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CHARACTERIZING  $F$ -RATIONALITY OF COHEN-MACAULAY RINGS VIA  
CANONICAL MODULES

by

SHUENN SIANG NG

Under the Direction of Yongwei Yao, PhD

ABSTRACT

This dissertation investigates the characterization of  $F$ -rationality. Much work has been done to characterize  $F$ -rationality. Here, we will assume that the underlying ring is commutative and Noetherian with unity of prime characteristic  $p$ . Juan Vélez has proved many results on rings that are closely related to Cohen-Macaulay rings via canonical modules. In this paper, we improve his results. Using Karen Smith's result in her dissertation work, we were able to obtain different approaches to prove some similar results of Juan Vélez. This

allows us to further analyze the relationship between  $F$ -rationality and canonical modules. Eventually, we obtain a few more interesting characterizations of  $F$ -rationality when  $R$  is reduced  $F$ -finite Cohen-Macaulay.

INDEX WORDS: algebra, duality, canonical module, Cohen-Macaulay, commutative,  $F$ -finite,  $F$ -rational, Frobenius, localization, local cohomology, Matlis Dual, module, Noetherian, tight closure, test element, test exponent

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by

SHUENN SIANG NG

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy  
in the College of Arts and Sciences  
Georgia State University

2018

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2018

CHARACTERIZING  $F$ -RATIONALITY OF COHEN-MACAULAY RINGS VIA  
CANONICAL MODULES

by

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## DEDICATION

This dissertation is dedicated to my parents.

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## CHAPTER 1

### INTRODUCTION

Throughout this dissertation, unless otherwise specified, all rings are assumed to be commutative, Noetherian, and with unity. By  $(R, \mathfrak{m}, k)$ , we mean that  $R$  is a local ring with  $\mathfrak{m}$  being the maximal ideal and  $k = R/\mathfrak{m}$  being the residue field of  $R$ .

The goal of this dissertation is to study the  $F$ -rationality of Cohen-Macaulay rings of prime characteristic  $p$  via canonical modules. Thus, for the rest of the introduction, we further assume that all rings have prime characteristic  $p$ . Since the ring  $R$  has prime characteristic  $p$  and  $M$  is an  $R$ -module, we can derive an  $R$ -module structure on the abelian group  $(M, +)$ , for any integer  $e \geq 0$  by  $r \cdot m := r^{p^e} m$  for any  $r \in R$  and  $m \in M$ . We denote the derived  $R$ -module by  ${}^e M$ . Similarly,  ${}^e R$  is used to denote the derived  $R$ -module structure on  $R$ .

In 1986, a very important concept, tight closure, was first introduced by Hochster and Huneke [HH88]. We first recall the definition of tight closure for ideals in a prime characteristic  $p$  Noetherian ring. Let  $I \leq R$  be an ideal of  $R$ . We say that  $x \in I$  is in the tight closure  $I^*$  of  $I$  if there exists an element  $c \in R^\circ$ , such that  $cx^{p^e} \in I^{[p^e]}$  for  $e \gg 0$ , where  $R^\circ$  is the complement of the union of all minimal primes of the ring  $R$  and  $I^{[p^e]}$  is the ideal of  $R$  generated by the  $p^e$ th powers of the elements of  $I$ . (Note that, given modules  $N \subseteq M$ , the tight closure of  $N$  in  $M$ ,  $N_M^*$  is defined in [HH90a]. Also, see Definition 2.1.2.) Notice that  $c$  can be dependent on both  $x$  and  $I$ . One hopes to have the same  $c$  that works when we “test” memberships in the tight closure. Such an element  $c$  was introduced in [HH90a], and is called a test element, if it exists. In the same paper, the authors showed the existence of test element under mild condition. (See Definition 2.1.3 for the definition and see [HH90a, §6] for more details on test element.)

The tight closure theory has given rise to many types of interesting rings of prime

characteristic  $p$ , such as rings, in which every ideal  $I$  is tightly closed (i.e.,  $I^* = I$ ), which are called weakly  $F$ -regular rings [HH90a]. However, in this dissertation, we are interested in the study of rings in which every ideal generated by parameters is tightly closed, which are called  $F$ -rational rings. This definition arose from the work of Fedder and Watanabe [FW89]. Some known characterizations of  $F$ -rationality can be found in [HH90b, §2]. Also, in [HY], Hochster and Yao defined the  $F$ -rational signature  $r(R)$  for local rings and they proved that  $r(R) > 0$  if and only if  $R$  is  $F$ -rational under mild conditions (see [HY, Definition 2.1, Theorem 4.1] for more details).

Our main result mainly comes from studying [Vél95]. In his paper [Vél95], Juan Vélez proved that if  $(R, \mathfrak{m})$  is a reduced  $F$ -finite Cohen-Macaulay local ring with a canonical module,  $\omega$ , then  $R$  is  $F$ -rational if and only if for each  $d \in R^\circ$ , there exists a  $q_0$  such that the map

$$\phi_{d,q}^* : \mathrm{Hom}_R({}^eR, \omega) \rightarrow \mathrm{Hom}_R(R, \omega)$$

is surjective for all  $q \geq q_0$ , where the map  $\phi_{d,q}^*$  is induced by  $\phi_{d,q} : R^e d \hookrightarrow {}^eR$ . (Compare this with Theorem 3.3.1 and see Chapter 2 for definitions.)

Another important characterization of  $F$ -rationality was discovered by Karen Smith in her dissertation work [Smi93], which roughly says that  $R$  is  $F$ -rational if and only if the tight closure of 0 in the highest cohomology of  $R$  is 0 (see Theorem 3.2.5 and [Smi93] for more details).

Using these two results and notation introduced in Chapter 3, we establish the following theorem:

**Theorem 3.3.4.** *Let  $(R, m, k)$  be a reduced  $F$ -finite Cohen-Macaulay ring with canonical module,  $\omega_R$ . Then, the following assertions are equivalent:*

- (1)  $R$  is  $F$ -rational.
- (2) For all  $c \in R^\circ$ , there exists an  $e_0 \in \mathbb{N}$  such that  $\forall e \geq e_0$ ,  $T_{eR, \omega}({}^e c) = \omega$ .
- (3) For all faithful finitely generated  $R$ -modules  $M$ , there exists an  $x \in M$  such that for all  $c \in R^\circ$ , there exists an  $e_0$  such that  $\forall e \geq e_0$ ,  $T_{eM, \omega}({}^e(cx)) = \omega$ .

(4) For all faithful finitely generated  $R$ -modules  $M$ , for all  $c \in R^\circ$ , there exists an  $e_0$  such that  $\forall e \geq e_0$ ,  $T_{eM, \omega}({}^e(cM)) = \omega$ .

(5) For all  $c \in R^\circ$ , there exists an  $e_0 \in \mathbb{N}$  such that  $\forall e \geq e_0$ ,  $T_{eR, \omega}({}^e(cR)) = \omega$ .

We then show the following lemma, which is crucial to our main theorem. (Note the similarity between the lemma and [Vél95, Remark 1.10].)

**Lemma 3.3.3.** *Let  $(R, \mathfrak{m}, k)$  be a reduced  $F$ -finite Cohen-Macaulay local ring with canonical module,  $\omega$ . Let  $c$  be a  $q_0$ -weak test element and  $q_1$  be a test exponent for  $c$  and  $H_{\mathfrak{m}}^d(R)$ . Then,*

(1) *If there exists an  $e \geq e_0$  such that  $T_{eR, \omega}({}^e c) = \omega$ , then  $R$  is  $F$ -rational.*

(2) *If  $R$  is  $F$ -rational, then  $T_{eR, \omega}({}^e c) = \omega$  for all  $e \geq e_1$ .*

Then, we prove the existence of a global test exponent under mild conditions in Chapter 4 using Matlis Duality.

**Corollary 4.1.10** (also see Lemma 4.1.8 and Lemma 4.1.9). *Let  $R$  be an  $F$ -finite Cohen-Macaulay ring with a canonical module and a locally stable test element,  $c$ . There exist an  $e \in \mathbb{N}$  such that  $q$  is a test exponent for  $c$  and  $H_{P_P}^{\dim(R_P)}(R_P)$ , for all  $P \in \text{Spec}(R)$ .*

Finally, we prove our main theorem. One statement is the following (see Theorem 4.2.9 for full details):

*Let  $R$  be an  $F$ -finite Cohen-Macaulay ring with a canonical module,  $\omega$ . Then,  $R$  is  $F$ -rational if and only if  $R_P$  is  $F$ -rational for every  $P \in \text{Spec}(R)$  such that  $\dim(R_P) < 2$ , and for every  $P \in \text{Spec}(R)$  such that  $\dim(R_P) \geq 2$ , there exists an  $e \in \mathbb{N}$  such that  $T_{e(P\omega_P), \omega_P}({}^e(P\omega_P)) = \omega_P$ .*

In [Vél95], the characterization focuses on homomorphisms from  ${}^e R$  to the canonical module for all  $e > 0$  while my characterization focuses on homomorphisms from  ${}^e M$  to the canonical module for various  $R$ -module  $M$  and for one single  $e > 0$ .

At the end of the dissertation, we discuss some remarks of the main theorem and possible future work.

## CHAPTER 2

### PRELIMINARIES

In this chapter, we set up notations and review some of the known results that will be used, directly or indirectly, in the following chapters. Again, all rings are assumed to be commutative, Noetherian, and with unity. Also, let  $\mathbb{N}$  denote the set  $\{1, 2, 3, \dots\}$ .

#### 2.1 Rings of prime characteristic $p$

In this section, we assume that the ring  $R$  has prime characteristic  $p$ . In this context, we will use the usual conventions  $q = p^e$ ,  $q_j = p^{e_j}$ , and  $q' = p^{e'}$ , etc. Then, for every integer  $e \geq 0$ , there exists the Frobenius homomorphism  $F^e : R \rightarrow R$  defined by  $r \mapsto r^q$  for all  $r \in R$ , where  $q := p^e$ .

Let  $M$  be an  $R$ -module. For every  $e \geq 0$ , there is a derived  $R$ -module structure on the abelian group  $(M, +)$  by  $r \cdot m := r^{p^e} m$  for any  $r \in R$  and  $m \in M$ . We denote the derived  $R$ -module by  ${}^e M$ . Let  $e', e \in \mathbb{N}$ . Note that  ${}^{e'}({}^e M)$  is simply  ${}^{e'+e} M$ .

Without knowing whether  $m$  is an element from  $M$ ,  ${}^e M$  or  ${}^{e'} M$ , the meaning of  $r \cdot m$  in last paragraph is unknown for  $r \in R$ . For this reason, we have the following conventions. By  ${}^e m$ , we consider it as an element in  ${}^e M$ . We have that  ${}^e r \cdot {}^e m = {}^e(rm)$  and  ${}^{e'} r \cdot {}^{e'+e} m = {}^{e'+e}(r^{q'} m)$ . In particular, we see that  $r \cdot {}^e m = {}^e(r^q m)$  (here, we can think of  $r$  as  ${}^0 r$ ). Also, for any subset  $A$  of  $M$ , we denote  ${}^e A$  as the set  $\{{}^e a \mid a \in A\} \subseteq {}^e M$ . Furthermore, for any  $h \in \text{Hom}_R(M, N)$ , where  $N$  is also an  $R$ -module, there exists an induced map  ${}^e h : {}^e M \rightarrow {}^e N$  such that  ${}^e h({}^e m) = {}^e(h(m))$ . It is easy to see that  ${}^e h$  is  ${}^e R$ -linear (hence,  $R$ -linear, i.e.  ${}^e h \in \text{Hom}_R({}^e M, {}^e N)$ ).

If  $R$  is reduced, the induced  $R$ -module structure  ${}^e R$  is isomorphic to  $R^{1/q} = \{r^{1/q} \mid r \in R\}$ , which is naturally an  $R$ -module as  $R \subseteq R^{1/q}$ .

We use  $R^\circ$  to denote the complement of the union of all minimal primes of the ring  $R$ .

We say  $R$  is  $F$ -finite if  ${}^1R$  is a finitely generated  $R$ -module (or equivalently  ${}^eR$  is finitely generated  $R$ -module for every  $e \geq 0$ ). Another equivalent definition of  $R$  being  $F$ -finite is that  $R$  is finite over its subring  $R^p$  (or equivalently  $R$  is finite over its subring  $R^{p^e}$  for every  $e \geq 0$ ).

Given an  $R$ -module  $M$ , we say that  $M$  is  $F$ -finite if  ${}^1M$  is a finitely generated  $R$ -module. Notice that this implies  ${}^eM$  is a finitely generated  $R$ -module for every  $e \geq 0$ . Also, if  $R$  is  $F$ -finite and  $M$  is finitely generated  $R$ -module,  $M$  is  $F$ -finite.

For any  $R$ -module  $M$  and  $e$ , we can always form a new  $R$ -module  $M \otimes_R {}^eR$  by scalar extension via the Frobenius homomorphism, denoted by  $F_R^e(M)$ . If  $M$  is finitely generated, then  $F_R^e(M)$  is also finitely generated for any integer  $e \geq 0$ .

We say that a Noetherian ring is *excellent* if the following conditions are satisfied (see, for example, [Mat70, Chapter 13] and [Kun76] for more details):

1. If  $S$  is an  $R$ -algebra of finite type and  $P \subseteq Q$  are prime ideals of  $S$ , then each saturated chain of prime ideals connecting  $P$  and  $Q$  has the same length.
2. For each  $P \in \text{Spec}(R)$ , the canonical homomorphism  $R_P \rightarrow \widehat{R}_P$ , where  $\widehat{R}_P$  is the completion of  $R_P$  with respect to  $P_P$ , is regular.
3. For each  $R$ -algebra  $S$  of finite type, the regular locus  $\text{Reg}(S)$  is an open subset of  $\text{Spec}(S)$ .

In 1976, Ernst Kunz showed a very important result:

**Theorem 2.1.1** ([Kun76, Theorem 2.5]). *Let  $R$  be a Noetherian ring of characteristic  $p$ .*

*If  $R$  is  $F$ -finite, then  $R$  is excellent.*

We rely on this fact quite a bit since a lot of results require that  $R$  is excellent.

### 2.1.1 Tight Closure

A very important concept in studying rings of characteristic  $p$  is tight closure. Tight closure was first studied and developed by Hochster and Huneke in the 1980s. In this

subsection, we will give definitions on tight closure, test element, and test exponent as well as state some results on their existence.

**Definition 2.1.2** ([HH90b, Definition 8.2]). Let  $R$  be a Noetherian ring of prime characteristic  $p$  and  $N \subseteq M$  be  $R$ -modules. The tight closure of  $N$  in  $M$ , denoted by  $N_M^*$ , is defined as follows: An element  $x \in M$  is said to be in  $N_M^*$  if there exists a  $c \in R^\circ$  such that  $x \otimes c \in N_M^{[q]} \subseteq M \otimes_R {}^eR$  for all  $e \gg 0$ , where  $N_M^{[q]}$  denotes the  $R$ -submodule of  $F_R^e(M)$  generated by  $\{x \otimes 1 \in M \otimes_R {}^eR \mid x \in N\}$ . (By convention, we denote  $cx^q := x \otimes c \in N_M^{[q]} \subseteq M \otimes_R {}^eR$ .)

In the introduction (Chapter 1), we gave the definition on tight closure of an ideal  $I$ . Note that  $I^* = I_R^*$  using above definition. As we can see from Definition 2.1.2,  $c$  depends on  $x$ ,  $N$  and  $M$ . Definition 2.1.3 gives a  $c$  that is independent of  $x$ ,  $N$  and  $M$ .

**Definition 2.1.3** ([HH90a, Definition 8.11]). Let  $R$  be a Noetherian ring of prime characteristic  $p$ ,  $q_0 = p^{e_0}$ , and  $N \subseteq M$  be  $R$ -modules. We say  $c \in R^\circ$  is a  $q_0$ -weak test element for  $N \subseteq M$  if  $c(N_M^*)^{[q]} \subseteq N_M^{[q]}$  for all  $q \geq q_0$ . In case  $N = 0$ , we may simply call it a test element for  $M$ . By a  $q_0$ -weak test element, we simply mean a  $q_0$ -weak test element for all  $R$ -modules. If a  $q_0$ -weak test element  $c$  remains a  $q_0$ -weak test element under every localization, then we call  $c$  a locally stable  $q_0$ -weak test element. Furthermore, in case  $q_0 = 1$ , we simply call  $c$  a test element or locally stable test element. Finally, we call  $c$  a completely stable test element if  $c$  is a test element in every completion of every localization of  $R$ .

Hochster and Huneke were able to prove the existence of this nice  $c$ , i.e., test element under mild-condition. See below.

**Lemma 2.1.4** ([HH90b, Corollary 6.26]). *Let  $R$  be a reduced algebra of finite type over an excellent local ring. Then, there exists a  $c \in R^\circ$  such that  $c$  is a locally stable test element for any  $R$ -module.*

*Remark 2.1.5.* If  $R$  is not reduced in Lemma 2.1.4, there exists a  $c \in R^\circ$  such that  $c$  is a locally stable  $q_0$ -weak test element.

**Lemma 2.1.6** ([HH90a, §6], [Sch11, Proposition 3.21]). *Let  $R$  be a reduced  $F$ -finite Noetherian ring of prime characteristic  $p$ . Then, there exists a  $c \in R^\circ$  such that for all  $d \in R^\circ$ , there exists an  $e > 0$ , and there exists an  $f \in \text{Hom}_R({}^eR, R)$  that maps  ${}^e d$  to  $c$ . Moreover, such a  $c$  is a completely stable test element for all  $R$ -modules (not necessarily finitely generated).*

*Remark 2.1.7.* Note that the element  $c$  in Lemma 2.1.6 exists in more general situations. For more details, see [HH90a, §6].

Hochster and Huneke introduced the notion of *test exponent* for tight closure. Loosely speaking, test exponents exist if and only if tight closure commutes with localization [HH02]. We can see how test exponents could be of great significance.

**Definition 2.1.8** ([HH02, Definition 2.2]). Let  $R$  be a Noetherian ring of prime characteristic  $p$ ,  $c \in R$ , and  $N \subseteq M$  (finitely generated)  $R$ -modules. We say that  $Q = p^E$  is a test exponent for  $c$  and  $N \subseteq M$  (over  $R$ ) if, for any  $x \in M$ , the occurrence of  $cx^q \in N_M^{[q]}$  for one single  $q \geq Q$  implies  $x \in N_M^*$ . In case  $N = 0$ , we may simply call it a test exponent for  $c$  and  $M$ .

Next, we define the Frobenius closure, which was used to prove the existence of test exponents for Artinian module in [HH02].

**Definition 2.1.9.** Let  $R$  be a Noetherian ring of prime characteristic  $p$  and  $N \subseteq M$  be  $R$ -modules. The Frobenius closure of  $N$  in  $M$ , denoted by  $N_M^F$ , is defined as follows: An element  $x \in M$  is said to be in  $N_M^F$  if  $\exists q$  such that  $x^q \in N_M^{[q]}$ .

**Proposition 2.1.10** ([HH02, Proposition 2.6]). *Let  $R$  be a Noetherian ring of prime characteristic  $p$  and  $N \subseteq M$  be  $R$ -modules such that  $M/N$  is Artinian. Assume there exists a  $d \in R^\circ$  that is a  $q_0$ -weak test element for  $N_M^{[q]} \subseteq F_R^e(M)$ , for all  $q \gg 0$ . Then, for any  $c \in R^\circ$ , there exists a test exponent for  $c$  and  $N \subseteq M$ .*

*Proof.* For every  $e \in \mathbb{N}$ , let  $N_e = \left\{ x \in M \mid cx^q \in (N_M^{[q]})_{F_R^e(M)}^F \right\}$ . Let  $x \in N_{e+1}$ . This means that  $cx^{qp} \in (N_M^{[qp]})_{F_R^e(M)}^F$ . Then,  $c^p x^{qp} \in (N_M^{[qp]})_{F_R^e(M)}^F$ . Thus, there exists a  $q'$  such that  $(c^p x^{qp})^{q'} \in N_M^{[qp]^{[q'()]}}$ . In other words,  $(cx^q)^{pq'} \in N_M^{[q]^{[pq'()]}}$ . Thus,  $x \in N_e$ . Since  $e$  was arbitrary,



we have the following descending chain  $N \supseteq N_1 \supseteq N_2 \supseteq \cdots \supseteq N_e \supseteq N_{e+1} \cdots$  and hence, there exists a  $Q = p^E$  such that  $N_e = N_E$  for all  $e \geq E$ .

Suppose  $q_1 \geq Q$  and  $cx^{q_1} \in N_M^{[q_1]}$ . Then,  $x \in N_{e_1}$ , which implies that  $x \in N_e$  for all  $e \geq E$ . This means  $cx^q \in (N_M^{[q]})_{F_R^e(M)}^F \subseteq (N_M^{[q]})_{F_R^e(M)}^*$  for all  $q \geq Q$ . Hence, there exists a  $d \in R^\circ$  such that  $d(cx^q)^{q_0} \in (N_M^{[q]})_{F_R^e(M)}^{[q_0]} = N_M^{[qq_0]}$  for all  $q \geq Q$ , which implies  $x \in N_M^*$ . Therefore,  $Q$  is a test exponent for  $c$  and  $N \subseteq M$ .  $\square$

## 2.2 Matlis Duality

Now, we switch gear to some homological algebra to talk about Matlis Duality, which was introduced by Matlis in the 1958. Let  $M, N, L$  be  $R$ -modules. We say that a sequence

$$M \xrightarrow{f} N \xrightarrow{g} L$$

is exact (at  $N$ ) if  $\ker(g) = \operatorname{im}(f)$ . Furthermore, we say the sequence

$$0 \longrightarrow M \longrightarrow N \longrightarrow L \longrightarrow 0$$

is a short exact sequence if it is exact everywhere, i.e., at  $M, N$ , and  $L$ .

In 1940, Baer introduced injective modules (see Definition 2.2.2), which leads to a module called the injective hull (see Definition 2.2.4).

**Proposition 2.2.1** ([Hum04, Proposition 2.1]). *Let  $R$  be a ring and  $E$  an  $R$ -module. Then the following assertions are equivalent:*

1. (Baer's Criterion) *Let  $I$  be an ideal in  $R$ . Every homomorphism from  $I$  to  $E$  extends to a homomorphism from  $R$  to  $E$ .*
2.  $\operatorname{Hom}_R(-, E)$  *preserves short exact sequences*
3. *For all  $R$ -module homomorphisms  $\phi : M \rightarrow N$  and  $\psi : M \rightarrow E$  where  $\phi$  is injective, there exists an  $R$ -module homomorphism  $\theta : N \rightarrow E$  such that  $\theta \circ \phi = \psi$ .*

**Definition 2.2.2** ([Hun04, Proposition 2.1]). An  $R$ -module  $E$  satisfies any (or all) of the above equivalent conditions is called an injective  $R$ -module.

**Definition 2.2.3** ([Hun04, Definition 2.2]). If  $N \subseteq M$  are  $R$ -modules, then  $M$  is said to be essential over  $N$  if every non-zero submodule  $T$  of  $M$  has a non-zero intersection with  $N$ .

**Definition 2.2.4** ([Hun04, Definition 2.4]). An injective  $R$ -module  $E$  that is an essential extension of an  $R$ -module  $M$  is called an injective hull of  $M$  and is denoted by  $E_R(M)$ .

Now, we are ready to define the Matlis dual of an  $R$ -module and show an important theorem, namely Matlis Duality. This theorem gives a one-to-one correspondence between finitely generated modules over the completion of a Noetherian ring  $R$  and Artinian modules over  $R$ .

**Definition 2.2.5** ([Hun04, Definition 3.4]). Let  $(R, \mathfrak{m}, k)$  be a local ring. The Matlis dual of an  $R$ -module  $M$  is the module  $M^\vee := \text{Hom}_R(M, E_R(k))$ .

**Theorem 2.2.6** ([Hun04, Theorem 3.5 (Matlis Duality)]). *Let  $(R, \mathfrak{m}, k)$  be a complete local ring and  $E = E_R(k)$  be the injective hull of  $k$ . Then, there is a 1-1 correspondence between finitely generated  $R$ -modules and Artinian  $R$ -modules. This correspondence is given as follows: If  $M$  is a finitely generated  $R$ -module, then  $M^\vee = \text{Hom}_R(M, E)$  is Artinian. If  $T$  is an Artinian  $R$ -module,  $T^\vee = \text{Hom}_R(T, E)$  is finitely generated over  $R$ . Moreover, if  $N$  is a finitely generated or Artinian  $R$ -module,  $N^{\vee\vee} \cong N$ . Also,  $E^\vee \cong R$  and  $R^\vee \cong E$ .*

The following lemma is a well-known fact involving Matlis Dual.

**Lemma 2.2.7.** *Let  $(R, \mathfrak{m}, k)$  be a local complete ring and let  $E = E_R(k)$ . Let  $H$  be Artinian or Noetherian  $R$ -module,  $A \leq H$  and  $T \leq \text{Hom}_R(H, E)$ . Then*

$$T^\vee \cong H/A \iff A = \{x \mid h(x) = 0, \forall h \in T\}.$$

For next theorem, given rings  $R$  and  $S$ ,  ${}_R M_S$  means that  $M$  is an  $R$ -left module and also an  $S$ -right module.

**Theorem 2.2.8** ([Rot09, Theorem 2.75 (Tensor-Hom Adjunction)]). *Given modules  $A_R$ ,  ${}_R B_S$ , and  $C_S$ , where  $R$  and  $S$  are rings, there is a natural isomorphism:*

$$\tau_{A,B,C} : \text{Hom}_S(A \otimes_R B, C) \rightarrow \text{Hom}_R(A, \text{Hom}_S(B, C)),$$

*such that for  $f : A \otimes_R B \rightarrow C$ ,  $a \in A$ , and  $b \in B$ ,*

$$\tau_{A,B,C}(f)(a)(b) = f(a \otimes b).$$

### 2.2.1 Local Cohomology

Local Cohomology was introduced by Grothendieck in the early 1960s. Other than the construction below, there are some other important ways to define local cohomology. For example, Gamma functor and Koszul Complex (see [Hun04, §2] for more details on these two constructions).

**Theorem 2.2.9** ([Rot09, pages 60–62]). *For a short exact sequence of  $R$ -modules*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

*and an  $R$ -module  $N$ , the sequence of  $R$ -modules*

$$0 \longrightarrow \text{Hom}_R(M'', N) \longrightarrow \text{Hom}_R(M, N) \longrightarrow \text{Hom}_R(M', N)$$

*is exact (we say that  $\text{Hom}_R(-, N)$  is left exact), but the right-most arrow need not be an epimorphism. Similarly, for a short exact sequence of  $R$ -modules*

$$0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0$$

and an  $R$ -module  $M$ , the sequence of  $R$ -modules

$$0 \longrightarrow \operatorname{Hom}_R(M, N') \longrightarrow \operatorname{Hom}_R(M, N) \longrightarrow \operatorname{Hom}_R(M, N'')$$

is exact (we say that  $\operatorname{Hom}_R(M, -)$  is left exact), but the right-most arrow need not be an epimorphism.

$\operatorname{Ext}$  measures the deviation from exactness. The following theorem gives some of its basic properties.

**Theorem 2.2.10** ([Wei94, Theorem 13.1, Theorem 25.2]). *Given short exact sequences of  $R$ -modules,*

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow N' \longrightarrow N \longrightarrow N'' \longrightarrow 0,$$

the  $\operatorname{Ext}$  functors satisfy the following properties:

1.  $\operatorname{Ext}_R^0(M, N)$  is naturally isomorphic to  $\operatorname{Hom}_R(M, N)$ .

2. The following sequences are exact:

$$\begin{aligned} \cdots \longrightarrow \operatorname{Ext}_R^n(M'', N) \longrightarrow \operatorname{Ext}_R^n(M, N) \longrightarrow \operatorname{Ext}_R^n(M', N) \longrightarrow \operatorname{Ext}_R^{n+1}(M'', N) \longrightarrow \cdots \\ \cdots \longrightarrow \operatorname{Ext}_R^n(M, N') \longrightarrow \operatorname{Ext}_R^n(M, N) \longrightarrow \operatorname{Ext}_R^n(M, N'') \longrightarrow \operatorname{Ext}_R^{n+1}(M, N') \longrightarrow \cdots \end{aligned}$$

Now that we have some background on  $\operatorname{Ext}$  functors, we can define cohomology module. Let  $R$  be a Noetherian ring,  $M$  be an  $R$ -module, and  $I \supseteq J$  be ideals of  $R$ . Notice that the surjection  $R/J \rightarrow R/I$  induces a map  $\operatorname{Ext}^i(R/I, M) \rightarrow \operatorname{Ext}^i(R/J, M)$ . Thus, a decreasing chain of ideals

$$I_1 \supseteq I_2 \supseteq \cdots \supseteq I_t \supseteq I_{t+1} \supseteq \cdots$$

gives us a direct limit system

$$\cdots \rightarrow \operatorname{Ext}_R^i(R/I_t, M) \rightarrow \operatorname{Ext}_R^i(R/I_{t+1}, M) \rightarrow \cdots$$

and we may form the direct limit of these Ext's.

**Definition 2.2.11** ([Hun04, §2.1]). We call the module,  $\varinjlim \text{Ext}^i(R/I^t, M)$ , the  $i$ th local cohomology module of  $M$  with support in  $I$  and denoted by  $H_I^i(M)$ .

The following is a consequence from Grothendieck local duality theorem, which roughly states that the local cohomology and Ext functors are Matlis duals of each other under the assumption that  $R$  is complete.

**Proposition 2.2.12** ([BH98, Proposition 3.5.4]). *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring and  $M$  a finitely generated  $R$ -module. The modules  $H_{\mathfrak{m}}^i(M)$  are Artinian.*

The fact that  $H_{\mathfrak{m}}^i(M)$  are Artinian allows us to show the existence of test exponent for the highest cohomology module, i.e.,  $H_{\mathfrak{m}}^d(R)$ .

**Corollary 2.2.13.** *Let  $(R, \mathfrak{m})$  be a Noetherian local ring of prime characteristic  $p$  and dimension  $d$ . Assume there exists a  $d \in R^\circ$  that is a  $q_0$ -weak test element for  $H_{\mathfrak{m}}^d(R)$ . Then, for any  $c \in R^\circ$ , there exists a test exponent for  $c$  and  $H_{\mathfrak{m}}^d(R)$ .*

*Proof.* By Proposition 2.2.12,  $H_{\mathfrak{m}}^d(R)$  is Artinian since  $R$  is finitely generated over itself. Notice that  $F_R^e(H_{\mathfrak{m}}^d(R)) = H_{\mathfrak{m}}^d(R)$  for all  $q \gg 1$ . Then, we simply let  $N = 0$  and  $M = H_{\mathfrak{m}}^d(R)$  in Proposition 2.1.10 and the statement follows.  $\square$

### 2.3 Cohen-Macaulay rings and modules

In this section, we will provide one of many definitions of Cohen-Macaulay ring and module. Cohen-Macaulay rings and modules play a huge role in commutative algebra, algebraic geometry, invariant theory, etc. Mel Hochster wrote ‘Life is really worth living in a Noetherian ring  $R$  when all the local rings have the property that every s.o.p. is an  $R$ -sequence’ in one of his papers. The Cohen-Macaulay ring is such a ring with this property, and it is named after Francis Sowerby Macaulay and Irvin Cohen as they proved that all such rings have the unmixedness property (see [Mat87, pages 136-137] for more details).

First we need to recall the definition of a regular element and a regular sequence. Let  $M$  be a module over a ring  $R$ . We say that  $x \in R$  is an  $M$ -regular element if  $x$  is not a zero-divisor on  $M$ , i.e.,  $xy = 0$  for  $y \in M$  implies  $y = 0$ . Then, we say that a sequence  $x_1, \dots, x_n \in R$  is called an  $M$ -regular sequence if  $x_i$  is an  $M/(x_1, \dots, x_{i-1})M$ -regular element for all  $i = 1, \dots, n$  and  $M/(x_1, \dots, x_n)M \neq 0$ . It is known that all maximal  $M$ -regular sequences in an ideal  $I$  with  $IM \neq M$  have the same length if  $M$  is finitely generated over  $R$  (see [BH98, §1.2]). Now, we are ready to define Cohen-Macaulay ring and module.

**Definition 2.3.1** ([BH98, Definition 2.1.1]). Let  $(R, \mathfrak{m})$  be a Noetherian local ring. A finitely generated  $R$ -module  $M \neq 0$  is a Cohen-Macaulay module if  $\text{depth } M = \dim M$ . If  $R$  itself is a Cohen-Macaulay module, then it is called a Cohen-Macaulay ring. A maximal Cohen-Macaulay module is a Cohen-Macaulay module  $M$  such that  $\dim M = \dim R$ . In general, if  $R$  is an arbitrary Noetherian ring, then  $M$  is a Cohen-Macaulay module if  $M_{\mathfrak{m}}$  is a Cohen-Macaulay module for all maximal ideals  $\mathfrak{m} \in \text{Supp } M$ . (We consider the zero module to be Cohen-Macaulay.) However, for  $M$  to be a maximal Cohen-Macaulay module, we require that  $M_{\mathfrak{m}}$  is such an  $R_{\mathfrak{m}}$ -module for each maximal ideal of  $\mathfrak{m}$  of  $R$ . As in the local case,  $R$  is a Cohen-Macaulay ring if it is a Cohen-Macaulay module.

By  $\text{depth } M$ , we mean  $\text{depth}(\mathfrak{m}, M)$ , which is the length of any maximal regular sequence on  $M$  in  $\mathfrak{m}$ . Also,  $\text{Supp } M$  denotes the set  $\{P \in \text{Spec}(R) \mid M_P \neq 0\}$ .

### 2.3.1 Canonical module

The canonical module was introduced by Grothendieck in connection with the local duality theorem that relates local cohomology with certain Ext functors. We will look at some interesting properties of canonical modules.

**Definition 2.3.2.** Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring of dimension  $d$  and  $M$  be a finitely generated  $R$ -module.  $M$  is said to be a canonical module for  $R$  if  $M^\vee = \text{Hom}_R(M, E_R(k)) \cong H_{\mathfrak{m}}^d(R)$ . The canonical module is usually denoted by  $\omega_R$  or  $\omega$  if the underlying ring is obvious. In general, let  $R$  be a Noetherian ring. We say that a finitely generated  $R$ -module  $\omega_R$  is a canonical module of  $R$  if  $(\omega_R)_P$  is a canonical module of  $R_P$  for prime ideals  $P$  of  $R$ .

It turns out that canonical modules are very nice when the ring is a local Cohen-Macaulay ring. The following theorem shows that in a Cohen-Macaulay local ring, canonical modules are unique up to isomorphism if they exist and that they commute with localization.

**Theorem 2.3.3** ([BH98, Theorem 3.3.4, Theorem 3.3.5]). *Let  $(R, \mathfrak{m}, k)$  be a Cohen-Macaulay local ring with canonical modules,  $\omega$  and  $\omega'$ . Then,*

1.  $\omega \cong \omega'$ , that is, canonical modules are unique up to isomorphism.
2.  $\text{Hom}_R(\omega, \omega') \cong R$ , and any generator  $\phi \in \text{Hom}_R(\omega, \omega')$  is an isomorphism.
3.  $(\omega_R)_P = \omega_{R_P}$ , for all  $P \in \text{Spec}(R)$ , that is, the canonical module localizes.

*Remark 2.3.4.* In particular,  $\text{Hom}_R(\omega, \omega) \cong R$ .

Another nice property of canonical module when  $R$  is local Cohen-Macaulay is that it is a faithful module, i.e.,  $\text{Ann}_R(\omega) = 0$ .

**Corollary 2.3.5.** *Let  $(R, \mathfrak{m}, k)$  be a Cohen-Macaulay local ring with canonical module,  $\omega$ . Then,  $\omega$  is a faithful  $R$ -module.*

*Proof.* Since  $1_R \in R$  can only be annihilated by 0,  $R$  is a faithful  $R$ -module. By Remark 2.3.4, we can conclude that  $\text{Hom}_R(\omega, \omega)$  is faithful  $R$ -module. Hence, it is easy to see that  $\omega$  is a faithful  $R$ -module. (If  $\omega$  is not a faithful  $R$ -module, then  $\text{Hom}_R(\omega, \omega)$  cannot be faithful  $R$ -module.) □

## 2.4 Completion, Flatness, and Localization

In this section, we mainly discuss how completion, flatness, and localization commutes with  $\text{Hom}_R(-, -)$ , when we are dealing with finitely presented modules, which is equivalent to finitely generated modules over Noetherian ring. We start with defining flat module and flat map.

**Definition 2.4.1** ([Mat87, page 45]). Let  $R$  be a ring and  $M$  an  $R$ -module. We say that  $M$  is flat over  $R$  if for all  $R$ -modules,  $N, L$ , and all injective maps  $\psi : N \rightarrow L$ ,  $0 \rightarrow N \otimes M \xrightarrow{\psi \otimes 1} L \otimes M$  is exact.

**Definition 2.4.2** ([Mat87, page 46]). Let  $R, S$  be rings. A ring map  $h : R \rightarrow S$  is called flat if  $S$  is flat over  $R$  as an  $R$ -module via  $h$ .

Using techniques from homological algebra, one can obtain the following result, which essentially says that flat commutes with  $\text{Hom}_R(-, -)$  under mild conditions.

**Theorem 2.4.3** ([Rot09, Lemma 4.85, Lemma 4.86]). *Let  $M, N$  be  $R$ -modules such that  $M$  is a finitely presented over  $R$ . If  $S$  is a flat  $R$ -algebra. then there exists a canonical isomorphism between  $S \otimes_R \text{Hom}_R(M, N)$  and  $\text{Hom}_S(S \otimes_R M, S \otimes_R N)$ , i.e.,*

$$S \otimes_R \text{Hom}_R(M, N) \cong \text{Hom}_S(S \otimes_R M, S \otimes_R N).$$

We now look at some facts about completion. We omit the details on the construction of completion and we refer the readers to [Mat87, §8]. By  $\widehat{\phantom{x}}$ , we mean the completion with respect to the maximal ideal, unless stated otherwise. It is shown that completion is a functor and that completion is a flat map. These facts lead to the commutativity between completion and  $\text{Hom}_R(-, -)$  (see Corollary 2.4.5 and Corollary 2.4.6).

**Lemma 2.4.4** ([Mat87, Theorem 8.7, Theorem 8.8]). *Let  $I$  be an ideal of a Noetherian ring  $R$ . Denote by  $\widehat{\phantom{x}}$  the completion with respect to  $I$ .*

1. *If  $M$  is a finitely generated  $R$ -module, then  $\widehat{M} \cong M \otimes_R \widehat{R}$ .*
2. *The natural map  $h : R \rightarrow \widehat{R}$  is flat, i.e.,  $\widehat{R}$  is flat over  $R$ .*

**Corollary 2.4.5** (Completion commutes with Hom). *Let  $R$  be a Noetherian ring and  $M, N$  be finitely generated  $R$ -modules. Also, let  $I$  and  $\widehat{\phantom{x}}$  be as Lemma 2.4.4. Then, we have the following isomorphisms*

$$\begin{aligned} \text{Hom}_R(\widehat{M}, N) &\cong \widehat{R} \otimes_R \text{Hom}_R(M, N) \\ &\cong \text{Hom}_{\widehat{R}}(\widehat{R} \otimes_R M, \widehat{R} \otimes_R N) \\ &\cong \text{Hom}_{\widehat{R}}(\widehat{M}, \widehat{N}). \end{aligned}$$



*Proof.* The first and third isomorphisms are due to Lemma 2.4.4. Now, we note that  $M$  is finitely presented since  $M$  is finitely generated over a Noetherian ring. Thus, applying Theorem 2.4.3 gives us the second isomorphism.  $\square$

**Corollary 2.4.6.** *Let  $R$  be a Noetherian ring,  $N$  be finitely generated  $R$ -module and  $M$  be  $\widehat{R}$ -module (e.g.  $M$  is Artinian). Then  $N \otimes_R M \cong \widehat{N} \otimes_{\widehat{R}} M$ .*

*Proof.* Using Lemma 2.4.4, we have that

$$\begin{aligned} \widehat{N} \otimes_{\widehat{R}} M &\cong (N \otimes_R \widehat{R}) \otimes_{\widehat{R}} M \\ &\cong N \otimes_R (\widehat{R} \otimes_{\widehat{R}} M) \\ &\cong N \otimes_R M. \end{aligned}$$

This completes the proof.  $\square$

Finally, we show that localization commutes with  $\text{Hom}_R(-, -)$  using Theorem 2.4.3.

**Lemma 2.4.7** ([Mat87, Theorem 4.4]). *Let  $R$  be a ring and  $S$  be a multiplicative set in  $R$ . Then,*

1. *For any  $R$ -module  $M$ ,  $S^{-1}M \cong S^{-1}R \otimes_R M$ .*
2. *The natural map  $h : R \rightarrow S^{-1}R$  is flat, i.e.,  $S^{-1}R$  is flat over  $R$ .*

**Corollary 2.4.8** (Localization commutes with Hom). *Let  $R$  be a Noetherian ring,  $M, N$  be finitely generated  $R$ -modules and  $S$  be a multiplicative set in  $R$ . Then, we have the following isomorphisms*

$$\begin{aligned} S^{-1} \text{Hom}_R(M, N) &\cong S^{-1}R \otimes_R \text{Hom}_R(M, N) \\ &\cong \text{Hom}_{S^{-1}R}(S^{-1}R \otimes_R M, S^{-1}R \otimes_R N) \\ &\cong \text{Hom}_{S^{-1}R}(S^{-1}M, S^{-1}N). \end{aligned}$$

*Proof.* Proof is analogous to the proof of Corollary 2.4.5.  $\square$

## CHAPTER 3

### PREPARATORY RESULTS

In this chapter, we first introduce a notation and discuss its different properties. Then, we provide some known results along with some new improved results of our own.

#### 3.1 A notation

**Definition 3.1.1.** Let  $R$  be a ring and  $M, N$  be  $R$ -modules. Let  $K \subseteq N$ . We define  $T_{N,M}^R(K) = \{h(k) \mid h \in \text{Hom}_R(N, M), k \in K\}$ . If the underlying ring is obvious, we will simply write  $T_{N,M}(K)$  instead of  $T_{N,M}^R(K)$ .

**Proposition 3.1.2.** Let  $R$  be a ring and  $M, N, K, L$  be  $R$ -modules. Let  $A \subseteq B \subseteq N$ ,  $n \in N$  and  $k \in K$ . Then, the following hold:

- (1)  $T_{N,M}(n)$  is a submodule of  $N$ .
- (2)  $\bigcup_{a \in A} T_{N,M}(a) = T_{N,M}(A)$ .
- (3) If  $T_{N,M}(n) \supseteq C$ , then  $T_{N,M}(n) \supseteq \langle C \rangle$ , where  $\langle C \rangle$  denotes the module generated by  $C$ .
- (4)  $T_{N,M}(A) \subseteq T_{N,M}(B)$ .
- (5) If there exists an  $h \in \text{Hom}_R(K, N)$  such that  $h(k) = n$ ,  $T_{N,M}(n) \subseteq T_{K,M}(k)$ .
- (6) If  $T_{N,M}(A) \supseteq C$  and  $T_{M,L}(C) \supseteq D$ , then  $T_{N,L}(A) \supseteq D$ .
- (7) If  $T_{N,M}(n) \supseteq C$ , then  $\bigcup_{r \in R} T_{N,M}(rn) \supseteq \bigcup_{r \in R} rC$ .
- (8)  $T_{N,M}(A)$  is annihilated by  $\text{Ann}_R(N)$ .

*Proof.* (1) First, we notice that  $T_{N,M}(n)$  is non-empty since  $\text{Hom}_R(N, M)$  is non-empty (the zero map). Let  $m_1, m_2 \in T_{N,M}(n)$ . Then, there exist  $h_1, h_2 \in \text{Hom}_R(N, M)$  such that

$h_1(n) = m_1$  and  $h_2(n) = m_2$ . Thus,  $m_1 - m_2 = h_1(n) - h_2(n) = (h_1 - h_2)(n) \in T_{N,M}(n)$  because  $h_1 - h_2 \in \text{Hom}_R(N, M)$ . Also, it is clear that  $rm_1 = rh_1(n) = (rh_1)(n) \in T_{N,M}(n)$ , for  $r \in R$ . Hence,  $T_{N,M}(n)$  is a submodule of  $N$ .

- (2) It just follows directly from the Definition 3.1.1.
- (3) By (1), we know that  $T_{N,M}(n)$  is a module that contains  $C$  and  $\langle C \rangle$  is the smallest module containing  $C$ . Hence, it is clear that  $T_{N,M}(n) \supseteq \langle C \rangle$ .
- (4) Let  $m \in T_{N,M}(A)$ . Then, there exists  $h \in \text{Hom}_R(N, M)$  such that  $h(a) = m$  for some  $a \in A$ . Since  $A \subseteq B$ , we have that  $a \in B$ . Thus,  $m = h(a)$  for some  $a \in B$ , which implies that  $m \in T_{N,M}(B)$ . Hence,  $T_{N,M}(A) \subseteq T_{N,M}(B)$ .
- (5) Assume there exists an  $h \in \text{Hom}_R(K, N)$  such that  $h(k) = n$ . Let  $m \in T_{N,M}(n)$  so there exists an  $f \in \text{Hom}_R(N, M)$  such that  $f(n) = m$ . Thus,  $(f \circ h)(k) = m$ , where  $f \circ h \in \text{Hom}_R(K, M)$ , which implies that  $m \in T_{K,M}(k)$ . Hence,  $T_{N,M}(n) \subseteq T_{K,M}(k)$ .
- (6) Assume that  $T_{N,M}(A) \supseteq C$  and  $T_{M,L}(C) \supseteq D$ . Let  $d \in D$ . Thus, there exists an  $h \in \text{Hom}_R(M, L)$  such that  $h(c) = d$  for some  $c \in C$ . For this  $c$ , there exists an  $f \in \text{Hom}_R(N, M)$  such that  $f(a) = c$  for some  $a \in A$ . Then, we have that  $(h \circ f)(a) = d$ , where  $h \circ f \in \text{Hom}_R(N, L)$ , for some  $a \in A$ . Thus,  $d \in T_{N,L}(A)$ . Hence,  $D \subseteq T_{N,L}(A)$ .
- (7) Let  $T_{N,M}(n) \supseteq C$ . It is enough to show that  $T_{N,M}(rn) \supseteq rC$  for some  $r \in R$ . Let  $x \in rC$ . Write  $x = rc$  for some  $c \in C$ . Since  $c \in C \subseteq T_{N,M}(n)$ , there exists an  $h \in \text{Hom}_R(N, M)$  such that  $h(n) = c$ . Thus,  $x = rh(n) = h(rn)$ , which implies that  $x \in T_{N,M}(rn)$ . Hence,  $\bigcup_{r \in R} T_{N,M}(rn) \supseteq \bigcup_{r \in R} rC$ .
- (8) In general, it is easy to see that  $\text{Hom}_R(N, M)$  is annihilated by either  $\text{Ann}_R(N)$  or  $\text{Ann}_R(M)$ . Thus, by Definition 3.1.1, the statement holds trivially.

□

Since  $T_{-, -}^R(-)$  is like  $\text{Hom}_R(-, -)$ , we expect it to behave like  $\text{Hom}_R(-, -)$  and it does!

We first show that flat commutes with  $T_{-, -}^R(-)$ .

**Proposition 3.1.3.** *Let  $R$  be a Noetherian ring,  $M, N$  be finitely generated  $R$ -modules and let  $n \in N$ . If  $\phi: R \rightarrow S$  is a flat homomorphism, i.e.,  $S$  is flat over  $R$ , then*

$$S \otimes_R T_{N,M}(n) = T_{S \otimes_R N, S \otimes_R M}(1 \otimes_R n)$$

*Proof.* Let  $x \in S \otimes_R T_{N,M}(n)$ . Thus,  $x = \sum_i s_i \otimes h_i(n)$ , for  $s_i \in S$  and  $h_i \in \text{Hom}_R(N, M)$ . Then, there exist an element

$$g = \sum_i s_i \otimes h_i \in S \otimes_R \text{Hom}_R(N, M) \cong \text{Hom}_S(S \otimes_R N, S \otimes_R M)$$

such that  $x = \sum_i (s_i \otimes h_i)(n)$ . Note that  $g$  is simply induced by  $h$  through  $\phi$ . Hence,  $x \in T_{S \otimes_R N, S \otimes_R M}(1 \otimes_R n)$ . Now, we let  $x \in T_{S \otimes_R N, S \otimes_R M}(1 \otimes_R n)$ . Then,

$$x = g(1 \otimes_R n), \text{ for some } g \in \text{Hom}_S(S \otimes_R N, S \otimes_R M) \cong S \otimes_R \text{Hom}_R(N, M).$$

Thus, we have that

$$x = \left( \sum_i s_i \otimes h_i \right) (1 \otimes_R n), \text{ for } s_i \in S \text{ and } h_i \in \text{Hom}_R(N, M).$$

Therefore, we get that  $x = \sum_i s_i \otimes h_i(n) \in S \otimes_R T_{N,M}(n)$ . This completes the proof.  $\square$

Now, with Proposition 3.1.3, we can easily show that completion commutes with  $T_{-, -}^R(-)$  similar to completion commutes with  $\text{Hom}_R(-, -)$  (see Corollary 2.4.5).

**Corollary 3.1.4.** *Let  $R$  be a Noetherian ring,  $M, N$  be finitely generated  $R$ -modules and  $I \leq R$ . Let  $A \subseteq N$  and let  $\widehat{\phantom{x}}$  denotes the completion with respect to  $I$ . Then,*

$$\widehat{T_{N,M}(n)} = T_{\widehat{N}, \widehat{M}}^{\widehat{R}}(\widehat{n}), \text{ where } \widehat{n} \text{ denotes the natural image of } n \in N.$$

*Proof.* By Lemma 2.4.4,  $\widehat{R}$  is flat over  $R$  and

$$T_{N,M}(\widehat{n}) \cong \widehat{R} \otimes_R T_{N,M}(n) \quad \text{and} \quad T_{\widehat{N},\widehat{M}}^{\widehat{R}}(\widehat{n}) \cong T_{\widehat{R} \otimes_R N, \widehat{R} \otimes_R M}^{\widehat{R}}(1 \otimes_R n).$$

Thus, the result follows from Proposition 3.1.3.  $\square$

Also, we can show that localization commutes with  $T_{-,-}^R(-)$  with ease, which is similar to showing that localization commutes with  $\text{Hom}_R(-, -)$ .

**Corollary 3.1.5.** *Let  $R$  be a Noetherian ring,  $M, N$  be finitely generated  $R$ -modules. Let  $A \subseteq N$  and  $S$  be a multiplicative closed set in  $R$ .*

$$S^{-1}(T_{N,M}(n)) = T_{S^{-1}N, S^{-1}M}(n/1).$$

*Proof.* By Lemma 2.4.4,  $S^{-1}R$  is flat over  $R$  and

$$S^{-1}T_{N,M}(n) \cong S^{-1}R \otimes_R T_{N,M}(n) \quad \text{and} \quad T_{S^{-1}N, S^{-1}M}(S^{-1}n) \cong T_{S^{-1}R \otimes_R N, S^{-1}R \otimes_R M}(1 \otimes_R n).$$

Thus, the result follows from Proposition 3.1.3.  $\square$

Frobenius works well with  $\text{Hom}_R(-, -)$  and hence, we can see that it works well with  $T_{-,-}^R(-)$  as expected.

**Lemma 3.1.6.** *Let  $R$  be a ring of characteristics  $p$  and  $M, N$  be  $R$ -modules. Let  $A \subseteq N$ . Then, for any  $e \in \mathbb{N}$ ,*

$$T_{N,M}(A) \supseteq B \implies T_{eN, eM}({}^eA) \supseteq {}^eB.$$

*Proof.* Assume that  $T_{N,M}(A) \supseteq B$ . It is very easy to see that for every  $h \in \text{Hom}_R(N, M)$ , there exists an  ${}^eh \in \text{Hom}_R({}^eN, {}^eM)$ . Hence,  ${}^eB = B \subseteq T_{N,M}(A) \subseteq T_{eN, eM}({}^eA)$  as sets.  $\square$

Recall that  ${}^eh$  above denotes the homomorphism induced by  $h \in \text{Hom}_R(N, M)$  after we “take frobenius power.” (See page 4.)

Here are a few more nice properties of  $\text{Hom}_R(-, -)$  that are needed later on.

**Proposition 3.1.7** ([DF04, Proposition 29 in § 10.5, Exercise 10 in Chapter 10]). *Let  $R$  be a ring and let  $M, N, L$  be  $R$ -modules. Then,*

1.  $\text{Hom}_R(M, N \oplus L) \cong \text{Hom}_R(M, N) \oplus \text{Hom}_R(M, L)$ .
2.  $\text{Hom}_R(N \oplus L, M) \cong \text{Hom}_R(N, M) \oplus \text{Hom}_R(L, M)$ .
3.  $\text{Hom}_R(R, M) \cong M$ .

Here, we look into how Matlis Dual interacts with  $T_{-, -}^R(-)$ . The first lemma below gives an interesting isomorphism.

**Lemma 3.1.8.** *Let  $(R, \mathfrak{m}, k)$  be a local ring and  $M, N$  be finitely generated  $R$ -modules. Let  $H = M^\vee = \text{Hom}_R(M, E_R(k))$ . Let  $c \in N$ . Then,*

$$(T_{N, M}(c))^\vee \cong H / \text{Ann}_H(c),$$

where  $\text{Ann}_H(c) = \{x \in H \mid c \otimes x = 0 \in N \otimes_R H\}$ .

*Proof.* First, we notice the following isomorphisms

$$\begin{aligned} \text{Hom}_{\widehat{R}}(\widehat{T}, E_R(k)) &\cong \text{Hom}_{\widehat{R}}(T \otimes_R \widehat{R}, E_R(k)) && \text{(Lemma 2.4.4)} \\ &\cong \text{Hom}_R(T, \text{Hom}_{\widehat{R}}(\widehat{R}, E_R(k))) && \text{(Theorem 2.2.8)} \\ &\cong \text{Hom}_R(T, E_R(k)) && \text{(Proposition 3.1.7)} \end{aligned}$$

$$\begin{aligned} (T_{N, M}(c))^\vee &\cong (\widehat{T_{N, M}(c)})^\vee && \text{(Corollary 2.4.5)} \\ &\cong H / \{x \in H \mid c \otimes x = 0 \in \widehat{N} \otimes_{\widehat{R}} H\} && \text{(Lemma 2.2.7)} \\ &\cong H / \{x \in H \mid c \otimes x = 0 \in N \otimes_R H\} && \text{(Corollary 2.4.6)} \\ &= H / \text{Ann}_H(c) \end{aligned}$$

With these observations, we can assume  $R = \widehat{R}$  without loss of generality, i.e.,  $R$  is complete.

Using Theorem 2.2.8 (i.e., Tensor-Hom Adjunction), we have the following isomorphism:

$$\mathrm{Hom}_R(H \otimes N, E_R(K)) \cong \mathrm{Hom}_R(N, \mathrm{Hom}_R(H, E_R(K))).$$

Using this canonical isomorphism, we see that

$$\begin{aligned} x \in \mathrm{Ann}_H(c) &\iff x \otimes c = 0 \\ &\iff h(x \otimes c) = 0, \forall h \in \mathrm{Hom}_R(H \otimes N, E_R(K)) \\ &\iff [g(c)](x) = 0, \forall g \in \mathrm{Hom}_R(N, \mathrm{Hom}_R(H, E_R(K))) \\ &\iff x \in \bigcap_{g \in \mathrm{Hom}_R(N, \mathrm{Hom}_R(H, E_R(K)))} \ker(g(c)) \\ &\iff x \in \bigcap_{f \in T_{N,M}(c)} \ker(f) \\ &\iff f(x) = 0, \forall f \in T_{N,M}(c) \leq \mathrm{Hom}_R(H, E_R(K)) \cong M. \end{aligned}$$

Thus,  $(T_{N,M}(c))^\vee \cong H / \mathrm{Ann}_H(c)$  by Lemma 2.2.7. □

Then, we can easily get the following correspondences.

**Corollary 3.1.9.** *Let  $(R, \mathfrak{m}, k)$  be a local ring and  $M, N$  be finitely generated  $R$ -modules.*

*Let  $H = M^\vee = \mathrm{Hom}_R(M, E_R(K))$ . Let  $c \in N$ . Then*

$$T_{N,M}(c) = M \iff \mathrm{Ann}_H(c) = 0.$$

**Corollary 3.1.10.** *Let  $(R, \mathfrak{m}, k)$  be a local ring with a canonical module,  $\omega$  and  $N$  be a finitely generated  $R$ -module. Let  $c \in N$  and  $H = \omega^\vee = H_{\mathfrak{m}}^d(R)$ . Then*

$$T_{N,\omega}(c) = \omega \iff \mathrm{Ann}_H(c) = \{x \in H_{\mathfrak{m}}^d(R) \mid c \otimes x = 0 \in N \otimes_R H_{\mathfrak{m}}^d(R)\} = 0.$$

### 3.2 $F$ -rationality

We now give a formal definition of  $F$ -rational rings and some of their important properties. In a local Noetherian ring, we say an ideal is a parameter ideal if and only if it is generated by part of a system of parameters.

A ring  $R$  is said to be reduced if it has no non-zero nilpotent elements, i.e.,

$$\text{for any } x \in R, x^n = 0 \text{ for } n \in \mathbb{N} \implies x = 0.$$

Then, we say that a ring is normal if it is reduced and it is integrally closed in its total ring of fractions.

**Definition 3.2.1** ([Smi02, Definition 2.2]). A local ring  $(R, \mathfrak{m})$  of prime characteristic,  $p$ , is  $F$ -rational if all parameter ideals are tightly closed. We say a ring  $R$  is  $F$ -rational if and only if  $R_P$  is  $F$ -rational for all  $P \in \text{Spec}(R)$ .

One can wonder how easy/hard it is to get a tightly closed ideal. The answer is that it is not difficult at all. It is shown that every ideal in a regular ring is tightly closed, and this well-known fact gives the following corollary.

**Corollary 3.2.2.** *Regular rings are  $F$ -rational rings.*

$F$ -rational rings are quite nice. For example,  $F$ -rational ring  $R$  is Cohen-Macaulay if  $R$  is a homomorphic image of a Cohen-Macaulay ring. Another nice property is stated in the theorem below.

**Theorem 3.2.3** ([HH90a, Theorem 4.2]).  *$F$ -rational rings are normal.*

**Corollary 3.2.4.** *Regular rings are normal.*

Karen Smith, in her dissertation [Smi93], has shown a very powerful characterization of  $F$ -rationality using tight closure theory, which says the following.



**Theorem 3.2.5** ([Smi93]). *Let  $(R, \mathfrak{m}, k)$  be a local Cohen-Macaulay ring of prime characteristic  $p$  and let  $\dim(R) = d$ . Then,*

$$R \text{ is } F\text{-rational} \iff 0_{H_{\mathfrak{m}}^d(R)}^* = 0.$$

### 3.3 Juan Vélez's results

In [Vél95], Juan Vélez gave a characterization of strongly  $F$ -rational rings, which was considered an analogue of the theory of strong  $F$ -regularity. As a result of that, he obtained a characterization of  $F$ -rationality. The following is one of his characterization of  $F$ -rational rings stated differently.

**Theorem 3.3.1** ([Vél95, Compare with Proposition 1.6]). *Assume that  $R$  is a reduced  $F$ -finite Cohen-Macaulay local ring with canonical module,  $\omega$ . Then  $R$  is  $F$ -rational if and only if for each  $c \in R^\circ$ , there exists an  $e_0 \in \mathbb{N}$  such that  $T_{eR, \omega}({}^e c) = \omega$ , for all  $e \geq e_0$ .*

After that, he obtained the following. Instead of checking all  $e \in \mathbb{N}$  for all  $c \in R^\circ$ , it is enough to check for one single  $e$  for one single  $c$ .

**Theorem 3.3.2** ([Vél95, Compare with Remark 1.10]). *Assume that  $R$  is a reduced  $F$ -finite Cohen-Macaulay local ring with canonical module,  $\omega$ . If  $T_{eR, \omega}({}^e c) = \omega$ , for one single value  $e$  and a fixed locally stable test element  $c \in R^\circ$ , then  $R$  is  $F$ -rational.*

We were then able to improve Theorem 3.3.2 by figuring out what  $e$  really works and we provide a different proof in Juan Vélez's paper, which relies on Karen's Smith's result (see Theorem 3.2.5). This lemma turns out to be very crucial to our main theorem because we only need to show  $T_{eR, \omega}({}^e c) = \omega$  for one  $e$  and one  $c$  to prove  $F$ -rationality.

**Lemma 3.3.3.** *Let  $(R, \mathfrak{m}, k)$  be a reduced  $F$ -finite Cohen-Macaulay local ring with canonical module,  $\omega$ . Let  $c$  be a  $q_0$ -weak test element (which exists) and  $q_1$  be a test exponent for  $c$  and  $H_{\mathfrak{m}}^d(R)$  (which exists). Then,*

(1) *If there exists an  $e \geq e_0$  such that  $T_{eR, \omega}({}^e c) = \omega$ , then  $R$  is  $F$ -rational.*

(2) If  $R$  is  $F$ -rational, then  $T_{eR,\omega}({}^e c) = \omega$  for all  $e \geq e_1$ .

*Proof.* (1) Assume that there exists an  $e \geq e_0$  such that  $T_{eR,\omega}({}^e c) = \omega$ . Then, for  $e \geq e_0$ ,

$$\begin{aligned} T_{eR,\omega}({}^e c) = \omega &\implies \{x \in H_{\mathfrak{m}}^d(R) \mid cx^q = 0\} = 0 \text{ by Corollary 3.1.10} \\ &\implies \text{if } cx^q = 0, \text{ then } x = 0. \end{aligned}$$

Since  $c$  is a  $q_0$ -weak test element, we have that  $0_{H_{\mathfrak{m}}^d(R)}^* = 0$ . Thus, by Theorem 3.2.5,  $R$  is  $F$ -rational.

(2) Assume that  $R$  is  $F$ -rational. Then, by Theorem 3.2.5,  $0_{H_{\mathfrak{m}}^d(R)}^* = 0$ . Now, we let  $e \geq e_1$  and we consider  $\{x \in H_{\mathfrak{m}}^d(R) \mid cx^q = 0\}$ . Since  $q_1$  is a test exponent for  $c$  and  $H_{\mathfrak{m}}^d(R)$ ,

$$cx^q = 0 \implies x \in 0_{H_{\mathfrak{m}}^d(R)}^* = 0 \implies x = 0.$$

Thus, we have that

$$\{x \in H_{\mathfrak{m}}^d(R) \mid cx^q = 0\} = 0, \text{ for all } e \geq e_1,$$

which is equivalent to

$$T_{eR,\omega}({}^e c) = \omega, \text{ for all } e \geq e_1, \text{ by Corollary 3.1.10.}$$

Therefore, this completes the proof.  $\square$

By studying Juan Vélez's result (see Theorem 3.3.1) carefully, we obtain the following equivalent statements.

**Theorem 3.3.4.** *Let  $(R, \mathfrak{m}, k)$  be reduced  $F$ -finite Cohen-Macaulay local ring with canonical module,  $\omega_R$ . Then, the following assertions are equivalent:*

(1)  $R$  is  $F$ -rational.

(2) For all  $c \in R^\circ$ , there exists an  $e_0 \in \mathbb{N}$  such that  $\forall e \geq e_0$ ,  $T_{eR,\omega}({}^e c) = \omega$ .

(3) For all faithful finitely generated  $R$ -modules  $M$ , there exists an  $x \in M$  such that for all  $c \in R^\circ$ , there exists an  $e_0$  such that  $\forall e \geq e_0$ ,  $T_{eM,\omega}({}^e cx) = \omega$ .

(4) For all faithful finitely generated  $R$ -modules  $M$ , for all  $c \in R^\circ$ , there exists an  $e_0$  such that  $\forall e \geq e_0$ ,  $T_{eM,\omega}({}^e cM) = \omega$ .

(5) for all  $c \in R^\circ$ , there exists an  $e_0 \in \mathbb{N}$  such that  $\forall e \geq e_0$ ,  $T_{eR,\omega}({}^e cR) = \omega$ .

*Proof.* For (1)  $\implies$  (2), let  $c \in R^\circ$  and let  $d$  be a  $q_0$ -weak test element (which exists). Then,  $cd$  is a  $q_0$ -weak test element. Then, by Lemma 3.3.3, there exists an  $e \geq e_0$  such that  $T_{eR,\omega}({}^e cd) = \omega$ . Since there exists a map  $h \in \text{Hom}_R({}^e R, {}^e R)$  such that  $h(c) = cd$ , we can apply Proposition 3.1.2(5) and obtain  $T_{eR,\omega}({}^e c) = \omega$ . Since  $c$  was arbitrary, (2) holds.

Now, for (2)  $\implies$  (3), we consider

$$(R^\circ)^{-1}M \text{ and } (R^\circ)^{-1}R \cong F_1 \times \cdots \times F_n,$$

where  $F_1, \dots, F_n$  are fields. Notice that  $(R^\circ)^{-1}M$  which is faithful over  $(R^\circ)^{-1}R$ . Then, we have that

$$(R^\circ)^{-1}M \cong F_1^{m_1} \times \cdots \times F_n^{m_n}, m_i \geq 1, \forall i = 1, 2, \dots, n.$$

Thus, there exists an onto map  $h'$  from  $(R^\circ)^{-1}M$  to  $(R^\circ)^{-1}R$  that sends  $\frac{x}{d}$  to  $\frac{1}{1}$ . Also,  $h' = \frac{h}{a}$ , where  $h \in \text{Hom}_R(M, R)$  and  $a \in R^\circ$ . Thus,  $h(x) = da$ , which implies that  $h(cx) = ch(x) = cda \in R^\circ$ . Thus, using Proposition 3.1.2(5),

$$\omega = T_{eR,\omega}({}^e c) \subseteq T_{eM,\omega}({}^e cx) \subseteq \omega,$$

which gives us (3).

(3)  $\implies$  (4) Notice that  $cx \in cM$ . By Proposition 3.1.2(4), we have

$$\omega = T_{eM,\omega}({}^e cx) \subseteq T_{eM,\omega}({}^e cM) \subseteq \omega,$$

which gives us (4).

(4)  $\implies$  (5) is trivial by letting  $M = R$ .

For (5)  $\implies$  (2), we only need to show that  $T_{eR,\omega}({}^e cR) \subseteq T_{eR,\omega}({}^e c)$ . Let  $x \in T_{eR,\omega}({}^e cR)$ . Thus,  $x = h({}^e cr)$  for  $r \in R$  and  $h \in \text{Hom}_R({}^e R, \omega)$ . Let  $g : {}^e R \rightarrow {}^e R$  such that  $g({}^e s) = {}^e sr$  for some  ${}^e s \in {}^e R$ .

**Claim:**  $g$  is a  ${}^e R$ -homomorphism.

Let  ${}^e x, {}^e y, {}^e z \in {}^e R$ . Then, the following holds.

$$\begin{aligned} g({}^e z {}^e x + {}^e y) &= {}^e ((zx + y)r) = {}^e (zxr + yr) \\ &= {}^e (zxr) + {}^e (yr) = {}^e z {}^e (xr) + {}^e (yr) = {}^e z g({}^e x) + g({}^e y), \end{aligned}$$

which proves the claim. In particular,  $g({}^e c) = {}^e (cr)$ . Therefore, Proposition 3.1.2(5) gives us the desired result.

Finally, for (2)  $\implies$  (1), we just apply Lemma 3.3.3(1). This completes the proof.  $\square$

## CHAPTER 4

### MAIN RESULTS

#### 4.1 A dual to tight closure theory

Neil Epstein and Karl Schwede introduced an operation called *tight interior* in their paper, A Dual To Tight Closure Theory [ES14]. We use this idea to prove Lemma 4.1.9, which was also observed by Mel Hochster and Yongwei Yao.

**Definition 4.1.1** ([ES14, §2]). Let  $R$  be a reduced  $F$ -finite Noetherian ring of prime characteristic  $p > 0$ . Let  $M$  be an  $R$ -module. We define the tight interior of  $M$ ,  $M_*$ , via:

$$M_* := \bigcap_{c \in R^\circ} \bigcap_{e_0 \geq 0} \sum_{e \geq e_0} T_{eR, M}({}^e c)$$

**Theorem 4.1.2** ([ES14, Corollary 3.6]). *Let  $(R, \mathfrak{m}, k)$  be a complete Noetherian local ring of characteristic  $p$ . Let  $L$  be either an Artinian or a finitely generated  $R$ -module (or any other Matlis-dualizable module). Then*

$$(L_*)^\vee \cong L^\vee / 0_{L^\vee}^* \quad \text{and} \quad (L^\vee)^* \cong (L / 0_L^*)^\vee$$

Before we can prove the lemma, we need a few results. Here, we provide the proof for Proposition 4.1.3, which shows that  $M_e$  is a descending chain.

**Proposition 4.1.3** ([HH02, Proposition 2.6]). *Let  $c \in R^\circ$ . For every  $R$ -module  $M$  and every integer  $e \geq 0$ , define*

$$M_e = \{x \in M \mid c \otimes x \in 0_{F_R^e(M)}^* \subseteq F_R^e(M) = {}^e R \otimes M\}.$$

*Then,  $M_0 \supseteq M_1 \supseteq \cdots \supseteq M_e \supseteq M_{e+1} \cdots$ , that is  $\{M_e\}_{e=0}^\infty$  is a descending chain of  $R$ -*

submodules of  $M$ .

*Proof.* Let  $d \in R^\circ$  be a  $q_0$ -weak test element (which exists).

$$\begin{aligned}
x \in M_e &\implies d^{1/q_1} \otimes_{eR} c \otimes_R x = 0 \text{ for } q_1 \geq q_0 \\
&\implies d^{1/q_1} c \otimes_R x = 0 \text{ for } q_1 \geq q_0 \\
&\implies c^{p-1} \cdot d^{1/q_1} c \otimes_R x = 0 \text{ for } q_1 \geq q_0 \\
&\implies d^{1/q_1} c^p \otimes_R x = 0 \text{ for } q_1 \geq q_0 \\
&\implies d^{1/q_1} \otimes_{eR} c^p \otimes_R x = 0 \text{ for } q_1 \geq q_0 \\
&\implies c^p \otimes x \in 0_{F_R^{e-1}(M)}^* \\
&\implies x \in M_{e-1}.
\end{aligned}$$

Thus,  $M_e \subseteq M_{e-1}$ . Since  $0 \in M_e$ , for all  $e \geq 0$ , they are non-empty and it is clear that they are all contained in  $M$ . Let  $x, y \in M_e$  and  $r \in R$ . Then, there exist  $d, f \in R^\circ$  such that  $d^{1/q_0} c \otimes x = 0$  and  $f^{1/q_1} c \otimes y = 0$  for  $e_0, e_1 \gg 0$ . Note that  $d_0^{1/q} c \otimes rx = r(d_0^{1/q} c \otimes x) = 0$ , which implies  $rx \in M_e$ . Now consider  $x - y$ . Since  $c \otimes x \in 0_{F_R^e(M)}^*$  and  $c \otimes y \in 0_{F_R^e(M)}^*$ ,  $c \otimes (x - y) \in 0_{F_R^e(M)}^*$  because  $0_{F_R^e(M)}^*$  is a module. Thus,  $x - y \in M_e$ . Therefore,  $\{M_e\}_{e=0}^\infty$  is a descending chain of  $R$ -submodules of  $M$ .  $\square$

In the proof of Lemma 4.1.8, we jump back and forth between the Noetherian side and Artinian side. The next proposition simply says that a descending chain in the Artinian side will correspond to an ascending chain in the Noetherian side. Notice that we are abusing notation when we write  $\left(\frac{M}{B}\right)^\vee \leq \left(\frac{M}{A}\right)^\vee$ . Consider the exact sequence

$$0 \longrightarrow A \longrightarrow M \longrightarrow \frac{M}{A} \longrightarrow 0.$$

Then, we take the dual of the exact sequence and obtain this exact sequence

$$0 \longleftarrow A^\vee \longleftarrow M^\vee \longleftarrow \left(\frac{M}{A}\right)^\vee \longleftarrow 0.$$

Thus, we can treat  $\left(\frac{M}{A}\right)^\vee$  as a submodule of  $M^\vee$ . Similarly, we can treat  $\left(\frac{M}{B}\right)^\vee$  as a submodule of  $M^\vee$ . Hence, we can compare these modules now as submodules of  $M^\vee$ .

**Proposition 4.1.4.** *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring and  $M$  be a finitely generated  $R$ -module. Let  $A, B \leq M$  be submodules of  $M$ . Then,*

$$A \leq B \iff \left(\frac{M}{B}\right)^\vee \leq \left(\frac{M}{A}\right)^\vee.$$

*Proof.* Let  $E$  denote the injective hull of  $k$ . Note that we have

$$A \leq B \iff \frac{M}{A} \xrightarrow{\text{canonical}} \frac{M}{B}.$$

Since  $\text{Hom}_R(-, E)$  preserves short exact sequence, we have that

$$A \leq B \iff \frac{M}{A} \xrightarrow{\text{canonical}} \frac{M}{B} \iff \left(\frac{M}{B}\right)^\vee \leq \left(\frac{M}{A}\right)^\vee.$$

This completes the proof. □

We study the following lemma that relates injective hulls of two local rings, which gives us Corollary 4.1.6.

**Lemma 4.1.5.** *Let  $\phi : (R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, \ell)$  be a local homomorphism of rings such that  $S$  is module-finite over  $R$ . Let  $E_R(k)$  and  $E_S(\ell)$  be the injective hulls of the residue fields of  $R$  and  $S$ , respectively. Then,*

$$\text{Hom}_R(S, E_R(k)) \cong E_S(\ell).$$

Using the above isomorphisms, we are able to show Corollary 4.1.6 and Corollary 4.1.7.

**Corollary 4.1.6.** *Let  $\phi : (R, \mathfrak{m}, k) \rightarrow (S, \mathfrak{n}, \ell)$  be a local homomorphism of complete rings such that  $S$  is module-finite over  $R$ . Let  $M$  be a finitely generated  $R$ -module and  $N = M^\vee$ . Let  $E_R(k)$  and  $E_S(\ell)$  be the injective hulls of the residue fields of  $R$  and  $S$ , respectively. Then,*

$$\text{Hom}_S(S \otimes_R N, E_S(\ell)) \cong \text{Hom}_R(S, M).$$

*Proof.* We have the following

$$\begin{aligned}
\mathrm{Hom}_S(S \otimes_R N, E_S(\ell)) &\cong \mathrm{Hom}_R(N, \mathrm{Hom}_S(S, E_S(\ell))) && \text{by Theorem 2.2.8} \\
&\cong \mathrm{Hom}_R(N, E_S(\ell)) && \text{by Proposition 3.1.7.3} \\
&\cong \mathrm{Hom}_R(N, \mathrm{Hom}_R(S, E_R(k))) && \text{by Lemma 4.1.5} \\
&\cong \mathrm{Hom}_R(S \otimes_R N, E_R(k)) && \text{by Theorem 2.2.8} \\
&\cong \mathrm{Hom}_R(S, \mathrm{Hom}_R(N, E_R(k))) && \text{by Theorem 2.2.8} \\
&\cong \mathrm{Hom}_R(S, M).
\end{aligned}$$

This completes the proof.  $\square$

**Corollary 4.1.7.** *Let  $(R, \mathfrak{m}, k)$  be a  $F$ -finite complete local ring of prime characteristic  $p$  with  $\dim(R) = d$ . Let  $M$  be a finitely generated  $R$ -module and  $N = M^\vee$ . Let  $E_R$  and  $E_{eR}$  be injective hulls of the residue fields of  $R$  and  ${}^eR$ , respectively, for all  $e \in \mathbb{N}$ . Then,*

$$\mathrm{Hom}_{{}^eR}({}^eR \otimes_R N, E_{{}^eR}) \cong \mathrm{Hom}_R({}^eR, M).$$

*Proof.* First, let fix an  $e \in \mathbb{N}$ . Then, we notice that  ${}^eR$  is also a local ring that is module-finite over  $R$  since  $R$  is  $F$ -finite. Thus, we simply apply Corollary 4.1.6.  $\square$

To simplify notation for the next lemma, let

$$E(R/P) = E_{R_P}(R_P/P_P) \text{ and } M_P^\vee = \mathrm{Hom}_{R_P}(M_P, E(R/P)).$$

For more clarification, if we take the dual  $(-)^\vee$  of a  $R_P$ -module, we apply  $\mathrm{Hom}_{R_P}(-, E(R/P))$ .

Now, we are ready to prove the existence of test exponent in a global sense.

**Lemma 4.1.8.** *Let  $R$  be a reduced  $F$ -finite ring of characteristic  $p$  with a locally stable test element  $c$  (which exists) and let  $M$  be a finitely generated  $R$ -module. There exist an  $E \in \mathbb{N}$  such that  $Q = p^E$  is a test exponent for  $c$  and  $M_P^\vee$ , for all  $P \in \mathrm{Spec}(R)$ .*



*Proof.* Fix  $e \in \mathbb{N}$  and a locally stable test element  $c$ . Also, fix a  $P \in \text{Spec}(R)$ . We consider the following

$$\begin{aligned} ({}^e R_P \otimes_{R_P} M_P^\vee)^\vee &= \text{Hom}_{{}^e R_P}({}^e R_P \otimes_{R_P} M_P^\vee, E({}^e(R/P))) \\ &\cong \text{Hom}_{{}^e R_P}({}^e R_P, M_P) \text{ by Corollary 4.1.7.} \end{aligned}$$

Now, we consider the following using definition 4.1.1

$$\text{Hom}_{{}^e R_P}({}^e R_P, M_P)_* = \bigcap_{d \in R^\circ} \bigcap_{e_0 \geq 0} \sum_{e_1 \geq e_0} T_{e_1({}^e R_P), \text{Hom}_{{}^e R_P}({}^e R_P, M_P)}({}^{e_1}(d)),$$

which corresponds to the  $0^*_{{}^e R_P \otimes_{R_P} M_P^\vee}$  because

$$0^*_{{}^e R_P \otimes_{R_P} M_P^\vee} = \bigcup_{d \in R^\circ} \bigcup_{e_0 \geq 0} \bigcap_{e_1 \geq e_0} \{x \in {}^e R_P \otimes_{R_P} M_P^\vee \mid dx^{e_1} = 0\}.$$

The aforementioned correspondence comes from the following isomorphism

$$(0^*_{{}^e R_P \otimes_{R_P} M_P^\vee})^\vee \cong \text{Hom}_{{}^e R_P}({}^e R_P, M_P) / \text{Hom}_{{}^e R_P}({}^e R_P, M_P)_*. \quad (4.1)$$

Since we have a locally stable test element  $c$ , we can rewrite it to

$$0^*_{{}^e R_P \otimes_{R_P} M_P^\vee} = \bigcup_{e_1 \geq 0} \{x \in {}^e R_P \otimes_{R_P} M_P^\vee \mid cx^{e_1} = 0\},$$

which corresponds to

$$\text{Hom}_{{}^e R_P}({}^e R_P, M_P)_* = \sum_{e_1 \geq 0} T_{e_1({}^e R_P), \text{Hom}_{{}^e R_P}({}^e R_P, M_P)}({}^{e_1}(c)).$$

as we use the same isomorphism in equation (4.1).

Now, we define  $M_{e'}(P)$ ,  $N_{e'}(P)$ , and  $N_{e'}$  as follows

$$M_{e'}(P) := \{x \in M_P^\vee \mid cx^{e'} \in 0^*_{{}^e R_P \otimes_{R_P} M_P^\vee}\}$$

$$N_{e'}(P) := \left( \text{Hom}_{R_P}({}^{e'}R_P, M_P)_* \right) ({}^{e'}c) = \left( \sum_{e_1 \geq 0} T_{e_1({}^{e'}R_P), \text{Hom}_{R_P}({}^{e'}R_P, M_P)} ({}^{e_1}({}^{e'}c)) \right) ({}^{e'}c)$$

$$N_{e'} := \left( \text{Hom}_R({}^{e'}R, M)_* \right) ({}^{e'}c) = \left( \sum_{e_1 \geq 0} T_{e_1({}^{e'}R), \text{Hom}_R({}^{e'}R, M)} ({}^{e_1}({}^{e'}c)) \right) ({}^{e'}c)$$

One can prove that  $M_{e'}(P)$  corresponds to  $N_{e'}(P)$ .

Also, notice that  $N_{e'}(P) = (N_{e'})_P$  since localization commutes with homomorphism by Corollary 2.4.8.

By Proposition 4.1.3,  $\{M_{e'}(P)\}_{e'=0}^\infty$  is a descending chain of  $R_P$ -submodules of  $M_P^\vee$ . Therefore, by Proposition 4.1.4,  $\{N_{e'}(P)\}_{e'=0}^\infty$  is an ascending chain of  $R_P$ -submodules of  $M_P$ , i.e.

$$N_0(P) \leq N_1(P) \leq \cdots N_{e'}(P) \leq N_{e'+1}(P) \leq \cdots \leq M_P \text{ for all } P \in \text{Spec}(R)$$

$$\implies (N_0)_P \leq (N_1)_P \leq \cdots (N_{e'})_P \leq (N_{e'+1})_P \leq \cdots \leq M_P \text{ for all } P \in \text{Spec}(R)$$

Since inclusion is a local property, we obtain

$$N_0 \leq N_1 \leq \cdots N_{e'} \leq N_{e'+1} \leq \cdots \leq M.$$

Since  $M$  is a finitely generated module over a Noetherian ring,  $M$  is Noetherian module. Thus, there exists an  $E$  such that

$$N_E = N_{E+e'} \text{ for all } e' \geq 0.$$

Then, we have that

$$(N_E)_P = (N_{E+e'})_P \text{ for all } e' \geq 0 \text{ and for all } P \in \text{Spec}(R)$$

because equality is also a local property. Therefore, correspondingly, we have

$$M_E(P) = M_{E+e'}(P) \text{ for all } e' \geq 0 \text{ and for all } P \in \text{Spec}(R)$$

**Claim:**  $Q = p^E$  is a test exponent for  $c$  and  $M_P^\vee$ , for all  $P \in \text{Spec}(R)$ . Let  $x \in M_P^\vee$  and let  ${}^{e_0}c \otimes x = 0$  for  $e_0 \geq E$ . We need to show that  $x \in 0_{M_P^\vee}^*$ . Let  $d$  be a locally stable test element (could be  $c$ ). Thus,

$$\begin{aligned} x \in M_{e_0}(P) &= M_{e_0+e'}(P) \text{ for all } e' \geq 0 \text{ and for all } P \in \text{Spec}(R) \\ \implies {}^{e_0+e'}c \otimes_{R_P} x &\in 0_{e_0+e'}^* \text{ for all } e' \geq 0 \text{ and for all } P \in \text{Spec}(R) \\ \implies {}^{e_0+e'}d {}^{e_0+e'}c \otimes_{R_P} x &= 0 \text{ for all } e' \geq 0 \text{ and for all } P \in \text{Spec}(R) \\ \implies {}^{e_0+e'}(dc) \otimes_{R_P} x &= 0 \text{ for all } e' \geq 0 \text{ and for all } P \in \text{Spec}(R) \\ \implies x \in 0_{M_P^\vee}^* &\text{ for all } P \in \text{Spec}(R) \text{ since } dc \in R^\circ \end{aligned}$$

Therefore,  $Q$  is a test exponent for  $c$  and  $M_P^\vee$ , for all  $P \in \text{Spec}(R)$ .  $\square$

We can refine Lemma 4.1.8 and remove the complete condition. Here, we provide a more direct proof:

**Lemma 4.1.9.** *Let  $R$  be a reduced  $F$ -finite ring of characteristic  $p$ . Let  $c$  be as in Lemma 2.1.6 and let  $M$  be a finitely generated  $R$ -module. There exists an  $E \in \mathbb{N}$  such that  $Q$  is a test exponent for  $c$  and  $M_P^\vee$ , for all  $P \in \text{Spec}(R)$ .*

*Proof.* Fix  $e \in \mathbb{N}$  and a locally stable test element  $c$ . We define  $N_e$  as in the proof of Lemma 4.1.8

$$N_e := (\text{Hom}_R({}^eR, M)_*)({}^e c) = \left( \sum_{e_1 \geq 0} T_{e_1({}^eR), \text{Hom}_R({}^eR, M)}({}^{e_1}({}^e c)) \right)({}^e c).$$

**Claim 1:**  $N_e = \left( \sum_{e_1 \geq 1} T_{e_1({}^eR), \text{Hom}_R({}^eR, M)}({}^{e_1}({}^e c)) \right)({}^e c)$ .

It is enough to show that

$$T_{eR, \text{Hom}_R({}^eR, M)}({}^e\mathcal{C}) \subseteq \left( \sum_{e_1 \geq 1} T_{e_1({}^eR), \text{Hom}_R({}^eR, M)}({}^{e_1}({}^e\mathcal{C})) \right) ({}^e\mathcal{C}).$$

We let  $x \in T_{eR, \text{Hom}_R({}^eR, M)}({}^e\mathcal{C})$ . Thus,

$$x = [h({}^e\mathcal{C})]({}^e\mathcal{C}), \text{ for } h \in \text{Hom}_{eR}({}^eR, \text{Hom}_R({}^eR, M)).$$

Then, using Theorem 2.2.8, we have that

$$\begin{aligned} \text{Hom}_{eR}({}^{e_1}({}^eR), \text{Hom}_R({}^eR, M)) &\cong \text{Hom}_R({}^{e_1}({}^eR) \otimes_{eR} {}^eR, M) \\ &\cong \text{Hom}_R({}^{e_1}({}^eR), M). \end{aligned} \quad (4.2)$$

Now, we let  $e_1 = 0$ . By tracing the canonical isomorphism above given by Theorem 2.2.8, we have that

$$x = f({}^e\mathcal{C} \cdot {}^e\mathcal{C}) \text{ for } f \in \text{Hom}_R({}^eR, M).$$

By Lemma 2.1.6, there exists a  $g \in \text{Hom}_R({}^{e_1}R, R)$  such that  $g({}^{e_1}c) = c$  for some  $e_1 > 0$ . Thus, there exists a  ${}^e g \in \text{Hom}_{eR}({}^{e_1}({}^eR), {}^eR)$  such that  ${}^e g({}^{e_1}({}^e\mathcal{C})) = {}^e\mathcal{C}$  for some  $e_1 > 0$ . Note that  ${}^e g$  is  ${}^eR$ -linear. Now we let  $\phi = f \circ {}^e g : {}^{e_1}({}^eR) \rightarrow M$ . Hence, we have the following:

$$\phi({}^{e_1}({}^e\mathcal{C}) \cdot {}^e\mathcal{C}) = f[{}^e g({}^{e_1}({}^e\mathcal{C}) \cdot {}^e\mathcal{C})] = f[{}^e\mathcal{C}({}^e g({}^{e_1}({}^e\mathcal{C})))] = f({}^e\mathcal{C} \cdot {}^e\mathcal{C}) = x.$$

Since  $\phi({}^{e_1}({}^e\mathcal{C}) \cdot {}^e\mathcal{C}) \in \text{Hom}_R({}^{e_1}({}^eR), M)$ ,

$$x \in \left( \sum_{e_1 \geq 1} T_{e_1({}^eR), \text{Hom}_R({}^eR, M)}({}^{e_1}({}^e\mathcal{C})) \right) ({}^e\mathcal{C}),$$

which proves Claim 1.

**Claim 2:**  $\{N_e\}_{e=0}^\infty$  is an ascending chain of  $R$ -submodules of  $M$ .

Let  $x \in N_e$ . Then, by Claim 1, we have that

$$x = \left[ \sum_{e_1 > 0} h_{e_1}({}^{e_1}({}^e c)) \right] ({}^e c), \text{ where } h_{e_1} \in \text{Hom}_{{}^e R}({}^{e_1}({}^e R), \text{Hom}_R({}^e R, M)).$$

By Equation (4.2), there exist

$$f_{e_1} \in \text{Hom}_R({}^{e_1}({}^e R), M) \text{ such that } x = \sum_{e_1 > 0} f_{e_1}({}^{e_1}({}^e c) \cdot {}^e c).$$

Now, we view  ${}^{e_1}({}^e R)$  as  ${}^{e_1-1}({}^{e+1}R)$ . Let  $\psi_{e_1} : {}^{e_1-1}({}^{e+1}R) \rightarrow {}^{e_1}({}^e R)$  such that  $\psi_{e_1}$  is multiplication by  ${}^{e_1}({}^e(c^{q_1 - \frac{q_1}{p}}))$ . Since  $q_1 - \frac{q_1}{p} > 0$ , it is easy to see that  $\psi_{e_1}$  is  ${}^{e_1}({}^e R)$ -linear (hence,  $R$ -linear). Also, notice that we have

$$\psi_{e_1}({}^{e_1-1}({}^{e+1}c) \cdot {}^{e+1}c) = {}^{e_1}({}^e(c^{q_1 - \frac{q_1}{p}})) \cdot {}^{e_1}({}^e(c \cdot c^{\frac{q_1}{p}})) = {}^{e_1}({}^e c) \cdot {}^e c.$$

(For readers who prefer  $R^{1/q}$  notation: we look at the following: Let  $\psi_{e_1} : R^{1/q_1 q} \rightarrow R^{1/q_1 q}$  such that  $\psi_{e_1}$  is multiplication by  $c^{\frac{p-1}{pq}}$ . Then, we have

$$\psi_{e_1}(c^{\frac{1}{q_1 q}} \cdot c^{\frac{1}{q_1 p}}) = c^{\frac{p-1}{pq}} \cdot c^{\frac{1}{q_1 q}} \cdot c^{\frac{1}{q_1 p}} = c^{\frac{1}{q_1 q}} \cdot c^{\frac{1}{q_1 p}}.)$$

Note that  $e_1 > 0$  gives that  $e_1 - 1 \geq 0$ . Thus,

$$x = \left[ \sum_{e_1 > 0} f_{e_1} \circ \psi_{e_1}({}^{e_1}({}^e c) \cdot {}^{e+1}c) \right],$$

where  $f_{e_1} \circ \psi_{e_1} \in \text{Hom}_{{}^{e+1}R}({}^{e_1-1}({}^{e+1}R), \text{Hom}_R({}^{e+1}R, M))$ . Thus,  $x \in N_{e+1}$ .

Therefore, the claim holds.

Then, everything will follow as in the proof of Lemma 4.1.8. □

For our purposes, we look at a special case of Lemma 4.1.9, which shows that test exponent exists for the highest cohomology of  $R_P$  for all prime ideals in  $R$ .

**Corollary 4.1.10.** *Let  $R$  be a reduced  $F$ -finite Cohen-Macaulay ring with a canonical module  $\omega$  and a locally stable test element  $c$ . There exists an  $e \in \mathbb{N}$  such that  $q$  is a test exponent for  $c$  and  $H_{P_P}^{\dim(R_P)}(R_P)$ , for all  $P \in \text{Spec}(R)$ .*

*Proof.* Notice that  $\omega$  is finitely generated over  $R$  and  $\omega_P^\vee \cong H_{P_P}^{\dim(R_P)}(R_P)$ . Therefore, just let  $M = \omega$  and  $M_P^\vee = H_{P_P}^{\dim(R_P)}(R_P)$  in Lemma 4.1.8 and the rest follows.  $\square$

## 4.2 The Main Theorem

In this section, we will use  $\bar{R}$  to denote the integral closure of  $R$  and  $\omega_R, \omega_S, \omega_{\bar{R}}$  to denote the canonical module for  $R, S$  and  $\bar{R}$ , respectively. We need some more facts about the canonical module and depth before we can prove the main theorem. First, we have that the integral closure of  $R$  is finitely generated as an  $R$ -module when  $R$  is a reduced excellent ring (e.g.,  $R$  is  $F$ -finite).

**Lemma 4.2.1.** *Let  $R$  be a reduced excellent ring and let  $\bar{R}$  be the integral closure of  $R$ . Then,  $\bar{R}$  is module-finite over  $R$ .*

The following two theorems show that  $\omega$  is isomorphic to  $R$  when  $R$  is Gorenstein and that  $\omega$  is isomorphic to  $R$  when  $R$  is Cohen-Macaulay. As we saw previously,  $\omega$  behaves nicely when the underlying ring is Cohen-Macaulay (local) ring.

**Theorem 4.2.2** ([BH98, Theorem 3.3.7]). *Let  $(R, \mathfrak{m}, k)$  be a Cohen-Macaulay local ring. Then,  $R$  is Gorenstein if and only if  $\omega_R$  exists and is isomorphic to  $R$ .*

**Theorem 4.2.3** ([BH98, Definition 3.3.1]). *Let  $R$  be a Cohen-Macaulay ring with a canonical module,  $\omega$ . Then,  $\omega$  is a maximal Cohen-Macaulay module.*

When we have a module finite ring homomorphism, we can realize the canonical module in the following way.

**Lemma 4.2.4** ([BH98, Theorem 3.3.7]). *Let  $\phi : (R, \mathfrak{m}) \rightarrow (S, \mathfrak{n})$  be a local homomorphism of Cohen-Macaulay local rings such that  $S$  is module-finite over  $R$ . If  $\omega_R$  exists, then  $\omega_S$  exists and*

$$\omega_S \cong \text{Ext}_R^t(S, \omega_R), \text{ where } t = \dim R - \dim S.$$

In particular, we can realize the canonical module of the integral closure of  $R$  this way.

**Corollary 4.2.5.** *Let  $R$  be a Cohen-Macaulay local ring with canonical module  $\omega$ . Then,*

$$\mathrm{Hom}_R(\bar{R}, \omega) \cong \omega_{\bar{R}}.$$

*Proof.* Note that  $\bar{R}$  is Cohen-Macaulay ring and that the natural inclusion  $\phi : R \subseteq \bar{R}$  is a local homomorphism. In fact,  $\phi$  is an integral extension and thus  $\dim(R) = \dim(\bar{R})$ . Also,  $\bar{R}$  is module-finite over  $R$  by Lemma 4.2.1. Therefore, we have

$$\begin{aligned} w_{\bar{R}} &\cong \mathrm{Ext}_R^0(\bar{R}, \omega_R) && \text{by Lemma 4.2.4} \\ &\cong \mathrm{Hom}_R(\bar{R}, R) && \text{by Theorem 2.2.10.} \end{aligned}$$

This completes the proof. □

Furthermore, we care about  $\mathrm{Ext}_R^1(-, \omega)$  being zero because we care about the ability to lift homomorphisms which is strongly related to  $T_{-, -}^R(-)$ . Before that, we look at a depth formula and another definition of depth using Ext.

**Proposition 4.2.6** ([BH98, Theorem 1.2.5, Exercise 1.4.19]). *Let  $(R, \mathfrak{m}, k)$  be a Noetherian local ring, and  $M, N$  finitely generated  $R$ -modules. Then,*

1.  $\mathrm{depth}[\mathrm{Hom}_R(M, N)] \geq \min\{2, \mathrm{depth}(N)\}$ , and
2.  $\mathrm{depth}(\mathrm{Ann}_R(M), N) = \inf\{n \mid \mathrm{Ext}_R^n(M, N) \neq 0\}$ .

Let  $R$  be a ring and let  $M$  and  $N$  be finitely generated  $R$ -modules. Then,

**Corollary 4.2.7.** *Let  $(R, \mathfrak{m})$  be a Noetherian local Cohen-Macaulay ring with canonical module,  $\omega$ . Let  $M$  be a finitely generated  $R$ -module such that  $\mathrm{Ann}_R(M) = \mathfrak{m}$ . Then,*

$$\mathrm{Ext}_R^1(M, \omega) = 0 \quad \text{if} \quad \dim(R) \geq 2.$$

*Proof.* By definition and Theorem 4.2.3,  $\omega$  is finitely generated and  $\omega$  is maximal Cohen-Macaulay. Using Proposition 4.2.6, we have that

$$2 \leq \dim(R) = \text{depth}(\omega) = \text{depth}(\text{Ann}_R(M), \omega) = \inf\{n \mid \text{Ext}_R^n(M, \omega) \neq 0\}.$$

Hence,  $\text{Ext}_R^1(M, \omega) = 0$ . This completes the proof.  $\square$

**Proposition 4.2.8** ([BH98, Theorem 3.3.10]). *Let  $R$  be a Cohen-Macaulay ring with canonical module,  $\omega$ . Then,  $\text{Ext}_R^i(\omega, \omega) = 0$ , for all  $i > 0$ .*

Now, we are ready to prove the main theorem.

**Theorem 4.2.9** (Main Theorem). *Let  $R$  be a  $F$ -finite Cohen-Macaulay ring with canonical module,  $\omega_R$ . Then, the following assertions are equivalent:*

- (1)  $R$  is  $F$ -rational.
- (2) For every finitely generated  $R$ -module  $M$  supported on  $\text{Spec}(R)$ ,  $\forall x \in M$  such that  $\text{Ann}_R(x) = 0$ , there exists an  $e_0$  such that  $\forall e \geq e_0$ ,  $T_{eM, \omega}({}^e x) = \omega$ .
- (3) For every finitely generated  $R$ -module  $M$  supported on  $\text{Spec}(R)$ , there exists an  $e_0$  such that  $\forall e \geq e_0$ ,  $T_{eM, \omega}({}^e M) = \omega$ .
- (4) For every  $M \in \{\bar{R}\} \cup \{P\omega \mid \dim(R_P) \geq 2\}$ , there exists an  $e \in \mathbb{N}$  such that  $T_{eM, \omega}({}^e M) = \omega$ .
- (5) There exists an  $e \in \mathbb{N}$  such that  $T_{e\bar{R}, \omega}({}^e \bar{R}) = \omega$ , and for every  $P \in \text{Spec}(R)$  such that  $\dim(R_P) \geq 2$ , there exists an  $e_1 \in \mathbb{N}$  such that  $T_{e_1(P\omega_P), \omega_P}({}^{e_1}(P\omega_P)) = \omega_P$ .
- (6) There exists an  $e \in \mathbb{N}$  such that  $T_{eN, \omega}({}^e N) = \omega$  for some  $\bar{R}$ -module  $N$ , and for every  $P \in \text{Spec}(R)$  such that  $\dim(R_P) \geq 2$ , there exists an  $e_1 \in \mathbb{N}$  such that  $T_{e_1(P\omega_P), \omega_P}({}^{e_1}(P\omega_P)) = \omega_P$ .
- (7)  $R_P$  is normal for every  $P \in \text{Spec}(R)$  such that  $\dim(R_P) < 2$ , and for every  $P \in \text{Spec}(R)$  such that  $\dim(R_P) \geq 2$ , there exists an  $e \in \mathbb{N}$  such that  $T_{e(P\omega_P), \omega_P}({}^e(P\omega_P)) = \omega_P$ .



(8)  $R_P$  is  $F$ -rational for every  $P \in \text{Spec}(R)$  such that  $\dim(R_P) < 2$ , and for every  $P \in \text{Spec}(R)$  such that  $\dim(R_P) \geq 2$ , there exists an  $e \in \mathbb{N}$  such that  $T_{e(P\omega_P), \omega_P}({}^e(P\omega_P)) = \omega_P$ .

*Remark 4.2.10.* Note that we do not assume that  $R$  is reduced in the Main Theorem. However, each statement in the Main Theorem implies that  $R$  is reduced. Here, we explain in detail:

In (1), assuming that  $R$  is  $F$ -rational implies that  $R$  is normal by Theorem 3.2.3, which implies that  $R$  is reduced by the definition of normal rings.

Now, we note that if  $R$  is not reduced and  $M$  is an  $R$ -module, then  ${}^eM$  cannot be a faithful  $R$ -module for any  $e \geq 1$ . Here is why. Since  $R$  is not reduced, there exists a nonzero nilpotent element. Hence, there exists a nonzero element  $r$  such that  $r^p = 0$ . Thus,  $r^q = 0$  and, hence  $r \cdot {}^eM = 0$ , for any  $e \geq 1$ . Therefore,  ${}^eM$  is not faithful, for any  $e \geq 1$ .

In each of (2), (3), (4), (5), and (6), assuming that  $T_{eM, \omega}({}^eM) = \omega$ , with  $e \geq 1$  implies that  ${}^eM$  is faithful, which then implies that  $R$  is reduced. Here is why. If  ${}^eM$  is not faithful, then  $T_{eM, \omega}({}^eM)$  is annihilated by  $\text{Ann}_R({}^eM) \neq 0$  by Proposition 3.1.2(8), which contradicts the fact that  $\omega$  is a faithful  $R$ -module.

Finally, in each of (7) and (8), we observe  $R_P$  is  $F$ -rational, hence  $R$  is reduced trivially, when  $\dim(R_P) < 2$ . Now, we let  $P \in \text{Spec}(R)$  such that  $\dim(R_P) \geq 2$ . With the same argument as above,  $T_{e(P\omega_P), \omega_P}({}^e(P\omega_P)) = \omega_P$  also forces that  $R_P$  to be reduced. Therefore,  $R_P$  is reduced for all  $P \in \text{Spec}(R)$ . Hence,  $R$  is reduced.

Now, we provide the proof of the Main Theorem.

*Proof of Theorem 4.2.9.* For (1)  $\implies$  (2), let  $x \in M$  such that  $\text{Ann}_R(x) = 0$  and we fix a  $c \in R^\circ$ . Thus, there exists an  $e_0 \in \mathbb{N}$  such that  $T_{eR, \omega}({}^ec) = \omega$ , for all  $e \geq e_0$  by Theorem 3.3.1. Thus, using similar approach while proving (2)  $\implies$  (3) in Theorem 3.3.4, we obtain

$$\omega = T_{eR, \omega}({}^ec) \subseteq T_{eM, \omega}({}^e(cx)) \subseteq T_{eM, \omega}({}^ex) \subseteq \omega,$$

as desired.

For (2)  $\implies$  (3), note that there exists an  $x \in M$  such that  $\text{Ann}_R(x) = 0$ . Thus, there exists an  $e_0$  such that  $\forall e \geq e_0$ ,  $T_{eM, \omega}({}^e x) = \omega$ . Since  $x \in M$ , we have

$$\omega = T_{eM, \omega}({}^e x) \subseteq T_{eM, \omega}({}^e M) \subseteq \omega,$$

as desired.

For (3)  $\implies$  (4), note that  $\bar{R}$  is supported on  $\text{Spec}(R)$  and it is finitely generated over  $R$  by Lemma 4.2.1. Now let  $P \in \text{Spec}(R)$  such that  $\dim(R_P) \geq 2$  and we consider  $P\omega$ . Clearly,  $P\omega$  is finitely generated over  $R$ . Let  $Q \in \text{Spec}(R)$ . If  $P \not\subseteq Q$ , then  $P_Q\omega_Q = \omega_Q \neq 0$ . If  $P \subseteq Q$ , then  $P_Q \neq 0$  as  $\dim(R_P) \geq 2$ . Hence,  $P_Q\omega_Q \neq 0$ . Thus,  $P\omega$  is supported on  $\text{Spec}(R)$ . Hence, (4) holds trivially.

For (4)  $\implies$  (5), we only need to show the second part of the statement. Here, we fix a  $P \in \text{Spec}(R)$  such that  $\dim(R_P) \geq 2$  and  $e_1 \in \mathbb{N}$ . Then, we localize  $T_{e_1(P\omega), \omega}({}^{e_1}(P\omega)) = \omega$  at  $P$  give us

$$[T_{e_1(P\omega), \omega}({}^{e_1}(P\omega))]_P = \omega_P.$$

Since  $R$  is  $F$ -finite,  ${}^{e_1}(P\omega)$  is finitely generated over  $R$ , and therefore, localization commutes with homomorphism. Hence, we get  $T_{e_1(P\omega_P), \omega_P}({}^{e_1}(P\omega_P)) = \omega_P$ .

For (5)  $\implies$  (6), let  $N = \bar{R}$  and everything follows.

For (6)  $\implies$  (7), we only need to show the first part of the statement. Without loss of generality, we can assume that  $R$  is a local ring. If  $\dim(R) = 0$ , then  $R$  is a regular local ring as  $R$  is reduced. Hence,  $R$  is normal. Now, assume  $\dim(R) = 1$ . Let  $e \in \mathbb{N}$  such that  $T_{eN, \omega}({}^e N) = \omega$ , for an  $\bar{R}$ -module  $N$ . Since  $\omega$  is finitely generated, we let  $y_1, \dots, y_\ell$  be generators of  $\omega$ . Thus, there exists  $h_1, \dots, h_\ell \in \text{Hom}_R({}^e N, \omega)$  such that  $h_1(n_1) = y_1, \dots, h_\ell(n_\ell) = y_\ell$ . Thus, let  $h = h_1 \oplus \dots \oplus h_\ell$ . Note that  $h \in \text{Hom}_R({}^e N^{\oplus \ell}, \omega)$  and  $h$  is onto. Then, we have that  $h(a_1) = y_1, \dots, h(a_\ell) = y_\ell$ , for some  $(a_1), \dots, (a_\ell) \in {}^e N^{\oplus \ell}$ . Since  ${}^e N^{\oplus \ell}$  is a  $\bar{R}$ -module, there exists an  $\bar{R}$ -linear map (hence,  $R$ -linear),  $g$ , from  $\bar{R}^{\oplus \ell}$  to  ${}^e N^{\oplus \ell}$  by sending each  $i$ -th basis,  $e_i$  to the tuples  $(a_i)$ , for all  $i = 1, \dots, \ell$ . Thus, the composition,  $f \circ g : \bar{R}^{\oplus \ell} \longrightarrow \omega$ , sends  $e_i$  to  $y_i$ , for all  $i = 1, \dots, \ell$ . Since  $y_1, \dots, y_\ell$  generates  $\omega$ , we can see

that  $f \circ g$  is surjective. Now, we consider the following short exact sequence:

$$0 \longrightarrow K \longrightarrow \bar{R}^{\oplus \ell} \longrightarrow \omega \longrightarrow 0,$$

where  $K = \ker(f \circ g)$ . Using long exact sequence, we obtain this exact sequence:

$$\mathrm{Ext}_R^1(\omega, \omega) \longleftarrow \mathrm{Hom}_R(K, \omega) \longleftarrow (\mathrm{Hom}_R(\bar{R}, \omega))^{\oplus \ell} \longleftarrow \mathrm{Hom}_R(\omega, \omega) \longleftarrow 0.$$

Since  $R$  is  $F$ -finite,  $R$  is excellent which implies that  $\bar{R}$  is module finite over  $R$  by Lemma 4.2.1. Thus,  $\mathrm{Hom}_R(\bar{R}, \omega) \cong \omega_{\bar{R}}$ . Furthermore, recall that  $\dim(R) = 1$  implies that  $\bar{R}$  is regular which implies that  $\bar{R}$  is Gorenstein. Thus,  $\mathrm{Hom}_R(\bar{R}, \omega) \cong \omega_{\bar{R}} \cong \bar{R}$  by Theorem 4.2.2. Also,  $\mathrm{Ext}_R^1(\omega, \omega) = 0$  by Proposition 4.2.8. Lastly, recall that  $\mathrm{Hom}_R(\omega, \omega) \cong R$  by Remark 2.3.4. With these, we have the following short exact sequence:

$$0 \longleftarrow \mathrm{Hom}_R(K, \omega) \longleftarrow \bar{R}^{\oplus \ell} \xleftarrow{\phi} R \longleftarrow 0.$$

By Proposition 4.2.6, we have

$$1 \geq \mathrm{depth}[\mathrm{Hom}_R(K, \omega)] \geq \min\{2, \mathrm{depth}(w)\} = 1.$$

Hence,  $\mathrm{Hom}_R(K, \omega)$  is a Cohen-Macaulay module. We need to show that  $R$  is normal, i.e.,  $R = \bar{R}$ . Assume that  $R \neq \bar{R}$ , which implies that  $\emptyset \neq \mathrm{Ass}\left(\frac{\bar{R}}{R}\right) \subseteq \mathrm{Supp}\left(\frac{\bar{R}}{R}\right)$ . Notice that

$$R_P = \bar{R}_P, \text{ for all } P \neq \mathfrak{m}$$

because  $\dim(R_P) = 0$  implies that  $R_P$  is regular as  $R_P$  is reduced. Thus, this implies that  $\mathrm{Supp}\left(\frac{\bar{R}}{R}\right) = \{\mathfrak{m}\}$ . Hence,  $\mathfrak{m} \in \mathrm{Ass}\left(\frac{\bar{R}}{R}\right)$ .

Next, let  $x = \phi(1)$  and let  $\psi : \bar{R} \longrightarrow \bar{R}x$  by  $\psi(a) = ax$ , for all  $a \in \bar{R}$ .

**Claim:**  $\psi$  is an  $\bar{R}$ -linear isomorphism (hence,  $R$ -linear isomorphism); in particular,  $\bar{R} \cong \bar{R}x$  as  $\bar{R}$ -modules (hence, as  $R$ -modules).

It is clear that  $\psi$  is an onto map. We just need to show that  $\psi$  is 1-1. Let  $ax = 0$  for some  $a \in \bar{R} \subseteq Q(R)$ . Write  $a = \frac{r}{s}$  for  $r \in R, s \in \text{NZD}(R)$ . Thus,  $\frac{r}{s}x = 0 \implies rx = 0 \implies \phi(rx) = 0 \implies r\phi(1) = 0 \implies r = 0 \implies a = 0$ , which proves the claim. Now we have the following:

$$\frac{\bar{R}}{R} \cong \frac{\bar{R}x}{Rx} \leq \frac{\bar{R}^{\oplus \ell}}{Rx} \cong \text{Hom}_R(K, \omega).$$

Since  $\mathfrak{m} \in \text{Ass}\left(\frac{\bar{R}}{R}\right) \subseteq \text{Ass}(\text{Hom}_R(K, \omega))$ ,  $\text{depth}\left(\frac{\bar{R}}{R}\right) = 0$ , which contradicts the fact that  $\text{Hom}_R(K, \omega)$  is Cohen-Macaulay. Hence,  $\bar{R} = R$ .

For (7)  $\implies$  (8), we again assume that  $R$  is local without loss of generality. If  $\dim(R) < 2$ ,  $R$  is regular. Therefore,  $R$  is  $F$ -rational by Corollary 3.2.2.

For (8)  $\implies$  (1), we assume that  $R$  is local without loss of generality since it is enough to show that  $R_P$  is  $F$ -rational for all  $P \in \text{Spec}(R)$ . Now, we let  $R = (R, \mathfrak{m}, k)$ . We induct on  $\dim(R)$ . By assumption,  $R$  is  $F$ -rational when  $\dim(R) = 0$  or 1. Now, we assume  $\dim(R_P) = d \geq 2$  and we assume that  $R_P$  is  $F$ -rational for all  $P \in \text{Spec}(R)$  such that  $\dim(R_P) \leq d - 1$ . Let  $c \in R^\circ$  be a test element (which exists). In light of Lemma 4.1.9, let  $e_0$  be a test exponent for  $\frac{c}{1} \in R_P$  and  $H_{R_P}^{\dim(R_P)}(R_P)$ , for all  $P \in \text{Spec}(R)$ . Since  $\dim(R_P) < d$  for all  $P \neq \mathfrak{m}$ ,  $R_P$  is  $F$ -rational, for all  $P \neq \mathfrak{m}$ . Thus,

$$T_{e_0(R_P), \omega_P} \left( e_0 \left( \frac{c}{1} \right) \right) = \omega_P, \forall P \neq \mathfrak{m},$$

which is equivalent to

$$(T_{e_0 R, \omega} (e_0 c))_P = \omega_P, \forall P \neq \mathfrak{m}$$

by Corollary 2.4.8. Thus,  $T_{e_0(R), \omega} (e_0 c)$  is  $\mathfrak{m}$ -primary in  $\omega$ . Thus, there exists  $\alpha \in \mathbb{N}$  such that  $T_{e_0(R), \omega} (e_0 c) \supseteq \mathfrak{m}^\alpha \omega$ . By assumption, there exists an  $e \in \mathbb{N}$  such that  $T_{\mathfrak{m}\omega, \omega} (e(\mathfrak{m}\omega)) = \omega$ .

**Claim:**  $T_{e\omega, \omega} (e(\mathfrak{m}\omega)) = \omega$ .

Consider this short exact sequence

$$0 \longrightarrow \mathfrak{m}\omega \longrightarrow \omega \longrightarrow \frac{\omega}{\mathfrak{m}\omega} \longrightarrow 0.$$

Then, we have this induced short exact sequence

$$0 \longrightarrow {}^e(\mathfrak{m}\omega) \longrightarrow {}^e\omega \longrightarrow {}^e\left(\frac{\omega}{\mathfrak{m}\omega}\right) \longrightarrow 0.$$

By long exact sequence, we obtain this exact sequence

$$\mathrm{Ext}_R^1\left({}^e\left(\frac{\omega}{\mathfrak{m}\omega}\right), \omega\right) \longleftarrow \mathrm{Hom}_R({}^e(\mathfrak{m}\omega), \omega) \longleftarrow \mathrm{Hom}_R({}^e\omega, \omega).$$

We note that  $\mathrm{Ann}_R({}^e(\frac{\omega}{\mathfrak{m}\omega})) = \mathfrak{m}$  and  $\mathrm{depth}(\omega) \geq 2$  since  $\dim(R) \geq 2$ . Thus, we can apply Corollary 4.2.7 to conclude that  $\mathrm{Ext}_R^1({}^e(\frac{\omega}{\mathfrak{m}\omega}), \omega) = 0$ . Thus, we get the following exact sequence

$$0 \longleftarrow \mathrm{Hom}_R({}^e(\mathfrak{m}\omega), \omega) \longleftarrow \mathrm{Hom}_R({}^e\omega, \omega).$$

Therefore, every homomorphism in  $\mathrm{Hom}_R({}^e(\mathfrak{m}\omega), \omega)$  can be extended to a homomorphism in  $\mathrm{Hom}_R({}^e\omega, \omega)$ . Thus, we can easily see the following by Definition 3.1.1.

$$\omega = T_{\mathfrak{m}\omega, \omega}({}^e(\mathfrak{m}\omega)) \subseteq T_{e\omega, \omega}({}^e(\mathfrak{m}\omega)) \subseteq \omega,$$

which proves the claim.

We use Proposition 3.1.2(7) and obtain the following:

$$\begin{aligned} T_{e\omega, \omega}({}^e(\mathfrak{m}\omega)) = \omega &\implies T_{e\omega, \omega}\left(\bigcup_{r_1 \in \mathfrak{m}} r_1 {}^e(\mathfrak{m}\omega)\right) \supseteq \bigcup_{r_1 \in \mathfrak{m}} r_1 \omega \\ &\implies T_{e\omega, \omega}\left(\bigcup_{r_1 \in \mathfrak{m}} {}^e(r_1^q \mathfrak{m}\omega)\right) \supseteq \bigcup_{r_1 \in \mathfrak{m}} r_1 \omega \\ &\implies T_{2e\omega, e\omega}\left(\bigcup_{r_1 \in \mathfrak{m}} {}^{2e}(r_1^q \mathfrak{m}\omega)\right) \supseteq \bigcup_{r_1 \in \mathfrak{m}} {}^e(r_1 \omega) \\ &\implies T_{2e\omega, e\omega}\left(\bigcup_{r_1, r_2 \in \mathfrak{m}} r_2 {}^{2e}(r_1^q \mathfrak{m}\omega)\right) \supseteq \bigcup_{r_1, r_2 \in \mathfrak{m}} r_2 {}^e(r_1 \omega) \end{aligned}$$

$$\implies T_{2e_\omega, e_\omega} \left( \bigcup_{r_1, r_2 \in \mathfrak{m}} 2e(r_2^{q^2} r_1^q \mathfrak{m}\omega) \right) \supseteq \bigcup_{r_1, r_2 \in \mathfrak{m}} e(r_2^q r_1 \omega).$$

Continue in this fashion, we will obtain that for all  $\ell \geq 1$ ,

$$T_{\ell e_\omega, (\ell-1)e_\omega} \left( \bigcup_{r_1, \dots, r_\ell \in \mathfrak{m}} \ell e(r_\ell^{q^\ell} \cdots r_2^{q^2} r_1^q \mathfrak{m}\omega) \right) \supseteq \bigcup_{r_1, \dots, r_\ell \in \mathfrak{m}} (\ell-1)e(r_\ell^{q^{\ell-1}} \cdots r_2^q r_1 \omega). \quad (\dagger_\ell)$$

We see that there exists an  $n$  such that

$$\bigcup_{r_1, \dots, r_n \in \mathfrak{m}} r_n^{q^n} \cdots r_2^{q^2} r_1^q \mathfrak{m}\omega \subseteq \mathfrak{m}^\alpha \omega \subseteq T_{e_0 R, \omega} ({}^{e_0} \mathcal{C}).$$

Using Lemma 3.1.6, we obtain

$$T_{ne({}^{e_0} R), ne_\omega} ({}^{ne}({}^{e_0} \mathcal{C})) \supseteq \bigcup_{r_1, \dots, r_n \in \mathfrak{m}} {}^{ne}(r_n^{q^n} \cdots r_2^{q^2} r_1^q \mathfrak{m}\omega). \quad (\ddagger_n)$$

Combining  $(\dagger_\ell)$  (with  $\ell = n$ ) and  $(\ddagger_n)$ , and applying Proposition 3.1.2(6), we obtain

$$T_{ne({}^{e_0} R), (n-1)e_\omega} ({}^{ne}({}^{e_0} \mathcal{C})) \supseteq \bigcup_{r_1, \dots, r_n \in \mathfrak{m}} (n-1)e(r_n^{q^{n-1}} \cdots r_2^q r_1 \omega). \quad (\star_{n-1})$$

Notice that  $T_{ne({}^{e_0} R), (n-1)e_\omega} ({}^{ne}({}^{e_0} \mathcal{C}))$  is a submodule of  $(n-1)e_\omega$  by Proposition 3.1.2(3). Also, notice

$$\left\langle \bigcup_{r_1, \dots, r_n \in \mathfrak{m}} (n-1)e(r_n^{q^{n-1}} \cdots r_2^q r_1 \omega) \right\rangle \supseteq \bigcup_{r_2, \dots, r_n \in \mathfrak{m}} (n-1)e(r_n^{q^{n-1}} \cdots r_2^q \mathfrak{m}\omega),$$

where  $\langle - \rangle$  denotes the module generated by  $-$ . Therefore, we obtain

$$\begin{aligned} T_{ne({}^{e_0} R), (n-1)e_\omega} ({}^{ne}({}^{e_0} \mathcal{C})) &\supseteq \left\langle \bigcup_{r_1, \dots, r_n \in \mathfrak{m}} (n-1)e(r_n^{q^{n-1}} \cdots r_2^q r_1 \omega) \right\rangle \\ &\supseteq \bigcup_{r_2, \dots, r_n \in \mathfrak{m}} (n-1)e(r_n^{q^{n-1}} \cdots r_2^q \mathfrak{m}\omega). \end{aligned}$$

Then, by relabeling, we obtain

$$T_{ne(e_0R), (n-1)e_\omega} ({}^{ne}e_0C) \supseteq \bigcup_{r_1, \dots, r_{n-1} \in \mathfrak{m}} ({}^{(n-1)e}e(r_{n-1}^{q^{n-1}} \cdots r_1^q \mathfrak{m}\omega)), \quad (\dagger_{n-1})$$

which is very similar to  $(\dagger_n)$ . Note that  $(\dagger_{n-1})$  is the following

$$T_{(n-1)e_\omega, (n-2)e_\omega} \left( \bigcup_{r_1, \dots, r_{n-1} \in \mathfrak{m}} ({}^{(n-1)e}e(r_{n-1}^{q^{n-1}} \cdots r_2^{q^2} r_1^q \mathfrak{m}\omega)) \right) \supseteq \bigcup_{r_1, \dots, r_{n-1} \in \mathfrak{m}} ({}^{(n-2)e}e(r_{n-1}^{q^{n-2}} \cdots r_2^q r_1 \omega)).$$

Again, combining  $(\dagger_{n-1})$  and  $(\dagger_{n-2})$ , and applying Proposition 3.1.2(6), we obtain

$$T_{ne(e_0R), (n-2)e_\omega} ({}^{ne}e_0C) \supseteq \bigcup_{r_1, \dots, r_{n-1} \in \mathfrak{m}} ({}^{(n-2)e}e(r_{n-1}^{q^{n-2}} \cdots r_2^q r_1 \omega)), \quad (\star_{n-2})$$

which is similar to  $(\star_{n-1})$ . As before, we note that  $T_{ne(e_0R), (n-2)e_\omega} ({}^{ne}e_0C)$  is a submodule of  $({}^{(n-2)e}e_\omega)$  by Proposition 3.1.2(3). Hence, we get that

$$\begin{aligned} T_{ne(e_0R), (n-2)e_\omega} ({}^{ne}e_0C) &\supseteq \left\langle \bigcup_{r_1, \dots, r_{n-1} \in \mathfrak{m}} ({}^{(n-2)e}e(r_{n-1}^{q^{n-2}} \cdots r_2^q r_1 \omega)) \right\rangle \\ &\supseteq \bigcup_{r_2, \dots, r_{n-1} \in \mathfrak{m}} ({}^{(n-1)e}e(r_{n-2}^{q^{n-2}} \cdots r_2^q \mathfrak{m}\omega)). \end{aligned}$$

Then, by relabeling, we obtain

$$T_{ne(e_0R), (n-2)e_\omega} ({}^{ne}e_0C) \supseteq \bigcup_{r_1, \dots, r_{n-2} \in \mathfrak{m}} ({}^{(n-1)e}e(r_{n-2}^{q^{n-1}} \cdots r_1^q \mathfrak{m}\omega)). \quad (\dagger_{n-2})$$

Therefore, we can now continue inductively and obtain  $(\dagger_0)$ , which is

$$T_{ne(e_0R), \omega} ({}^{ne}e_0C) \supseteq \mathfrak{m}\omega.$$

Then, by Lemma 3.1.6, we have that

$$T_{(n+1)e(e_0R), e_\omega} ({}^{(n+1)e}e_0C) \supseteq e(\mathfrak{m}\omega).$$

Also, recall that we have  $T_{e,\omega}(e(\mathfrak{m}\omega)) = \omega$  from claim above. Hence, by Proposition 3.1.2(6), we get  $T_{e_1R,\omega}(e_1c) = \omega$ , where  $e_1 = (n+1)e + e_0$ . Therefore,  $R$  is  $F$ -rational by Corollary 4.1.10 and this completes the proof.  $\square$

Many theorems can be reduced to the  $F$ -finite situation. For example, the existence of test element was reduced to the  $F$ -finite case (see [HH90a, §6] for details on gamma construction and see [Vél95, §2] for details on gamma construction done on  $F$ -rationality).

**Question 4.2.11.** Can we prove the Main Theorem without the  $F$ -finite assumption?

*Remark 4.2.12.* Here, we provide details on where  $F$ -finite assumption is not needed.

For (2)  $\implies$  (3), (4)  $\implies$  (5), (5)  $\implies$  (6), and (7)  $\implies$  (8), it is easy that we do not need the ring to be  $F$ -finite.

For (3)  $\implies$  (4) and (6)  $\implies$  (7), we can get rid of the  $F$ -finite assumption but we do need to assume  $\bar{R}$  to be finitely generated over  $R$  (e.g.,  $R$  is excellent).

However, we are not sure how to replace the  $F$ -finite assumption with something weaker for (1)  $\implies$  (2) and (8)  $\implies$  (1).

Other questions we are considering are the following:

**Question 4.2.13.** Instead of assuming that  $R$  is Cohen-Macaulay, can we replace that assumption by  $R$  having the  $S_2$  property in the Main Theorem?

**Question 4.2.14.** Can we replace  $T_{-, -}(-)$  by  $\langle T_{-, -}(-) \rangle$  in the Main Theorem?

For these two questions, we are quite hopeful that it will work out and we are currently working on them.



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