method of moments (GMM) and the representative agent's Euler equations as in Hansen and Singleton (1982). To facilitate the estimation, we show that the SDF in our model can be re-expressed in terms of the growth rates of the average productivity of firms and the aggregate nominal consumption. We employ productivity data over the period 1987-2020 provided by the Bureau of Labor Statistics (BLS) to construct the empirical measure of the SDF. Consistent with the model, we use the capital productivity to measure the average productivity of firms and the average sales-to-earnings ratio to calibrate the product substitutability. We measure aggregate consumption by the expenditure on nondurable personal consumption and services.

Following the previous literature we use the Fama-French 25 and 2 x 3 portfolios defined using Size and Book-to-Market (B/M), and the Fama-French 3 factor portfolios, as well as Fama-French portfolios with lagged instruments such as lagged dividend-price ratio (Campbell and Shiller (1988)), lagged CAY (Lettau and Ludvigson (2001)), lagged productivity growth and lagged product variety growth. Our GMM estimates of the relative risk aversion of the representative agent average about 5 across specifications and the average estimate of the annual discount factor is 0.94. Hence, the incorporation of product heterogeneity, even in the basic model with a single-industry economy, is able to reconcile the observed equity premium and risk-free rate for relative risk aversion levels and discount factors that are much closer to empirical values. We also compare the aggregate price index computed from the data with the CPI. The aggregate price index is much more volatile and its covariance with the market return is significantly more negative, thereby providing empirical support for the reasons why our model generates a higher equity premium and lower risk-free rate for standard risk aversions and discount factors.

In reality, the economy has multiple industries with products of firms within an industry being closer substitutes than those of firms in different sectors. Relatedly, the impact of product market competition is more pronounced among firms operating within the same industry rather than across industries. The distinctions between *intra-industry* and *inter-industry* product substitutabilities, as well as intra-industry competition, are likely to significantly influence the SDF and, thereby, asset prices. Accordingly, we extend the basic model to a multi-industry economy. As in the basic model, there is a unique equilibrium. The SDF has the same form as in the basic model except that the aggregate consumption index incorporates intra-industry and inter-industry product substitutabilities.

Our GMM analysis of the multi-industry model leads to lower relative risk aversion estimates of around 2.6 and average annual discount factors of 0.95. Hence, the extended model does an even better job than the basic model in reconciling the equity premium and risk-free rate for empirically reasonable relative risk aversion levels and discount factors. Although the GMM estimation exercise has the advantage that it does not rely on parametric assumptions, it does not exploit the structure of our fully specified general equilibrium model. We next employ the Simulated Method of Moments (SMM) to structurally estimate the model by matching asset price and product market moments, and exploit the estimated model to generate additional testable empirical predictions. We choose standard parametric specifications for the aggregate consumption process and the productivity distribution of firms. The estimated model matches all relevant moments related to the equity premium, the risk-free rate, and the dispersion of industrial risk premia. The estimated relative risk aversion, annual discount factor and inter-industry product substitutability are approximately 5.97, 0.99 and -1.21, respectively. Interestingly, while goods are substitutes within industries, they are complements between industries. Although the risk aversion parameter is higher than in the multi-industry GMM specification, the results of the structural estimation are consistent with those of the GMM estimation.

The rich specification of the multi-industry model generates a broader set of predictions. The market risk premium decreases with the average *intra-industry* product substitutability, while the risk-free rate remains stable. The results arise from the fact that the SDF's sensitivity to the market return, the SDF's volatility, and its expectation all decline with the average intrasector product substitutability. The market risk premium decreases with the *inter-industry* product substitutability. Indeed, an increase in the inter-industry product substitutability has an analogous but stronger effect on the SDF relative to the intra-industry case, thereby causing the market risk premium to increase. The differing effects of intrasector and intersector product substitutabilities on the market risk premium imply that there are subtle differences between the impacts of product heterogeneity within and across industries. Our analysis also leads to the novel cross-sectional prediction that industries with higher product substitutabilities have higher excess returns. In line with the theory, we use the industrial sales to profit ratio as the empirical proxy for the intra-industry product substitutability and provide empirical support for the prediction.

We contribute to the literature by building a general-equilibrium model that incorporates endogenous product variety and studying its implications for asset prices. A few earlier studies analyze the CCAPM in a multi-good framework. Piazzesi, Schneider, and Tuzel (2007) incorporates a consumption bundle of non-housing consumption and housing services into the CCAPM. Assuming that both components are nonseparable in utility, the authors show that cyclical variation in the housing share raises the expected equity premium, while long-run trends and volatility in the housing share reduce the risk-free rate. Pakoš (2004) and Yogo (2006) focus on a consumption bundle comprising nondurable and durable components and show how the interaction of both components can increase aggregate consumption risk. Aït-Sahalia, Parker, and Yogo (2004) introduces luxury goods into the CCAPM and shows how the covariance of the consumption of luxury goods with equity returns raises the equity premium. In a different vein, Savov (2011) highlights the impact of the mismeasurement of real consumption in empirical analyses of the CCAPM. He shows that real consumption as measured by garbage generation is much more volatile and covaries more strongly with the market than traditional measures of consumption. In the sam light, we highlight the mismeasurement of real consumption in traditional empirical analyses by showing that real consumption should be measured by nominal consumption deflated by the aggregate price index, rather than by the CPI.

A number of studies modify the traditional CCAPM by altering the preferences of the representative agent to incorporate recursive preferences (Epstein and Zin (1989, 1991)) or habit formation (Campbell and Cochrane (1999)). We complement these studies by developing a general-equilibrium model of an economy with multiple, imperfectly substitutable products. We show that the explicit incorporation of preferences for (endogenous) product variety has a quantitatively significant impact on asset prices even with otherwise standard CRRA utility functions for agents. A few recent studies employ different perspectives to examine the impact of investors' preferences for different goods on asset prices. Van Binsbergen (2016) introduces habit formation for individual good varieties ("deep" habit) and studies cross-sectional returns. In his model, the price elasticity of demand differs across goods, leading to cross-sectional variation in expected returns. He shows that firms with high relative product prices have low expected returns. Product variety, however, is fixed in his framework. Opp, Parlour and Walden (2012) focus on the strategic competition between a fixed number of firms in each sector. They do not structurally estimate their model to obtain quantitative implications for asset prices. Loualiche (2021) demonstrates the impact of heterogeneity in industry entry costs and concentration on the cross-sectional variation in sectoral premia. He assumes recursive Epstein-Zin (1989, 1991) and translog preferences and homogeneous productivity across (the four) sectors in a stylized calibrated example. Scanlon (2019) also allows for multiple products, but *assumes* that an agent's utility exogenously depends on variety and quality growth in addition to real consumption, which he identifies with nominal consumption deflated by the CPI. He shows that cyclical product variety growth can influence asset prices, generating not only risk-free rates in line with observed historical averages, but also equity premia of about one-third of the observed values. Corhay, Kung and Schmid (2020) analyze a general-equilibrium model with recursive Epstein-Zin preferences and a single final consumption good that is produced using heterogeneous intermediate goods. The authors show that higher industry markups, which vary across industries because of differing entry costs, are associated with higher expected returns over time and across industries.

We complement the above studies in the following aspects. First, we incorporate *endogenous* product variety, and focus on explaining the equity premium and risk-free rate in addition to cross-sectional returns. As we discussed earlier, the endogenous determination of the number of goods/firms plays an important role in explaining the market equity premium and risk-free rate.

Second, we develop a parsimonious, analytically tractable model with standard CRRA preferences and constant price elasticity of demand across goods within each sector (CES). The closed-form characterizations of the product and asset markets equilibrium greatly facilitate the estimation of the model. The tractability of the model makes it potentially useful as a tool for further explorations into the asset pricing implications of product variety. Third, we carry out a comprehensive estimation of our model in two ways. 1) We use GMM to estimate the model using productivity data from the Bureau of Labor Statistics with the benefit of the approach being that it only employs the expressions of the pricing kernel in our model and does not depend on parametric assumptions. 2) We conduct a full-fledged structural estimation of our model using SMM by matching product market and asset price moments. In both the GMM and SMM estimations of the model, we are able to reconcile the equity premium and risk-free rate for empirically plausible values of the relative risk aversion and discount factor. Moreover, our estimates of the relative risk aversion and discount factor are consistent across the two estimation approaches. Fourth, we exploit our structurally estimated model to derive novel implications for the effects of intersector and intrasector product substitutabilities on asset prices.

Finally, we also differ from the aforementioned studies in that we emphasize the important distinction between the consumer price index (CPI) and the aggregate price index in measuring real consumption. As in Dixit and Stiglitz (1977), the aggregate real consumption—the consumptions of individual goods aggregated using the CES aggregator—is the aggregate nominal consumption weighted by the product-variety-based aggregate price index that differs in general from the CPI. While the CPI is the average price of a specific, *exogenous* basket of commodities, the aggregate price index is a nontrivial moment (that depends on the product substitutability) of the distribution of prices of the *endogenously determined* products in the economy. In other words, real returns are obtained from nominal returns using the aggregate price index (rather than the CPI) as the deflator. The impact of product heterogeneity on asset prices and returns thus stems from the distinction between the exogenously specified CPI and the endogenously determined aggregate price index. Consequently, we do not use the CPI to deflate nominal returns as in the aforementioned studies, but instead employ productivity data to construct our empirical measure of the aggregate price index. The distinction between the CPI and the aggregate price index is intimately tied to the composition risk across consumption goods that is a key feature of our model and plays a quantitatively important role in reconciling the observed equity premium and risk-free rate for plausible risk aversion levels and discount factors.

2 The Basic Model

We first present a basic model of a discrete-time, single-industry economy. The analysis of this model illustrates the intuition for our results in a simple setting. In Section 4, we develop the more general model of a multi-industry economy.

2.1 Model Overview

We consider an economy with heterogeneous, monopolistically competitive firms producing imperfectly substitutable products (Dixit and Stiglitz (1977)). The economy has an infinite horizon with discrete dates, $0, 1, 2, 3, \ldots$ For simplicity, we refer to the period $[t, t+1]$ as period t. There is a continuum of identical agents in each period with measure one. At date 0, each agent is endowed with K units of capital. Capital cannot be consumed so that agents derive no *direct* utility from it, but it can be used to produce multiple consumption goods.

In each period, firms rent capital from agents. At the beginning of a period, the aggregate state of the economy is observed by all market participants. To operate during the period, each firm must supply a *fixed* amount of capital that depends on the aggregate state as in studies such as Comin and Gertler (2006) and Jaimovich and Floetotto (2008). *After* the capital is supplied, firms experience idiosyncratic productivity shocks that are independently and identically distributed across firms with a distribution that depends on the aggregate state. Firms then make their production decisions for the period. Production requires additional *variable* capital that firms rent from agents. If a firm chooses not to produce, its owners can reinvest its capital stock in other active firms in the economy.

An important aspect of the model is that the mass of active firms (and, therefore, the mass of goods) in the economy is *endogenous* and varies with the aggregate state. We assume for simplicity that capital cannot be augmented or depleted so that the output produced by firms is consumed by agents in each period. Economic growth, therefore, arises from growth in the mass and variety

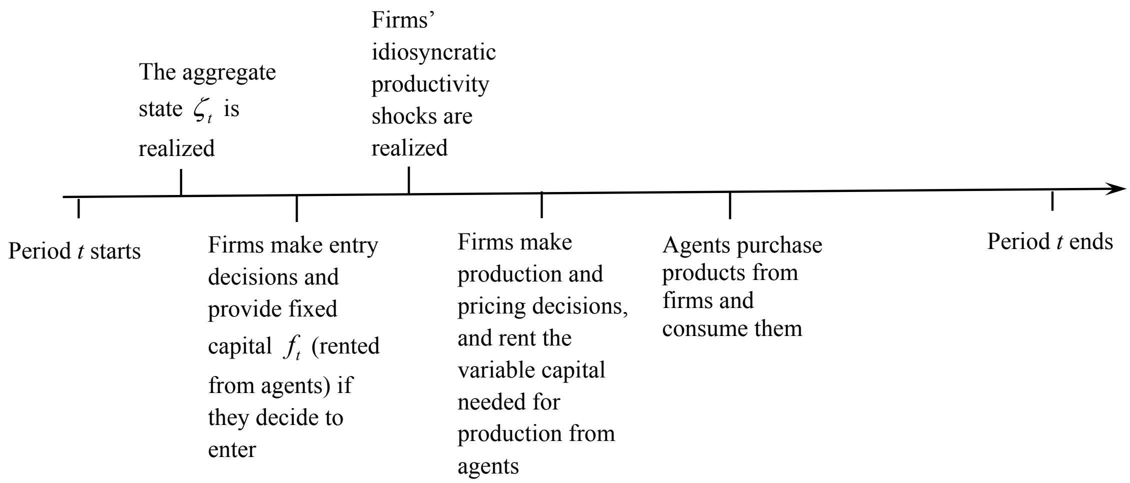

Figure 1: **Model Timeline**

of products in the economy. In Appendix B, we show that our main implications are robust to the incorporation of capital depreciation and investment. In Appendix C, we allow for long-lived firms who incur sunk entry costs rather than fixed costs in each period. Further, firms can experience "death" shocks in any period that force them to exit the market (Melitz (2003), Bilbiie et al (2012)). The extensions do not alter the main results of our analysis.

Figure 1 shows the timeline of events in the model. We now describe the elements of the model in more detail.

2.1.1 Preferences

At any date s, the representative consumer has preferences for consumption of the continuum of goods produced by the economy in future periods that are described by

$$\mathcal{U}_s = \mathbb{E}_s \sum_{t=s}^{\infty} \beta^{t-s} U(\Lambda_t), \tag{1}$$

where $\beta > 0$ is the consumer's time discount factor, and $\mathbb{E}_s[\cdot]$ denotes the expectation conditional on the information available to agents at date s. The function U is strictly increasing and concave and satisfies the Inada conditions, $U'(0) = \infty$ and $U'(\infty) = 0$. In (1),

$$\Lambda_t = \left[\int_{\Omega_t} q_t(\omega_t)^{\rho} d\omega_t \right]^{\frac{1}{\rho}}; \; 0 < \rho < 1, \tag{2}$$

where Ω_t is the set of available goods in the economy in period t, and ω_t is a finite measure on the Borel σ-algebras of Ω_t. The set of available goods, Ω_t, in period t is *endogenous* and could vary over time.

We write all prices in *nominal terms* (dollars). As we discuss in more detail later, this clarifies a central message of our study that the appropriate incorporation of product variety implies that the consumer price index (CPI) is *not* the appropriate numeraire to express prices in *real terms.* Money plays no other role in the economy. If $p_t(\omega_t)$ is the price of good ω_t in period t then, as shown by Dixit and Stiglitz (1977), the optimal consumption and expenditure decisions for individual goods are

$$q_t(\omega_t) = \Lambda_t \left[\frac{p_t(\omega_t)}{P_t}\right]^{-\sigma}, \tag{3}$$

$$\xi_t(\omega_t) = Y_t \left[\frac{p_t(\omega_t)}{P_t}\right]^{1-\sigma} . \; t \in \{0, 1, 2, 3, ...\} . \tag{4}$$

In the above,

$$\sigma = \frac{1}{1-\rho} > 1 \tag{5}$$

is the elasticity of substitution between products or the product substitutability. Y_t is the aggregate expenditure of the representative consumer in period t, and

$$P_t = \left[\int_{\Omega_t} p_t(\omega_t)^{1-\sigma} d\omega_t\right]^{\frac{1}{1-\sigma}} \tag{6}$$

is the *aggregate price index* in period t—a moment of the distribution of product prices—that determines the consumption and expenditure decisions by (3) and (4). The aggregate consumption index, Λ_t, can also be expressed as

$$\Lambda_t = \frac{Y_t}{P_t}. \tag{7}$$

Each active firm produces a single product in which the firm has a monopoly. However, the firms compete monopolistically in the sense that they take the aggregate price index P_t as given in making the output and pricing decisions for their individual products.

2.1.2 Production

In each period, agents rents their capital to firms. At the beginning of period t, the aggregate state of the economy, which is observed by all agents, is $\zeta_t \in (0, \infty)$. Production requires fixed and variable amounts of capital, where the variable capital depends on the quantity of a firm's output. Specifically, after observing the aggregate state ζ_t, each firm rents a fixed amount of capital f_t that can vary with the aggregate state. Firms subsequently experience idiosyncratic productivity shocks that are i.i.d. across firms and drawn from a continuous distribution with density $\mathfrak{g}_t$. The distribution, $\mathfrak{g}_t$, also depends on the aggregate state. To simplify notation, we do not explicitly indicate the dependence of f_t, $\mathfrak{g}_t$ and other variables on the aggregate state.

If a firm with productivity α chooses to produce q units of its good, the amount of variable capital it requires is

$$k = \frac{q}{\alpha}. \tag{8}$$

Let r_t be the capital rental rate in period $[t, t+1]$. As shown by Dixit and Stiglitz (1977), the optimal price set by the firm, as well as its output, revenue and profit (revenue less capital rental costs) are, respectively

$$\text{Price } p_t(\alpha) = \frac{r_t}{\rho\alpha}, \tag{9}$$

$$\text{Output } q_t(\alpha) = Y_t \left(P_t\right)^{\sigma-1} \left(\frac{\rho\alpha}{r_t}\right)^{\sigma}, \tag{10}$$

$$\text{Revenue } \xi_t(\alpha) = p_t(\alpha) q_t(\alpha) = Y_t (P_t \rho\alpha)^{\sigma-1} r_t^{1-\sigma}, \tag{11}$$

$$\text{Profit } \pi_t(\alpha) = \xi_t(\alpha) - r_t k_t(\alpha) = \frac{\xi_t(\alpha)}{\sigma}. \tag{12}$$

2.2 Product Market Equilibrium and Asset Prices

We now derive the product market equilibrium.

2.2.1 Equilibrium Conditions

There is free entry of firms in every period, and each firm is required to supply the fixed capital f_t at the beginning of period t. Note that the fixed capital must be rented *before* firms undertake production for the period. The *free entry equilibrium condition* implies that the expected profit of an active firm must equal the cost of renting the fixed amount of capital f_t:

$$\overbrace{r_t f_t}^{\text{capital rental cost}} = \overbrace{\int_0^\infty \pi_t(\alpha)\mathfrak{g}_t(\alpha)d\alpha}^{\text{expected profit of active firm}} . \tag{13}$$

Second, *product market clearing* requires that the aggregate revenue of producing firms in each period equal the aggregate expenditure by consumers.

$$\overbrace{Y_t}^{\text{aggregate expenditure}} = \overbrace{M_t \int_0^\infty \xi_t(\alpha)\mathfrak{g}_t(\alpha)d\alpha}^{\text{aggregate revenue}} = \sigma M_t \int_0^\infty \pi_t(\alpha)\mathfrak{g}_t(\alpha)d\alpha, \tag{14}$$

where the last equality above follows from (11) and (12). M_t is the mass of producing firms that is endogenously determined by the above condition. Further, the aggregate expenditure of consumers must equal their capital rental income:

$$Y_t = r_t K. \tag{15}$$

2.2.2 Equilibrium Characterization

Dividing (13) by (14) and using (15),

$$M_t = \frac{K}{f_t \sigma}. \tag{16}$$

The mass of firms (and, therefore, the mass of goods) declines with the fixed capital f_t and the elasticity of substitution σ. In particular, because f_t varies with the aggregate state, ζ_t, of the economy, the mass of firms also varies. In particular, if the required fixed capital investment declines as the aggregate state of the economy improves (as one might expect), it follows from (16) that the mass of firms/products then increases with the aggregate state. That is, better aggregate states are associated with a greater variety of products that is consistent with empirical evidence for procyclical product variety growth (Axerloglou (2003), Broda and Weinstein (2007)).

By (11), (12) and (13), we have

$$r_t f_t = \int_0^\infty \frac{Y_t \left(P_t \rho \alpha\right)^{\sigma-1} r_t^{1-\sigma}}{\sigma} \mathfrak{g}_t(\alpha)d\alpha. \tag{17}$$

Substituting (15) in (17), using (16) and re-arranging terms, we get‘

$$\frac{r_t}{P_t} = \rho M_t^{\frac{1}{\sigma-1}} \bar{\alpha}_t, \tag{18}$$

where $\bar{\alpha}_t$ represents the “average productivity”,

$$\bar{\alpha}_t := \left(\int \alpha^{\sigma-1} \mathfrak{g}_t(\alpha) d\alpha \right)^{\frac{1}{\sigma-1}}. \tag{19}$$

The quantity, $\frac{r_t}{P_t}$, in (18) is the *real* rental rate of capital. Equivalently, from (9) and (18), the *real price* of the product of the firm with average productivity, $\bar{\alpha}_t$, is

$$\frac{p_t(\bar{\alpha}_t)}{P_t} = M_t^{\frac{1}{\sigma-1}}. \tag{20}$$

As the productivity distribution $\mathfrak{g}_t$ varies with the aggregate state of the economy, it follows from (18) that the real rental rate of capital also varies. From (7), (18) and (20), we see that the aggregate price index, P_t, is the appropriate numeraire with respect to which nominal prices and returns should be normalized to express them in real terms. Indeed, a key message of our study is that the aggregate price index, P_t, differs in general from the consumer price index (CPI) in the data. The CPI is computed using the prices of an *exogenously specified* basket of products. Hence, the CPI does not correspond to the moment of the distribution of prices of the *endogenously determined* products in the economy that determines P_t in (6). We explore the implications of this crucial distinction for asset prices below.

2.2.3 Asset Markets and Asset Prices

In what follows, all period t variables (carrying a subscript t) are viewed as the realizations at time t of stochastic processes adapted to the filtration generated by the aggregate state process (ζ_t). Expectations conditional with respect to information at t are denoted by $\mathbb{E}_t[\cdot]$. At each date, there is a complete set of (nominal) one-period Arrow securities promising one dollar next period contingent on the future aggregate state. The pricing kernel generated by the prices of these Arrow securities is denoted by the process (Q_t). Since all agents are identical, we can focus on the representative agent who holds capital K and enters period t with holdings of a_t units of the Arrow security purchased in period $t-1$. Initial financial wealth is $a_0 = 0$, and rental income in period t is $r_t K$. In addition to consuming, the agent acquires a portfolio a_{t+1} of one-period Arrow securities traded at t corresponding to every possible aggregate state in the next period,leading to the sequence of budget constraints

$$P_t \Lambda_t + \mathbb{E}_t \left(\frac{Q_{t+1}}{Q_t} a_{t+1} \right) \leq a_t + r_t K, \ \forall t. \tag{21}$$

To rule out Ponzi schemes, debt is bounded by the present value of future income,

$$a_{t+1} \geq -\frac{1}{Q_{t+1}} \mathbb{E}_{t+1} \left(\sum_{s>t+1} Q_s r_s K \right), \quad \forall t. \tag{22}$$

The agent chooses a consumption plan and holdings of Arrow securities to maximize (1) subject to the sequence of budget constraints (21) and the borrowing constraints (22), or equivalently (Hernandez and Santos, 1996), subject to the intertemporal budget constraint

$$\mathbb{E} \left(\sum_{t \geq 0} Q_t P_t \Lambda_t \right) \leq \mathbb{E} \left(\sum_{t \geq 0} Q_t r_t K \right). \tag{23}$$

From the first order conditions, the SDF for valuing nominal assets is

$$\Phi_{t+1} := \frac{Q_{t+1}}{Q_t} = \beta \frac{U'(\Lambda_{t+1}) P_t}{U'(\Lambda_t) P_{t+1}} = \beta \frac{U'(\Lambda_{t+1}) \Lambda_{t+1} Y_t}{U'(\Lambda_t) \Lambda_t Y_{t+1}}, \tag{24}$$

where the aggregate consumption index Λ_t satisfies

$$\Lambda_t = \frac{Y_t}{P_t} = K \frac{r_t}{P_t} = K \rho M_t^{\frac{1}{\sigma - 1}} \bar{\alpha}_t. \tag{25}$$

In (25), the first equality follows from (7), the second follows from (15), and the third follow from the expression (18) for the real rental rate of capital, $\frac{r_t}{P_t}$.

By (24), the *real SDF* (for valuing *real returns*) is $\beta \frac{U'(\Lambda_{t+1})}{U'(\Lambda_t)}$. In other words, as in traditional consumption-based asset pricing models, the real SDF is the marginal rate of substitution of the representative agent at the "aggregate consumption" process. In traditional models, which assume a single consumption good (or, alternatively, *perfectly substitutable* consumption goods), aggregate consumption is simply the linear aggregation of the consumptions of individual goods. In contrast, with heterogeneous consumption goods that are imperfectly substitutable, the aggregate consumption is determined by the consumptions of individual goods aggregated using the CES aggregator. Further, as discussed earlier, real returns are obtained from nominal returns using the aggregate price index (rather than the CPI) as the deflator. In other words, the rate of inflation is the growth rate of the aggregate price index rather than the CPI. In our empirical and quantitative analysis, we use firm productivity data from the Bureau of Labor Statistics to infer the aggregate price index and, thereby, construct the SDF that correctly incorporates product heterogeneity and imperfect product substitutability.

2.2.4 Equity Premium and Risk-free Rate Puzzles: Qualitative Analysis

Consider any asset in the economy. If R_{t+1} is the realized nominal return of the asset over the period $[t, t+1]$, then the following Euler equation must be satisfied:

$$1 = \mathbb{E}_t \left[\Phi_{t+1} R_{t+1}\right] = \mathbb{E}_t \left[\beta \frac{U'(\Lambda_{t+1})\Lambda_{t+1}}{U'(\Lambda_t)\Lambda_t} \frac{Y_t}{Y_{t+1}} R_{t+1}\right], \tag{26}$$

where the second equality above follows from (7). As in the traditional asset pricing literature, assume that

$$U(x) = \frac{x^{1-\gamma}}{1-\gamma}; \gamma > 0. \tag{27}$$

By (24) and (25), the nominal SDF can be expressed as

$$\Phi_{t+1} = \beta \left(\frac{\Lambda_{t+1}}{\Lambda_t}\right)^{-\gamma} \frac{P_t}{P_{t+1}}. \tag{28}$$

Since the aggregate consumption index, $\Lambda_t = \frac{Y_t}{P_t}$, by (7), we have

$$\Phi_{t+1} = \beta \left(\frac{Y_{t+1}}{Y_t}\right)^{-\gamma} \left(\frac{P_t}{P_{t+1}}\right)^{1-\gamma} \tag{29}$$

It is instructive to qualitatively compare the key properties of the SDF in our model and the SDF in the traditional consumption-based asset pricing model (CCAPM). The nominal SDF in the

CCAPM is

$$\Phi_{t+1} = \beta \left(\frac{Y_{t+1}}{Y_t}\right)^{-\gamma} \left(\frac{CPI_t}{CPI_{t+1}}\right)^{1-\gamma}, \tag{30}$$

where CPI_t is the consumer price index at date t. The central sources of the "equity premium" and "risk-free rate" puzzles in the traditional consumption-based asset pricing model (CCAPM) is that the growth rates in aggregate consumption and the CPI are not volatile enough and do not covary sufficiently with the market. Consequently, we need very high values of γ to ensure that the volatility of the SDF in the CCAPM is high enough, and its covaration with the market return is sufficiently negative. This worsens the risk-free rate puzzle as the expectation of the SDF decreases with high γ (e.g., Cochrane (2005)).

In contrast, by (29), the SDF in our model contains the factor, $\left(\frac{P_t}{P_{t+1}}\right)^{1-\gamma}$, instead of the factor, $\left(\frac{CPI_t}{CPI_{t+1}}\right)^{1-\gamma}$. The consumer price index is the price of an *exogenously specified* basket of commodities whose variation over time is relatively smooth. In contrast, the aggregate price, P_t,is determined by a moment of the firm productivity distribution that depends on the product substitutability as well as the *endogenously determined* mass of products. Both the firm productivity distribution and the mass of products vary with the aggregate shock. To understand the impact of these key differences, we can use (25) to re-express the term, $\left(\frac{P_t}{P_{t+1}}\right)^{1-\gamma}$, as

$$\left(\frac{P_t}{P_{t+1}}\right)^{1-\gamma} = \left(\frac{Y_t}{Y_{t+1}}\right)^{1-\gamma} \left(\frac{M_{t+1}}{M_t}\right)^{\frac{1-\gamma}{\sigma-1}} \left(\frac{\bar{\alpha}_{t+1}}{\bar{\alpha}_t}\right)^{1-\gamma} \tag{31}$$

The term $\left(\frac{M_{t+1}}{M_t}\right)$ in (31) is the growth in the mass of products, while $\frac{\bar{\alpha}_{t+1}}{\bar{\alpha}_t}$ is the growth in the average productivity of firms. If, as is quite plausible, firms' productivity distribution increases with the aggregate state (in the sense of first order stochastic dominance), then the number of products and the average productivity of firms both covary with the market return. Further, if σ is small (products are imperfectly substitutable), the term, $\left(\frac{M_{t+1}}{M_t}\right)^{\frac{1-\gamma}{\sigma-1}}$,appearing in (31) could be large and negative for even moderate levels of the relative risk aversion coefficient γ.

In the next section, we estimate the parameters of the model using firm productivity data from the Bureau of Labor Statistics and data on the number of establishments from the Bureau of Labor Statistics to infer the aggregate price index and, thereby, construct the SDF in our model. We illustrate the impact of the distinction between the aggregate price index and the consumer price index that drives the difference between the SDFs in our model and the CCAPM.

3 Empirical Analysis of Basic Model

In this section, we examine whether the basic model is able to reconcile the empirically observed equity risk premium and risk-free rate. We use the Generalized Method of Moments (GMM) to estimate the risk aversion and discount factor using the Euler equations (26) corresponding to various portfolios of assets (see Hansen and Singleton (1983)). We employ annual capital productivity measures provided by the Bureau of Labor Statistics (BLS) over the period 1987-2020. We begin in 1987 because the BLS provides productivity measures for multiple industries starting in 1987 and we use these data in the analysis of our more general, multi-industry model developed in Section 4.

Consistent with the model, we use the annual *capital productivity* reported by BLS as the proxy for the average productivity $\bar{\alpha}_t$ for firms appearing in the SDF by (28) and (31). We use the number of establishments reported by the BLS to represent M_t, and the *aggregate expenditure on nondurable personal consumption and services* from the National Income and Product Accounts (NIPAs) published by the BEA to represent Y_t. Note that, by (29) and (31), the SDF depends only on the ratios, $\frac{Y_{t+1}}{Y_t}, \frac{P_{t+1}}{P_t}, \frac{\bar{\alpha}_{t+1}}{\bar{\alpha}_t}$. The summary statistics for the key variables are presented in Table 1.As we noted in the previous section, the key difference between the pricing kernels in our model and the CCAPM ((29) and (30)) arises from the difference in the price indices in the models. To illustrate the contrast in the data, we first compare the price indices in our model and the CCAPM and their correlations with the market risk premium in Table 2. The table shows that the growth in the aggregate price index, $\dfrac{P_{t+1}}{P_t}$, in our model (computed using BLS/BEA data) is more than twice as volatile as the growth in the CPI, $\frac{CPI_{t+1}}{CPI_t}$. Further, $\dfrac{P_{t+1}}{P_t}$ is negatively related to the excess market return with a correlation of -0.42, whose magnitude is much larger than the correlation, 0.06, of the CPI growth, $\frac{CPI_{t+1}}{CPI_t}$, with the excess market return. In Figure 2, we display the time series of the growths in the two price indices and the excess market return. It is clear that the growth in the aggregate price index is not only more volatile but significantly more negatively correlated with the excess market return. These results provide support for our theoretical arguments in the previous section suggesting that our model has the potential to explain the equity premium and risk-free rate.

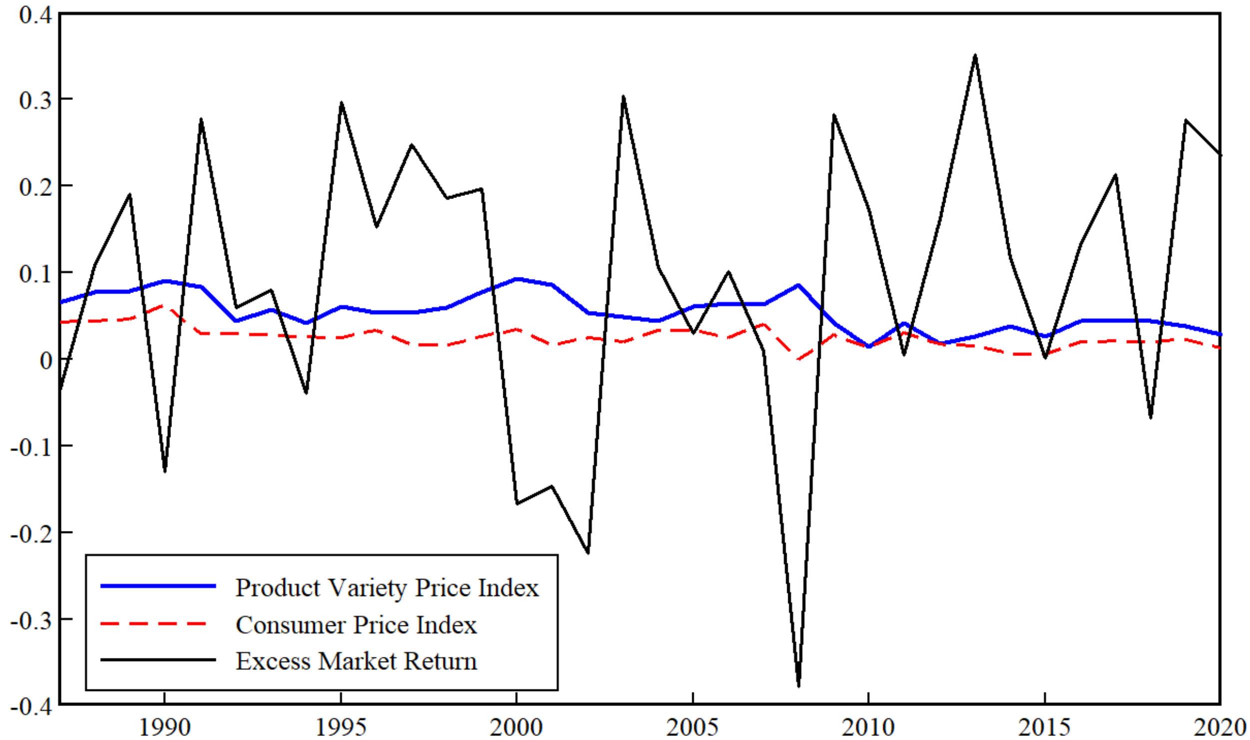

Figure 2: **Aggregate Price Index with Product Variety and CPI**

We next carry out a formal GMM estimation. Let the return over period $[t, t+1]$ for an arbitrary portfolio of assets be R_{t+1} and the risk-free return be R_t^{rf}. By (28) and (31), the asset Euler equation corresponding to the return on the portfolio and the risk-free return are

$$\mathbb{E}_t \left[\beta \left(\frac{M_{t+1}}{M_t} \right)^{\frac{1-\gamma}{\sigma-1}} \left(\frac{\bar{\alpha}_{t+1}}{\bar{\alpha}_t} \right)^{1-\gamma} \frac{Y_t}{Y_{t+1}} R_{t+1} \right] = 1, \tag{32}$$

$$\mathbb{E}_t \left[\beta \left(\frac{M_{t+1}}{M_t} \right)^{\frac{1-\gamma}{\sigma-1}} \left(\frac{\bar{\alpha}_{t+1}}{\bar{\alpha}_t} \right)^{1-\gamma} \frac{Y_t}{Y_{t+1}} R_t^{rf} \right] = 1. \tag{33}$$

Furthermore, following Hansen and Singleton (1993), we can also include a set of variables I_t that encode information available at the start of period t as instruments and consider the following Euler equation:

$$\mathbb{E}_t \left[\left(\beta \left(\frac{M_{t+1}}{M_t} \right)^{\frac{1-\gamma}{\sigma-1}} \left(\frac{\bar{\alpha}_{t+1}}{\bar{\alpha}_t} \right)^{1-\gamma} \frac{Y_t}{Y_{t+1}} R_{t+1} - 1 \right) \otimes I_t \right] = 0. \tag{34}$$

We choose $\sigma = 6.27$, which corresponds to the average industry markup (Sales/EBIT) in our sample. We estimate the relative risk aversion γ and the discount factor β simultaneously using the sample analogues of the moment conditions, (32) and (33), where we employ several groups of portfolios. In particular, consistent with previous literature, we use the Fama-French 25 portfolios defined using Size and Book-to-Market (B/M), the Fama-French 3 factor returns and the Fama-French 2x3 portfolios , as well as Fama-French portfolios with lagged instruments such as lagged dividend-price ratio (Campbell and Shiller (1988)), lagged CAY (Lettau and Ludvigson (2001)), lagged average productivity growth $\frac{\bar{\alpha}_t}{\bar{\alpha}_{t-1}}$ and lagged firm/product variety growth $\frac{M_t}{M_{t-1}}$. In all specifications, we also include the market return and risk-free return as test asset returns in (32) and (33).

Table 3 reports the results, where we adjust standard errors for heteroskedasticity and autocorrelation following Newey and West. Table 3 shows that, when we use the pricing kernel, (28), of our basic model, the estimated risk aversion coefficients are 5.77, 3.60, 2.43, 5.34 and 6.75, when we use the FF25 portfolios, FF3, FF3 with instrument, FF2x3 and FF2x3 with instruments, respectively. The estimates of the discount factor β are 0.90, 0.97, 0.99, 0.92 and 0.90 , respectively, for the five sets of GMM estimations. We also report Hansen's J-statistics for overidentifying restrictions and the associated p-values. With the exception of the FF3 specification, the J-statistics from the moment conditions are statistically indistinguishable from zero. Therefore, the model fits the moments well.

As the actual economy has many different industries/sectors, the basic model is a simplification. Therefore, the above results, although encouraging, should be viewed as a preliminary attempt to reconcile the equity premium and risk-free rate puzzles. Further, the GMM estimation approach does not exploit the full structure of our general equilibrium model. As we will see shortly, the GMM and structural estimations of the extended model with multiple industries generate significantly lower estimates of γ and higher estimates of β that are much closer to empirically plausible values.

4 The Extended Model

In reality, the economy has multiple industries with products of firms within an industry being closer substitutes than those of firms in different industries. For example, products within the "Retail" or

“Accomodation” industries are closer substitutes than products across these industries. Relatedly, intra-industry product market competition is more intense than competition between firms across industries. In fact, products between industries can be complements, such as is the case with tourism, in which different industries (i.e., “Transportation”, “Accomodation”, “Entertainment” and “Retail”) have a complementary relationship. The incorporation of differing inter- and intra-industry product substitutabilities could have a significant impact on the SDF and asset prices. Accordingly, we extend the basic model by modeling an economy with multiple industries and differing intra-industry and inter-industry product substitutabilities.

4.1 Model Description

There are N industries $1, 2, ..., N$. The preferences of the representative consumer are given by

$$\mathcal{U}_s = \mathbb{E}_s \sum_{t=s}^{\infty} \beta^{t-s} U(\Lambda_t), \; 0 < \beta < 1 \tag{35}$$

where

$$\Lambda_t = \begin{cases} \left[\sum_{i=1}^{N} \Lambda_{it}^{\delta}\right]^{\frac{1}{\delta}}, & -\infty < \delta < 1, \delta \neq 0 \\ \prod_{i=1}^{N} \Lambda_{it}, & otherwise \end{cases} \tag{36}$$

$$\Lambda_{it} = \left[\int_{\Omega_{it}} q_{it}(\omega_{it})^{\rho_i} d\omega_{it}\right]^{\frac{1}{\rho_i}}; 0 < \rho_i < 1; \; 1 \leq i \leq N, \tag{37}$$

In (36) and (38), Λ_{it} is the contribution to the consumer’s total consumption index, Λ_t, from the goods produced by industry i, δ determines the degree of product substitutability across industries, while ρ_i determines the degree of product substitutability within industry i.

If $p_{it}(\omega_{it})$ is the price of good ω_{it} produced by industry i in period t, then we can show that the optimal consumption and expenditure decisions for individual goods are

$$q_{it}(\omega_{it}) = \Lambda_{it} \left[\frac{p_{it}(\omega_{it})}{P_{it}}\right]^{-\sigma_i}, \tag{38}$$

$$\xi_{it}(\omega_{it}) = Y_{it} \left[\frac{p_{it}(\omega_{it})}{P_{it}}\right]^{1-\sigma_i}, \; t \in \{0, 1, 2, 3, ...\}. \tag{39}$$

In the above $Y_{it} = P_{it}\Lambda_{it}$ is the aggregate expenditure of the representative consumer on industry i (or the size of industry i) in period t, and

$$P_{it} = \left[\int_{\Omega_{it}} p_{it}(\omega_{it})^{1-\sigma_i} d\omega_{it}\right]^{\frac{1}{1-\sigma_i}} \tag{40}$$

is the *aggregate price index* of industry i in period t, where

$$\sigma_i = \frac{1}{1-\rho_i} \tag{41}$$

is the *intra-industry product substitutability* of the products in industry i.

The consumer's expenditure minimization program leads to the following expression for the industrial sizes:

$$Y_{it} = \begin{cases} Y_t \frac{P_{it}^{1-\tau}}{P_t^{1-\tau}}, & \tau \neq 1 \\ \frac{1}{N} Y_t, & otherwise \end{cases}, \tag{42}$$

where

$$P_t = \begin{cases} \left[\sum_{i=1}^{N} P_{it}^{1-\tau}\right]^{\frac{1}{1-\tau}}, & \tau \neq 1 \\ \prod_{i=1}^{N} P_{it}, & otherwise \end{cases} \tag{43}$$

is the aggregate price index for the entire economy, and

$$\tau = \frac{1}{1-\delta} > 0 \tag{44}$$

is the *inter-industry product substitutability.*

The information structure is as described in the basic model. The initial fixed capital investment, however, could vary across industries and is denoted as f_{it} for industry i. The productivity distribution of firms in industry i in period t is $\mathfrak{g}_{it}$. If r_t denotes the capital rental rate then, by the same arguments used in Section 2, the price of the good produced by a firm with productivity α in industry i is

$$p_{it}(\alpha) = \frac{r_t}{\rho_i \alpha}, \tag{45}$$

and its output is

$$q_{it}(\alpha) = Y_{it}\left(P_{it}\right)^{\sigma_i-1}\left(\frac{\rho_i\alpha}{r_t}\right)^{\sigma_i}, \tag{46}$$

The firm's revenue and profit are

$$\xi_{it}(\alpha) = p_{it}(\alpha)q_{it}(\alpha) = Y_{it}(P_{it}\rho_i\alpha)^{\sigma_i-1}r_t^{1-\sigma_i}, \tag{47}$$

$$\pi_{it}(\alpha) = \frac{\xi_{it}(\alpha)}{\sigma_i}. \tag{48}$$

4.2 Product Market Equilibrium

We can characterize the product market equilibrium that we summarize in the following proposition.[1]

Proposition 1 (Product Market Equilibrium in Extended Model)**.** *The aggregate price index, P_t, solves*

$$1 = \sum_i \left(\frac{P_t}{r_t}\right)^{(\tau-1)\frac{\sigma_i-1}{\sigma_i-\tau}} \left(\frac{Y_t\left(\rho_i\bar{\alpha}_{it}\right)^{\sigma_i-1}}{r_t f_{it}\sigma_i}\right)^{\frac{\tau-1}{\sigma_i-\tau}}, \tag{49}$$

where

$$\bar{\alpha}_{it} := \left[\int_0^\infty \alpha^{\sigma_i-1}\mathfrak{g}_{it}(\alpha)d\alpha\right]^{\frac{1}{\sigma_i-1}}. \tag{50}$$

The total revenue/size of industry i is given by

$$Y_{it} = Y_t^{\frac{\sigma_i-1}{\sigma_i-\tau}}\left(\frac{(\rho_i\bar{\alpha}_{it})^{\sigma_i-1}}{r_t f_{it}\sigma_i}\right)^{\frac{\tau-1}{\sigma_i-\tau}}\left(\frac{P_t}{r_t}\right)^{(\tau-1)\frac{\sigma_i-1}{\sigma_i-\tau}}. \tag{51}$$

The mass of firms and aggregate price index for industry i (in units of capital) are given by

$$M_{it} = \frac{Y_{it}}{r_t f_{it}\sigma_i}, \tag{52}$$

$$\frac{P_{it}}{r_t} = \left(M_{it}^{\frac{1}{\sigma_i-1}}\rho_i\bar{\alpha}_{it}\right)^{-1}. \tag{53}$$

This characterization of the product market equilibrium facilitates the structural estimation of the extended model.

[1]The proof is presented in Appendix A.

4.3 Asset Markets and Asset Prices

The asset market structure is as described in Section 2.2.3 except that we now have multiple industries. The economy in any period t is determined by the aggregate state history ζ^t, the distribution of fixed capital investments across industries, $\{f_{it}(\zeta^t); i = 1, ..., N\}$, and the distributions of firm productivities in each industry, $\{\mathfrak{g}_{it}(\zeta^t); i = 1, ..., N\}$. The budget constraints of the representative agent now are

$$\sum_{i=1}^{N} P_{it}\Lambda_{it} + \mathbb{E}_t \frac{Q_{t+1}}{Q_t} a_{t+1} \leq a_t + r_t K, \ \forall t \geq 0, \tag{54}$$

where Q_t is the price of the nominal one-period Arrow security at date t as in the basic model.

Maximizing (35) subject to the budget constraints (54), the non-binding debt constraints (22), and using (42), (44), we obtain the same expression for the SDF as in (24),

$$\Phi_{t+1} := \frac{Q_{t+1}}{Q_t} = \beta \frac{U'(\Lambda_{t+1})P_t}{U'(\Lambda_t)P_{t+1}}, \tag{55}$$

where the consumption indices, Λ_t, Λ_{t+1} in periods $t, t+1$, respectively, now determined in terms of the industrial consumption indices and given by (36). For CRRA utility with coefficient of relative risk aversion γ, it follows directly from (55) that the SDF is

$$\Phi_{t+1} = \beta \frac{\Lambda_{t+1}^{1-\gamma} Y_t}{\Lambda_t^{1-\gamma} Y_{t+1}}. \tag{56}$$

5 Empirical and Quantitative Analysis of Extended Model

To facilitate a direct comparison with our analysis of the single-industry model, we first use GMM to estimate the risk aversion and discount factor parameters in the extended model using the Euler equations corresponding to various asset portfolios.

5.1 GMM Estimation

Using similar arguments as in Section 3, we have

$$\Lambda_{it} = \rho_i Y_{it} M_{it}^{\frac{1}{\sigma_i - 1}} \bar{\alpha}_{it} \frac{1}{r_t}$$

and

$$\Phi_{t+1} = \begin{cases} \beta \dfrac{\left[\sum_{i=1}^{N}\left(\rho_i Y_{i,t+1} M_{i,t+1}^{\frac{1}{\sigma_i-1}} \bar{\alpha}_{i,t+1}\right)^{\delta}\right]^{\frac{1-\gamma}{\delta}}}{\left[\sum_{i=1}^{N}\left(\rho_i Y_{it} M_{it}^{\frac{1}{\sigma_i-1}} \bar{\alpha}_{it}\right)^{\delta}\right]^{\frac{1-\gamma}{\delta}}} \left(\frac{Y_t}{Y_{t+1}}\right)^{2-\gamma}, & \delta \neq 0 \\ \beta \prod_{i=1}^{N} \left[\frac{Y_{i,t+1}}{Y_{it}} \left(\frac{M_{i,t+1}}{M_{it}}\right)^{\frac{1}{\sigma_i-1}} \frac{\bar{\alpha}_{i,t+1}}{\bar{\alpha}_{it}}\right]^{1-\gamma} \left(\frac{Y_t}{Y_{t+1}}\right)^{2-\gamma}, & otherwise \end{cases}. \tag{57}$$

Analogous to Section 3, we use the aggregate nondurable consumption and services from the BEA as a proxy for Y_t and the number of establishments in each industry as a proxy for M_{it}. The BLS provides capital productivity data for firms in different industries since 1987[2]. Based on (47) and (48), we use the average sales to profits ratio in industry i over our sample period as empirical analogues for the intra-industry product substitutability parameters σ_i and set $\rho_i = \frac{\sigma_i - 1}{\sigma_i}$. We report calibrated values of σ_i using this procedure in Table 4. Our final sample consists of data for 14 industries over 34 years from 1987 to 2020.[3]

To construct the proxy for industry output Y_{it}, we use Gross Domestic Product Value Added from the BEA. We classify a firm as a "final consumption" firm if it produces categories of the consumption goods surveyed in the Consumer Expenditure Surveys (CEX). We show the classification (based on the Fama-French 48 industries of firms) in Table 5.

The inter-industry product substitutability, $\delta = \frac{\tau - 1}{\tau}$, cannot be directly calibrated from the data. Accordingly, we employ the estimate of the inter-industry product substitutability from our structural estimation exercise in Section 5.2. Our baseline estimate is $\delta = -1.21$, which means that inter-industry products exhibit a complementary relationship. As in the GMM analysis of the basic model in Section 3, we use the Euler equations corresponding to the market return, risk-free return, FF 25, FF 3 and FF2x3 portfolio returns, as well as lagged instruments, to estimate the relative risk aversion, γ and the discount factor, β.

Table 6 shows the results of the GMM estimation. The estimates of the relative risk aversion γ are around 2.44, 1.99, 1.09, 4.03 and 3.51 for the FF25, FF3, FF3 with instruments, FF2x3, FF2x3

[2]Given that the capital productivity measures are in index form and the inter-industry scaling does matter for the SDF, we use the GDP Value Added / Private Fixed Assets by industry in 1987 as a proxy for the initial capital productivity level. The data for the Private Fixed Assets was obtained from the BEA.

[3]We exclude Agriculture, Mining, Wholesale, Finance and Insurance and Government from out sample.

with instruments, portfolio returns, respectively. The corresponding estimates of the discount factor, β, are between 0.9103 and 0.9999 We conclude that the extended model is compatible with the observed market risk premium, risk-free return, and the selected Fama-French portfolio excess returns for empirically plausible values of the risk aversion and discount factor. We also report Hansen's J-statistics for overidentifying restrictions and the associated p-values. With the exception of the FF3 specification, the J-statistics are not significantly different from zero, suggesting that the model fits the moment conditions well.

5.2 Structural Estimation of the Extended Model

The GMM analysis does not exploit the structure of our fully specified general equilibrium model. We next proceed to structurally estimate the model using the simulated method of moments. We then exploit the estimated model to generate additional novel testable empirical predictions that pertain to the effects of product market competition—specifically, intra-industry and inter-industry product substitutabilities—on asset prices and returns. We exploit the closed-form characterization of the equilibrium in Proposition 1. For the structural estimation, it is necessary to choose parametric forms for the aggregate sales process, the industrial productivity distributions and the entry costs.

5.2.1 Parametric Specifications

We assume that the nominal aggregate revenue/sales follows the process

$$\ln Y_t = z_t + \lambda_z^{(1)} + \lambda_z^{(2)} t, \tag{58}$$

where the z_t term is an ARIMA(1,1,0) process with

$$\Delta z_t = a_z \Delta z_{t-1} + \varepsilon_{zt} \tag{59}$$

and $\varepsilon_{zt} \sim N(0, \sigma_z^2)$ are normally distributed shocks that are independent across time. For each industry i, the entry cost follows

$$\ln f_{it} = c_i w_t + \lambda_{f_i}^{(1)} + \lambda_{f_i}^{(2)} t, \tag{60}$$

where w_t is an ARIMA(1,1,0) process with

$$\Delta w_t = a_w \Delta w_{t-1} + \varepsilon_{wt}, \quad \varepsilon_{tw} \sim N(0, \sigma_w^2). \tag{61}$$

The shocks ε_{wt} are i.i.d. across time. The shocks ε_{zt} and ε_{wt} have a contemporaneous correlation ρ_{zw} at each date t. The process w_t can be viewed as an aggregate shock process that affects the entry costs of industries with differing sensitivities captured by the parameters, $\{c_i\}$.

The productivities of firms in industry i are distributed as

$$\ln \alpha_{it} = b_i w_t + \lambda_{\alpha_i}^{(1)} + \lambda_{\alpha_i}^{(2)} t + \varepsilon_{\alpha_i t}, \; \varepsilon_{\alpha_i t} \sim N(0, \sigma_{\alpha_i}^2). \tag{62}$$

The shock process, $\varepsilon_{\alpha_i t}$, is independent of the processes ε_{zt} and ε_{wt}.

From Proposition 1 and (56), the SDF in the extended model can be written as

$$\Phi_t = \begin{cases} \beta \dfrac{\left[\sum_{i=1}^N \left(Y_{i,t+1}^{\frac{\sigma_i}{\sigma_i - 1}} \rho_i \bar{\alpha}_{i,t+1} (\sigma_i r_{t+1} f_{i,t+1})^{\frac{1}{1-\sigma_i}} \right)^{\delta} \right]^{\frac{1-\gamma}{\delta}}}{\left[\sum_{i=1}^N \left(Y_{it}^{\frac{\sigma_i}{\sigma_i - 1}} \rho_i \bar{\alpha}_{it} (\sigma_i r_t f_{it})^{\frac{1}{1-\sigma_i}} \right)^{\delta} \right]^{\frac{1-\gamma}{\delta}}} \left(\frac{Y_t}{Y_{t+1}}\right)^{2-\gamma}, & \delta \neq 0 \\ \beta \prod_{i=1}^N \left[\left(\frac{Y_{i,t+1}}{Y_{it}}\right)^{\frac{\sigma_i}{\sigma_i - 1}} \frac{\bar{\alpha}_{i,t+1}}{\bar{\alpha}_{it}} \left(\frac{r_{t+1} f_{i,t+1}}{r_t f_{it}}\right)^{\frac{1}{1-\sigma_i}} \right]^{1-\gamma} \left(\frac{Y_t}{Y_{t+1}}\right)^{2-\gamma}, & otherwise \end{cases}. \tag{63}$$

Due to the complexity of the above formula, we do not have simple closed-form representations of the risk-free return and market return in this setting. Therefore, we use (63) and simulation to compute the unconditional asset pricing moments implied by the model.

5.2.2 Calibrated and Estimated Parameters

We use aggregate consumption data to calibrate the parameters of the aggregate processes a_z, σ_z, $\lambda_z^{(1)}$, and $\lambda_z^{(2)}$. We use the 2-digit NAICS 14 industry GDP Value Added data for final consumption firms, the number of establishments in the industries, and the equilibrium conditions in Proposition 1 to calibrate the industrial parameters $(\sigma_i, a_w, \sigma_w, \rho_{zw}, c_i, \lambda_{fi}^{(1)}, \lambda_{fi}^{(2)}, \sigma_{\alpha_i})$. Specifically, we use the average sales to profit margin in each industry to calibrate the intra-industry product substitutability σ_i. We calibrate the entry cost f_{it} using the GDP Value Added, private fixed assets, and the number of establishments in (52):

$$f_{it} = \frac{Y_{it}K}{\sigma_i M_{it} Y_t}. \tag{64}$$

Next, we detrend the process f_{it} and obtain the first principal component to extract the aggregate shock process w_t and obtain the parameters $\lambda_{fi}^{(1)}, \lambda_{fi}^{(2)}, a_w, \sigma_w, \rho_{zw}$, and c_i. We use the BLS data on capital productivity as the measure of $\bar{\alpha}_{it}$ and use linear regressions to directly estimate b_i, $\lambda_{\alpha_i}^{(1)}$,$\lambda_{\alpha_i}^{(2)}$ and σ_{α_i} in (62).

We employ the Simulated Method of Moments (SMM) to estimate the remaining free parameters β, γ and δ to match moments associated to the risk-free rate and the risk premium, as well as the dispersion of industry risk permia across different industries. We describe further details of the simulation and estimation procedure in Appendix D.

Table 7 shows the results of the SMM estimation as well as the standard errors of estimated parameters and simulated moments that we compute using bootstrapping as we describe in Appendix D. Table 8 reports the calibrated values of the industrial parameters $(\sigma_i, \sigma_{\alpha_i}, b_i, c_i, \lambda_{f_i}^{(1)}, \lambda_{f_i}^{(2)}, \lambda_{\alpha_i}^{(1)}, \lambda_{\alpha_i}^{(2)})$. The estimated values of the discount factor and risk aversion are 0.99985 and 5.97, respectively, which are comparable to the GMM estimates obtained in Section 5.1. We note that all moments are matched well, including the mean of the market risk premium, the mean and volatility of the risk-free return and the dispersion of industrial risk premia. This is verified using Hansen's J test for overidentifying restrictions, which confirms the validity of the model with a p-value of 0.9534.

For robustness purposes we run the GMM exercises for both single and multi-industry models, as well as the SMM exercise, using the number of firms instead of the number of establishments. As Table 9 shows, the parameter estimates under the different specifications are broadly consistent with the previous results using the total and industry number of establishments as measures for M_t and M_{it}, respectively. This is unsurprising given the high correlation (i.e., above 0.95) between the number of firms and the number of establishments.

5.2.3 Results

Figure 3 presents the comparative statics of risk-free returns and market equity premia in the multi-industry model when we allow parameters to vary around their baseline values in Table 7.

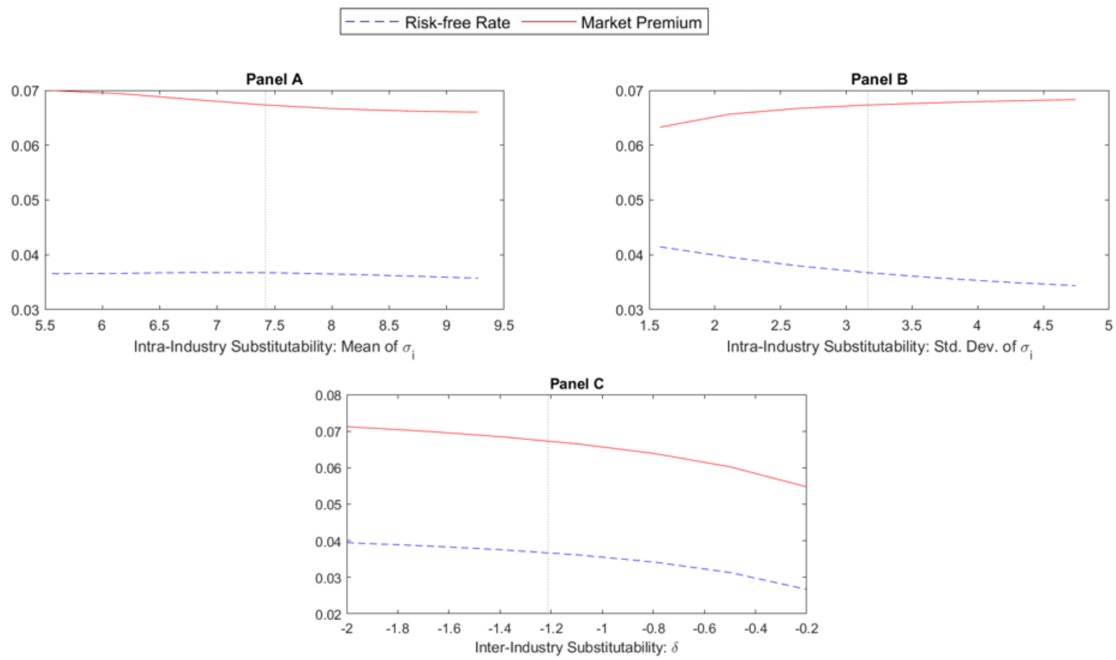

Figure 3: **Comparative Statics of Risk-Free Rate and Market Risk Premia: Multis-Industry Model** The vertical dotted lines correspond to the baseline values of the parameters.

The market risk premium declines with the average intra-industry product substitutability σ_i, while the risk-free return exhibits a relatively flat behavior. The intuition hinges on the fact that the magnitude of the SDF's sensitivity to the market return, the SDF's volatility, and its expectation all decline with the average intra-industry product substitutability. The market risk premium increases with the standard deviation of intra-industry substitutabilities. An increase in the standard deviation of intra-industry product substitutabilities increases *composition risk* stemming from the variability of the basket of consumption goods, thereby increasing the volatility of the SDF and increasing the market risk premium. Therefore, the market risk premium depends not only on the mean but also the distribution of sector product substitutabilities.

The equity premium is decreasing in the *inter-industry* product substitutability $\tau = \frac{1}{1-\delta}$. An increase in the inter-industry product substitutability has a similar but stronger effect on the SDF relative to the intra-industry substitutability case, thereby causing the market risk premium to decrease. The differing effects of intra-industry and inter-industry product substitutabilities on the market risk premium imply that there are subtle differences between the effects of product heterogeneity *within* and *across* industries.

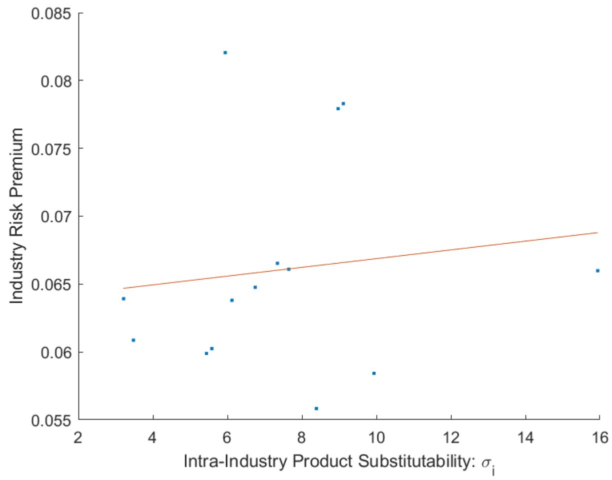

Figure 4: **Industry Risk Premia and Intra-industry Product Substitutabilities**

Our model also generates cross-sectional predictions that relate industry risk premia to industrial product market characteristics. Figure 4 shows that the expected risk premia of the 14 industries in the simulated economies depend positively on (intra-industry) product substitutabilities (σ_i).

We can understand the intuition for the positive relation between industrial risk premia and intra-industry product substitutabilities as follows. Using $Y_t = Kr_t$, (51) in Proposition 1 can be rewritten as

$$\left(\frac{Y_{it}}{r_t}\right)^{\frac{\tau-\sigma_i}{\sigma_i-1}} = (f_{it}\sigma_i)^{\frac{\tau-1}{\sigma_i-1}} (\rho_i \bar{\alpha}_{it})^{1-\tau} \left(\frac{P_t}{r_t}\right)^{1-\tau} \frac{1}{K}. \tag{65}$$

From the above, the elasticity of (real) industry i revenues (and profits) with respect to real aggregate consumption is

$$\frac{d\ln Y_{it}/P_t}{d\ln Y_t/P_t} = \frac{d\ln Y_{it}/P_t}{d\ln \frac{Y_t/r_t}{P_t/r_t}} = -\frac{d\ln \frac{Y_{it}/r_t}{P_t/r_t}}{d\ln P_t/r_t} = 1 - \frac{d\ln Y_{it}/r_t}{d\ln P_t/r_t} = 1 + (1-\tau)\left(1 - \frac{1-\tau}{\sigma_i - \tau}\right). \tag{66}$$

The elasticity measures the sensitivity of the profits in industry i to changes in the stochastic discount factor, and is increasing in σ_i. The elasticity is akin to an asset pricing beta for the returns in industry i. This suggests that risk premia are higher for industries with high elasticities of substitution as the profits of such industries are more responsive to aggregate shocks.

6 Conclusions

We develop a parsimonious and tractable general-equilibrium model to show how product heterogeneity and product market competition affect asset prices. We estimate the basic and extended versions of the model using a nonparametric GMM approach as well as a parametric structural approach. Our analysis shows that product heterogeneity, indeed, substantially influences asset prices. The observed market risk premium and risk-free rate can be reconciled for moderate relative risk aversion levels around 4 and empirically plausible annual discount factors . We employ our structurally estimated model to derive novel implications for how product market characteristics influence asset prices. Interestingly, we find that while products are substitutes within industries, they are complements between industries. The market risk premium *decreases* with the average *intra-industry* product substitutability, but *increases* with the *inter-industry* product substitutability. We also derive the novel cross-sectional prediction that industries with higher product substitutabilities have higher excess returns that we verify empirically. In summary, our study shows that composition risk arising from the imperfect substitutability of multiple heterogeneous consumption goods is an important determinant of asset prices and reconciles the observed equity premium and risk-free rate for reasonable risk aversion levels and discount factors.

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Appendix A Proof of 1

Using the logic in the derivation of (16), we have

$$M_{it} = \frac{K}{f_{it}\sigma_i}, \forall i,$$

which can be rewritten, using (15), as

$$M_{it} = \frac{Y_{it}}{r_t f_{it}\sigma_i}. \tag{A1}$$

Applying the logic used in the derivation of (18),we obtain

$$\frac{P_{it}}{r_t} = \left(M_{it}^{\frac{1}{\sigma_i-1}}\rho_i\bar{\alpha}_{it}\right)^{-1}. \tag{A2}$$

Now we proceed to replace (A1) into (A2), and replace P_{it} using (42)

$$Y_{it} = Y_t\left(\left(\frac{Y_{it}}{r_t f_{it}\sigma_i}\right)^{\frac{1}{\sigma_i-1}}\rho_i\bar{\alpha}_{it}\frac{P_t}{r_t}\right)^{\tau-1}.$$

Next we solve for Y_{it}

$$\begin{aligned}
Y_{it} &= Y_t\left(\left(\frac{Y_{it}}{r_t f_{it}\sigma_i}\right)^{\frac{1}{\sigma_i-1}}\rho_i\bar{\alpha}_{it}\frac{P_t}{r_t}\right)^{\tau-1} \\
&= Y_{it}^{\frac{\tau-1}{\sigma_i-1}}Y_t\left(\frac{(\rho_i\bar{\alpha}_{it})^{\sigma_i-1}}{r_t f_{it}\sigma_i}\right)^{\frac{\tau-1}{\sigma_i-1}}\left(\frac{P_t}{r_t}\right)^{\tau-1} \\
&= Y_t^{\frac{\sigma_i-1}{\sigma_i-\tau}}\left(\frac{(\rho_i\bar{\alpha}_{it})^{\sigma_i-1}}{r_t f_{it}\sigma_i}\right)^{\frac{\tau-1}{\sigma_i-\tau}}\left(\frac{P_t}{r_t}\right)^{(\tau-1)\frac{\sigma_i-1}{\sigma_i-\tau}}.
\end{aligned}$$

To derive (49) we obtain $\sum_i Y_{it}$

$$\sum_i Y_{it} = Y_t = \sum_i Y_t^{\frac{\sigma_i-1}{\sigma_i-\tau}}\left(\frac{(\rho_i\bar{\alpha}_{it})^{\sigma_i-1}}{r_t f_{it}\sigma_i}\right)^{\frac{\tau-1}{\sigma_i-\tau}}\left(\frac{P_t}{r_t}\right)^{(\tau-1)\frac{\sigma_i-1}{\sigma_i-\tau}}.$$

Finally, we simplify to arrive at the following expression

$$1 = \sum_i\left(\frac{P_t}{r_t}\right)^{(\tau-1)\frac{\sigma_i-1}{\sigma_i-\tau}}\left(\frac{Y_t\,(\rho_i\bar{\alpha}_{it})^{\sigma_i-1}}{r_t f_{it}\sigma_i}\right)^{\frac{\tau-1}{\sigma_i-\tau}}.$$

Appendix B Capital Depreciation and Investment

In this Appendix, we present a variation of the model in which capital depreciates, and consumers can invest in the accumulation of capital. To simplify the exposition, we consider the basic model although the extended model can be analogously modified to incorporate capital depreciation and investment. As in Section IV and Appendix E of Bilbiie et al. (2012), investment in capital (I_t) requires the same composite of available product varieties as the consumption basket. The aggregate capital accumulation equation is

$$K_{t+1} = (1 - \delta^K)K_t + I_t,$$

where δ^K is the depreciation rate, and I_t is the investment at period t. Denote by a_t the (nominal) Arrow securities holdings the representative agent starts period t with. We assume that the Arrow security holdings are subject to the same "no Ponzi" conditions as before (debt bounded by the present value of future income). Capital is nonnegative. These nonnegativity constraints do not bind due to the Inada conditions imposed on U. The budget constraint of the representative agent is

$$P_t\Lambda_t + P_tI_t + \mathbb{E}_t\Phi_{t+1}a_{t+1} = a_t + r_tK_t.$$

From the above equation, we see that the total income from renting the capital stock, r_tK_t, and the agent's holdings in Arrow securities, a_t, are used to finance the purchase of the consumption "basket", Λ_t, the investment "basket", I_t, and the next period Arrow securities. The first order conditions for Arrow security holdings give the stochastic discount factor,

$$\Phi_{t+1} = \beta\frac{U'(\Lambda_{t+1})P_t}{U'(\Lambda_t)P_{t+1}},$$

which has the same expression as in (24). The first order conditions with respect to capital investment lead to

$$1 = \mathbb{E}_t\Phi_{t+1}\frac{P_{t+1}}{P_t}\left(1 - \delta^K + \frac{r_{t+1}}{P_{t+1}}\right),$$

or equivalently, to

$$1 = \beta\mathbb{E}_t\frac{U'(\Lambda_{t+1})}{U'(\Lambda_t)}\left(1 - \delta^K + \frac{r_{t+1}}{P_{t+1}}\right).$$

The market clearing condition $a_t = 0$ implies the aggregate accounting equation

$$P_t\Lambda_t + P_tI_t = r_tK_t.$$

We can use the Euler equations to estimate the risk aversion and discount factor as in the basic model. The difference, however, is that aggregate revenue equals aggregate consumption plus investment. Recall, however, that we also carried out estimation exercises in the main body restricting consideration to the sales of final "consumption goods," and obtained similar estimates for the relative risk aversion and discount factor parameters. Those results are, therefore, directly applicable to the extended model with capital depreciation and investment. In summary, the incorporation of capital depreciation and investment does not significantly alter the results of our analysis of the basic model (see Section 3).

Appendix C Sunk Entry Costs and Firm Exit

In this Appendix, we present a variation of the model that allows firms to incur sunk costs upon entry, but no fixed costs in each period. Further, firms can experience a "death" shock in any period that forces them to exit. Again, we consider the basic model for simplicity.

A firm must incur an exogenous sunk entry cost f_t^e (in units of capital) if it decides to enter in period t. There are no fixed production costs, and thus entering firms produce every period until they are hit by a death shock that occurs with probability η in every period. Entrants start producing only next period, while the death shock can occur even at the end of the entry period. The expected value of discounted profits of a new entrant at t (and also the ex-dividend value of a firm at t) at t is

$$v_t := \mathbb{E}_t\left[\sum_{s>t}(1-\eta)^{s-t}\frac{Q_s}{Q_t}d_s\right],$$

where $d_s := E_\alpha \pi_s(\alpha)$ are the average dividends (profits) of a firm active at s and Q is the nominal pricing kernel. Denoting by M_t^e the mass of new entrants at t, the evolution of the number of firms is

$$M_{t+1} = (1-\eta)(M_t + M_t^e).$$

The free entry condition implies

$$v_t \begin{cases} = f_t^e r_t & \text{if } M_t^e > 0, \\ \leq f_t^e r_t & \text{otherwise.} \end{cases}$$

The agent can buy a fraction x_t of the total market index (total firms) priced at a unit price v_t and invest in nominal one-period ahead Arrow securities with face values a_{t+1}, hence his budget is

$$P_t\Lambda_t + \mathbb{E}_t\left[\frac{Q_{t+1}}{Q_t}a_{t+1}\right] + v_t(M_t + M_t^e)x_t = r_tK + a_t + (v_t + d_t)M_tx_{t-1}.$$

In equilibrium, $x_t = 1$. The optimality conditions for Arrow securities imply that the SDF is

$$\Phi_{t+1} := \frac{Q_{t+1}}{Q_t} = \beta(1-\eta)\frac{U'(\Lambda_{t+1})P_t}{U'(\Lambda_t)P_{t+1}} = \beta(1-\eta)\frac{U'(\Lambda_{t+1})\Lambda_{t+1}Y_t}{U'(\Lambda_t)\Lambda_tY_{t+1}},$$

while the optimality condition for holding shares in the market index is

$$1 = E_t\left[\Phi_{t+1}\frac{v_{t+1}+d_{t+1}}{v_t}\right].$$

From the standpoint of GMM empirical analysis, this model is equivalent to our basic model with fixed costs in each period. The only difference is that the estimate for the discount factor of an agent is now in fact an estimate for $\beta(1-\eta)$, rather than only of β. Incorporating the possibility of firm exits leads to *higher* discount factors than those in the basic model, thereby bringing them even closer to empirically plausible values.

Appendix D Details of the Structural Estimation Procedure

As we discussed in the Section 5.2 on the estimation of the multi-industry model, the remaining three parameters that we need to estimate are $\mathbf{\Theta} = \{\beta, \gamma, \delta\}$. We use the Simulated Method of Moments (SMM) to estimate these parameters. For a given candidate set of parameters $\mathbf{\Theta}$, we simulate the model to create 1,000 artificial economies. Each economy contains a time-series of 100 years of data. Specifically, we first begin with a set of seed values $\Theta = \{\beta_0, \gamma_0, \delta_0\}$ and condition on the initial values of the aggregate shocks $\{\Delta z_t, \Delta w_t\}$. We then use equations (58) to (62) and the equilibrium conditions (49) to (53) in Proposition 1 to generate the fundamental states of the simulated economy and determine industry sales, entry costs, and all other variables. Since Δz_t and Δw_t are normally distributed, we use a Gauss-Hermite (GH) Quadrature grid composed of 25 [4] different combinations of shocks, and repeat the previous procedure for each grid point.

We then use the simulated economies to compute "simulated moments", and estimate the parameters that best match the simulated moments with the empirical moments. In particular, for each simulated economy, we calculate the means and standard deviations of the risk-free returns and market risk premia as well as the dispersion of industry risk premia. For each set of 1,000 artificial economies with 100 years of data, we obtain the risk-free rate, market return and industry returns. In what follows, j denotes both the market return case (i.e., $j = M$) and the different industry returns cases (i.e., $j = \{1, 2, ..., N\}$):

$$R_t^{rf} = \frac{1}{\mathbb{E}_t\left[\Phi_{t+1}\right]},$$

$$R_{j,t+1} = \frac{v_{j,t+1} + d_{j,t+1}}{v_{j,t}},$$

where

$$v_{j,t} = \sum_{s>t+1} \mathbb{E}_t\left[\beta^{s-t}\frac{Q_s'}{Q_t'}Y_{j,s}\right],$$

$$d_{j,t+1} = Y_{j,t+1},$$

[4] We choose 5 nodes for each aggregate shock variable and then obtain a grid with all possible combinations.

$$Q_t' = \left[\sum_{i=1}^{N} \eta_i^{1-\delta} \left(Y_{it}^{\frac{\sigma_i}{\sigma_i - 1}} \rho_i \bar{\alpha}_{it} \left(\sigma_i r_t f_{it}\right)^{\frac{1}{1-\sigma_i}}\right)^{\delta}\right]^{\frac{1-\gamma}{\delta}} Y_t^{\gamma-2}.$$

We then compute the following moments using the GH quadrature, with each combination of initial aggregate shocks containing 1,000 simulated economies:

- Unconditional mean of risk-free rate: $\mathbb{E}\left[R_t^{rf}\right]$.
- Unconditional standard deviation of risk-free rate: $\sqrt{\mathbb{V}\left[R_t^{rf}\right]}$.
- Unconditional mean of market risk premium: $\mathbb{E}\left[R_{M,t+1} - R_t^{rf}\right]$.
- Unconditional dispersion of expected industry risk premia: $\sqrt{\mathbb{E}\left[\overline{R}_i^{(P)2}\right] - \mathbb{E}\left[\overline{R}_i^{(P)}\right]^2}$,where $\overline{R}_i^{(P)} = \mathbb{E}\left[R_{i,t+1} - R_t^{rf}\right]$.

We do not calculate the standard deviation of the market risk premium, since this would imply doing a nested simulation that increases the computing time exponentially. While the law of total expectation allows for a straightforward calculation of the expected value of the market return, this is not necessarily true for higher moments. We denote by $\hat{\boldsymbol{V}}(\boldsymbol{\Theta})$ the 4×1 vector of the average simulated moments, and $\boldsymbol{V}$ the vector of empirical values of these moments in the data. We determine the parameter set $\boldsymbol{\theta}^*$ that solves the following optimization problem,

$$\boldsymbol{\Theta}^* = \arg\min_{\boldsymbol{\Theta}} (\boldsymbol{V} - \hat{\boldsymbol{V}}(\boldsymbol{\Theta}))^T \boldsymbol{W} (\boldsymbol{V} - \hat{\boldsymbol{V}}(\boldsymbol{\Theta})), \tag{A3}$$

where $\boldsymbol{W}$ is the identity matrix[5]. The empirical and theoretical moments can be expressed in the following form:

$$\boldsymbol{V} = \frac{1}{T} \sum_{t=1}^{T} g_t,$$

$$\hat{\boldsymbol{V}}(\boldsymbol{\Theta}) = \frac{1}{T_{sim}} \sum_{t=1}^{T_{sim}} \hat{g}_t(\boldsymbol{\Theta}),$$

where T is the number of observations in the empirical data and T_{sim} is the total number of simulations.

The asymptotic distribution of $\boldsymbol{\Theta}$ is given by

$$\sqrt{T}\left(\hat{\boldsymbol{\Theta}} - \boldsymbol{\Theta}_0\right) \to N\left(\boldsymbol{0}, (1+\bar{\tau})\left(\boldsymbol{G}^T \boldsymbol{W} \boldsymbol{G}\right)^{-1} \boldsymbol{G}^T \boldsymbol{W} \boldsymbol{\Omega} \boldsymbol{W}^T \boldsymbol{G}\left(\boldsymbol{G}^T \boldsymbol{W} \boldsymbol{G}\right)^{-1}\right), \tag{A4}$$

with

[5] This is consistent with Herzkovic et al. (2020).

$$\frac{T}{T_{sim}} \to \bar{\tau},$$

$$\boldsymbol{G} = \mathbb{E}\left[\nabla_{\boldsymbol{\Theta}} \hat{g}_t\left(\boldsymbol{\Theta}\right)|_{\boldsymbol{\Theta}=\boldsymbol{\Theta}_0}\right],$$

$$\boldsymbol{\Omega} = \sum_{t=-\infty}^{\infty} \mathbb{E}\left[\left(g_t - \mathbb{E}\left(g_t\right)\right)\left(g_t - \mathbb{E}\left(g_t\right)\right)^T\right].$$

To compute the standard errors of the estimated parameters we estimate the necessary inputs (i.e., $\hat{\boldsymbol{G}}$ and $\hat{\boldsymbol{\Omega}}$) and take the squared root of the main diagonal elements of $\mathbb{V}\left[\hat{\boldsymbol{\Theta}}\right]$, which is obtained directly from (A4). Model validity is addressed by using the Hansen J test for overidentifying restrictions.

Table 1: Summary Statistics

The table below shows summary statistics for the key variables in the model, from 1987 to 2020. The variables considered are both the levels and relative changes in the aggregate consumption expenditures (Y_t), number of establishments (M_t) and capital productivity (α_t).

	Obs	Mean.	Std. Dev.	Min	p25	p50	p75	Max
Y_t (\$ b)	34	7267	3197	2634	4387	6937	9829	12882
$\frac{\Delta Y_t}{Y_{t-1}}$ (%)	34	4.91	2.29	-3.2	3.89	5.3	6.35	8.38
M_t (mm)	34	5.50	0.53	4.47	5.09	5.62	5.90	6.26
$\frac{\Delta M_t}{M_{t-1}}$ (%)	34	1.07	1.01	-1.94	0.64	1.15	1.78	2.71
α_t	34	0.30	0.02	0.26	0.28	0.30	0.32	0.33
$\frac{\Delta \alpha_t}{\alpha_{t-1}}$ (%)	34	-0.62	1.78	-5.85	-1.3	-0.23	0.37	2.49

Table 2: Comparison of Price Indices

This table compares the growth in the product-variety-based aggregate price index (P_t) and the growth of the Consumer Price Index (CPI_t), and their covariance with the market return R_M. We consider annual returns over the period 1987 to 2020.

	Mean	Std. Dev.	Corr(R_M)	Cov(R_M)
$\frac{P_{t+1}}{P_t}$	1.0537	0.0207	-0.4188	-0.001468
$\frac{CPI_{t+1}}{CPI_t}$	1.0257	0.0126	-0.0647	-0.000138

Table 3: GMM Estimates of Risk Aversion γ and Discount Factor β in Basic Model

This table reports the results of the GMM estimation of the risk aversion and discount factor in the basic model. Our sample consists of annual data from 1987 to 2020. In each column, the moments for the corresponding portfolio returns, market return, and risk-free return are used. We use aggregate nondurable consumption and services from NIPA in the GMM estimation. "FF25" means that the returns of Fama-French 25 portfolios by size and book-to-market are used in the moment conditions. "FF3" means that the returns of Fama-French 3 factors are used in the moment conditions. . "FF3 and Instruments" indicates that additional instruments, including lagged consumption growth, lagged CAY, lagged dividend-price ratio, lagged productivity growth, and lagged growth in number of establishments, are used in the moment conditions. "FF2x3" means that the returns of the Fama-French 2x3 portfolios by size and book-to-market value are used in the moment conditions. "FF2x3 and Instruments" is analogous to the "FF3 and Instruments" case, in which additional instruments are included in the moment conditions. Standard errors in the GMM estimation are calculated following Newey and West and reported in parentheses.

	GMM Estimation: Single-Industry Model with Product Variety				
	FF25	FF3	FF3 and Instruments	FF2x3	FF2x3 and Instruments
β	0.9038	0.9749	0.9999	0.9242	0.902
	(0.0304)	(0.0835)	(0.0085)	(0.0627)	(0.0054)
γ	5.7751	3.6031	2.427	5.3412	6.7456
	(7.3836)	(16.4775)	(1.3037)	(12.9192)	(0.5608)
J-test	11.8615	9.7873	10.8747	10.2107	11.7242
p-value	0.9876	0.0205	0.5397	0.1161	0.9828

Table 4: Intra-Industry Product Substitutability Parameters in the Extended Model

This table shows the values of intra-industry product substitutability parameters estimated by the mean sales to profits ratios in the 14 2-digit NAICS industries. We only consider firms that produce final consumption goods.

NAICS (2 digit)	Industry Name	Mean Sales/Profits Ratio (σ_i)
22	Utilities	5.44
23	Construction	8.97
31-33	Manufacturing	6.75
44-45	Retail	15.94
48-49	Transportation	8.39
51	Information	3.21
53	Real Estate	3.48
54	Professional Services	5.94
56	Administrative Services	9.11
61	Education	6.12
62	Health	7.65
71	Entertainment	5.58
72	Accomodation	9.94
81	Other Services	7.34

Table 5: Classification of Final Consumption Goods in Fama-French 48 Industries

This table reports the classification of Fama-French 48 industries into industries that produce final consumption goods (1) or that do not (0). The classification is based on the categories of consumption goods in the Consumption Expenditure Survey by the Bureau of Labor Statistics.

Number	Abbreviation	Description	Consumption Goods
1	Agric	Agriculture	0
2	Food	Food Products	1
3	Soda	Candy & Soda	1
4	Beer	Beer & Liquor	1
5	Smoke	Tobacco Products	1
6	Toys	Recreation	1
8	Books	Printing and Publishing	1
9	Hshld	Consumer Goods	1
10	Clths	Apparel	1
11	Hlth	Healthcare	1
12	MedEq	Medical Equipment	0
13	Drugs	Pharmaceutical Products	1
14	Chems	Chemicals	0
16	Txtls	Textiles	0
17	BldMt	Construction Materials	0
18	Cnstr	Construction	1
19	Steel	Steel Works Etc	0
20	FabPr	Fabricated Products	0
21	Mach	Machinery	0
22	ElcEq	Electrical Equipment	1
23	Autos	Automobiles and Trucks	1
24	Aero	Aircraft	0
25	Ships	Shipbuilding, Railroad Equipment	0
26	Guns	Defense	0
27	Gold	Precious Metals	0
28	Mines	Non-Metallic and Industrial Metal Mining	0
29	Coal	Coal	0
30	Oil	Petroleum and Natural Gas	0
31	Util	Utilities	1
32	Telcm	Communication	1
33	PerSv	Personal Services	1
34	BusSv	Business Services	0
35	Comps	Computers	1
36	Chips	Electronic Equipment	0
37	LabEq	Measuring and Control Equipment	0
38	Paper	Business Supplies	0
39	Boxes	Shipping Containers	1
40	Trans	Transportation	1

(Continued)

Number	Abbreviation	Description	Final Goods
41	Whlsl	Wholesale	0
42	Rtail	Retail	1
43	Meals	Restaurants, Hotels, Motels	1
44	Banks	Banking	0
45	Insur	Insurance	0
46	RlEst	Real Estate	1
48	Other	Almost Nothing	0

Table 6: GMM Estimates of Risk Aversion γ and Discount Factor β in Extended Multi-Industry Model

This table reports the results of GMM estimation of the risk aversion and discount factor in the extended model. Our sample consists of annual data from 1987 to 2020. We use Gross Domestic Product Value Added of final consumption firms to generate the SDF in the extended model. In each column, the moments for the corresponding portfolio returns, market return, and risk-free return are used. “FF25” means that the returns of Fama-French 25 portfolios by size and book-to-market are used in the moment conditions. “FF3” means that the returns of the Fama-French 3 factor model are used in the moment conditions. “FF3 and Instruments” indicates that additional instruments, including lagged consumption growth, lagged CAY, lagged dividend-price ratio, lagged productivity growth, and lagged growth in number of establishments, are used in the moment conditions. “FF 2x3” means that the returns of Fama-French 2x3 portfolios by size and book-to-market value are used in the moment conditions. “FF2x3 and Instruments” is analogous to the “FF3 and Instruments” case, in which additional instruments are included in the moment conditions. Standard errors in the GMM estimation are calculated following Newey and West and reported in parentheses.

	GMM Estimation: Multi-Industry Model with Product Variety				
	FF25	FF3	FF3 and Instruments	FF2x3	FF2x3 and Instruments
β	0.9152	0.9881	0.9999	0.9187	0.9103
	(0.0145)	(0.0398)	(0.0031)	(0.0249)	(0.0027)
γ	2.4363	1.9907	1.0888	4.0273	3.512
	(3.0471)	(6.348)	(0.3339)	(4.1264)	(0.1521)
J-test	12.2094	10.6658	11.9598	10.917	12.2288
p-value	0.9848	0.0137	0.4489	0.091	0.9772

Table 7: Structural Estimation of the Extended Multi-Industry Model

This table reports the results of the structural estimation of the extended model using the Simulated Method of Moments (SMM). In Panel A, we report the directly estimated parameters. In Panel B, we report the indirectly estimated structural parameters using SMM with the standard errors of the estimated parameters in parentheses. In Panel C, we report the empirical and simulated moments along with the corresponding standard errors of the simulated moments in parentheses.

Panel A: Directly Estimated Parameters

	Parameter	Estimate
Directly Estimated Parameters		
Persistence of aggregate consumptions expenditures	a_z	0.6249
Intercept of aggregate consumption expenditures	$\lambda_z^{(1)}$	12.67
Growth rate of aggregate consumption expendittures	$\lambda_z^{(2)}$	0.0668
Volatility of aggregate consumption expenditures shocks	σ_z	0.022
Persistence of entry costs	a_w	0.5199
Volatility of entry costs	σ_w	0.049
Correlation of shocks	ρ_{zw}	-0.3164

Panel B: Indirectly Estimated Parameters using SMM

	Parameter	Estimate
Discount Factor	β	0.9998
		(0.0033)
Risk Aversion	γ	5.97
		(0.4113)
Inter-industry Product Substitutability	δ	-1.21
		(0.5987)

Table 7: Continued

Panel C: Empirical and Simulated Moments

	Empirical Moments	Simulated Moments
Mean of Market Risk Premia	6.55%	6.72%
Mean Risk-free Return	3.19%	3.63%
Std. Dev. Risk-free Return	2.53%	2.76%
Dispersion of Industry Risk Premia	1.86%	0.81%
Model Validity		
J-test	0.0034	
p-value	0.9534	

Table 8: Calibrated Industrial Parameters in Structural Estimation of the Extended Model

This table reports the calibrated industrial parameters in the structural estimation of the extended model.

Industry	σ_i	σ_{α_i}	$\lambda_{f_i}^{(1)}$	$\lambda_{f_i}^{(2)}$	$\lambda_{\alpha}^{(1)}$	$\lambda_{\alpha}^{(2)}$	b_i	c_i
Utilities	5.44	0.0735	0.83	-0.0258	-1.48	-0.0057	-0.4795	0.7424
Construction	8.97	0.1045	-2.79	-0.0068	1.12	-0.0295	0.023	-0.1114
Manufacturing	6.75	0.0408	-0.64	-0.0082	-0.29	-0.0093	-0.2133	0.251
Retail	15.94	0.0571	-3.53	-0.0056	0.17	-0.0102	-0.1494	-0.1024
Transportation	8.39	0.0682	-1.71	-0.0148	-1.08	0.0052	-0.0813	0.3027
Information	3.21	0.0407	0.15	-0.0133	-0.82	-0.0118	-0.1806	0.37
Real Estate	3.48	0.0644	0.06	-0.0111	-2.26	-0.0032	0.1908	0.1818
Professional Services	5.94	0.1208	-2.03	-0.0056	0.82	-0.0359	0.3484	0.1948
Administrative Services	9.11	0.0979	-2.72	0.0004	0.43	-0.029	0.0715	-0.0632
Education	6.12	0.0654	-1.64	-0.007	-0.88	-0.0075	-0.2134	0.1547
Health	7.65	0.0556	-2.31	-0.0064	-0.38	-0.0096	0.1883	0.1352
Entertainment	5.58	0.1053	-1.91	-0.0076	-0.45	-0.0015	-0.1927	0.0443
Accomodation	9.94	0.0582	-3.15	-0.0101	-0.42	0.0024	0.0868	-0.0929
Other Services	7.34	0.0551	-3.16	-0.0129	-0.2	-0.013	-0.2999	-0.0808

Table 9: GMM and SMM Robustness Exercises Using Number of Firms

Panel A: GMM Single Industry

This table reports the results of the GMM estimation of the risk aversion and discount factor in the basic model using the number of firms as the measure of M_t.

	GMM Estimation: Single-Industry Model with Product Variety				
	FF25	FF3	FF3 and Instruments	FF2x3	FF2x3 and Instruments
β	0.9026	0.972	0.9999	0.9258	0.9011
	(0.0331)	(0.0908)	(0.0078)	(0.0677)	(0.0042)
γ	5.6344	3.7674	2.3611	4.7645	6.2594
	(7.3026)	(16.292)	(1.1536)	(12.9145)	(0.5268)
J-test	11.8621	9.7701	10.8685	10.2166	11.7958
p-value	0.9876	0.0206	0.5402	0.1158	0.9821

Panel B: GMM Multi Industry

This table reports the results of the GMM estimation of the risk aversion and discount factor in the extended model using the number of firms as the measure of M_{it}, for all industries $i \in \{1, 2, ..., N\}$.

	GMM Estimation: Multi-Industry Model with Product Variety				
	FF25	FF3	FF3 and Instruments	FF2x3	FF2x3 and Instruments
β	0.9106	0.9756	0.9999	0.908	0.906
	(0.02)	(0.0574)	(0.0034)	(0.0343)	(0.0026)
γ	2.7642	3.0037	1.1518	4.6569	3.7191
	(3.1594)	(6.5697)	(0.3742)	(4.4372)	(0.2002)
J-test	12.2104	10.6444	11.8348	10.9257	12.2253
p-value	0.9848	0.0138	0.459	0.0907	0.9772

Panel C: Indirectly Estimated Parameters using SMM

	Parameter	Estimate
Discount Factor	β	0.9998
		(0.0029)
Risk Aversion	γ	5.66
		(0.3141)
Inter-industry Product Substitutability	δ	-1.20
		(0.6918)