Preservice Teachers’ Understanding Of Geometric Definitions And Their Use In The Concept Of Special Quadrilaterals

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The purpose of this study is to see how preservice teachers understand mathematical definitions within a geometry context. Yet, within the collegiate mathematics coursework, many preservice teachers do have struggles with some of the basic geometric concepts. Consequently, this study specifically looks at the geometric definitions of quadrilaterals and how preservice teachers use those definitions to form a holistic understanding of quadrilaterals.

Using the Action-Process-Object-Schema (APOS) theory as the theoretical framework, the study proposes a preliminary genetic decomposition for the concept of the hierarchical properties of special quadrilaterals. Data is analyzed from interviews and class documents of
twenty-six preservice teachers as to whether they used the constructions from the preliminary genetic decomposition or other constructions not considered.

Due to the importance of mathematical definitions in preservice teachers’ background preparation for future field work, this study proposes the following questions:

1) What are preservice teachers’ understandings of geometric definitions?
   i. What are preservice teacher’s personal definitions for special quadrilaterals?
   ii. How do preservice teachers apply the distinction between necessary and sufficient conditions for a mathematical definition?

2) How does the understanding of geometric definitions contribute to preservice teachers’ understanding of special quadrilaterals?
   i. Are preservice teachers able to perceive and use the hierarchical nature of special quadrilaterals?
   ii. Are preservice teachers able to discern equivalent definitions for special quadrilaterals?

Based on the results of the data analysis, the genetic decomposition is revised. Finally, the study concludes with pedagogical recommendations for teaching the concept of special quadrilaterals and suggestions for further research on this topic.

INDEX WORDS: APOS Theory, Hierarchical Definitions, Preservice teachers, Special Quadrilaterals, Pedagogy
PRESERVICE TEACHERS’ UNDERSTANDING OF GEOMETRIC DEFINITIONS AND THEIR USE IN THE CONCEPT OF SPECIAL QUADRILATERALS

by

JEFFREY E. MCCAMMON

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy in the College of Arts and Sciences Georgia State University 2018
PRESERVICE TEACHERS’ UNDERSTANDING OF GEOMETRIC DEFINITIONS AND
THEIR USE IN THE CONCEPT OF SPECIAL QUADRILATERALS

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College of Arts and Sciences
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May 2018
DEDICATION

In honor of

Teresa Fletcher
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I give all praise, honor, and glory to God through Jesus Christ for the opportunity to do this work and the abilities He has provided me to make it possible.

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1 INTRODUCTION

The purpose of this study is to explore how preservice teachers understand mathematical definitions within a geometry context. Teacher preparation courses are designed to help future teachers in their own comprehension of the subject matter they will eventually teach in the field. Yet, within the collegiate mathematics coursework, many preservice teachers do have struggles with some of the basic geometric concepts. Consequently, this study will specifically look at the geometric definitions of quadrilaterals and how preservice teachers use those definitions to form a holistic understanding of quadrilaterals.

Mathematical definitions are one of the cornerstones of mathematics. Proofs, which may be the essence of advanced mathematical thought, are built on the deductive analysis between definitions, axioms, and previously proven theorems (Brown, 1998). To build on prior mathematical knowledge, the mathematician uses these new theorems to construct new mathematical objects. Thus, with each new object, and for convenience of representation to the mathematical community, a definition follows which clarifies the essence of the object for future use and study.

In every mathematical system, a simple set of primitives can be used to create defined terms (Brown, 1998). The primitive terms, sometimes referred to as undefined terms, may be thoroughly described within the system. For example, in Set Theory, the primitives of set and is a member of can be used to build all the remaining objects and axioms of the system (Brown, 1998). Yet, no specific definition of set or membership exists within the system.

The criteria for mathematical definitions are both eliminability and non-creativity. Any defined term can be replaced with an expression in terms of primitive terms (eliminability) and
definitions cannot help prove new theorems than what could have been proven without the definitions (non-creativity) (Brown, 1998). In this instance, definitions represent mathematical shorthand to describe new objects. Yet, when a definition is introduced, it cannot produce new theorems just with the introduction of the new term alone. Brown shows that these concepts were developed from the debates of Hilbert and Frege. Hilbert's work in geometry was to demonstrate how axioms built on each other and each axiom was consistent based on the consistency of the original axioms. Thus, definitions came from the context of the axioms. "Terms are not explicitly and independently defined, but rather pick up their meaning by figuring in the axioms" (Brown, 1998, p. 115). Frege felt that terms should already have meaning before the axioms. "If terms are being defined by the axioms, then taking an axiom to be true in one setting and false in another setting changes the very meaning of any terms involved" (Brown, 1998, p. 119). Thus, consistency proofs that Hilbert proposed as his formal approach would not be necessary since definitions are separate from the axioms.

Lakatos offers yet another view of definition. He believed that definitions are theoretical and should not be fixed (Lakatos, 1976). Instead definitions are changed through the proofs of mathematical theorems. Brown argues that Lakatos felt that mathematics does not have a foundation and so primitives would not exist. Thus, the difference between definitions and theorems is only that theorems must be proved whereas definitions would adapt based on the contextual need and the conventions within the mathematical community.

Yet, a mathematical object or symbol could have multiple definitions based on the context of the definition. Wilhelmi, Godino and Lacasta explain their notion of “holistic meaning” when dealing with the different categorical definitions of absolute value notion (Wilhelmi, Godino, & Lacasta, 2007). For example, the absolute value of a real number is based on whether the number
is originally negative or positive which can be easily determined in the real numbers. Yet, the absolute value of a complex number is defined as a measure of the distance from zero on the complex coordinate system. Furthermore, Euclidean distance can be called the absolute value. In this context, each definition is based on a higher level of abstraction of the use of absolute value.

In mathematics education, definitions can have various roles and uses. Definitions introduce objects to a mathematical theory and help express the properties of the defined objects (Mariotti & Fischbein, 1997). Definitions can also be a fundamental part of concept formation (Tall & Vinner, 1981; Vinner, 1976). In addition, a definition can help categorize the difference between examples of an object and nonexamples (Tsamir, Tirosh, & Levenson, 2008).

In the remaining sections of this chapter, I will establish why a study regarding preservice teachers’ understanding of mathematical definitions in geometry is necessary; state the goals of this study; and provide a theoretical framework that will guide the data analysis.

1.1 Statement of the Problem

Preservice teachers encounter mathematical definitions not only throughout their preparatory coursework but in the field. The difficulties that teachers have in their geometric understanding can eventually affect their practice with students (Mayberry, 1983; Quinn, 1997). Quinn found that methods courses that used manipulatives, technology, and cooperative learning brought a significant change in the attitudes of preservice elementary teachers towards mathematics (Quinn, 1997). Quinn suggests that teaching Geometry in courses in a less abstract fashion while modeling appropriate teaching pedagogies would be of great benefit to the teachers.

Cunningham and Roberts tested preservice teachers on their understanding of the altitude of a triangle and the diagonals of a polygon (Cunningham & Roberts, 2010). Teachers could answer the questions well until the non-prototypical altitude of a right triangle and the non-
A prototypical diagonal of a concave polygon resulted in significant decrease in correct answers even when the concept definition was given to the teachers. Their research shows that there were disconnections between the definition and the actual understanding of the concept.

Mayberry applied a test designed to analyze which van Hiele Level of understanding (see section 2.1) preservice teachers exhibited for different geometric concepts (Mayberry, 1983). Her research showed that many of the preservice teachers who had even taken high school geometry were below Level III. Thus, these future teachers were not ready for a more formal geometry course. Furthermore, the tests revealed that students were on different levels for different concepts.

Consequently, research has attempted to address curricular and pedagogical changes in educational coursework for teachers (Graeber, 1999; Shriki, 2010). Graeber was concerned with how university students had difficulty in grasping the ‘big ideas’ of a course and what areas should preservice teachers be covering in their methods courses (Graeber, 1999). One of the conclusions in her research is that preservice teachers ought to have the knowledge of the difference between procedural knowledge and conceptual knowledge. For example, students might understand the procedure of multiplying two fractions but not conceptually understand fractions or multiplication. Likewise, preservice teachers who can only do procedural mathematics will have difficulty with explaining mathematics on a conceptual level to their students.

Shriki gave preservice teachers an opportunity to explore the realm of creativity in their mathematics by creating a new geometric concept and finding appropriate properties of this new concept (Shriki, 2010). The results led to a significant development of the preservice teachers’ mathematical knowledge, the meaning of definitions, and how mathematical objects related to each other. However, Shriki found that several of the preservice teachers did not enjoy the process.
and suggested that further research is needed to determine the inhibitions by learners of using creativity in mathematics education.

In the *Principles and Standards for School Mathematics* (*NCTM, 2000*), the National Council of Teachers of Mathematics has emphasized that all grades 3-5 students should:

- Classify two – and three – dimensional shapes according to their properties and develop definitions of classes of shapes;
- Make and test conjectures about geometric properties and relationships and develop logical arguments to justify conclusions.

Furthermore, in grades 6-8 all students should:

- Precisely describe, classify, and understand relationships among two – and three – dimensional objects using their defining properties.

Furthermore, the Common Core State Standards Initiative ("Common Core Standards," 2014) for mathematics state the following:

Grade 3: Reason with shapes and their attributes.

- Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attribute (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.

Grade 5: Classify two – dimensional figures into categories based on their properties.

- Understand that attributes belonging to a category of two – dimensional figures also belong to all subcategories of that category. For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.
• Classify two – dimensional figures in a hierarchy based on properties.

Both NCTM and the Common Core State Standards Initiative emphasize the importance of special quadrilaterals in their respective lists of standards. Teachers of elementary mathematics will need to understand these concepts well to adequately instruct students.

1.1.1 Elementary School Teacher Preparation

This section overviews the process of elementary school teacher preparation. Since teacher certification is based on state certification requirements instead of federal regulations, I will present the process of certification required by the state of Georgia of which this study took place. After an overview of the general process of elementary school teacher preparation, I will look further at specific standards in mathematics and geometry that teachers must master before taking their certification examination.

1.1.1.1 Elementary School Teacher Preparation in the State of Georgia

The Georgia Professional Standards Commission manages all the regulations and procedures toward teacher certification ("Teacher Certification Degrees," 2015). A teacher must hold a bachelor’s degree and obtain teacher certification from an accredited institute. Certificates come in two forms: Clear Renewable and Non-Renewable. The Clear Renewable Certificate is obtained through a more traditional route of a university teacher preparation program in coordination with a bachelor’s degree. Under special circumstances, an educator can receive a non-renewable certificate which may reflect an alternative teacher preparation program. In addition, all teachers must pass the appropriate assessment from the Georgia Assessments for the Certification of Educators (GACE).
Alternative routes for gaining teacher certification are available, typically for those holding a bachelor’s degree or higher in addition to the desire to teach in a “high need” area. Specifically, teachers can commence teaching with a Non-Renewable Non-Professional certificate or through the Georgia Teacher Academy for Preparation and Pedagogy (GaTAPP) certification pathway. Upon completion of this process, the teacher will be eligible for the Clear Renewable certificate. Approved fields of certification for elementary school teachers are the Elementary Childhood Education (Birth – Grade 5) and Middle Grades Education (Grades 4-8).

On a traditional pathway, the future educator may attend an institute that offers an accredited Georgia teacher certification program. These programs must be approved by the Georgia Professional Standards Commission (GaPSC) as well as the Southern Association of Colleges and Schools (SACS). Although not mandatory, future educators should also consider a program that is accredited by the Council for Accreditation of Educator Preparation (CAEP). This national agency is recognized by the US Department of Education and is known for approving a distinguished standard of teaching excellence in the educational field.

In the next section, I will discuss further the traditional pathway for elementary school teacher preparation using a college with an approved accredited program for elementary school teacher preparation.

1.1.1.2 Elementary Teacher Preparation in the Collegiate Level

The Georgia Profession Standards Commission (GaPSC) oversees and approves educator preparation programs in Georgia ("Georgia Standards for the Approval of Professional Education Units and Educator Preparation Programs," 2008). There are 8 standards that the program must adhere to to be approved by the GaPSC of which two standards are relevant to this research:
Standard 1: Candidates preparing to work in schools as teachers or other school professionals know and demonstrate the content knowledge, pedagogical content knowledge and skills, pedagogical and professional knowledge and skills and professional dispositions necessary to help all students learn. Assessments indicate that candidates meet professional, state, and institution/agency standards.

Standard 4: The professional education unit designs, implements, and evaluates curriculum and provides experiences for candidates to acquire and demonstrate the knowledge, skills, and professional dispositions necessary to help all students learn. Assessments indicate that candidates can demonstrate and apply proficiencies related to diversity. Experiences provided for candidates include working with diverse populations, including higher education and P-12 school faculty, candidates, and students in P-12 schools.

In addition, CAEP has aligned with the National Council for Accreditation of Teacher Education regarding standards for educator preparation. Standard 1 specifically states that “Candidates demonstrate knowledge, skills, and professional dispositions for effective work in schools” ("CAEP Standards for Educator Preparation Aligned with the NCATE Standards," 2013). This standard is aligned with the GaPSC’s first standard for approved teacher preparation programs.

The emphasis on teacher knowledge and skills for future educators provides the motivation for this research. Elementary school teachers are to excel in the content areas of their respective subjects. Likewise, an educator preparatory program must design a curriculum (Standard 4) that emphasizes teacher content knowledge (Standard 1). Specifically, in preparation for Common Core Mathematics standards, the teacher must understand the material being taught and know multiple methods to reach the diversity of learning styles in the student population.
The next section reviews the necessary knowledge of mathematics content that elementary teachers must master. This material may be covered on the GACE examination required for certification. In addition, this mathematics will be in the standards that potentially will be taught in the regular education classroom.

### 1.1.2 Preparation of Mathematics Content

Elementary school teachers must know and understand the mathematics they are required to teach. With the advent of the Common Core Standards, educators should focus on the following domains in elementary schools: Counting and Cardinality, Operations and Algebraic Thinking, Number and Operations in Base Ten, Measurement and Data, Geometry, and Numbers and Operations with Fractions. Consequently, educator preparation programs expect evidence of content understanding, methods preparation, or a combination of both in the forms of coursework, assessments, and passing the state certification examination.

In Georgia, the GACE examination for Early Childhood Education has two parts and emphasizes six major topics: Reading and Language Arts, Social Studies, Analysis, Mathematics, Science, and Health Education/Physical Education/the Arts. The mathematics portion of the test is 53% of the second test. Specifically, each section has test objectives that are aligned with the questions.

- **Objective 1**: Understand and applies knowledge of counting and cardinality
- **Objective 2**: Understands and applies knowledge of operations and algebraic thinking
- **Objective 3**: Understands and applies knowledge of numbers and operations in base 10
- **Objective 4**: Understands and applies knowledge of numbers and fractions
- **Objective 5**: Understands and applies knowledge of measurement concepts and data
Objective 6: Understands and applies knowledge of geometry.

Under Objective 6, the GACE exam also focuses to:

The beginning Early Childhood Education teacher:

A. Knows how to reason with shapes and their attributes

B. Knows how to graph points on the coordinate plane to solve real-world and mathematical problems

C. Knows how to draw and identify lines and angles and can classify shapes by properties of their lines and angles

Although the GACE exam can only cover certain topics to be tested, the test design provides an emphasis of study that the examinee must know before the test. Furthermore, the material is in alignment with Common Core Standards.

In the next section, I will go more in depth with the Geometry standards that elementary school teachers must know and prepare for in the classroom.

1.1.3 Preparation in Geometry

Several Common Core standards prepare students for problem solving with special quadrilaterals.

Grade 2: Recognize and draw shapes having specified attributes, such as a given number of angles or a given number of equal faces. Identify triangles, quadrilaterals, pentagons, hexagons, and cubes.

Grade 3: Understand that shapes in different categories (e.g., rhombuses, rectangles, and others) may share attributes (e.g., having four sides), and that the shared attributes can define a larger category (e.g., quadrilaterals). Recognize rhombuses, rectangles, and squares as examples
of quadrilaterals, and draw examples of quadrilaterals that do not belong to any of these subcategories.

Grade 5: Understand that attributes belonging to a category of two-dimensional figures also belong to all subcategories of that category. For example, all rectangles have four right angles and squares are rectangles, so all squares have four right angles.

Classify two-dimensional figures in a hierarchy based on properties.

Preservice teachers must acquire understanding in these areas before teaching the concept of special quadrilaterals in the elementary education classroom. University programs may require their students to take math classes that cover most of the mathematics they will need in their practice. In this study, the participants involved were from such a math class that prepared future elementary school teachers for the Geometry content they would eventually teach.

This section concludes the general outline for elementary school teacher preparatory programs with emphasis on mathematics and geometry. In the following section, I will review the research questions for this study.

1.2 Research Questions

Due to the importance of mathematical definitions in preservice teachers’ background preparation for future field work, this study proposes the following research questions:

1) What are preservice teachers’ understandings of geometric definitions?
   i. What are preservice teacher’s personal definitions for special quadrilaterals?
   ii. How do preservice teachers apply the distinction between necessary and sufficient conditions for a mathematical definition?
2) How does the understanding of geometric definitions contribute to preservice teachers’ understanding of special quadrilaterals?

i. Are preservice teachers able to perceive and use the hierarchical nature of special quadrilaterals?

ii. Are preservice teachers able to discern equivalent definitions for special quadrilaterals?

A research framework is needed to guide the design of the study and analysis of data regarding the understanding of mathematics. This study uses the Action-Process-Object-Schema (APOS) theory (Arnon et al., 2014; Asiala et al., 1996; Cottrill et al., 1996). Each of these constructs will be elaborated in the following section.

1.3 Theoretical Perspective

APOS Theory, which is based on the ideas of Piaget, states that the process of learning a concept involves a construction of mental structures and certain mental mechanisms that are applied to the particular concept (Dubinsky, Weller, McDonald, & Brown, 2005). These mechanisms are called *interiorization, encapsulation, coordination, and generalization* and the structures are *actions, processes, objects, and schemas*. Understanding a concept requires a construction of the schema corresponding to that concept (Hamdan, 2006). The description of how students may learn a concept through this construction process is called a *genetic decomposition* of the concept.

APOS Theory is based on two hypotheses. The first hypothesis is as follows:

An individual’s mathematical knowledge is her or his tendency to respond to perceived mathematical problem situations by reflecting on problems and their solutions in a social
context and by constructing or reconstructing mathematical actions, processes and objects and organizing these in schemas to use in dealing with situations. (Asiala et al., p. 5)

This hypothesis acknowledges that what a person knows and her capabilities are not always readily available to her in each situation. The issue is two-fold: learning the concept and being able to access it when needed. The reconstructing of mathematical knowledge may hinge on initial perception of that knowledge and the reflection of that work.

The second hypothesis is centered on learning and teaching. Put simply, learners do not learn mathematical concepts linearly. When individuals are considering a new mathematical concept, not all the relevant mental constructions are remembered in this situation. Consequently, students cannot be expected to learn mathematics in the logical, axiomatic system that may be organized for the mathematical community (Arnon et al., 2014; Asiala et al., 1996). Rather, students need a holistic approach to the presentation of the concept. Even so, the student’s understanding may grow in spurts and stops; the student may even develop only partial understanding of the concept.

In APOS Theory, a learner’s level of mathematical knowledge can be represented as one of four general stages: actions, processes, objects, and schema. According to the theory, the development of a mathematical concept originates as an individual applies a transformation on one of these types of mathematical knowledge. The initial conception of the mathematical concept is an action in that the learner is only able to perform under the reaction of an external stimulus (Cottrill et al., 1996). The learner might be able to recall a fact from memory or is able to perform a task by rote skill. For example, a student may understand that a rectangle is a quadrilateral with congruent angles. A student has an action conception of a rectangle if he/she can only identify its angle properties by physically drawing a rectangle and labeling it.
When the action is repeated and reflected upon, it may be interiorized into a mental process. The learner has adapted the concept so that he does not have to actually perform the action explicitly but rather mentally (Trigueros & Martinez-Planell, 2010). The process is perceived as an internal construction and does not depend on the external nature of an action. Using the rectangle example, the student is at a process conception of understanding of a rectangle when she can state that all rectangles have congruent angles by imagining in her mind different rectangles and observing that the angles are congruent. Through reflection on those actions in her mind without drawing the rectangle she can conclude that all rectangles would have congruent angles.

Once a learner developed a process conception, the process can be transformed in several ways. Processes can be reversed (e.g. given the property that a quadrilateral has congruent angles, it must in fact be a rectangle) and/or can be coordinated with two or more other processes to form another process or even a schema (Cottrill et al., 1996).

If an individual becomes aware of a process as a totality and can construct transformations explicitly or mentally on it then the individual has encapsulated the process into an object. The learner can also de-encapsulate the object back to the process from which it came from (Arnon et al., 2014; Asiala et al., 1996). In the concept of a rectangle, students are operating at an object understanding when the student can compare a rectangle to other objects, such as kites. That is, to compare a rectangle to a kite, the student needs to think of both entities as objects. Further, to perform the comparison, the student needs to de-encapsulate each object back to its original process including the properties. For example, the object of a rectangle (and kite) would need to be de-encapsulated back to the properties of the rectangle (and kite). The student would compare these different sets of properties and then encapsulate these properties that are similar or different on the two objects.
Once an individual has constructed objects and processes, then an interconnection can link more than one process together. All the actions, objects, and coordinated processes related to the concept make up the schema of the mathematical concept. A schema refers to all the mental objects and operations which the learner is able to develop for the mathematical concept (Dubinsky, 1986). A schema can be treated as an object and included in an organization of a “higher level” schema (Arnon et al., 2014; Asiala et al., 1996). A student who can take properties of rectangles and apply them in other contexts (such as advanced area or volume problems) is working at a schema level.

The paradigm for APOS theory consists of a three-step cycle: Theoretical Analysis, Design and Implement Instruction, and Observations/Assessments (Arnon et al., 2014; Asiala et al., 1996; Cottrill et al., 1996). Research begins with the Theoretical Analysis component by evaluating the epistemology of the mathematical concept, the researcher’s own understanding of the concept, and the Literature Review. Based on this analysis, the initial (hypothetical) genetic decomposition is developed. For this study, the concept of understanding special quadrilaterals will be broken down into elemental mental constructs that an individual would need to develop to understand this concept.

This initial genetic decomposition leads to the component of Designing and Implementing Instruction. The design of the instruction should address how the learner moves through stages of cognitive growth through each of the mental constructs as proposed in the genetic decomposition. By implementing the instruction, the cycle moves to a stage of Observation and Assessments. Data is gathered and analyzed which leads to a reconsideration of the initial theoretical analysis. In this study, a revised genetic decomposition of the concept of special quadrilaterals will emerge that will lead to future pedagogical implications. Of interest in this study will be the role of
mathematical definitions and how students’ understandings of this notion impact their concepts of special quadrilaterals.

In short, this study will begin with an initial genetic decomposition of the concept of quadrilaterals. A key component of that genetic decomposition is the notion of mathematical definitions and how hierarchical definitions provide a systematic organization of quadrilaterals relating the properties of one object to the properties of another more generalized description of the object. For example, a hierarchical definition is used when the properties of a rectangle are noted to also be included as some of the properties of a square.

1.4 Outline of the Study

In the following chapters, this study will attempt to answer the research questions (as stated above in Section 1.2) regarding the understanding of mathematical definitions in a geometry context. Chapter 2 presents a review of the literature to the field of research in analyzing student conceptions in geometry. The literature review includes prior research conducted on students’ understanding of special quadrilaterals and the notion of mathematical definitions. Chapter 3 describes how APOS theory will be used throughout the analysis. A preliminary genetic decomposition will be discussed. This chapter also gives details to the methodology for data collection and analysis. Chapter 4 reports the results of the data analysis pertaining to each research question. Chapter 5 provides conclusions of the research. This chapter also presents implications for pedagogical applications, limitations of the study, and concludes with suggestions for future research on the topic of special quadrilaterals.
2 LITERATURE REVIEW

In the previous chapter, I gave an introduction and a brief overview of literature that justifies the need for this study. This chapter provides a more comprehensive literature review that shows relevant research in this field of study. Section 2.1 describes an epistemological analysis of the concept of special quadrilaterals. Section 2.2 reviews literature related to different frameworks for analyzing geometrical reasoning.

2.1 Epistemological Analysis of the Concept of Special Quadrilaterals

In the following sections, I will focus on studies that regard the nature of learning the concept of special quadrilaterals. Section 2.1.1 discusses the nature of mathematical definitions. Section 2.1.2 will present the standard mathematical definition of polygon and special quadrilaterals. Finally, Section 2.1.3 will give the definitions as presented by the textbook used in the Geometry course the students took.

2.1.1 The Nature of Mathematical Definitions

A definition cannot be proven but instead it introduces a new expression or word that can describe the mathematical context succinctly. All mathematical systems are built on "primitives" or terms that cannot be defined within the system. A definition uses primitives or other definitions to construct a new term that can be useful in proving theorems within the system.

In Euclidean Geometry, "points" and "lines" are undefined in the system. These terms would be primitives. Yet, a term like "collinear points" which is defined as points on the same line would be considered a definition based on the primitives. This new term is used as an abbreviation or shortcut to theories that need collinear points.
The concept of definition has evolved over time. A modern view would be that definitions must exhibit the characteristics of eliminability and non-creativity (Brown, 1998). A defined term can be explicitly replaced with undefined terms, known as primitives (eliminability), and no theorems can be proven with definitions that could not be proven without them (non-creativity). These concepts were developed from the debates of Hilbert and Frege. Hilbert's work in geometry was to demonstrate how axioms built on each other and each axiom was consistent based on the consistency of the original axioms. Thus definitions came from the context of the axioms (p. 113). "Terms are not explicitly and independently defined, but rather pick up their meaning by figuring in the axioms" (p. 115). Frege felt that terms should already have meaning before the axioms. "If terms are being defined by the axioms, then taking an axiom to be true in one setting and false in another setting changes the very meaning of any terms involved" (p. 119). Thus, according to Frege, consistency proofs that Hilbert proposed as his formal approach would not be necessary since definitions are separate from the axioms.

Definitions are one of the foundations of mathematical proof and theory. Zaslavsky and Shir describe the main roles of mathematical definitions as follows:

1) Introducing the objects of a theory and capturing the essence of a concept by conveying its characterizing properties

2) Constituting fundamental components for concept formation

3) Establishing the foundation for proofs and problem solving

4) Creating uniformity in the meaning of concepts, which allows us to communicate mathematical ideas more freely (Zaslavsky & Shir, 2005)

Furthermore, Vinner suggests five characteristics of the role of definitions in mathematics:

1) Concepts are mainly acquired by means of their definitions.
2) Students will use definitions to solve problems and prove theorems when necessary from a mathematical point of view.

3) Definitions should be minimal.

4) It is desirable that definitions will be elegant.

5) Definitions are arbitrary. (Vinner, 1991)

In terms of arbitrary, one could define a trapezoid as a quadrilateral having at least one pair of opposite sides which are parallel. Another definition of a trapezoid could be a quadrilateral having only one pair of opposite sides which are parallel. In which case, the former definition implies that parallelograms are trapezoids whereas in the later definition they are not. To differentiate the two definitions, I will refer to the first definition as the *inclusive* definition of a trapezoid and the second definition as the *exclusive* definition of a trapezoid.

Similarly, Winicki-Landman and Leikin suggest the following mathematical characteristics of a definition:

1) Defining is giving a name. The name of the new concept is presented in the statement used as a definition and appears only once in this statement.

2) For defining the new concept, only previously defined concepts may be used.

3) A definition establishes necessary and sufficient conditions for the concept.

4) The set of conditions should be minimal.

5) A definition is arbitrary. (Winicki-Landman & Leikin, 2000)

The necessary conditions would be the properties of the concept. The sufficient conditions are indications of the concept. Any statement that provides both necessary and sufficient conditions can define the concept. Moreover any statement that belongs to the class of definitions could be arbitrarily used as a definition while the other statements become theorems of the concept.
(Winicki-Landman & Leikin, 2000; Zazkis & Leikin, 2008). For example, a square could be defined as a rhombus with congruent angles. A square could also be defined as a rectangle with congruent sides. Whichever definition is used the other statement can be proven using the definition and the given properties of a rhombus or rectangle.

Yet another choice in the creation of definitions is whether a concept will follow in a hierarchical relationship with another concept or whether a partitional system is employed. The hierarchical classification implies that a concept represents a subset of another concept. For example, the definition that a rhombus is a kite with congruent sides classifies all rhombuses with the characteristics of kites. Yet a partitional system makes the concept disjoint from another object. In this case, although unconventional, a kite could be defined as a quadrilateral with two distinct pairs of adjacent congruent sides and all sides cannot be congruent. By this definition rhombuses and kites are separate objects. De Villiers suggest several reasons that a hierarchical definition be considered over the partitional system:

1) It leads to more economical definitions of concepts and formulation of theorems.
2) It simplifies the deductive systematization and derivation of the properties of more special concepts
3) It often provides a useful conceptual schema during problem solving
4) It sometimes suggests alternative definitions and new propositions
5) It provides a useful global prospective. (De Villiers, 1994)

In general, hierarchical definitions are shorter than partitional definitions since many times the partitional definition must specifically exclude characteristics of other objects. Hierarchical inclusion also assists in proving properties of an object especially when that object is in a subset classification of another object. For example, if a rhombus is defined in terms of being a special
kite then all the properties of kite are also properties of a rhombus. In terms of global perspectives, hierarchical definitions can also lead to greater connectivity between objects. As an example, since a rhombus is an intersection of a kite and a parallelogram it follows that the diagonals of a rhombus must be perpendicular bisectors since all the properties of the diagonals of a kite (perpendicular) and the properties of the diagonals of a parallelogram (bisect each other) must be in a rhombus.

Lakatos offers yet another view of definition. He believed that definitions are theoretical and should not be fixed. Instead definitions are changed through the proofs of mathematical theorems. Lakatos felt that mathematics does not have a foundation and so primitives would not exist (Brown, 1998). Thus, the difference between definitions and theorems is only that theorems must be proved whereas definitions would adapt based on the contextual need and the conversations within the mathematical community. The Lakatosian viewpoint establishes definitions as generated from proofs (Ouvrier-Buffet, 2006). However, this viewpoint is difficult to bring to the classroom. Pimm states:

(This notion) seems particularly problematic in terms of teaching mathematics, because of needing to perceive the definition as a tool custom-made to do a particular job that cannot be known by those trying to learn it, certainly not with an order of presentation that seems to require definitions to come first (Pimm, 1993, p. 272).

Yet Lakatos’ approach does provide a model of mathematical discovery which combined both the social and conceptual aspects (Ouvrier-Buffet, 2006).

Teachers of mathematics must be proficient in mathematical definitions especially when textbooks and curriculum materials may differ in the conventions used to define geometric objects.
If teachers do not have deeper understanding of these conventions and their implications, they will have limitations in their guidance of student learning.

The next section covers standard geometric definitions of polygons and special quadrilaterals.

2.1.2 Standard Geometric Definitions

The following sections will present definitions of polygons and special quadrilaterals. These definitions have been created under the same criteria listed in the previous section. Of interest in this study are the following criteria for a definition of a special quadrilateral:

i) Identifies that object is a quadrilateral (or closed four-sided polygon or figure)

ii) Identifies properties correctly

iii) Establishes necessary and sufficient conditions

iv) The set of conditions should be minimal

Since each of the figures in this study are special quadrilaterals, the definition given by students needs to at least identify that the figure is a quadrilateral (criterion (i)). Also, a definition must have correctly listed properties according to mathematical convention (criterion (ii)). As teachers of elementary students, preservice teachers must convey correct mathematical knowledge. Criterion (iii) employs Winicki-Landman and Leikin’s condition that definitions must establish necessary and sufficient conditions (Winicki-Landman & Leikin, 2000). Finally, Vinner, Winicki-Landman, and Leikin recommend that definitions should be minimal (Vinner, 1991; Winicki-Landman & Leikin, 2000). Minimal means using the least amount of properties to sufficiently define the figure. For example, the following definition is not minimal: a rhombus is a
parallelogram with congruent sides and perpendicular diagonals. A minimal definition could be the following: a rhombus is a quadrilateral with congruent sides.

I will start with a definition of a polygon and then the remaining definitions will be specific quadrilaterals.

2.1.2.1 Definition of a Polygon

A polygon can be defined as a geometric object "consisting of a number of points (called vertices) and an equal number of line segments (called sides), namely a cyclically ordered set of points in a plane, with no three successive points collinear, together with the line segments joining consecutive pairs of the points. In other words, a polygon is closed broken line lying in a plane" (Coxeter, 1967, p. 51). Another definition is “a polygon is a closed, two-dimensional figure that consists of three or more straight line segments” (Salomon, 2011, p. 88) Both of these definitions do not define what is meant by a closed figure in the plane and they both imply that the polygon is the border of the figure and not the interior (Weisstein, 1999-2015a).

The complexities of the polygon shape show up in different forms. Polygons can be convex, concave, or star (Weisstein, 1999-2015a). A planar polygon is convex if it contains all the line segments that connect any two points on the polygon. If a planar polygon is not convex, it must be concave. The star polygon is formed by connecting with straight lines every qth point out of p regularly spaced points lying on a circumference (Weisstein, 1999-2015b).

2.1.2.2 Definition of a Square

In a study by Zazkis and Leikin, participants were asked to give as many examples for the definition of a square (Zazkis & Leikin, 2008). The following are 13 examples from an expert example space:
1) A regular quadrilateral
2) A quadrilateral with all the angles and all the sides equal
3) A quadrilateral with all the sides equal and an angle of 90°
4) A rectangle with equal sides
5) A rectangle with perpendicular diagonals
6) A rhombus with equal angles
7) A rhombus with equal diagonals
8) A parallelogram with equal adjacent angles and equal adjacent sides
9) A parallelogram with equal and perpendicular diagonals
10) A quadrilateral having 4 symmetry axes
11) A quadrilateral symmetric under rotation by 90°
12) The locus of all the points in a plane for which the sum of the distances from two given perpendicular lines is constant
13) The locus of all the points in a plane for which the maximum of the distances from two given perpendicular lines is constant

2.1.2.3 Definition of a Rectangle

The following definitions for a rectangle are acceptable in this study:
1) A quadrilateral with four right angles.
2) A quadrilateral with congruent angles.
3) A parallelogram with at least one right angle.
4) A parallelogram with four right angles.
5) A parallelogram with congruent angles.
6) A parallelogram with congruent diagonals.
2.1.2.4 Definition of a Parallelogram

The following definitions for a parallelogram are acceptable in this study:

1) A quadrilateral with opposite sides that are parallel.
2) A quadrilateral with opposite sides that are congruent.
3) A quadrilateral with a pair of congruent parallel sides.
4) A quadrilateral with diagonals that bisect each other.
5) A quadrilateral with congruent opposite angles.

2.1.2.5 Definition of a Rhombus

The following definitions for a rhombus are acceptable in this study:

1) A quadrilateral with congruent sides.
2) A quadrilateral whose diagonals are perpendicular bisectors of each other.
3) A parallelogram with congruent sides.
4) A quadrilateral with both diagonals are lines of symmetry.

2.1.2.6 Definition of a Kite

The following definitions for a kite are acceptable in this study:

1) A quadrilateral with two distinct pairs of congruent adjacent sides.
2) A quadrilateral with at least one diagonal that is a line of symmetry.
3) A quadrilateral with at least one diagonal that is a perpendicular bisector of the other diagonal.
2.1.2.7 **Definition of a Trapezoid (inclusive)**

Trapezoids can be defined inclusively or exclusively. An inclusive definition allows for other special quadrilaterals to be special cases of the trapezoid. The inclusive definition of a trapezoid is a quadrilateral that has at least one pair of parallel sides. Consequently, parallelograms, rectangles, rhombuses, and squares are all special cases of a trapezoid.

2.1.2.8 **Definition of a Trapezoid (exclusive)**

The exclusive definition of a trapezoid removes the hierarchical nature and partitions trapezoids as an object of its own kind. The exclusive definition of a trapezoid is a quadrilateral with only one pair of parallel sides. Consequently, no other special quadrilateral listed above is a special case of a trapezoid.

2.1.3 **Textbook Definitions**

The preservice teachers who took the one semester Geometry course were given two textbooks for their Geometry studies. One textbook (Tussy, 2010) used in the course defines polynomials as a “closed geometric figure with at least three line segments for its sides.” A quadrilateral is a “polygon with four sides.” The text then proceeds to mention common quadrilaterals and gives a brief description/definition:

- Parallelogram (Opposite sides parallel)
- Rectangle (Parallelogram with four right angles)
- Square (Rectangle with sides of equal length)
Rhombus (Parallelogram with sides of equal length)

Trapezoid (Exactly two sides parallel).

The other textbook (Aichele, 2008) gives the following definitions for some special quadrilaterals:

Inclusive Trapezoid: a quadrilateral with at least one pair of parallel sides.

Exclusive Trapezoid: a quadrilateral with only two sides parallel.

Kite: A quadrilateral with two separate pairs of equal adjacent sides.

Students were familiar with these definitions from the class instruction.

In the next section, I will introduce prior studies in special quadrilaterals.

2.2 Prior Studies on Special Quadrilaterals

There are three significant frameworks that have been used in geometric mathematical research. A predominant framework in geometric mathematical research since 1976 has been the van Hiele Levels of Understanding (Burger & Shaughnessy, 1986; Currie & Pegg, 1998; Fujita, 2012; Fuys, Geddes, & Tischler, 1988; Gutiérrez, Jaime, & Fortuny, 1991; Mayberry, 1983; Unal, Jakubowski, & Corey, 2009; Usiskin, 1982; Wilson, 1990). Another framework that has impacted mathematical research is the Concept Image – Concept Definition framework (Cunningham & Roberts, 2010; Roh, 2008; Tall & Vinner, 1981; Vinner & Dreyfus, 1989). A third approach includes Figural Concepts (Fischbein, 1993; Fujita, 2012; Mariotti & Fischbein, 1997).

In the following sections, I will introduce these frameworks to provide context to the geometric studies from previous research. In each section, I will also show what some of the significant research has been accomplished in the field of special quadrilaterals.
2.2.1 van Hiele Levels of Understanding

The van Hiele framework of development in Geometry was created to address the difficulties students encountered with secondary school Geometry (Fuys et al., 1988). The model was first developed by Dina van Hiele-Geldof and Pierre van Hiele in 1957 as a structure of thought levels of student understanding. According to the framework, the student, going through appropriate instructional experiences, passes through the following five hierarchical levels:

Level Zero: The student identifies, names, compares and operates on geometric figures (e.g., triangles, angles, intersecting or parallel lines) according to their appearance.

Level One: The student analyzes figures in terms of their components and relationships among components and discovers properties/rules of a class of shapes empirically (e.g., by folding, measuring, using a grid or diagram).

Level Two: The student logically interrelates previously discovered properties/rules by giving or following informal arguments.

Level Three: The student proves theorems deductively and establishes interrelationships among networks of theorems.

Level Four: The student establishes theorems in different postulational systems and analyzes/compares these systems. (Fuys, et al., pg 5)

At each stage, the levels are characterized by how the student perceives the object. At Level Zero, the objects are purely geometric figures through their appearance. At Level One, the geometric figures of Level Zero can now be manipulated as a class of objects and certain properties of the classes can be discovered. At Level Two, the properties of Level One objects become the objects that the student can work through and logically order. At Level Three, the relationships between the properties become the objects. Finally, at Level Four, the systems created by the
theorems are the objects that the student handles. Crowley proposes the following names for the van Hiele levels: visualization, analysis, informal deduction, formal deduction, and rigor (Crowley, 1987).

Conceptually students could perform tasks on a particular level and not the level above and should be able to perform tasks at all lower levels (Mayberry, 1983). Yet Mayberry discovered that students performed on different levels for different concepts. As reported by Mayberry, “Van Hiele states two implications of his theory: (a) A student cannot function adequately at a level without having had experiences that enable the student to think intuitively at each preceding level. (b) If the language of instruction is at a higher level than a student’s thought processes are, the student will not understand the instruction.” (p. 67) Mayberry’s study gives support for these statements. The concern is that most high school geometry textbooks appear to be geared for level two thought processes and the students of her study (who were preservice elementary teachers) were showing evidence of lower level thought processes.

Many researchers have used a written test to determine a student’s van Hiele level (Gutiérrez et al., 1991; Mayberry, 1983; Usiskin, 1982). Another approach is through students working on activities and being interviewed (Burger & Shaughnessy, 1986; Fuys et al., 1988). In these specific studies with interviews, students were assessed and assigned a van Hiele level. Burger and Shaughnessy recognized that students seemed to be in transition between two levels, so they solved the issue by seeking a consensus of the evaluator’s opinions. Fuys et al. assigned students to intermediary levels (e.g. Level 1-2 to indicate the student uses thought processes from both Levels 1 and 2). Gutiérrez et al. propose a different approach. They quantify the acquisition of a level through a graduated scale of zero to hundred. Within this scale there are five levels of acquisition that are determined through an assigned numerical score. The questions are open ended and scores
are averaged for items assigned to measure a level. On the test that they developed for the study, they did find a peculiar result where some students had a better acquisition of Level Three than of Level Two which either contradicts the van Hiele theoretical framework or shows an issue to a fault in their test.

The framework has been used to investigate students’ understanding for class inclusion of quadrilaterals (Currie & Pegg, 1998). This study among secondary students focuses on hierarchical understanding of quadrilaterals. Open ended activities included designing tree diagrams which link the different quadrilaterals. The researchers provided a coding of six different classifications of response to how the quadrilaterals could be classified hierarchically.

Using Mayberry’s protocol (Mayberry, 1983), Unal, Jakubowski, and Corey investigated the geometric thinking of four preservice middle and secondary mathematics teachers (Unal et al., 2009). Spatial ability scores were examined and the learners in the mid-range spatial ability showed the most change after instruction. The student in the lowest van Hiele level showed the smallest growth from pre-test to post-test. For questions that did not contain a figure, the student was not able to draw an appropriate figure to analyze. Her lack of spatial ability directly affected her responses. The researchers suggest that the materials used in the instruction were of a higher van Hiele level than that of the understanding of the student. Consequently, growth in understanding using the van Hiele level is difficult to assess when the student is on a different level than the material presented in the assessment.

In one study, (Usiskin, 1982), the conclusion was that the highest van Hiele level is not testable. Usiskin provided a battery of tests to 2699 U.S. students of Geometry at the beginning and end of a school year from thirteen different schools. The tests consisted of an Entering Geometry Test, a van Hiele Level Test (both fall and spring semesters), a Comprehensive
Assessment Program Geometry Test, and a Proof Test. The van Hiele tests were divided into questions of different van Hiele levels. When students reached a pre-determined amount of correct responses on that level, the student received a weighted score which was then compiled and assigned a van Hiele overall level. Some of the research goals were to find out how students are distributed with respect to van Hiele levels and what changes in van Hiele levels take place after one year’s study in Geometry. Usiskin also concluded that arbitrary decisions regarding the number of correct responses to attain a level could affect the level assigned by the evaluator. Wilson (1990) reanalyzed Usiskin’s results and affirmed that the designed test was not able to measure the highest van Hiele level with accuracy.

The van Hiele Levels of Learning framework has been documented to be testable up to Level Three (Usiskin, 1982). Levels Zero and One in where students identify geometric objects by appearance and empirical measurement correlate to an action understanding of the concept. Level Two implies that students can use informal arguments to build understanding of geometric objects. Likewise, a student with a process understanding of using conditional logic and hierarchical definitions can determine properties of special quadrilaterals. Van Hiele’s Level Three shows that students can use deductive logic to prove theorems. When a student has reached an object understanding of special quadrilaterals, the student can accomplish Level Three activities. Finally, Level Four says that students can see the entire geometric system and create theorems. The van Hiele levels will help as a reference point in understanding the appropriate conception of student understanding in the APOS framework.
2.2.2 Concept Image – Concept Definition

The van Hiele Levels of Understanding provides specific levels of attainment for geometric concepts. On the other hand, the Concept Image – Concept Definition framework focuses on the difference between a student’s concept image and the student’s concept definition. The concept image is the total sum of all the mental constructs and images associated with a particular concept (Tall & Vinner, 1981). The experiences of the learner contribute to any change and fluctuation of the original concept image. Thus, a concept image can evolve into misconception if there is no conflict to disrupt the mental image of the concept. A student’s image is formed from examples and even non-examples of the concept. However, this formed concept image may be erroneous as compared to the more formal definition of the concept (Vinner & Dreyfus, 1989). Consequently, the student’s work and understanding of a concept would be different than what the teacher expects.

A concept definition is the verbal explanation of the student’s concept image by the student (Tall & Vinner, 1981). The possibility exists that a student may not be able to articulate the entirety of his concept image in words. Other times the concept image and the concept definition may be in conflict. In these circumstances, the learner may be at unease with the concept in general. In time, the student works out the difference as either the concept image or concept definition changes to eliminate the conflict factor.

In a formal geometry class, students might be asked to memorize a set of geometric definitions. However, in recalling the concept, the student’s concept image might be connected to an incorrect example (Cunningham & Roberts, 2010). Thus, part of growth of a concept might be approached through the teaching strategy of Concept Attainment. This model encourages students to form a definition for a concept by seeing examples and nonexamples (Cunningham & Roberts,
Revisions to the definition are appropriate as more examples and nonexamples are presented to the students. The purpose is to create the cognitive conflict so that students’ conflict image moves closer to the formal definition of the concept (Tall & Vinner, 1981).

Hasegawa notes that in a classroom, a concept may be introduced with only its proper examples (Hasegawa, 1997). Incidentally, a mathematical definition may be included within a mathematical structure or system which is hidden from the children targeted at instruction. For example, the definition of a quadrilateral may contain the term “segment” but this term cannot be defined mathematically on an elementary level. Instead this definition presupposes a system of Euclidean geometry. This hidden system may lead students to have a pseudo-conceptual level of understanding for quadrilaterals. Hasegawa defines pseudo-concepts as which students identify objects with their concept image that are close in shape to the concept definition of the object but are not mathematically accepted as a representation of the object.

Students (and teachers alike) may have a resilient concept image that keeps them from considering the correct concept definition (Groth, 2006). A concept image may have taken years to build up from prior schooling. So, if a student has had several years of encountering one definition for an object, then the student may struggle with the idea that an object may have another definition that is accepted, especially in the case of the trapezoid. This problem may be exasperated when geometric concepts have been taught as rote memorization without further explanation for many student experiences (Fuys et al., 1988). Consequently, many students leave their study of geometry in grades K-12 with deficient concept images for geometric ideas.

The framework for Concept Image – Concept Definition applies a connection between seeing where the student’s own concept definition is in potential conflict with the student’s concept image. In this framework, the student should define in his own words the concept definition that
the student is using as an operational definition for the concept. In a similar approach, this study will require students to define quadrilaterals in their own words and use those definitions to solve various problems.

The Concept Image – Concept Definition framework does complement the APOS framework. In APOS, students interiorize a process when moving from an action conception. At this level, a student’s concept image is the internalized thought process to how the student understands the concept. During the interview process of this study, students will express their concept definitions allowing for an analysis of whether the interiorization of the concept has truly occurred. If a student’s concept definition is inadequate or incorrect to the formal definition then the student may still be at a pre-action conception with incorrect rote memorization.

2.2.3 **Figural Concepts Framework**

Figural concepts are “mental entities…which reflect spatial properties (shape, position, magnitude) and at the same time, possess conceptual qualities - like ideality, abstractness, generality, perfection” (Fischbein, 1993). This framework positions that geometric concepts have two aspects, the figural and the conceptual (Mariotti & Fischbein, 1997). The figural aspects refer to spatial contexts while the conceptual aspect refers to the more abstract and theoretical nature. Although both aspects, in principle, should interact harmoniously, conflicts and difficulties may arise between the two aspects.

Operations can be performed on figural concepts that would not be considered with real objects. Fischbein (1993) explains that much of geometry is dealt as a general representation of a concept. Points, lines, and planes as conceptually described cannot exist. Consequently, many of the objects used in mathematics stem from an ideal construct that is not based on perceived reality.
Yet, because of the axiomatic system of geometry, these objects must behave according to their defined properties.

Conceptualized geometry problems involve abstract representation. For example, the idea of roundness may be described by a wheel, but abstractly represented by a circle. Manipulating the circle as a mental construct provides a generalization for appropriate calculations to solve a specific problem. Thus, the figural aspect of a circle may also be used to imagine an ideal circle but not one represented in concrete form on paper, but rather as a concept in the mind. Consequently, if the mental image demonstrates a faulty concept, then the definition of the object must be reassessed to regain a new figural concept. Monaghan states that students may fall into error when asked to differentiate between objects where there may be no necessary differences (Monaghan, 2000). For example, a rectangle may have all four sides equal where a square must have all four sides equal. Yet the students tend to only consider rectangles as objects that are oblong. Fischbein (1993) contends that exposing students to geometrical problems with loci can help fuse the conceptual image and the figural concept. On the other hand, Hasegawa’s findings suggest that students at the beginning stage of geometric learning may take drawings in textbooks and on the chalkboard as concrete objects themselves without understanding the abstract definition of the object (Hasegawa, 1997).

Fujita (2011) uses this framework as a basis for his exploration of the prototype phenomenon. Students are prone to be influenced by the prototypical examples of geometric concepts rather than focusing on the definitions or properties associated with the concept (Fujita, 2012). In dealing with the inclusive nature of quadrilaterals, learners tend to add false conceptual attributes to the more general quadrilateral. For example, in the case of parallelograms, learners might consider that parallelograms do not have right angles, which may be a result of the
prototypical image of a “slanted” parallelogram. The researcher assesses 85 year 9 (age 14) U.K. students about their identification of specific quadrilaterals and basic definitions of special quadrilaterals. The assessment asks students to take visuals of quadrilaterals and classify them in their respective special quadrilateral classes (e.g. parallelogram family, rectangle family, rhombus family, etc.) In addition, students are asked to clarify which of a list of characteristics of a quadrilateral are true. In conclusion, Fujita (2011) suggests that definitions of objects need to be taught with their respective image. Also, the roles of concept examples and relevant non-examples along with critical attributes are important aspects to work with students who have a limited figural concept for a specific quadrilateral.

Duval has constructed a framework for studying diagrams and visuals in regards to what operations students perform when confronted with geometric figures (Duval, 1995). His apprehensions of diagrams refer to how students can observe and comprehend a geometric diagram while solving a problem. The four types of diagram apprehensions are perceptual, sequential, discursive, and operative. As a geometric figure, the diagram must have perceptual apprehension in addition to one of the other three apprehensions (Deliyianni, 2009). Perceptual apprehension refers to a person’s ability to name figures and the ability to recognize sub-figures within the figure. Sequential apprehension requires a sequence of steps to be followed with tools (e.g. compass and straightedge) while making a diagram (González, 2013). Discursive apprehension regards the use of propositions or concepts that justify the different operations that are performed on the diagram and that mathematical properties in the diagram cannot be determined by perceptual apprehension. Operative apprehensions are the specific operations that are performed on the diagram. The subtypes of operative apprehension include the mereologic, optic, and position. The mereologic involves adding or deleting geometric objects. The optic refers
to the size of the diagram and how it can be scaled. The position refers to the orientation of the diagram by translation or rotation. Any of these operations can be performed mentally or physically.

The Figural Concept Framework extends the concept image into two aspects, figural and conceptual. The figural aspect focuses on how students work through a concept in spatial terms. Fujita’s (2012) study recognizes the issue of prototypical images that students use which may cause conflict with the formal definition of the object. For example, a student may say a rectangle can never have four congruent sides because that student has a prototypical image of an oblong figure. In this study, the genetic decomposition uses the element of visualization as an essential component to the understanding of special quadrilaterals. The figural concept that students use to work through assigned classwork and interview problems will be observed through the lens of the Figural Concept Framework to see if prototypical images are disrupting the concept image of students.

Under the APOS framework, the genetic decomposition of the concept of hierarchical definitions of special quadrilaterals uses the schema of Visualization. Since students use their initial visualization to combine with the hierarchical definitions, students may have an incorrect prototypical image and in turn leads to incorrect conclusions about properties of special quadrilaterals. The interiorization of the concept would thus have errors and students will be on a pre-action conception of their understanding of the concept of special quadrilaterals.

2.3 Overview

This chapter focused on literature on the appropriate criteria for mathematical definitions (2.1.1). Applying these criteria, several definitions for special quadrilaterals were constructed
I also shared the definitions that were used in the Geometry course the students took.

Prior research has given three frameworks which provide a lens into geometric understanding. Van Hiele Levels of Understanding (2.2.1) give indicators of students’ levels of comprehension toward Geometry. The levels correspond well with APOS structures and can be useful to identify where students are conceptually in the APOS framework. The Concept Image-Concept Definition framework (2.2.2) dissects how students’ concept images may cause misunderstandings when there is conflict with their concept definitions and the formal definitions. In the APOS framework, interiorizing an action into a process involves students having appropriate concept definitions. The Figural Concepts framework (2.2.3) shows that students struggle with prototypical images of figures that may conflict with the formal definition of a figure. In this study, visualization of a figure is essential for students to accurately find equivalent definitions of a figures. Analysis of students’ responses in the data with prototypical drawings of figures may help explain errors in students’ thoughts.

Throughout the literature, research is lacking on student understandings of special quadrilaterals with an APOS lens. Also, preservice teachers’ understandings of their subject properly are extremely important for their future students’ well-being. This study is designed to address the needs of helping preservice teachers improve their understanding of special quadrilaterals.
3 METHODOLOGY

In Chapter One, I described how APOS Theory would be the guiding framework for this study. In this chapter, I will show how the APOS Theory framework will guide the methodology for my research. Specifically, I will focus on the context (3.1), the procedure (3.2), and the preliminary genetic decomposition (3.3).

3.1 The Context

Students taking an elementary geometry course during Spring 2013 were recruited to participate in this study. The course is a required course in the sequence of mathematics courses for the Early Childhood Education program. The scope of the course covers topics from Geometry that are normally taught in the upper elementary and middle school levels. The instructor provided content and pedagogical tools that the students can implement in their own classrooms when they are in the teaching profession.

There were two sections of the class that were invited to participate. The study aimed to ask all students from the course to participate yet a subgroup of twenty-six students out of sixty volunteered based on scheduling and availability. The participating students have had widely varying academic backgrounds, knowledge, and skills of mathematical literacy, and ability to express their mathematical and content pedagogical knowledge. All these components are important and may impact their future teaching of elementary mathematics, so they were good candidates for inclusion in the study.

After permission from the IRB, another researcher explained the project to the students and asked for voluntary participation in the project. Course instructors were not involved in the recruitment process. The benefit of the project to the individual student was presented. It was made
clear that participation would not affect any person's grade directly however it may affect their grade only in a positive way since some learning may occur during the interview which then might affect their class performance. An alternative assignment was available for non-participants that required similar time, effort, and grade value to the course. The content and purpose of the study was explained including the consent forms and the rights of students to stop participating at any time during the problem-solving session or the interview (See Appendix A). Students also received an email notification regarding the consent to use their written work in the class for the study.

3.2 Procedure

Eight interview sessions were conducted with one to four participants attending each session. Participants worked individually on a written problem-solving session for 30 minutes. This session was followed by an audio-taped and video recorded hour-long interview with questions related to the problems solved by the students. Students each took turns who would share first on a problem. The interviewer encouraged students to elaborate on their answers to help understand students’ conceptual understanding on the problem. A common protocol was used (see Appendix B) to maintain consistency between the interview sessions. Since I was the instructor of the course, I did not attend any of the sessions to avoid influencing students’ responses.

Additional data consisted of students' written work completed as part of the coursework, specifically problems from two quizzes and the final examination. In addition, course instruction was audio-recorded over the topic of special quadrilaterals to affirm that proper definitions were used and explained during class-time.
3.3 Genetic Decomposition for the Concept of Special Quadrilaterals

As noted in Chapter One, one of the first steps in APOS Theory research is to create an initial genetic decomposition. This genetic decomposition is to show how learners may proceed through the development of understanding the concept. After data has been collected and analyzed a revised genetic decomposition may be created based on the results of this study, if necessary.

The following step is the initial genetic decomposition for the comprehension of the hierarchical properties of special quadrilaterals:

![Diagram of Genetic Decomposition](image)

**Figure 1: Preliminary Genetic Decomposition of Special Quadrilaterals**
1) The schema of Mathematical Definition is interiorized to conceptualize hierarchical definitions. At this step, students understand that mathematical definitions can be used to differentiate geometric shapes into their appropriate quadrilateral classifications.

2) Hierarchical definitions involve conditional statements. The schema of Logical Reasoning in encapsulated in treating the process of conditional reasoning for if-then statements and applies that to inclusive definitions. At this step students can conceptually reason that quadrilaterals with properties may belong to a higher classification of properties (e.g. if a quadrilateral is a rhombus then it is also a kite).

3) Visualization must be generalized from a prototypical concept image of special quadrilaterals to a more inclusive context. At this step, students use their concept image of a quadrilateral to visualize the properties that overlap with other quadrilaterals (e.g. a student may visualize a square and focus on the properties of four congruent sides to consider that a square has the same properties as a rhombus.)

4) Properties of special quadrilaterals emerge as the inclusive definitions are de-encapsulated back to the characteristics of the more general quadrilaterals. At this step, students take the generalized visualization from Step 3 and the Hierarchical Definitions of Inclusion from Steps 1 and 2 to identify the properties of the quadrilateral they are investigating.

5) The application of the properties of special quadrilaterals to solve problems comes from a de-encapsulation of the general properties of these quadrilaterals. Students can now apply the discovered properties within the context of problem solving.
This preliminary genetic decomposition of the concept of special quadrilaterals above will be investigated through the data collected and analyzed in this study. The following chapter breaks down each of the data that pertains to the main research questions (Section 1.2).
4 DATA ANALYSIS AND RESULTS

This chapter discusses data analysis and results taken from class assignments and interviews with participating preservice teachers enrolled in a college Geometry course. Since all the preservice teachers were students in the course, I will use the terms “preservice teachers” and “students” interchangeably throughout the rest of this analysis. The data analysis in this chapter has four subsections, each focusing on one research question (see Section 1.2). In Section 4.1 the focus of analysis is on preservice teachers’ personal definitions of special quadrilaterals (research question 1i). Section 4.2 analyzes how preservice teachers applied the distinction between necessary and sufficient conditions for their personal definitions (research question 1ii). Section 4.3 focuses on how preservice teachers perceived and used the hierarchical nature of the definitions of special quadrilaterals (research question 2i). Finally, Section 4.4 analyzes the ability of preservice teachers to discern equivalent definitions between special quadrilaterals (research question 2ii).

Data for this research was collected through students’ responses on specific questions on class assessments (quizzes and the final exam) and through a semi-structured interview. The responses, both written and oral, were coded for correctness and if wrong or incomplete, the type of error committed. Using the APOS framework (Section 1.3), I analyzed all student responses to see if the students showed evidence of pre-action, action, or process conception of the specific topic that was being assessed.
The following table summarizes the data sources that were analyzed for each specific research question (Section 1.2):

<table>
<thead>
<tr>
<th>Section</th>
<th>Research Question (RQ)</th>
<th>Data Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>4.1</td>
<td>RQ 1i</td>
<td>Interview Part 1, Questions 1a-g</td>
</tr>
<tr>
<td>4.2</td>
<td>RQ 1ii</td>
<td>Quiz Question #13, #14</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Quiz Question #3, #4</td>
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<td></td>
<td></td>
<td>Final Exam Question #31</td>
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<tr>
<td>4.3</td>
<td>RQ 2i</td>
<td>Interview Part 1, Questions 1a-g: special cases</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Interview Part 1, Questions 2a-j</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Final Exam Question #22 a-e</td>
</tr>
<tr>
<td>4.4</td>
<td>RQ 2ii</td>
<td>Interview Part 2, Questions 3-5</td>
</tr>
</tbody>
</table>

**Table 1: Data Sources**

Each of these sections listed in Table 1 relate to the preliminary genetic decomposition on the concept of special quadrilaterals. Students’ responses in terms of APOS may indicate different levels of understanding for the concept depending on the circumstance as was evident in this study. The focus is to determine if the genetic decomposition is accurate or whether it needs to be adjusted based on students’ understanding. Consequently, as students’ responses are analyzed, the students might have different APOS conceptions as indicated based on the question or situation.

The next section begins the analysis of each research question. The first section lays the foundation for this study in finding what are the students’ initial definition of each special quadrilateral. Misconceptions of the definitions could provide difficulties for the students’ responses in later questions.
4.1 Preservice Teachers’ Understandings of Geometric Definitions

The first research question is about preservice teachers’ understandings of geometric definitions. Before the oral part of the interview began, students were asked to write down their responses to the questions on paper. The first question asked the students to define each of seven quadrilaterals: squares, rectangles, parallelograms, rhombuses, kites, trapezoids (inclusive definition), and trapezoids (exclusive definition). The question also asked them to list any special cases of the object, if any.

When the interview began, students were asked to share their definitions and, at times, to elaborate on their definitions and explanations. The analysis in this section is based on the transcription of the interviews along with their written responses to construct a more complete picture for the students’ understanding of the object they are defining. To characterize a good definition, the following criteria were used:

i) Identifies that object is a quadrilateral (or closed four-sided polygon or figure)

ii) Identifies properties correctly

iii) Establishes necessary and sufficient conditions

iv) The set of conditions should be minimal

Since each of the figures in this study are special quadrilaterals, the definition given by students needs to at least identify that the figure is a quadrilateral (criterion (i)). Also, a definition must have correctly listed properties according to mathematical convention (criterion (ii)). As teachers of elementary students, preservice teachers must convey correct mathematical knowledge. Criterion (iii) employs Winicki-Landman and Leikin’s condition that definitions must establish necessary and sufficient conditions (Winicki-Landman & Leikin, 2000). Finally, Vinner, Winicki-
Landman, and Leikin recommend that definitions should be minimal (Vinner, 1991; Winicki-Landman & Leikin, 2000). Minimal means using the least amount of properties to sufficiently define the figure. For example, the following definition is not minimal: a rhombus is a parallelogram with congruent sides and perpendicular diagonals. A minimal definition could be the following: a rhombus is a quadrilateral with congruent sides.

Even though all the special quadrilaterals were defined in the college Geometry class with minimal hierarchical definitions, students rarely gave such explicit answers. Yet, all responses could be categorized based on the criteria of a good definition as listed above. Consequently, students’ definitions were grouped into three categories based on what conception they exhibited: pre-action, action, and process conceptions.

When a student could give a definition that satisfied criteria (i) – (iii) of the above, then he can at least identify the quadrilateral correctly even though his definition does not meet criterion (iv). Without the minimal criterion, the students’ definitions would be classified as indication of action conception. On the other hand, if the student’s definition does not satisfy one or more criteria (i) – (iii), then the student does not understand the specific quadrilateral concept. These definitions would be classified as indication of pre-action conception. Any student’s definition that met all four criteria (i) – (iv) could be categorized as process level depending how the student used his/her definition in further situations.

The following subsections provide detailed analysis of students’ responses related to the definition of a special quadrilateral. Specifically, the students were asked to define the following figures: squares (4.1.1), rectangles (4.1.2), parallelograms (4.1.3), rhombuses (4.1.4), kites (4.1.5), trapezoids - inclusive definition (4.1.6), and trapezoids – exclusive definition (4.1.7).
4.1.1 Squares

Section 2.1.3 presented thirteen appropriate definitions for a square. Of which, the following three definitions would be expected from the students of a preservice teacher Geometry course:

- A rectangle with congruent sides.
- A rhombus with congruent angles.
- Both a rectangle and a rhombus.

In the Geometry course, the instructor shared how a figure may have different definitions based on the best context of use for the figure. Even though the choice of the definition may be arbitrary, any other definition of that figure would be considered as properties of the figure. Also, the instructor taught the difference between defining a figure with necessary and sufficient conditions as opposed to listing several properties of the figure.

Out of all the figures, the participants in the study had the most success in defining the square. Seventeen out of twenty-six students gave definitions that satisfied all four criteria. Specifically, their responses defined squares as quadrilaterals (i), used properties of squares correctly (ii), used necessary and sufficient conditions (iii), and were minimal (iv). Seven students gave responses that met criteria (i) – (iii) but not criterion (iv). Two students gave a response that did not meet criterion (iii). The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory definition.

Anna’s definition of a square satisfies all four criteria. Her definition is as follows:

*Quadrilateral that has four sides with equal length and four angles of equal measure.*

Her definition lists appropriate characteristics of a square that were emphasized in the instruction of the course. This definition is appropriate for instruction to elementary students as this teacher
will teach in the future. By listing only the necessary and sufficient characteristics of a square, Anna has also minimally defined a square. Consequently, Anna gave a definition that is indicative of the process conception of the definition of squares.

Mary gives a definition that satisfies all criteria except (iv). Her definition is the following:

\[
[\text{A square is a} \ \text{rectangle and rhombus; 4 right angles; 2 sets of parallel lines and 4 congruent sides.}]
\]

Mary does not use a minimal definition, but instead lists several properties for a square. All of properties that Mary listed are already implied with the rectangle and rhombus part of her definition. She does say “parallel lines” when “parallel sides” would have been a better descriptor. Yet, since parallel sides are parts of lines that are indeed parallel, this property is not technically incorrect. Mary did not include any drawings with her definition. Consequently, Mary’s definition fits criteria (i) – (iii), which is regarded in the category of action conception for the definition of a square.

The following definition by Cheryl unsatisfactorily defines a square:

\[
\text{All sides are equal length.}
\]

Cheryl’s definition gives a necessary but not sufficient condition which means she is missing criterion (iii). Indeed, all squares have sides of equal lengths. Yet, the angles must also be congruent in a square. In addition, Cheryl never explains that the figure described must be a quadrilateral or 4-sided figure. She also does not include any drawings in her paperwork that imply that the figure must be 4-sided. Cheryl’s definition is an example of a pre-action conception of the definition of a square.

The next section is a report on the analysis of the students’ responses to defining a rectangle.
4.1.2 Rectangles

Rectangles may be defined in various ways of which the following two are the most common:

A rectangle is a quadrilateral with four right angles.

A rectangle is a parallelogram with at least one right angle.

Five students’ definitions of rectangles met all four criteria (i) – (iv). Specifically, their responses defined rectangles as quadrilaterals (i), used properties of rectangles correctly (ii), used necessary and sufficient conditions (iii), and were minimal (iv). An additional eleven students’ definitions met criteria (i) – (iii). Of the remaining ten responses, six of them left off that the figure must be 4-sided (criterion (i)). One student had a necessary but not sufficient definition (criterion (iii)). The remaining three students had definitions that had incorrect properties or used the definition of a square to define a rectangle (criterion (ii)). The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory definition.

Anne applies a hierarchical approach in her definition. She said:

[A rectangle is a] parallelogram with 4 right angles.

Anne also included a drawing with her definition:

Figure 2: Anne's definition of a rectangle
This definition meets all the requirements of the four criteria. Even though she could have alternatively expressed a rectangle as a parallelogram with at least one right angle, Anne’s definition is consistent with the definition she will have to teach to elementary students. Consequently, this definition illustrates a process conception of the definition of rectangles.

Jennifer gave a definition that was almost satisfactory. She said:

*A rectangle is* a quadrilateral with two congruent sides that form four right angles.

If Jennifer had only described a rectangle as a quadrilateral with four right angles, she would have had a good definition. However, she added the property of two congruent sides making the definition no longer minimal (criterion (iv)). Also by stating that two of the sides are congruent, she leaves an ambiguous characterization of rectangles. Furthermore, it is unclear whether she meant to imply that at least two sides must be congruent or say two pairs of congruent sides. A further interpretation is that only two sides must be congruent. In this case, her definition would be wrong by use of an incorrect property (criterion (iii)). Consequently, without a visual aid to clarify, Jennifer’s definition falls into the category of pre-action conception of the definition of rectangles.

Likewise, Jamie defined a rectangle with the definition for a square:

*A rectangle* has 90° right angles and four equal sides.

Jamie’s definition was accompanied with a drawing, which is included below:

![Figure 3: Jamie's definition of a rectangle](image-url)
The implication of four-sided figure was presumed with the drawing. She only needed to state that the rectangle has only 90-degree angles (the redundancy of the word “right” is unnecessary). However, she lists a property that is not always true: four equal sides. Interestingly, she drew a prototypical rectangle with one two sides longer than the others which contradicts her description. This confusion is further evident when she later explains her special case of a square:

Jamie: I said that a rectangle has 90-degree right angles and four equal sides and so a special case of a rectangle would be a square because a square does have 90-degree right angles and two pairs of congruent sides.

Interviewer: And it is special because…

Jamie: because they have 90-degree angles and then parallel to each the sides

Interviewer: And in addition it has…

Jamie: two pairs of congruent sides

Jamie states the rectangle property (two pairs of congruent sides) but affirms a property that is not true (four equal sides) in the general case of the rectangle. This property combined with the rest of her definition describes a square and eliminates the cases when rectangles are not squares. Jamie’s response is therefore an indication of Jamie’s pre-action conception of understanding of rectangles.

Cheryl’s definition is an example of a definition that has a necessary but not sufficient condition. She said:

Opposites are equal lengths.

Cheryl’s definition also did not include any specification of four-sided figure (criterion (i)) and she did not have a drawing. Even if it had mention a quadrilateral, her definition could have been descriptive of a parallelogram since there is no mention of the congruent angles. This definition
misses both criteria (i) (not mentioning that the figure is four-sided) and (iii) (sufficient and necessary conditions) and is therefore a pre-action conception of the definition of rectangles.

Margaret’s definition correctly describes a rectangle but is not minimal:

*Quadrilateral, sides form right angles, opposite sides are congruent and parallel.*

Margaret also included a drawing with her definition:

![Figure 4: Margaret's definition of a rectangle](image)

Margaret’s definition would have been appropriate with saying a quadrilateral with right angles. The other two properties, opposite sides are congruent and parallel, characterize that a rectangle is also a parallelogram. However, these extra properties are inferred from the fact that a quadrilateral with four right angles must also be a parallelogram. Consequently, Margaret’s definition illustrates an action conception of the definition of rectangles.

The next section is an analysis of students’ responses to the definition of a parallelogram.

### 4.1.3 Parallelograms

Parallelograms could be described as any of following:

- Quadrilaterals whose opposite sides are parallel.
- Quadrilaterals whose opposite sides are congruent.
- Quadrilaterals with at least one pair of parallel, congruent sides.
Quadrilaterals whose opposite angles are congruent.

Quadrilaterals whose diagonals bisect each other.

Consequently, there are many choices that students could have used to define parallelograms appropriately. Twelve of the responses given out of twenty-six defined a parallelogram that met criteria (i) – (iv). Specifically, their responses defined parallelograms as quadrilaterals (i), used properties of parallelograms correctly (ii), used necessary and sufficient conditions (iii), and were minimal (iv). Two additional responses met criteria (i) – (iii) but did not keep a minimal definition of a parallelogram. Of the remaining twelve responses, six students did not mention or draw a picture that the figure must be 4-sided (criterion (i)). Four students had necessary but not sufficient conditions to defining a parallelogram (criterion (iii)). Two students’ responses had a satisfactory drawing but unsatisfactorily described the definition of a parallelogram verbally (criterion (ii)). The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory definition.

Margaret’s definition meets all four criteria. Her work is shown below:

Figure 5: Margaret's definition of a parallelogram

Margaret gives an accurate definition of a parallelogram and employs the minimal condition (iv). Margaret’s response is indicative of a process conception for the definition of a parallelogram.

Jack’s response gave much more information than was necessary to define a parallelogram:

[A] quadrilateral with two sets of opposite congruent sides- both sets are parallel.
Jack also included a drawing with his definition:

![Figure 6: Jack's definition of a parallelogram](image)

Jack mentions that the two sets of opposite sides are congruent and parallel. Optimally, he should stop with just a quadrilateral with two sets of opposite congruent sides or a quadrilateral with two sets of parallel sides. However, by describing both characteristics, he was not applying the minimal criterion (iv). Jack’s response would fall in the category of having an action conception of the definition of parallelograms.

Mary’s definition lacks any description that a parallelogram is a quadrilateral:

*It must have two sets of parallel sides.*

Mary includes no drawing to illustrate her definition. Strictly stated, her definition allows for figures with more than four sides, which disqualifies her definition (criterion (i)). If her definition only applies to quadrilaterals, then she would be completely correct. Consequently, Mary’s definition is on a pre-action conception for the definition of parallelograms.

Tammie’s response demonstrates the difficulty of distinguishing figures based on their hierarchical definitions:

*A figure or quadrilateral with at least one pair of opposite sides that are parallel.*

This definition does have necessary but not sufficient conditions (criterion (iii)) in that at least one pair of opposite sides are parallel instead of both pairs of opposite sides are parallel. Tammie also did not include any drawings with her definition. Incidentally, Tammie’s definition would be
appropriate for an inclusive definition of a trapezoid. However, because this definition is unsatisfactory for a parallelogram, her response falls in the pre-action conception for the definition of a parallelogram.

Amity combines several properties when she defines a parallelogram as:

*Parallelogram have congruent sides, opposite sides are equal, angles are equal, it has ASA property.*

Amity did include a picture of a four-sided figure with her written work:

![Figure 7: Amity's definition of a parallelogram](image)

There are several issues with Amity’s definition. First, her definition presupposes that all the sides are congruent, which is a characteristic of a rhombus instead of the more generic parallelogram. Second, the description of congruent sides in addition to parallel opposite sides is unnecessarily when considering the minimal criterion (iv). Finally, Amity adds a reference to the ASA property. The instructor for the class referred to the ASA postulate for proving congruent triangles. Amity could be referring to the characteristic that a diagonal of a parallelogram creates two congruent triangles, which usually are shown to be congruent by the ASA postulate. In this case, Amity may be referencing to the two congruent triangles formed by a diagonal of a parallelogram. Since Amity’s ASA property was not defined as a characteristic definition of a parallelogram, her definition falls under the pre-action conception for defining parallelograms.
The next section reports on the analysis of students’ responses to the definition of a rhombus.

4.1.4 Rhombuses

A rhombus is a quadrilateral with four congruent sides. Another variation for the definition of a rhombus is a parallelogram with four congruent sides. Eleven students’ responses met all four criteria (i-iv). Specifically, their responses defined rhombuses as quadrilaterals (i), used properties of rhombuses correctly (ii), used necessary and sufficient conditions (iii), and were minimal (iv). Six of the students gave more information than needed by listing additional properties, thus missing the minimal criterion (iv). Of the remaining nine responses, four responses failed to mention that the figure must be 4-sided (criterion (i)). Three other responses have necessary but not sufficient conditions for the definition of a rhombus (criterion (iii)). The remaining two responses contain unsatisfactory information (criterion (ii)). The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory definition.

Jennifer provides a response that meets all four criteria (i-iv):

*Quadrilateral with four congruent sides.*

Her response is concise, accurate, and appropriate for elementary school age students to learn. Consequently, this definition is an example of a process conception for the definition of a rhombus.

Tammie’s definition listed an extra property of rhombuses:

*Four congruent sides; diagonals form a perpendicular bisector*

Tammie starts her definition accurately with four congruent sides. Yet, she also includes the property that the diagonals are perpendicular bisectors of each other. This characteristic is
unnecessary for the minimal definition (criterion (iv)). Consequently, her definition illustrates an action-conception for the definition of a rhombus.

Amity’s definition she shared orally is an example of a necessary but not sufficient condition for a rhombus:

\[ \text{Opposite sides are congruent} \]

In Amity’s written work, she references four sides, so her intention of 4-sided figure is implied. Below is a copy of her written work for the definition of a rhombus:

\[ \text{Rhombus has (4 equal sides) diagonals bisect; lines across parallel; (kite + square)} \]

Her definition would be minimal if it only included four congruent sides. Below is a copy of her written work:
However, the extra property that the lines [sides] opposite are parallel is unnecessary. The final characteristic included in Sonam’s definition is an error: “kite + square.” A rhombus could accurately be called a kite that is also a parallelogram. In other words, all the characteristics of kites and parallelograms are found in a rhombus. Sonam’s error was that she mentions that a rhombus is a square when all squares are rhombuses instead. Because of this error, Sonam’s definition illustrates pre-action conception for the definition of a rhombus.

The next section is an analysis of students’ responses to the definition of a kite.

4.1.5 Kites

A kite is a quadrilateral with two distinct pairs of congruent adjacent sides. A kite could also be defined as a quadrilateral with a diagonal that is a line of symmetry. In this study, nine students gave definitions that met criteria (i – iv). Specifically, their responses defined kites as quadrilaterals (i), used properties of kites correctly (ii), used necessary and sufficient conditions (iii), and were minimal (iv). Four responses are worded in a way that removes rhombuses as special types of kites (criterion (iii)). On the other hand, three responses more accurately defined a rhombus than a kite (criterion (iii)). Of the remaining ten definitions, seven definitions had other necessary but not sufficient conditions (criterion (iii)). The final three definitions had other mistakes that led to an incorrect definition (criterion (ii)). The following are representative...
examples from each group of responses. I will start with a representative example of a student who gave a satisfactory definition.

Tammie gives a satisfactory definition of a kite:

*Two pair of congruent adjacent sides and a diagonal form a perpendicular bisector*

By adding the criteria of the diagonal forms, a perpendicular bisector (with the other diagonal), then Tammie’s definition does not have to accommodate the non-overlapping criteria of the two pairs of congruent adjacent sides. Her definition meets all four criteria (i – iv). Tammie’s definition would fall in the category of a process conception for the definition of a kite.

Jack’s definition is almost correct except for one technicality:

*Quadrilateral with two sets of non-parallel congruent sides.*

By using non-parallel instead of distinct and adjacent congruent sides, Jack describes kites that are distinct from being parallelograms. Thus, this definition separates rhombuses as not being a specific type of kite. However, Jack does draw a picture that adequately represents the characteristics of a kite (see below):

![Figure 10: Jake's definition of a kite](image)

Based on his drawing, Jack may have been using “non-parallel” interchangeably with “adjacent.” Although adjacent sides must be non-parallel, non-parallel sides must not necessarily be adjacent. Consequently, Jack’s definition classifies as a pre-action conception for the definition of a kite.

Mary’s definition goes to the other extreme by describing a rhombus as compared to Jack’s definition:
Two sets of congruent adjacent sides and then perpendicular bisectors for diagonals.

A kite only needs one diagonal to be a perpendicular bisector of the other diagonal. By stating that both diagonals are perpendicular bisectors of each other, Mary only describes figures that would technically be rhombuses as kites. This eliminates the prototypical kite which visually corresponds to two distinct pairs of congruent adjacent sides. Furthermore, Mary did not include a drawing with her definition. Consequently, Mary’s mistake classifies as a pre-action conception for the definition of a kite.

Sophie’s definition is close to the standard definition of a kite:

\[ \text{A quadrilateral that have pairs of adjacent equal sides} \]

Sophie’s work is included below:

\[ \text{Figure 11: Sophie's definition of a kite} \]

Sophie does include the characteristic that the pairs of congruent sides must be adjacent. However, by not mentioning the characteristic of distinct congruent sides, her definition could also describe any quadrilateral with at least three congruent sides. This definition contains necessary but not sufficient conditions (criterion (iii)). Therefore, Sophie’s definition illustrates a pre-action conception for the definition for a kite.

Cassie’s spoken definition during the interview also incorrectly applies the wrong hierarchical approach:
Parallelogram with two pairs of adjacent congruent sides and perpendicular diagonals.

Cassie also drew a picture on her paper with a slightly different definition on her paper:

![Diagram of a parallelogram with two pairs of congruent sides.]

Figure 12: Cassie's definition of a kite

Cassie’s picture is an appropriate representation of a kite. Yet, the only figure that satisfies her oral definition is a rhombus, which can be classified as a specific type of kite. On her written response, the only parallelogram that satisfies the conditions of a kite is a rhombus. By connecting a kite with a parallelogram, Cassie has classified a kite as a special kind of parallelogram, which is not true in the general case. Consequently, Cassie’s definition illustrates a pre-action conception of the definition for a kite.

The next section discusses the analysis of students’ responses to the inclusive definition of a trapezoid.

4.1.6 Trapezoids (inclusive)

An inclusive trapezoid is a quadrilateral with at least one pair of parallel sides. This definition is inclusive in the sense that parallelograms are included in the classifications of trapezoids since parallelograms have two pairs of parallel sides. Twelve responses met all four criteria (i–iv). Specifically, their responses defined trapezoids (inclusive) as quadrilaterals (i), used properties of trapezoids (inclusive) correctly (ii), used necessary and sufficient conditions (iii), and were minimal (iv). Four responses left out the criterion (i) that a trapezoid is a 4-sided figure or a quadrilateral. Of the remaining ten responses, five of the students used characteristics that would describe either an
isosceles trapezoid or a parallelogram (criterion (ii)). The remaining five definitions replaced the word “parallel” with “congruent” in the definition (criterion (ii)). The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory definition.

Jennifer’s definition meets all four criteria:

Quadrilateral with at least one pair of parallel sides

Her written work is below:

f) Trapezoid (Inclusive)

Figure 13: Jennifer's definition of a trapezoid (inclusive)

Jennifer has an accurate definition that even meets the minimal criterion (iv). Jennifer’s response is indicative of a process conception for the definition of a trapezoid (inclusive).

Mary’s definition leaves out criterion (i):

[An inclusive trapezoid] must have at least one pair of parallel sides.

Mary does not have any indication that a trapezoid must be a four-sided figure. She also does not have any drawings to demonstrate her definition and imply that the figure must be four-sided. Consequently, her definition illustrates a pre-action level definition for inclusive trapezoids.

Heather’s definition describes a more specific type of trapezoid, the isosceles trapezoid, in which the legs are congruent:

Two parallel lines w/ two congruent sides.

Heather’s definition alone would technically be ambiguous if not for her drawing of an isosceles trapezoid on her paper:
The drawing presupposes that the figure must have four sides. Since this definition is too specific and does not need the two congruent sides to be an inclusive trapezoid, the definition illustrates a pre-action conception of the definition of an inclusive trapezoid.

Cheryl’s definition puts attention on congruency instead of parallel sides:

(At least one) one side is congruent from the other.

Her written work is shown below:

Cheryl does not have any implication in her definition that the figure must be 4-sided (criterion (i)). Cheryl also did not include any drawings with her definition. Replacing the word “congruent” with “parallel” would be an appropriate change to what a trapezoid could be defined. She might
have confused the two words in her definition. Yet, since trapezoids do not need any congruent sides, this definition would be incorrect and would fall under the pre-action conception of the definition of an inclusive trapezoid.

The following section reports on the analysis of the responses for the exclusive definition of a trapezoid.

4.1.7 Trapezoids (exclusive)

An exclusive trapezoid is a quadrilateral with only one pair of parallel sides. This definition excludes a hierarchical structure with parallelograms. Twelve of the students’ definitions satisfy criteria (i – iv). Specifically, their responses defined trapezoids (exclusive) as quadrilaterals (i), used properties of trapezoids (exclusive) correctly (ii), used necessary and sufficient conditions (iii), and were minimal (iv). Nine responses did not meet criterion (i). The remaining five definitions have mistaken properties and do not adequately define exclusive trapezoids (criterion (ii)). The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory definition.

Margaret gives an example of a definition that meets all four criteria (i – iv):

*Quadrilateral with only one pair of opposite sides that are parallel.*

Her definition affirms the exclusivity of a trapezoid since there is only one pair of sides that are parallel instead of at least one pair of sides that are parallel. Margaret did not need to draw a picture with her definition. Therefore, Margaret’s definition illustrates a process conception for the definition of an exclusive trapezoid.

Tammie’s definition is only missing criterion (i):

*Has one pair of parallel sides.*
Tammie does not include any drawings with her written work. Without criterion (i), this definition could be used for figures with more than 4 sides. Consequently, Tammie’s definition demonstrates a pre-action conception for the definition of exclusive trapezoids.

Anne’s definition replaces the word “parallel” with “congruent”:

Exactly one pair of congruent sides.

The following is the picture that Anne drew with her definition:

Figure 16: Anne's definition of a trapezoid (exclusive)

Anne does have a drawing that shows a four-sided figure, however she uses congruency tic marks on the legs of the trapezoid. This drawing implies that she is considering an isosceles trapezoid for her definition. Whether she means to say “parallel” instead of “congruent” or is thinking of an isosceles trapezoid as her exclusive trapezoid is uncertain. In either case, Anne’s definition exemplifies a pre-action level for defining exclusive trapezoids.

The next section summarizes the analyses for all seven definitions.

4.1.8 Summary for Research Question 1i

The first research question concerns preservice teachers’ understandings of geometric definitions. In the previous data students were asked to give a definition for various quadrilaterals, specifically squares (4.1.1), rectangles (4.1.2), parallelograms (4.1.3), rhombuses (4.1.4), kites
(4.1.5), trapezoids – inclusive definition (4.1.6), and trapezoids – exclusive definition (4.1.7). To characterize a good definition, the following criteria were used:

i) Identifies that object is a quadrilateral (or closed four-sided polygon or figure)

ii) Identifies properties correctly

iii) Establishes necessary and sufficient conditions

iv) The set of conditions should be minimal

When a student could give a definition that satisfied criteria (i) – (iii) of the above, then he can at least identify the quadrilateral correctly even though his definition does not meet criterion (iv). Without the minimal criterion, the students’ definitions would be classified as indication of action conception. On the other hand, if the student’s definition does not satisfy one or more criteria (i) – (iii), then the student does not understand the specific quadrilateral concept. These definitions would be classified as indication of pre-action conception. Any student’s definition that met all four criteria (i) – (iv) could be categorized as process level depending how the student used her definition in further situations.
The following table is a summary of the results for this research question:

<table>
<thead>
<tr>
<th>Figure</th>
<th>Satisfactory</th>
<th>Missing quadrilateral in definition or without a picture (criterion (i))</th>
<th>Invalid definition (criterion (ii))</th>
<th>Unsatisfactory response for necessary and sufficient conditions (criterion (iii))</th>
<th>Not minimal (criterion (iv))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square</td>
<td>17 (65%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>2 (8%)</td>
<td>7 (27%)</td>
</tr>
<tr>
<td>Rectangle</td>
<td>5 (19%)</td>
<td>6 (23%)</td>
<td>3 (12%)</td>
<td>1 (4%)</td>
<td>11 (42%)</td>
</tr>
<tr>
<td>Parallelogram</td>
<td>12 (46%)</td>
<td>6 (23%)</td>
<td>2 (8%)</td>
<td>4 (15%)</td>
<td>2 (8%)</td>
</tr>
<tr>
<td>Rhombus</td>
<td>11 (42%)</td>
<td>4 (15%)</td>
<td>2 (8%)</td>
<td>3 (12%)</td>
<td>6 (23%)</td>
</tr>
<tr>
<td>Kite</td>
<td>9 (34%)</td>
<td>0 (0%)</td>
<td>3 (12%)</td>
<td>14 (54%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>Trapezoid (inclusive)</td>
<td>12 (46%)</td>
<td>4 (15%)</td>
<td>10 (39%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>Trapezoid (exclusive)</td>
<td>12 (46%)</td>
<td>9 (35%)</td>
<td>5 (19%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
</tr>
</tbody>
</table>

Table 2: Summary of Results from Research Question 1i

Most of the students could satisfactorily define a square (seventeen) with only seven students missing the minimal criterion (iv). Sixteen students could define rectangles satisfactorily but only five of the sixteen considered the minimal criterion (iv). Students were equally able to satisfactorily define parallelograms (twelve), rhombuses (eleven), trapezoids – inclusive (twelve) and trapezoids – exclusive (twelve). The kite turned out to be the most difficult quadrilateral to define satisfactorily with only nine students.

The next section reports on the analyses of data for the second research question in this study. This question sees if students can take a definition that is incomplete for a special quadrilateral and reason what special quadrilaterals could also be included in the definition. The data for this section will come primarily from questions from a quiz and the final exam.
4.2 Preservice Teachers’ Application of the Distinction of Necessary and Sufficient Conditions

The second research question (1iii) focuses on how preservice teachers apply the distinction between necessary and sufficient conditions for a mathematical definition. Students from a preservice teacher Geometry class released their responses to several assessments including two quizzes and the final exam. Two questions from both quizzes and one question from the final exam that is a duplicate question from one of the quizzes addresses this research question and is analyzed. Furthermore, the results from the analysis of the prior research question (Section 4.1) will also contribute to our exploration of this research question.

The four specific questions (one is a duplicate on a separate assessment) all have the same type of structure. Each question asks about a property of a specific quadrilateral that is necessary but not sufficient to define that quadrilateral. Consequently, the student needs to show an example of a different quadrilateral that shares the given property but is also not a special case of the given quadrilateral in the question. Students must be aware of the properties of special quadrilaterals, know what differentiates one quadrilateral from another, and know how to avoid hierarchical classifications of quadrilaterals in their answers.

In this section I will take each of the four separate questions and analyze all the students’ responses. The first two questions (4.2.1 and 4.2.2) came from the same quiz given to the students. The next two questions (4.2.3 and 4.2.4) came from a second quiz that was given a few weeks later in the semester. A fifth question (4.2.5) is duplicated from one of the previous questions (specifically question four). The question is from the students’ final exam in the Geometry course and will be used to identify if there was any improvement in student’s understanding over this
concept. All the questions were written in the same style as problems that were given in one of the textbooks for the course (Aichele, 2008). The final section (4.2.6) is a summary of all the analyses for research question 1ii.

Based on the APOS framework, students who give an incorrect answer would demonstrate pre-action conception of applying necessary and sufficient conditions. Students were asked to draw pictures for their counterexamples. Since drawings of figures usually indicate an action conception, it is difficult to determine whether students needed to draw a picture to solve the problem or whether they could have interiorized the process. Consequently, students who correctly drew a counterexample would at least be at the action conception of applying necessary and sufficient conditions and may potentially be at a higher conception.

The following section reports on the analysis of written responses to question one given on a quiz to the students who were in a college geometry course.

4.2.1 Question One

The first question analyzed came from a quiz given to a class of preservice elementary teachers taking a college geometry course. The question reads as follows:

*A ‘bad’ definition of a kite is given next. Show how it is bad by drawing and marking a picture of a quadrilateral that fits the bad definition, but is clearly not a kite.*

*Bad definition: A kite is a quadrilateral that has at least one pair of congruent opposite angles.*

The purpose of the question is to test whether students can identify the difference between a property and a definition. Specifically, one of the properties of a kite, “has at least one pair of congruent opposite angles” is a necessary condition but is not sufficient to differentiate a kite from other quadrilaterals that may fit that description. For example, a parallelogram has two pairs of
congruent opposite angles. Likewise, any quadrilateral that has parallelogram properties, such as a rectangle, rhombus, and a square, would also fit under that definition. Yet, a rhombus and a square also have kite properties as well.

In this analysis, a satisfactory answer would be an answer stating any quadrilateral that has a pair of congruent opposite angles, but is not a kite. Most students chose to use prototypical quadrilaterals so the answer of a parallelogram or a rectangle would be the norm. Out of twenty-six responses, thirteen students gave the correct answer of a parallelogram or a rectangle. Eleven out of twenty-six responses were unsatisfactory responses. Six of these eleven incorrect responses were special cases of kites such as a “rhombus” or a “square” which would be incorrect since those figures have inherited properties of kites. The remaining five incorrect responses gave figures which did not fit the criterion given in the problem to be considered a counterexample. Two responses were left blank. The following are representative examples from each group of responses. I will start with a representative example of student who gave a satisfactory response.

Mary gave a response that was satisfactory. The following is her work:

![Figure 17: Mary's counterexample for a "bad" definition of a kite](image)

Mary drew a quadrilateral that appears to be a prototypical rectangle. She marks two opposite angles that are congruent and she uses right angle markings. Her figure is clearly not a kite since it lacks distinct congruent adjacent sides. Moreover, she clarifies that her image is indeed a
rectangle by stating so in her written work. Mary’s answer exemplifies at least action conception for necessary and sufficient conditions.

Anna’s response to this question is representative of students with partial understanding of the conditions in the “bad” definition. She drew a quadrilateral with one pair of opposite congruent sides marked. The following is her drawing:

![Figure 18: Anna's counterexample for a "bad" definition of a kite](image)

The quadrilateral is similar in shape to a prototypical kite but there are no other distinguishing marks (such as perpendicular diagonals) that would signify it is a kite. The congruent sides are the only markings even though one side visually looks slightly larger than the other. Anna may have misunderstood the difference between marking congruent sides and congruent angles as the problem requested. This response exemplifies pre-action conception for necessary and sufficient conditions.

Susan’s response is an example of students’ responses indicating a lack of interpretation of the given conditions. She drew a trapezoid for her answer. The following is the drawing she gave for this problem:
Susan marked each angle with a different degree measure. Susan also noted that there were “no congruent opposite angles.” She drew a figure that was “clearly not a kite” but it also did not adhere to the property of “at least one pair of congruent opposite angles.” Instead she created a figure that had no angles congruent. Susan’s counterexample did not disprove why the original definition was a “bad” definition but neither did it correctly use the properties. Therefore, Susan’s response illustrates a pre-action conception of using necessary and sufficient conditions. Susan’s response was one of three responses out of the twenty-six that were trapezoids.

Some students (six) responded with figures that are special cases of kites and are therefore unsatisfactory. Sophie drew a rhombus and marked a pair of congruent opposite angles:

Figure 19: Susan's counterexample for a "bad" definition of a kite

Figure 20: Sophie's counterexample of a "bad" definition of a kite
Since a rhombus has the properties of a parallelogram, both pairs of opposite angles are indeed congruent. However, a rhombus also can be classified as a special type of kite. Therefore, this answer is incorrect because it does not differentiate from a classification as a kite. Consequently, Sophie’s response illustrates a pre-action conception of necessary and sufficient conditions.

The next section reports on the analysis of the second of five questions that use necessary and sufficient conditions to find counterexamples to “bad” definitions.

**4.2.2 Question Two**

Responses for the second question came from the same quiz as question one above. The question is worded as follows:

*A “bad” definition of a parallelogram is given next. Show how it is bad by drawing and marking a picture that fits the bad definition, but is clearly not a parallelogram.*

*Bad Definition: A quadrilateral is a parallelogram if at least one diagonal cuts the figure into two congruent halves.*

This question compares a significant difference between a parallelogram and a kite. Parallelograms and kites both have at least one diagonal that cuts itself into two congruent halves. A diagonal of a parallelogram cuts it into two congruent triangles that are rotationally symmetric. One of the diagonals of a kite is a line of symmetry that creates two congruent triangles which are reflections of each other. Therefore, a correct answer to this question is the kite. Answers that included rhombuses, rectangles, and squares all have the characteristics of parallelograms and would be incorrect to this problem.

Jennifer’s answer is an example of a satisfactory response. The following is her written work:
Figure 21: Jennifer's counterexample of a “bad” definition of a parallelogram

Jennifer shows a correctly drawn kite with a diagonal that cuts the figure into two congruent halves. Her figure is also not a parallelogram and is labeled in such a way that the rhombus is not an option. Her response illustrates at least the action conception of necessary and sufficient conditions.

This problem was particularly problematic for the students on this quiz. Only six out of twenty-six responses provided satisfactory responses. One student drew a picture that resembled a kite but there were no markings or written explanations to verify if the student was indeed drawing a kite so that response is considered incorrect or at least incomplete. Two of the remaining nineteen responses were blank and all the other seventeen responses were unsatisfactory. Eleven out of the seventeen unsatisfactory responses had figures that were either special cases of parallelograms or were still parallelograms. The remaining six incorrect responses were figures that did not meet the criteria of the “bad” definition and therefore cannot be considered as counterexamples. Below are representative examples of each group of responses that were unsatisfactory. I will start with a representative response of a special case of a parallelogram.

Tammie’s response is unsatisfactory because it is a special case of a parallelogram. The following is the figure she drew:
Figure 22: Tammie's counterexample of a "bad" definition of a parallelogram

Tammie does draw a quadrilateral whose diagonal cuts it into two congruent halves. However, her congruent sides marked by tic marks also imply that the figure is a rhombus. Since a rhombus is a special case of a parallelogram, this response is unsatisfactory. Tammie drew an example of the given conditions and not a true counterexample. A proper answer for this problem would have been a kite. Tammie’s response is indicative of a pre-action conception of necessary and sufficient conditions.

Susan’s response did not meet the criteria required in the “bad” definition. Her picture is below:

Figure 23: Susan's counterexample of a "bad" definition of a parallelogram

Susan’s response of a trapezoid intentionally does not have congruent halves as she has said so on her paper. Consequently, it appears that she has interpreted the intention of the question to find a figure that does not have congruent halves created by a diagonal. Her figure cannot be used to
explain why the original definition was “bad.” Her response exemplifies pre-action conception of necessary and sufficient conditions.

Anne’s answer is also a representative of responses that did not meet the criterion of the given “bad” definition. Her work is below:

Anne’s response is ambiguous since it has no markings of congruency or parallel sides. The triangles are not even drawn as congruent figures. The figure does have two sides that appear to be parallel which may be indicative of a trapezoid. Anne’s response illustrates a pre-action conception of necessary and sufficient conditions.

The next section is a report of the analysis for question three which comes from a different quiz than the previous two problems.

4.2.3 Question Three

This question gives another “bad” definition of a parallelogram. The question is worded as follows:

*A “bad” definition of a parallelogram is given next. Show how it is bad by drawing and marking a picture that fits the bad definition, but is clearly not a parallelogram.*

*Bad Definition: A quadrilateral is a parallelogram if it has two pairs of congruent sides.*
This question also focuses on the difference between parallelograms and kites. Parallelograms have two pairs of opposite congruent sides. Kites have two pairs of adjacent congruent sides. A kite would be the appropriate satisfactory answer since it is a counterexample that is not a parallelogram. Since a rhombus has characteristics of both the parallelogram and kite, it would not be considered correct for this question. Any quadrilateral that is a special case of a parallelogram (e.g. rhombus, rectangle, and square) would be an unsatisfactory response.

Tiffani’s response is an example of a satisfactory response. The following is the drawing that she drew for her answer:

![Figure 25: Tiffani's counterexample for a "bad" definition of a parallelogram](image)

Tiffani draws a figure that has two pairs of congruent sides but is not a parallelogram. Consequently, her response indicates at least an action conception of necessary and sufficient conditions.

Eleven out of twenty-six responses were satisfactory. Out of the remaining fifteen responses, ten of them had unsatisfactory responses and five of them were not attempted. Three students gave an answer of “rectangle” which is incorrect because rectangles are a special type of parallelogram. Five other answers were an isosceles trapezoid, which only has one pair of congruent sides. The final two responses were 1) a general quadrilateral with no special quadrilateral properties and 2) a parallelogram. Both unsatisfactory responses do not meet the criteria of having two pairs of congruent sides. Below are representative examples of each group
of responses that were unsatisfactory. I will start with a representative response of a special case of a parallelogram.

Ebony’s response is an example of students giving a special case of a parallelogram as a counterexample. The following is her drawing:

![Image](image1.png)

**Figure 26: Ebony's counterexample of a "bad" definition of a parallelogram**

Ebony does draw a quadrilateral with two pairs of congruent sides. No other writing is included with her response. She marks the pairs with congruency tic marks. She also denotes a rectangle with the four right angles that are marked. However, since a rectangle is a special case of a parallelogram, she has not found an adequate counterexample to this “bad” definition. Ebony’s response illustrates a pre-action conception to necessary and sufficient conditions.

Tammie’s response exemplifies an unsatisfactory response for misinterpreting the requirements for a counterexample. The following is her picture:

![Image](image2.png)

**Figure 27: Tammie's counterexample of a "bad" definition of a parallelogram**
Tammie finds an example of a quadrilateral with only one pair of congruent sides. However, to find an adequate counterexample, the quadrilateral needs to have two pairs of congruent sides while still not being a parallelogram. She did not mark the parallel sides of her trapezoid on her picture, but she describes her figure as an isosceles trapezoid. Tammie’s response is indicative of a pre-action conception of necessary and sufficient conditions.

Lydia gives an example of an answer that also does not meet the criteria of a counterexample. The following is her response:

Figure 28: Lydia's counterexample of a "bad" definition for a parallelogram

Lydia does mark two pairs of congruent sides on her quadrilateral. However, any quadrilateral with opposite sides that are congruent is a parallelogram. Consequently, her counterexample is not a new figure, but rather an example of the definition of a parallelogram. Her goal is to draw a figure that is not a parallelogram but instead she draws a parallelogram. Lydia’s response illustrates a pre-action conception of necessary and sufficient conditions.

The next section is a report of the analysis for the fourth question on “bad” definitions.

4.2.4 Question Four

The following question comes from the same quiz given as question three to the same class of preservice elementary teachers. The question is worded as follows:

*A “bad” definition of a rhombus is given next. Show how it is bad by drawing and*
marking a picture that fits the bad definition, but is clearly not a rhombus.

**Bad Definition:** A quadrilateral is a rhombus if the diagonals are perpendicular to each other.

This question determines if students can differentiate the properties between a rhombus and a kite. Even though rhombuses share the properties of kites since a rhombus is a special kind of kite, this definition is not sufficient to describe rhombuses alone. In fact, kites also have diagonals that are perpendicular to each other. A general name for quadrilaterals that have perpendicular diagonals is orthodiagonal quadrilaterals, in which case a kite, a rhombus, and a square are special cases. Yet, there are orthodiagonal quadrilaterals that do not classify as any of these special figures (see Figure 24).

![Figure 29: Example of orthodiagonal figure that is not a kite](image)

Stephani’s response illustrates a satisfactory answer to this question. Her work is shown below:

![Figure 30: Stephani's counterexample of a "bad" definition of a rhombus](image)
Stephani draws a quadrilateral whose diagonals are perpendicular to each other but clearly is not a rhombus. Her use of congruency tic marks emphasize that she is indeed drawing a kite instead of a rhombus. Stephani’s response illustrates at least the action conception of necessary and sufficient conditions.

Seven out of twenty-six responses satisfactorily described a kite for the answer. Out of the nineteen unsatisfactory responses, twelve responses were of figures that were incorrect counterexamples, two responses were examples instead of counterexamples, and five students did not answer the question. Below are representative examples of each group of responses that were unsatisfactory responses. I will start with a representative response of a special case of a rhombus.

Mary’s response is an example of a special case of a rhombus. Her work is shown below:

![Figure 31: Mary's counterexample of a "bad" definition of a rhombus](image)

Mary’s response is a figure whose diagonals are perpendicular. However, she did not draw the perpendicular diagonals. Rather, she shows a square that has perpendicular sides. She may be misinterpreting what a diagonal is and instead focused on perpendicular sides. The result is that she still creates a square that is a special case of a rhombus. Categorically, Mary gave an example instead of a true counterexample to the “bad” definition. Consequently, Mary’s response illustrates a pre-action conception of the necessary and sufficient conditions.
Madison’s response is a figure that does not fit the conditions of the “bad” definition. Her work is shown below:

**Figure 32: Madison's counterexample of a "bad" definition of a rhombus**

Madison drew a rectangle as her counterexample. She did not draw any diagonals but instead showed perpendicular sides. Similarly, to Mary’s response, she might have misinterpreted what a diagonal was and confused diagonals for sides. If diagonals were drawn, Madison’s figure could not have perpendicular diagonals. Consequently, Madison’s response exemplifies a pre-action conception of necessary and sufficient conditions.

Leah’s response is an example of responses that did not have anything perpendicular in the figure. Her work is shown below:

**Figure 33: Leah's counterexample to a "bad" definition of a rhombus**
Leah’s drawing appears to be an isosceles trapezoid. The legs are congruent and the bases appear to be parallel, although they are not marked with the standard arrows used as parallel tic marks. However, the congruent base angles imply that the bases must be parallel. She does not draw any diagonals and if she had, in this figure, the diagonals would not necessarily be perpendicular. Therefore, this figure cannot be used as an appropriate counterexample. Leah’s response also indicates a pre-action conception for necessary and sufficient conditions.

The next section reports of the analysis on the same question as question four, but the problem was given at the end of the semester on the Final Examination.

4.2.5 Question Five

The final question analyzed was a duplicate question given to the same Geometry class as question four above but on the Final Examination. The repetition of this question is to provide additional evidence in a different setting for students’ responses to this type of question. For convenience, this question is repeated below:

A “bad” definition of a rhombus is given next. Show how it is bad by drawing and marking a picture that fits the bad definition, but is clearly not a rhombus.

Bad Definition: A quadrilateral is a rhombus if the diagonals are perpendicular to each other.

On this second attempt at this problem, the rate of success for answering the correct answer of kite increased to seventeen out of twenty-three attempted responses. Of the six unsatisfactory responses, three responses were rectangles. Another two unsatisfactory responses were trapezoids. Finally, one student responded with a rhombus which does not meet the criteria for the answer to the question. In addition, three responses were left unanswered. In the original quiz given earlier in the semester, seven students had satisfactory responses (see Section 4.2.4) and all seven
maintained their satisfactory responses on the Final Exam. Consequently, ten additional students could satisfactorily answer the problem on the second attempt. Below are representative examples of each group of responses that were categorized. I will start with a representative response of a satisfactory response.

Heather’s response correctly finds an appropriate counterexample. Her work is shown below:

**Figure 34: Heather's counterexample of a "bad" definition for a rhombus**

Heather correctly draws a quadrilateral with perpendicular diagonals. Her figure does not fall into any of the traditional special quadrilaterals and can be more generalized as an orthodiagonal quadrilateral. Her response indicates at least an action conception of necessary and sufficient conditions.

Ebony gave an incorrect response with a unique mistake. The following is her drawing:

**Figure 35: Ebony's counterexample of a "bad" definition for a rhombus**
Ebony appears to have drawn a rectangle. However, she has drawn extra lines that are perpendicular through the figure. Presumably, these lines are her interpretation of the diagonals of the figure. Instead of connecting opposite vertices for the diagonals, she connected opposite sides. Ebony’s response indicates a pre-action conception of necessary and sufficient conditions.

Bailey’s response is also incorrect. Her work is below:

![Bad Definition: A quadrilateral is a rhombus if the diagonals are perpendicular to each other.](image)

**Figure 36: Bailey's counterexample of a "bad" definition for a rhombus**

Bailey’s drawing does show intersecting diagonals; however, they do not appear to be perpendicular. Her angles on the trapezoid appear to have right angle marks even though the angles should not be right angles. She may have been confused over what needed to be perpendicular to be considered a counterexample for this problem. Consequently, Bailey’s response is indicative of a pre-action conception for necessary and sufficient conditions.

The next section is a summary of the analyses over the problems given for research question 1ii.

### 4.2.6 Summary of responses for research question 1ii

The second research question (1ii) focuses on how preservice teachers apply the distinction between necessary and sufficient conditions for a mathematical definition. Based on the APOS framework, students who give an unsatisfactory answer would demonstrate pre-action conception of applying necessary and sufficient conditions. Students were asked to draw pictures for their
counterexamples. Since drawings of figures usually indicate an action conception, it is difficult to
determine whether students needed to draw a picture to solve the problem or whether they could
have interiorized the process. Consequently, students who correctly drew a counterexample would
at least be at the action conception of applying necessary and sufficient conditions and may
potentially be at a higher conception.

For convenience the five questions (“bad definitions”) are listed below:

**Question One:** A kite is a quadrilateral that has at least one pair of congruent opposite
angles.

**Question Two:** A quadrilateral is a parallelogram if at least one diagonal cuts the figure
into two congruent halves.

**Question Three:** A quadrilateral is a parallelogram if it has two pairs of congruent sides.

**Questions Four & Five:** A quadrilateral is a rhombus if the diagonals are perpendicular
to each other.
The following table is a summary of the results for this research question:

<table>
<thead>
<tr>
<th>Question</th>
<th>Satisfactory</th>
<th>Counterexample is special case of given figure</th>
<th>Counterexample does not fit given criterion</th>
<th>No response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question One</td>
<td>13 (50%)</td>
<td>6 (23%)</td>
<td>5 (19%)</td>
<td>2 (8%)</td>
</tr>
<tr>
<td>Question Two</td>
<td>6 (23%)</td>
<td>11 (42%)</td>
<td>7 (27%)</td>
<td>2 (8%)</td>
</tr>
<tr>
<td>Question Three</td>
<td>11 (42%)</td>
<td>3 (12%)</td>
<td>7 (27%)</td>
<td>5 (19%)</td>
</tr>
<tr>
<td>Question Four</td>
<td>7 (27%)</td>
<td>2 (8%)</td>
<td>12 (46%)</td>
<td>5 (19%)</td>
</tr>
<tr>
<td>Question Five</td>
<td>17 (65%)</td>
<td>1 (4%)</td>
<td>5 (19%)</td>
<td>3 (12%)</td>
</tr>
</tbody>
</table>

**Table 3: Student Responses to Research Question 1ii**

According to the table above, most students could find a proper counterexample to Question One (thirteen) and Question Five (seventeen). Question Two gave the most difficulty with only six satisfactory responses with Question Four closely following with only seven satisfactory responses. Several students (eleven) also struggled with Question Two by listing special cases to the parallelogram instead of considering the kite as a proper counterexample. A significant number of students in all five questions found counterexamples that did not meet the criterion of the given problem. A smaller selection of students for each question did not attempt the problem.

In the analysis of research question 1i (Section 4.1), twenty-six students from interview groups defined each of the special quadrilaterals. One of the criteria to determine a satisfactory definition was criterion (iii): whether the student’s definition had appropriate necessary and sufficient conditions for the quadrilateral. In this study, students had the greatest struggle with criterion (iii) in defining a kite. Fourteen out of twenty-six responses were unsatisfactory because of missing a necessary property or having insufficient requirements. These students left off key
parts of the definition that made a kite a unique figure. On the other hand, the other figures had better success with this criterion (iii). Instead criterion (ii) which is whether the definition is even valid (ten out of twenty-six responses) was more the issue for trapezoids than criterion (iii).

The next section reports on the analysis of data for the third research question in this study. This question explores how students used the hierarchical nature of special quadrilaterals to solve problems. The data for this section will come primarily from questions from the interview and the final exam given to students in their Geometry class.

4.3 Preservice Teachers’ Perception and Use of Hierarchical Nature of Special Quadrilaterals

The third research question (2i) focuses on how preservice teachers use geometric definitions to understand the properties of special quadrilaterals. Specifically, were students able to use the hierarchical nature of special quadrilaterals to solve problems appropriately?

Data for this section came from two sources: the interviews of the students and the final exam given to the students in their Geometry course. The first questions in the interview asked students to define each special quadrilateral and then determine if there are other quadrilaterals that could be classified as a special case of the quadrilateral they had defined. The first section (4.3.1) analyzes the students’ responses for the special cases of each quadrilateral. The second section (4.3.2) involves a series of true/false questions where students had to determine if a quadrilateral can be also classified as a special case for another quadrilateral. The third section (4.3.3) also has five true/false questions that were used on the final exam for the Geometry course the students were enrolled in. These questions were a subset of the questions used in the interview.
4.3.1 Questions Regarding Special Cases

During the interview, students were asked to define seven different quadrilaterals: parallelograms, rectangles, rhombuses, kites, squares, trapezoids (inclusive) and trapezoids (exclusive). For each quadrilateral, students were also asked to provide a list of quadrilaterals, if any, that would be considered special cases of the defined quadrilateral. For example, the special cases for a parallelogram are rectangles, rhombuses, and squares since each of these figures can be defined as a parallelogram with additional characteristics. This table includes all the standard special cases for each designated quadrilateral (Aichele, 2008).

<table>
<thead>
<tr>
<th>Special Cases</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parallelogram</td>
</tr>
<tr>
<td>Rectangle, Rhombus, Square</td>
</tr>
<tr>
<td>Rectangle</td>
</tr>
<tr>
<td>Square</td>
</tr>
<tr>
<td>Rhombus</td>
</tr>
<tr>
<td>Square</td>
</tr>
<tr>
<td>Kite</td>
</tr>
<tr>
<td>Rhombus, Square</td>
</tr>
<tr>
<td>Square</td>
</tr>
<tr>
<td>None</td>
</tr>
<tr>
<td>Trapezoid (Inclusive)</td>
</tr>
<tr>
<td>Parallelogram, Rectangle, Rhombus, Square</td>
</tr>
<tr>
<td>Trapezoid (Exclusive)</td>
</tr>
<tr>
<td>None</td>
</tr>
</tbody>
</table>

Table 4: Summary of Special Cases of Quadrilaterals

At first many of the students were confused with the instructions of finding the special cases. Throughout the interview, several students amended their answers orally as they received clarification what the instructions for finding a special case meant. Therefore, students’ spoken responses were counted as their final answer instead of just their written work. In summary, out of
twenty-six participants, students seemed to understand well that a square was a special case of a rectangle (twenty-one correct responses) and a square is also a special case of a rhombus (twenty-three correct responses). Most of the students (fifteen) also correctly stated that a square does not have a special case. However, most of the students had difficulties finding all the special cases of a parallelogram, kite, and both types of trapezoids. A complete summary of the types of responses is provided at the end of this section (4.3.1.8).

The concept of determining a special case of a quadrilateral requires three steps:

1) Students must be able to identify the correct definition of the quadrilateral.

2) Students must know the properties of other quadrilaterals that could be potential candidates to be classified as a special case.

3) Students must connect the correct properties from the given quadrilateral to the special case(s).

A student who misses any of these criteria would have a pre-action conception of the classification of a special case for a specific quadrilateral. Students who need to draw pictures for their special cases with pictures of their defined quadrilateral would have an action conception. Students who can interiorize the process and determine the special cases through mental constructs and no drawings could have a process conception.

The next section begins a series of reports on the analyses of the special cases for each of the following quadrilaterals: parallelograms (4.3.1.1), rectangles (4.3.1.2), rhombuses (4.3.1.3), kites (4.3.1.4), squares (4.3.1.5), trapezoids – inclusive (4.3.1.6), and trapezoids – exclusive (4.3.1.7). A summary (4.3.1.8) of these results can be found at the end of this section.
4.3.1.1 Parallelograms

Parallelograms have three special cases: rectangles, rhombuses, and squares. The difficulty in this question is increased from other problems because there are three special cases to remember. Also, the hierarchical nature of special quadrilaterals means that the square is also a special case of both the rhombus and the rectangle. Thus, a square could be considered not a direct special case but rather another level down on the hierarchical development of the special quadrilaterals. Consequently, students must consider special cases (e.g. the square) of special cases (e.g. rhombus or rectangle).

Only eight students correctly stated all the special cases for a parallelogram: rectangle, rhombus, and square. Many students (fourteen) gave an incomplete answer for the special case of a parallelogram. Seven students gave a single response instead of listing all three figures: Rectangle (two), Rhombus (four), and Square (one). A few students recognized two of the figures: Rectangle and Square (six) and Rectangle and Rhombus (one). One student said there were no special cases for a parallelogram. Three students did not answer the question.

All the students who responded to this question correctly did not draw any pictures on their papers. Two of the students who were correct in the oral response of the interview did have some errors on their paper; one student added trapezoids and the other student left off rhombuses. Furthermore, all eight students had a correct definition of a parallelogram. Consequently, the responses of these eight students are indicative of a process conception of the special case of a parallelogram. The responses of the remaining eighteen students in the interview who had various errors exemplify a pre-action conception of the special case of a parallelogram.

The next section reports on the analysis for the special case of a rectangle.
4.3.1.2 Rectangles

A rectangle has only one special case out of the standard options for special quadrilaterals: square. Since there is only one correct answer to the special case of a rectangle, there were no incomplete answers. Twenty-one students gave the correct response of square. However, there were four wrong answers: Parallelogram or trapezoid (1), Square and some kites (1), None (1), and Rhombus (1). Only one student did not answer the question.

All the students who correctly answered squares did not draw any pictures with their work. The responses of these twenty-one students are indicative of process conceptions of a special case of a rectangle. The incorrect response of “parallelogram or trapezoid” reversed the idea of special case and the student finds a more generic quadrilateral. The response of “square and some kites” accurately lists the square as a special case, but then the student mentions “some kites” which are not hierarchically related to the rectangle. The student who said “rhombus” also tried to connect a figure that is not directly related to a kite. The fourth incorrect response of “none” gives an example of a student who could not discern the characteristics of a square that are inherited from a rectangle. All four of these responses are indicative of a pre-action conception for the special case of a rectangle.

The next section is a report of the analysis of the special case of a rhombus.

4.3.1.3 Rhombuses

Like rectangles, there is only one special case of a rhombus (a square), so there were no incomplete answers given. Twenty-three students gave the correct answer without any drawings. Their responses are indicative of a process conception for the special case of a rhombus. Two students gave the answer of a kite as their special case for a rhombus. Their responses show a reversal of not finding a true special case but rather a more general quadrilateral. These two
incorrect responses exemplify a pre-action conception for the special case of a rhombus. One student chose not to respond.

The next section reports on the analyses for the special cases of a kite.

4.3.1.4 Kites

Kites have two special cases: rhombuses and squares. Like rectangles, students have to think of two hierarchical levels of special cases since squares are the special case of rhombuses. Consequently, several students struggled answering the special case of a kite. Only nine students gave correct responses (Rhombus and Squares). The nine correct responses were indicative of the process conception of the special cases of a kite. Some students (7) did give incomplete responses: Rhombus (5) and Square (2). These students did not consider all the possible special cases that were allowed for kites. Of the remaining six incorrect responses, five of the students replied with “none” and one student said “ASA,” which may refer to the Angle Side Angle Theorem for Triangle Congruence. All thirteen incorrect responses exemplify a pre-action conception of the special cases of a kite. Four of the responses were left unanswered.

The next section reports on the analysis of the special case of a square.

4.3.1.5 Squares

Many students (fifteen) answered that the square had no special case. Their responses exhibit evidence of a process conception of the special case of a square. Of the eight incorrect responses, six students mentioned “rhombus”, one student mentioned “rectangle”, and one student listed “rhombus, kite, rectangle, trapezoid, and parallelogram.” All eight students show a misinterpretation of the meaning of special case and they found more general quadrilaterals. These responses demonstrate a pre-action conception of the special case of a square. Three students did not have a response.
The next section reports on the analysis of the special case of an inclusively defined trapezoid.

4.3.1.6 Trapezoids (Inclusive)

The inclusive definition of a trapezoid includes parallelograms and all figures that are special cases of parallelograms: rhombuses, rectangles, and squares. Students must conceptually consider three hierarchical levels of special cases: squares are special cases of rhombuses and rectangles; rhombuses and rectangles are special cases of parallelograms. Consequently, the inclusive definition of a trapezoid was the most difficult for students to find all correct special cases; only three students answered this problem correct. These three responses indicate a process conception of the special cases of an inclusively defined trapezoid. Yet fourteen students had some various partially correct answers: five students stated “parallelograms”, two students only said “rectangle”, two students included both “parallelogram and rectangle”, one student said “square and rectangle”, one student said “rectangle, rhombus, and square”, one student said “all trapezoids, rectangle, square, and rhombus”, one student said “parallelogram, rhombus, and square”, and one student said “parallelogram, rectangle, and square.” An additional four students had completely incorrect responses: three students said “none” and one student said “isosceles trapezoids.” All eighteen incorrect responses exhibit a pre-action conception of the special cases of an inclusively defined trapezoid. Five students had no response.

4.3.1.7 Trapezoids (Exclusive)

Students were more successful (twelve) with naming the lack of a special case for an exclusive definition of a trapezoid than finding all the special cases of an inclusively defined trapezoid. The responses of “none” of the twelve students that were correct indicated a process conception of the special case for an exclusively defined trapezoid. Of the four incorrect answers,
two students said “parallelograms”, one student said “trapezoid,” and one student said “parallelogram and rectangle.” These four responses exemplify a pre-action conception of the special case of a trapezoid (exclusive). Ten students did not have a response to this question. The unusually high number of non-responses may show the confusion students had toward what is an exclusively defined trapezoid.

The next section is a summary for all the results for special cases of quadrilaterals.

4.3.1.8  Summary of the results for special cases of quadrilaterals

The concept of determining a special case of a quadrilateral requires three steps:

1) Students must be able to identify the correct definition of the quadrilateral.

2) Students must know the properties of other quadrilaterals that could be potential candidates to be classified as a special case.

3) Students must connect the correct properties from the given quadrilateral to the special case(s).

A student who misses any of these criteria would have a pre-action conception of the classification of a special case for a specific quadrilateral. Students who need to draw pictures for their special cases with pictures of their defined quadrilateral would have an action conception. Students who can interiorize the process and determine the special cases through mental constructs and no drawings could have a process conception.

The following table summarizes the responses for all the special cases of quadrilaterals.

<table>
<thead>
<tr>
<th></th>
<th>Satisfactory Without Drawings</th>
<th>Incomplete</th>
<th>Incorrect</th>
<th>No response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Parallelogram</td>
<td>8 (31%)</td>
<td>14 (54%)</td>
<td>1 (4%)</td>
<td>3 (11%)</td>
</tr>
<tr>
<td>Rectangle</td>
<td>21 (81%)</td>
<td>0 (0%)</td>
<td>4 (15%)</td>
<td>1 (4%)</td>
</tr>
</tbody>
</table>
The majority of the twenty-six students could satisfactorily find the special cases of a rectangle (twenty-one), a rhombus (twenty-three) and a square (fifteen). The special case for a rectangle and a rhombus is a square and a square does not have a special case. More difficult for the students was finding all the special cases for the parallelogram (eight), kite (nine), inclusive trapezoid (three) and the exclusive trapezoid (twelve). The first three quadrilaterals (parallelogram, kite, and inclusive trapezoid) have multiple special cases but the last one (exclusive trapezoid) has no special cases. No drawings were included in any of these satisfactory responses so all the students’ responses were indicative of a process conception of understanding special cases of quadrilaterals. Several students had an incomplete answer to finding all the special cases of a parallelogram (fourteen), kite (seven), and inclusive trapezoid (fourteen). These students left off one or more of the special cases from their lists. These students’ responses have a pre-action conception of understanding special cases of quadrilaterals. The remaining groups of students either had incorrect special cases lists or did not respond. These students’ responses (or lack of response) is indicative of a pre-action conception of understanding special cases of quadrilaterals.

The next set of questions also came from the interviews of the twenty-six students. These questions were true or false questions. The students were instructed to explain why if the statement

<table>
<thead>
<tr>
<th>Quadrilateral</th>
<th>Number (%)</th>
<th>Number (%)</th>
<th>Number (%)</th>
<th>Number (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rhombus</td>
<td>23 (88%)</td>
<td>0 (0%)</td>
<td>2 (8%)</td>
<td>1 (4%)</td>
</tr>
<tr>
<td>Kite</td>
<td>9 (35%)</td>
<td>7 (27%)</td>
<td>6 (23%)</td>
<td>4 (15%)</td>
</tr>
<tr>
<td>Square</td>
<td>15 (58%)</td>
<td>0 (0%)</td>
<td>8 (31%)</td>
<td>3 (11%)</td>
</tr>
<tr>
<td>Trapezoid (I)</td>
<td>3 (11%)</td>
<td>14 (54%)</td>
<td>4 (15%)</td>
<td>5 (20%)</td>
</tr>
<tr>
<td>Trapezoid (E)</td>
<td>12 (46%)</td>
<td>0 (0%)</td>
<td>4 (15%)</td>
<td>10 (39%)</td>
</tr>
</tbody>
</table>

Table 5: Summary of Students' Responses about Special Quadrilaterals
is true and give a counterexample if the statement was false. These questions specifically require understanding of the hierarchical nature of the special quadrilaterals to adequately respond.

4.3.2 True/False Questions from Interview

These questions are designed as a rewording of the questions in section 4.3.1. However, the terms “special case” is not used in any of the questions. Instead, the student should take a specific quadrilateral and determine if it could be classified under the properties of a more general class of quadrilateral. For example, a problem could state “all squares are rectangles.” The student should respond with true or, if it were possible, provide a counterexample of a square that would not be a rectangle. Conceptually, the student must determine if the square is indeed a special case of a rectangle.

To answer these questions correctly, the student must complete these three steps:

1) Understand the definition and some properties of the special case quadrilateral.

2) Understand the definition and some properties of the second quadrilateral.

3) Determine if the second quadrilateral is a more general classification of the special case quadrilateral.

Students who are not aware of the proper definitions and at least some basic properties of the figures would indicate a pre-action conception for this hierarchical concept of special quadrilaterals. Interiorizing the process and correctly answering the question with appropriate rationale would indicate a process conception. Students who drew pictures may be at a process conception and their drawings were for clarification of what they interiorized for the solution or they may have needed the drawing to work out the solution which would indicate an action conception. Because of limitations of knowing when the student drew the picture, I will classify that a response with a drawing as indicative of at least an action conception.
Below is a report of students’ answers to ten ‘true/false’ statements about special cases of quadrilaterals. The final section (4.3.2.11) is a summary of the results of all ten statements.

4.3.2.1 Question 2A: All rectangles are parallelograms.

This statement is a true statement since a rectangle can be defined as a parallelogram with a right angle (or four right angles). All twenty-six students answered correctly with true. Eight of the students also included drawings with their answers. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Tammie gave a response that is an example of interiorizing the concept of hierarchical nature of rectangles and parallelograms. Her work is shown below:

![Figure 37: Tammie's work for All Rectangles are Parallelograms](image)

Tammie keeps her explanation for true to the main point: “has 2 pair of parallel sides.” Both rectangles and parallelograms have two pairs of parallel sides. This characteristic is the primary reason why all rectangles are parallelograms. Without a drawing, Tammie has interiorized that a characteristic of rectangles is the same definition of a parallelogram. Consequently, Tammie’s response is indicative of a process conception of the hierarchical nature of rectangles and parallelograms.

Jennifer uses her picture to back up her logic for the answer. The following is her work:
Jennifer shows a rectangle with the four right angles. The drawing also has the tic marks for parallel sides which implies that it is a parallelogram. In her written explanation, Jennifer restates that a rectangle is with “4 right angles & 2 pair of congruent sides.” She also shows the logical connection that “b/c the lines intersect at a right angle, they are parallel.” This property presumes the contrapositive of the Same Sides Interior Theorem. The contrapositive of this Theorem states that if two lines are cut by a transversal have supplementary same sides interior angles, then the lines are parallel. A corollary of the theorem would be if the same side interior angles formed by two lines and a transversal are right angles, then the lines are parallel. Jennifer reasons that the angles are right angles so the lines must be parallel. Jennifer’s logic succinctly verifies the statement that all rectangles are parallelograms. Since she drew a rectangle, her response is indicative of at least an action conception of the hierarchical nature of rectangles and parallelograms.

The next question comes from the same interview and deals with how rhombuses and parallelograms relate to each other hierarchically.

4.3.2.2 Question 2B: All rhombuses are parallelograms.

This statement is also a true statement since a rhombus can be defined as a parallelogram with congruent sides. Twenty-two students said that the statement was true. Of these twenty-two
students, six of them drew pictures along with their answers. Four students thought that the statement was false. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Anna gave a response that was correct without any drawings. The following is her written work:

![Anna's response](image)

**Figure 39: Anna's response to All Rhombuses are Parallelograms**

Anna succinctly shares the most pertinent characteristic of rhombuses for this problem: “a rhombus has 2 set of parallel sides.” Since all parallelograms have two sets of parallel sides, her logic holds. Without a drawing, Anna has interiorized the characteristics of a rhombus and parallelograms. Her response is indicative of a process conception of the hierarchical nature of rhombuses and parallelograms.

Mary’s answer is an example of a response that uses drawings. Her work is below:

![Mary's response](image)

**Figure 40: Mary's response to All Rhombuses are Parallelograms**
Mary uses a hierarchical definition that “a rhombus is a kite + parallelogram.” This definition means that all the properties of a kite and a parallelogram reside in a rhombus. Therefore, a rhombus must also be a special case of a parallelogram. Mary uses a drawing to illustrate how all four sides must be congruent in this special parallelogram. Mary’s response is indicative of at least an action conception of the hierarchical nature of rhombuses and parallelograms.

Three of the students who answered “false” came from the same interview group and two of them built a similar argument for why they thought the statement was false but eventually changed their minds:

Lydia: I put not because a kite could also be a rhombus, right? So it could be similar to a kite and it doesn’t have parallel sides?

Interviewer: Ok

Julie: I put false because all rhombuses are kites and a rhombus is a special kind of kite. Some rhombus…. rhombuses could be parallelograms but not all of them. I said some but not all.

Amity: I said false. It can be a square or a rectangle.

Interviewer: A square?

Amity: Uh huh

Interviewer: And is a square a parallelogram?

Amity: I’m not sure

Interviewer: So how did you define a parallelogram? What did you say?

Amity: At least on pair…

Interviewer: Parallelogram?

Amity: …of parallel sides, yeah.

Interviewer: and what about rhombus?
Amity: Rhombus does not have parallel sides. It doesn’t.

Interviewer: Does not?

Amity: No

Lydia: I…

Interviewer: ok

Julie: Parallel means that they don’t meet and rhombuses they tend to meet…ok…no never mind. Hold on.

Lydia: A Kite

Julie: Yeah. Kites meet each other but rhombuses…I mean look at a rhombus a different way it looks like a parallelogram so never mind, they are parallelograms.

Lydia and Julie built their arguments around how rhombuses are also kites. In their responses, they considered kites as separate objects from parallelograms. However, they did not consider that the rhombus was an object that was both a kite and a parallelogram. Similarly, Amity argues that rhombuses do not have parallel sides, so they cannot be parallelograms. Yet when Julie turns her drawing of a rhombus around she realizes that rhombuses also look like parallelograms as well. Julie’s work is below:

Figure 41: Julie's work on All Rhombuses are Parallelograms
Julie’s original picture of a rhombus is a prototypical diamond-shaped figure. Her parallelogram is also a prototypically oriented parallelogram. However, when she turned her rhombus in the same orientation as her parallelogram, she realized that all rhombuses are parallelograms. Her image of a rhombus was rigid to a specific orientation. Consequently, her final correct response indicates at least an action conception of the hierarchical relationship between rhombuses and parallelograms.

Sarah also thought that the statement was false. Like the other students above she thought that rhombuses do not have parallel sides:

Interviewer: ok the next…all rhombuses are parallelograms

Sarah: I said false because rhombuses don’t have… not all the sides are parallel of a rhombus.

Sarah also included a counterexample for a drawing:

![Figure 42: Sarah's response for All Rhombuses are Parallelograms](image)

Sarah’s counterexample is a rectangle drawn with tic marks for parallel sides. Perhaps she thought she needed a figure that had parallel sides which was obviously not a rhombus. Her idea that the opposite sides of a rhombus are not parallel may stem from her prior definition of a rhombus: “a special type of kite resembles a diamond all lengths are the same.” Sarah included a picture with her definition:
Sarah’s drawing is a prototypical diamond. However, her drawing focuses only on the four congruent sides and is not drawn carefully to show opposite parallel sides. Consequently, Sarah does not consider that the opposite sides need to be parallel. Sarah’s response illustrates a pre-action conception of the hierarchical nature of rhombuses and parallelograms.

The next section reports on the analysis of a question concerning rhombuses and kites.

### 4.3.2.3 Question 2C: All rhombuses are kites.

This statement is also a true statement. A rhombus can be defined as a kite with congruent sides. Twenty-two students had satisfactory responses while four students had unsatisfactory responses. Nine of the twenty-two correct responses had drawings to accompany their answers. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Julia’s response is an example of a correct response without any drawings. Her work is shown below:

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**Figure 43: Sarah's definition of a rhombus**

A special type of kite resembles a diamond all lengths are the same.
Figure 44: Julia's response for All Rhombuses are Kites

Julia states that the statement is “true because a rhombus has at least two pairs that are congruent just like a kite.” Presumably the two pairs refer to the two pairs of adjacent congruent sides that are characteristic of a kite. As such, Julia is stating what properties of rhombuses that coincide with kites. Since Julia does not have to draw a picture, her response is indicative of a process conception of the hierarchical nature of rhombuses and kites.

Mary’s response is an example of a correct response that uses a drawing. Her work follows below:

Figure 45: Mary's response for All Rhombuses are Kites

Mary states that the “definition of kite is that it must have 2 pairs of congruent sides.” In her definition section, she did say that the kite has “2 sets of congruent adjacent sides.” Presumably and based on her picture, she might have implied that the congruent sides must be adjacent as well. Mary also clarifies by saying that “in case of a rhombus the lengths just happen to be the same
length.” Her drawing collaborates with her reasoning by showing both the characteristics of a kite and a rhombus simultaneously. Since Mary drew a picture, her response is indicative of at least an action conception of the hierarchical nature of rhombuses and kites.

The following three students in the same interview group changed their minds on this problem after they had reasoned through the definition of a kite from an earlier question. Below is a transcribing of the conversation of the three students who reasoned from false to true:

Interviewer: Ok so we go to part C right? …all rhombuses are kites

Bailey: I put false but looking around I don’t know I’m not going to go by everyone else, I put false because I put rhombuses can also be a special case of a rectangle so oh wait, yeah, so and a rectangle is not a kite

Interviewer: So it is a special case, so how would you test that if it’s a kite, what was your definition if it’s a kite? What was your definition that it is a rhombus and what was your definition if it’s a kite?

Bailey: For rhombus I put two sets of parallel lines but kite is two sets of congruent, congruent sides

Interviewer: But is there anything else that was said about rhombuses; is it more than just two sets of parallel lines?

Bailey: Congruent sides as well

Interviewer: All sides are congruent

Bailey: But in a kite there’s not all sides are congruent

Interviewer: But your definition of kite is what?

Bailey: It has two sets of congruent sides

Interviewer: So would a rhombus fit that?
Bailey: I get confused because when you draw a kite you have two sides that are longer than the other two so you have two different sides or different sets of sides that are congruent but not all the sides are congruent

Interviewer: Right, but it does not say anything whether they are all sides could be or not. It’s important that you have two sets that are congruent. It’s not, your definition does not say that it’s not possible.

Bailey: So the answer is true then, ok

Jennifer: I put false but I guess I was wrong. I think about it now. I said a square is a rhombus but not kite but a square cannot be a kite so I guess it’s true

Megan: Yeah, I did the same thing; I put no, false, but now I understand why. The reason I said no is because I was thinking of what she said about the two sets. I say kites have two sets that are congruent and a rhombus all of them congruent, but I but…now I understand after I guess after y’all talked about it.

Bailey started with a false statement that rhombuses can be a special case of a rectangle and a rectangle cannot be a kite so therefore a rhombus is not a kite. However, as she goes through her definitions of kite and rhombus she begins to see the connection between the two objects. Yet, she admits she struggles with how a kite has two sides that are longer than the other two sides when a kite is drawn. Her personal concept image of a kite conflicted with the concept definition of the properties of a kite.

Jennifer likewise thought of a square that is an example of a rhombus but incorrectly is not an example of a kite. She does not go into details but she does assert that she thinks a square cannot be a kite. On her written work, she drew a prototypical square. The following is her work:
However, had Jennifer drawn a square that was tilted she may have seen the connection of a square to a kite. Only during the interview and discussions of the special cases of kites did Jennifer change her mind about the connection of squares and kites.

Megan looked at the logic of the definitions. She reasoned that two sets of congruent sides excluded the possibility that all sides could be congruent. Thus, she thought that rhombuses cannot be kites. In the discussion, she realized that she had to be more inclusive in her definition of kite.

Another student, Madison, decided that the answer was false. She reasoned that kites had to be equal on both sides. This property suggests not only the symmetry of kites, but that not all the sides of the kite can be congruent. On her paperwork, Madison suggested a rectangle as her counterexample. Unfortunately, she never explained why she suggested that counterexample nor did she use drawings to supplement her answer. Madison’s response is indicative of a pre-action conception of the hierarchical nature of rhombuses and kites.

The following section reports on the analysis to the question concerning the relationship of kites and parallelograms.

4.3.2.4 Question 2D: All kites are parallelograms.

This statement is a false statement. The only kite that can double as a parallelogram is the special case of a rhombus. Otherwise, the opposite sides of the kite are not parallel. Twenty-two students correctly identified this statement as false. Four students thought the statement was true.
Of the twenty-two correct responses (“false”), ten students used drawings to clarify their answers. Of the four incorrect responses (“true”), two students used drawings. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Lydia’s response is an example of a correct answer without drawings. Her work is below:

![Figure 47: Lydia's Response to All Kites are Parallelograms](image)

Lydia correctly infers how a kite is not characterized with parallel sides. Therefore, all kites are not parallelograms. Her wording could be more precise by stating “all kites do not have parallel sides.” However, she only needs to affirm that a single kite exists that does not have parallel sides in order to properly show a counterexample to the statement. Without any drawings, Lydia interiorized the characteristics of kites and parallelograms and deduced that the statement was false. Consequently, Lydia’s response exemplifies a process conception of the non-hierarchical nature of kites and parallelograms.

Sophie’s response correctly answers the question while also using drawings. Her complete answer is below:
Sophie has accurately responded “false” and given a reason that “some kites don’t have all opposite sides that are parallel to each other.” She even has an answer of accurate precision by stating “some” instead of “all” kites. Sophie’s picture is an appropriate counterexample of a kite whose opposite sides are not parallel. Since she did use a drawing, her response is indicative of at least an action conception of the non-hierarchical nature of kites and parallelograms.

Heather and Bailey were in the same interview group and answered the question similarly. Bailey explained that “kites are special case of rhombus and rhombuses are parallelograms.” Bailey’s work is below:

Bailey’s argument is based on the false premise that kites are special cases of rhombuses. Instead, rhombuses are special cases of kites. This reversal of the idea of special case may stem where kites have only two sets of congruent sides, rhombuses have all four congruent sides. The student may
be thinking that two is less than four so a kite must be a special case for a rhombus. Heather said she was “thinking about the rhombus” as well during this problem. She did not elaborate any more specifically. Both Bailey’s and Heather’s responses are indicative of a pre-action conception of the non-hierarchical nature of kites and parallelograms.

Amity also said “true” because “opposite sides are congruent.” Her work is the following:

Presumably, Amity was saying that both kites and parallelograms have congruent opposite sides. However, in general, kites do not have congruent opposite sides besides the special case of a rhombus. Amity’s response is an example of a pre-action conception of the non-hierarchical nature of kites and parallelograms.

The next section reports on the analysis of the question concerning parallelograms and the inclusive definition of trapezoids.

4.3.2.5 Question 2E: All parallelograms are trapezoids (inclusive).

Students who understood this question had to know the inclusive definition of a trapezoid which states that a trapezoid is a quadrilateral with at least one pair of parallel sides. Since parallelograms have two pairs of parallel sides, all parallelograms must be trapezoids (inclusive). Twenty-four students said “true” while only two students said the statement was false. Of the twenty-four correct responses, nine responses had drawings with them. The following are
representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Jennifer gave a correct response without any drawings. Her work is below:

![Figure 51: Jennifer's Response to All Parallelograms are Trapezoids (Inclusive)](image)

Jennifer correctly notes that “all parallelograms have at least one pair of parallel sides.” Her explanation contains the proper inclusive definition of a trapezoid. Since she did not draw any pictures, her response shows that she interiorized the characteristics of a parallelogram and an inclusively defined trapezoid. Consequently, her response is indicative of a process conception of the hierarchical nature of parallelograms and inclusively defined trapezoids.

Mary’s response shows a correct answer with drawings. Her work is shown below:

![Figure 52: Mary's Response to All Parallelograms are Trapezoids (Inclusive)](image)

Mary repeats the definition of trapezoids: “trapezoids have at least one pair of parallel sides.” She also affirms the characteristic of parallelograms, which have “2 pairs.” Mary’s picture of a
parallelogram with two pairs of tic marks for parallel sides shows that she has verified that all parallelograms are indeed inclusively defined trapezoids. Since she used a drawing, Mary’s response exemplifies at least an action conception of the hierarchical nature of parallelograms and inclusively defined trapezoids.

Lydia responded with “false” because “trapezoids are a type of parallelogram.” She even clarified in the interview “not parallelograms are trapezoids.” Her work is given below:

![Figure 53: Lydia's Response to All Parallelograms are Trapezoids (Inclusive)](image)

Although Lydia has reversed the true logic, her response does exhibit a common mistake of reversing the general figure with the special case. A parallelogram has a stricter requirement of two pairs of parallel sides, whereas a trapezoid only needs one pair of parallel sides. In these regards, Lydia sees the less restrictive requirement of one pair of parallel sides as a subset of the requirement of two pairs of parallel sides. Lydia’s response exemplifies a pre-action conception of the hierarchical nature of parallelograms and inclusively defined trapezoids.

The next section is a report of the analysis of the relationship between parallelograms and exclusively defined trapezoids.
4.3.2.6 **Question 2F: All parallelograms are trapezoids (exclusive).**

This question goes with Question 2E (section 4.3.2.6) since it depends on the difference between the two definitions of a trapezoid. The exclusive definition says that a trapezoid has *only* one pair of parallel sides. The inclusive definition says that a trapezoid has *at least* one pair of parallel sides. Consequently, parallelograms cannot be trapezoids based on this definition. Twenty-four students responded correctly with “false” whereas only two students thought the statement was true. Of the twenty-four students who responded correctly (false), eleven students used drawings to support their answers. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Jack’s response is an example of a correct answer with no pictures. His work is given below:

![Jack's Response](image)

**Figure 54: Jack's Response to All Parallelograms are Trapezoids (Exclusive)**

Jack shows that parallelograms have “2 sets of parallel lines.” The two pairs of parallel sides disqualify a parallelogram from being an exclusively defined trapezoid. Without any need for a picture, Jack interiorized the properties of parallelograms and trapezoids to answer this question. Consequently, his response is indicative of a process conception of the non-hierarchical nature of exclusively defined trapezoids and parallelograms.

Jennifer also had a correct response and included a drawing to supplement her answer. Her work is shown below:
Figure 55: Jennifer’s Response to All Parallelograms are Trapezoids (Exclusive)

Jennifer rightly states that “parallelograms can have more than one set of parallel lines.” In fact, all parallelograms have two pairs of parallel sides. She drew a picture of a parallelogram and marked the pairs of parallel sides with tic marks. By drawing a picture, her response indicates at least an action conception of the non-hierarchical nature of parallelograms and exclusively defined trapezoids.

Lydia’s response is an unsatisfactory response; she said “true one pair of sides must be parallel.” Her work is shown below:

Figure 56: Lydia's Response to All Parallelograms are Trapezoids (Exclusive)

Even though Lydia does not expound any more on her answer, there is evidence that she realized that the exclusive definition of a trapezoid results in only one pair of sides must be parallel. One possible explanation for her error is that she reversed the implications of the definitions of inclusive and exclusive trapezoids. Moreover, based on her answer for Question 2E (Section 4.3.2.5), her response indicates that she may have thought that a trapezoid was a special case for a
parallelogram. Consequently, Lydia’s response is indicative of a pre-action conception of non-hierarchical nature of parallelograms and exclusively defined trapezoids.

The next section reports on the analysis of the question on the relationship of trapezoids and kites.

### 4.3.2.7 Question 2G: All trapezoids are kites.

Trapezoids must have one or at least one pair of parallel sides. Kites only have parallel sides if they are specifically rhombuses. Yet, trapezoids do not necessarily have four congruent sides so not all trapezoids are kites. Twenty-five students correctly answered “false” and one student answered “true.” Of the twenty-five correct responses, sixteen responses included drawings. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Mary has a correct response without a drawing. Her work is below:

![Figure 57: Mary's Response to All Trapezoids are Kites](image)

Mary correctly states that “trapezoids must have 1 pair of parallel sides” as a property of trapezoids. Her strict use of only one pair implies that she presumed the exclusive definition of a trapezoid. Mary also points out that kites do not have one pair of parallel sides. Without a drawing, Mary interiorized the properties of trapezoids and compared that to the properties of kites. Consequently, Mary’s response indicates a process conception of the non-hierarchical nature of trapezoids and kites.
Jennifer also had a correct response but she included a drawing with her answer. The following is her work:

![Figure 58: Jennifer's Response to All Trapezoids are Kites](image)

Jennifer draws a trapezoid (shown by two opposite sides with parallel tic marks) and states that “none of the adjacent sides are congruent.” She found an example that did not have adjacent congruent sides, which is a characteristic of kites. By drawing a picture, her response is indicative of at least an action conception of the non-hierarchical nature of trapezoids and kites.

Amity was the one student who answered “true.” In the interview she admits she does not know why yet on her paperwork she wrote “its [sic] inclusive.” Her work is given below:

![Figure 59: Amity's Response to All Trapezoids are Kites](image)

The term “inclusive” may have referenced to the inclusive definition of a trapezoid. Amity may have thought that an inclusive trapezoid may “include” kites. Her response is characteristic of a pre-action conception of the non-hierarchical nature of trapezoids and kites.

According to the interview, Lydia’s logic for “false” was justified by “looking at the picture.” Although this method is not always the best way to generalize a concept, she did find a
counterexample through visualization of a trapezoid that did not have the properties of a kite. Her work is given below:

![Image of Lydia's Response to All Trapezoids are Kites]

**Figure 60: Lydia's Response to All Trapezoids are Kites**

Lydia’s logic is based on her statement that “trapezoids have bases” and “kites do not.” In a trapezoid, the parallel sides are called bases. Lydia labels the correct pair of sides in her drawing. Lydia may have been referring to the property of trapezoids that at least one pair of opposite sides needs to be parallel. Consequently, Lydia’s response indicates at least an action conception of the non-hierarchical relationship of trapezoids and kites.

The next section reports on the analysis on the question regarding the relationship between rectangles and kites.

### 4.3.2.8 Question 2H: All rectangles are kites.

To answer this question, a student would need a counterexample such as a rectangle whose consecutive sides were not congruent. Incidentally, the only rectangle that can be classified also as a kite is the square. All twenty-six students correctly identified this answer with “false.” Seventeen out of these twenty-six students used drawings to complement their answers. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Jack gave a correct response without using any drawings. His work is shown below:
Jack explains in his response that “rectangles have congruent opposite sides; kites have congruent adjacent sides.” Jack uses the parallelogram properties of rectangles (congruent opposite sides) instead of the definition of a rectangle (quadrilateral with four right angles) to solve this problem. Without a drawing, Jack has interiorized the properties of a rectangle and the properties of a kite. Consequently, Jack’s response is indicative of the non-hierarchical relationship between kites and rectangles.

Megan also gave a correct response and she included drawings to go with her answer. Her work is shown below:

Megan adequately explains that a “kite is not a parallelogram.” Since all rectangles are parallelograms, then if the statement was true, then a kite would have to exhibit properties of parallelograms. However, in general, kites do not have parallel opposite sides. Only rhombuses,
a special case of the kite, have the properties of both kites and parallelograms. Megan draws a rectangle which does not have the rhombus characteristic of congruent sides so she has given an example of a rectangle that is not a kite. Since she needed a picture to answer the question, Megan’s response indicates at least an action conception of the non-hierarchical relationship between rectangles and kites.

Julia also gives a correct response with a picture but uses a different logic to solve the problem. The following is Julia’s work:

Figure 63: Julia's Response to All Rectangles are Kites

Julia’s response focuses in on the 90-degree angles that rectangles have while kites do not have all 90-degree angles. Her answer could have been more precise with the inclusion of the word “all” for 90-degree angles. The only kite that does have the all right angles is the special case of a square. Yet Julia’s picture of a rectangle is not a square. Consequently, she shows an example of a rectangle in her drawing that is not a kite. Julia’s response exemplifies at least an action conception of the non-hierarchical relationships of rectangles and kites.

The next section reports on the analysis of the relationship between squares and kites.

4.3.2.9 Question 2I: All squares are kites.

This question incorporates a transitive property of inclusive figures. All squares are rhombuses and all rhombuses are kites so therefore all squares are kites. Students could also use
the definition of a kite which is a quadrilateral with two non-overlapping pairs of adjacent congruent sides to verify that squares fit this classification. This problem caused more difficulties for the students than the other problems. Twelve students answered satisfactorily with “true” while fourteen students answered unsatisfactorily. Seven students who answered correctly also had drawings with their answers. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Mary’s response is an example of a correct response with a drawing. Her work follows below:

![Figure 64: Mary's Response to All Squares are Kites](image)

She correctly says that squares have “2 pairs of adjacent sides.” She is also correct that squares have “perpendicular bisectors.” The former descriptor is a characteristic of kites; the later descriptor is not quite correct for kites. Kites have one diagonal that is a perpendicular bisector to the other diagonal. However, her answer is regarding squares, which is correct to say that both diagonals are perpendicular bisectors to each other. With her picture, Mary’s response is indicative of at least an action conception of the hierarchical nature of squares and kites.

The most common mistake was reversing the special case and the general case. For example, Cheryl said in the interview, “I put like I don’t know why but I know it’s false because squares have all equal sides and kites don’t.” Her comment shows that she thought that since
squares have a more specific qualification (“all equal sides”) then a kite could not be a square. However, the opposite is true; squares are special cases of kites. Likewise, Ebony used a similar idea and suggested that a counterexample could be a rectangle. She said in her interview, “since a square is a special case of a rectangle, it can’t be…kites don’t have parallel sides.” Her logic is that since all squares are rectangles and since not all rectangles are kites, all squares cannot be kites. Her second argument connects to why all rectangles are not kites since kites do not have parallel sides. Her implication is that for a square to be a kite, then all kites would need to have pairs of parallel sides. Yet the direction of the question is not asking if all kites are squares but rather if all squares are kites. Both Ebony’s and Cheryl’s responses are examples of a pre-action conception of the hierarchical nature of squares and kites.

The next section reports on the analysis of the question regarding the relationship between rectangles and isosceles trapezoids.

4.3.2.10 Question 2J: All rectangles are isosceles trapezoids (inclusive).

This question assesses whether students understand the nature of isosceles trapezoids with the inclusive definition of a trapezoid. Since an isosceles trapezoid has at least one pair of parallel sides and congruent legs, rectangles would fall into this category. Twenty-four students answered the question satisfactorily with “true” and only two students had unsatisfactory answers. Of the twenty-four correct responses, sixteen responses had drawings with them. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Ebony’s response was correct and had a drawing. Her work is shown below:
Ebony correctly states that a “rectangle has @ least one pair of || sides & so does a iso[see] trapezoid.” On her written expression she focuses on how a rectangle is a special case of an inclusively defined trapezoid. She does not mention the pair of congruent sides in a rectangle that correspond to the pair of congruent sides in an isosceles trapezoid. Yet, her drawings of a rectangle and an isosceles trapezoid both have the pair of congruent legs marked with congruency tic marks. With her drawings, she has a thorough answer to the question. Since she needed to draw a picture to respond, Ebony’s response is indicative of at least an action conception of the hierarchical nature of rectangles and isosceles trapezoids.

Susan was one of the students who answered incorrectly. The following is her written work:
She said in the interview, “false because all the angles of a rectangle are 90 degrees but isosceles trapezoids aren’t.” Later in the interview she admits that she was not sure what “isosceles” meant for trapezoids. In either case, she originally focused on the right angles being an issue as to whether a rectangle is an isosceles trapezoid. This logic implies that isosceles trapezoids cannot have right angles which is an incorrect idea. Consequently, her response is indicative of a pre-action conception of the hierarchical nature of rectangles and isosceles trapezoids.

The next section is a summary of the data analyzed from these ten questions.

4.3.2.11 Summary of Responses to True/False Questions from Interview

For convenience, the following are the questions from this section:

2A: All rectangles are parallelograms.
2B: All rhombuses are parallelograms.
2C: All rhombuses are kites.
2D: All kites are parallelograms.
2E: All parallelograms are trapezoids (inclusive).
2F: All parallelograms are trapezoids (exclusive).
2G: All trapezoids are kites.
2H: All rectangles are kites.
2I: All squares are kites.

2J: All rectangles are isosceles trapezoids (inclusive).

To answer these questions correctly, the student must complete these three steps:

1) Understand the definition and some properties of the special case quadrilateral.

2) Understand the definition and some properties of the second quadrilateral.

3) Determine if the second quadrilateral is a more general classification of the special case quadrilateral.

Students who are not aware of the proper definitions and at least some basic properties of the figures would indicate a pre-action conception for this hierarchical concept of special quadrilaterals. The use of drawings with the proper definition would indicate at least an action conception. Interiorizing the process and correctly answering the question with appropriate rationale would indicate a process conception.
The following table is a summary of the responses from the data of the ten true/false questions:

<table>
<thead>
<tr>
<th></th>
<th>Satisfactory with no drawings</th>
<th>Satisfactory with drawings</th>
<th>Unsatisfactory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 2A</td>
<td>18 (69%)</td>
<td>8 (31%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>Question 2B</td>
<td>16 (62%)</td>
<td>6 (23%)</td>
<td>4 (15%)</td>
</tr>
<tr>
<td>Question 2C</td>
<td>13 (50%)</td>
<td>9 (35%)</td>
<td>4 (15%)</td>
</tr>
<tr>
<td>Question 2D</td>
<td>12 (46%)</td>
<td>10 (39%)</td>
<td>4 (15%)</td>
</tr>
<tr>
<td>Question 2E</td>
<td>15 (57%)</td>
<td>9 (35%)</td>
<td>2 (8%)</td>
</tr>
<tr>
<td>Question 2F</td>
<td>13 (50%)</td>
<td>11 (42%)</td>
<td>2 (8%)</td>
</tr>
<tr>
<td>Question 2G</td>
<td>9 (35%)</td>
<td>16 (62%)</td>
<td>1 (3%)</td>
</tr>
<tr>
<td>Question 2H</td>
<td>9 (35%)</td>
<td>17 (65%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>Question 2I</td>
<td>5 (19%)</td>
<td>7 (27%)</td>
<td>14 (54%)</td>
</tr>
<tr>
<td>Question 2J</td>
<td>8 (30%)</td>
<td>16 (62%)</td>
<td>2 (8%)</td>
</tr>
</tbody>
</table>

*Table 6: Students' Responses for True/False Questions given during the Interview*

Students were successful with these ten questions except for Question 2I: All squares are kites. Fourteen out of twenty-six students decided that squares are not kites. One issue is that squares are two levels lower hierarchically for special cases. More specifically, a rhombus is a special case of a kite and a square is a special case of a rhombus. Students must work through a rhombus to solve this problem. The other issue is that students have a tendency of reversing the general case with the special case. A common error observed in this study is that some students seem to think that all kites are squares because the square has more congruent sides than a kite does.
Question 2A had the highest rate of students who could satisfactorily answer the question without any pictures drawn. The responses of these eighteen students indicated a process conception of understanding the hierarchical relationship between rectangles and parallelograms. An additional eight students also had a satisfactory answer but had to draw pictures. The responses of these students exemplify at least an action conception of understanding the hierarchical relationship between rectangles and parallelograms.

Question 2H also had a high rate of satisfactory answers. This question asserts that “all rectangles are kites” which is a false statement. Nine students could determine the falsity of this statement without drawing pictures. The responses of these students indicate a process conception for the non-hierarchical relationship between rectangles and kites. An additional seventeen students drew a picture to accompany their answers. The responses of these students exemplify at least an action conception for the non-hierarchical relationship between rectangles and kites.

The next section is a summary of five questions from this previous section that were also given on the students’ Final Examination for the Geometry course. The interviews for this research were conducted about six weeks before the final exam. This analysis will help determine if the students were able to apply a satisfactory approach to special quadrilaterals for longevity to their understanding.

4.3.3 True/False Questions from Class Documents

The questions analyzed in this section are taken from the Final Examination of the Geometry course for preservice teachers. The questions are also a subset of the ones used during the interview section for data analyzed in the previous section. Below are the questions that were analyzed:
Final Exam questions:

a) All rhombuses are parallelograms
b) All rhombuses are kites
c) All kites are parallelograms
d) All parallelograms are trapezoids (inclusive definition of trapezoids)
e) All rectangles are kites

To answer these questions correctly, the student must complete these three steps:

1) Understand the definition and some properties of the special case quadrilateral.
2) Understand the definition and some properties of the second quadrilateral.
3) Determine if the second quadrilateral is a more general classification of the special case quadrilateral.

Students who are not aware of the proper definitions and at least some basic properties of the figures would indicate a pre-action conception for this hierarchical concept of special quadrilaterals. Interiorizing the process and correctly answering the question with appropriate rationale would indicate a process conception. The use of drawings with the proper definition would indicate at least an action conception. Since the questions were taken after the students took the final, I could not determine whether the students drew pictures to help solve the problem or to use as reference of explaining what they interiorized to solve the problem. Consequently, I will classify their responses as at least an action conception.

The following section reports on the analysis of the responses to each of the five questions. At the end of this section (4.3.3.6) is a summary of the five responses. The last section (4.3.3.7) is
a comparison of the responses that students gave during the interview and during the Final Examination.

4.3.3.1 Question (a): All rhombuses are parallelograms

This statement is also a true statement since a rhombus can be defined as a parallelogram with congruent sides. Twenty-four students said that the statement was true. Of these twenty-four students, five of them drew pictures along with their answers. Two students thought that the statement was false. Most of the students (nineteen) only put “true” with no other remarks or pictures. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Jessica had a satisfactory answer and drew a picture with her work. The following is her complete response:

Jessica does not explain in words why she said the problem is “true.” However, she did draw a rhombus (with four congruent sides marked with congruency tic marks) and a parallelogram (with opposite parallel sides marked with parallel tic marks). Since Jessica needed to draw a picture to solve this problem, her response exemplifies at least an action conception of the hierarchical nature of rhombuses and parallelograms.

Lydia gave a response that was unsatisfactory. Her work is below:

Figure 67: Jessica's Response to All Rhombuses are Parallelograms
Lydia is correct when she states that “a rhombus is a kite.” She even draws an accurate picture of a rhombus with four congruent sides and a kite with two distinct pairs of adjacent congruent sides. However, Lydia’s argument fails when she says that a rhombus is not a parallelogram. Presumptively, Lydia is connecting a fact that kites are not special cases of parallelograms. Lydia fails to recognize that the rhombus is a figure that is both a kite and a parallelogram. Consequently, Lydia’s response is an example of a pre-action conception of the hierarchical nature of rhombuses and parallelograms.

The next section gives a report on the analysis on question (b).

4.3.3.2 Question (b): All rhombuses are kites.

This question is also a true statement. Rhombuses are special cases of kites. Nine students answered the question satisfactorily with “true.” Of these nine responses only one student responded with a diagram. Seventeen students answered unsatisfactorily with “false” and gave a counterexample that would not appropriately work in this situation. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Stephani gave a correct response without any drawings. Her work is shown below:

![Figure 68: Lydia's Response to All Rhombuses are Parallelograms]

Lydia is correct when she states that “a rhombus is a kite.” She even draws an accurate picture of a rhombus with four congruent sides and a kite with two distinct pairs of adjacent congruent sides. However, Lydia’s argument fails when she says that a rhombus is not a parallelogram. Presumptively, Lydia is connecting a fact that kites are not special cases of parallelograms. Lydia fails to recognize that the rhombus is a figure that is both a kite and a parallelogram. Consequently, Lydia’s response is an example of a pre-action conception of the hierarchical nature of rhombuses and parallelograms.

The next section gives a report on the analysis on question (b).
Stephani’s response gives two reasons why she thinks that all rhombuses are kites. She mentions that “two pair of separate sides are congruent.” Presumably she is referring to the characteristics of a kite that a rhombus also has. Stephani also mentions that a “diagonal bisect opposite angles.” This characteristic is a sufficient argument since the diagonal does divide the kite and rhombus into two congruent halves. Technically, she did not need both descriptions because either one would have been sufficient as evidence of her claim. Her response shows a process conception of the hierarchical nature of rhombuses and kites.

Cheryl gave a response that is an example of an unsatisfactory answer without any drawings. Her work is shown below:

Cheryl does correctly state that “rhombuses are equal length on all sides.” She is pointing out that the sides of a rhombus are congruent. In general, she is correct that “kites are not” congruent on all sides. However, since a rhombus is a special case of a kite, Cheryl’s logic does not hold up.
Consequently, Cheryl’s response is indicative of a pre-action conception of the hierarchical nature of rhombuses and kites.

Tammie also gives a response that is an unsatisfactory answer but includes a drawing. Her work is shown below:

![Figure 71: Tammie’s Response to All Rhombuses are Kites](image)

Tammie’s reason for why all rhombuses are not kites goes back to the definition of a rhombus. She is correct that “all four sides are congruent” in a rhombus. Tammie may be thinking that there are kites whose sides are not all congruent. However, a rhombus is a special case of a kite and Tammie’s logic does not allow for kites to have all congruent sides. Her response is indicative of a pre-action conception of the hierarchical nature of rhombuses and kites.

The next section reports on the analysis of the relationship between kites and parallelograms.

4.3.3.3 Question (c): All kites are parallelograms.

This statement is a false statement. The only kite that can double as a parallelogram is the special case of a rhombus. Otherwise, the opposite sides of the kite are not parallel. Seventeen students correctly identified this statement as false. Nine students thought the statement was true. Of the seventeen satisfactory responses (“false”), fifteen students used drawings to clarify their answers. Only two of the nine unsatisfactory responses (“true”) used drawings with their answers.
The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Mary’s answer is an example of a correct answer that did not have any drawings. Her work is below:

![Mary's Response to All Kites are Parallelograms](image1)

**Figure 72: Mary's Response to All Kites are Parallelograms**

Mary explains correctly that “kites must have 2 pairs of congruent adjacent sides.” She implies that parallelograms do not have two pairs of congruent adjacent sides. The only exception are rhombuses which are a special case of both kites and parallelograms. However, since the general case of a kite does not have all sides congruent, Mary’s argument still has credibility. Mary’s response is an example of a process conception of the non-hierarchical relationship of kites and parallelograms.

Margaret’s response is an example of a response that satisfactorily answers the question with a drawing. The following is her work:

![Margaret's Response to All Kites are Parallelograms](image2)

**Figure 73: Margaret's Response to All Kites are Parallelograms**
Margaret succinctly answers the question by saying that “some kites do not have parallel sides.” Since there are kites that do not have opposite parallel sides, these kites cannot be parallelograms. Margaret’s response does use a picture to show the non-parallel sides. Margaret’s response is indicative of at least an action conception of the non-hierarchical relationship of kites and parallelograms.

Jessica gave a response that was incorrect and she drew pictures. Her work is shown below:

![Figure 74: Jessica's Response to All Kites are Parallelograms](image)

Jessica incorrectly states “true” and draws a kite and a parallelogram. Her pictures do not follow the logic of a counterexample because her kite does not appear to have parallel sides. Perhaps the student did not understand the scope of the question. Jessica’s response is an example of a pre-action conception of the non-hierarchical relationship between kites and parallelograms.

The next section reports on the analysis of parallelograms and inclusively defined trapezoids.

4.3.3.4 Question (d): All parallelograms are trapezoids (inclusive).

Students who understood this question had to know the inclusive definition of a trapezoid which states that a trapezoid is a quadrilateral with at least one pair of parallel sides. Since parallelograms have two pairs of parallel sides, all parallelograms must be trapezoids (inclusive). Fifteen students said “true” while eleven students said the statement was false. Of the fifteen correct responses, four responses had drawings with them. The following are representative
examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Margaret’s response is an example of a satisfactory answer without any drawings. Her work is shown below:

Figure 75: Margaret's Response to All Parallelograms are Trapezoids (Inclusive)

Margaret summarized the inclusive definition of a trapezoid with “at least 1 pair of parallel sides.” Since parallelograms have at least one pair of parallel sides, all parallelograms must be trapezoids. Consequently, her response is indicative of a process conception of the hierarchical nature of parallelograms and inclusively defined trapezoids.

Stephani’s response is also satisfactory yet with a drawing. Her work is included below:

Figure 76: Stephani’s Response to All Parallelograms are Trapezoids (Inclusive)

Stephani also states the inclusive definition of trapezoids with “at least one pair of sides are parallel.” She then draws a trapezoid and a parallelogram and shows with parallel lines tic marks that at least one pair of sides are parallel in both figures. With the pictures, Stephani’s response is
an example of at least an action conception to the hierarchical nature of parallelograms and inclusively defined trapezoids.

Anne gave a response that unsatisfactorily answers the question. Her work is shown below:

Figure 77: Anne's Response to All Parallelograms are Trapezoids (Inclusive)

Anne states that “all trapezoids are parallelograms.” She has reversed the special case of a trapezoid which is a parallelogram and now implies that a trapezoid is a special case of a parallelogram. Anne does not explain why she thinks that all trapezoids are parallelograms but she does draw two pictures. She correctly shows that a rectangle is a parallelogram and she draws an isosceles trapezoid. There is no indication why she drew a rectangle as a specific case for the more generic parallelogram or why the rectangle justifies her answer. Consequently, Anne’s response exemplifies a pre-action conception to the hierarchical nature of trapezoids and parallelograms.

Sophie also gave an unsatisfactory response. Her response is given below:
Sophie reasons that the trapezoids only have “one opposite sides [that] are parallel to each other” and parallelograms must have both pairs of sides be parallel to each other. She therefore concludes that if a trapezoid did have two pairs of sides parallel to each other, then the quadrilateral would be called a parallelogram instead of a trapezoid. She did consider that the statement might be true but her complete answer in written form affirm that her answer is false. Sophie’s answer does not account for the hierarchical relationship implied by the inclusive definition of a trapezoid. Sophie’s response is indicative of a pre-action conception of the hierarchical nature of parallelograms and trapezoids.

The next section reports from the analysis of the relationship between rectangles and kites.

4.3.3.5 Question (e): All rectangles are kites.

To answer this question, a student would need a counterexample such as a rectangle whose consecutive sides were not congruent. A square, which is a special case of a rectangle, would not be sufficient since it is also a special case of a kite. Twenty-four students satisfactorily identified this answer with “false.” Twenty-two out of these twenty-four students used drawings to complement their answers. The remaining two students answered unsatisfactorily with “true” and did not include any other explanations. The following are representative examples from each
group of responses. I will start with a representative example of a student who gave a satisfactory response.

Madison gave a correct response without drawing a picture. Her work is shown below:

![Madison's Response to All Rectangles are Kites](image)

**Figure 79: Madison's Response to All Rectangles are Kites**

Madison affirms that all “rectangles have 4 right angles.” She also states that “all kites do not have 4 right angles.” In fact, the only kites that do have four right angles would technically be called squares, since squares are a special case of a kite. So any general rectangle that does not have all congruent sides (e.g. a square) cannot be classified as a kite. Madison could have explained her logic more in depth, however, she satisfactorily presents the conditions needed to justify her case. Without a drawing, Madison interiorizes the properties of rectangles and kites to determine her solution for this problem. Her response exemplifies a process conception of the non-hierarchical relationship between rectangles and kites.

Susan used drawings to clarify her response. Susan’s work is shown below:
Susan correctly shared that “rectangles have angles that are right angle[s] and opposite sides [are] congruent.” In general, kites do not have all right angles and opposites that are congruent. The exception is the special case of a square, a kite with all right angles and opposite congruent sides. Yet, Susan uses a prototypical rectangle to show that she is not using the square as her counterexample. Susan’s drawing is an example of a rectangle that cannot be classified as a kite. Consequently, Susan’s response is indicative of at least an action conception of the non-hierarchical relationship between kites and rectangles.

Jennifer also uses a drawing to clarify her reasoning to this problem. Her work is shown below:

Figure 81: Jennifer's Response to All Rectangles are Kites
Jennifer uses one of the most succinct arguments that students gave for this problem. She states that “adjacent sides are not congruent” with her drawing of a prototypical rectangle. Jennifer focuses directly on how her example of a rectangle does not meet the qualifications of a kite (e.g. “adjacent sides are not congruent”). Jennifer’s response exemplifies at least an action conception of the non-hierarchical relationship between kites and rectangles.

The next section is a summary of the responses for these five questions from the Final Examination.

### 4.3.3.6 Summary of True/False Questions from Class Documents

For convenience, the following are the questions from this section:

a) All rhombuses are parallelograms  
b) All rhombuses are kites  
c) All kites are parallelograms  
d) All parallelograms are trapezoids (inclusive definition of trapezoids)  
e) All rectangles are kites

To answer these questions correctly, the student must complete these three steps:

1) Understand the definition and some properties of the special case quadrilateral.  
2) Understand the definition and some properties of the second quadrilateral.  
3) Determine if the second quadrilateral is a more general classification of the special case quadrilateral.

Students who are not aware of the proper definitions and at least some basic properties of the figures would indicate a pre-action conception for this hierarchical concept of special
quadrilaterals. The use of drawings with the proper definition would indicate at least an action conception. Interiorizing the process and correctly answering the question with appropriate rationale would indicate a process conception.

The following is a summary of the responses given for these five questions:

<table>
<thead>
<tr>
<th>Question</th>
<th>Satisfactory with no drawings</th>
<th>Satisfactory with drawings</th>
<th>Unsatisfactory</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question (a)</td>
<td>19 (73%)</td>
<td>5 (19%)</td>
<td>2 (8%)</td>
</tr>
<tr>
<td>Question (b)</td>
<td>8 (31%)</td>
<td>1 (4%)</td>
<td>17 (65%)</td>
</tr>
<tr>
<td>Question (c)</td>
<td>2 (8%)</td>
<td>15 (58%)</td>
<td>9 (34%)</td>
</tr>
<tr>
<td>Question (d)</td>
<td>11 (42%)</td>
<td>4 (16%)</td>
<td>11 (42%)</td>
</tr>
<tr>
<td>Question (e)</td>
<td>2 (8%)</td>
<td>22 (84%)</td>
<td>2 (8%)</td>
</tr>
</tbody>
</table>

Table 7: Summary of Responses for True/False Questions given from the Final Examination

Question (a) and Question (e) had a high success rate of twenty-four satisfactory responses. Students could discern that all rhombuses were parallelograms and that all rectangles were not kites. On Question (a), nineteen students could answer the question without working out the problem with a drawing. The responses of these students indicate a process conception for understanding the hierarchical nature of rhombuses and parallelograms. Five of the students on Question (a) had to draw a picture so their responses exemplify at least an action conception for understanding the hierarchical nature of rhombuses and parallelograms. Conversely only two students could determine a counterexample to Question (e) without drawing a picture. These two responses show a process conception for understanding the non-hierarchical nature of rectangles and kites. Instead most of the students (twenty-two) drew a picture for a counterexample to
Question (e) so their responses show at least an action conception for understanding the non-hierarchical nature of rectangles and kites.

Questions (b), (c), and (d) had more students struggle with finding satisfactory responses. For Question (b): All rhombuses are kites, students showed a tendency to reverse the special case and the general case. A rhombus is a special case of a kite, but since it has four congruent sides, students in this study tend to think a kite should be a type of rhombus. The seventeen unsatisfactory responses indicate a pre-action conception for the understanding of the hierarchical nature of rhombuses and kites.

For Question (c): All kites are parallelograms, students were more successful with a satisfactory answer than they were for Question (b). The nine students who wrote an unsatisfactory answer thought that a kite was a special case of a parallelogram. These responses indicate a pre-action conception for the understanding of the non-hierarchical relationship of kites and parallelograms. More students (fifteen) could draw a counterexample but because they depended on their drawings, their responses exemplify at least an action conception for the understanding of the non-hierarchical relationship of kites and parallelograms.

For Question (d): All parallelograms are trapezoids (inclusive definition for trapezoid), eleven students could agree with the statement without using drawings. These responses are indicative of a process conception of the understanding of the hierarchical nature of parallelograms and inclusively defined trapezoids. On the contrary, eleven students disagreed with the statement even though a parallelogram is a special case of an inclusively defined trapezoid. These eleven unsatisfactory responses exemplify a pre-action conception of the understanding of the hierarchical nature of parallelograms and inclusively defined trapezoids.
The next section compares the results of the five questions that showed up on the interview and the Final Examination of the Geometry Course the students were taking.

4.3.3.7 Comparison of Five Questions

The following table summarizes the responses of the twenty-six students who participated in the interview and responded on the final exam:

<table>
<thead>
<tr>
<th>Question</th>
<th>Correlation to Interview question</th>
<th>Both Correct Responses</th>
<th>Incorrect interview but correct final</th>
<th>Both incorrect responses</th>
<th>Correct interview but incorrect final</th>
</tr>
</thead>
<tbody>
<tr>
<td>a)</td>
<td>2B</td>
<td>22 (84%)</td>
<td>2 (8%)</td>
<td>2 (8%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>b)</td>
<td>2C</td>
<td>8 (31%)</td>
<td>1 (4%)</td>
<td>3 (11%)</td>
<td>14 (54%)</td>
</tr>
<tr>
<td>c)</td>
<td>2D</td>
<td>14 (54%)</td>
<td>3 (11%)</td>
<td>1 (4%)</td>
<td>8 (31%)</td>
</tr>
<tr>
<td>d)</td>
<td>2E</td>
<td>14 (54%)</td>
<td>1 (4%)</td>
<td>1 (4%)</td>
<td>10 (38%)</td>
</tr>
<tr>
<td>e)</td>
<td>2H</td>
<td>24 (92%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>2 (8%)</td>
</tr>
</tbody>
</table>

Table 8: Comparison of Questions on Interview and Final Examination

Questions (a) and (e) overwhelmingly had a positive response for students getting a satisfactory answer both during the interview and during the final exam. Since a rhombus can be defined as a parallelogram with congruent sides, question (a) focuses on the definition of a rhombus. Question (e) is false since not all rectangles are kites. An elongated rectangle can be considered enough visual evidence as a counterexample. All the students were correct on this problem for the interview based on visualization of prototypical rectangles and knowing basic characteristics of a kite.
Questions (b), (c) and (d) had strong reversals from the correct answers given at the interview and then the student answered incorrectly on the final exam. Question (b) had twenty-two satisfactory answers at the interview but diminished to only eight satisfactory answers on the final exam. At the interview, several students originally wrote “false” on their documents and then changed it to “true” after discussion of the definitions of rhombuses and kites from prior questions. These students’ answers show that they had not interiorized the concept of the hierarchical nature of kites and rhombuses. Instead they had a pre-action conception of this relationship. Similarly, for Question (c), twenty-two students had satisfactory answers at the interview but only fourteen had satisfactory answers on the final exam. This question is “false” and could have been verified with a kite whose opposite sides are obviously not congruent (a characteristic of a parallelogram). Unfortunately, the limitation of the final exam is not being able to ask the students to give the rationale behind their answers. Presumably, the students who had difficulty expressing a concise definition of a kite could not connect a kite’s characteristics with the properties of a parallelograms. Question (d) had twenty-four satisfactory answers during the interview and only fourteen satisfactory answers on the final exam. All parallelograms are indeed trapezoids when considering the inclusive definition of a trapezoid. Many students verbally shared their difficulties with the difference between inclusive and exclusive definitions of a trapezoid during the interview. Again, their answers show a pre-action conception of understanding this hierarchical relationship.

Of interest are the few students who originally missed the question on the interview, but they have now expressed a correct answer on the final exam. Although there was not a large pool of students who missed the original problems (no more than four for a specific question), having a few students correctly answer the question shows some progress. These students have at least an action conception of the problem, especially if they drew pictures to answer the question.
In the next section, I will analyze data for the fourth research question of whether students are able to discern equivalent definitions for special quadrilaterals.

4.4 Preservice Teachers’ Discernment of Equivalent Definitions for Special Quadrilaterals

The fourth research question (2ii) continues the theme of how preservice teachers’ understanding of geometric definitions contribute to their understanding of special quadrilaterals. Specifically, are students able to discern equivalent definitions for special quadrilaterals?

During the interview, students were asked seven additional questions regarding equivalent definitions of three specific quadrilaterals. These questions did not have standard definitions of the figures, but rather the student had to understand which properties were necessary and sufficient for each quadrilateral. If the definition was not complete, the student would explain why by using a counterexample, either through a name of another special quadrilateral or through a picture. These problems were similar to problems that were introduced in one of the textbooks for the Geometry course (Aichele, 2008).

The questions are as follows:

3a: A quadrilateral is a parallelogram if at least one diagonal cuts the figure into two congruent halves.

3b: A quadrilateral is a parallelogram if it has two pairs of equal sides.

4a: A quadrilateral is a rhombus if the diagonals are perpendicular bisectors of each other.

4b: A quadrilateral is a rhombus if the diagonals are perpendicular to each other.

5a: A kite is a quadrilateral that has at least one pair of congruent opposite angles.

5b: A kite is a quadrilateral that has perpendicular diagonals.
5c: *A kite is a quadrilateral in which at least one diagonal is a perpendicular bisector of the other.*

Based on the APOS framework, students who give an unsatisfactory answer would demonstrate pre-action conception of equivalent definitions. Students who did not need to draw a picture to name a counterexample or to verify an equivalent definition would have had to interiorize the properties of the figure being defined and would thus exemplify a process conception of equivalent definitions. Students who needed to draw a picture to find a counterexample or to verify an equivalent definition would show at least an action conception of equivalent definitions since a limitation of the data collection is verifying whether the student drew the picture to solve the problem or afterwards to clarify their reasoning.

The next section begins the analysis of the seven questions and their possible equivalent definitions. The final section (4.4.8) is a summary of the responses to all seven questions.

### 4.4.1 Question 3a

*A quadrilateral is a parallelogram if at least one diagonal cuts the figure into two congruent halves.*

One of the characteristics of a parallelogram is that each diagonal can cut the parallelogram into two congruent triangles. However, a kite, which is not a parallelogram, has a similar characteristic in that one of its diagonals can cut itself into two congruent triangles. The definition given in this problem has a necessary condition for a parallelogram but not a sufficient condition. Most students can identify that the necessary condition is relevant to parallelograms. However, to determine whether it is sufficient requires the skill of determining if there are any counterexamples, such as a kite. The definition would be an equivalent definition had it been worded as “a quadrilateral is a parallelogram if both of its diagonals cuts the figure into two congruent halves.”
Out of twenty-six responses, thirteen students responded satisfactorily with “no” and used kite as a counterexample. Twelve out of these thirteen students drew pictures to explain their counterexample. Twelve students also had an unsatisfactory response of “yes” or “true” and one student said “no” but had an unsatisfactory counterexample. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Ebony found a counterexample in the form of a kite. Her work is below:

Figure 82: Ebony's Response to Question 3a

Ebony drew several quadrilaterals to see which ones had a diagonal that cuts the figure into two congruent halves. She has what appears to be a rectangle, a parallelogram, two trapezoids, and a kite. The rectangle and parallelogram would not be counterexamples since they both have the properties of parallelograms. The trapezoids did not work since that figure cannot be cut by a diagonal into two congruent halves. The kite was the only figure that satisfied the condition of the definition while still not being a parallelogram. Ebony’s multiple drawings affirm that her response is indicative of an action conception of equivalent definitions.
Susan was the one student who said “no” yet gave an incorrect counterexample. Her work is shown below:

**Figure 83: Susan's Response to Question 3a**

During the interview she said, “I put false, but I got confused, what when it says counterexample; what are we supposed to be giving? Something that proves it right or wrong?... I drew a trapezoid, but then a trapezoid is a parallelogram so I don’t know what I did, but I put false.” Susan’s counterexample does not fit the criterion mentioned in the definition, so it is possible that she interpreted “counterexample” as an example that was false for the statement. In this case, her misunderstanding led to a logical fallacy. Susan’s response exemplifies a pre-action conception of equivalent definitions.

Of the twelve students who said “yes” or “true”, eight of them drew pictures of a parallelogram on their papers and verified with a diagonal that the diagonal cuts the figure into two congruent halves. No student used any markings to presume that they logically verified the two halves were congruent by any specific triangle congruency postulate (e.g. SAS, SSS, ASA, or AAS). The drawings were presumed to have two congruent triangles when the diagonal was
included. These unsatisfactory responses exemplify a pre-action conception of equivalent definitions.

The next section reports on the analysis of another possible equivalent definition of a parallelogram.

4.4.2 Question 3b

A quadrilateral is a parallelogram if it has two pairs of equal sides.

A parallelogram does have two pairs of equal opposite sides. Similarly, a kite has two pairs of equal adjacent sides. Since not all kites are parallelograms, this definition has a necessary but not sufficient condition. Students sometimes look at the “equal sides” and assume the characteristic of “equal opposite sides.” The error is either presuming a narrow definition of “equal sides” to only opposite sides or not carefully reading the problem statement in its entirety.

Out of twenty-six students, fifteen students had a satisfactory answer with a kite as a counterexample. Twelve out of the fifteen satisfactory answers drew pictures to explain their counterexamples. Four students gave an unsatisfactory counterexample and seven students incorrectly stated that the definition was equivalent for a parallelogram. Four out of the seven students who had incorrect responses drew a picture of a parallelogram to affirm their answers. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Tiffani’s response was an example of a satisfactory answer with a picture. Her work is shown below:
Tiffani's need to draw a picture indicates at least an action conception of equivalent definitions.

Madison did not explain during the interview why she thought the definition was not sufficient. However, she did draw a trapezoid in her problem as a counterexample. The following is her work:

Figure 85: Madison's Response to Question 3b
Madison’s response of a trapezoid does not comply with the parameters of the given definition. The trapezoid does not have two pairs of equal sides. Consequently, Madison’s response is indicative of a pre-action conception of equivalent definitions.

Heather also drew a trapezoid and she also explained why in the interview: “I said no, trapezoids, right? You can draw a trapezoid and none of the sides are equal but it has parallel lines.” The following is her work:

**Figure 86: Heather’s Response to Question 3b**

Later in the interview she defends her answer because “it is a parallelogram.” She had mistakenly thought that a trapezoid was a parallelogram. By this acclamation (she asked later “isn’t every trapezoid a parallelogram?”) she reasoned that the problem’s definition was too specific requiring two pairs of equal sides. Heather’s response is indicative of a pre-action conception of equivalent definitions.

The next section reports on the analysis of a possible equivalent definition of a rhombus.

4.4.3 **Question 4a**

*A quadrilateral is a rhombus if the diagonals are perpendicular bisectors of each other.*

This definition is an equivalent definition of a rhombus. This problem can be solved quickly by showing that both the diagonals of a rhombus are lines of symmetry, which implies that
all the sides of the quadrilateral must be congruent. An alternate route is by showing that since the diagonals are perpendicular bisectors of each other, the diagonals are creating four congruent right triangles which implies that the sides of the quadrilateral must be congruent. The discussions in the interview did not seek to prove or justify why the student considered this definition an equivalent definition to the standard definition (a quadrilateral with congruent sides).

Of twenty-six student responses, twenty-three of them affirmed that the definition was appropriate. Sixteen of the twenty-three satisfactory responses had pictures drawn. Three students said that the definition was not equivalent. Two of the three unsatisfactory responses had pictures. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Jack’s response is an example of a satisfactory answer with a picture. His work is shown below:

![Figure 87: Jack's Response to Question 4a](image)

Jack used a drawing to show the diagonals in a quadrilateral that are perpendicular bisectors of each other. His figure has the congruency tic marks for the segments of the diagonals that are congruent. Technically, one of the diagonals should have had three congruence tic marks to show that the diagonals are different lengths. With the markings of the right angles, Jack has constructed four congruent triangles which imply that the quadrilateral must be a rhombus since all the sides are congruent. Jack needed a picture to explain his reasons that the definition was an equivalent
definition of a rhombus. Jack’s response exemplifies at least an action conception of equivalent definitions.

Sophie responded directly in the interview with “I said false; rhombus diagonals don’t have perpendicular bisectors.” On her paper she drew a figure that resembles more like a prototypical parallelogram and the diagonals are not perpendicular to each other. The following is her work:

![Figure 88: Sophie's Response to Question 4a](image)

Sophie missed a main characteristic of a rhombus (the diagonals are perpendicular to each other) and that the sides are congruent to each other since her picture of a rhombus does not support that characteristic well. Sophie’s response demonstrates a pre-action conception of equivalent definitions.

Cassie did not answer directly on her interview concerning this question. However, she did respond on her paperwork with “no” and drew a square. The following is her work:
Cassie’s square is not a sufficient counterexample since squares are special cases of rhombuses. A counterexample should show a figure that has diagonals that are perpendicular bisectors of each other and yet is not a rhombus. Consequently, Cassie’s response is indicative of a pre-action conception of equivalent definitions.

Amity did not draw any pictures on her paperwork. However, she did have the following conversation during the interview of her team when asked about her responses:

Interviewer: Any why did, um Amity, why did you put no?
Amity: Um a rectangle can be a perpendicular bisector too…um, I’m not sure.
Interviewer: So umm…what does the word perpendicular mean?
Amity: It bisects perpendicularly…90 degree angles
Interviewer: 90 degree angles and what does it mean “bisects”?
Amity: Goes in half
Julie: If you fold it in half
Amity: Congruent
Interviewer: They are the same. So is there any other object that has diagonals bisecting each other?

Amity: I said a rectangle and…

Interviewer: and perpendicular?

Amity: Do both…perpendicular and bisector?

Lydia: (jots a picture on her paper) A square

Julie: Well we know this would be a perpendicular bisector because all the sides are equal so if you fold it…if you fold it like this then that’s equal and if you fold it up then it’s equal as well so that’s how I see it.

At first Amity justifies her answer by saying a rectangle has diagonals that are perpendicular bisectors of each other. Later in the interview, she hesitates when she clarifies that the figure must have perpendicular diagonals and they must also bisect each other. Amity’s response exemplifies a pre-action conception of equivalent definitions. Julie then explains why she feels that the definition works by folding paper. Julie’s response is shown below:

Figure 90: Julie's Response to Question 4a
Julie shows by paperfolding that the diagonals are perpendicular bisectors to each other and the figure must be a rhombus because all the sides are congruent. Julie’s response is at least an action conception of equivalent definitions.

The next section reports on the analysis of another possible equivalent definition of a rhombus.

4.4.4 Question 4b

A quadrilateral is a rhombus if the diagonals are perpendicular to each other.

A few students in the interviews were a little surprised about the difference between Question 4b and Question 4a. The key word of “bisector” is missing from Question 4b. A special quadrilateral that has perpendicular diagonals is a kite, however if the diagonals are constructed first, an orthodiagonal quadrilateral (see Section 4.2.4) can be constructed. Students studied kites in the Geometry class but did not specifically consider orthodiagonal quadrilaterals as a named class of quadrilaterals. Therefore, the answer of this question is “no” with the kite being an expected counterexample.

Out of twenty-six students, there were sixteen satisfactory responses of “no” with a counterexample of a kite and one satisfactory response of “no” with an orthodiagonal quadrilateral as a response. Of the seventeen “no” responses, fourteen responses had pictures drawn. Six students responded with “yes”, two students said “no” with an incorrect counterexample, and one student did not answer the question. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Anne’s response is an example of a correct response with a drawing. Her work is shown below:
Anne answers that “no, it would be a kite.” Her drawing is a kite with perpendicular diagonals shown. This counterexample is correct. Anne’s response is indicative of at least an action conception of equivalent definitions.

Jennifer gave a unique counterexample for her response. Her work is shown below:

Jennifer took the conditions of the definition (diagonals are perpendicular to each other) and drew a picture that had perpendicular diagonals but was not a rhombus. Her specific figure falls into the class of orthodiagonal quadrilaterals (See Section 4.2.4). Since this class of quadrilaterals was not discussed in the Geometry class Jennifer took, it would have been difficult for her to describe this quadrilateral without a picture. She may have interiorized the properties of rhombuses and
orthodiagonal quadrilaterals but needed to draw the picture since she did not know the name of this class of quadrilaterals. Consequently, her response is at least an action conception of equivalent definitions.

Susan and Julia were both in the same interview group and both said “yes” while also listing a kite as another example in their interviews. Neither student had any pictures or explanations written down.

Susan: I put yes but I put a kite no, I put yes

Interviewer: Ok.

Julia: I put yes because if you see this picture that is perpendicular the diagonals of a kite can be so I put yes.

Interviewer: So a kite is the example?

Julia: No no no, I mean yes, I put yes

Interviewer: Why did you put yes?

Julia: Because when I was reading it, I thought of a kite, if you drew a kite like the diagonals are perpendicular to each other.

Both Susan and Julia considered a kite, which is a good counterexample for this question. However, they both answered the question that the definition was a good definition for a rhombus. Perhaps they thought that a kite is a special case of a rhombus. This reversal can happen if the student thinks that the more restrictive definition (in this case a rhombus) characterizes the less restrictive definition (in this case a kite). Both responses exemplify pre-action conceptions of equivalent definitions.

Lydia answered “yes” but then stated in the interview that even though a kite was an example of the definition, “a kite is a type of rhombus.” The following is her written work:
Figure 93: Lydia's Response to Question 4b

On Lydia’s paper, she further reasoned this way: “if all sides are going to be of equal length, the diagonals will make them perpendicular to each other.” She also had picture drawn of a rhombus and a kite with perpendicular diagonals. She is correct in her written statement, saying that a rhombus will have perpendicular diagonals. However, she is not complete with her definition, since this condition is necessary but not sufficient. Lydia’s response exemplifies a pre-action conception of equivalent definitions.

Amity gives an example of a response that provides an unsatisfactory counterexample. Her work is shown below:

Figure 94: Amity's Response to Question 4b
Amity appears to draw several figures including a prototypical rhombus (diamond shaped), a rectangle, and a square. She concludes that a square is the appropriate counterexample. Unfortunately, a square is a special case of a rhombus, so Amity’s picture is an example and not a counterexample. Amity’s response shows a pre-action conception for equivalent definitions.

The next question reports on the analysis of a potential equivalent definition of a kite.

4.4.5 Question 5a

A kite is a quadrilateral that has at least one pair of congruent opposite angles.

This question deals with kites which in this study have been one of the more difficult figures for students to define accurately. Since kites have at least one diagonal which also serves as a line of symmetry, then kites do have at least one pair of congruent opposite angles. However, parallelograms also have two pairs of congruent opposite angles and not all parallelograms are kites. Therefore, this definition has a necessary but not sufficient condition.

Out of twenty-six students, seventeen responses were satisfactory with a “no” response and a counterexample of a kite. Nine of the seventeen satisfactory responses gave a picture to help clarify their answers. Seven responses were unsatisfactory with a “yes” response and two responses were “no” but with an unsatisfactory counterexample. The following are representative examples from each group of responses. I will start with a representative example of a student who gave a satisfactory response.

Mary’s response has a satisfactory answer and gives a drawing. Her work is shown below:
Mary draws a picture of a rectangle which she affirms has at least one pair of congruent opposite angles since all the angles are right angles. She also explains that the rectangle she drew “does not have adjacent congruent sides.” Consequently, her rectangle is not a kite and she has provided an appropriate counterexample. Mary’s response is indicative of at least an action conception of equivalent definitions.

Julia’s response gives an example of an unsatisfactory counterexample. Her work is shown below:
Julia states that “a trapezoid is a quadrilateral that has at least one pair of congruent opposite angles.” Her picture drawn on her paper has a prototypical trapezoid except with tic marks designating congruent opposite angles. Unfortunately, her tic marks should be the same on the base angles and not the opposite angles. In general, trapezoids do not have congruent opposite angles, but parallelograms do have congruent opposite angles. An inclusively defined trapezoid does include parallelograms as special cases, yet Julia does not draw a prototypical parallelogram. Julia’s response is indicative of a pre-action conception of equivalent definitions.

The seven students who responded with “yes” did not give further explanations for their answers during their interviews. Five of the seven students drew a kite to show that a kite does fit within the necessary condition (at least one pair of congruent opposite angles) of the definition. However, by not considering a parallelogram as a counterexample, these students’ responses exemplify a pre-action conception of equivalent definitions.

The next section reports on the analysis of another potential equivalent definition of a kite.

### 4.4.6 Question 5b

A *kite is a quadrilateral that has perpendicular diagonals.*

This question is like question 4b: A quadrilateral is a rhombus if the diagonals are perpendicular to each other. An orthodiagonal quadrilateral (see Section 4.2.4) that is not a kite is the appropriate counterexample. Since the classification of orthodiagonal quadrilaterals were not discussed in the Geometry class, students would need to use a drawing to show a counterexample. Since drawings of figures usually indicates an action conception, it is difficult to determine whether students needed to draw a picture to solve the problem or whether they could have
interiorized the process. Consequently, students who correctly drew a counterexample would at least be at the action conception for equivalent definitions.

Out of twenty-six student responses, only one student had a satisfactory answer. Eight students said “no” but did not find an appropriate counterexample. Seventeen students said “yes” that the definition is an equivalent definition to a kite. The following are representative examples from each group of responses. I will start with the example of a student who gave a satisfactory response.

Tiffani was the only student who provided a satisfactory answer. Her work is shown below:

![Figure 97: Tiffani’s Response to Question 5b](image)

Tiffani at first did not get the answer drawn on her work. At first, she named a parallelogram as a counterexample in her interview. After several incorrect answers, the interviewer prompted her team to consider drawing any quadrilateral that the diagonals are perpendicular.

Interviewer: So, so can you draw any quadrilateral that [the diagonals] are perpendicular?

Tammie: A rectangle…no it’s not it….not a rectangle…what about a rhombus…a rhombus is a kite

Interviewer: So can you start like drawing those diagonals that are perpendicular and then…

Tammie: Any kind of shape
Tiffani: A ummm trapezoid

Interviewer: A trapezoid does not have...

Tiffani: And does it? Well, I don’t know what I drew…it could be like the kite that we drew earlier where you are looking at this little…yeah like this…would this match?

Interviewer: But if it is a kite, well, it really depends if you drew a kite…is that a kite that you drew or not?

Tiffani: I’m thinking of a trapezoid

Interviewer: Oh, I see.

Tiffani’s drawing does have an appearance of a trapezoid. It appears that she started with the perpendicular diagonals and when she connected all the sides, she thought she had made a trapezoidal figure. Although prompted, Tiffani created her own quadrilateral that met the conditions of the definition while not being a kite. Tiffani’s response is indicative of an action conception for equivalent definitions.

Julie said the definition given in the problem is incorrect, but she could not give an adequate counterexample. Her work is shown below:

---

Figure 98: Julie's Response to Question 4b
Instead of finding a counterexample, Julie attempted to correct the definition. She stated that “a kite is a quadrilateral that has one diagonal going straight down.” Julie focused on the diagonal that forms a line of symmetry for a kite. However, Julie did not acknowledge that kites do have two diagonals and that they are perpendicular to each other. Consequently, Julie’s response exemplifies a pre-action conception of equivalent definitions.

Cassie’s response is a typical response for students who did think that the definition was an equivalent definition of a kite. Her work is shown below:

![Image of Cassie's response]

**Figure 99: Cassie's Response to Question 4b**

Cassie does not attempt to show a counterexample because she does not think that one exists. She does draw her version of a kite and the diagonals do appear to be perpendicular. Since Cassie does not consider any other quadrilateral as a counterexample, her response is indicative of a pre-action conception of equivalent definitions.

The next section summarizes the analysis on another potential equivalent definition of a kite.

### 4.4.7 Question 5c

*A kite is a quadrilateral in which at least one diagonal is a perpendicular bisector of the other.*
This definition is an equivalent definition of a kite. A quadrilateral that has one diagonal perpendicularly bisects the other means the quadrilateral has a line of symmetry that coincides with that diagonal. A kite is defined as a quadrilateral that has at least one diagonal that is a line of symmetry (see Section 4.1.5).

Twenty-five out of twenty-six students responded satisfactorily with “yes.” Nine of the twenty-five students with satisfactory responses had drawings with their answers. Only one student did not have a response to the question. The following is a representative example of a student who gave a satisfactory response.

Lydia gives a typical response for a satisfactory answer with a drawing. Her work is shown below:

Figure 100: Lydia’s Response to Question 5c

Lydia has a drawing of a kite with perpendicular diagonals. She does not mark that the diagonal also is the bisector of the other diagonal. She also does not mention that this diagonal is a line of symmetry. She affirms the definition by observation of a kite. Lydia’s response is indicative of at least an action conception of equivalent definitions.

The next section is a summary of the responses for the seven questions about equivalent definitions.
4.4.8 **Summary of Responses for Seven Equivalent Definitions**

The fourth research question (2ii) focuses if students can discern equivalent definitions for special quadrilaterals. For convenience the questions are listed below:

3a: *A quadrilateral is a parallelogram if at least one diagonal cuts the figure into two congruent halves.*

3b: *A quadrilateral is a parallelogram if it has two pairs of equal sides.*

4a: *A quadrilateral is a rhombus if the diagonals are perpendicular bisectors of each other.*

4b: *A quadrilateral is a rhombus if the diagonals are perpendicular to each other.*

5a: *A kite is a quadrilateral that has at least one pair of congruent opposite angles.*

5b: *A kite is a quadrilateral that has perpendicular diagonals.*

5c: *A kite is a quadrilateral in which at least one diagonal is a perpendicular bisector of the other.*

Based on the APOS framework, students who give an unsatisfactory answer would demonstrate pre-action conception of equivalent definitions. Students who needed to draw a picture to find a counterexample or to verify an equivalent definition would show at least an action conception of equivalent definitions. Students who did not need to draw a picture to name a counterexample or to verify an equivalent definition would have had to interiorize the properties of the figure being defined and would thus exemplify a process conception of equivalent definitions.
The table below is a summary of the responses for the seven equivalent definitions:

<table>
<thead>
<tr>
<th>Question 3a</th>
<th>Satisfactory with no pictures</th>
<th>Satisfactory with pictures</th>
<th>Incorrect counterexample</th>
<th>Unsatisfactory response</th>
<th>No Response</th>
</tr>
</thead>
<tbody>
<tr>
<td>Question 3a</td>
<td>1 (4%)</td>
<td>12 (46%)</td>
<td>1 (4%)</td>
<td>12 (46%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>Question 3b</td>
<td>3 (11%)</td>
<td>12 (46%)</td>
<td>4 (16%)</td>
<td>7 (27%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>Question 4a</td>
<td>7 (27%)</td>
<td>16 (61%)</td>
<td>0 (0%)</td>
<td>3 (12%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>Question 4b</td>
<td>3 (11%)</td>
<td>14 (54%)</td>
<td>2 (8%)</td>
<td>6 (23%)</td>
<td>1 (4%)</td>
</tr>
<tr>
<td>Question 5a</td>
<td>8 (31%)</td>
<td>9 (34%)</td>
<td>2 (8%)</td>
<td>7 (27%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>Question 5b</td>
<td>0 (0%)</td>
<td>1 (4%)</td>
<td>8 (31%)</td>
<td>17 (65%)</td>
<td>0 (0%)</td>
</tr>
<tr>
<td>Question 5c</td>
<td>16 (62%)</td>
<td>9 (34%)</td>
<td>0 (0%)</td>
<td>0 (0%)</td>
<td>1 (4%)</td>
</tr>
</tbody>
</table>

Table 9: Summary of Response for Equivalent Definitions

Question 5c had the most satisfactory answers (twenty-five) followed closely with Question 4a (twenty-three). Both problems are equivalent definitions for a kite is a quadrilateral in which at least one diagonal is a perpendicular bisector of the other and a rhombus is a quadrilateral in which the diagonals are perpendicular bisectors of each other. Sixteen students did not need to use drawings to explain Question 5c and seven students did not need to use drawings for Question 4a. These responses without drawings are indicative of a process conception for equivalent definitions.

Question 4b and Question 5a both had seventeen satisfactory answers. Students could identify a counterexample of a kite when the diagonals are perpendicular to each other. Students also could identify a counterexample of a parallelogram for a quadrilateral that has at least one pair of congruent opposite angles. Both questions had more students who needed to draw pictures to find an appropriate counterexample. The fourteen satisfactory responses with pictures for
Question 4b and the nine satisfactory responses with pictures for Question 5a are indicative of at least an action conception for understanding equivalent definitions.

Question 3a and 3b had similar results to each other. Thirteen students had satisfactory answers for Question 3a and fifteen students had satisfactory answers for Question 3b. Both questions had a kite as a counterexample and twelve students who answered each question needed to draw a picture of a kite. These twelve responses for each question indicate at least an action conception for equivalent definitions.

The question with the least number of satisfactory answers is Question 5b. Only one student could draw an orthodiagonal quadrilateral to satisfy the condition of perpendicular diagonals. Many students had answered a kite for the similar Question 4b so they also thought that a kite must be the answer for this question, too. The twenty-five unsatisfactory responses exemplify pre-action conception for equivalent definitions of a kite.
5 DISCUSSION AND CONCLUSIONS

This study explored the understanding of special quadrilaterals by twenty-six students from a Geometry class on a preservice teacher track. Using APOS theory, this study investigated (1) preservice teachers’ understanding of geometric definitions, specifically special quadrilaterals; (2) preservice teachers’ application of necessary and sufficient conditions in definitions; (3) preservice teachers’ perception and use of the hierarchical nature of special quadrilaterals; and (4) preservice teachers’ discernment of equivalent definitions for special quadrilaterals. Ultimately, this study aims to confirm whether the preliminary genetic decomposition proposed in Section 3.3 appropriately conveys how an individual understands the concept of special quadrilaterals.

This chapter presents a discussion and conclusions from this study. Section 5.1 covers each research question from Section 1.2 using a summary of the data analysis from Chapter 4. Section 5.2 focus on a revision of the genetic decomposition for the concept of special quadrilaterals. In Section 5.3, I will include pedagogical suggestions for the teaching of the concept of special quadrilaterals. Section 5.4 discusses the limitations of this study and Section 5.5 gives suggestions for future research on the topic of special quadrilaterals.

5.1 Discussion of the Results

Using the APOS Theory Framework, this study aimed to answer specific research questions (see Section 1.2). The data presented in Chapter 4 are analyzed and discussed in this section as insights to each of the research questions. In addition, the findings of this study are juxtaposed with related literature on the concept of special quadrilaterals.
5.1.1 Research Question 1i

This question asked what are students’ personal definitions for the special quadrilaterals. During an interview, students were asked to write down in their own words definitions of a parallelogram, rectangle, rhombus, kite, square, trapezoid (inclusive), and trapezoid (exclusive).

To characterize a good definition, the following criteria were used:

i) Identifies that object is a quadrilateral (or closed four-sided polygon or figure)

ii) Identifies properties correctly

iii) Establishes necessary and sufficient conditions

iv) The set of conditions should be minimal

When a student could give a definition that satisfied criteria (i) – (iii) of the above, then he/she can at least identify the quadrilateral correctly even though his definition does not meet criterion (iv). Without the minimal criterion, the students’ definitions would be classified as indication of action conception. On the other hand, if the student’s definition does not satisfy one or more criteria (i) – (iii), then the student does not understand the specific quadrilateral concept. These definitions would be classified as indication of pre-action conception. Any student’s definition that met all four criteria (i) – (iv) could be categorized as process level depending how the student used her definition in further situations.

Out of twenty-six students most of them could satisfactorily define a square (seventeen) with only seven students missing the minimal criterion (iv). Sixteen students could define rectangles satisfactorily but only five of the sixteen considered the minimal criterion (iv). Students were about on the same ability to satisfactorily define parallelograms (twelve), rhombuses (eleven), trapezoids – inclusive (twelve) and trapezoids – exclusive (twelve). The kite turned out to be the most difficult quadrilateral to define satisfactorily with only nine students.
Criterion (iv) proved to be difficult for several students for certain quadrilaterals. Students listed multiple properties of squares (seven), rectangles (eleven), parallelograms (two), and rhombuses (six). These students seemed to interpret a definition as listing everything they could say for certainty was a property of the figure. These students followed a typical pattern of students remembering prior experiences of diagrams and properties that are associated with the concept instead of the concept definition (Cunningham & Roberts, 2010). Some researchers (Leikin & Winicki-Landman, 2000; Vinner, 1991) insist on the minimality requirement for definitions while others (Pimm, 1993) suggest that context might require some redundancy in definitions. However, in a potential teaching environment such as an elementary school setting, an optimal strategy is keeping definitions as short as possible while maintaining the proper characteristics of the figure.

Another concern are the responses by students that either had an invalid definition (criterion ii) or unsatisfactory response for necessary and sufficient conditions (criterion iii). For example, the kite had three responses that did not meet criterion (ii) and fourteen responses that were insufficient for criterion (iii) out of twenty-six total responses. Tall and Vinner (1981) discuss how different concept images may be brought to the minds of students at different times. These images can conflict with the formal definition of the concept. Consequently, the students who struggled with criterion (ii) or (iii) had insufficient concept images of the figure they were attempting to define. Furthermore, the way a definition is presented to the students can affect the students’ concept images (Leikin & Zazkis, 2010). Unfortunately, if preservice teachers have difficulties with faulty definitions or misuse the language of mathematics, then the consequences might lead to further misconceptions in the classroom (Guner & Gulten, 2016). Of course, special attention should always be given to pedagogical approach for introducing and explaining a
concept. In Section 5.3, I will give some pedagogical suggestions for improving student comprehension of the concept of special quadrilaterals.

5.1.2 Research Question i

Research question i asked how do preservice teachers apply the distinction between necessary and sufficient conditions for a mathematical definition of a special quadrilateral. Students from a preservice teacher Geometry class released their responses to several assessments including two quizzes (two questions each) and the final examination (one question).

The four specific questions on two quizzes all have the same type of structure. The fifth question from the final examination is a duplicate of one of the questions from a quiz. Each question asks about a property of a specific quadrilateral that is necessary but not sufficient to define that quadrilateral. Consequently, the student needed to show an example of a different quadrilateral that shares the given property but is also not a special case of the given quadrilateral in the question. Students must be aware of the properties of special quadrilaterals, know what differentiates one quadrilateral from another, and know to avoid hierarchical classifications of quadrilaterals in their answers.

Based on the APOS framework, students who gave an unsatisfactory answer demonstrated pre-action conception of applying necessary and sufficient conditions. Students were asked in the problems to draw pictures for their counterexamples. Since drawings of figures usually indicate an action conception, it was difficult to determine whether students needed to draw a picture to solve the problem or whether they could have interiorized the process. Consequently, students who correctly drew a counterexample would at least be at the action conception of applying necessary and sufficient conditions and may potentially be at a higher conception.
For convenience the questions are listed below:

**Question One**: A kite is a quadrilateral that has at least one pair of congruent opposite angles.

**Question Two**: A quadrilateral is a parallelogram if at least one diagonal cuts the figure into two congruent halves.

**Question Three**: A quadrilateral is a parallelogram if it has two pairs of congruent sides.

**Questions Four & Five**: A quadrilateral is a rhombus if the diagonals are perpendicular to each other.

Out of twenty-six students, most of them could find a proper counterexample to Question One (thirteen) and Question Five (seventeen). Question Two gave the most difficulty with only six satisfactory responses and Question Four closely following with only seven satisfactory responses. Several students (eleven) also struggled with Question Two by listing special cases to the parallelogram instead of considering the kite as a proper counterexample. A significant number of students in all five questions found counterexamples that did not meet the criterion of the given problem.

For the successful completion of this type of problem, students had to overcome several obstacles. First, students had to know the proper definition of the figure given. Second, students had to understand what the description for the definition was about in the quadrilateral. For example, if the definition mentioned “congruent opposite angles,” students must understand what congruent opposite angles are. Third, students had to recognize characteristics that were shared in the definition with other figures that would not be classified as special cases of the original figure. For example, a counterexample for question one which stated that a kite is a quadrilateral that has at least one pair of congruent opposite angles is a parallelogram. Students could not use a rhombus...
or a square as counterexamples since they are special cases of a kite. Finally, in justifying their picture of a counterexample, students had to use proper notation and vocabulary.

In each of the steps listed above, students ran into difficulties. As mentioned in the discussion for research question 1i, some students had a difficulty in knowing the formal definition of the figure. For example, in the interview nine out of twenty-six students could satisfactorily define a kite. Yet, the first question in this section asks about the definition of a kite which was already problematic for many students. However, a few students could overcome this inadequacy as they reasoned that the current definition given in the problem does not have sufficient conditions. Consequently, the students could focus more on finding a counterexample that was not a special case of the given quadrilateral than knowing the exact sufficient conditions to be a correct definition of the quadrilateral.

The second step involving knowing the description given in the definition. Students had to discern what each of the following properties meant:

- at least one pair of congruent opposite angles
- at least one diagonal cuts the figure into two congruent halves
- has two pairs of congruent sides
- diagonals are perpendicular to each other

Each of these properties contains geometric terms that students would need to have correct concept definitions so that the problems could be solved. For example, students do not always understand that a diagonal must be a segment connecting two nonconsecutive vertices (Duatepe-Paksu, Iymen, & Sinem Pakmak, 2013; Ozkan & Bal, 2017). Instead students may take a segment that is not horizontal or vertical as a diagonal. In this study, some students would interchange the words
“congruent” and “parallel.” These misconceptions complicated the process of solving these problems (Guner & Gulten, 2016).

The third step was recognizing the characteristics given with other quadrilaterals that were not special cases of the original figure. For example, six students mistakenly found a special case of a kite for question one and eleven students used special cases of parallelograms for question two. In addition, some students struggled by naming a quadrilateral that did not meet the characteristics given in the definition. Either the students did not understand the given characteristic or they did not know that the quadrilateral they named does not share that same characteristic. For question four, twelve students named a counterexample that did not fit with the property named in the problem.

As students construct their concept image of geometrical objects, students may remember examples or properties that belong to the concept from prior experiences. The goal is for the student’s concept image to be used without error to associate appropriate properties of the figure in the context of a problem (Gutierrez & Jaime, 1999). However, the tendency is for students to meticulously copy down figures in a classroom setting that are memorized without the students understanding the basic properties that define that figure (Kuzniak & Rauscher, 2011).

5.1.3 Research Question 2i

The third research question (2i) focuses on how preservice teachers use the hierarchical nature of special quadrilaterals. Data for this section came from two sources: the interviews of the students and the final examination given to the students in their Geometry course. The first questions in the interview asked students to define each special quadrilateral and then determine if there are other quadrilaterals that could be classified as a special case of the quadrilateral they
had defined. The students were also asked during the interview a series of true/false questions where students had to determine if a quadrilateral can be also classified as a special case for another quadrilateral. Finally, the final examination had five true/false questions that were used. These questions were a subset of the questions used in the interview.

5.1.3.1 Special Cases

During the interview, students were asked to define seven different quadrilaterals: parallelograms, rectangles, rhombuses, kites, squares, trapezoids (inclusive) and trapezoids (exclusive). For each quadrilateral, students were also asked to provide a list of quadrilaterals, if any, that would be considered special cases of the defined quadrilateral. For example, the special cases for a parallelogram are rectangles, rhombuses, and squares since each of these figures can be defined as a parallelogram with additional characteristics.

At first many of the students were confused with the instructions of finding the special cases. Throughout the interview, several students amended their answers orally as they received clarification what the instructions for finding a special case meant. Therefore, students’ spoken responses were counted as their final answer instead of just their written work. In summary, out of twenty-six participants, students seemed to understand well that a square was a special case of a rectangle (twenty-one correct responses) and a square is also a special case of a rhombus (twenty-three correct responses). Most of the students (fifteen) also correctly stated that a square does not have a special case. However, most of the students had difficulties finding all the special cases of a parallelogram, kite, and both types of trapezoids.

The concept of determining a special case of a quadrilateral requires three steps:

1) Students must be able to identify the correct definition of the quadrilateral.
2) Students must know the properties of other quadrilaterals that could be potential candidates to be classified as a special case.

3) Students must connect the correct properties from the given quadrilateral to the special case(s).

A student who misses any of these criteria would have a pre-action conception of the classification of a special case for a specific quadrilateral. Students who need to draw pictures for their special cases with pictures of their defined quadrilateral would have at an action conception. Students who can interiorize the process and determine the special cases through mental constructs and no drawings could have a process conception.

The majority of the twenty-six students could satisfactorily find the special cases of a rectangle (twenty-one), a rhombus (twenty-three) and a square (fifteen). The special case for both a rectangle and a rhombus is a square and a square does not have a special case. More difficult for the students was finding all the special cases for the parallelogram (eight), kite (nine), inclusive trapezoid (three) and the exclusive trapezoid (twelve). The first three quadrilaterals (parallelogram, kite, and inclusive trapezoid) have multiple special cases but the last one (exclusive trapezoid) has no special cases. No drawings were included in any of these satisfactory responses so all the students’ responses were indicative of a process conception of understanding special cases of quadrilaterals. Several students had incomplete answers to finding all the special cases of a parallelogram (fourteen), kite (seven), and inclusive trapezoid (fourteen). These students left off one or more of the special cases from their lists. These students’ responses have a pre-action conception of understanding special cases of quadrilaterals. The remaining groups of students either had incorrect special cases lists or did not respond. These students’ responses (or lack of response) is indicative of a pre-action conception of understanding special cases of quadrilaterals.
The results of this study coincide with the results of other studies (Butuner & Filiz, 2017; Fujita, 2012; Guner & Gulten, 2016) in that students have difficulties in identifying quadrilaterals that are special cases of other quadrilaterals. Since a definition of a figure may not mention the features of its special cases, students would need to deduce these features through visualization or from the properties of the specific quadrilaterals. For example, a parallelogram is defined as a quadrilateral with two sets of parallel sides while a rectangle may be defined as a quadrilateral with four congruent angles. The definition of a rectangle does not mention parallel sides, but these can be inferred from the visual of a rectangle or knowing some properties about rectangles.

5.1.3.2 True/False Questions

The next set of questions are designed as a rewording of the questions used above. However, the terms “special case” is not used in any of the questions. Instead, the student should take a specific quadrilateral and determine if it could be classified under the properties of a more general class of quadrilateral. For example, a problem could state “all squares are rectangles.” The student should respond with true or, if it were possible, provide a counterexample of a square that would not be a rectangle. Conceptually, the student must determine if the square is indeed a special case of a rectangle.

To answer these questions correctly, the student must complete these three steps:

1) Understand the definition and some properties of the special case quadrilateral.

2) Understand the definition and some properties of the second quadrilateral.

3) Determine if the second quadrilateral is a more general classification of the special case quadrilateral.
Students who are not aware of the proper definitions and at least some basic properties of the figures would indicate a pre-action conception for this hierarchical concept of special quadrilaterals. The use of drawings with the proper definition would indicate at least an action conception. Interiorizing the process and correctly answering the question with appropriate rationale would indicate a process conception.

For convenience, the following are the questions from this section:

2A: All rectangle are parallelograms.
2B: All rhombuses are parallelograms.
2C: All rhombuses are kites.
2D: All kites are parallelograms.
2E: All parallelograms are trapezoids (inclusive).
2F: All parallelograms are trapezoids (exclusive).
2G: All trapezoids are kites.
2H: All rectangles are kites.
2I: All squares are kites.
2J: All rectangles are isosceles trapezoids (inclusive).

Students were successful with these ten questions except for Question 2I: All squares are kites. Fourteen out of twenty-six students decided that squares are not kites. One issue is that squares are two levels lower hierarchically for special cases. More specifically, a rhombus is a special case of a kite and a square is a special case of a rhombus. Students must work through a rhombus to solve this problem. The other issue is that students have a tendency of reversing the general case with the special case. A common error observed in this study is that some students
seem to think that all kites are squares because the square has more congruent sides than a kite does.

Question 2A (all rectangles are parallelograms) had the highest rate of students who could satisfactorily answer the question without any pictures drawn. The responses of these eighteen students indicated a process conception of understanding the hierarchical relationship between rectangles and parallelograms. An additional eight students also had a satisfactory answer but had to draw pictures. The responses of these students exemplify at least an action conception of understanding the hierarchical relationship between rectangles and parallelograms.

Question 2H also had a high rate of satisfactory answers. This question asserts that “all rectangles are kites” which is a false statement. Nine students could determine the falsity of this statement without drawing pictures. The responses of these students indicate a process conception for the non-hierarchical relationship between rectangles and kites. An additional seventeen students drew a picture to accompany their answers. The responses of these students exemplify at least an action conception for the non-hierarchical relationship between rectangles and kites.

5.1.3.3 Final Examination True/False Questions

The questions analyzed in this section are taken from the Final Examination of the Geometry course for preservice teachers. The questions are also a subset of the ones used during the interview section for data analyzed in the previous section. Below are the questions that were analyzed:

a) All rhombuses are parallelograms
b) All rhombuses are kites
c) All kites are parallelograms
d) All parallelograms are trapezoids (inclusive definition of trapezoids)

e) All rectangles are kites

To answer these questions correctly, the student must complete these three steps:

1) Understand the definition and some properties of the special case quadrilateral.

2) Understand the definition and some properties of the second quadrilateral.

3) Determine if the second quadrilateral is a more general classification of the special case quadrilateral.

Students who are not aware of the proper definitions and at least some basic properties of the figures would indicate a pre-action conception for this hierarchical concept of special quadrilaterals. The use of drawings with the proper definition would indicate at least an action conception. Interiorizing the process and correctly answering the question with appropriate rationale would indicate a process conception.

Question (a) and Question (e) had a high success rate of twenty-four satisfactory responses. Students could discern that all rhombuses were parallelograms and that all rectangles were not kites. On Question (a), nineteen students could answer the question without working out the problem with a drawing. The responses of these students indicate a process conception for understanding the hierarchical nature of rhombuses and parallelograms. Five of the students on Question (a) had to draw a picture so their responses exemplify at least an action conception for understanding the hierarchical nature of rhombuses and parallelograms. Conversely only two students could determine a counterexample to Question (e) without drawing a picture. These two responses show a process conception for understanding the non-hierarchical nature of rectangles and kites. Instead most of the students (twenty-two) drew a picture for a counterexample to
Question (e) so their responses show at least an action conception for understanding the non-hierarchical nature of rectangles and kites.

Questions (b), (c), and (d) had more students struggle with finding satisfactory responses. For Question (b): All rhombuses are kites, students showed a tendency to reverse the special case and the general case. A rhombus is a special case of a kite, but since it has four congruent sides, students in this study tend to think a kite should be a type of rhombus. The seventeen unsatisfactory responses indicate a pre-action conception for the understanding of the hierarchical nature of rhombuses and kites.

For Question (c): All kites are parallelograms, students were more successful with a satisfactory answer than they were for Question (b). The nine students who wrote an unsatisfactory answer thought that a kite was a special case of a parallelogram. These responses indicate a pre-action conception for the understanding of the non-hierarchical relationship of kites and parallelograms. More students (fifteen) could draw a counterexample but because they depended on their drawings, their responses exemplify at least an action conception for the understanding of the non-hierarchical relationship of kites and parallelograms.

For Question (d): All parallelograms are trapezoids (inclusive definition for trapezoid), eleven students could agree with the statement without using drawings. These responses are indicative of a process conception of the understanding of the hierarchical nature of parallelograms and inclusively defined trapezoids. On the contrary, eleven students disagreed with the statement even though a parallelogram is a special case of an inclusively defined trapezoid. These eleven unsatisfactory responses exemplify a pre-action conception of the understanding of the hierarchical nature of parallelograms and inclusively defined trapezoids.
The results of this study confirm the results of previous studies that students have difficulties with the characteristics of kites (Çontay & Paksu, 2012) and trapezoids (Erdogan, 2014; Türnüklü, 2015). Students struggled by finding special cases of the quadrilateral given instead of counterexamples of quadrilaterals with the property given in the definition. This difficulty mirrors the study by Fujita (2012) in which students were hesitant to consider a square a rectangle because of prototypical imagery of both figures. The students in this study may have had similar confusions in thinking that special cases of quadrilaterals were not special cases at all but separate quadrilaterals. So, a rhombus in the minds of some of the students would be a completely different figure than a parallelogram since all the sides of the rhombus are congruent.

One study (Butuner & Filiz, 2017) proposes that changing the definition of a kite to a quadrilateral formed with two isosceles triangles sharing a common base and having students study the various properties of diagonals of kites may help lay a foundation for understanding inclusion properties. By comparing the diagonal properties of kites (e.g. perpendicular) with diagonal properties of parallelograms (e.g. bisectors) then students may connect that parallelograms do not belong in the inclusion family of kites. Yet this study may show some of the difficulty of that approach when students struggled with a consistent understanding of what a diagonal of a quadrilateral was. Instead, the study parallels the study on preservice elementary teachers’ perceptions of the diagonal of a quadrilateral in that many students did not know the meaning of diagonal or even if the students could identify some properties with diagonals, the properties were from memory and not from general understanding (Duatepe-Paksu et al., 2013).
5.1.4 Research Question 2ii

The fourth research question (2ii) continues the theme of how preservice teachers’ understanding of geometric definitions contribute to their understanding of special quadrilaterals. Specifically, are students able to discern equivalent definitions for special quadrilaterals?

During the interview, students were asked seven additional questions regarding equivalent definitions of three specific quadrilaterals. These questions did not have standard definitions of the figures, but rather the student had to understand which properties were necessary and sufficient for each quadrilateral. If the definition was not complete, the student would explain why by using a counterexample, either through a name of another special quadrilateral or through a picture.

For convenience, the questions are as follows:

3a: A quadrilateral is a parallelogram if at least one diagonal cuts the figure into two congruent halves.

3b: A quadrilateral is a parallelogram if it has two pairs of equal sides.

4a: A quadrilateral is a rhombus if the diagonals are perpendicular bisectors of each other.

4b: A quadrilateral is a rhombus if the diagonals are perpendicular to each other.

5a: A kite is a quadrilateral that has at least one pair of congruent opposite angles.

5b: A kite is a quadrilateral that has perpendicular diagonals.

5c: A kite is a quadrilateral in which at least one diagonal is a perpendicular bisector of the other.

Based on the APOS framework, students who give an unsatisfactory answer would demonstrate pre-action conception of equivalent definitions. Students who needed to draw a picture to find a counterexample or to verify an equivalent definition would show at least an action conception of equivalent definitions. Students who did not need to draw a picture to name a
counterexample or to verify an equivalent definition would have had to interiorize the properties of the figure being defined and would thus exemplify a process conception of equivalent definitions.

Question 5c had the most satisfactory answers (twenty-five) followed closely with Question 4a (twenty-three). Sixteen students did not need to use drawings to explain Question 5c and seven students did not need to use drawings for Question 4a. These responses without drawings are indicative of a process conception for equivalent definitions.

Question 4b and Question 5a both had seventeen satisfactory answers. Students could identify a counterexample of a kite when the diagonals are perpendicular to each other. Students also could identify a counterexample of a parallelogram for a quadrilateral that has at least one pair of congruent opposite angles. Both questions had more students who needed to draw pictures to find an appropriate counterexample. The fourteen satisfactory responses with pictures for Question 4b and the nine satisfactory responses with pictures for Question 5a are indicative of at least an action conception for understanding equivalent definitions.

Question 3a and 3b had similar results to each other. Thirteen students had satisfactory answers for Question 3a and fifteen students had satisfactory answers for Question 3b. Both questions had a kite as a counterexample and twelve students who answered each question needed to draw a picture of a kite. These twelve responses for each question indicate at least an action conception for equivalent definitions.

The question with the least number of satisfactory answers is Question 5b. Only one student could draw an orthodiagonal quadrilateral to satisfy the condition of perpendicular diagonals. The key to solving this scenario was to draw the perpendicular diagonals first and then connect the sides of the quadrilateral. Many students had answered a kite for the similar Question
4b so they may also have thought that a kite must be the answer for this question, too. Unfortunately drawing an example is not sufficient as drawing a counterexample for a kite. The twenty-five unsatisfactory responses exemplify pre-action conception for equivalent definitions of a kite. The results of this study align with the results of Creager (2015) which suggests that students might consider the special cases of quadrilaterals as the only types of quadrilaterals that exist. Consequently, students may have difficulty considering general types of quadrilaterals that do not have the parameters that define special quadrilaterals.

Engaging students with alternative definitions of a concept can lead to refinements of their understandings of a concept (Zaslavsky & Shir, 2005). However, geometric learning that promotes memorization and recall, and that is teacher-centric cannot help students elevate their geometric understanding (Abdullah & Zakaria, 2013). Instead, environments of discovery learning that encourage the acquisition of geometric concepts should be employed (Günhan, 2014). As teachers consider alternative definitions in their classroom, a danger is for teachers to seek what appears to be the most uncomplicated definition for their students. These definitions may be what the teacher perceives to be the simplest terms or the least challenging to a students’ current conceptual understanding. However, the problem is that important characteristics of the figure may be overlooked that could be used in future mathematics (Salinas, Lynch-Davis, Mawhinney, & Crocker, 2014).

The results of this study have affirmed prior research concerning the difficulties of students with defining and using special quadrilaterals. Specifically, preservice teachers struggled with hierarchical properties of special quadrilaterals if there was already a difficulty in 1) defining the special quadrilaterals, 2) knowing the special cases of the special quadrilaterals, 3) knowing the properties of the special quadrilaterals, and 4) understanding the hierarchical relationships (and
possible non-relationships) between the special quadrilaterals. A common misconception was the reversal between the general case and the special case of a special quadrilateral. Preservice teachers also had difficulties with properties of special quadrilaterals because of misconceptions of terms such as “diagonal,” “congruent”, “parallel”, and “adjacent.” Furthermore, preservice teachers needed to visualize special quadrilaterals beyond their prototypical shapes to interiorize the properties of the special quadrilaterals.

In the next section I propose a new genetic decomposition of the concept of hierarchical properties of special quadrilaterals.
5.2 Revision of the Genetic Decomposition

According to APOS Theory, after data is analyzed, the original genetic decomposition (see Section 3.3) should be revised as needed. The results of this study seem to support the general constructions of the genetic decomposition, yet there are some steps that need to be refined and additional steps should be added. This section proposes a newly revised genetic decomposition for the comprehension of the hierarchical properties of special quadrilaterals.

The steps that are revised or are new will be in bold:

![Diagram of Revised Genetic Decomposition of Special Quadrilaterals]

**Figure 101: Revised Genetic Decomposition of Special Quadrilaterals**

1) The schema of Mathematical Definition is interiorized to conceptualize hierarchical definitions. At this step, students understand that mathematical definitions can be used to differentiate geometric shapes into their appropriate quadrilateral classifications.
2) Hierarchical definitions involve conditional statements. The schema of Logical Reasoning in encapsulated in treating the process of conditional reasoning for if-then statements and applies that to inclusive definitions. At this step students can conceptually reason that quadrilaterals with properties may belong to a higher classification of properties (e.g. if a quadrilateral is a rhombus then it is also a kite).

3) Visualization must be generalized from a prototypical concept image of special quadrilaterals to a more inclusive context. At this step, students use their concept image of a quadrilateral to visualize the properties that overlap with other quadrilaterals (e.g. a student may visualize a square and focus on the properties of four congruent sides to consider that a square has the same properties as a rhombus.)

4) Properties of special quadrilaterals emerge as the inclusive definitions are de-encapsulated back to the characteristics of the more general quadrilaterals. At this step, students take the generalized visualization from Step 3 and the Hierarchical Definitions of Inclusion from Steps 1 and 2 to identify the properties of the quadrilateral they are investigating.

5) Special cases of the quadrilateral are interiorized from the properties of the quadrilateral under investigation (e.g. a student recognizes that all the properties of a parallelogram also pass to a rectangle, rhombus, and a square.)

6) The application of the properties of special quadrilaterals to solve problems comes from a de-encapsulation of both the general properties of these quadrilaterals and the special cases of the quadrilateral. Students can now apply the discovered properties within the context of problem solving.

According to the APOS framework (Arnon et al., 2014; Asiala et al., 1996), this revised genetic decomposition should be tested for its accuracy. First, instructional activities should be
designed to help students through the constructions of the genetic decomposition. Second, new data should be taken from student work to determine if the instruction has effectively guided students through these steps. Finally, the genetic decomposition can be further revised, as necessary.

5.3 Pedagogical Recommendations

Based on the results of this study, the following recommendations are made:

1. Prior to teaching special quadrilaterals:
   - Create tasks for students to classify abstract objects based on definitions.
   - Create tasks for students to use conditional reasoning to make valid arguments.

2. Create tasks for students to classify both prototypical and non-prototypical quadrilaterals into the appropriate general class of quadrilateral. For example, with various quadrilaterals available, students would find all quadrilaterals that would represent a kite, parallelogram, etc.

3. Create tasks for students to draw multiple examples of each kind of special quadrilateral, encouraging deviation from the prototypical images. For squares, students could draw squares with different tilt angles.

4. Create tasks where students develop and list all the properties of a special quadrilateral, including properties inherited from other special quadrilaterals.

5. Create opportunities for students to use the properties of quadrilaterals in real life applications.

5.4 Limitations of this Study

There were three limitations for this study:
• The number of participants was limited to twenty-six, all of which came from two sections of College Geometry for preservice teachers in the same semester. Incorporating more students to the population of the study could have overcome this limitation.

• This study did not address the longevity of students’ understanding of the concept of special quadrilaterals after the completion of the semester. If this study did not have a time consideration, the follow up with activities that have been suggested would have been appropriate.

• The results of this study can only be generalized to similar situations with Geometry courses of preservice teachers. Further investigation is needed to students in other levels (primary and secondary) regarding their understanding of special quadrilaterals from an APOS framework.

5.5 Suggestions for Further Research

As shown in the literature review, a significant amount of research has investigated students’ understanding of the concept of special quadrilaterals. Specifically, studies have used the Van Hiele levels of understanding, Concept Image – Concept Definition, or the Figural Concepts frameworks regarding students’ conceptions of special quadrilaterals. This study applied the APOS Theory Framework and developed a genetic decomposition for special quadrilaterals which was unique to the body of research on this topic. Based on the results of this study and what we learned about students’ conceptions and misconceptions of special quadrilaterals, the following topics could be explored in future research:
1) How do students perceive the relationships of the two triangles formed by diagonals of special quadrilaterals?

2) How do students define and use auxiliary vocabulary such as “diagonal”, “congruent sides”, and “parallel sides” regarding special quadrilaterals?

3) How does the study of special quadrilaterals contribute to students’ understanding of area of quadrilaterals?

In the true nature of the APOS Theory Framework, the findings of this study are just part of a cycle to meet the continual need for improvement of student understanding of special quadrilaterals. The newly developed genetic decomposition for special quadrilaterals should be explored with student instructional activities and further collection of data for analysis. Also, a study with the implementation of the pedagogical recommendations and their impact to student understanding of special quadrilaterals would be appropriate. In addition, this study did not incorporate the use of dynamic software and so future research regarding the impact of technology on student learning is suggested.
REFERENCES


CAEP Standards for Educator Preparation Aligned with the NCATE Standards. (2013).
http://www.corestandards.org/


APPENDICES

Appendix A Sample Consent Form

Title: Understanding of Geometric Definitions of Preservice Teachers

Principal Investigator: Draga Vidakovic
Jeff McCammon

Sponsor: Not funded

I. Purpose:

The purpose of the study is to investigate about the understanding of mathematical definitions of preservice teachers. You are invited to participate because you are taking one of the required mathematics courses (MATH 3050/7050) this semester. A total of 40 participants will be invited to participate in the study.

II. Procedures:

There will be two sets/ways of data collection: (1) homework assignments and other individual work from all students who agree to participate and (2) individual interviews with students who volunteer to participate in the interviews. Regarding the first set of data collection, students enrolled in MATH 3050/7050 will be sent an email giving students the option to consent or not for their homework and other individual work to be used in this study. In the case of consent, in order to protect students’ identities, all real names will be replaced by pseudo-names.

The second set of data collection involves participation in an interview. If you decide to participate, you will be interviewed once. The interview will focus on a problem-solving
situation. That is, we would ask you to solve in writing and then explain your solution verbally several problems that you have seen in MATH 3050/7050. The interviews will be audio taped and videotaped and will require up to 60 minutes of your time. The Principal Investigators listed above will conduct the interviews.

After the interview, participants will be asked to complete a Student Evaluation of Instructor Form. The purpose of this form is to insure that students are treated fairly and without coercion when asked to participate in research projects in the classroom. These evaluations will not be seen by your instructor until after the semester is completed and all final grades submitted. The results will be presented in a summarized group format.

III. Risks:

In this study, you will not have any more risks than you would in a normal day of life. However, if you become uncomfortable in responding to questions, you may choose not to answer the questions or stop at any time. Students will not have any penalty if they do not participate in this study. Non-participation in the study will not make the teacher upset nor will participants have an advantage in grade over non-participants. An alternative assignment will be available for non-participants that will require similar time, effort, and grade value.

IV. Benefits:
Participation in this study may not benefit you personally. Overall, we hope to gain information about students' understanding of mathematical definitions in a Geometric context. This knowledge will enable us to possibly modify our curriculum and include more emphases on mathematical definitions in future mathematics courses that are required in the teacher preparation program.

V. Voluntary Participation and Withdrawal:

Participation in research is voluntary. You do not have to be in this study. If you decide to be in the study and change your mind, you have the right to drop out at any time. You may skip questions or stop participating at any time. Whatever you decide, you will not lose any benefits to which you are otherwise entitled.

VI. Confidentiality:

We will keep your records private to the extent allowed by law. Draga Vidakovic and Jeff McCammon will have access to the information you provide. Information may also be shared with those who make sure the study is done correctly (GSU Institutional Review Board, the Office for Human Research Protection (OHRP)). We will use a pseudo-name rather than your name on study records. The information you provide will be stored on a password and firewall protected computer. The key linking students with the pseudo-name will be kept in a locked cabinet. Your name and other facts that might point to you will not appear when we present this study or publish its results. The findings will be summarized and reported in group form. You will not be identified personally. No information regarding student grades of
assignments will be released. The key and interview recordings will be destroyed 2 years after the study is concluded.

VII. Contact Persons:

Contact Draga Vidakovic at 404-413-6451 or dvidakovic@gsu.edu if you have questions about this study. You can also call if think you have been harmed by the study. Call Susan Vogtner in the Georgia State University Office of Research Integrity at 404-413-3513 or svogtner1@gsu.edu if you want to talk to someone who is not part of the study team. You can talk about questions, concerns, offer input, obtain information, or suggestions about the study. You can also call Susan Vogtner if you have questions or concerns about your rights in this study.

VIII. Copy of Consent Form to Subject:

We will give you a copy of this consent form to keep.

If you are willing to volunteer for this research and be audio and video recorded please sign below.

____________________________________________  __________________
Participant                                      Date

____________________________________________  __________________
Principal Investigator                           Date
Appendix B Questions used during interview

Name: _______________________________ Date: ___________________ Time: __________

Question: What prior experiences have you had with Geometry before taking MATH 3050?

Question: In Part 1, were there any terms that you have not experienced in a prior mathematical course before MATH 3050?

Prompt: (If student did not write out definition in words) Did you draw a picture? Can you describe in words how your picture represents the term?

Part 1

Please answer the following questions:

1. In your own words define the following terms. If there are any special cases, please list.
   a) Parallelogram:
      
      A quadrilateral whose opposite sides are parallel.
      
      Special cases of Parallelograms: Rectangles, Rhombuses, Squares
      
      Prompt: (If there is difficulty for the term “special cases”) Are there any terms below that would could be classified as a parallelogram?
   
   b) Rectangle
      
      A parallelogram with 4 right angles (or with one right angle). OR
      
      A quadrilateral with 4 congruent angles.
      
      Special cases of Rectangles: Squares
c) Rhombus

A parallelogram 4 congruent sides. (Or a quadrilateral with 4 congruent sides).

Special cases of Rhombuses: Squares

d) Kite

A quadrilateral with two separate pairs of adjacent congruent sides

Special cases of Kites: Rhombuses, Squares

e) Square

A rectangle with congruent sides OR A rhombus with congruent angles

OR A quadrilateral with congruent sides and angles

Special cases of Squares: None

Prompt: (If student lists terms here) Are you saying that all __________ are also squares?

f) Trapezoid (Inclusive)

A quadrilateral with at least one pair of parallel sides.

Special cases of Trapezoid (inclusive): Parallelogram, rectangle, rhombus, square

g) Trapezoid (Exclusive)

A quadrilateral with only one pair of parallel sides

Special cases of Trapezoid (Exclusive): None
Prompt: (If student has difficulty between inclusive and exclusive) What is your general understanding of a trapezoid?

2. True or False. If true, explain why. If false, give a counterexample (you may draw a figure to help your explanation).

A) All rectangles are parallelograms
   True

   Prompt: Would a rectangle be a special case for a parallelogram?

B) All rhombuses are parallelograms
   True

   Prompt: Would a rhombus be a special case for a parallelogram?

C) All rhombuses are kites
   True

   Prompt: Would a rhombus be a special case for a kite?

D) All kites are parallelograms
   False

   Prompt: Would a kite be a special case for a parallelogram?

E) All parallelograms are trapezoids (inclusive)
   True

   Prompt: Would a parallelogram fall under the category of the inclusive definition of a trapezoid?

F) All parallelograms are trapezoids (exclusive)
False

Prompt: Would a parallelogram fall under the category of the exclusive definition of a trapezoid?

Prompt: (If the answer is true for both E and F), how does the difference in the definition of a trapezoid make any impact to what other objects (such as parallelograms) could be classified as trapezoids, too?

G) All trapezoids are kites
False

Prompt: Would a trapezoid be a special case of a kite?

H) All rectangles are kites
False

Prompt: Would a rectangle be a special case of a kite?

I) All squares are kites
True

Prompt: Would a square be a special case of a kite?

J) All rectangles are isosceles trapezoids (inclusive)
True

Prompt: Would a rectangle fall under the category of the inclusive definition of isosceles trapezoids?
Part 2

3. For each possible definition below, determine if it is an equivalent definition of a

Parallelogram. If NO, then give a counterexample:

Prompt: Can you draw a picture of any object that would work with the definition given but is not the object currently being defined?

A) Possible Definition A: A quadrilateral is a parallelogram if at least one diagonal cuts the figure into two congruent halves.

No, counterexample: Kite

Prompt: What object is constructed if only one diagonal cuts the figure into two congruent halves? ...if both diagonals cut the figure into two congruent halves?

B) Possible Definition B: A quadrilateral is a parallelogram if it has two pairs of equal sides.

No; Counterexample: Kite

Prompt: Do all parallelograms have two pairs of equal sides? Does any other object have two pairs to equals sides that is not a parallelogram?

4. For each possible definition below, determine if it is an equivalent definition of a

Rhombus. If NO, then give a counterexample:
A) Possible Definition A: A quadrilateral is a rhombus if the diagonals are perpendicular bisectors of each other.

Yes

*Prompt:* What does perpendicular mean? What does a bisector do? Is there any other object that has its diagonals as perpendicular bisectors of each other?

B) Possible Definition B: A quadrilateral is a rhombus if the diagonals are perpendicular to each other.

No, Counterexample: kite (there are other figures that can be drawn with perpendicular diagonals that are not rhombuses)

*Prompt:* How is Definition B differ from Definition A? Does being a bisector make a difference on what objects are created when the diagonals are perpendicular to each other?

5. For each possible definition below, determine if it is an equivalent definition of a **Kite**. If NO, then give a counterexample:

A) Possible Definition A: A kite is a quadrilateral that has at least one pair of congruent opposite angles.

No, Counterexample: Parallelogram

*Prompt:* What objects have only one pair of congruent opposite angles? What objects have two pairs of congruent opposite angles? Do all of these objects fall under the category of kites?

B) Possible Definition B: A kite is a quadrilateral that has perpendicular diagonals.
No, student would have to draw a picture. Incorrect answers: rectangle, rhombus, square

Prompt: Are there any figures other than kites than have perpendicular diagonals?

***Note…probably hardest question since there is not a proper name for the picture drawn.

C) Possible Definition C: A kite is a quadrilateral in which at least one diagonal is a perpendicular bisector of the other.

Yes

Prompt: What objects have only one diagonal that is a perpendicular bisector of the other? What objects have both diagonals that are perpendicular bisectors of each other? Do all of these objects fall in the category of kites?