4-30-2018

Capacities and Cancellation

Robin Baidya

CAPACITIES AND CANCELLATION

by

ROBIN BAIDYA

Under the Direction of Yongwei Yao

ABSTRACT

This dissertation investigates the existence of surjective and split surjective maps between modules. A classic result in this direction is Serre's Splitting Theorem, which gives a lower bound on the maximum rank of a free summand of a finitely generated projective module. Here, the underlying ring is assumed to be commutative and Noetherian with a finite-dimensional maximal spectrum. De Stefani, Polstra, and Yao generalize this theorem by removing the projective condition. Bass extends Serre's Splitting Theorem by considering a module-finite algebra over the ring and replacing the Noetherian condition on the ring with a Noetherian condition on the maximal spectrum. We generalize all of these results by replacing the free summand with a direct sum of copies of a finitely presented module. This requires us to replace rank with a more abstract notion that we call *splitting capacity*. We generalize Serre, De Stefani–Polstra–Yao, and Bass in a second way by studying the number of summands isomorphic to one module that can appear in a quotient of another module. This is related to the notion of *surjective capacity*. In the case of finitely generated modules over a Dedekind domain, we show that we can even characterize when a surjective or splitting capacity is equal to a fixed nonnegative integer. In a separate case, we can guarantee when there exists a split surjective map of a special form. This allows us to extend cancellation theorems by Bass and De Stefani–Polstra–Yao, which provide criteria for when two modules with isomorphic direct-sum complements in a larger module are isomorphic. The complement in each of these theorems is assumed to be finitely generated and projective, although each theorem easily reduces to the case in which the complement is a rank-one free module. We show that we can replace the rank-one free module with a finitely presented homothetic module. As a consequence, our work reveals a *cancellation property* shared by ideals of finite-dimensional Noetherian normal domains and canonical modules of finite-dimensional Cohen–Macaulay rings.

INDEX WORDS: algebra, basic set, Bass, cancellation, canonical module, Cohen–Macaulay, commutative, direct sum, endomorphism, Dedekind, homothetic, homothety, integrally closed, module, Noetherian, normal, projective, ring, Serre, splitting capacity, stable range, stable rank, surjective capacity

CAPACITIES AND CANCELLATION

by

ROBIN BAIDYA

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

in the College of Arts and Sciences

Georgia State University

2018

CAPACITIES AND CANCELLATION

by

ROBIN BAIDYA

Committee Chair: Yongwei Yao

Committee: Florian Enescu

Zhongshan Li

Mariana Montiel

Electronic Version Approved:

Office of Graduate Studies

College of Arts and Sciences

Georgia State University

May 2018

DEDICATION

To my sister Sophia, who changed the course of my life

ACKNOWLEDGEMENTS

Insofar as this dissertation marks the culmination of my academic career thus far, many deserve mention here, although I can only name a few.

There are those to whom I owe my interest in mathematics. My mother and father, who dazzled me when I was very young with their ability to perform lightning-quick computations, first cultured my fascination with the subject. I attribute my mathematical curiosity above all else to the problems that they assigned to me during the summer and the extra mathematics courses that they drove me to and from during the school year. Tom Farquhar, my fifth-grade mathematics teacher, can perhaps claim the status of the next most important player, as he encouraged me to advance through mathematics courses in an accelerated fashion. This, and his impeccably drafted classroom decorations on problem-solving techniques, which remain fresh in my mind, offered me a glimpse of the vast world of mathematics at an early age.

I have had the fortune of spending time with many outstanding mathematicians since then. To Ronald Solomon, I owe much of my knowledge of, and enthusiasm for, abstract algebra. To David Clampitt, I owe my graduate education, for it was due to his suggestion that I applied to Georgia State University in the first place. For their leadership of the Department of Mathematics and Statistics at Georgia State University, Guantao Chen and Alexandra Smirnova merit my admiration and gratitude. For many wonderful seminar talks and discussions on algebra, I thank Alan Dills, Irina Ilioaea, Siang Ng, Felipe Pérez, and Kevin Slote. There are many others whom I have neglected to mention, though they remain in my thoughts.

The fact that this dissertation has reached its final form is largely due to four particular professors at Georgia State University. For serving on my dissertation committee, I offer heartfelt thanks to Florian Enescu, Zhongshan Li, Mariana Montiel, and my advisor Yongwei Yao. This dissertation is better because of their time, their energy, their diligence, and their insights. I would also like to thank my advisor for fostering my growth as a mathematician and for helping me develop my mathematical work. It was my advisor who pointed me to the problems that I address in this dissertation, and it was at his urging that I decided to commit myself to this project. My advisor's exceptional ability to generalize statements and simplify proofs was an inspiration at many stages of the writing process.

There are a few additional people without whose support I could not have progressed to this point in my life. To Ned Behrman, Joan Herbers, Rocco Di Pietro, and all of the other musicians with whom I used to play in Columbus, Ohio, I give thanks, for their mentorship, their sensitivity, and their optimism. To my sisters, I give thanks, for their resourcefulness, their creativity, and their unending generosity. And to Lee Shropshire, who has provided a constant source of humor, assurance, and love, I give thanks.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Determining the direct summands of a finitely generated module over a principal ideal domain is a local problem: Such a module is a direct sum of cyclic coprimary modules, and the summands associated to a given prime ideal of the ring can be identified by localizing the module at that prime ideal. Moreover, every summand of the module already appears in such a decomposition, and two summands are isomorphic if and only if their complements are isomorphic. (See Section 2.7.)

It is to be expected that some of the above properties will fail if we replace the principal ideal domain with a more general commutative ring. Indeed, it is possible for all of the above properties to fail, even over a one-dimensional hypersurface domain. Wiegand shows, for example, that the ring $R := k[x^2, x^3]$, where k is a field, does not satisfy any of the above properties [37, page 447]: This ring has a nonprincipal invertible ideal I such that $\overline{R} \oplus I \cong \overline{R} \oplus R$, where $\overline{R}$ denotes $k[x]$, the integral closure of R in the function field $k(x)$. Since $\overline{R}$ is a finitely generated R-module, so is $\overline{R} \oplus I$. Being torsion-free of rank one, the R-modules $\overline{R}$ and I are indecomposable, and yet neither one is cyclic. As I is invertible, it is locally indistinguishable from R.

This example can be used to generate a similar example in any finite dimension $d \geqslant 2$: If S is the d-dimensional hypersurface domain $R[y_1, \ldots, y_{d-1}]$, then IS is a nonprincipal invertible ideal of S such that $\overline{S} \oplus IS \cong \overline{S} \oplus S$, where $\overline{S} := k[x, y_1, \ldots, y_{d-1}]$.

These examples might suggest that, if local information can characterize a global summand of a finitely generated module, then the underlying ring must not differ very much from a principal ideal domain. There might even be an implication that such a ring would have dimension at most one. On the contrary, this dissertation will show that affirmative statements are available in many cases if certain local phenomena occur in abundance. These statements will apply to a large class of modules over a large class of rings, and collectively the rings will cover every possible finite dimension.

1.1 Surjective and splitting capacities

Serre's Splitting Theorem [28, *Théorème 1*] is heralded as one of the earliest theorems to require the existence of sufficiently many local split surjective maps between modules to ensure the existence of global split surjective maps. This result from algebraic K-theory gives a lower bound on the maximum rank of a free summand of a finitely generated projective module over a certain type of commutative Noetherian ring.

To state this theorem precisely, we must first review some fundamentals from commutative algebra. For a commutative ring R, the symbol $\operatorname{Spec}(R)$ refers to the set of all prime ideals of R. Called the *prime spectrum of* R, the set $\operatorname{Spec}(R)$ can be viewed as a topological space whose closed sets are precisely the sets of the form $\operatorname{Var}(I) := \{\mathfrak{p} \in \operatorname{Spec}(R) : I \subseteq \mathfrak{p}\}$, where I is an ideal of R. The symbol $j\text{-}\operatorname{Spec}(R)$ refers to the set of all $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{p}$ can be expressed as an intersection of maximal ideals of R; this is called the *j-spectrum of* R. Similarly, the *maximal spectrum of* R, denoted $\operatorname{Max}(R)$, refers to the set of all maximal ideals of R. We view $j\text{-}\operatorname{Spec}(R)$ and $\operatorname{Max}(R)$ as subspaces of $\operatorname{Spec}(R)$ in the natural way. The space $j\text{-}\operatorname{Spec}(R)$ was introduced by Swan in [32] and was used by Eisenbud and Evans in [10] and by De Stefani, Polstra, and Yao in [7].

We need some notions from general topology here as well. We say that a topological space is *Noetherian* if every nonempty collection of closed sets has a minimal member. In the corollary following Proposition 1 in [32], Swan notes that $j\text{-}\operatorname{Spec}(R)$ is Noetherian if and only if $\operatorname{Max}(R)$ is Noetherian. (On the other hand, the prime spectrum of a commutative Noetherian ring and all of its subspaces are always Noetherian.) A nonempty subset of a topological space is *irreducible* if it is not the union of two of its proper closed subsets. For example, the closed irreducible subsets of $\operatorname{Spec}(R)$ are precisely the sets $\operatorname{Var}(\mathfrak{p})$, where $\mathfrak{p} \in \operatorname{Spec}(R)$. The *Krull dimension* of a topological space Y, denoted $\dim(Y)$, is the supremum of the nonnegative integers t such that there exists a chain of $t+1$ distinct closed irreducible subsets of Y. In [32], Swan shows that $\dim(j\text{-Spec}(R)) = \dim(\operatorname{Max}(R))$. For a subspace X of $\operatorname{Spec}(R)$ and a prime $\mathfrak{p}$ of X, the symbol $\dim_X(\mathfrak{p})$ refers to the Krull dimension of $\operatorname{Var}(\mathfrak{p}) \cap X$. Occasionally, for an arbitrary ideal I of R, we will abuse notation and write $\dim(R/I)$ to mean $\dim(\operatorname{Var}(I))$, and we will call this number the *dimension of* I.

At this point, we could state Serre's Splitting Theorem, but we choose to defer this for just another moment in order to introduce a new definition. The purpose of this definition is to generalize the concept of rank. This will help us unify several theorems, some our own, that build off of Serre's work. In the following definition, $N^{\oplus t}$ refers to the direct sum of t copies of a right module N over a ring S. We do not assume that the ring S is commutative; however, since we would like to exploit the topological properties of the prime spectra of commutative rings, we fix a ring homomorphism from a commutative ring R to the ring S such that the image of R is in the center of S. Relative to this ring homomorphism, we may view every right S-module as a right R-module, and we may localize at any prime ideal of R accordingly. The ring S is then an R*-algebra*, and one can verify that, for every $\mathfrak{p} \in \operatorname{Spec}(R)$, defining

$$\frac{s}{u} \cdot \frac{t}{v} = \frac{st}{uv} \qquad \text{for all } \frac{s}{u}, \frac{t}{v} \in S_{\mathfrak{p}}$$

renders $S_{\mathfrak{p}}$ an $R_{\mathfrak{p}}$-algebra in the natural way. In particular, localization commutes with restriction of scalars for every $\mathfrak{p} \in \operatorname{Spec}(R)$.

Definition 1.1.1. Let R be a commutative ring, S an R-algebra, and M and N right S-modules. We let $\operatorname{sur}_S(M, N)$ denote the supremum of the nonnegative integers t such that there exists a surjective S-linear map from M to $N^{\oplus t}$, and we refer to $\operatorname{sur}_S(M, N)$ as the *global surjective capacity of* M *with respect to* N *over* S.

Let $\mathfrak{p} \in \operatorname{Spec}(R)$. We refer to $\operatorname{sur}_{S_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ as the *local surjective capacity of* M *with respect to* N *over* S *at* $\mathfrak{p}$.

Under the hypotheses of the previous definition, we observe that we can always get an upper bound on a global surjective capacity in terms of local surjective capacities: Defining

$\mathrm{Supp}_R(N) := \{\mathfrak{p} \in \mathrm{Spec}(R) : N_\mathfrak{p} \neq 0\}$ (this is called the *support of N over R*), we may write

$$\mathrm{sur}_S(M, N) \leqslant \inf\{\mathrm{sur}_{S_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) : \mathfrak{m} \in \mathrm{Max}(R) \cap \mathrm{Supp}_R(N)\}.$$

Serre's Splitting Theorem then gives a lower bound on $\mathrm{sur}_S(M, N)$ for certain choices of R, S, M, and N.

Theorem 1.1.2 (Serre's Splitting Theorem [28, *Théorème 1*])**.** *Let R be a commutative Noetherian ring, and let P be a finitely generated projective R-module. Let $X := j\text{-}\mathrm{Spec}(R)$, and suppose that $\dim(X) < \infty$. Then*

$$\mathrm{sur}_R(P, R) \geqslant \inf\{\mathrm{sur}_{R_\mathfrak{p}}(P_\mathfrak{p}, R_\mathfrak{p}) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in X\}.$$

As promised, this theorem gives a lower bound on the maximum rank of a free summand of P: Since every surjective map onto a free module splits, $\mathrm{sur}_R(P, R)$ is exactly the maximum rank of a free summand of P. Combining Serre's lower bound with the trivial upper bound mentioned earlier, we can now express $\mathrm{sur}_R(P, R)$ with an error of at most $\dim(X)$ when $R \neq 0$: If $t := \inf\{\mathrm{sur}_{R_\mathfrak{p}}(P_\mathfrak{p}, R_\mathfrak{p}) : \mathfrak{p} \in X\}$, then

$$t - \dim(X) \leqslant \mathrm{sur}_R(P, R) \leqslant t.$$

The lower bound $t - \dim(X)$ is, of course, generally worse than the lower bound in Serre's Splitting Theorem; the purpose of the last display is only to clarify Serre's main point.

Several generalizations and extensions of Serre's Splitting Theorem can be found in the literature. De Stefani, Polstra, and Yao show that the projective condition can be removed from Serre's Splitting Theorem:

Theorem 1.1.3 (De Stefani–Polstra–Yao Splitting Theorem [7, Theorem 3.12])**.** *Let R be a commutative Noetherian ring, and let M be a finitely generated R-module. Let $X :=$*

j-Spec(R)*, and suppose that* $\dim(X) < \infty$*. Then*

$$\operatorname{sur}_R(M, R) \geqslant \inf\{\operatorname{sur}_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in X\}.$$

Bass extends Serre's Splitting Theorem by replacing the Noetherian condition on R with a Noetherian condition on $\operatorname{Max}(R)$, considering a module-finite algebra S over R, and replacing P with a direct summand of a direct sum of finitely presented right S-modules:

Theorem 1.1.4 (Bass's Splitting Theorem [3, Theorem 8.2])**.** *Let R be a commutative ring, S a module-finite R-algebra, and M a direct summand of a direct sum of finitely presented right S-modules. Suppose that $Y := \operatorname{Max}(R)$ is Noetherian with $\dim(Y) < \infty$. Suppose also that $\operatorname{sur}_{S_\mathfrak{m}}(M_\mathfrak{m}, S_\mathfrak{m}) \geqslant 1 + \dim(Y)$ for every $\mathfrak{m} \in Y$. Then $\operatorname{sur}_S(M, S) \geqslant 1$.*

Bass's Splitting Theorem is an extension of Serre in the sense that it implies Serre *under the conditions of Serre*. However, it should be noted that Bass does not express $\operatorname{sur}_S(M, S)$ with the same level of accuracy as Serre; Bass gives a sufficient criterion only for a *positive* global surjective capacity. De Stefani, Polstra, and Yao, on the other hand, are able to maintain the same level of accuracy as Serre but in a setting different from Bass.

There are some other variations on Serre's Splitting Theorem worth mentioning. Stafford [29, Proposition 5.5] proves a result for Noetherian rings that are not necessarily module-finite algebras over commutative rings with finite-dimensional Noetherian maximal spectra. For this, Stafford defines an analogue of surjective capacity called *r-rank*, and he replaces Krull dimension with another notion that accommodates noncommutative Noetherian rings. Eisenbud and Evans extend Serre's Splitting Theorem in another direction via their Basic Element Theorem [10, Theorem A]. Mixing Serre's conditions with those of Bass, Eisenbud and Evans consider a finitely generated right module M over a ring S that is a module-finite algebra over a commutative ring R, where R has a finite-dimensional Noetherian j-spectrum. Working under these conditions and one additional condition, they show that it is possible to find an element x of M that is part of a minimal generating set for M after localizing at any prime ideal in j-Spec$(R) \cap \operatorname{Supp}_R(M)$. Heitmann [18, Theorem 2.5] and Coquand, Lombardi, and Quitté [6, Corollary 3.2] then generalize the Basic Element Theorem in various ways.

The majority of our efforts in Chapter 2 will be directed toward generalizing the splitting theorems of De Stefani–Polstra–Yao and Bass. We generalize the De Stefani–Polstra–Yao Splitting Theorem in the following manner: We let S be a module-finite R-algebra; we allow M to be a right S-module; and we replace the module R with an arbitrary finitely generated right S-module N:

Theorem 1.1.5. *Let R be a commutative Noetherian ring, S a module-finite R-algebra, and M and N finitely generated right S-modules. Let $X := j\text{-}\mathrm{Spec}(R) \cap \mathrm{Supp}_R(N)$, and suppose that $\dim(X) < \infty$. Then*

$$\mathrm{sur}_S(M, N) \geqslant \inf\{\mathrm{sur}_{S_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in X\}.$$

Assuming the truth of a certain statement that we call the *Surjective Lemma* (Lemma 2.1.14), we can prove Theorem 1.1.5 rather quickly, and we do so in Section 2.1.

With a little more work, we can generalize Bass's Splitting Theorem as well. For this, we replace the module S with an arbitrary finitely generated right S-module N, and we change the number *1* to an arbitrary positive integer t. With the help of these modifications, we get two new conclusions:

Theorem 1.1.6. *Let R be a commutative ring, S a module-finite R-algebra, M a direct summand of a direct sum of finitely presented right S-modules, and N a finitely generated right S-module. Suppose that $Y := \mathrm{Max}(R) \cap \mathrm{Supp}_R(N)$ is Noetherian with $\dim(Y) < \infty$. Then the following statements hold:*

(1) Let t be a positive integer, and suppose that $\mathrm{sur}_{S_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) \geqslant t + \dim(Y)$ for every $\mathfrak{m} \in Y$. Then $\mathrm{sur}_S(M, N) \geqslant t$.

(2) Having an infinite surjective capacity is a local property. In fact, $\mathrm{sur}_S(M, N) = \infty$ if and only if $\mathrm{sur}_{S_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) = \infty$ for every $\mathfrak{m} \in Y$.

(3) Suppose that $\operatorname{sur}_{S_{\mathfrak{n}}}(M_{\mathfrak{n}}, N_{\mathfrak{n}}) < \infty$ *for some* $\mathfrak{n} \in Y$*. Then*

$$\operatorname{sur}_S(M, N) \geqslant \min\{\operatorname{sur}_{S_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) : \mathfrak{m} \in Y\} - \dim(Y).$$

We prove Theorem 1.1.6 in Section 2.2, again assuming the truth of the Surjective Lemma. Sections 2.3–2.5 are then dedicated to proving the Surjective Lemma. In Section 2.3, we show that proving the Surjective Lemma, which may involve infinitely many prime ideals of R, reduces to proving a statement about a finite subset Λ of $\operatorname{Spec}(R)$. In Section 2.4, we study the maximal ideals of R in Λ and continue working toward a proof of the Surjective Lemma. We finish our proof of the Surjective Lemma in Section 2.5.

In Section 2.6, we show that we can prove split surjective analogues of Theorems 1.1.5 and 1.1.6 with just a few modifications to the techniques of Sections 2.1–2.5. This requires us to give the following definition:

Definition 1.1.7. Let R be a commutative ring, S an R-algebra, and M and N right S-modules. We let $\operatorname{spl}_S(M, N)$ denote the supremum of the nonnegative integers t such that there exists a split surjective S-linear map from M to $N^{\oplus t}$, and we refer to $\operatorname{spl}_S(M, N)$ as the *global splitting capacity of* M *with respect to* N *over* S.

Let $\mathfrak{p} \in \operatorname{Spec}(R)$. We refer to $\operatorname{spl}_{S_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$ as the *local splitting capacity of* M *with respect to* N *over* S *at* $\mathfrak{p}$.

As with surjective capacities, we can always get an upper bound on a global splitting capacity in terms of local splitting capacities:

$$\operatorname{spl}_S(M, N) \leqslant \inf\{\operatorname{spl}_{S_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) : \mathfrak{m} \in \operatorname{Max}(R) \cap \operatorname{Supp}_R(N)\}.$$

Next, we notice that, in Theorems 1.1.3 and 1.1.4, we can replace every instance of the symbol *sur* with the symbol *spl* to get a lower bound on a global splitting capacity using local splitting capacities. Our next two theorems ensure that we can make the same modifications to Theorems 1.1.5 and 1.1.6 if we assume that N is finitely presented over S.

Theorem 1.1.8. *Let R be a commutative Noetherian ring, S a module-finite R-algebra, and M and N finitely generated right S-modules. Let $X := j\text{-Spec}(R) \cap \operatorname{Supp}_R(N)$, and suppose that* $\dim(X) < \infty$. *Then*

$$\operatorname{spl}_S(M, N) \geqslant \inf\{\operatorname{spl}_{S_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in X\}.$$

Theorem 1.1.9. *Let R be a commutative ring, S a module-finite R-algebra, M a direct summand of a direct sum of finitely presented right S-modules, and N a finitely presented right S-module. Suppose that $Y := \operatorname{Max}(R) \cap \operatorname{Supp}_R(N)$ is Noetherian with $\dim(Y) < \infty$. Then the following statements hold:*

(1) Let t be a positive integer, and suppose that $\operatorname{spl}_{S_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) \geqslant t + \dim(Y)$ for every $\mathfrak{m} \in Y$. Then $\operatorname{spl}_S(M, N) \geqslant t$.

(2) Having an infinite splitting capacity is a local property. In fact, $\operatorname{spl}_S(M, N) = \infty$ if and only if $\operatorname{spl}_{S_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) = \infty$ for every $\mathfrak{m} \in Y$.

(3) Suppose that $\operatorname{spl}_{S_\mathfrak{n}}(M_\mathfrak{n}, N_\mathfrak{n}) < \infty$ for some $\mathfrak{n} \in Y$. Then

$$\operatorname{spl}_S(M, N) \geqslant \min\{\operatorname{spl}_{S_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) : \mathfrak{m} \in Y\} - \dim(Y).$$

To close Chapter 2, we show in Section 2.7 that, for finitely generated modules over a Dedekind domain, we can do more than provide lower and upper bounds on surjective and splitting capacities: In this case, we can characterize when a surjective or splitting capacity is equal to a fixed nonnegative integer. Our proofs rely heavily on cancellation properties available under these circumstances.

1.2 Cancellation of homothetic modules

The *Cancellation Problem* asks whether two modules with isomorphic direct-sum complements in a larger module are isomorphic. In other words, if K, L, and M are right modules over a ring S such that $K \oplus L \cong K \oplus M$, the question is whether $L \cong M$. If the answer is *yes*, then we say that *cancellation holds*; otherwise, we say that *cancellation fails*. Wiegand's example from the beginning of this chapter shows that cancellation can fail for finitely generated modules over a one-dimensional hypersurface domain. Additional counterexamples to cancellation are due to Eilenberg [2, page 24]; Chase [5, Proposition 4.2]; Guralnick, Levy, and Wiegand [14]; Hassler and Wiegand [16, Example 6.20]; Klingler and Levy [20]; Levy and Wiegand [22, Example 6.8]; Swan [31, Theorem 3]; and Wiegand [36] [37, Theorem 3.1 and Corollary 3.3].

On the other hand, it is well known that cancellation holds for finitely generated modules over a Dedekind domain. (See Section 2.7.) To describe some additional examples in which cancellation holds, we first give a definition.

Definition 1.2.1. Let S be a ring. The *stable rank of* S, denoted $\operatorname{sr}(S)$, is the infimum of the positive integers t such that, for every integer $u \geqslant t+1$ and for all $r_1, \dots, r_u \in S$ satisfying $r_1S+\cdots+r_uS = S$, there exist $s_1, \dots, s_{u-1} \in S$ with $(r_1+r_us_1)S+\cdots+(r_{u-1}+r_us_{u-1})S = S$.

Using the concept of stable rank, Evans derives the following cancellation theorem in [12]:

Theorem 1.2.2 (Evans's Cancellation Theorem [12, Theorem 2])**.** *Let S be a ring, and let K, L, and M be right S-modules. Suppose that* $\operatorname{sr}(\operatorname{End}_S(K)) = 1$ *and that $K \oplus L \cong K \oplus M$. Then $L \cong M$.*

There are a few known classes of rings that can serve as the endomorphism ring in Evans's Cancellation Theorem. If R is a zero-dimensional commutative ring, for example, then R has stable rank one [21, Proposition 2.12], as does the ring of $n \times n$ matrices with entries in R for every positive integer n [34, Theorem 3]. A classic result of Bass states that a ring has stable rank one if factoring out its Jacobson radical produces an Artinian ring [3, Corollary 6.5]. This of course implies that every commutative Noetherian ring with only finitely many maximal ideals has stable rank one. There are also commutative rings of stable rank one with positive-dimensional maximal spectra. In fact, in [17], Heinzer proves that, for every positive integer d, there exists a non-Noetherian Bézout domain of stable rank one whose maximal spectrum is d-dimensional and Noetherian. These examples give some sense of the diversity of rings with stable rank one.

Nevertheless, there are many common rings with stable rank greater than one. By a result of Estes and Ohm [11, Corollary 7.7], the ring of integers of an algebraic number field always has stable rank two. In particular, the ring $\mathbb{Z}$ of all rational integers has stable rank two. By another result of Estes and Ohm (partially attributed to Heinzer in [11, page 361]), every non-Artinian affine domain over a field has stable rank at least two. For example, $\mathrm{sr}(k[x_1, \ldots, x_d]) \geqslant 2$, where k is a field and $x_1, \ldots, x_d$ are indeterminates. In fact, if k is any nonzero commutative ring, then $\mathrm{sr}(k[x_1, \ldots, x_d]) \geqslant \max\{2, 1 + \mathrm{floor}(d/2)\}$ by a result of Gabel [13]. If k is a subfield of the field $\mathbb{R}$ of all real numbers, then we can say even more: In this case, $\mathrm{sr}(k[x_1, \ldots, x_d]) = 1 + d$ by a result of Vasershtein [34, Theorem 8]. These examples illustrate some limitations of the stable rank condition in Evans's Cancellation Theorem. Since Warfield [35, Theorem 2.1] shows that this stable rank condition is equivalent to the so-called *substitution* and *common complement* properties, the limitations of the latter properties are also highlighted by the examples of Estes–Ohm, Estes–Heinzer–Ohm, Gabel, and Vasershtein.

Our goal in Chapter 3 is to prove cancellation theorems that do not rely on any stable rank conditions. Here, Bass's Cancellation Theorem [3, Theorem 9.3] serves as our guide.

Theorem 1.2.3 (Bass's Cancellation Theorem [3, Theorem 9.3]). *Let R be a commutative ring for which $Y := \mathrm{Max}(R)$ is Noetherian with $\dim(Y) < \infty$, and let S be a module-finite R-algebra. Let M be a right S-module, and suppose that M has a projective direct summand M' over S such that $\mathrm{spl}_{S_\mathfrak{m}}(M'_\mathfrak{m}, S_\mathfrak{m}) \geqslant 1 + \dim(Y)$ for every $\mathfrak{m} \in Y$. Let P be a finitely generated projective right S-module, and let L be a right S-module for which $P \oplus L \cong P \oplus M$. Then $L \cong M$.*

Stafford proves an analogue of this result in the case that S is an arbitrary (possibly noncommutative) Noetherian ring [29, Corollary 5.11]. De Stefani, Polstra, and Yao extend Bass's Cancellation Theorem in a different manner [7, Theorem 3.13]:

Theorem 1.2.4 (De Stefani–Polstra–Yao Cancellation Theorem [7, Theorem 3.13])**.** *Let R be a commutative Noetherian ring for which $X := j\text{-Spec}(R)$ is finite-dimensional. Let M be a finitely generated R-module, and suppose that $\operatorname{spl}_{R_\mathfrak{p}}(M_\mathfrak{p}, R_\mathfrak{p}) \geqslant 1 + \dim_X(\mathfrak{p})$ for every $\mathfrak{p} \in X$. Let P be a finitely generated projective R-module, and let L be an R-module for which $P \oplus L \cong P \oplus M$. Then $L \cong M$.*

In Section 3.1, we simultaneously extend Bass's Cancellation Theorem and generalize the De Stefani–Polstra–Yao Cancellation Theorem. To state our main cancellation theorem, we give the following definition:

Definition 1.2.5. Let R be a commutative ring, and let N be an R-module. A map $f \in \operatorname{End}_R(N)$ is a *homothety* if there exists $a \in R$ such that $f(x) = xa$ for every $x \in N$. We will say that N is *homothetic over* R if every member of $\operatorname{End}_R(N)$ is a homothety or, equivalently, if the natural ring homomorphism from R to $\operatorname{End}_R(N)$ is surjective.

Our main cancellation theorem gives a criterion for cancelling a finitely presented homothetic module. The most basic example of such a module is a commutative ring considered as a module over itself. We can also provide the following three classes of examples:

Example 1.2.6. Every ideal of a Noetherian normal domain is finitely presented and homothetic.

Proof. Let R be a Noetherian normal domain with fraction field T, and let I be a nonzero ideal of R. Since R is Noetherian, I is finitely presented over R. Let $f \in \operatorname{Hom}_R(I, I)$, and let $x, y \in I - 0$. Then $f(x)y = f(xy) = f(y)x$, so that

$$a := \frac{f(x)}{x} = \frac{f(y)}{y} \in T.$$

Thus $f(z) = za$ for every $z \in I$. Moreover, for every $b \in T$, we have $xa = xb$ if and only if $a = b$, and so $\operatorname{Hom}_R(I, I)$ can be identified with an R-submodule J of T that contains R. Since I is finitely generated over R, there exists $c \in I - 0$ such that $cJ \subseteq I$, and so J is a fractional ideal of R. Since R is Noetherian, J is then finitely generated over R. As a result,

J is a module-finite extension of R in T. Since R is normal, $R = J \cong \mathrm{Hom}_R(I, I)$. Hence I is homothetic. $\square$

Example 1.2.7. Let R be a Cohen–Macaulay ring with a canonical module ω. Then ω is finitely presented and homothetic.

Proof. See [4, Chapters 1–3] for an introduction to Cohen–Macaulay rings and canonical modules. By definition, ω is finitely presented over R. For proof that ω is homothetic when R is local, see [4, Proposition 3.3.11(c)(ii)]. The non-local case is similar. $\square$

Example 1.2.8. Let R be an integral domain, and let I be an ideal of R generated by a finite regular sequence of length at least two. Then I is finitely presented and homothetic.

Proof. Suppose that I can be generated by a regular sequence of finite length $n \geqslant 2$. Then the *Koszul complex* of the sequence indicates that I has a finite presentation

$$R^{\oplus \binom{n}{2}} \longrightarrow R^{\oplus n} \longrightarrow I \longrightarrow 0$$

(see [4, Section 1.6]). To show that I is homothetic, first let x, y be any regular sequence of length two in I. Let $f \in \mathrm{Hom}_R(I, I)$. Then, just as in Example 1.2.6, we have $f(x)y = f(xy) = f(y)x$. Since x, y is a regular sequence, $f(x) = xa$ for some $a \in R$. Now let z be any element of I. Then $f(z)x = f(xz) = f(x)z = xaz$. Since $x \neq 0$, we have $f(z) = za$. Hence f is a homothety, and so I is homothetic. $\square$

These examples provide us with the proper context in which to state our main cancellation theorem.

Theorem 1.2.9 (Main Cancellation Theorem)**.** *Let R be a commutative ring, and let N be a finitely presented homothetic R-module for which $Y := \mathrm{Max}(R) \cap \mathrm{Supp}_R(N)$ is Noetherian with $\dim(Y) < \infty$. Let M be an R-module, and suppose that one of the following conditions holds:*

(1) M is a direct summand of a direct sum of finitely presented R-modules such that $\mathrm{spl}_{R_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) \geqslant 1 + \dim(Y)$ for every $\mathfrak{m} \in Y$.

(2) *R is Noetherian; M is finitely generated over R; and* $\operatorname{spl}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \geqslant 1 + \dim_X(\mathfrak{p})$ *for every* $\mathfrak{p} \in X := j\text{-}\operatorname{Spec}(R) \cap \operatorname{Supp}_R(N)$.

Let K be a direct summand of a direct sum of finitely many copies of N, and let L be an R-module for which $K \oplus L \cong K \oplus M$. Then $L \cong M$.

Our main cancellation theorem extends Bass's Cancellation Theorem in the case that S equals R—and generalizes the De Stefani–Polstra–Yao Cancellation Theorem completely—by replacing the module R with a finitely presented homothetic R-module N. A direct summand of a direct sum of finitely many copies of N is then an obvious abstraction of a finitely generated projective R-module. Hence the module K in our main cancellation theorem replaces the module P in the cancellation theorems of Bass and De Stefani–Polstra–Yao. We prove our main cancellation theorem in Section 3.1.

We would like to conclude this chapter by demonstrating that our main cancellation theorem presents new information relative to Evans, Bass, and De Stefani–Polstra–Yao. We will give a simple example that meets the criteria of our main cancellation theorem, and we will show that Evans, Bass, and De Stefani–Polstra–Yao do not apply to this example.

Example 1.2.10. Let $R = k[x, y, z]$, where k is a field; let $K = N = xR + yR + zR$; and let $M := N \oplus R^{\oplus 3}$. Then, for every R-module L such that $K \oplus L \cong K \oplus M$, we have $L \cong M$.

Proof. Since R is a Noetherian normal domain, the ideal N of R is finitely presented and homothetic. Since R is an affine domain, $X := j\text{-}\operatorname{Spec}(R) \cap \operatorname{Supp}_R(N) = \operatorname{Spec}(R)$ is Noetherian and finite-dimensional; in fact, $\dim(X) = 3$ here. Of course, M is finitely generated over R. It remains to show that $\operatorname{spl}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \geqslant 1 + \dim_X(\mathfrak{p})$ for every $\mathfrak{p} \in X$. For this, simply note that $M_N \cong N_N \oplus R_N^{\oplus 3}$ and that $M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus 4} \cong N_{\mathfrak{p}}^{\oplus 4}$ for every $\mathfrak{p} \in X - \{N\}$. Our main cancellation theorem now finishes the proof. $\square$

Evans does not apply to this example since $\operatorname{sr}(\operatorname{End}_R(K)) = \operatorname{sr}(R) \geqslant 2$ by Gabel [13]. Bass and De Stefani–Polstra–Yao do not apply since K is nonprincipal and, hence, is non-projective by the Quillen–Suslin Theorem [27] [30].

In light of Bass's Cancellation Theorem, a tempting approach to the above example would be to apply $\operatorname{Hom}_R(-,R)$ to the isomorphism $K \oplus L \cong K \oplus N \oplus R^{\oplus 3}$ to yield the new isomorphism $R \oplus \operatorname{Hom}_R(L,R) \cong R \oplus R^{\oplus 4}$. Bass's Cancellation Theorem would then tell us that $\operatorname{Hom}_R(L,R) \cong R^{\oplus 4}$. However, at this point, we would have reached a dead end: Neither L nor M is isomorphic to $R^{\oplus 4}$ since, by our main cancellation theorem, $L \cong M = N \oplus R^{\oplus 3}$, which is not even flat, much less free.

Using the De Stefani–Polstra–Yao Cancellation Theorem in a similar way would also prove to be inconclusive: Certainly, we could apply $\operatorname{Hom}_R(-,N)$ to the isomorphism $K \oplus L \cong K \oplus N \oplus R^{\oplus 3}$ to yield $R \oplus \operatorname{Hom}_R(L,N) \cong R \oplus R \oplus N^{\oplus 3}$. The De Stefani–Polstra–Yao Cancellation Theorem would then tell us that $\operatorname{Hom}_R(L,N) \cong R \oplus N^{\oplus 3}$. Applying $\operatorname{Hom}_R(-,N)$ once again, we would get $\operatorname{Hom}_R(\operatorname{Hom}_R(L,N),N) \cong N \oplus R^{\oplus 3} \cong M$. Now, unless we knew *a priori* that $L \cong \operatorname{Hom}_R(\operatorname{Hom}_R(L,N),N)$, this approach would not verify that $L \cong M$. The point here is that our main cancellation theorem *implies* $L \cong \operatorname{Hom}_R(\operatorname{Hom}_R(L,N),N)$ as a corollary of the fact that $L \cong M$.

In Section 3.2, we chronicle a few more cancellation theorems from the literature and show that the above example is also not covered by these other theorems. We close the dissertation with a vast generalization of the above example and two additional cancellation examples unacknowledged by older cancellation theorems. Until then, it is our hope that the above example provides sufficient motivation to prove our main cancellation theorem. Indeed, since this theorem is an application of our work on splitting capacities and since our work on splitting capacities is an adaptation of our work on surjective capacities, any example that motivates this theorem motivates the entire dissertation.

CHAPTER 2

SURJECTIVE AND SPLITTING CAPACITIES

In this chapter, we abide by the following conventions: We let R be a commutative ring; we let S be a module-finite R-algebra; we let M denote a right S-module; and we let N denote a finitely generated right S-module. We view every left (respectively, right) S-module as a left (respectively, right) R-module in the natural way.

We begin by proving Theorem 1.1.5, modulo a lemma (Lemma 2.1.14), in Section 2.1. We call this lemma the *Surjective Lemma*, and we prove it in several stages: Over the course of Sections 2.2 and 2.3, we first reduce the proof of the Surjective Lemma to a verification that a certain condition holds on a finite set Λ of prime ideals of R. Along the way, we prove Theorem 1.1.6, modulo the Surjective Lemma. In Section 2.4, we restrict our attention to the maximal ideals of R in Λ and complete the base case of an induction necessary to prove the Surjective Lemma. In Section 2.5, we address the remaining members of Λ and finish a proof of the Surjective Lemma. Finally, Section 2.6 presents split surjective versions of our surjective theorems, and Section 2.7 covers characterizations of surjective and splitting capacities available for finitely generated modules over Dedekind domains.

2.1 Proof of Theorem 1.1.5

In this section, we prove Theorem 1.1.5, modulo the Surjective Lemma (Lemma 2.1.14). The statement of the Surjective Lemma requires the following definition, which establishes analogues of local and global surjective capacities for arbitrary R-submodules F of $\mathrm{Hom}_S(M, N)$.

Definition 2.1.1. Let F be an R-submodule of $\mathrm{Hom}_S(M, N)$, and let $\mathfrak{p} \in \mathrm{Spec}(R)$. We let $\partial(F)$ denote the supremum of the nonnegative integers t such that there exists a surjective $f \in F^{\oplus t} \subseteq \mathrm{Hom}_S(M, N^{\oplus t})$. We define $L^{\oplus 0} = 0$ for every R-module L so that there always exists a nonnegative integer t (namely, zero) such that $F^{\oplus t}$ contains a surjective map. We let $\partial_{\mathfrak{p}}(F)$ denote the supremum of the nonnegative integers t such that there exists $f \in F^{\oplus t}$ with the property that $f_{\mathfrak{p}}$ is surjective.

Let n be a positive integer. We will think of a member of $\operatorname{Hom}_S(M, N^{\oplus n})$ as a column

$$\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix},$$

where $f_1, \dots, f_n \in \operatorname{Hom}_S(M, N)$. When we refer to such an n-tuple without using a display, we write $(f_1, \dots, f_n)^\top$ to denote the transpose of a row of functions.

Remark 2.1.2. By the previous definition, $\partial(\operatorname{Hom}_S(M, N)) = \operatorname{sur}_S(M, N)$. Also, if $\operatorname{Hom}_{S_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \cong (\operatorname{Hom}_S(M, N))_{\mathfrak{p}}$ as $R_{\mathfrak{p}}$-modules, then $\partial_{\mathfrak{p}}(\operatorname{Hom}_S(M, N)) = \operatorname{sur}_{S_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$. In general, though, $\partial_{\mathfrak{p}}(\operatorname{Hom}_S(M, N))$ may not be equal to $\operatorname{sur}_{S_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}})$.

Next we show that, for every finitely generated R-submodule F of $\operatorname{Hom}_S(M, N)$, it is the case that $\partial(F) = \infty$ if and only if $N = 0$. For this, we need a lemma that is usually attributed to Nakayama (despite the fact that Nakayama attributes this result to Jacobson in [24]).

Lemma 2.1.3 (Nakayama's Lemma [24])**.** *Suppose that R is a commutative ring with a unique maximal ideal $\mathfrak{m}$, and suppose that N is a finitely generated right R-module such that $N\mathfrak{m} = N$. Then $N = 0$.*

Proof. Suppose that $N \neq 0$. Let $x_1, \dots, x_n$ be elements of N that form a minimal generating set for N. Then there exist $a_1, \dots, a_n \in \mathfrak{m}$ such that $x_n = x_1a_1 + \cdots + x_na_n$. Hence $x_n(1 - a_n) = x_1a_1 + \cdots + x_{n-1}a_{n-1}$. Since $a_n \in \mathfrak{m}$, we see that $1 - a_n$ is a unit of R. Hence $x_n \in x_1R + \cdots + x_{n-1}R$. Thus $x_1, \dots, x_n$ do not form a minimal generating set for N, a contradiction. So $N = 0$. $\square$

We now use Nakayama's Lemma to reveal an important fact about finitely generated R-submodules of $\operatorname{Hom}_S(M, N)$:

Remark 2.1.4. Let F be a finitely generated R-submodule of $\operatorname{Hom}_S(M,N)$. We observe that $\partial(F)=\infty$ if and only if $N=0$: Certainly, if $N=0$, then $\partial(F)=\infty$. Suppose that $\partial(F)=\infty$, and let $f_1,\ldots,f_n\in F$ such that $F=Rf_1+\cdots+Rf_n$. Let $f:=(f_1,\ldots,f_n)^\top$. Since $\partial(F)=\infty>n+1$, there exists a surjective map $g:=(g_1,\ldots,g_{n+1})^\top\in F^{\oplus(n+1)}$. Since $g_1,\ldots,g_{n+1}$ are R-linear combinations of $f_1,\ldots,f_n$, there exists an $(n+1)\times n$ matrix B with entries in R such that $g=Bf$. Hence B represents a surjective R-linear map from $N^{\oplus n}$ to $N^{\oplus(n+1)}$. Let $\mathfrak{m}\in\operatorname{Max}(R)$. Then the $(R/\mathfrak{m})$-linear transformation $B\otimes 1_{R/\mathfrak{m}}$ maps $(N/N\mathfrak{m})^{\oplus n}$ onto $(N/N\mathfrak{m})^{\oplus(n+1)}$. Since N is finitely generated over R, we see that $N/N\mathfrak{m}$ is finitely generated over $R/\mathfrak{m}$. Hence $N_\mathfrak{m}/N_\mathfrak{m}\mathfrak{m}_\mathfrak{m}\cong N/N\mathfrak{m}=0$. Now, Nakayama's Lemma tells us that $N_\mathfrak{m}=0$. Since our choice of $\mathfrak{m}\in\operatorname{Max}(R)$ was arbitrary, we have shown that $N=0$.

Let $\mathfrak{p}\in\operatorname{Spec}(R)$. Then, as a result of the discussion above, $\partial_\mathfrak{p}(F)=\infty$ if and only if $\mathfrak{p}\notin\operatorname{Supp}_R(N)$.

Using the symbol ∂, we now describe a condition weaker than surjectivity that we can impose on a member of $\operatorname{Hom}_S(M,N^{\oplus n})$.

Definition 2.1.5. Let n,t be positive integers with $n\geqslant t$; let $\mathfrak{p}\in X\subseteq\operatorname{Spec}(R)$; and let $f:=(f_1,\ldots,f_n)^\top\in\operatorname{Hom}_S(M,N^{\oplus n})$. We say that f is $(t,X,\mathfrak{p})$-*surjective* if $\partial_\mathfrak{p}(Rf_1+\cdots+Rf_n)\geqslant\min\{n,t+\dim_X(\mathfrak{p})\}$.

Let $Y\subseteq X$. We say that f is (t,X,Y)-*surjective* if f is $(t,X,\mathfrak{q})$-surjective for every $\mathfrak{q}\in Y$.

When t and X are understood, we use the less cumbersome terms $\mathfrak{p}$-*surjective* and Y-*surjective* in place of $(t,X,\mathfrak{p})$-*surjective* and (t,X,Y)-*surjective*, respectively.

Remark 2.1.6. Maintaining the hypotheses in the previous definition, we see that f is $(n,X,\mathfrak{p})$-surjective if and only if $f_\mathfrak{p}$ is surjective: Suppose that f is $(n,X,\mathfrak{p})$-surjective. Then there exists an $n\times n$ matrix B with entries in R such that $B_\mathfrak{p}f_\mathfrak{p}=(Bf)_\mathfrak{p}$ is surjective. Now $B_\mathfrak{p}$ represents a surjective $S_\mathfrak{p}$-linear map from $N_\mathfrak{p}^{\oplus n}$ to itself. Since N is finitely generated over R, we see that $N_\mathfrak{p}$ is finitely generated over $R_\mathfrak{p}$. Hence $B_\mathfrak{p}$ is bijective, and so $f_{\mathfrak{p}}$ is surjective. Conversely, if $f_{\mathfrak{p}}$ is surjective, then $\partial_{\mathfrak{p}}(Rf_1 + \cdots + Rf_n) \geqslant n$, and so f is $(n, X, \mathfrak{p})$-surjective.

We need one more definition to state the Surjective Lemma, and this concerns the notion of a *basic set for R*. De Stefani, Polstra, and Yao use basic sets to prove [7, Theorems 3.9 and 4.5]. These theorems have conclusions that are weaker than those of [7, Theorems 3.12 and 4.8], respectively, but the hypotheses of [7, Theorems 3.9 and 4.5] are more general. Later in this section, we state an analogue of [7, Theorems 3.9 and 4.5] and a generalization of [7, Theorems 3.12 and 4.8] in Theorem 2.1.15. We prove another analogue of [7, Theorems 3.9 and 4.5] in Chapter 3 (Theorem 3.1.2).

Here is the definition of a basic set for R, along with a few examples:

Definition 2.1.7. Let $X \subseteq \operatorname{Spec}(R)$. We say that X is a *basic set for R* if X is Noetherian and if, for every $\mathfrak{p} \in \operatorname{Spec}(R)$ that can be written as an intersection of members of X, it is the case that $\mathfrak{p} \in X$.

Example 2.1.8. Every finite subset of $\operatorname{Spec}(R)$ is a basic set for R.

Example 2.1.9. If R is Noetherian, then $\operatorname{Spec}(R)$ is a basic set for R. If R is a *Jacobson ring*, then $j\text{-}\operatorname{Spec}(R) = \operatorname{Spec}(R)$. Hence, if R is a Noetherian Jacobson ring, then $j\text{-}\operatorname{Spec}(R)$ is a basic set for R. In fact, a more general statement holds, as the next example shows.

Example 2.1.10. Let $X := j\text{-}\operatorname{Spec}(R) \cap \operatorname{Supp}_R(N)$, and suppose that X is Noetherian. Then X is a basic set for R: Let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{p}$ is an intersection of members of X. Since every member of $j\text{-}\operatorname{Spec}(R)$ is an intersection of maximal ideals of R, so is $\mathfrak{p}$. Hence $\mathfrak{p} \in j\text{-}\operatorname{Spec}(R)$. Since N is finitely generated over R, we see that $\operatorname{Supp}_R(N) = \operatorname{Var}(\operatorname{Ann}_R(N))$. Hence $\mathfrak{p}$ is an intersection of prime ideals that contain $\operatorname{Ann}_R(N)$, and so $\mathfrak{p}$ must itself contain $\operatorname{Ann}_R(N)$. Thus $\mathfrak{p} \in \operatorname{Var}(\operatorname{Ann}_R(N)) = \operatorname{Supp}_R(N)$. We have proved then that $\mathfrak{p} \in X$. Thus X is a basic set for R.

Example 2.1.11. By Example 2.1.10, if $j\text{-}\operatorname{Spec}(R)$ is Noetherian, then $j\text{-}\operatorname{Spec}(R)$ is a basic set for R. If R is Artinian, then $\operatorname{Max}(R) = j\text{-}\operatorname{Spec}(R) = \operatorname{Spec}(R)$. If R is *semilocal* (Noetherian with only finitely many maximal ideals) but not Artinian, then $\mathrm{Max}(R) = j\text{-Spec}(R) \subsetneq \mathrm{Spec}(R)$, and $\dim(j\text{-Spec}(R)) = 0 < \dim(\mathrm{Spec}(R))$.

Example 2.1.12. Suppose that R is a one-dimensional Noetherian domain with infinitely many maximal ideals. Then $\mathrm{Max}(R)$ is not a basic set for R: If $\mathrm{Max}(R)$ is a basic set for R, then $\mathrm{Jac}(R) \neq 0$, and so $\mathrm{Min}(\mathrm{Jac}(R)) = \mathrm{Max}(R)$ is a finite set, a contradiction.

Notice that, in this example, $\mathrm{Max}(R) \subsetneq j\text{-Spec}(R) = \mathrm{Spec}(R)$.

Example 2.1.13. Let d be an integer with $d \geqslant 2$, and let T be the domain

$$k[x_1, \ldots, x_d, y_1, y_2, y_3, \ldots],$$

where k is a field and $x_1, \ldots, x_d, y_1, y_2, y_3, \ldots$ are indeterminates. Invert every element of T outside of the set

$$(x_1T + \cdots + x_dT) \cup y_1T \cup y_2T \cup y_3T \cup \cdots$$

to form a new domain, and suppose that R is this new domain. Example 3.3 in [26] then indicates that R is a d-dimensional Noetherian ring such that

$$\mathrm{Max}(R) = \{x_1R + \cdots + x_dR, y_1R, y_2R, y_3R, \ldots\}$$

and such that every nonzero prime ideal of R is contained a unique maximal ideal.

We will prove that $\mathrm{Max}(R) \subsetneq j\text{-Spec}(R) \subsetneq \mathrm{Spec}(R)$. To prove that the first inclusion is strict, we will show that $0 \in j\text{-Spec}(R) - \mathrm{Max}(R)$. Of course, $0 \notin \mathrm{Max}(R)$. Suppose that $0 \notin j\text{-Spec}(R)$. Then $\mathrm{Jac}(R) \neq 0$. Since R is Noetherian, $\mathrm{Min}(\mathrm{Jac}(R))$ is a finite set. On the other hand, $\mathrm{Min}(\mathrm{Jac}(R))$ contains the infinite set $\{y_1R, y_2R, y_3R, \ldots\}$, a contradiction. Hence $0 \in j\text{-Spec}(R)$.

To prove that $j\text{-Spec}(R) \neq \mathrm{Spec}(R)$, we will first show that $j\text{-Spec}(R) = \{0\} \cup \mathrm{Max}(R)$. We have already shown that $0 \in j\text{-Spec}(R)$, and $\mathrm{Max}(R) \subseteq j\text{-Spec}(R)$ by definition, so it remains to show that every nonzero member $\mathfrak{p}$ of $j\text{-Spec}(R)$ is in $\mathrm{Max}(R)$. As we mentioned above, $\mathfrak{p}$ is contained in a unique maximal ideal of R. Since $\mathfrak{p}$ is an intersection of maximal ideals of R, it must then be the case that $\mathfrak{p} \in \operatorname{Max}(R)$. Hence j-Spec$(R) = \{0\} \cup \operatorname{Max}(R)$, and so $x_1R \in \operatorname{Spec}(R) - j\text{-}\operatorname{Spec}(R)$. Thus j-Spec$(R) \neq \operatorname{Spec}(R)$.

Notice that, in this example, $\dim(j\text{-}\operatorname{Spec}(R)) = 1 < d = \dim(\operatorname{Spec}(R))$. In particular, if $X := \operatorname{Spec}(R)$ and $Y := j\text{-}\operatorname{Spec}(R)$, then $\dim_Y(0) = 1 < d = \dim_X(0)$.

We are now prepared to state the Surjective Lemma.

Lemma 2.1.14 (Surjective Lemma)**.** *Let n, t be positive integers with $n \geqslant 1+t$, and let X be a subspace of $\operatorname{Supp}_R(N)$ that is a basic set for R. Let $f := (f_1, \ldots, f_n)^\top \in \operatorname{Hom}_S(M, N^{\oplus n})$, and suppose that f is (t, X, X)-surjective. Then there exist $f'_1, \ldots, f'_{n-1} \in Rf_1 + \cdots + Rf_n$ such that $f' := (f'_1, \ldots, f'_{n-1})^\top$ is (t, X, X)-surjective.*

We prove this lemma in Section 2.5. Given this lemma, we can prove Theorem 2.1.15 below. Part (1) of Theorem 2.1.15 provides an analogue of [7, Theorems 3.9 and 4.5]; and Part (2) generalizes [7, Theorems 3.12 and 4.8] and [28, *Théorème 1*].

Theorem 2.1.15. *Let L be an S-submodule of M; let F be a finitely generated R-submodule of $\operatorname{Hom}_S(L, N)$; and let G be an R-submodule of $\operatorname{Hom}_S(M, N)$. Suppose that every member of F can be extended to a member of G. Let X be a subspace of $\operatorname{Supp}_R(N)$ that is a basic set for R, and suppose that $\dim(X) < \infty$. Then the following statements hold:*

(1) Let t be a positive integer, and suppose that $\partial_{\mathfrak{p}}(F) \geqslant t + \dim_X(\mathfrak{p})$ for every $\mathfrak{p} \in X$. Then there exists $g \in G^{\oplus t}$ such that $g_{\mathfrak{p}}$ is surjective for every $\mathfrak{p} \in X$.

(2) Suppose that $\operatorname{Max}(R) \cap \operatorname{Supp}_R(N) \subseteq X$. Then

$$\partial(G) \geqslant \inf\{\partial_{\mathfrak{p}}(F) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in X\}.$$

Proof of Theorem 2.1.15, modulo the Surjective Lemma. (1) We may assume that $X \neq \varnothing$. Let $n := \mu_R(F)$. Since t is a positive integer, n is a positive integer. Let $f_1, \ldots, f_n \in F$ such that $F = Rf_1 + \cdots + Rf_n$; let $f := (f_1, \ldots, f_n)^\top$; and let $\mathfrak{q} \in X$. Then there exists a $t \times n$ matrix B with entries in R such that $(Bf)_{\mathfrak{q}} = B_{\mathfrak{q}} f_{\mathfrak{q}}$ is surjective. Hence $B \otimes 1_{\kappa(\mathfrak{q})}$ represents a surjective $\kappa(\mathfrak{q})$-linear map from $N^{\oplus n} \otimes \kappa(\mathfrak{q})$ to $N^{\oplus t} \otimes \kappa(\mathfrak{q})$. Since $N \otimes \kappa(\mathfrak{q})$ is finitely generated over $\kappa(\mathfrak{q})$, we see that $n \geqslant t$. Thus f is (t, X, X)-surjective. Now, after $n - t$ applications of the Surjective Lemma, we obtain $f'_1, \ldots, f'_t \in F$ such that $f' := (f'_1, \ldots, f'_t)^\top$ is (t, X, X)-surjective. By Remark 2.1.6, we see that $f'_\mathfrak{p}$ is surjective for every $\mathfrak{p} \in X$. By assumption, the members $f'_1, \ldots, f'_t$ of F can be extended to members $g_1, \ldots, g_t$ of G, respectively. Let $g := (g_1, \ldots, g_t)^\top \in G^{\oplus t}$. Since $f'_\mathfrak{p}$ is surjective for every $\mathfrak{p} \in X$, we see that $g_\mathfrak{p}$ is surjective for every $\mathfrak{p} \in X$.

(2) Let

$$t := \inf\{\partial_\mathfrak{p}(F) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in X\}.$$

If $t \leqslant 0$, then there is nothing to prove. If $t = \infty$, then $N = 0$ by Remark 2.1.4, and so $\partial(G) = t$. Suppose then that t is a positive integer. By Part (1) of this theorem, there exists $g \in G^{\oplus t}$ such that $g_\mathfrak{p}$ is surjective for every $\mathfrak{p} \in X$. Since $\operatorname{Max}(R) \cap \operatorname{Supp}_R(N) \subseteq X$, we see that g is surjective. Hence $\partial(G) \geqslant t$. □

We now show that Theorem 1.1.5 is an easy corollary of Theorem 2.1.15.

Proof of Theorem 1.1.5, modulo the Surjective Lemma. Let $L = M$, and let $F = G$ denote $\operatorname{Hom}_S(M, N)$. By our hypotheses, F is finitely generated over R. By Example 2.1.10, we see that X is a subspace of $\operatorname{Supp}_R(N)$ that is a basic set for R. Also, $\operatorname{Max}(R) \cap \operatorname{Supp}_R(N) \subseteq X$. Hence, by Part (2) of Theorem 2.1.15, we get

$$\begin{aligned} \operatorname{sur}_S(M, N) &= \partial(G) \\ &\geqslant \inf\{\partial_\mathfrak{p}(F) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in X\} \\ &= \inf\{\operatorname{sur}_{S_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in X\}. \quad \square \end{aligned}$$

With a little more work, we can also prove Theorem 1.1.6, modulo the Surjective Lemma. We accomplish this goal in the next section.

2.2 Proof of Theorem 1.1.6

Under the hypotheses of Theorem 1.1.5, we see that $\mathrm{Hom}_S(M,N)$ is finitely generated over R, and we see that $(\mathrm{Hom}_S(M,N))_{\mathfrak{p}}$ and $\mathrm{Hom}_{S_{\mathfrak{p}}}(M_{\mathfrak{p}},N_{\mathfrak{p}})$ are isomorphic as $R_{\mathfrak{p}}$-modules for every $\mathfrak{p}\in\mathrm{Spec}(R)$. Hence, a finite number of global S-linear maps govern all local surjective capacities.

This property does not follow from the more general hypotheses of Theorem 1.1.6. As a result, we are not immediately in a position to apply the Surjective Lemma (Lemma 2.1.14) when attempting to prove Theorem 1.1.6.

In this section, we show that we can circumvent the issue by obtaining a finitely generated R-submodule F of $\mathrm{Hom}_S(M,N)$ such that $\partial_{\mathfrak{m}}(F)$ is sufficiently large for every $\mathfrak{m}\in\mathrm{Max}(R)\cap\mathrm{Supp}_R(N)$. We can then prove Theorem 1.1.6, modulo the Surjective Lemma.

With this goal in mind, we first show that, for every R-submodule F of $\mathrm{Hom}_S(M,N)$, the function on $\mathrm{Spec}(R)$ taking $\mathfrak{p}$ to $\partial_{\mathfrak{p}}(F)$ is lower semicontinuous.

Lemma 2.2.1. *Let F be an R-submodule of $\mathrm{Hom}_S(M,N)$, and let t be a nonnegative integer. Then the set $\{\mathfrak{p}\in\mathrm{Spec}(R):\partial_{\mathfrak{p}}(F)>t\}$ is open, and so the set $\{\mathfrak{p}\in\mathrm{Spec}(R):\partial_{\mathfrak{p}}(F)\leqslant t\}$ is closed. Hence, for every subspace X of $\mathrm{Spec}(R)$, the set $Y_t:=\{\mathfrak{p}\in X:\partial_{\mathfrak{p}}(F)\leqslant t\}$ is closed in X.*

Proof. Let $\mathfrak{p}\in\mathrm{Spec}(R)$ such that $\partial_{\mathfrak{p}}(F)>t$; let $f\in F^{\oplus(t+1)}$ such that $f_{\mathfrak{p}}$ is surjective; and let $C:=\operatorname{coker} f$. Since $C_{\mathfrak{p}}=0$ and since C is finitely generated over R, there is an element $s\in R-\mathfrak{p}$ such that $sC=0$. Let $U:=\{\mathfrak{q}\in\mathrm{Spec}(R):s\notin\mathfrak{q}\}$. Then $C_{\mathfrak{q}}=0$ for every $\mathfrak{q}\in U$. Hence U is an open neighborhood of $\mathfrak{p}$ such that $\partial_{\mathfrak{q}}(F)>t$ for every $\mathfrak{q}\in U$. Thus the set $\{\mathfrak{p}\in\mathrm{Spec}(R):\partial_{\mathfrak{p}}(F)>t\}$ is open. This proves the first claim of the lemma. The last two claims of the lemma follow from the first claim. □

Using the previous lemma, we can show that, under the hypotheses of Part (1) of Theorem 1.1.6, there exists a finitely generated R-submodule F of $\mathrm{Hom}_S(M,N)$ such that $\partial_{\mathfrak{m}}(F)\geqslant t+\dim(Y)$ for every $\mathfrak{m}\in Y$. This is, in fact, a consequence of a more general statement:

Lemma 2.2.2. *Suppose that M is a direct summand of a direct sum of finitely presented right S-modules. Let X be a Noetherian subspace of* $\operatorname{Supp}_R(N)$ *such that* $\dim(X) < \infty$*. Let t be a positive integer, and suppose that* $\operatorname{sur}_{S_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \geqslant t + \dim(X)$ *for every* $\mathfrak{p} \in X$*. Then there exists a finitely generated R-submodule F of* $\operatorname{Hom}_S(M, N)$ *such that* $\partial_\mathfrak{p}(F) \geqslant t + \dim(X)$ *for every* $\mathfrak{p} \in X$*.*

Proof. Let $u := (t-1) + \dim(X)$, and let $\mathscr{G}$ denote the collection of all finitely generated R-submodules of $\operatorname{Hom}_S(M, N)$. For every $G \in \mathscr{G}$, let $Y(G) := \{\mathfrak{p} \in X : \partial_\mathfrak{p}(G) \leqslant u\}$, and let $\mathscr{Y} := \{Y(G) : G \in \mathscr{G}\}$. We aim to prove that $\varnothing \in \mathscr{Y}$. Suppose not. Then X is nonempty. By Lemma 2.2.1, we see that $Y(G)$ is closed in X for every $G \in \mathscr{G}$. Since X is Noetherian and nonempty, there is $G' \in \mathscr{G}$ such that $Y(G')$ is a minimal member of $\mathscr{Y}$. By assumption, $Y(G') \neq \varnothing$, so let $\mathfrak{q} \in Y(G')$. By our hypothesis on M, there exist $h_1, \ldots, h_{u+1} \in \operatorname{Hom}_S(M, N)$ such that $(h_1, \ldots, h_{u+1})_\mathfrak{q}^\top$ is surjective. Let $H := G' + Rh_1 + \cdots + Rh_{u+1}$. Then $H \in \mathscr{G}$, and $\partial_\mathfrak{q}(H) \geqslant u + 1 = t + \dim(X)$. Thus $\mathfrak{q} \in Y(G') - Y(H)$, and so $Y(H) \subsetneq Y(G')$, contradicting the minimality of $Y(G')$ in $\mathscr{Y}$. Thus $\varnothing \in \mathscr{Y}$, and so there exists a finitely generated R-submodule F of $\operatorname{Hom}_S(M, N)$ such that $Y(F) = \varnothing$. Consequently, $\partial_\mathfrak{p}(F) \geqslant u + 1 = t + \dim(X)$ for every $\mathfrak{p} \in X$. ☐

We can now prove Theorem 1.1.6, modulo the Surjective Lemma.

Proof of Theorem 1.1.6, modulo the Surjective Lemma. Let $X := j\text{-}\operatorname{Spec}(R) \cap \operatorname{Supp}_R(N)$. By the corollary following Proposition 1 in [32], we see that $\dim(X) = \dim(Y) < \infty$ and that X is Noetherian since Y is Noetherian. By Example 2.1.10, we see that X is a basic set for R.

(1) Since $\operatorname{sur}_{S_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) \geqslant t + \dim(Y)$ for every $\mathfrak{m} \in Y$, we see that $\operatorname{sur}_{S_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \geqslant t + \dim(X)$ for every $\mathfrak{p} \in X$. By Lemma 2.2.2, there exists a finitely generated R-submodule F of $\operatorname{Hom}_S(M, N)$ such that $\partial_\mathfrak{p}(F) \geqslant t + \dim(X) \geqslant t + \dim_X(\mathfrak{p})$ for every $\mathfrak{p} \in X$. Now, let $L = M$ and $G = \operatorname{Hom}_S(M, N)$. Then Part (2) of Theorem 2.1.15 tells us that $\operatorname{sur}_S(M, N) = \partial(G) \geqslant t$.

(2) Certainly, if $\operatorname{sur}_S(M,N)=\infty$, then $\operatorname{sur}_{S_\mathfrak{m}}(M_\mathfrak{m},N_\mathfrak{m})=\infty$ for every $\mathfrak{m}\in Y$. Suppose then that $\operatorname{sur}_{S_\mathfrak{m}}(M_\mathfrak{m},N_\mathfrak{m})=\infty$ for every $\mathfrak{m}\in Y$. Then, $\operatorname{sur}_{S_\mathfrak{m}}(M_\mathfrak{m},N_\mathfrak{m})\geqslant t+\dim(Y)$ for every $\mathfrak{m}\in Y$ and for every positive integer t. Hence, by Part (1) of this theorem, $\operatorname{sur}_S(M,N)\geqslant t$ for every positive integer t. Thus $\operatorname{sur}_S(M,N)=\infty$.

(3) Let

$$t := \min\{\operatorname{sur}_{S_\mathfrak{m}}(M_\mathfrak{m},N_\mathfrak{m}) : \mathfrak{m}\in Y\} - \dim(Y).$$

Then $\operatorname{sur}_{S_\mathfrak{m}}(M_\mathfrak{m},N_\mathfrak{m})\geqslant t+\dim(Y)$ for every $\mathfrak{m}\in Y$. If $t\leqslant 0$, then certainly $\operatorname{sur}_S(M,N)\geqslant t$. Suppose then that t is a positive integer. Then, by Part (1) of this theorem, $\operatorname{sur}_S(M,N)\geqslant t$. $\square$

We can also prove the following variations of Lemma 2.2.2 and Theorems 2.1.15 and 1.1.6. These variations are noteworthy in the sense that they do not require M to be a direct summand of a direct sum of finitely presented right S-modules. We omit the proofs since they are similar to the proofs of the analogous statements from earlier.

Lemma 2.2.3. *Let F be an R-submodule of $\operatorname{Hom}_S(M,N)$. Let X be a Noetherian subspace of $\operatorname{Supp}_R(N)$, and suppose that $\dim(X)<\infty$. Let t be a positive integer, and suppose that $\partial_\mathfrak{p}(F)\geqslant t+\dim(X)$ for every $\mathfrak{p}\in X$. Then there exists a finitely generated R-submodule F' of F such that $\partial_\mathfrak{p}(F')\geqslant t+\dim(X)$ for every $\mathfrak{p}\in X$.*

Theorem 2.2.4. *Let L be an S-submodule of M; let F be an R-submodule of $\operatorname{Hom}_S(L,N)$; and let G be an R-submodule of $\operatorname{Hom}_S(M,N)$. Suppose that every member of F can be extended to a member of G. Then the following statements hold:*

(1) Let X be a subspace of $\operatorname{Supp}_R(N)$ that is a basic set for R with $\dim(X)<\infty$. Let t be a positive integer, and suppose that $\partial_\mathfrak{p}(F)\geqslant t+\dim(X)$ for every $\mathfrak{p}\in X$. Then there exists $g\in G^{\oplus t}$ such that $g_\mathfrak{p}$ is surjective for every $\mathfrak{p}\in X$.

(2) Suppose that $Y := \operatorname{Max}(R)\cap\operatorname{Supp}_R(N)$ is Noetherian with $\dim(Y)<\infty$. Then the following statements hold:

(a) *Let t be a positive integer, and suppose that $\partial_{\mathfrak{m}}(F) \geqslant t + \dim(Y)$ for every $\mathfrak{m} \in Y$. Then $\partial(G) \geqslant t$.*

(b) *If $\partial_{\mathfrak{m}}(F) = \infty$ for every $\mathfrak{m} \in Y$, then $\partial(G) = \infty$. Hence $\partial(F) = \infty$ if and only if $\partial_{\mathfrak{m}}(F) = \infty$ for every $\mathfrak{m} \in Y$.*

(c) *Suppose that $\partial_{\mathfrak{n}}(F) < \infty$ for some $\mathfrak{n} \in Y$. Then*

$$\partial(G) \geqslant \min\{\partial_{\mathfrak{m}}(F) : \mathfrak{m} \in Y\} - \dim(Y).$$

In the next section, we continue working toward a proof of the Surjective Lemma.

2.3 The set Λ

Assuming the hypotheses of the Surjective Lemma (Lemma 2.1.14), we show in this section that there is a finite subset Λ of X with a crucial property: If there exists an invertible $n \times n$ matrix A with entries in R such that the first $n-1$ components of $Af := (f'_1, \ldots, f'_n)^\top$ form a map $f' := (f'_1, \ldots, f'_{n-1})^\top$ that is (t, X, Λ)-surjective, then f' is (t, X, X)-surjective. In other words, we show that proving the Surjective Lemma, which may involve infinitely many prime ideals of R, reduces to proving a statement about Λ, a finite set of prime ideals.

We begin by covering two properties of basic sets for R with Proposition 2.3.1. Property (1), which holds not only for every basic set but also for every Noetherian topological space, is proved, for example, in [15, Proposition 1.5]. Property (2) is proved for the special case of $X := j\text{-Spec}(R)$ in [32, Proposition 2], although the more general result here can be proved by similar means.

Proposition 2.3.1. *Let X be a basic set for R. Then X has the following properties:*

(1) *Every closed subset of X is a union of finitely many closed irreducible subsets of X.*

(2) *Every closed irreducible subset of X has a unique generic point.*

In the proof of the next lemma, we define the set Λ. The proof of this lemma is similar to the proofs of [7, Lemmas 3.6 and 4.2].

Lemma 2.3.2. *Let F be a finitely generated R-submodule of $\operatorname{Hom}_S(M,N)$, and let X be a subspace of $\operatorname{Supp}_R(N)$ that is a basic set for R. Then there exists a finite subset Λ of X such that, for every $\mathfrak{p} \in X - \Lambda$, there exists $\mathfrak{q} \in \Lambda$ with the properties that $\mathfrak{q} \subsetneq \mathfrak{p}$ and $\partial_{\mathfrak{q}}(F) = \partial_{\mathfrak{p}}(F)$.*

Proof. For every nonnegative integer t, let $Y_t := \{\mathfrak{p} \in X : \partial_{\mathfrak{p}}(F) \leqslant t\}$. Proposition 2.3.1 tells us that X has a finite number of minimal points. Since $X \subseteq \operatorname{Supp}_R(N)$, Remark 2.1.4 tells us that every minimal point of X belongs to a set $Y_t := \{\mathfrak{p} \in X : \partial_{\mathfrak{p}}(F) \leqslant t\}$ for some nonnegative integer t. Hence, for all t sufficiently large, $Y_t = X$. Also, for every nonnegative integer t, the set Y_t is closed by Lemma 2.2.1, and so the set $\operatorname{Min}(Y_t)$ of all minimal points of Y_t is finite by Proposition 2.3.1. Accordingly, we define

$$\Lambda := \bigcup_{t=0}^{\infty} \operatorname{Min}(Y_t)$$

and observe that Λ is finite.

Let $\mathfrak{p} \in X - \Lambda$, and let $u := \partial_{\mathfrak{p}}(F)$ so that $\mathfrak{p} \in Y_u$ by definition. By Proposition 2.3.1, there is $\mathfrak{q} \in \operatorname{Min}(Y_u)$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. Since $\mathfrak{q} \in \Lambda$, we see that $\mathfrak{q} \neq \mathfrak{p}$. Also, $u \geqslant \partial_{\mathfrak{q}}(F) \geqslant \partial_{\mathfrak{p}}(F) = u$, and so $\partial_{\mathfrak{q}}(F) = \partial_{\mathfrak{p}}(F)$. $\square$

Assume the hypotheses of Part (2) of Theorem 2.1.15, and define Λ as in Lemma 2.3.2 with respect to F and X. In Part (2) of Theorem 2.1.15, we give a lower bound on $\partial(G)$ whose expression involves all of the members of X, where X could be an infinite set. In Part (1) of the following corollary, we express this same lower bound using only the members of the finite set Λ. As a result, we can improve our expression of the lower bound on $\operatorname{sur}_S(M,N)$ in Theorem 1.1.5 as well, and this is the content of Part (2) of the following corollary. Of course, since Theorems 2.1.15 and 1.1.5 rely on the Surjective Lemma, we still need to assume the truth of the Surjective Lemma for the following corollary.

Corollary 2.3.3. *We make the following improvements to Theorems 2.1.15 and 1.1.5:*

(1) Assume the hypotheses of Part (2) of Theorem 2.1.15, and let Λ be defined as in Lemma 2.3.2 with respect to F and X. Then

$$\partial(G) \geqslant \inf\{\partial_{\mathfrak{p}}(F) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in \Lambda\}.$$

(2) Assume the hypotheses of Theorem 1.1.5, and let Λ be defined as in Lemma 2.3.2 with respect to $F := \operatorname{Hom}_S(M, N)$ and X. Then

$$\operatorname{sur}_S(M, N) \geqslant \inf\{\operatorname{sur}_{S_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in \Lambda\}.$$

Proof of Corollary 2.3.3, modulo the Surjective Lemma. (1) Let

$$t := \inf\{\partial_{\mathfrak{p}}(F) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in X\},$$

and let

$$u := \inf\{\partial_{\mathfrak{p}}(F) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in \Lambda\}.$$

We will show that $t = u$. Certainly $t \leqslant u$, so it remains to show that $t \geqslant u$. If $t = \infty$, then certainly $t \geqslant u$. Suppose then that t is an integer so that $X \neq \varnothing$. Let $\mathfrak{p}_0 \in X$ such that $t = \partial_{\mathfrak{p}_0}(F) - \dim_X(\mathfrak{p}_0)$. Suppose, by way of contradiction, that $\mathfrak{p}_0 \notin \Lambda$. By Lemma 2.3.2, there exists $\mathfrak{q}_0 \in \Lambda$ such that $\mathfrak{q}_0 \subsetneq \mathfrak{p}_0$ and $\partial_{\mathfrak{q}_0}(F) = \partial_{\mathfrak{p}_0}(F)$. Hence

$$t = \partial_{\mathfrak{p}_0}(F) - \dim_X(\mathfrak{p}_0) > \partial_{\mathfrak{q}_0}(F) - \dim_X(\mathfrak{q}_0) \geqslant t,$$

a contradiction. Thus $\mathfrak{p}_0 \in \Lambda$, and so $t \geqslant u$. Thus $t = u$. Now, by Part (2) of Theorem 2.1.15, we see that $\partial(G) \geqslant t = u$.

(2) Let $L = M$, and let $G = F$. Since F is finitely generated over R, Part (1) of this corollary tells us that

$$\begin{aligned}\operatorname{sur}_S(M, N) &= \partial(G)\\ &\geqslant \inf\{\partial_{\mathfrak{p}}(F) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in \Lambda\}\\ &= \inf\{\operatorname{sur}_{S_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in \Lambda\}. \quad \square\end{aligned}$$

We now return to the task of reducing the proof of the Surjective Lemma to the study of a finite subset Λ of X. The following definition will be useful in subsequent work:

Definition 2.3.4. Let n be a positive integer. The symbol $\mathbf{GL}(n, R)$ refers to the group of all invertible $n \times n$ matrices with entries in R. This group is called the *general linear group of degree n over R.*

Remark 2.3.5. With respect to the previous definition, an $n \times n$ matrix with entries in R is in $\mathbf{GL}(n, R)$ if and only if its determinant is a unit of R.

In the following definition, we establish a large amount of notation that we will use in the remainder of this section and in the next three sections:

Definition 2.3.6. Let n be an integer with $n \geqslant 2$; let $f := (f_1, \ldots, f_n)^\top \in \operatorname{Hom}_S(M, N^{\oplus n})$; and let $\mathfrak{p} \in \operatorname{Supp}_R(N)$. Fix the following notation relative to n, f, and $\mathfrak{p}$: Let $F := Rf_1 + \cdots + Rf_n$; let $\overline{}$ denote the functor $- \otimes_R \kappa(\mathfrak{p})$; and, for every matrix $\Xi := (\xi_{i,j})$ with entries in R or $R_{\mathfrak{p}}$, let $\overline{\Xi} := \left(\overline{\xi_{i,j}}\right)$. If a matrix $A \in \mathbf{GL}(n, R)$ is given, then let $f^* := Af := (f'_1, \ldots, f'_n)^\top$; let $f' := (f'_1, \ldots, f'_{n-1})^\top$; and let $F' := Rf'_1 + \cdots + Rf'_{n-1}$.

Upon proving the following lemma, we will be prepared to achieve the goal of this section.

Lemma 2.3.7. *Let n be an integer with $n \geqslant 2$; let $f := (f_1, \ldots, f_n)^\top \in \operatorname{Hom}_S(M, N^{\oplus n})$; let $A \in \mathbf{GL}(n, R)$; and let $\mathfrak{p} \in \operatorname{Supp}_R(N)$. Then, relative to Definition 2.3.6, we have $\partial_{\mathfrak{p}}(F') \geqslant \partial_{\mathfrak{p}}(F) - 1$.*

Proof. If $\partial_{\mathfrak{p}}(F) \leqslant 1$, then $\partial_{\mathfrak{p}}(F') \geqslant 0 \geqslant \partial_{\mathfrak{p}}(F) - 1$, and so we are done. Suppose then, for the rest of the proof, that $\partial_{\mathfrak{p}}(F) \geqslant 2$.

Let $d := \partial_{\mathfrak{p}}(F)$, and let B be a $d \times n$ matrix with entries in R such that $(Bf)_{\mathfrak{p}}$ is surjective. Let $C \in \mathbf{GL}(d, R_{\mathfrak{p}})$ such that CBA^{-1} can be represented as a matrix $(b_{i,j})$ with entries in R, where $b_{1,n}, \ldots, b_{d-1,n} \in \mathfrak{p}$. Hence

$$\overline{CBA^{-1}} = \begin{pmatrix} \overline{b_{1,1}} & \cdots & \overline{b_{1,n-1}} & \overline{0} \\ \vdots & \ddots & \vdots & \vdots \\ \overline{b_{d-1,1}} & \cdots & \overline{b_{d-1,n-1}} & \overline{0} \\ \overline{b_{d,1}} & \cdots & \overline{b_{d,n-1}} & \overline{b_{d,n}} \end{pmatrix}.$$

Let

$$B' := \begin{pmatrix} b_{1,1} & \cdots & b_{1,n-1} \\ \vdots & \ddots & \vdots \\ b_{d-1,1} & \cdots & b_{d-1,n-1} \end{pmatrix}.$$

Then $B'f' \in (F')^{\oplus(d-1)}$, and $\overline{B'f'}$ is surjective. Nakayama's Lemma then tells us that $(B'f')_{\mathfrak{p}}$ is surjective. Hence $\partial_{\mathfrak{p}}(F') \geqslant d - 1 = \partial_{\mathfrak{p}}(F) - 1$. $\square$

Lemma 2.3.8. *Assume the hypotheses of the Surjective Lemma, and define Λ as in Lemma 2.3.2 with respect to $F := Rf_1 + \cdots + Rf_n$ and X. Let $A \in \mathbf{GL}(n, R)$, and suppose that, with respect to Definition 2.3.6, we have that f' is (t, X, Λ)-surjective. Then f' is (t, X, X)-surjective.*

Proof. Since t and X are understood, we can use the terms $\mathfrak{p}$-*surjective* and Y-*surjective* for any $\mathfrak{p} \in X$ and for any $Y \subseteq X$ without the risk of confusion. Hence, we aim to prove that f' is X-surjective.

Let $\mathfrak{p} \in X - \Lambda$. By Lemma 2.3.7, we have $\partial_{\mathfrak{p}}(F') \geqslant \partial_{\mathfrak{p}}(F) - 1$. By Lemma 2.3.2, there exists $\mathfrak{q} \in \Lambda$ such that $\mathfrak{q} \subsetneq \mathfrak{p}$ and $\partial_{\mathfrak{q}}(F) = \partial_{\mathfrak{p}}(F)$. Since f is $\mathfrak{q}$-surjective by assumption, we now have

$$\begin{aligned} \partial_{\mathfrak{p}}(F') &\geqslant \partial_{\mathfrak{p}}(F) - 1 \\ &= \partial_{\mathfrak{q}}(F) - 1 \\ &\geqslant \min\{n, t + \dim_X(\mathfrak{q})\} - 1 \\ &\geqslant \min\{n - 1, t + \dim_X(\mathfrak{p})\}, \end{aligned}$$

and so f' is $\mathfrak{p}$-surjective. Thus f' is $(X - \Lambda)$-surjective. Since f' is Λ-surjective by assumption, we conclude that f' is X-surjective. $\square$

Assume the hypotheses of the Surjective Lemma. By Lemma 2.3.8, proving the Surjective Lemma now reduces to finding a matrix $V \in \mathbf{GL}(n, R)$ such that the first $n - 1$ components of $Vf := (g_1, \ldots, g_n)^\top$ form a map $(g_1, \ldots, g_{n-1})^\top$ that is (t, X, Λ)-surjective, where Λ is defined as in Lemma 2.3.2 with respect to $F := Rf_1 + \cdots + Rf_n$ and X. We compute such a matrix V over the course of the next two sections, completing a proof of the Surjective Lemma in Section 2.5.

2.4 The maximal ideals of R in Λ

Assume the hypotheses of the Surjective Lemma, and define Λ as in Lemma 2.3.2 with respect to $F := Rf_1 + \cdots + Rf_n$ and X so that, for every $\mathfrak{p} \in X - \Lambda$, there exists $\mathfrak{q} \in \Lambda$ for which $\mathfrak{q} \subsetneq \mathfrak{p}$ and $\partial_{\mathfrak{q}}(F) = \partial_{\mathfrak{p}}(F)$. In this section, we find a matrix $V \in \mathbf{GL}(n, R)$ such that the first $n - 1$ components of $Vf := (g_1, \ldots, g_n)^\top$ form a map $(g_1, \ldots, g_{n-1})^\top$ that is $(t, X, \mathfrak{m})$-surjective for every $\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R)$. (If every maximal ideal of R avoids Λ, then the results of this section may be ignored.) The following definition will be useful in the work to come:

Definition 2.4.1. Let n be an integer with $n \geqslant 2$. For every $i \in \{1, \ldots, n-1\}$, let P_i be the $n \times n$ permutation matrix obtained by switching the ith row and the nth row of the $n \times n$ identity matrix, and let P_n denote the $n \times n$ identity matrix itself.

We use the matrices $P_1, \ldots, P_n$ in the following lemma to prove the existence of a matrix V exhibiting the properties described at the beginning of this section.

Lemma 2.4.2. *Assume the hypotheses of the Surjective Lemma; define Λ as in Lemma 2.3.2 with respect to $F := Rf_1 + \cdots + Rf_n$ and X; and let $\mathfrak{m} \in \Lambda \cap \mathrm{Max}(R)$. There exists a number $\mathscr{L}_{\mathfrak{m}} \in \{1, \ldots, n\}$ such that, for all $s_1, \ldots, s_n \in R - \mathfrak{m}$, there exist $r_{\mathfrak{m},1}, \ldots, r_{\mathfrak{m},n-1} \in R$ such that, for every $n \times n$ matrix V with entries in R, if*

$$V \equiv \begin{pmatrix} 1 & 0 & \cdots & 0 & r_{\mathfrak{m},1} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & r_{\mathfrak{m},n-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} P_{\mathscr{L}_{\mathfrak{m}}} \begin{pmatrix} s_1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & s_n \end{pmatrix} \pmod{\mathfrak{m}},$$

then the first $n-1$ components of $Vf := (g_1, \ldots, g_n)^\top$ form a map $(g_1, \ldots, g_{n-1})^\top$ that is $(t, X, \mathfrak{m})$-surjective.

Proof. Define every object in Definition 2.3.6 with respect to our current hypotheses, with $\mathfrak{m}$ taking the place of $\mathfrak{p}$ and with A defined as the $n \times n$ identity matrix. Then we have $f' := (f_1, \ldots, f_{n-1})^\top$. Let $d := \partial_{\mathfrak{m}}(F)$, and let B be a $d \times n$ matrix with entries in R such that $(Bf)_{\mathfrak{m}}$ is surjective. Let $C \in \mathbf{GL}(d, R_{\mathfrak{m}})$ such that CB can be represented by a matrix $(b_{i,j})$ with entries in R and such that $\overline{CB}$ is in the following reduced row echelon form, where the nonzero entries are clustered toward the top right corner of the matrix:

$$\overline{CB} = \begin{pmatrix} \overline{0} & \cdots & \overline{0} & \overline{1} & \vdots & \overline{0} & \vdots & \overline{0} & \vdots \\ \overline{0} & \cdots & \overline{0} & \overline{0} & \vdots & \overline{1} & \vdots & \overline{0} & \vdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \overline{0} & \cdots & \overline{0} & \overline{0} & \vdots & \overline{0} & \vdots & \overline{1} & \vdots \end{pmatrix}.$$

Here, the vertical and horizontal ellipses denote possible omissions of entries, and the zero columns on the left may not be present. For every $i \in \{1, \ldots, d\}$, let j_i be the smallest number in the set $\{1, \ldots, n\}$ such that $\overline{b_{i,j_i}} \neq \overline{0}$. We assume that, for every $i \in \{1, \ldots, d\}$, the entry $\overline{b_{i,j_i}}$ is the only nonzero entry in the (j_i)th column of $\overline{CB}$.

Below, we consider several cases and prove the lemma in each case. Since t and X are understood, we may use the term $\mathfrak{m}$-*surjective* for the rest of the proof without the risk of ambiguity. For every $i \in \{1, \ldots, n\}$, let $t_i \in R - \mathfrak{m}$ such that $\overline{s_i t_i} = \overline{1}$.

First suppose that f' is $\mathfrak{m}$-surjective. Then, by Nakayama's Lemma, we may let $\mathscr{L}_{\mathfrak{m}} = n$ and $r_{\mathfrak{m},1} = \cdots = r_{\mathfrak{m},n-1} = 0$, regardless of the values of $s_1, \ldots, s_n \in R - \mathfrak{m}$.

Suppose next that $j_d \leqslant n-1$. Then, by Nakayama's Lemma, we may take $\mathscr{L}_{\mathfrak{m}} = n$, and we may define $r_{\mathfrak{m},j}$ for every $j \in \{1, \ldots, n-1\}$ as follows: If $j = j_i$ for some $i \in \{1, \ldots, d\}$, then let $r_{\mathfrak{m},j} = s_j b_{i,n} t_n$; otherwise, let $r_{\mathfrak{m},j} = 0$.

Now suppose, for the rest of the proof, that f' is not $\mathfrak{m}$-surjective and that $j_d = n$. If $\partial_{\mathfrak{m}}(F) = n$, then $\partial_{\mathfrak{m}}(F') \geqslant \partial_{\mathfrak{m}}(F) - 1 = n - 1$ by Lemma 2.3.7, and so f' is $\mathfrak{m}$-surjective, a contradiction. Hence $\partial_{\mathfrak{m}}(F) \leqslant n - 1$.

Since $j_d = n$ and since $d = \partial_{\mathfrak{m}}(F) \leqslant n-1$, there exists $k \in \{1, \ldots, n-1\} - \{j_1, \ldots, j_{d-1}\}$. Accordingly, by Nakayama's Lemma, we may let $\mathscr{L}_{\mathfrak{m}}$ be any such k, and we may define $r_{\mathfrak{m},j}$ for every $j \in \{1, \ldots, n-1\}$ as follows: If $j = j_i$ for some $i \in \{1, \ldots, d-1\}$, then let $r_{\mathfrak{m},j} = s_j b_{i,k} t_k$; otherwise, let $r_{\mathfrak{m},j} = 0$. $\square$

For the second lemma of this section, we must recall the Chinese Remainder Theorem, which identifies an important congruence property of commutative rings.

Theorem 2.4.3 (Chinese Remainder Theorem [8, Theorem 17, Section 7.6])**.** *Let R be a commutative ring; let $a_1, \ldots, a_u \in R$; and let $\mathfrak{m}_1, \ldots, \mathfrak{m}_u \in \mathrm{Max}(R)$. Then there exists $s \in R$ such that $s \equiv a_i \pmod{\mathfrak{m}_i}$ for every $i \in \{1, \ldots, u\}$. Moreover, the coset $s + \mathfrak{m}_1 \cdots \mathfrak{m}_u$ is the collection of all elements $r \in R$ such that $r \equiv a_i \pmod{\mathfrak{m}_i}$ for every $i \in \{1, \ldots, u\}$.*

In the proof of the next lemma, we use the Chinese Remainder Theorem to find a matrix $Q \in \mathbf{GL}(n, R)$ and elements

$$s_1, \ldots, s_n \in R - \bigcup_{\mathfrak{m} \in \Lambda \cap \mathrm{Max}(R)} \mathfrak{m}$$

such that

$$Q \equiv P_{\mathscr{L}_{\mathfrak{m}}} \begin{pmatrix} s_1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & s_n \end{pmatrix} \pmod{\mathfrak{m}}$$

for every $\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R)$. We phrase the next lemma using language more general than this because we will use the lemma under different circumstances later in the dissertation.

Lemma 2.4.4. *Let R be a commutative ring, and let n be an integer with $n \geqslant 2$. Let $\Lambda_1, \ldots, \Lambda_n$ be finite, pairwise disjoint subsets of $\operatorname{Max}(R)$. (Here, we allow some, or even all, of these sets to be empty.) Then there exist a matrix $Q \in \mathbf{GL}(n, R)$ and elements*

$$s_1, \ldots, s_n \in R - \bigcup_{\mathfrak{m} \in \Lambda_1 \cup \cdots \cup \Lambda_n} \mathfrak{m}$$

such that, for every $i \in \{1, \ldots, n\}$ and for every $\mathfrak{m} \in \Lambda_i$, the matrix Q satisfies the congruence

$$Q \equiv P_i \begin{pmatrix} s_1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & s_n \end{pmatrix} \pmod{\mathfrak{m}}.$$

Moreover, for any

$$a \in R - \bigcup_{\mathfrak{m} \in \Lambda_1} \mathfrak{m},$$

we can arrange for the first row of Q to be of the form

$$\begin{pmatrix} 1 - ab & 0 & \cdots & 0 & ab \end{pmatrix}$$

for some $b \in R$.

Remark 2.4.5. If we did not require Q to be invertible over R, then the first statement of Lemma 2.4.4 would be trivial: We could take $s_1 = \cdots = s_n = 1$, and we could simply apply the Chinese Remainder Theorem once for each entry of Q. Invertibility, then, is what makes the first statement nontrivial, and of course the second statement requires even more care. It should be noted, however, that we do not use the second statement of Lemma 2.4.4 in this chapter. Instead, we use it in Chapter 3.

Proof of Lemma 2.4.4. Let

$$a \in R - \bigcup_{\mathfrak{m} \in \Lambda_1} \mathfrak{m}.$$

For every $i \in \{1, \ldots, n\}$, let

$$I_i := \bigcap_{\mathfrak{m} \in \Lambda_i} \mathfrak{m} \qquad \text{and} \qquad J_i := \bigcap_{j \in \{1,\ldots,n\} - \{i\}} \left(\bigcap_{\mathfrak{m} \in \Lambda_j} \mathfrak{m} \right),$$

and let

$$U_i := \bigcup_{\mathfrak{m} \in \Lambda_i} \mathfrak{m} \qquad \text{and} \qquad V_i := \bigcup_{j \in \{1,\ldots,n\} - \{i\}} \left(\bigcup_{\mathfrak{m} \in \Lambda_j} \mathfrak{m} \right).$$

We would like to prove that there exist

$$\begin{array}{lllll} a_1 \in I_1 - V_1, & a_2 \in I_2 - V_2, & \ldots, & a_{n-1} \in I_{n-1} - V_{n-1}, & \\ b_1 \in aJ_1 - U_1, & b_2 \in J_2 - U_2, & \ldots, & b_{n-1} \in J_{n-1} - U_{n-1}, & \\ c_1 \in J_1 - U_1, & c_2 \in J_2 - U_2, & \ldots, & c_{n-1} \in J_{n-1} - U_{n-1}, & c_n \in J_n - U_n \end{array}$$

such that $a_1 = 1 - b_1$ and such that the $n \times n$ matrix

$$Q := \begin{pmatrix} a_1 & 0 & \cdots & 0 & b_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & a_{n-1} & b_{n-1} \\ c_1 & \cdots & \cdots & c_{n-1} & c_n \end{pmatrix}$$

has determinant 1 and is thus invertible. After we have accomplished this goal, we can appeal to the Chinese Remainder Theorem to produce

$$s_1, \ldots, s_n \in R - \bigcup_{\mathfrak{m} \in \Lambda_1 \cup \cdots \cup \Lambda_n} \mathfrak{m}$$

such that the following conditions hold:

(1) For every $i \in \{1, \ldots, n-1\}$ and for every $\mathfrak{m} \in (\Lambda_1 \cup \cdots \cup \Lambda_n) - \Lambda_i$, the element s_i satisfies the congruence $s_i \equiv a_i \pmod{\mathfrak{m}}$.

(2) For every $i \in \{1, \ldots, n-1\}$ and for every $\mathfrak{m} \in \Lambda_i$, the element s_n satisfies the congruence $s_n \equiv b_i \pmod{\mathfrak{m}}$.

(3) For every $i \in \{1, \ldots, n\}$ and for every $\mathfrak{m} \in \Lambda_i$, the element s_i satisfies the congruence $s_i \equiv c_i \pmod{\mathfrak{m}}$.

Finally, since $b_1 \in aJ_1$, we can choose $b \in J_1$ such that $b_1 = ab$. Hence $a_1 = 1 - b_1 = 1 - ab$. The matrix Q and the elements $s_1, \ldots, s_n$ of R will then jointly satisfy all of the conditions described in the lemma.

We note that, for any comaximal ideals K, L of R and for any $\alpha \in K$ and $\beta \in L$ such that $\alpha + \beta = 1$, it is the case that $R = \sqrt{\alpha R + \beta R} = \sqrt{\alpha^2 R + \beta^2 R} \subseteq \sqrt{\alpha K + \beta L}$, and so $\alpha K + \beta L = R$. We will use this observation shortly.

Next, we prove that $aJ_1 + J_2 + \cdots + J_n = R$. Suppose not. Then there exists $\mathfrak{n} \in \operatorname{Max}(R)$ such that $aJ_1 + J_2 + \cdots + J_n \subseteq \mathfrak{n}$, and so $aJ_1, J_2, \ldots, J_n \subseteq \mathfrak{n}$. Since $aJ_1 \subseteq \mathfrak{n}$, we see that $a \in \mathfrak{n}$ or $J_1 \subseteq \mathfrak{n}$. Either way, $\mathfrak{n} \notin \Lambda_1$. On the other hand, $J_2, \ldots, J_n \subseteq \mathfrak{n}$, and so $\mathfrak{n} \in \Lambda_1$, a contradiction. Hence $aJ_1 + J_2 + \cdots + J_n = R$, and so $J_1 + J_2 + \cdots + J_n = R$ as well.

Let $i \in \{1, \ldots, n-1\}$, and suppose that we have defined

$$\begin{array}{llll} a_1 \in I_1 - V_1, & a_2 \in I_2 - V_2, & \ldots, & a_{i-1} \in I_{i-1} - V_{i-1}, \\ b_1 \in aJ_1 - U_1, & b_2 \in J_2 - U_2, & \ldots, & b_{i-1} \in J_{i-1} - U_{i-1} \end{array}$$

and the ideals

$$\begin{aligned}K_{i-1} \quad := \quad & (b_1 a_2 \cdots a_{i-2} J_1) + \cdots + (a_1 \cdots a_{i-3} b_{i-2} J_{i-2}) \\ & + (a_1 \cdots a_{i-2} J_i) + \cdots + (a_1 \cdots a_{i-2} J_n)\end{aligned}$$

and

$$L_{i-1} := a_1 \cdots a_{i-2} J_{i-1}$$

of R so that $a_{i-1} K_{i-1} + b_{i-1} L_{i-1} = R$. Let

$$\begin{aligned}K_i \quad := \quad & (b_1 a_2 \cdots a_{i-1} J_1) + \cdots + (a_1 \cdots a_{i-2} b_{i-1} J_{i-1}) \\ & + (a_1 \cdots a_{i-1} J_{i+1}) + \cdots + (a_1 \cdots a_{i-1} J_n)\end{aligned}$$

and

$$L_i := a_1 \cdots a_{i-1} J_i$$

so that $K_i + L_i = a_{i-1} K_{i-1} + b_{i-1} L_{i-1} = R$. If $i = 1$, then let $a_1 \in K_1$ and $b_1 \in aJ_1 \subseteq L_1$ with $a_1 + b_1 = 1$ so that $R = a_1 K_1 + b_1 a J_1 \subseteq a_1 K_1 + b_1 L_1$ and, hence, so that $a_1 K_1 + b_1 L_1 = R$. If $i \geqslant 2$, then simply let $a_i \in K_i$ and $b_i \in L_i$ with $a_i + b_i = 1$ so that $a_i K_i + b_i L_i = R$.

We will prove that $a_i \in I_i - V_i$ and that $b_i \in J_i - U_i$ (in fact, $b_1 \in aJ_1 - U_1$). Certainly $a_i \in K_i \subseteq I_i$, and $b_i \in L_i \subseteq J_i$ (in fact, $b_1 \in aJ_1$). Suppose that $a_i \in V_i$. Then there is $\mathfrak{n} \in (\varLambda_1 \cup \cdots \cup \varLambda_n) - \varLambda_i$ such that $a_i \in \mathfrak{n}$. Since $b_i \in J_i \subseteq \mathfrak{n}$, we have $1 = a_i + b_i \in \mathfrak{n}$, a contradiction. Hence $a_i \in I_i - V_i$. Similarly, $b_i \in J_i - U_i$ (in fact, $b_1 \in aJ_1 - U_1$).

By induction on i, we can thus define

$$\begin{aligned}& a_1 \in I_1 - V_1, \quad a_2 \in I_2 - V_2, \quad \ldots, \quad a_{n-1} \in I_{n-1} - V_{n-1}, \\ & b_1 \in aJ_1 - U_1, \quad b_2 \in J_2 - U_2, \quad \ldots, \quad b_{n-1} \in J_{n-1} - U_{n-1}\end{aligned}$$

and ideals

$$K_{n-1} := (b_1 a_2 \cdots a_{n-2} J_1) + \cdots + (a_1 \cdots a_{n-3} b_{n-2} J_{n-2}) + (a_1 \cdots a_{n-2} J_n)$$

and

$$L_{n-1} := a_1 \cdots a_{n-2} J_{n-1}$$

of R so that $a_{n-1}K_{n-1} + b_{n-1}L_{n-1} = R$. Hence

$$(b_1 a_2 \cdots a_{n-1} J_1) + \cdots + (a_1 \cdots a_{n-2} b_{n-1} J_{n-1}) + (a_1 \cdots a_{n-1} J_n) = R.$$

Accordingly, we can choose

$$c_1 \in J_1, \quad \ldots, \quad c_{n-1} \in J_{n-1}, \quad c_n \in J_n$$

such that the determinant

$$-(b_1 a_2 \cdots a_{n-1} c_1) - \cdots - (a_1 \cdots a_{n-2} b_{n-1} c_{n-1}) + (a_1 \cdots a_{n-1} c_n)$$

of the matrix

$$Q := \begin{pmatrix} a_1 & 0 & \cdots & 0 & b_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & a_{n-1} & b_{n-1} \\ c_1 & \cdots & \cdots & c_{n-1} & c_n \end{pmatrix}$$

is equal to 1.

It remains to show that $c_i \notin U_i$ for every $i \in \{1, \ldots, n\}$. Suppose that $c_1 \in U_1$. Then there is $\mathfrak{n} \in \Lambda_1$ such that $c_1 \in \mathfrak{n}$. Since $J_2, \ldots, J_n \subseteq \mathfrak{n}$, we have $c_2, \ldots, c_n \in \mathfrak{n}$, and so $1 = \det(Q) \in Rc_1 + \cdots + Rc_n \subseteq \mathfrak{n}$, a contradiction. Hence $c_1 \notin U_1$. Similarly, $c_i \notin U_i$ for every $i \in \{2, \ldots, n\}$. □

We combine the results of the last two lemmas to achieve the goal of this section:

Lemma 2.4.6. *Assume the hypotheses of the Surjective Lemma, and define Λ as in Lemma 2.3.2 with respect to $F := Rf_1 + \cdots + Rf_n$ and X. Then there exists a matrix*

$V \in \mathbf{GL}(n,R)$ such that the first $n-1$ components of $Vf := (g_1,\dots,g_n)^\top$ form a map $(g_1,\dots,g_{n-1})^\top$ that is $(t,X,\mathfrak{m})$-surjective for every $\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R)$.

Proof. For every $\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R)$, choose $\mathscr{L}_{\mathfrak{m}} \in \{1,\dots,n\}$ so that it satisfies the conclusion of Lemma 2.4.2; and, for every $i \in \{1,\dots,n\}$, let

$$\Lambda_i = \{\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R) : \mathscr{L}_{\mathfrak{m}} = i\}.$$

Then $\Lambda_1,\dots,\Lambda_n$ are finite, pairwise disjoint subsets of $\operatorname{Max}(R)$. Hence, by Lemma 2.4.4, there exist a matrix $Q \in \mathbf{GL}(n,R)$ and elements

$$s_1,\dots,s_n \in R - \bigcup_{\mathfrak{m}\in\Lambda_1\cup\cdots\cup\Lambda_n} \mathfrak{m}$$

such that, for every $i \in \{1,\dots,n\}$ and for every $\mathfrak{m} \in \Lambda_i$, we have

$$Q \equiv P_i \begin{pmatrix} s_1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & s_n \end{pmatrix} \pmod{\mathfrak{m}}.$$

Choose $r_{\mathfrak{m},1},\dots,r_{\mathfrak{m},n-1} \in R$ so that they satisfy the conclusion of Lemma 2.4.2 relative to $\mathscr{L}_{\mathfrak{m}}$ and $s_1,\dots,s_n$. Apply the Chinese Remainder Theorem to find $r_1,\dots,r_{n-1} \in R$ such that $r_i \equiv r_{\mathfrak{m},i} \pmod{\mathfrak{m}}$ for every $i \in \{1,\dots,n-1\}$ and for every $\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R)$. Let

$$U := \begin{pmatrix} 1 & 0 & \cdots & 0 & r_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & r_{n-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \in \mathbf{GL}(n,R),$$

and let $V := UQ$. Then

$$V \equiv \begin{pmatrix} 1 & 0 & \cdots & 0 & r_{\mathfrak{m},1} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & r_{\mathfrak{m},n-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} P_{\mathscr{L}_\mathfrak{m}} \begin{pmatrix} s_1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & s_n \end{pmatrix} \pmod{\mathfrak{m}}$$

for every $\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R)$. Now, Lemma 2.4.2 tells us that the first $n-1$ components of $Vf := (g_1, \ldots, g_n)^\top$ form a map $(g_1, \ldots, g_{n-1})^\top$ that is $(t, X, \mathfrak{m})$-surjective for every $\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R)$. $\square$

In the next section, we complete our proof of the Surjective Lemma.

2.5 Proof of the Surjective Lemma

Throughout this section, we assume the hypotheses of the Surjective Lemma (that is, Lemma 2.1.14), and we let Λ be defined as in Lemma 2.3.2 with respect to $F := Rf_1 + \cdots + Rf_n$ and X. Recall that Λ is a finite subset of X such that, for every $\mathfrak{p} \in X - \Lambda$, there exists $\mathfrak{q} \in \Lambda$ for which $\mathfrak{q} \subsetneq \mathfrak{p}$ and $\partial_\mathfrak{q}(F) = \partial_\mathfrak{p}(F)$. Here, since t and X are understood, we may use the terms $\mathfrak{p}$-*surjective* and Y-*surjective* for any $\mathfrak{p} \in X$ and for any $Y \subseteq X$ without the risk of confusion.

In this section, we find a matrix $V \in \mathbf{GL}(n, R)$ such that the first $n-1$ components of $Vf := (g_1, \ldots, g_n)^\top$ form a map $g := (g_1, \ldots, g_{n-1})^\top$ that is Λ-surjective. Lemma 2.3.8 will then tell us that g is X-surjective and, hence, that we have proved the Surjective Lemma.

Proof of the Surjective Lemma. Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be the distinct members of $\Lambda - \operatorname{Max}(R)$, and arrange $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ so that, for every $\ell \in \{1, \ldots, m\}$, the prime $\mathfrak{q}_\ell$ is a minimal member of the set $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_\ell\}$. We prove, by induction on $\ell \geqslant 0$, that there exists $V \in \mathbf{GL}(n, R)$ such that the first $n-1$ components of $Vf := (g_1, \ldots, g_n)^\top$ form a map $(g_1, \ldots, g_{n-1})^\top$ that is $\mathfrak{p}$-surjective for every $\mathfrak{p} \in \Lambda - \{\mathfrak{q}_{\ell+1}, \ldots, \mathfrak{q}_m\}$.

Lemma 2.4.6 proves the case in which $\ell = 0$. Suppose then that $1 \leqslant \ell \leqslant m$ and that there exists $A \in \mathbf{GL}(n, R)$ such that the first $n-1$ components of $f^* := Af := (f'_1, \ldots, f'_n)^\top$ form a map $f' := (f'_1, \ldots, f'_{n-1})^\top$ that is $\mathfrak{p}$-surjective for every $\mathfrak{p} \in \varLambda - \{\mathfrak{q}_\ell, \ldots, \mathfrak{q}_m\}$. If f' happens to be $\mathfrak{q}_\ell$-surjective as well, then we may set $V = A$ to finish the inductive step. Suppose then that f' is not $\mathfrak{q}_\ell$-surjective. Define every object in Definition 2.3.6 with respect to our current hypotheses, with $\mathfrak{q}_\ell$ taking the place of $\mathfrak{p}$.

Let

$$J := \bigcap_{\mathfrak{p} \in \varLambda - \{\mathfrak{q}_\ell, \ldots, \mathfrak{q}_m\}} \mathfrak{p}.$$

It suffices to find $r_1, \ldots, r_{n-1} \in J$ such that, if

$$U := \begin{pmatrix} 1 & 0 & \cdots & 0 & r_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & r_{n-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

and if $Uf^* := (g_1, \ldots, g_n)^\top$, then $G := Rg_1 + \cdots + Rg_{n-1}$ satisfies $\partial_{\mathfrak{q}_\ell}(G) = \partial_{\mathfrak{q}_\ell}(F)$: Given such $r_1, \ldots, r_{n-1} \in J$, we will see that the first $n-1$ components of $Uf^* = UAf := (g_1, \ldots, g_n)^\top$ form a map $(g_1, \ldots, g_{n-1})^\top$ that is not only $\mathfrak{q}_\ell$-surjective but, by Nakayama's Lemma, also $\mathfrak{p}$-surjective for every $\mathfrak{p} \in \varLambda - \{\mathfrak{q}_\ell, \ldots, \mathfrak{q}_m\}$. Thus we will be able to take $V := UA$ to finish the inductive step and, thus, the proof overall. Before we find such $r_1, \ldots, r_{n-1} \in J$, though, we must complete some more preparatory work.

To simplify notation, let $\mathfrak{q} := \mathfrak{q}_\ell$ from now on. First we show that $\partial_{\mathfrak{q}}(F') = \partial_{\mathfrak{q}}(F) - 1$ and that $\partial_{\mathfrak{q}}(F) \leqslant n - 1$. By Lemma 2.3.7 and by our assumption that f' is not $\mathfrak{q}$-surjective, we have

$$\partial_{\mathfrak{q}}(F) - 1 \leqslant \partial_{\mathfrak{q}}(F') < \min\{n-1, t + \dim_X(\mathfrak{q})\} \leqslant \partial_{\mathfrak{q}}(F),$$

and so $\partial_{\mathfrak{q}}(F') = \partial_{\mathfrak{q}}(F) - 1$. Now, if $\partial_{\mathfrak{q}}(F) = n$, then $\partial_{\mathfrak{q}}(F') = \partial_{\mathfrak{q}}(F) - 1 = n - 1$, and so f' is $\mathfrak{q}$-surjective, a contradiction. Hence $\partial_{\mathfrak{q}}(F) \leqslant n - 1$.

Let $d := \partial_{\mathfrak{p}}(F)$, and let B be a $d \times n$ matrix with entries in R such that $(Bf^*)_{\mathfrak{q}}$ is surjective. Let $C \in \mathbf{GL}(d, R_{\mathfrak{q}})$ such that CB can be represented by a matrix $(b_{i,j})$ with entries from R and such that $\overline{CB}$ is in the following row echelon form with the nonzero entries clustered toward the top right corner of the matrix and with $s \in R - \mathfrak{q}$:

$$\overline{CB} = \begin{pmatrix} \overline{0} & \cdots & \overline{0} & \overline{s} & \vdots & \overline{0} & \vdots & \overline{0} & \vdots \\ \overline{0} & \cdots & \overline{0} & \overline{0} & \vdots & \overline{s} & \vdots & \overline{0} & \vdots \\ \multicolumn{9}{c}{\dotfill} \\ \overline{0} & \cdots & \overline{0} & \overline{0} & \vdots & \overline{0} & \vdots & \overline{s} & \vdots \end{pmatrix}.$$

Here, the vertical and horizontal ellipses denote possible omissions of entries, and the zero columns on the left may not be present. Now, for every $i \in \{1, \ldots, d\}$, let j_i be the smallest number in the set $\{1, \ldots, n\}$ such that $\overline{b_{i,j_i}} \neq \overline{0}$. We assume that, for every $i \in \{1, \ldots, d\}$, the entry $\overline{b_{i,j_i}}$ is the only nonzero entry in the (j_i)th column of $\overline{CB}$. Let

$$B^* := (b^*_{i,j}) := \begin{pmatrix} 0 & \cdots & 0 & s & \vdots & 0 & \vdots & 0 & \vdots \\ 0 & \cdots & 0 & 0 & \vdots & s & \vdots & 0 & \vdots \\ \multicolumn{9}{c}{\dotfill} \\ 0 & \cdots & 0 & 0 & \vdots & 0 & \vdots & s & \vdots \end{pmatrix}$$

be a $d \times n$ matrix with entries in R that satisfies the following conditions:

(1) $\overline{B^*} = \overline{CB}$.

(2) For every $i \in \{1, \ldots, d\}$ and for every $j \in \{1, \ldots, n\}$, if $\overline{b_{i,j}} = \overline{0}$, then $b^*_{i,j} = 0$.

(3) For every $i \in \{1, \ldots, d\}$, we have $b^*_{i,j_i} = s$.

Hence $B^* f^* \in F^{\oplus d}$, and $\overline{B^* f^*}$ is surjective. Nakayama's Lemma then tells us that $(B^* f^*)_{\mathfrak{q}}$ is surjective. Thus, we assume, without loss of generality, that $B = CB = (b_{i,j})$ and that B already has the desirable form of B^*.

Since $(Bf^*)_{\mathfrak{q}}$ is surjective, there exists a finitely generated R-submodule L of M such that the restriction of $(Bf^*)_{\mathfrak{q}}$ to $L_{\mathfrak{q}}$ is surjective. We may assume, then, without loss of generality, that $M_{\mathfrak{q}}$ is a finitely generated $R_{\mathfrak{q}}$-module.

Let $\mu := \mu_{R_{\mathfrak{q}}}(M_{\mathfrak{q}})$, and let $\nu := \mu_{R_{\mathfrak{q}}}(N_{\mathfrak{q}})$. Since $\mathfrak{q} \in \mathrm{Supp}_R(N)$, we see that $\nu \geqslant 1$. Since f is $\mathfrak{q}$-surjective, $d \geqslant t \geqslant 1$, and so Nakayama's Lemma tells us that $\mu \geqslant d\nu \geqslant 1$. In fact, without loss of generality, we may assume that $\mu = d\nu$.

Let $E := (\varepsilon_1, \ldots, \varepsilon_{d\nu})^\top$ be an ordered $d\nu$-tuple of elements of $M_{\mathfrak{q}}$ such that $\{\varepsilon_1, \ldots, \varepsilon_{d\nu}\}$ is a minimal generating set for $M_{\mathfrak{q}}$ over $R_{\mathfrak{q}}$, and let $Z := (\zeta_1, \ldots, \zeta_\nu)^\top$ be an ordered ν-tuple of elements of $N_{\mathfrak{q}}$ such that $\{\zeta_1, \ldots, \zeta_\nu\}$ is a minimal generating set for $N_{\mathfrak{q}}$ over $R_{\mathfrak{q}}$. For every $i \in \{1, \ldots, n\}$, let $\varphi_i' := (f_i')_{\mathfrak{q}}$, and let $\varPhi_i'$ be a $\nu \times d\nu$ matrix with entries in $R_{\mathfrak{q}}$ that represents φ_i' with respect to E and Z in the following sense: For every $j \in \{1, \ldots, d\nu\}$, if $\theta_{1,j}, \ldots, \theta_{\nu,j} \in R_{\mathfrak{q}}$ such that $\varphi_i'(\varepsilon_j) = \theta_{1,j}\zeta_1 + \cdots + \theta_{\nu,j}\zeta_\nu$, then we may define the jth column of $\varPhi_i'$ to be

$$\begin{pmatrix} \theta_{1,j} \\ \vdots \\ \theta_{\nu,j} \end{pmatrix}.$$

Now let $\varPhi^*$ be the $n\nu \times d\nu$ matrix whose ith $\nu \times d\nu$ block is $\varPhi_i'$. Hence

$$\varPhi^* = \begin{pmatrix} \varPhi_1' \\ \vdots \\ \varPhi_n' \end{pmatrix}.$$

Finally, we let $\mathrm{rank}(\varXi)$ denote the rank of a matrix $\varXi$ with entries in $\kappa(\mathfrak{q})$.

We now return to the task of finding $r_1, \ldots, r_{n-1} \in J$ that satisfy the criteria described earlier. We consider two cases.

Case 1: $j_d \leqslant n-1$. In this case, B has the following form:

$$B = \begin{pmatrix} \vdots & s & \vdots & 0 & \vdots & 0 & \vdots & b_{1,n} \\ \vdots & 0 & \vdots & s & \vdots & 0 & \vdots & b_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & 0 & \vdots & 0 & \vdots & s & \vdots & b_{d,n} \end{pmatrix}.$$

Let $r_j = 0 \in J$ for every $j \in \{1, \ldots, n-1\} - \{j_1, \ldots, j_d\}$. Let $i \in \{1, \ldots, d\}$, and suppose that we have defined $r_{j_1}, \ldots, r_{j_{(i-1)}} \in J$. Let

$$B_i := \begin{pmatrix} \vdots & s & \vdots & 0 & \vdots & 0 & \vdots & 0 & \vdots & sr_{j_1} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & 0 & \vdots & s & \vdots & 0 & \vdots & 0 & \vdots & sr_{j_{(i-1)}} \\ \vdots & 0 & \vdots & 0 & \vdots & s & \vdots & 0 & \vdots & 0 \\ \vdots & 0 & \vdots & 0 & \vdots & 0 & \vdots & s & \vdots & b_{i+1,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

be the $d \times n$ matrix obtained from B by replacing $b_{1,n}, \ldots, b_{i-1,n}, b_{i,n}$ with $sr_{j_1}, \ldots, sr_{j_{(i-1)}}$, 0, respectively. Let $\Omega_i := (B_i \otimes I_\nu)\Phi^*$, where I_ν denotes the $\nu \times \nu$ identity matrix, and let

$$\Omega_i' := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \Phi_n' \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

be the $d\nu \times d\nu$ matrix obtained by replacing the ith $\nu \times d\nu$ block of the zero $d\nu \times d\nu$ matrix with Φ_n'.

Suppose, as an inductive hypothesis, that rank $\left(\overline{\Omega_i + b_{i,n}\Omega_i'}\right) = d\nu$. We will prove that there exists $r_{j_i} \in J$ such that rank $\left(\overline{\Omega_i + sr_{j_i}\Omega_i'}\right) = d\nu$.

Let $\mathscr{I}$ denote the ideal $(sJ+\mathfrak{q})/\mathfrak{q}$ of $R/\mathfrak{q}$. Since $\mathfrak{q}$ is a nonmaximal prime ideal of R, we see that $R/\mathfrak{q}$ is an infinite domain. Since $s \in R - \mathfrak{q}$ and since $J \not\subseteq \mathfrak{q}$, the ideal $\mathscr{I}$ is nonzero, hence infinite.

Let

$$\mathscr{S}_i := \left\{\sigma \in \kappa(\mathfrak{q}) : \operatorname{rank}\left(\overline{\Omega_i} + \sigma\overline{\Omega_i'}\right) \leqslant d\nu - 1\right\}.$$

We will show that $\mathscr{I}$ contains an element ρ_i that avoids $\mathscr{S}_i$. Let $\mathscr{D}_i(x)$ denote the determinant of $\overline{\Omega_i} + x\overline{\Omega_i'}$, where x is a variable. Since rank $\left(\overline{\Omega_i + b_{i,n}\Omega_i'}\right) = d\nu$, we see that $\mathscr{D}_i\left(\overline{b_{i,n}}\right) \neq \overline{0}$. Hence $\mathscr{D}_i(x)$ is a nonzero polynomial. Since the degree of $\mathscr{D}_i(x)$ is at most ν, we see that $|\mathscr{S}_i| \leqslant \nu$. Since $\mathscr{I}$ is infinite, $\mathscr{I}$ must then contain an element ρ_i that avoids $\mathscr{S}_i$.

Now let $r_{j_i} \in J$ such that $\overline{sr_{j_i}} = \rho_i$. Then rank $\left(\overline{\Omega_i + sr_{j_i}\Omega_i'}\right) = d\nu$, as promised.

By induction, then, we can define matrices $B_1, \Omega_1, \Omega_1', \ldots, B_d, \Omega_d, \Omega_d'$ and $r_{j_1}, \ldots, r_{j_d} \in J$ such that rank $\left(\overline{\Omega_d + sr_{j_d}\Omega_d'}\right) = d\nu$. Now, let B' be the $d\times(n-1)$ matrix obtained by deleting the nth column of B; let

$$U := \begin{pmatrix} 1 & 0 & \cdots & 0 & r_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & r_{n-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix};$$

and let Γ denote the $(n-1)\nu \times d\nu$ matrix obtained by deleting the nth $\nu \times d\nu$ block of $(U \otimes I_\nu)\Phi^*$. Then $(B' \otimes I_\nu)\Gamma = \Omega_d + sr_{j_d}\Omega_d'$, and so rank $\left[\overline{(B' \otimes I_\nu)\Gamma}\right] = d\nu$. Let $Uf^* := (g_1, \ldots, g_n)^\top$, and let $G := Rg_1 + \cdots + Rg_{n-1}$. Then, by Nakayama's Lemma, $(B' \otimes I_\nu)\Gamma$ represents a surjection in $G_\mathfrak{q}^{\oplus d}$ from $M_\mathfrak{q}$ to $N_\mathfrak{q}^{\oplus d}$, and so $\partial_\mathfrak{q}(G) = d = \partial_\mathfrak{q}(F)$, as desired.

Case 2: $j_d = n$. Here, we have

$$B = \begin{pmatrix} \vdots & s & \vdots & 0 & \vdots & 0 \\ \vdots & 0 & \vdots & s & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \vdots & 0 & \vdots & 0 & \vdots & s \end{pmatrix}.$$

Since $d \leqslant n-1$ and since $j_d = n$, there is $k \in \{1, \ldots, n-1\} - \{j_1, \ldots, j_{d-1}\}$.

Let $\Omega := (B \otimes I_\nu)\Phi^*$, and let

$$\Omega' := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \Phi'_k \end{pmatrix}$$

be the $d\nu \times d\nu$ matrix obtained by replacing the dth $\nu \times d\nu$ block of the zero $d\nu \times d\nu$ matrix with Φ'_k.

Let $\mathscr{J}$ denote the ideal $(J + \mathfrak{q})/\mathfrak{q}$ of $R/\mathfrak{q}$. Since $\mathscr{I} \subseteq \mathscr{J}$, we see that $\mathscr{J}$ is infinite.

Let

$$\mathscr{S} := \left\{\sigma \in \kappa(\mathfrak{q}) : \operatorname{rank}\left(\overline{\Omega} + \sigma\overline{\Omega'}\right) \leqslant d\nu - 1\right\}.$$

We will show that $\mathscr{J}$ contains a nonzero element ρ such that ρ^{-1} avoids $\mathscr{S}$. Let $\mathscr{D}(x)$ be the determinant of $\overline{\Omega} + x\overline{\Omega'}$, where x is a variable. Since $\operatorname{rank}\left(\overline{\Omega + 0\Omega'}\right) = \operatorname{rank}\left(\overline{\Omega}\right) = d\nu$, we see that $\mathscr{D}\left(\overline{0}\right) \neq \overline{0}$. Hence $\mathscr{D}(x)$ is a nonzero polynomial. Since the degree of $\mathscr{D}(x)$ is at most ν, we see that $|\mathscr{S}| \leqslant \nu$. Since $\mathscr{J}$ is infinite, $\mathscr{J}$ must then contain a nonzero element ρ such that ρ^{-1} avoids $\mathscr{S}$.

Now let $r \in J - \mathfrak{q}$ such that $\overline{r} = \rho$, and let $1/r$ denote the multiplicative inverse of the element $r/1$ of $R_{\mathfrak{q}}$ so that $\overline{(1/r)} = \rho^{-1}$. Let

$$B_1 = \begin{pmatrix} \vdots & b_{1,k} & \vdots & 0 \\ \vdots & b_{2,k} & \vdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ \vdots & b_{d-1,k} & \vdots & 0 \\ \vdots & 1/r & \vdots & s \end{pmatrix}$$

be the $d \times n$ matrix obtained from B by replacing $b_{d,k} = 0$ with $1/r$. Note that $(B_1 \otimes I_\nu)\varPhi^* = \varOmega + (1/r)\varOmega'$ so that $\operatorname{rank}\left[\overline{(B_1 \otimes I_\nu)\varPhi^*}\right] = d\nu$.

Next, let

$$B_2 := \begin{pmatrix} 1 & 0 & \cdots & 0 & -rb_{1,k} \\ 0 & \ddots & \ddots & \vdots & -rb_{2,k} \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & -rb_{d-1,k} \\ 0 & \cdots & \cdots & 0 & rs \end{pmatrix} \in \mathbf{GL}(d, R_{\mathfrak{q}}).$$

Then $\operatorname{rank}\left[\overline{(B_2B_1 \otimes I_\nu)\varPhi^*}\right] = d\nu$. Also,

$$B_2B_1 = \begin{pmatrix} \vdots & 0 & \vdots & s(-rb_{1,k}) \\ \vdots & 0 & \vdots & s(-rb_{2,k}) \\ \cdots & \cdots & \cdots & \cdots \\ \vdots & 0 & \vdots & s(-rb_{d-1,k}) \\ \vdots & s & \vdots & s(rs) \end{pmatrix},$$

where the column $(0, 0, \ldots, 0, s)^\top$ displayed above is the kth column of B_2B_1. Now permute the rows of B_2B_1 to yield a matrix B_3 such that $\overline{B_3}$ is in row echelon form. Then $\operatorname{rank}\left[\overline{(B_3 \otimes I_\nu)\varPhi^*}\right] = d\nu$, and so we have reduced to Case 1.

This completes the main inductive step of our proof. □

Now that we have worked through our proof of the Surjective Lemma, we can reveal why we address the members of $\mathit{\Lambda} \cap \operatorname{Max}(R)$ separately in Section 2.4. Let $\mathfrak{q}$ be defined as in the main inductive step of this section. Since $\mathfrak{q}$ is a nonmaximal prime ideal of R, we see that $R/\mathfrak{q}$ is an infinite domain and, hence, that the nonzero ideals $\mathscr{I}$ and $\mathscr{J}$ of $R/\mathfrak{q}$ are also infinite. In Case 1, we find that, for every $i \in \{1, \ldots, d\}$, there is an element of $\mathscr{I}$ that avoids $\mathscr{S}_i$ since $|\mathscr{S}_i| \leqslant \mu_{R_\mathfrak{q}}(N_\mathfrak{q}) < \infty$. In Case 2, we find that $\mathscr{J}$ must have a nonzero element whose multiplicative inverse avoids $\mathscr{S}$ since $|\{\overline{0}\} \cup \mathscr{S}| \leqslant 1 + \mu_{R_\mathfrak{q}}(N_\mathfrak{q}) < \infty$. These lines of reasoning, *mutatis mutandis*, are not necessarily available for a given $\mathfrak{m} \in \mathit{\Lambda} \cap \operatorname{Max}(R)$. In particular, it is not always the case that $R/\mathfrak{m}$ is infinite or even that $|R/\mathfrak{m}| \geqslant 2 + \mu_{R_\mathfrak{m}}(N_\mathfrak{m})$. Hence, in general, we cannot mimic the method that we use on the members of $\mathit{\Lambda} - \operatorname{Max}(R)$ to treat the members of $\mathit{\Lambda} \cap \operatorname{Max}(R)$. The possibility that $|R/\mathfrak{m}| \leqslant 1 + \mu_{R_\mathfrak{m}}(N_\mathfrak{m})$ for some $\mathfrak{m} \in \mathit{\Lambda} \cap \operatorname{Max}(R)$ is what compelled us to find a special method for dealing with the members of $\mathit{\Lambda} \cap \operatorname{Max}(R)$, and it is this method that we present in Section 2.4.

This is not the only method that works. In fact, there is an alternative to the method of Section 2.4 and this section that automatically simplifies our proof of the Surjective Lemma in a special case: With respect to Lemma 2.4.2, we can define

$$\mathit{\Lambda}_i := \{\mathfrak{m} \in \mathit{\Lambda} \cap \operatorname{Max}(R) : \mathscr{L}_\mathfrak{m} = i \text{ and } |R/\mathfrak{m}| \leqslant 1 + \mu_{R_\mathfrak{m}}(N_\mathfrak{m})\}$$

for every $i \in \{1, \ldots, n\}$, use Lemmas 2.4.2 and 2.4.4 to account for the members of $\mathit{\Lambda}_1 \cup \cdots \cup \mathit{\Lambda}_n$ only, and then proceed by induction on the remaining members of $\mathit{\Lambda}$ as in the current section. The benefit of this approach is that, if $|R/\mathfrak{m}| \geqslant 2 + \mu_{R_\mathfrak{m}}(N_\mathfrak{m})$ for every $\mathfrak{m} \in \mathit{\Lambda} \cap \operatorname{Max}(R)$, then $\mathit{\Lambda}_1, \ldots, \mathit{\Lambda}_n$ are all empty, and so it suffices to use the method of this section for the entirety of $\mathit{\Lambda}$.

In another special case, the proof of the Surjective Lemma does not go through as quickly, but the result of the inductive step is still comparable to the one in this section: If $\mu_{R_\mathfrak{m}}(N_\mathfrak{m}) = 1$ for every $\mathfrak{m} \in \mathit{\Lambda} \cap \operatorname{Max}(R)$, then we can mimic the method used in the inductive step of the proof of [7, Lemma 3.8] for the members of $\mathit{\Lambda} \cap \operatorname{Max}(R)$, and we can apply the method of this section to the remaining members of Λ. This case is noteworthy in the following sense: Let $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ be the distinct members of Λ, and arrange $\mathfrak{q}_1, \ldots, \mathfrak{q}_m$ so that, for every $\ell \in \{1, \ldots, m\}$, the prime $\mathfrak{q}_\ell$ is a minimal member of the set $\{\mathfrak{q}_1, \ldots, \mathfrak{q}_\ell\}$. Also, list the members of $\Lambda \cap \operatorname{Max}(R)$ first. Let $\ell \in \{1, \ldots, m\}$, and let

$$J := \bigcap_{i=1}^{\ell-1} \mathfrak{q}_i.$$

Suppose that there exists $A \in \mathbf{GL}(n, R)$ such that the first $n-1$ components of $f^* := Af := (f_1', \ldots, f_n')^\top$ form a map $(f_1', \ldots, f_{n-1}')^\top$ that is $\mathfrak{q}_i$-surjective for every $i \in \{1, \ldots, \ell - 1\}$. Then there exist $r_1, \ldots, r_{n-1} \in J$ such that, if

$$U := \begin{pmatrix} 1 & 0 & \cdots & 0 & r_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & r_{n-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix},$$

then the first $n-1$ components of $Uf^* := (g_1, \ldots, g_n)^\top$ form a map $(g_1, \ldots, g_{n-1})^\top$ that is $\mathfrak{q}_i$-surjective for every $i \in \{1, \ldots, \ell\}$.

We can combine the two special cases that we have mentioned to yield the following corollary of the Surjective Lemma:

Corollary 2.5.1. *Assume the hypotheses of the Surjective Lemma. Define Λ as in Lemma 2.3.2 with respect to $F := Rf_1 + \cdots + Rf_n$ and X. Suppose that, for every $\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R)$, it is the case that $|R/\mathfrak{m}| \geqslant 2 + \mu_{R_\mathfrak{m}}(N_\mathfrak{m})$ or $\mu_{R_\mathfrak{m}}(N_\mathfrak{m}) = 1$. (For example, we may suppose that every residue field of R is infinite or that N is a locally cyclic R-module.) Then there exist $r_1, \ldots, r_{n-1} \in R$ such that $(f_1 + r_1 f_n, \ldots, f_{n-1} + r_{n-1} f_n)^\top$ is (t, X, X)-surjective.*

We leave it to the reader to spell out the ramifications of this corollary for Theorems 1.1.5, 1.1.6, 2.1.15, and 2.2.4.

Despite their benefits, the alternative approaches to the Surjective Lemma have an obvious drawback: Conditions on the sizes of residue fields and minimal generating sets do not receive proper context until the middle of the current section. For this reason, we decided to present a method that avoids specific reference to the sizes of residue fields and minimal generating sets when handling the members of $\Lambda \cap \operatorname{Max}(R)$. We remain faithful to this decision in our presentation of the analogous results of the next section.

2.6 Proofs of Theorems 1.1.8 and 1.1.9

Throughout this section, let R denote a commutative ring; let S denote a module-finite R-algebra; let M denote a right S-module; and let N denote a finitely presented right S-module. As before, we view every left (respectively, right) S-module as a left (respectively, right) R-module in the natural way.

In this section, we prove Theorems 1.1.8 and 1.1.9. Since many of the techniques here are similar to those that we use in Sections 2.1–2.5, we do not provide as much detail here as before. Still, we state all of the necessary definitions and lemmas, and we indicate the major differences between the proofs here and their earlier analogues.

Of note is the fact that there are only four results in this section that do not require N to be finitely presented over S: For Remark 2.6.2, Remark 2.6.4, Lemma 2.6.13, and Lemma 2.6.15, it suffices for N to be finitely generated over S. Every other result in this section ultimately relies on Lemma 2.6.7, and Lemma 2.6.7 relies on the finite presentation of N over S.

We begin with the following definitions and remarks:

Definition 2.6.1. Let F be an R-submodule of $\operatorname{Hom}_S(M, N)$, and let $\mathfrak{p} \in \operatorname{Spec}(R)$. We let $\delta(F)$ denote the supremum of the nonnegative integers t such that there exists $f \in F^{\oplus t} \subseteq \operatorname{Hom}_S(M, N^{\oplus t})$ that is split surjective over S. There always exists a nonnegative integer t such that $F^{\oplus t}$ contains a split surjective map since, as in Definition 2.1.1, we define $L^{\oplus 0} = 0$ for every R-module L. We let $\delta_{\mathfrak{p}}(F)$ denote the supremum of the nonnegative integers t such that there exists $f \in F^{\oplus t}$ with the property that $f_{\mathfrak{p}}$ is split surjective over $S_{\mathfrak{p}}$.

Remark 2.6.2. Let F be a finitely generated R-submodule of $\operatorname{Hom}_S(M,N)$. We observe that $\delta(F)=\infty$ if and only if $N=0$: Certainly, if $N=0$, then $\delta(F)=\infty$. On the other hand, if $\delta(F)=\infty$, then $\partial(F)=\infty$, and so $N=0$ by Remark 2.1.4.

Let $\mathfrak{p}\in\operatorname{Spec}(R)$. Then, by the preceding discussion, $\delta_{\mathfrak{p}}(F)=\infty$ if and only if $\mathfrak{p}\notin\operatorname{Supp}_R(N)$.

Definition 2.6.3. Let n,t be positive integers with $n\geqslant t$; let $\mathfrak{p}\in X\subseteq\operatorname{Spec}(R)$; and let $f:=(f_1,\ldots,f_n)^\top\in\operatorname{Hom}_S(M,N^{\oplus n})$. We say that f is $(t,X,\mathfrak{p})$-*split* if $\delta_{\mathfrak{p}}(Rf_1+\cdots+Rf_n)\geqslant\min\{n,t+\dim_X(\mathfrak{p})\}$.

Let $Y\subseteq X$. We say that f is (t,X,Y)-*split* if f is $(t,X,\mathfrak{q})$-split for every $\mathfrak{q}\in Y$.

When t and X are understood, we use the terms $\mathfrak{p}$-*split* and Y-*split* in place of $(t,X,\mathfrak{p})$-*split* and (t,X,Y)-*split*, respectively.

Remark 2.6.4. Maintaining the hypotheses in the previous definition, we see that f is $(n,X,\mathfrak{p})$-split if and only if $f_{\mathfrak{p}}$ is split surjective over $S_{\mathfrak{p}}$. The reasoning is basically the same as in Remark 2.1.6.

We now state an analogue of the Surjective Lemma (Lemma 2.1.14).

Lemma 2.6.5 (Splitting Lemma)**.** *Let n,t be positive integers with $n\geqslant 1+t$, and let X be a subspace of $\operatorname{Supp}_R(N)$ that is a basic set for R. Let $f:=(f_1,\ldots,f_n)^\top\in\operatorname{Hom}_S(M,N^{\oplus n})$, and suppose that f is (t,X,X)-split. Then there exist $f'_1,\ldots,f'_{n-1}\in Rf_1+\cdots+Rf_n$ such that $f':=(f'_1,\ldots,f'_{n-1})^\top$ is (t,X,X)-split.*

We defer the proof of the Splitting Lemma to the end of this section. Assuming the truth of the Splitting Lemma, we could prove the following theorem at this point, but we omit the proof since it is basically the same as the proof of Theorem 2.1.15. Still, we would like to make one note about the proof. We use the finite presentation of N over S more than just through the use of Lemma 2.6.7: When applying Part (1) of Theorem 2.6.6 to prove Part (2), we use the fact that a map $g\in\operatorname{Hom}_S(M,N^{\oplus t})$ is split surjective over S if and only if $g_{\mathfrak{m}}$ is split surjective over $S_{\mathfrak{m}}$ for every $\mathfrak{m}\in\operatorname{Max}(R)\cap\operatorname{Supp}_R(N)$.

Theorem 2.6.6. *Let L be an S-submodule of M; let F be a finitely generated R-submodule of $\operatorname{Hom}_S(L,N)$; and let G be an R-submodule of $\operatorname{Hom}_S(M,N)$. Suppose that every member of F can be extended to a member of G. Let X be a subspace of $\operatorname{Supp}_R(N)$ that is a basic set for R, and suppose that $\dim(X)<\infty$. Then the following statements hold:*

(1) Let t be a positive integer, and suppose that $\delta_{\mathfrak{p}}(F)\geqslant t+\dim_X(\mathfrak{p})$ for every $\mathfrak{p}\in X$. Then there exists $g\in G^{\oplus t}$ such that $g_{\mathfrak{p}}$ is split surjective over $S_{\mathfrak{p}}$ for every $\mathfrak{p}\in X$.

(2) Suppose that $\operatorname{Max}(R)\cap\operatorname{Supp}_R(N)\subseteq X$. Then

$$\delta(G)\geqslant\inf\{\delta_{\mathfrak{p}}(F)-\dim_X(\mathfrak{p}):\mathfrak{p}\in X\}.$$

We could prove Theorem 1.1.8 at this point, but we omit the proof since it is similar to the proof of Theorem 1.1.5.

As with the Surjective Lemma, we can reduce the proof of the Splitting Lemma to a consideration of a finite subset Λ of X. The following lemma, which is analogous to Lemma 2.2.1, helps us reach this goal. As we mention above, this lemma marks the main point in this section that relies on the finite presentation of N over S.

Lemma 2.6.7. *Let F be an R-submodule of $\operatorname{Hom}_S(M,N)$, and let t be a nonnegative integer. Then the set $\{\mathfrak{p}\in\operatorname{Spec}(R):\delta_{\mathfrak{p}}(F)>t\}$ is open, and so the set $\{\mathfrak{p}\in\operatorname{Spec}(R):\delta_{\mathfrak{p}}(F)\leqslant t\}$ is closed. Hence, for every subspace X of $\operatorname{Spec}(R)$, the set $Y_t:=\{\mathfrak{p}\in X:\delta_{\mathfrak{p}}(F)\leqslant t\}$ is closed in X.*

Proof. Let $\mathfrak{p}\in\operatorname{Spec}(R)$ such that $\delta_{\mathfrak{p}}(F)>t$; let $f\in F^{\oplus(t+1)}$ such that $f_{\mathfrak{p}}$ is split surjective over $S_{\mathfrak{p}}$; and let $L=N^{\oplus(t+1)}$. Then there exists $g\in\operatorname{Hom}_{S_{\mathfrak{p}}}(L_{\mathfrak{p}},M_{\mathfrak{p}})$ such that $1_{L_{\mathfrak{p}}}=f_{\mathfrak{p}}\circ g$. Since N is finitely presented over S, we see that L is finitely presented over S, and so $\operatorname{Hom}_{S_{\mathfrak{p}}}(L_{\mathfrak{p}},M_{\mathfrak{p}})\cong(\operatorname{Hom}_S(L,M))_{\mathfrak{p}}$ as $R_{\mathfrak{p}}$-modules. Hence g can be written as h/u for some $h\in\operatorname{Hom}_S(L,M)$ and $u\in R-\mathfrak{p}$. Since $1_{L_{\mathfrak{p}}}=f_{\mathfrak{p}}\circ g=f_{\mathfrak{p}}\circ h/u$, we see that there exists $v\in R-\mathfrak{p}$ such that $vu\cdot 1_L=v(f\circ h)=f\circ vh$. Let $U:=\{\mathfrak{q}\in\operatorname{Spec}(R):vu\notin\mathfrak{q}\}$. Since $vh\in\operatorname{Hom}_S(L,M)$, we see that $vh/vu\in\operatorname{Hom}_{S_{\mathfrak{q}}}(L_{\mathfrak{q}},M_{\mathfrak{q}})$ for every $\mathfrak{q}\in U$. Now, for every

$\mathfrak{q} \in U$, we see that $1_{L_{\mathfrak{q}}} = f_{\mathfrak{q}} \circ vh/vu$ so that $f_{\mathfrak{q}}$ is split surjective over $S_{\mathfrak{q}}$. Hence U is an open neighborhood of $\mathfrak{p}$ such that $\delta_{\mathfrak{q}}(F) > t$ for every $\mathfrak{q} \in U$. Thus the set $\{\mathfrak{p} \in \mathrm{Spec}(R) : \delta_{\mathfrak{p}}(F) > t\}$ is open. This proves the first claim of the lemma. The last two claims of the lemma follow from the first claim. $\square$

We state the following lemma with an eye toward Theorem 1.1.9. We omit the proof of this lemma since it is similar to the proof of Lemma 2.2.2.

Lemma 2.6.8. *Suppose that M is a direct summand of a direct sum of finitely presented right S-modules. Let X be a Noetherian subspace of $\mathrm{Supp}_R(N)$ such that $\dim(X) < \infty$. Let t be a positive integer, and suppose that $\mathrm{spl}_{S_{\mathfrak{p}}}(M_{\mathfrak{p}}, N_{\mathfrak{p}}) \geqslant t + \dim(X)$ for every $\mathfrak{p} \in X$. Then there exists a finitely generated R-submodule F of $\mathrm{Hom}_S(M, N)$ such that $\delta_{\mathfrak{p}}(F) \geqslant t + \dim(X)$ for every $\mathfrak{p} \in X$.*

We would now be in a position to prove Theorem 1.1.9, modulo the Splitting Lemma. We omit the proof since it is similar to the proof of Theorem 1.1.6.

We would now also be able to prove the following variations of Lemma 2.6.8 and Theorems 2.6.6 and 1.1.9, but we omit the proofs. As with Lemma 2.2.3 and Theorem 2.2.4, the following variations are noteworthy in the sense that they do not require M to be a direct summand of a direct sum of finitely presented right S-modules.

Lemma 2.6.9. *Let F be an R-submodule of $\mathrm{Hom}_S(M, N)$. Let X be a Noetherian subspace of $\mathrm{Supp}_R(N)$, and suppose that $\dim(X) < \infty$. Let t be a positive integer, and suppose that $\delta_{\mathfrak{p}}(F) \geqslant t + \dim(X)$ for every $\mathfrak{p} \in X$. Then there exists a finitely generated R-submodule F' of F such that $\delta_{\mathfrak{p}}(F') \geqslant t + \dim(X)$ for every $\mathfrak{p} \in X$.*

Theorem 2.6.10. *Let L be an S-submodule of M; let F be an R-submodule of $\mathrm{Hom}_S(L, N)$; and let G be an R-submodule of $\mathrm{Hom}_S(M, N)$. Suppose that every member of F can be extended to a member of G. Then the following statements hold:*

(1) Let X be a subspace of $\mathrm{Supp}_R(N)$ that is a basic set for R with $\dim(X) < \infty$. Let t be a positive integer, and suppose that $\delta_{\mathfrak{p}}(F) \geqslant t + \dim(X)$ for every $\mathfrak{p} \in X$. Then there exists $g \in G^{\oplus t}$ such that $g_{\mathfrak{p}}$ is split surjective over $S_{\mathfrak{p}}$ for every $\mathfrak{p} \in X$.

(2) *Suppose that* $Y := \mathrm{Max}(R) \cap \mathrm{Supp}_R(N)$ *is Noetherian with* $\dim(Y) < \infty$. *Then the following statements hold:*

   (a) *Let* t *be a positive integer, and suppose that* $\delta_{\mathfrak{m}}(F) \geqslant t + \dim(Y)$ *for every* $\mathfrak{m} \in Y$. *Then* $\delta(G) \geqslant t$.

   (b) *If* $\delta_{\mathfrak{m}}(F) = \infty$ *for every* $\mathfrak{m} \in Y$, *then* $\delta(G) = \infty$. *Hence* $\delta(F) = \infty$ *if and only if* $\delta_{\mathfrak{m}}(F) = \infty$ *for every* $\mathfrak{m} \in Y$.

   (c) *Suppose that* $\delta_{\mathfrak{n}}(F) < \infty$ *for some* $\mathfrak{n} \in Y$. *Then*

$$\delta(G) \geqslant \min\{\delta_{\mathfrak{m}}(F) : \mathfrak{m} \in Y\} - \dim(Y).$$

Let F be a finitely generated R-submodule of $\mathrm{Hom}_S(M, N)$, and let X be a subspace of $\mathrm{Supp}_R(N)$ that is a basic set for R. The next lemma shows that there is a finite subset Λ of X that completely determines the function on X taking $\mathfrak{p}$ to $\delta_{\mathfrak{p}}(F)$. We omit the proof since it is basically the same as the one for Lemma 2.3.2.

Lemma 2.6.11. *Let* F *be a finitely generated* R*-submodule of* $\mathrm{Hom}_S(M, N)$, *and let* X *be a subspace of* $\mathrm{Supp}_R(N)$ *that is a basic set for* R. *Then there exists a finite subset* Λ *of* X *such that, for every* $\mathfrak{p} \in X - \Lambda$, *there exists* $\mathfrak{q} \in \Lambda$ *with the properties that* $\mathfrak{q} \subsetneq \mathfrak{p}$ *and* $\delta_{\mathfrak{q}}(F) = \delta_{\mathfrak{p}}(F)$.

This lemma yields the following analogue of Corollary 2.3.3. We omit the proof.

Corollary 2.6.12. *We make the following improvements to Theorems 2.6.6 and 1.1.8:*

(1) *Assume the hypotheses of Part (2) of Theorem 2.6.6, and let* Λ *be defined as in Lemma 2.6.11 with respect to* F *and* X. *Then*

$$\delta(G) \geqslant \inf\{\delta_{\mathfrak{p}}(F) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in \Lambda\}.$$

(2) Assume the hypotheses of Theorem 1.1.8, and let Λ be defined as in Lemma 2.6.11 with respect to $F := \mathrm{Hom}_S(M, N)$ and X. Then

$$\mathrm{spl}_S(M, N) \geqslant \inf\{\mathrm{spl}_{S_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) - \dim_X(\mathfrak{p}) : \mathfrak{p} \in \Lambda\}.$$

We can also reduce the proof of the Splitting Lemma to a consideration of the finite set Λ. To this end, we present the following analogues of Lemmas 2.3.7 and 2.3.8. We omit the proofs.

Lemma 2.6.13. *Let n be an integer with $n \geqslant 2$; let $\mathfrak{p} \in \mathrm{Supp}_R(N)$; let $f := (f_1, \ldots, f_n)^\top \in \mathrm{Hom}_S(M, N^{\oplus n})$; and let $A \in \mathbf{GL}(n, R)$. Then, with respect to Definition 2.3.6, we have $\delta_\mathfrak{p}(F') \geqslant \delta_\mathfrak{p}(F) - 1$.*

Lemma 2.6.14. *Assume the hypotheses of the Splitting Lemma, and define Λ as in Lemma 2.6.11 with respect to $F := Rf_1 + \cdots + Rf_n$ and X. Let $A \in \mathbf{GL}(n, R)$, and suppose that, with respect to Definition 2.3.6, we have that f' is (t, X, Λ)-split. Then f' is (t, X, X)-split.*

For the rest of this section, we assume the hypotheses of the Splitting Lemma, and we let Λ be defined as in Lemma 2.6.11 with respect to $F := Rf_1 + \cdots + Rf_n$ and X. Since t and X are understood, we can use the terms $\mathfrak{p}$-*split* and Y-*split* for any $\mathfrak{p} \in X$ and for any $Y \subseteq X$ without the risk of confusion.

The next two lemmas will allow us to find a matrix $V \in \mathbf{GL}(n, R)$ such that the first $n-1$ components of $Vf := (g_1, \ldots, g_n)^\top$ form a map $(g_1, \ldots, g_{n-1})^\top$ that is $\mathfrak{m}$-split for every $\mathfrak{m} \in \Lambda \cap \mathrm{Max}(R)$. The proofs are similar to those of Lemmas 2.4.2 and 2.4.6.

Lemma 2.6.15. *Let $\mathfrak{m} \in \Lambda \cap \mathrm{Max}(R)$. There exists a number $\mathscr{L}_\mathfrak{m} \in \{1, \ldots, n\}$ such that, for all $s_1, \ldots, s_n \in R - \mathfrak{m}$, there exist $r_{\mathfrak{m},1}, \ldots, r_{\mathfrak{m},n-1} \in R$ such that, for every $n \times n$ matrix*

V with entries in R, if

$$V \equiv \begin{pmatrix} 1 & 0 & \cdots & 0 & r_{\mathfrak{m},1} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & r_{\mathfrak{m},n-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} P_{\mathscr{L}_\mathfrak{m}} \begin{pmatrix} s_1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & s_n \end{pmatrix} \pmod{\mathfrak{m}},$$

then the first $n-1$ components of $Vf := (g_1, \ldots, g_n)^\top$ form a map $(g_1, \ldots, g_{n-1})^\top$ that is $\mathfrak{m}$-split.

Lemma 2.6.16. *There exists a matrix $V \in \mathbf{GL}(n, R)$ such that the first $n-1$ components of $Vf := (g_1, \ldots, g_n)^\top$ form a map $(g_1, \ldots, g_{n-1})^\top$ that is $\mathfrak{m}$-split for every $\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R)$.*

To prove the Splitting Lemma, it remains to find a matrix $V \in \mathbf{GL}(n, R)$ such that the first $n-1$ components of $Vf := (g_1, \ldots, g_n)^\top$ form a map $(g_1, \ldots, g_{n-1})^\top$ that is Λ-split, for Lemma 2.6.14 will then finish the proof.

Proof of the Splitting Lemma. The beginning of the proof is basically the same as the proof of the Surjective Lemma up to, and including, the point where we reduce to the case in which a matrix B has a certain desirable form with $(Bf^*)_\mathfrak{q}$ surjective. Of course, here, we need $(Bf^*)_\mathfrak{q}$ to be not only surjective but also split surjective over $S_\mathfrak{q}$.

Let $d := \delta_\mathfrak{q}(F)$. Since $(Bf^*)_\mathfrak{q}$ is split surjective over $S_\mathfrak{q}$, there exists an S-submodule L of M such that the restriction of $(Bf^*)_\mathfrak{q}$ to $L_\mathfrak{q}$ is an isomorphism. We may assume, then, without loss of generality, that $M_\mathfrak{q} = N_\mathfrak{q}^{\oplus d}$.

It remains to find $r_1, \ldots, r_{n-1} \in J$ such that, if

$$U := \begin{pmatrix} 1 & 0 & \cdots & 0 & r_1 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & r_{n-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix}$$

and if $Uf^* := (g_1, \ldots, g_n)^\top$, then $G := Rg_1 + \cdots + Rg_{n-1}$ satisfies $\delta_{\mathfrak{q}}(G) = d$. On the other hand, since we have reduced to the case in which $M_{\mathfrak{q}} = N_{\mathfrak{q}}^{\oplus d}$, it suffices to verify that $\partial_{\mathfrak{q}}(G) = d$, for this will imply that $\delta_{\mathfrak{q}}(G) = d$. Thus, from here, we may proceed once more as in the proof of the Surjective Lemma. □

Just as with the Surjective Lemma, there is a special case that admits a stronger version of the Splitting Lemma:

Corollary 2.6.17. *Assume the hypotheses of the Splitting Lemma. Define Λ as in Lemma 2.6.11 with respect to $F := Rf_1 + \cdots + Rf_n$ and X. Suppose that, for every $\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R)$, it is the case that $|R/\mathfrak{m}| \geqslant 2 + \mu_{R_\mathfrak{m}}(N_\mathfrak{m})$ or $\mu_{R_\mathfrak{m}}(N_\mathfrak{m}) = 1$. Then there exist $r_1, \ldots, r_{n-1} \in R$ such that $(f_1 + r_1 f_n, \ldots, f_{n-1} + r_{n-1} f_n)^\top$ is (t, X, X)-split.*

This corollary yields improved versions of Theorems 1.1.8, 1.1.9, 2.6.6, and 2.6.10, but we omit the details.

In the next section, we consider another special case in which we can improve upon our previous results.

2.7 Finitely generated modules over Dedekind domains

In this section, we characterize global surjective and splitting capacities of finitely generated modules over Dedekind domains. We define a *Dedekind domain* to be an integral domain in which every ideal is projective. Since every ideal of a Dedekind domain is projective, every ideal is finitely generated [8, pages 760–762] and locally free, and so a Dedekind domain is either a field or a one-dimensional regular domain.

A *fractional ideal* of a Dedekind domain R is an R-submodule I of the fraction field of R such that there exists a nonzero $a \in R$ with aI an ideal of R. Hence every fractional ideal of R is isomorphic to an ideal of R. We define an equivalence relation $\sim$ on the set $\mathscr{F}$ of all nonzero fractional ideals of R by letting $I \sim J$ if and only if there exist nonzero $a, b \in R$ such that $aI = bJ$. The set of all equivalence classes of $\mathscr{F}$ with respect to $\sim$ forms an abelian group under multiplication with $[I][J] := [IJ]$ for all nonzero ideals I and J of R. This group, denoted $\operatorname{Pic}(R)$, is called the *Picard group of* R or the *class group of* R, and its identity is $[R]$, the class of all principal fractional ideals of R. See [1, pages 457–460]; [8, pages 760–762]; or [9, pages 253–258] for more details.

For every module M over a Dedekind domain R, we define $\operatorname{Tor}_R(M)$ to be the torsion R-submodule of M. We omit the subscript R if the underlying ring is understood. Indeed, in this section, we will often omit subscripts referring to rings in notation such as $\operatorname{Ass}_R(N)$, $\operatorname{Supp}_R(N)$, $\operatorname{sur}_R(M, N)$, and $\operatorname{spl}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p})$ since, in many cases, the underlying ring will be understood. In particular, we do not consider arbitrary module-finite algebras over Dedekind domains here.

We now recall the structure theorem for finitely generated modules over Dedekind domains. The various parts of this theorem can be found in [8, pages 771 and 774]. Alternatively, Parts (1), (3), and (4) can be found in [1, page 463] or [9, pages 484–485], and Part (2) can be deduced from results in [1, page 458] or [9, page 258].

Theorem 2.7.1 ([1, pages 458 and 463]; [8, pages 771 and 774]; [9, pages 258 and 484–485])**.** *Let M be a finitely generated module over a Dedekind domain R. Then the following hold:*

(1) $M \cong \operatorname{Tor}(M) \oplus M/\operatorname{Tor}(M)$.

(2) $\operatorname{Tor}(M)$ *is a direct sum of R-modules, each of which has the form $R/\mathfrak{m}^i$ for some $\mathfrak{m} \in \operatorname{Spec}(R) - \{0\}$ and some positive integer i. This decomposition is unique up to a permutation of factors.*

(3) There is an alternative decomposition of $\operatorname{Tor}(M)$ *as* $(R/I_1) \oplus \cdots \oplus (R/I_u)$ *for some nonzero proper ideals* $I_1, \ldots, I_u$ *of* R *such that* $I_1 \subseteq \cdots \subseteq I_u$*. This decomposition is unique.*

(4) Suppose that $M \neq \operatorname{Tor}(M)$*. Then there is a nonzero ideal* I *of* R *such that* $M/\operatorname{Tor}(M) \cong R^{\oplus(r-1)} \oplus I$*, where* $r := \operatorname{rank}(M)$*. The ideal* I *is unique up to isomorphism.*

Part (2) of the preceding theorem gives the *elementary divisor decomposition* of $\operatorname{Tor}(M)$, and Part (3) gives the *invariant factor decomposition* of $\operatorname{Tor}(M)$. If $M \neq \operatorname{Tor}(M)$, then the class $[I]$ of the ideal I from Part (4) is called the *Steinitz class of* M and is denoted by $[M]$. See [8, page 773].

We collect a few more properties of Dedekind domains in the following lemma.

Lemma 2.7.2. *Let* R *be a Dedekind domain. Then* R *satisfies the following properties:*

(1) Let I *and* J *be nonzero ideals of* R*. Then* $I \oplus J \cong R \oplus IJ$*.*

(2) Let I *and* J *be nonzero ideals of* R*. Then there is a nonzero ideal* K *of* R *such that* $I \cong KJ$*.*

(3) Let I *be a nonzero ideal of* R*, and let* M *be a cyclic torsion* R*-module. Then there is a surjective* R*-linear map from* I *to* M*.*

(4) Let M *be a finitely generated torsion* R*-module, and let* u *be a positive integer. Suppose that* $\mu_{R_\mathfrak{m}}(M_\mathfrak{m}) \leqslant u$ *for every* $\mathfrak{m} \in \operatorname{Ass}(M)$*. Then* $\mu_R(M) \leqslant u$*.*

Proof. (1) See [1, pages 461–462]; [8, page 769]; or [9, page 484].

(2) Since $\operatorname{Pic}(R)$ is a group, there is an ideal K of R such that $[I] = [K][J] = [KJ]$. Hence $I \cong KJ$.

(3) The result is obvious if $M = 0$, so suppose that $M \neq 0$. Then there is a nonzero proper ideal J of R such that $M \cong R/J$. From [8, page 765], we learn that I and J can be written as $I = \mathfrak{m}_1^{v_1} \cdots \mathfrak{m}_u^{v_u}$ and $J = \mathfrak{m}_1^{w_1} \cdots \mathfrak{m}_u^{w_u}$, where $\mathfrak{m}_1, \ldots, \mathfrak{m}_u$ are distinct nonzero prime ideals of R and where $v_1, \ldots, v_u, w_1, \ldots, w_u$ are nonnegative integers. Now let

$K := \mathfrak{m}_1^{v_1+w_1} \cdots \mathfrak{m}_u^{v_u+w_u}$. By results from [8, page 768], we see that $\mathfrak{m}_i^{v_i}/\mathfrak{m}_i^{v_i+w_i} \cong R/\mathfrak{m}_i^{w_i}$ for every $i \in \{1, \ldots, u\}$ and, hence, that $I/K \cong R/J$.

(4) This follows from the invariant factor decomposition of $\operatorname{Tor}(M)$ in Theorem 2.7.1. $\square$

Here is our first main result on Dedekind domains:

Proposition 2.7.3. *Let M and N be finitely generated modules over a Dedekind domain R; let $X := \operatorname{Ass}(N) - \{0\}$; and let t be a positive integer. Then $\operatorname{sur}(M, N) \geqslant t$ if and only if $\operatorname{sur}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) \geqslant t$ for every $\mathfrak{m} \in X$ and one of the following conditions holds:*

(1) $\operatorname{rank}(N) = 0$.

(2) $\operatorname{rank}(M) \geqslant 1 + t \cdot \operatorname{rank}(N)$.

(3) $\operatorname{rank}(M) = t \cdot \operatorname{rank}(N) \geqslant t$, *and* $[M] = [N]^t$.

Moreover, if $\operatorname{sur}(M, N) \geqslant t$ and (3) holds, then we have that $\operatorname{sur}(\operatorname{Tor}(M), \operatorname{Tor}(N)) \geqslant t$ and $\operatorname{sur}(M, N) = t$.

Proof. Let $r := \operatorname{rank}(M)$, and let $s := \operatorname{rank}(N)$.

Suppose first that $\operatorname{sur}(M, N) \geqslant t$. Certainly $\operatorname{sur}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) \geqslant t$ for every $\mathfrak{m} \in X$. Suppose that neither (1) nor (2) holds. Then $r = st \geqslant t$, and so $\operatorname{sur}(M, N) = t$. Let I, J be ideals of R that represent $[M], [N]$, respectively. Then $N^{\oplus t}/\operatorname{Tor}(N^{\oplus t}) \cong R^{\oplus(st-1)} \oplus J^t$ by Part (1) of Lemma 2.7.2. Let e be a surjective R-linear map from M to $N^{\oplus t}$. Note that $\operatorname{Hom}_R(\operatorname{Tor}(M), R^{\oplus(st-1)} \oplus J^t) = 0$. Hence there exist R-linear maps

$$\begin{array}{rclcl} f & : & \operatorname{Tor}(M) & \to & \operatorname{Tor}(N)^{\oplus t} \quad , \\ g & : & R^{\oplus(st-1)} \oplus I & \to & \operatorname{Tor}(N)^{\oplus t} \quad , \\ h & : & R^{\oplus(st-1)} \oplus I & \to & R^{\oplus(st-1)} \oplus J^t \end{array}$$

such that the following matrix represents e:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{Tor}(M) & R^{\oplus(st-1)} \oplus I \end{array} \\ \begin{array}{c} \mathrm{Tor}(N)^{\oplus t} \\ R^{\oplus(st-1)} \oplus J^t \end{array} & \begin{pmatrix} f & g \\ 0 & h \end{pmatrix} \end{array}.$$

Clearly, h is surjective. Since $R^{\oplus(st-1)} \oplus I$ is torsion-free, $\ker h$ is torsion-free. Since $\operatorname{rank}(\ker h) = \operatorname{rank}(M) - \operatorname{rank}(N^{\oplus t}) = 0$, we see that $\ker h = 0$. Hence h is an isomorphism, and so Theorem 2.7.1 tells us that $I \cong J^t$. Thus $[M] = [N]^t$, proving (3). As a result, the Snake Lemma tells us that f is surjective. Hence $\operatorname{sur}(\operatorname{Tor}(M), \operatorname{Tor}(N)) \geqslant t$.

Next suppose that $\operatorname{sur}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) \geqslant t$ for every $\mathfrak{m} \in X$. If (1) holds, then clearly it is the case that $\operatorname{sur}(M, N) \geqslant t$.

Suppose then that (1) does not hold but that (2) does hold. Let I, J be ideals of R that represent $[M], [N]$, respectively. By Part (2) of Lemma 2.7.2, there exists a nonzero ideal K of R such that $I \cong KJ^t$. By Part (1) of Lemma 2.7.2, we may then write $M/\operatorname{Tor}(M)$ as

$$R^{\oplus(r-1)} \oplus I \cong R^{\oplus(r-st-1)} \oplus K \oplus R^{(st-1)} \oplus J^t.$$

If $X = \varnothing$, then immediately we see that $\operatorname{sur}(M, N) \geqslant t$ since $N^{\oplus t}/\operatorname{Tor}(N^{\oplus t}) \cong R^{\oplus(st-1)} \oplus J^t$ by Part (1) of Lemma 2.7.2. Suppose then that $X \neq \varnothing$. Let $\mathfrak{m} \in X$, and let $e(\mathfrak{m})$ be a surjective R-linear map from $M_{\mathfrak{m}}$ to $N_{\mathfrak{m}}^{\oplus t}$. Note that $\operatorname{Hom}_R(\operatorname{Tor}(M_{\mathfrak{m}}), R_{\mathfrak{m}}^{\oplus st}) = 0$. Hence there exist R-linear maps

$$\begin{array}{rclcl} f(\mathfrak{m}) & : & \mathrm{Tor}(M_{\mathfrak{m}}) & \to & \mathrm{Tor}(N_{\mathfrak{m}})^{\oplus t} \quad , \\ g(\mathfrak{m}) & : & R_{\mathfrak{m}}^{\oplus r} & \to & \mathrm{Tor}(N_{\mathfrak{m}})^{\oplus t} \quad , \\ h(\mathfrak{m}) & : & R_{\mathfrak{m}}^{\oplus r} & \to & R_{\mathfrak{m}}^{\oplus st} \end{array}$$

such that the following matrix represents $e(\mathfrak{m})$:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{Tor}(M_{\mathfrak{m}}) & R_{\mathfrak{m}}^{\oplus r} \end{array} \\ \begin{array}{l} \mathrm{Tor}(N_{\mathfrak{m}})^{\oplus t} \\ R_{\mathfrak{m}}^{\oplus st} \end{array} & \begin{pmatrix} f(\mathfrak{m}) & g(\mathfrak{m}) \\ 0 & h(\mathfrak{m}) \end{pmatrix} \end{array}.$$

Clearly, $h(\mathfrak{m})$ is surjective. Since $R_{\mathfrak{m}}^{\oplus st}$ is free over $R_{\mathfrak{m}}$, we see that $\ker h(\mathfrak{m}) \cong R_{\mathfrak{m}}^{\oplus(r-st)}$. As a result, the Snake Lemma tells us that $\mu_{R_{\mathfrak{m}}}(\operatorname{coker} f(\mathfrak{m})) \leqslant r - st$. Lift a generating set of $\operatorname{coker} f(\mathfrak{m})$ to a subset $\mathscr{C}(\mathfrak{m})$ of $\mathrm{Tor}(N_{\mathfrak{m}})^{\oplus t} \subseteq \mathrm{Tor}(N)^{\oplus t}$. Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_u$ be the distinct members of X, and let C be the R-module generated by $\mathscr{C}(\mathfrak{m}_1) \cup \cdots \cup \mathscr{C}(\mathfrak{m}_u)$. Then, by Part (4) of Lemma 2.7.2, we see that $\mu_R(C) \leqslant r - st$. Hence, by Part (3) of Lemma 2.7.2, there is a surjective R-linear map from $R^{\oplus(r-st-1)} \oplus K$ to C. As a result, there is a surjective R-linear map from $\mathrm{Tor}(M) \oplus R^{\oplus(r-st-1)} \oplus K$ to $\operatorname{im} f(\mathfrak{m}_1) + \cdots + \operatorname{im} f(\mathfrak{m}_u) + C = \mathrm{Tor}(N)^{\oplus t}$. Altogether, then, we find that $\mathrm{sur}(M, N) \geqslant t$.

Finally, suppose that (3) holds. If $X = \varnothing$, then immediately we see that $\mathrm{sur}(M, N) = t$. Suppose then that $X \neq \varnothing$. Let $\mathfrak{m} \in X$, and let $e(\mathfrak{m})$ be a surjective R-linear map from $M_{\mathfrak{m}}$ to $N_{\mathfrak{m}}^{\oplus t}$. Note that $\mathrm{Hom}_R(\mathrm{Tor}(M_{\mathfrak{m}}), R_{\mathfrak{m}}^{\oplus st}) = 0$. Hence there exist R-linear maps

$$\begin{array}{lclcl} f(\mathfrak{m}) & : & \mathrm{Tor}(M_{\mathfrak{m}}) & \to & \mathrm{Tor}(N_{\mathfrak{m}})^{\oplus t} \quad , \\ g(\mathfrak{m}) & : & R_{\mathfrak{m}}^{\oplus st} & \to & \mathrm{Tor}(N_{\mathfrak{m}})^{\oplus t} \quad , \\ h(\mathfrak{m}) & : & R_{\mathfrak{m}}^{\oplus st} & \to & R_{\mathfrak{m}}^{\oplus st} \end{array}$$

such that the following matrix represents $e(\mathfrak{m})$:

$$\begin{array}{cc} & \begin{array}{cc} \mathrm{Tor}(M_{\mathfrak{m}}) & R_{\mathfrak{m}}^{\oplus st} \end{array} \\ \begin{array}{l} \mathrm{Tor}(N_{\mathfrak{m}})^{\oplus t} \\ R_{\mathfrak{m}}^{\oplus st} \end{array} & \begin{pmatrix} f(\mathfrak{m}) & g(\mathfrak{m}) \\ 0 & h(\mathfrak{m}) \end{pmatrix} \end{array}.$$

Clearly, $h(\mathfrak{m})$ is surjective. Since $R_{\mathfrak{m}}^{\oplus st}$ is finitely generated over $R_{\mathfrak{m}}$, we see that $h(\mathfrak{m})$ is an isomorphism. As a result, the Snake Lemma tells us that $f(\mathfrak{m})$ is surjective. Hence

$\operatorname{sur}(\operatorname{Tor}(M_\mathfrak{m}), \operatorname{Tor}(N_\mathfrak{m})) \geqslant t$, and so $\operatorname{sur}(\operatorname{Tor}(M), \operatorname{Tor}(N)) \geqslant t$. Moreover, since $\operatorname{rank}(M) = \operatorname{rank}(N^{\oplus t})$ and since $[M] = [N]^t$, Theorem 2.7.1 and Part (1) of Lemma 2.7.2 tell us that $M/\operatorname{Tor}(M) \cong N^{\oplus t}/\operatorname{Tor}(N^{\oplus t})$. Altogether then, $\operatorname{sur}(M, N) = t$. □

Using the previous proposition, we can characterize the global surjective capacity of a finitely generated module M with a respect to a finitely generated module N over a Dedekind domain R: To see this, let u be a positive integer. Then $\operatorname{sur}(M, N) = u$ if and only if $\operatorname{sur}(M, N) \geqslant u$ and $\operatorname{sur}(M, N) < u + 1$. We can apply the previous proposition with $t = u$ to characterize the statement that $\operatorname{sur}(M, N) \geqslant u$, and we can apply the previous proposition with $t = u + 1$ to characterize the statement that $\operatorname{sur}(M, N) < u + 1$. We also have the following corollary:

Corollary 2.7.4. *Let M and N be finitely generated modules over a Dedekind domain R, and let $X := \operatorname{Ass}(N) - \{0\}$. Then $\operatorname{sur}(M, N) = 0$ if and only if one of the following conditions holds:*

(1) $\operatorname{sur}(M_\mathfrak{m}, N_\mathfrak{m}) = 0$ *for some* $\mathfrak{m} \in X$.

(2) $\operatorname{rank}(N) \geqslant 1 + \operatorname{rank}(M)$.

(3) $\operatorname{rank}(M) = \operatorname{rank}(N) \geqslant 1$, *and* $[M] \neq [N]$.

We can give a result analogous to Proposition 2.7.3 for splitting capacities:

Proposition 2.7.5. *Let M and N be finitely generated modules over a Dedekind domain R; let $X := \operatorname{Ass}(N) - \{0\}$; and let t be a positive integer. Then $\operatorname{spl}(M, N) \geqslant t$ if and only if $\operatorname{spl}(M_\mathfrak{m}, N_\mathfrak{m}) \geqslant t$ for every $\mathfrak{m} \in X$ and one of the following conditions holds:*

(1) $\operatorname{rank}(N) = 0$.

(2) $\operatorname{rank}(M) \geqslant 1 + t \cdot \operatorname{rank}(N)$.

(3) $\operatorname{rank}(M) = t \cdot \operatorname{rank}(N) \geqslant t$, *and* $[M] = [N]^t$.

Moreover, if $\operatorname{spl}(M, N) \geqslant t$ and (3) holds, then $\operatorname{spl}(\operatorname{Tor}(M), \operatorname{Tor}(N)) \geqslant t$, and $\operatorname{spl}(M, N) = t$.

Proof. Let $r := \operatorname{rank}(M)$, and let $s := \operatorname{rank}(N)$.

Suppose first that $\operatorname{spl}(M, N) \geqslant t$. Certainly $\operatorname{spl}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) \geqslant t$ for every $\mathfrak{m} \in X$. Suppose that neither (1) nor (2) holds. Then $r = st \geqslant t$, and so $\operatorname{spl}(M, N) = t$. Let I, J be ideals of R that represent $[M], [N]$, respectively. Then, by Theorem 2.7.1 and Part (1) of Lemma 2.7.2, we see that $I \cong J^t$, and so $[M] = [N]^t$, proving (3). Theorem 2.7.1 also implies that $\operatorname{spl}(\operatorname{Tor}(M), \operatorname{Tor}(N)) \geqslant t$.

Next suppose that $\operatorname{spl}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) \geqslant t$ for every $\mathfrak{m} \in X$. If (1) holds, then it must be the case that $\operatorname{spl}(M, N) \geqslant t$.

Suppose then that (1) does not hold but that (2) does hold. Let I, J be ideals of R that represent $[M], [N]$, respectively. By Part (2) of Lemma 2.7.2, there exists a nonzero ideal K of R such that $I \cong KJ^t$. By Part (1) of Lemma 2.7.2, we may write $M/\operatorname{Tor}(M)$ as

$$R^{\oplus(r-1)} \oplus I \cong R^{\oplus(r-st-1)} \oplus K \oplus R^{(st-1)} \oplus J^t.$$

Now Theorem 2.7.1 implies that $\operatorname{spl}(M, N) \geqslant t$ since $N^{\oplus t}/\operatorname{Tor}(N^{\oplus t}) \cong R^{\oplus(st-1)} \oplus J^t$ by Part (1) of Lemma 2.7.2.

Finally, suppose that (3) holds. Then Theorem 2.7.1 and Part (1) of Lemma 2.7.2 immediately imply that $\operatorname{spl}(\operatorname{Tor}(M), \operatorname{Tor}(N)) \geqslant t$ and that $\operatorname{spl}(M, N) = t$. $\square$

In light of the previous proposition, we could now characterize global splitting capacities of finitely generated modules over a Dedekind domain in a manner similar to the case for global surjective capacities. We omit the details.

To close this section, we mention a few more cases in which we can characterize global surjective and splitting capacities. We have already covered some of these cases. For example, a characterization of an infinite global surjective capacity can be found in Part (2) of Theorem 1.1.6, and Part (2) of Theorem 1.1.9 offers a characterization of an infinite global splitting capacity. Under the hypotheses of Part (3) of Theorem 1.1.6, if $\dim(Y) = 0$, then

$$\operatorname{sur}_S(M, N) = \min\{\operatorname{sur}_{S_{\mathfrak{m}}}(M_{\mathfrak{m}}, N_{\mathfrak{m}}) : \mathfrak{m} \in Y\}$$

since we always have

$$\mathrm{sur}_S(M,N) \leqslant \inf\{\mathrm{sur}_{S_\mathfrak{m}}(M_\mathfrak{m},N_\mathfrak{m}) : \mathfrak{m} \in Y\}.$$

We can draw an analogous conclusion from Part (3) of Theorem 1.1.9 when $\dim(Y) = 0$: In this case,

$$\mathrm{spl}_S(M,N) = \min\{\mathrm{spl}_{S_\mathfrak{m}}(M_\mathfrak{m},N_\mathfrak{m}) : \mathfrak{m} \in Y\}.$$

In particular, if our underlying commutative ring is *quasisemilocal* (the ring has only finitely many maximal ideals), then global surjective capacities are completely determined by local surjective capacities, and we can say the same for splitting capacities.

We now observe that, whenever we have a result concerning surjective or splitting capacities over finitely many commutative rings $R_1, \ldots, R_u$, we have an analogous result for the direct product $R_1 \times \cdots \times R_u$ of these rings. For example, since a *commutative Noetherian hereditary ring* is a direct product of finitely many Dedekind domains, we can characterize global surjective and splitting capacities of finitely generated modules over any commutative Noetherian hereditary ring, given Propositions 2.7.3 and 2.7.5. In light of our additional results on global surjective and splitting capacities over commutative quasisemilocal rings, we can characterize global surjective and splitting capacities over any direct product of finitely many commutative quasisemilocal rings and Dedekind domains. One historically significant example of such a direct product is given by Hungerford's Theorem [19, Theorem 1]: A *commutative principal ideal ring* (that is, a commutative ring in which every ideal is principal) is a direct product of finitely many quotients of principal ideal domains. Direct products thus provide a way to extend some of our results to larger classes of rings.

CHAPTER 3

CANCELLATION OF HOMOTHETIC MODULES

Our goals in this chapter are to prove our main cancellation theorem (Theorem 1.2.9) and to illustrate the new information that it provides. We accomplish the first goal in Section 3.1 and the second goal in Section 3.2.

3.1 Proof of Theorem 1.2.9

Throughout this section, let R be a commutative ring, M an R-module, and N a finitely presented R-module. We begin with the statement of the following lemma. We defer the proof of the lemma to the end of this section for the purpose of advancing more quickly to the proof of the main cancellation theorem. The following result extends the Splitting Lemma (Lemma 2.6.5).

Lemma 3.1.1 (Cancellation Lemma)**.** *Let n be an integer with $n \geqslant 2$, and let X be a subspace of $\operatorname{Supp}_R(N)$ that is a basic set for R. Suppose that $N_{\mathfrak{p}}$ is homothetic over $R_{\mathfrak{p}}$ for every $\mathfrak{p} \in X$ such that $\dim_X(\mathfrak{p}) \geqslant 1$. Let $f := (f_1, \ldots, f_n)^\top \in \operatorname{Hom}_R(M, N^{\oplus n})$, and suppose that f is $(1, X, X)$-split. Then there exist $f'_1, \ldots, f'_{n-1} \in Rf_1 + \cdots + Rf_n$ such that $f' := (f'_1, \ldots, f'_{n-1})^\top$ is $(1, X, X)$-split. Moreover, for every $a \in R$ such that $(a, f_1) \in \operatorname{Hom}_R(N \oplus M, N)$ is $(1, X, X)$-split, we can arrange for f'_1 to be in $f_1 + a(Rf_1 + \cdots + Rf_n)$ so that (a, f'_1) is also $(1, X, X)$-split.*

This lemma immediately implies the following theorem, which complements [7, Theorems 3.9 and 4.5] and establishes a criterion for determining when a given coset of $\operatorname{Hom}_R(M, N)$ contains a map that is split surjective over R.

Theorem 3.1.2. *Let L be an R-submodule of M; let F be a finitely generated R-submodule of $\operatorname{Hom}_R(L, N)$; and let G be an R-submodule of $\operatorname{Hom}_R(M, N)$. Suppose that every member* *of F can be extended to a member of G. Let X be a subspace of $\operatorname{Supp}_R(N)$ that is a basic set for R, and suppose that $\dim(X) < \infty$. Suppose that $N_{\mathfrak{p}}$ is homothetic over $R_{\mathfrak{p}}$ for every $\mathfrak{p} \in X$ such that $\dim_X(\mathfrak{p}) \geqslant 1$. Suppose also that $\delta_{\mathfrak{p}}(F) \geqslant 1 + \dim_X(\mathfrak{p})$ for every $\mathfrak{p} \in X$. Let $a \in R$ and $f_1 \in F$ such that $(a, f_1) \in \operatorname{Hom}_R(N \oplus L, N)$ is $(1, X, X)$-split. Then there exists $g \in G$ such that $g_{\mathfrak{p}}$ is split surjective over $R_{\mathfrak{p}}$ for every $\mathfrak{p} \in X$ and such that $g|_L \in f_1 + aF$. If $\operatorname{Max}(R) \cap \operatorname{Supp}_R(N) \subseteq X$, then g is split surjective over R.*

Proof. Lemma 3.1.1 can be used to prove the first claim; the proof is similar to the proof of Theorem 2.1.15. The second claim follows from the fact that, since N is finitely presented over R, the map g is split surjective if and only if $g_{\mathfrak{m}}$ is split surjective for every $\mathfrak{m} \in \operatorname{Max}(R) \cap \operatorname{Supp}_R(N)$. $\square$

Theorem 3.1.2, in turn, immediately yields the following cancellation result.

Theorem 3.1.3. *Suppose that $X := j\text{-}\operatorname{Spec}(R) \cap \operatorname{Supp}_R(N)$ is Noetherian with $\dim(X) < \infty$. Suppose also that $N_{\mathfrak{p}}$ is homothetic over $R_{\mathfrak{p}}$ for every $\mathfrak{p} \in X$ such that $\dim_X(\mathfrak{p}) \geqslant 1$. Let F be a finitely generated R-submodule of $\operatorname{Hom}_R(M, N)$, and suppose that $\delta_{\mathfrak{p}}(F) \geqslant 1 + \dim_X(\mathfrak{p})$ for every $\mathfrak{p} \in X$. Let L be an R-module, and suppose that there exist $a \in R$ and $f_1 \in F$ for which*

$$\theta_0 := \begin{pmatrix} a & f_1 \\ * & * \end{pmatrix} \in \operatorname{Hom}_R(N \oplus M, N \oplus L)$$

is an isomorphism. Then $L \cong M$.

Proof. Note that $(a, f_1) \in \operatorname{Hom}_R(N \oplus M, N)$ is $(1, X, X)$-split. Let $\{f_1, \ldots, f_n\}$ be a generating set for F over R. By Theorem 3.1.2, there exists $f_0 \in F$ for which $f := f_1 + af_0$ is split surjective. Accordingly, let

$$\theta_1 := \begin{pmatrix} 1_N & f_0 \\ 0 & 1_M \end{pmatrix} \in \operatorname{Aut}_R(N \oplus M),$$

and note that $(a, f_1) \circ \theta_1 = (a, f)$. Since f is split surjective, there exists $g \in \operatorname{Hom}_R(N, M)$

for which $f \circ g = 1_N$. Accordingly, let

$$\theta_2 := \begin{pmatrix} 1_N & 0 \\ g - ag & 1_M \end{pmatrix} \in \mathrm{Aut}_R(N \oplus M),$$

and note that $(a, f_1) \circ \theta_1 \circ \theta_2 = (1_N, f)$. Next, let

$$\theta_3 := \begin{pmatrix} 1_N & -f \\ 0 & 1_M \end{pmatrix} \in \mathrm{Aut}_R(N \oplus M),$$

and note that $(a, f_1) \circ \theta_1 \circ \theta_2 \circ \theta_3 = (1_N, 0)$. Now let

$$\theta := \theta_0 \circ \theta_1 \circ \theta_2 \circ \theta_3 = \begin{pmatrix} 1_N & 0 \\ * & * \end{pmatrix} \in \mathrm{Hom}_R(N \oplus M, N \oplus L),$$

and note that θ is an isomorphism. By the Five Lemma, $L \cong M$. $\square$

With the help of the previous theorem and Lemmas 2.6.8 and 2.6.9, we can prove the following generalization of our main cancellation theorem (Theorem 1.2.9):

Theorem 3.1.4. *Suppose that $Y := \mathrm{Max}(R) \cap \mathrm{Supp}_R(N)$ is Noetherian with $\dim(Y) < \infty$, and suppose that N is homothetic over R. Suppose also that one of the following conditions holds:*

(1) M is a direct summand of a direct sum of finitely presented R-modules such that $\mathrm{spl}_{R_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m}) \geqslant 1 + \dim(Y)$ for every $\mathfrak{m} \in Y$.

(2) R is Noetherian; M is finitely generated over R; and $\mathrm{spl}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \geqslant 1 + \dim_X(\mathfrak{p})$ for every $\mathfrak{p} \in X := j\text{-}\mathrm{Spec}(R) \cap \mathrm{Supp}_R(N)$.

(3) There exists an R-submodule F of $\mathrm{Hom}_R(M, N)$ such that $\delta_\mathfrak{m}(F) \geqslant 1 + \dim(Y)$ for every $\mathfrak{m} \in Y$.

Let K be a direct summand of a direct sum of finitely many copies of N, and let L be an R-module for which $K \oplus L \cong K \oplus M$. Then $L \cong M$.

Proof. There exist an R-module K' and a positive integer i for which $K' \oplus K \cong N^{\oplus i}$. Hence $N^{\oplus i} \oplus L \cong N^{\oplus i} \oplus M$, and so, by induction on i, we may assume that $N \oplus L \cong N \oplus M$. Now, using the fact that N is homothetic, we see that there exist $a \in R$ and $f_1 \in \operatorname{Hom}_R(M, N)$ for which

$$\theta_0 := \begin{pmatrix} a & f_1 \\ * & * \end{pmatrix} \in \operatorname{Hom}_R(N \oplus M, N \oplus L)$$

is an isomorphism. Since N is homothetic, N is locally homothetic. Since Y is Noetherian with $\dim(Y) < \infty$, we see that X is Noetherian and that $\dim(X) = \dim(Y) < \infty$ by [32, Proposition 1]. Hence, in light of Theorem 3.1.3, it suffices to prove that Conditions (1), (2), and (3) all imply the following condition:

(4) There exists a finitely generated R-submodule F' of $\operatorname{Hom}_R(M, N)$ such that $f_1 \in F'$ and such that $\delta_{\mathfrak{p}}(F') \geqslant 1 + \dim_X(\mathfrak{p})$ for every $\mathfrak{p} \in X$.

Indeed, if Condition (1) holds, then Lemma 2.6.8 implies Condition (4); if Condition (2) holds, then we may take $F' = \operatorname{Hom}_R(M, N)$ to satisfy Condition (4); and, if Condition (3) holds, then Lemma 2.6.9 implies Condition (4). $\square$

We now begin the task of proving the Cancellation Lemma (Lemma 3.1.1) stated at the beginning of this section. To prove the Cancellation Lemma, we must first extend Lemmas 2.6.15 and 2.6.16.

Lemma 3.1.5. *Assume the hypotheses of the Cancellation Lemma, and choose $a \in R$ such that the map (a, f_1) is $(1, X, X)$-split. Define Λ as in Lemma 2.6.11 with respect to $F := Rf_1 + \cdots + Rf_n$ and X so that, for every $\mathfrak{p} \in X - \Lambda$, there exists $\mathfrak{q} \in \Lambda$ for which $\mathfrak{q} \subsetneq \mathfrak{p}$ and $\delta_{\mathfrak{q}}(F) = \delta_{\mathfrak{p}}(F)$. Let $\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R)$. Then there exists a number $\mathscr{L}_{\mathfrak{m}} \in \{1, \ldots, n\}$ such that, for all $s_1, \ldots, s_n \in R - \mathfrak{m}$, there exist $r_{\mathfrak{m},1}, \ldots, r_{\mathfrak{m},n-1} \in R$ such that, for every $n \times n$*

matrix V with entries in R, if

$$V \equiv \begin{pmatrix} 1 & 0 & \cdots & 0 & r_{\mathfrak{m},1} \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & 1 & r_{\mathfrak{m},n-1} \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} P_{\mathscr{L}_{\mathfrak{m}}} \begin{pmatrix} s_1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & s_n \end{pmatrix} \pmod{\mathfrak{m}},$$

then the first $n-1$ components of $Vf := (g_1, \ldots, g_n)^\top$ form a map $(g_1, \ldots, g_{n-1})^\top$ that is $(1, X, \mathfrak{m})$-split. Moreover, we can arrange for $r_{\mathfrak{m},1}$ to be in aR.

Proof. Since X is understood and since it is understood that we are taking $t = 1$, we can use the term $\mathfrak{m}$-*split* without the risk of confusion.

The first claim of the lemma is a special case of Lemma 2.6.15. For the second claim, we may begin as in the proof of Lemma 2.4.2 and proceed up to the point where we assume that f' is not $\mathfrak{m}$-surjective. Of course, here we will assume that f' is simply not $\mathfrak{m}$-split. We will show that $a \notin \mathfrak{m}$. Suppose the contrary. Since (a, f_1) is $\mathfrak{m}$-split, we see that $(f_1)_{\mathfrak{m}}$ is split surjective over $R_{\mathfrak{m}}$. Hence f' is $\mathfrak{m}$-split, a contradiction. Thus $a \notin \mathfrak{m}$. Now, mimicking the proof of Lemma 2.4.2 once more, we may arrange for $r_{\mathfrak{m},1}$ to be in aR, and we may choose $r_{\mathfrak{m},2}, \ldots, r_{\mathfrak{m},n-1}$ just as before. ☐

Lemma 3.1.6. *Assume the hypotheses of the Cancellation Lemma, and choose $a \in R$ such that the map (a, f_1) is $(1, X, X)$-split. Define Λ as in Lemma 2.6.11 with respect to $F := Rf_1 + \cdots + Rf_n$ and X so that, for every $\mathfrak{p} \in X - \Lambda$, there exists $\mathfrak{q} \in \Lambda$ for which $\mathfrak{q} \subsetneq \mathfrak{p}$ and $\delta_{\mathfrak{q}}(F) = \delta_{\mathfrak{p}}(F)$. There exists a matrix $V \in \mathbf{GL}(n, R)$ such that the first $n-1$ components of $Vf := (g_1, \ldots, g_n)^\top$ form a map $(g_1, \ldots, g_{n-1})^\top$ that is $(1, X, \mathfrak{m})$-split for every $\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R)$. Moreover, we can arrange for g_1 to be in $f_1 + aF$.*

Proof. As in the proof of Lemma 3.1.5, since $t = 1$ and since X is understood, the term $\mathfrak{m}$-*split* is unambiguous for a fixed choice of $\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R)$.

The first claim of the present lemma is but a particular instance of Lemma 2.6.16, so it remains only to prove the second claim. For every $\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R)$, choose $\mathscr{L}_{\mathfrak{m}} \in \{1, \ldots, n\}$ so that it satisfies the conclusion of Lemma 3.1.5.

For every $i \in \{1, \ldots, n\}$, let

$$\Lambda_i = \{\mathfrak{m} \in \Lambda \cap \operatorname{Max}(R) : \mathscr{L}_{\mathfrak{m}} = i\}.$$

Then $\Lambda_1, \ldots, \Lambda_n$ are finite, pairwise disjoint subsets of $\operatorname{Max}(R)$. We would like to show that

$$a \in R - \bigcup_{\mathfrak{m} \in \Lambda_1} \mathfrak{m}.$$

Suppose not. Let $\mathfrak{n} \in \Lambda_1$ such that $a \in \mathfrak{n}$. Working under the hypotheses given, define all objects in Definition 2.3.6 with $\mathfrak{p} = \mathfrak{n}$ and with A as the $n \times n$ identity matrix. Since $a \in \mathfrak{n}$ and since (a, f_1) is $\mathfrak{n}$-split, we see that $(f_1)_{\mathfrak{n}}$ is split surjective over $R_{\mathfrak{n}}$. Hence f' is $\mathfrak{n}$-split. On the other hand, since $\mathscr{L}_{\mathfrak{n}} = 1$, the definition of $\mathscr{L}_{\mathfrak{n}}$ in Lemma 2.6.15, which mirrors the definition of $\mathscr{L}_{\mathfrak{n}}$ in Lemma 2.4.2, tells us that f' is not $\mathfrak{n}$-split, a contradiction. So a avoids every member of Λ_1.

Now, by Lemma 2.4.4, there exist a matrix $Q \in \mathbf{GL}(n, R)$; elements

$$s_1, \ldots, s_n \in R - \bigcup_{\mathfrak{m} \in \Lambda_1 \cup \cdots \cup \Lambda_n} \mathfrak{m};$$

and $b \in R$ such that, for every $i \in \{1, \ldots, n\}$ and for every $\mathfrak{m} \in \Lambda_i$, we have that

$$Q \equiv P_i \begin{pmatrix} s_1 & 0 & \cdots & 0 & 0 \\ 0 & \ddots & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & 0 & s_{n-1} & 0 \\ 0 & \cdots & \cdots & 0 & s_n \end{pmatrix} \pmod{\mathfrak{m}}$$

and such that the first row of Q has the form

$$\begin{pmatrix} 1-ab & 0 & \cdots & 0 & ab \end{pmatrix}.$$

Let $r_{\mathfrak{m},1}, \ldots, r_{\mathfrak{m},n-1} \in R$ so that they satisfy the conclusion of Lemma 3.1.5 relative to $\mathscr{L}_\mathfrak{n}$ and $s_1, \ldots, s_n$, with $r_{\mathfrak{m},1} \in aR$. Define U as in the proof of Lemma 2.4.6 so that $V := UQ$ satisfies the first claim in the present lemma. The fact that the first component of Vf is in $f_1 + aF$ can be verified by direct computation. $\square$

We show in Section 2.6 that, to prove the Splitting Lemma, it suffices to find a matrix $V \in \mathbf{GL}(n, R)$ such that the first $n-1$ components of $Vf := (g_1, \ldots, g_n)^\top$ form a map $(g_1, \ldots, g_{n-1})^\top$ that is (t, X, Λ)-split, for Lemma 2.6.14 will then finish the proof. Proving the Cancellation Lemma, however, requires more care: For every $a \in R$ such that the map (a, f_1) in $\mathrm{Hom}_R(N \oplus M, N)$ is $(1, X, X)$-split, we must also show that we can arrange for g_1 to be in $f_1 + aF$. For the latter task, we need two lemmas, comparable to [7, Lemmas 4.6 and 4.7], that will help us deal with the members $\mathfrak{p} \in X$ for which $\dim_X(\mathfrak{p}) \geqslant 1$. For such a prime $\mathfrak{p}$, if $N_\mathfrak{p}$ is homothetic over $R_\mathfrak{p}$, Lemma 3.1.8 gives a case in which we can determine $\delta_\mathfrak{p}(J)$ for a distinguished R-submodule J of F. This fact explains the homothetic assumption in the Cancellation Lemma. We can prove Lemma 3.1.8 easily once we establish the following result. We denote the length of an R-module D by $\lambda_R(D)$.

Lemma 3.1.7. *Let R be a commutative ring with a unique maximal ideal $\mathfrak{m}$; let M be an R-module; and let N be a nonzero, finitely presented, homothetic R-module. Let F be a finitely generated R-submodule of* $\mathrm{Hom}_R(M, N)$*, and let $I(F)$ denote the set of all $f \in F$ such that* $\mathrm{im}(f \circ e) \subseteq N\mathfrak{m}$ *for every $e \in$* $\mathrm{Hom}_R(N, M)$*. Then $I(F)$ is an R-submodule of F, and*

$$\delta(F) = \lambda_R\left(\frac{F}{I(F)}\right).$$

Proof. The first claim is routine to prove. It remains, then, to prove the second claim. Let $d := \delta(F)$. Since $N \neq 0$ and since F is finitely generated over R, Remark 2.6.2 tells us that $d < \infty$. Now, there exist R-submodules K and L of M such that $K \cong N^{\oplus d}$ and such that $M = K \oplus L$. Moreover, there exist an R-submodule G of F and an R-submodule H of $\operatorname{Hom}_R(M, N)$ such that $G \cong \operatorname{Hom}_R(K, N)$, such that $H \cong \operatorname{Hom}_R(L, N)$, and such that $\operatorname{Hom}_R(M, N) = G \oplus H$.

We will prove that $I(F) = \mathfrak{m}G \oplus (H \cap F)$. Clearly $\mathfrak{m}G \subseteq I(F)$. Let $f \in H \cap F$, and suppose that $f \notin I(F)$. Then there exists $e \in \operatorname{Hom}_R(N, M)$ such that $\operatorname{im}(f \circ e) \nsubseteq N\mathfrak{m}$. Since N is homothetic, $f \circ e$ is then an isomorphism, and so f is split surjective. Hence $\delta(F) \geqslant d + 1$, a contradiction. Thus $f \in I(F)$, and so $\mathfrak{m}G \oplus (H \cap F) \subseteq I(F)$. Now let $f' \in I(F)$. Since $\operatorname{Hom}_R(M, N) = G \oplus H$, there exist $g \in G$ and $h \in H$ such that $f' = g + h$. Since $h = f' - g \in I(F) + G \subseteq F$, we see that $h \in H \cap F$. Suppose that $g \notin \mathfrak{m}G$. Then, since N is homothetic, g is split surjective. As a result, there exists $e' \in \operatorname{Hom}_R(N, M)$ such that $\operatorname{im}(g \circ e') = N$. On the other hand, since $h \in H \cap F \subseteq I(F)$, we see that $\operatorname{im}(g \circ e') = \operatorname{im}((f' - h) \circ e') \subseteq N\mathfrak{m}$, a contradiction. Hence $g \in \mathfrak{m}G$, and so $I(F) \subseteq \mathfrak{m}G \oplus (H \cap F)$.

Now, since $\operatorname{End}_R(N)$ is nonzero cyclic and since $F = (G \oplus H) \cap F = G \oplus (H \cap F)$, we see that

$$\delta(F) = \lambda_R\left(\frac{G \oplus (H \cap F)}{\mathfrak{m}G \oplus (H \cap F)}\right) = \lambda_R\left(\frac{F}{I(F)}\right). \qquad \square$$

Lemma 3.1.8. *Let R be a commutative ring with a unique maximal ideal $\mathfrak{m}$; let M be an R-module; and let N be a nonzero, finitely presented, homothetic R-module. Let F be a finitely generated R-submodule of $\operatorname{Hom}_R(M, N)$, and let $f \in F$. Suppose that f is split surjective with section e, where $e \in \operatorname{Hom}_R(N, M)$. Let $J := \{h - (h \circ e \circ f) : h \in F\}$. Then J is an R-submodule of F, and $\delta(J) = \delta(F) - 1$.*

Proof. Once again, the first claim is easy to prove, so we prove only the second claim here. Let $K = \operatorname{im}(e)$, and let $L := \{x - (e \circ f)(x) : x \in M\}$ so that L is an R-submodule of M with $M = K \oplus L$. Let $G = Rf \cong \operatorname{Hom}_R(K, N) \cong \operatorname{End}_R(N)$, and let $H := \{h - (h \circ e \circ f) : h \in \operatorname{Hom}_R(M, N)\}$ so that H is an R-submodule of $\operatorname{Hom}_R(M, N)$ with $\operatorname{Hom}_R(M, N) = G \oplus H$ and $J = H \cap F$. Define $I(F)$ as in Lemma 3.1.7, and define $I(G)$ and $I(J)$ similarly. Since

$F = (G \oplus H) \cap F = G \oplus (H \cap F) = G \oplus J$, we see that $I(F) = I(G) \oplus I(J)$. Hence we get the following split short exact sequence:

$$0 \longrightarrow \frac{G}{I(G)} \longrightarrow \frac{F}{I(F)} \longrightarrow \frac{J}{I(J)} \longrightarrow 0.$$

Since $F = G \oplus J$ is finitely generated, J is finitely generated. Also, $I(G) = \mathfrak{m}G$. Hence, by Lemma 3.1.7 and by the fact that G is nonzero cyclic,

$$\delta(J) = \lambda_R\left(\frac{J}{I(J)}\right) = \lambda_R\left(\frac{F}{I(F)}\right) - \lambda_R\left(\frac{G}{\mathfrak{m}G}\right) = \delta(F) - 1. \qquad \square$$

We are now ready to prove the Cancellation Lemma.

Proof of the Cancellation Lemma. Define Λ as in Lemma 2.6.11 with respect to $F := Rf_1 + \cdots + Rf_n$ and X. We remind the reader that Λ is a finite subset of X such that, for every $\mathfrak{p} \in X - \Lambda$, there exists $\mathfrak{q} \in \Lambda$ for which $\mathfrak{q} \subsetneq \mathfrak{p}$ and $\delta_{\mathfrak{q}}(F) = \delta_{\mathfrak{p}}(F)$. Since X is understood and since it is understood that we are taking $t = 1$, we can use the terms $\mathfrak{p}$-*split* and Y-*split* for any $\mathfrak{p} \in X$ and for any $Y \subseteq X$ without the risk of confusion.

Let $a \in R$ such that (a, f_1) is $\mathfrak{m}$-split. We may begin as in the proof of the Splitting Lemma, making small changes involving the element a of R as necessary. For example, we now want r_1 to be in aJ rather than simply in J. Here, Lemma 3.1.6 covers the base case of the induction on ℓ. We may proceed until we find that we must distinguish two cases involving the index j_d. Just as in the proof of the Surjective Lemma, the case in which $j_d = n$ reduces to the case in which $j_d \leqslant n - 1$. Since this reduction does not rely on any of the extra conditions in the Cancellation Lemma, we simply deal with the case in which $j_d \leqslant n - 1$ here.

We begin this case as in the Surjective Lemma, again accounting for a as needed. After setting up the induction on the index i, we treat the cases $i = 1$ and $i \geqslant 2$ separately.

First suppose that $i = 1$. Within this case, we will deal with the subcases $a \in \mathfrak{q}$ and $a \notin \mathfrak{q}$ separately.

First suppose that $a \in \mathfrak{q}$. Since (a, f_1') is $\mathfrak{q}$-split, $(f_1')_{\mathfrak{q}}$ is split surjective. If $\dim_X(\mathfrak{q}) = 0$, then f' is $\mathfrak{q}$-split, a contradiction. Hence, $\dim_X(\mathfrak{q}) \geqslant 1$, and so $N_{\mathfrak{q}}$ is homothetic over $R_{\mathfrak{q}}$ by hypothesis. Now, by Lemma 3.1.8, we may assume that $j_1 = 1$ and that $b_{1,2} = \cdots = b_{1,n} = 0$. As a result, we may take $r_{j_1} = 0$.

Now suppose that $a \notin \mathfrak{q}$. Let $\mathscr{I}_1$ denote the ideal $(saJ + \mathfrak{q})/\mathfrak{q}$ of $R/\mathfrak{q}$. Let

$$\mathscr{S}_1 := \left\{\sigma \in \kappa(\mathfrak{q}) : \operatorname{rank}\left(\overline{\Omega_1} + \sigma\overline{\Omega_1'}\right) \leqslant d\nu - 1\right\}.$$

As in the proof of the Surjective Lemma, we can show that there is $\rho_1 \in \mathscr{I}_1 - \mathscr{S}_1$. Select an element r_{j_1} in aJ for which $\overline{sr_{j_1}} = \rho_1$. Then $\operatorname{rank}\left(\overline{\Omega_1 + sr_{j_1}\Omega_1'}\right) = d\nu$, as desired.

Now, starting with the case in which $i \geqslant 2$, we may continue as in the proof of the Surjective Lemma. $\square$

In the next section, we give some examples illustrating the content of our cancellation theorems.

3.2 Cancellation examples

In Section 1.2, we discuss three cancellation theorems in addition to our own: Evans's Cancellation Theorem (Theorem 1.2.2) states that a module whose endomorphism ring has stable rank one can always be cancelled. Bass's Cancellation Theorem (Theorem 1.2.3) and the De Stefani–Polstra–Yao Cancellation Theorem (Theorem 1.2.4) give criteria for when a finitely generated projective module can be cancelled. In Section 3.1, we extend the theorems of Bass and De–Stefani–Polstra–Yao by delineating conditions under which a finitely presented homothetic module can be cancelled. An easy induction then allows us to cancel a direct summand of a direct sum of finitely many copies of such a module as well.

At the end of Section 1.2, we give a cancellation example that can be proved by our main cancellation theorem but not by Evans, Bass, or De Stefani–Polstra–Yao. In this section, we would like to mention a few more cancellation theorems and supply cancellation examples that are also not covered by these theorems.

Although we have made it clear that our cancellation theorems apply to some nonprojective modules, we would like to highlight three cancellation theorems concerning projective modules here, mostly to deliver the point that projective modules have cancellation properties beyond the ones proved by Bass and De Stefani–Polstra–Yao. Eisenbud and Evans [10, page 302], for example, show that, if R is a commutative ring with a finite-dimensional Noetherian j-spectrum and S is a module-finite R-algebra, then a finitely generated right S-module K of finite projective dimension can be cancelled from the isomorphism $K \oplus L \cong K \oplus M$ if L and M are finitely generated projective right S-modules and $\operatorname{rank}(M) \geqslant 1 + \dim(j\text{-Spec}(R))$. Kaplansky [10, page 302] shows that, if every finitely generated ideal of a commutative ring R is projective (R is *semihereditary*) and K is a projective R-module, then, for all R-modules L and M, the isomorphism $K \oplus L \cong K \oplus M$ implies that $L \cong M$. Murthy and Swan [23, Theorem 1] show that, if R is an affine algebra of dimension at most two over an algebraically closed field and if K, L, and M are finitely generated projective R-modules, then $K \oplus L \cong K \oplus M$ implies that $L \cong M$. There are many other cancellation results on projective modules, but these three give a good survey of the most common types: There are those that extend Bass's topological approach, those that appeal to the special arithmetic properties of certain integral domains, and those that rely on the geometry of vector bundles.

Given the wealth of cancellation results on finitely generated projective modules, it is natural to consider what statements might be available for finitely generated torsion-free modules. In the one-dimensional case, a number of results exploit the cancellation properties of finitely generated modules over Dedekind domains to yield results for reduced one-dimensional Noetherian rings with module-finite normalizations. Articles by Wiegand [36] [37] and Hassler and Wiegand [16] prove general results in this direction as well as special results for certain quadratic orders, integral group rings, and affine curves over algebraically closed fields. Levy and Wiegand [22, Theorem 6.2] prove a cancellation theorem for modules over a *Bass ring*, that is, a reduced commutative ring whose ideals are all two-generated and whose normalization is module-finite. In particular, Levy and Wiegand show that, if R is a Bass ring, K is a finitely generated projective R-module, and L and M are finitely generated torsion-free R-modules such that $K \oplus L \cong K \oplus M$, then $L \cong M$.

There are also results on finitely generated torsion-free modules over rings of dimension two. Chase [5, Theorem 3.6] shows that, if $R = k[x, y]$, where k is a field, and L and M are finitely generated torsion-free R-modules such that $K \oplus L \cong K \oplus M$ for some finitely generated R-module K, then $R \oplus L \cong R \oplus M$ in particular. If, moreover, k is algebraically closed of characteristic zero, Chase shows that $L \cong M$. Wiegand [36, Theorem 1.2] extends the last result to the case in which R is any regular affine domain of dimension two over an algebraically closed field of characteristic zero.

In addition, there are results that do not rely on the dimension of the underlying ring. A result of Kaplansky [10, page 302] states that, if R is any commutative ring and M is an ideal of R, then, for every R-module L, the isomorphism $R \oplus L \cong R \oplus M$ implies that $L \cong M$. Vasconcelos [33] proves that, if R is a Noetherian normal domain, then $K \oplus L \cong K \oplus M$ implies that $L \cong M$ if K is a finitely generated R-module and L and M are ideals of R.

We would like to mention one final cancellation theorem that exists in a category of its own:

Theorem 3.2.1 (Warfield [35, Theorems 1.2 and 1.6])**.** *Let S be a (possibly noncommutative) ring, and let K, L, and M be right S-modules such that* $\mathrm{spl}_S(M, K) \geqslant \mathrm{sr}(\mathrm{End}_S(K))$ *and* $K \oplus L \cong K \oplus M$*. Then* $L \cong M$*.*

The breathtaking generality of this theorem cannot be understated. To illustrate the scope of this result, we first recall Bass's Stable Range Theorem, which gives an upper bound on the stable rank of a certain type of ring. We use the term *stable range* instead of *stable rank* in the name of this theorem to observe tradition: In Bass's original terminology [3, page 14], a *stable range for* $\mathbf{GL}(S)$ referred to an upwards-closed set of positive integers $\{n, n+1, n+2, \ldots\}$ satisfying a certain property. In modern terminology, we can describe this property simply by saying that $\mathrm{sr}(S) \leqslant n$. Bass would then say that n *defines a stable range for* $\mathbf{GL}(S)$. Hence, the stable rank of S is the least integer that defines a stable range for $\mathbf{GL}(S)$.

Theorem 3.2.2 (Bass's Stable Range Theorem [3, Theorem 11.1])**.** *Let S be a module-finite algebra over a commutative ring with a Noetherian j-spectrum of finite dimension d. Then* $\mathrm{sr}(S) \leqslant 1 + d$*.*

This fact about stable rank allows us to compare Warfield's Cancellation Theorem more easily with our main cancellation theorem. For this, we consider the special case in which $R = S = \mathrm{End}_S(K)$ is a commutative ring with a Noetherian j-spectrum of finite dimension d and stable rank $1+d$. (For example, by a result of Vasershtein [34, Theorem 8], we may take $R := k[x_1, \ldots, x_d]$, where k is a subfield of the field $\mathbb{R}$ of all real numbers.) Now, assuming the conditions in Warfield's Cancellation Theorem, we note that our cancellation theorem is inconclusive: Here, it could be the case that K is not finitely presented.

Nevertheless, there are many cases in which our main cancellation theorem reveals new information relative to Warfield's Cancellation Theorem—and relative to the other cancellation theorems that we have mentioned as well. We begin with the following generalization of Example 1.2.10.

Example 3.2.3. Let R be an affine domain of dimension $d \geqslant 3$ over a field k. Suppose that R satisfies the (S_2) *property*; in other words, suppose that I contains a regular sequence of length two for every proper ideal I of R of dimension at most $d-2$. (For example, we may suppose that R is an affine domain of dimension at least three that is normal or Cohen–Macaulay.) Let $K = N$ be a proper ideal of R such that $\dim(R/N) \leqslant \min\{d, \mathrm{sr}(R)\} - 2$. Let $M := N^{\oplus t} \oplus R^{\oplus u}$, where t, u are integers such that $1 + \dim(R/N) \leqslant t \leqslant \mathrm{sr}(R) - 1$ and $u \geqslant 1 + d - t$. Then, for every R-module L such that $K \oplus L \cong K \oplus M$, we have $L \cong M$.

More generally, we may take $M := N^{\oplus t} \oplus I_1 \oplus \cdots \oplus I_u$, where $I_1, \ldots, I_u$ are ideals of R such that there exists $\mathfrak{q} \in \mathrm{Var}(N)$ with $\mathfrak{q} \subsetneq \sqrt{I_i}$ for every $i \in \{1, \ldots, u\}$.

Remark 3.2.4. Note that, since $\mathrm{sr}(R) \geqslant 2$ by Estes–Heinzer–Ohm [11, page 361], there always exists an ideal N of R such that $\dim(R/N) \leqslant \min\{d, \mathrm{sr}(R)\} - 2$: For example, N could be any zero-dimensional ideal of R. Also note that, by Vasershtein [34, Theorem 8], if k is a subfield of the field $\mathbb{R}$ of all real numbers, then $\mathrm{sr}(R) = 1 + d$, and so we can take N to be any proper ideal of R of dimension at most $d - 2$ in this case.

Proof of Example 3.2.3. Since R is an affine domain, we see that N is finitely presented and that $X := j\text{-Spec}(R) \cap \mathrm{Supp}_R(N) = \mathrm{Spec}(R)$ is Noetherian and finite-dimensional. Since R is (S_2) and $\dim(R/N) \leqslant d-2$, we see that N contains a regular sequence of length two and, hence, is homothetic. Also, it is clear that M is finitely generated over R. The fact that $\mathrm{spl}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \geqslant 1 + \dim(R/\mathfrak{p})$ for every $\mathfrak{p} \in X$ follows directly from the choices of the integers t, u and the ideals $I_1, \ldots, I_u$ of R. By our main cancellation theorem, then, we are done. □

Certainly, this example is not covered by any of the theorems that we have mentioned concerning projective modules, nor is it covered by any theorems concerning rings of dimension one or two. Also, neither L nor M is an ideal of R, and so neither Kaplansky nor Vasconcelos applies. It remains to show that Warfield's Cancellation Theorem does not apply.

For this, it suffices to show that $\mathrm{spl}_R(M, N) = t$ since $t \leqslant \mathrm{sr}(R) - 1$ by definition. Of course, $\mathrm{spl}_R(M, N) \geqslant t$. Suppose, by way of contradiction, that $M = N^{\oplus(t+1)} \oplus G$ for some R-module G. Since $\mathrm{rank}(M) = t + u$, we see that $\mathrm{rank}(G) = u - 1$. By assumption, there exists $\mathfrak{q} \in \mathrm{Var}(N)$ with $\mathfrak{q} \subsetneq \sqrt{I_i}$ for every $i \in \{1, \ldots, u\}$. Since $\dim(R/N) \leqslant d-2$, the Krull Principal Ideal Theorem tells us that $\mu_{R_\mathfrak{q}}(N_\mathfrak{q}) \geqslant 2$. Double-counting $\mu_{R_\mathfrak{q}}(M_\mathfrak{q})$ then yields

$$t \cdot \mu_{R_\mathfrak{q}}(N_\mathfrak{q}) + u = \mu_{R_\mathfrak{q}}(N_\mathfrak{q}^{\oplus t} \oplus R_\mathfrak{q}^{\oplus u}) = \mu_{R_\mathfrak{q}}(N_\mathfrak{q}^{\oplus(t+1)} \oplus G_\mathfrak{q}) = (t+1) \cdot \mu_{R_\mathfrak{q}}(N_\mathfrak{q}) + \mu_{R_\mathfrak{q}}(G_\mathfrak{q})$$

so that $\mu_{R_\mathfrak{q}}(G_\mathfrak{q}) = u - \mu_{R_\mathfrak{q}}(N_\mathfrak{q}) \leqslant u - 2$. Now

$$u - 1 = \mathrm{rank}(G) \leqslant \mu_{R_\mathfrak{q}}(G_\mathfrak{q}) \leqslant u - 2,$$

a contradiction. Thus $\mathrm{spl}_R(M, N) = t \leqslant \mathrm{sr}(R) - 1$, and so Warfield's Cancellation Theorem does not apply to this example.

If $R := k[x_1, \ldots, x_d]$ for a field k and an integer $d \geqslant 3$, then we can also form a statement about the nonprincipal ideals of dimension $d-1$ whose nonprincipal minimal primes have sufficiently small dimension.

Example 3.2.5. Let $R := k[x_1, \ldots, x_d]$, where k is a field and d is an integer with $d \geqslant 3$. Let $K = N$ be a nonprincipal ideal of R of dimension $d-1$ such that

$$s := \max\{1 + \dim(R/\mathfrak{p}) : \mathfrak{p} \in \mathrm{Min}(N), \dim(R/\mathfrak{p}) \leqslant d-2\} \leqslant \mathrm{sr}(R) - 1,$$

where $\mathrm{Min}(N)$ denotes the set of minimal members of $\mathrm{Var}(N)$. Let $M := N^{\oplus t} \oplus R^{\oplus u}$, where t, u are integers such that $s \leqslant t \leqslant \mathrm{sr}(R) - 1$ and $u \geqslant 1 + d - t$. Then, for every R-module L such that $K \oplus L \cong K \oplus M$, we have $L \cong M$.

Proof. The only condition in our main cancellation theorem that perhaps needs verification is the one on local splitting capacities. To this end, we must note that, for every $\mathfrak{p} \in X := \mathrm{Spec}(R)$, it is the case that $N_\mathfrak{p} \cong R_\mathfrak{p}$ if and only if every member of $\mathrm{Min}(N)$ contained in $\mathfrak{p}$ has dimension $d-1$. This follows from the fact that $R_\mathfrak{p}$ is a Noetherian unique factorization domain for every $\mathfrak{p} \in X$. Now, $M_\mathfrak{p} \cong R_\mathfrak{p}^{\oplus(t+u)} \cong N_\mathfrak{p}^{\oplus(t+u)}$ for every $\mathfrak{p} \in X$ such that every member of $\mathrm{Min}(N)$ contained in $\mathfrak{p}$ has dimension $d-1$, and of course $t + u \geqslant 1 + d$ by our choice of u. On the other hand, for every $\mathfrak{p} \in X$ such that $\mathfrak{p}$ contains a member of $\mathrm{Min}(N)$ of dimension at most $d-2$, it is the case that $M_\mathfrak{p} \cong N_\mathfrak{p}^{\oplus t} \oplus R_\mathfrak{p}^{\oplus u}$ with $t \geqslant s \geqslant 1 + \dim(R/\mathfrak{p})$. $\square$

Again, we may not apply any of the older cancellation theorems that we have mentioned. To show, in particular, that Warfield's Cancellation Theorem does not apply, we may argue mostly as before; however, since we now cannot use the Krull Principal Ideal Theorem, we may use the fact that N is nonprojective to ensure that there exists $\mathfrak{q} \in \mathrm{Var}(N)$ such that $\mu_{R_\mathfrak{q}}(N_\mathfrak{q}) \geqslant 2$.

Our main cancellation theorem reveals a cancellation property of another type of $(d-1)$-dimensional ideal, namely, a canonical ideal of a reduced non-Gorenstein Cohen–Macaulay affine ring. We study an ideal of this type in the next example.

Example 3.2.6. Let d, n be integers with $2 \leqslant d \leqslant n-2$, and let $R := k[x_1, \ldots, x_n]/I$, where k is a subfield of $\mathbb{R}$ and I is the ideal of $k[x_1, \ldots, x_n]$ generated by all squarefree monomials of degree $d+1$. (Here, R is called the *d-dimensional squarefree Veronese ring in n variables over k.*) Let N be a canonical ideal of R, and let $M := N^{\oplus d} \oplus R^{\oplus u}$, where u is a positive integer. Then, for every R-module L such that $K \oplus L \cong K \oplus M$, we have $L \cong M$.

This example relies on the fact that the affine ring R described therein is in fact reduced and Cohen–Macaulay. To show that R is reduced, we may simply point to the fact that I is the intersection of all prime ideals of $k[x_1, \ldots, x_n]$ generated by $n-d$ variables. Since each of these prime ideals has dimension d in $k[x_1, \ldots, x_n]$, the intersection of all of these prime ideals must form the unique minimal primary decomposition of I. To show that R is Cohen–Macaulay, we may show that R is a *Stanley–Reisner ring* of a *shellable simplicial complex* Δ [4, Theorem 5.1.13]. We direct the reader to [4, Section 5.1] for an introduction to the algebra of simplicial complexes and simply note here that Δ is *pure of dimension* $d-1$ and that ordering the *facets* of Δ lexicographically produces a *shelling*.

Now that we have verified that R is a reduced Cohen–Macaulay affine ring, we can appeal to [4, Proposition 3.3.18] to conclude that R has an ideal N that is a canonical module. To show that N is not projective, we may show that R is not Gorenstein. For this, we use the *h-vector* $(h_0, \ldots, h_d)$ of Δ, which is defined in [4, page 205]. Formulas for $h_0, \ldots, h_d$ in terms of the facets of Δ are given in [4, Lemma 5.1.8]. From these formulas, we find that $h_0 = 1 \neq \binom{n-1}{d} = h_d$, which shows that Δ is not an *Euler complex* [4, Theorem 5.4.2]. Combining this observation with the fact that Δ is its own *core*, we may apply [4, Theorem 5.5.2] to conclude that R is not Gorenstein.

Given that R is a reduced non-Gorenstein Cohen–Macaulay affine ring, we can appeal to [4, Proposition 3.3.18] to find that N has dimension $d-1$ in R. Since it is clear that $\operatorname{spl}_{R_\mathfrak{p}}(M_\mathfrak{p}, N_\mathfrak{p}) \geqslant d \geqslant 1 + \dim(R/\mathfrak{p})$ for every nonminimal prime ideal $\mathfrak{p}$ of R, proving Example 3.2.6 now simply amounts to verifying that $M_\mathfrak{p} \cong N_\mathfrak{p}^{\oplus(t+u)} \cong R_\mathfrak{p}^{\oplus(t+u)}$ for every minimal prime $\mathfrak{p}$ of R. The last statement follows from the fact that N has dimension $d-1$ in R.

It should now be clear that none of the older cancellation theorems we have mentioned apply to this example. For instance, to show that Warfield's Cancellation Theorem does not apply, we must prove that $\operatorname{spl}_R(M, N) \leqslant \operatorname{sr}(R) - 1$. For this, we can use a rank-based argument as before, noting that there exists $\mathfrak{q} \in \operatorname{Var}(N)$ such that $\mu_{R_\mathfrak{q}}(N_\mathfrak{q}) \geqslant 2$ since N is nonprojective. This will show that $\operatorname{spl}_R(M, N) = d$. It remains to show that $\operatorname{sr}(R) = 1 + d$. For this, we note that, for every minimal prime $\mathfrak{p}$ of R, the ring $R/\mathfrak{p}$ is isomorphic to a polynomial ring over k in d variables. Such a ring has stable rank $1 + d$ by Vasershtein [34, Theorem 8]. Next we note that $\operatorname{sr}(R/\mathfrak{p}) \leqslant \operatorname{sr}(R)$; this follows from [21, Proposition 1.5(1)]. Now, using Bass's Stable Range Theorem, we find that $\operatorname{sr}(R) \leqslant 1 + d$. Combining our observations, we find that $1 + d = \operatorname{sr}(R/\mathfrak{p}) \leqslant \operatorname{sr}(R) \leqslant 1 + d$ so that $\operatorname{sr}(R) = 1 + d$. Hence $\operatorname{spl}_R(M, N) = d = \operatorname{sr}(R) - 1$.

For a final comment on this example, we would like to mention that R is not normal and that R is not a domain: Since the underlying field is perfect and since every minimal prime of I in $k[x_1, \ldots, x_n]$ has dimension d, we can use the Jacobian criterion [9, Corollary 16.20] to show that R contains a $(d-1)$-dimensional prime ideal $\mathfrak{p}$ such that $R_\mathfrak{p}$ is not regular. This demonstrates that R is not normal. To show that R is not a domain, we simply remark that R has more than one minimal prime; in fact, $|\operatorname{Min}(R)| = \binom{n}{n-d}$.

Surely there are many more examples of cancellation. We invite the reader to consider, for instance, how the examples above might be generalized or adapted to accommodate a non-Noetherian domain with finite-dimensional Noetherian maximal spectrum and stable rank greater than one. The difficulty in producing an interesting example of this type is not in finding such a ring: For every positive integer d, Heinzer furnishes a non-Noetherian Jacobson Bézout domain B with a d-dimensional Noetherian prime spectrum [17]. Since B has stable rank one, we may take our ring to be $B[x]$: By Gabel [13], the ring $B[x]$ has stable rank at least two. Now, since B is not Noetherian, $B[x]$ is also not Noetherian. Since B has finite dimension, so does $B[x]$. By a result of Ohm and Pendleton [25, Theorem 2.5], since B has a Noetherian spectrum, so does $B[x]$. The ring $B[x]$, which we will now call R, also has a finitely presented nonprojective homothetic ideal N: We may take N to be $Rx + Rb$, where b is a nonzero nonunit of B, since x, b is a regular sequence of length two in R. The issue would then be to construct, for instance, a finitely presented R-module M such that $\mathrm{spl}_R(M, N) \leqslant \mathrm{sr}(R) - 1$ and such that $1 + \dim(R) \leqslant \mathrm{spl}_{R_\mathfrak{m}}(M_\mathfrak{m}, N_\mathfrak{m})$ for every $\mathfrak{m} \in \mathrm{Max}(R)$. We have not been able to accomplish this.

There is also the question of whether our main cancellation theorem can be extended to module-finite algebras over commutative rings with finite-dimensional Noetherian maximal spectra. For this, we encourage the reader to consider what an appropriate generalization of a homothetic module might be over a noncommutative ring, given that right-multiplication by an element of a noncommutative ring is not necessarily a linear endomorphism of a module.

We would like to close with the following conjectures, the first two of which remove the conditions on residue fields and generating sets from Corollaries 2.5.1 and 2.6.17:

Conjecture 3.2.7. *Assume the hypotheses of the Surjective Lemma. Then there exist* $r_1, \ldots, r_{n-1} \in R$ *such that* $(f_1 + r_1 f_n, \ldots, f_{n-1} + r_{n-1} f_n)^\top$ *is* (t, X, X)*-surjective.*

Conjecture 3.2.8. *Assume the hypotheses of the Splitting Lemma. Then there exist* $r_1, \ldots, r_{n-1} \in R$ *such that* $(f_1 + r_1 f_n, \ldots, f_{n-1} + r_{n-1} f_n)^\top$ *is* (t, X, X)*-split.*

Conjecture 3.2.9. *Assume the hypotheses of the Cancellation Lemma. Let* $a \in R$ *such that* $(a, f_1) \in \mathrm{Hom}_R(N \oplus M, N)$ *is* $(1, X, X)$*-split. Then there exist* $r_1, \ldots, r_{n-1} \in R$ *such that* $(f_1 + r_1 f_n, f_2 + r_2 f_n, \ldots, f_{n-1} + r_{n-1} f_n)^\top$ *is* $(1, X, X)$*-split. Moreover, we can arrange for* r_1 *to be in* aR*.*

These conjectures hold under the hypotheses of Bass in [3] and under the hypotheses of De Stefani–Polstra–Yao in [7]. Accordingly, these conjectures, if proved, would establish more faithful generalizations of older work while confirming our suspicions that the permutation techniques from Sections 2.4, 2.6, and 3.1 can be avoided.

REFERENCES

[1] M. Auslander and D. Buchsbaum, *Groups, rings, modules*, Harper and Row, 1974.

[2] H. Bass, *Big projective modules are free*, Illinois J. Math. **7** (1963), 24–31.

[3] ______, *K-theory and stable algebra*, Publ. Math. Inst. Hautes Études Sci. **22** (1964), 5–60.

[4] W. Bruns and J. Herzog, *Cohen–Macaulay rings*, Cambridge University Press, 1993.

[5] S. U. Chase, *Torsion-free modules over $K[x, y]$*, Pacific J. Math. **12** (1962), 437–447.

[6] T. Coquand, H. Lombardi, and C. Quitté, *Generating non-Noetherian modules constructively*, Manuscripta Math. **115** (2004), 513–520.

[7] A. De Stefani, T. Polstra, and Y. Yao, *Generalizing Serre's Splitting Theorem and Bass's Cancellation Theorem via free-basic elements*, Proc. Amer. Math. Soc. **146** (2018), 1417–1430.

[8] D. Dummit and R. Foote, *Abstract algebra*, third ed., John Wiley and Sons, 2004.

[9] D. Eisenbud, *Commutative algebra with a view toward algebraic geometry*, Springer, 2004.

[10] D. Eisenbud and E. G. Evans, Jr., *Generating modules efficiently: theorems from algebraic K-theory*, J. Algebra **27** (1973), 278–305.

[11] D. Estes and J. Ohm, *Stable range in commutative rings*, J. Algebra **7** (1967), 343–362.

[12] E. G. Evans, Jr., *Krull–Schmidt and cancellation over local rings*, Pacific J. Math. **46** (1973), 115–121.

[13] M. Gabel, *Lower bounds on the stable range of polynomial rings*, Pacific J. Math. **61** (1975), 117–120.

[14] R. M. Guralnick, L. S. Levy, and R. B. Warfield, Jr., *Cancellation counterexamples in Krull dimension 1*, Proc. Amer. Math. Soc. **109** (1990), 323–326.

[15] R. Hartshorne, *Algebraic geometry*, Springer, 1977.

[16] W. Hassler and R. Wiegand, *Direct sum cancellation for modules over one-dimensional rings*, J. Algebra **283** (2005), 93–124.

[17] W. J. Heinzer, *J-Noetherian integral domains with 1 in the stable range*, Proc. Amer. Math. Soc. **19** (1968), 1369–1372.

[18] R. Heitmann, *Generating non-Noetherian modules efficiently*, Michigan Math. J. **31** (1984), 167–180.

[19] T. Hungerford, *On the structure of principal ideal rings*, Pacific J. Math. **25** (1968), 543–547.

[20] L. Klingler and L. S. Levy, *Direct-sum cancellation: modules versus lattices*, Comm. Algebra **18** (1990), 1857–1868.

[21] T. Y. Lam, *A crash course on stable range, cancellation, substitution and exchange*, J. Algebra Appl. **3** (2004), 301–343.

[22] L. S. Levy and R. Wiegand, *Dedekind-like behavior of rings with 2-generated ideals*, J. Pure Appl. Algebra **37** (1985), 41–58.

[23] M. P. Murthy and R. G. Swan, *Vector bundles over affine surfaces*, Invent. Math. **36** (1976), 125–165.

[24] T. Nakayama, *A remark on finitely generated modules*, Nagoya Math. J. **3** (1951), 139–140.

[25] J. Ohm and R. L. Pendleton, *Rings with noetherian spectrum*, Duke Math. J. **35** (1968), 631–639.

[26] B. Olberding, *Contributions to module theory: in memory of A. L. S. Corner*, ch. Characterizations and constructions of h-local domains, pp. 385–406, Walter de Gruyter, 2008.

[27] D. Quillen, *Projective modules over polynomial rings*, Invent. Math. **36** (1976), 167–171.

[28] J.-P. Serre, *Modules projectifs et espaces fibrés à fibre vectorielle*, Séminaire Dubreil. Algèbre et théorie des nombres **11** (1957/1958), no. 2, 1–18.

[29] J. T. Stafford, *Generating modules efficiently: algebraic K-theory for noncommutative Noetherian rings*, J. Algebra **69** (1981), no. 2, 312–346.

[30] A. A. Suslin, *Projective modules over polynomial rings are free*, Dokl. Akad. Nauk SSSR **229** (1976), no. 5, 1063–1066, Russian.

[31] R. G. Swan, *Vector bundles and projective modules*, Trans. Amer. Math. Soc. **105** (1962), 264–277.

[32] ———, *The number of generators of a module*, Math. Z. **102** (1967), 318–322.

[33] W. Vasconcelos, *Ideals and cancellation*, Math. Z. **102** (1967), 353–355.

[34] L. N. Vasershtein, *Stable rank of rings and dimensionality of topological spaces*, Funct. Anal. Appl. **5** (1971), 102–110.

[35] R. B. Warfield, Jr., *Cancellation of modules and groups and stable range of endomorphism rings*, Pacific J. Math. **91** (1980), 457–485.

[36] R. Wiegand, *Cancellation over commutative rings of dimension one and two*, J. Algebra **88** (1984), 438–459.

[37] ———, *Direct sum cancellation over Noetherian rings*, Abelian groups and modules (R. Gobel, C. Metelli, A. Orsatti, and L. Salce, eds.), CISM Courses and Lectures, vol. 267, International Centre for Mechanical Sciences, Springer-Verlag Wien, 1984, pp. 441–466.