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ACCEPTANCE

This dissertation, THE INTERPLAY AMONG PROSPECTIVE SECONDARY MATHEMATICS TEACHERS'AFFECT, METACOGNITION, AND MATHEMATICAL COGNITION IN A PROBLEM-SOLVING CONTEXT, by BELINDA PICKETT EDWARDS, was prepared under the direction of the candidate's Dissertation Advisory Committee. It is accepted by the committee members in partial fulfillment of the requirements for the degree Doctor of Philosophy in the College of Education, Georgia State University.

The Dissertation Advisory Committee and the student's Department Chair, as representatives of the faculty, certify that this dissertation has met all standards of excellence and scholarship as determined by the faculty. The Dean of the College of Education concurs.

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ABSTRACT

THE INTERPLAY AMONG PROSPECTIVE SECONDARY MATHEMATICS TEACHERS'AFFECT, METACOGNITION, AND MATHEMATICAL COGNITION IN A PROBLEM-SOLVING CONTEXT
by
Belinda P. Edwards

The purpose of this grounded theory study was to explore the interplay of prospective secondary mathematics teachers' affect, metacognition, and mathematical cognition in a problem-solving context. From a social constructivist epistemological paradigm and using a constructivist grounded theory approach, the main research question guiding the study was: What is the characterization of the interplay among prospective teachers' mathematical beliefs, mathematical behavior, and mathematical knowledge in the context of solving mathematics problems? I conducted four interviews with four prospective secondary mathematics teachers enrolled in an undergraduate mathematics course. Participant artifacts, observations, and researcher reflections were regularly recorded and included as part of the data collection.

The theory that emerged from the study is grounded in the participants' mathematics problem-solving experiences and it depicts the interplay among affect, metacognition, and mathematical cognition as meta-affect, persistence and autonomy, and meta-strategic knowledge. For the participants, "Knowing How and Knowing Why" mathematics procedures work and having the ability to justify their reasoning and problem solutions represented mathematics knowledge and understanding that could

empower them to become productive problem-solvers and effective secondary mathematics teachers. The results of the study also indicated that the participants interpreted their experiences with difficult, challenging problem-solving situations as opportunities to learn and understand mathematics deeply. Although they experienced fear, frustration, and disappointment in difficult problem-solving and mathematics-learning situations, they viewed such difficulty with the expectation that feelings of satisfaction, joy, pride, and confidence would occur because of their mathematical understanding. In problem-solving situations, affect, metacognition, and mathematics cognition interacted in a way that resulted in mathematics understanding that was productive and empowering for these prospective teachers. The theory resulting from this study has implications for prospective teachers, teacher education, curriculum development, and mathematics education research.

THE INTERPLAY AMONG PROSPECTIVE SECONDARY MATHEMATICS TEACHERS'AFFECT, METACOGNITION, AND MATHEMATICAL COGNITION IN A PROBLEM-SOLVING CONTEXT

by
Belinda P. Edwards

A Dissertation

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ACKNOWLEDGMENTS

As a little girl growing up on a farm in Boston Georgia, I spent a lot of time dreaming about becoming anything other than a peanut or corn farmer. Neither of my parents graduated from high school, but they both valued education. Although there was always work to do on the farm, my parents always emphasized the importance of studying and earning good grades in school. Schoolwork always took precedence over farm work. My siblings and I were expected to work hard and earn good grades. The fulfillment of my dream to earn a doctorate degree can be attributed to the work ethic they passed on to me. My father had a saying, “work while you rest”. He taught me about hard work—stay active, not idle. Whenever I did not think I could make it, Mother would remind me that with the Lord’s help all things are possible. I have consistently maintained that belief. I wish to thank my mother, Ollie Mae, and my father, Curtis Pickett, may he rest in peace, for everything they taught me about what it means to work hard and put my trust in God.

When I married my husband, more than twenty-two years ago, I delayed my goals and dreams and instead traveled with him on his journey to reach his naval career goals. During those years of traveling, I learned a lot about myself. He taught me how to stay strong and overcome the most difficult of circumstances. Over our 22-year marriage, he played a major role in helping me realize my dream and along the way, two beautiful children came into my life.

Karen and Nolan, you are two of the most wonderful people I know. I am so glad that you are my children. You are beautiful, loving, and have been very understanding over the past six years. Thank you for hanging in there with mommy and supporting me along the way. Your hugs and kisses always made me realize how lucky I am to have you in my life. On our next trip to Disney World, I promise I will bring no books and I will ride Space Mountain with my eyes open.

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CHAPTER 1

INTRODUCTION

This study focused on the mathematical education of prospective secondary mathematics teachers and their problem-solving experiences. The National Council of Teachers of Mathematics (NCTM, 2000) *Problem Solving Standard* has recommended that "by the end of grade 12 students should be able to: build new mathematical knowledge through problem solving, solve problems that arise in mathematics and in other contexts, apply and adapt a variety of appropriate strategies to solve problems, and monitor and reflect on the process of mathematical problem-solving" (p. 52). Likewise, the Georgia Performance Standards (GPS, 2005) has emphasized problem-solving throughout the curriculum and encouraged teachers to provide opportunities for students to learn mathematics through the perspectives and methods of problem-solving. Prospective secondary mathematics teachers need to have opportunities to develop substantial deep mathematics understanding for teaching in a problem-solving context to implement the curriculum envisioned by the NCTM and GPS (Ball, Bass, & Hill, 2005; CBMS, 2001; Even, 1993; Ma, 2004; Usiskin, 2001).

Problem-solving is an important part of teaching and learning mathematics (NCTM, 2000). However, prospective secondary mathematics teachers often have limited opportunities (within a problem-solving environment) to connect their advanced college-level mathematics with the mathematics they will teach (Usiskin, 2001). Burkhardt (1995) found that teacher education programs expect prospective secondary mathematics

teachers to teach mathematics in a way unlike the way their teachers' taught (Burkhardt, 1988; Schoenfeld, 1992; Thompson, 1989). Many prospective secondary mathematics teachers learned mathematics in a rule-based classroom and thus have had no experience learning mathematics in a true problem-solving environment (Fennema & Frank, 1992). Schoenfeld (1992) suggested that it is up to the teacher to guide students through the problem-solving process and prospective secondary mathematics teachers must themselves have the knowledge and disposition of effective problem solvers to support students in a problem-solving environment (NCTM, 2000).

Helping prospective secondary mathematics teachers develop and learn ways in which they can improve their problem-solving competence, deepen their knowledge and understanding of mathematics, and enhance their mathematical thinking is a goal of many mathematicians and mathematics educators (Ball, et al, 2005; Ma, 1999). Based on my experience teaching mathematics and mathematics methods in a problem-solving context, I understand the difficulties associated with achieving this goal. Prospective teachers come to teacher preparation with beliefs about the nature of mathematics, mathematics learning and teaching, and attitudes toward problem-solving that can interfere with their cognitive and metacognitive behavior (Emenaker, 1988; Thompson, 1992). When helping prospective teachers and practicing teachers improve their problem-solving competence and enhance their mathematics thinking skills, Thompson (1992) found that prospective and practicing teachers often encounter a number of hindrances such as beliefs, values, and attitudes toward problem-solving. Other researchers (e.g., DeBellis & Goldin, 1997; Lester, 1994; McLeod, 1992; Schoenfeld, 1989) have substantiated that affective variables such as beliefs, attitudes, and emotions have a powerful influence on cognitive

behavior. Schoenfeld (1992) has suggested that purely cognitive behavior is very rare, and that most learners approach and carry out mathematical tasks based on how they view those tasks.

Few studies in mathematics education focus on the intersection among prospective secondary mathematics teachers, affective behavior, and problem-solving (McLeod, 1992; Phillipp, 2007; Schoenfeld, 1992). Traditionally, mathematics education research has focused on cognitive aspects of mathematics learning and understanding (Malmivuori, 2001; McLeod, 1992). Because of the attention given to studies on cognition, there have been significant gains and progress in the field of cognitive science (Malmivuori, 2001). Only a few studies give attention to understanding the interrelationship between affect and cognitive processes during mathematics learning and problem-solving (Carlson & Bloom, 2005; McLeod, 1992; Schoenfeld, 1992). Those studies that address cognitive and affective responses during mathematics learning and problem-solving do so at the prospective elementary teacher or K-12 level (Phillipp, 2007). Schoenfeld (1992) suggested that the affective and cognitive domain interrelationship is under-conceptualized. He explained, “We are a long way from a unified perspective that allows for the meaningful integration of cognition and affect” (p. 364).

To establish the background and rationale for the study, I begin with a quote from the NCTM (1991) *Professional Standards for Teaching Mathematics* document and later share some of the challenges associated with preparing prospective secondary mathematics teachers. I continue with a discussion of my experience as a mathematics educator who teaches prospective secondary mathematics teachers in a mathematics

methods course at a southern university. I discuss some of the elements known to influence success in problem-solving such as affective behavior along with cognitive and metacognitive behavior. I state the research questions, along with the significance of the study, followed by operational definitions and I explain the framework guiding the study.

Background

Mathematics Education Research and Reform

> Effective teachers of problem-solving must themselves have the knowledge and dispositions of effective problem-solvers (NCTM, 2000, p. 341).

The above National Council of Teachers of Mathematics (NCTM, 2000) statement makes a strong case for a rethinking of the mathematical education of prospective secondary mathematics teachers. The statement addresses the need for them to have opportunities during teacher preparation to develop deep knowledge and understanding of school mathematics concepts and a productive disposition in a problem-solving context. The NCTM has identified problem-solving as the most important topic in the mathematics curriculum and it is central to learning and understanding mathematics deeply. It is especially important that prospective and practicing secondary mathematics teachers understand that mathematics presented in a problem-solving context gives meaning to the mathematics at hand and is a motivation tool in the classroom (Sharp & Adams, 2002). Part of the problem is that prospective mathematics teachers, like other mathematics learners, encounter hindrances during the problem-solving process (Borko & Putnam, 1996; Ball & Wilson, 1990; DeBellis & Goldin, 1997). Beliefs, emotions, planning, monitoring, and attitudes toward problem-solving are some of those hindrances (Carlson & Bloom, 2005; McLeod, 1992; Schoenfeld, 1992; Thompson, 1992).

As a mathematics teacher educator teaching in the mathematics department of a southern suburban university, my overall goal in my work is to improve mathematics teaching. I believe one of the best ways to raise student achievement is to ensure a quality teacher in each classroom. I do this by supporting prospective teachers in developing an understanding of mathematics that is deep, well-connected, and conceptually grounded. For the past six years, I have taught mathematics and mathematics education courses through the perspective and methods of problem-solving. In these courses, I often encounter students whose beliefs, emotions, and attitudes about mathematics range anywhere from feelings of discomfort and panic to feelings of satisfaction, passion, and pride as they engage in the mathematics learning and problem-solving process.

When teaching mathematics and engaging students in mathematical tasks, I have noticed that students often demonstrate negative affective behavior when they find their problem-solving efforts or mathematics understanding unproductive. In my six years of teaching mathematics and mathematics methods, I have noticed that students' attitudes and emotional behavior seems to be a determining factor in (a) how they approach mathematics problems, (b) how much time they spent solving a problem, and (c) whether or not they ask for assistance when they lack mathematical understanding. I am unsure about the extent to which my students' emotions, beliefs, and attitudes have an effect on their mathematics learning and problem-solving competence, but I believe that their negative affective behavior can be a negative force in their mathematics understanding and problem-solving efforts. Research has substantiated that affective variables have a powerful influence on problem-solvers' behavior (McLeod, 1992; Schoenfeld, 1992). My

goal in this study was to gain a better understanding of the interaction of affective, cognitive, and metacognitive behavior during mathematics learning and problem-solving.

Statement of the Problem

Researchers (Ball & McDiarmid, 1989; Ball & Wilson, 1990; Carlson & Bloom, 2005; Garofalo & Lester, 1985; Kloosterman, 2002; McLeod, 1989; Schoenfeld, 1992; Shaughnessy, 1985) have linked affective behavior and metacognitive behavior to success or failure in mathematics learning, understanding, and problem-solving. These researchers have suggested that successful cognitive performance depends on having not only adequate mathematical knowledge but also an awareness and control over that knowledge. They also point to negative beliefs and attitudes about mathematics as a limiting factor in a learner's problem-solving performance.

Beliefs influence one's view of mathematics, constrain one's choice of strategies used to solve mathematics problems, and even restrict the type of problems one perceives as mathematics (McLeod, 1992). In my experience teaching mathematics I have noticed that negative attitudes toward mathematics and problem-solving can act as a negative force in one's problem-solving efforts. Some elementary education students I teach, who demonstrate negative mathematics attitudes about mathematics in general, are more concerned about obtaining getting a good grade than they are about understanding the mathematics deeply. Secondary mathematics education students enrolled in my methods class have positive attitudes about mathematics, but some of them hold beliefs about conceptual mathematics teaching and learning that sometimes influences how they approach and solve mathematics problems.

Emenaker (1996) found that teachers' beliefs and attitudes have a strong influence on their approach to teaching mathematics and on their students' belief systems. Thompson (1992) suggested that teachers' views of mathematics play a significant role in shaping their instructional practice. Hirsch (1986) found that "one's conception of what mathematics is affects one's conception of how it should be presented" (p. 13). It seems reasonable to conclude, drawing from the research on affect, prospective teachers' beliefs and other affective factors could possibly hinder their cognitive problem-solving processes and those of their students. When considered as a whole, these findings suggest that there needs to be more research examining the integration of affective behavior, cognitive, and metacognitive behavior relative to problem-solving.

Rationale for the Study

Traditionally, mathematics education research has focused on the cognitive aspects of mathematics learning and understanding (Malmivuori, 2001; McLeod, 1992). Because of the attention given to studies on cognition, there have been significant gains and progress in the field of cognitive science (Malmivuori, 2001). Other than beliefs, few studies give attention to understanding the role of affective factors and cognitive processes during mathematical learning and problem-solving (Carlson & Bloom, 2005; McLeod, 1992; Schoenfeld, 1992). Even fewer studies in mathematics education integrate prospective secondary mathematics teachers' cognition and affect. Most studies addressing the integration of cognition and affect during mathematics learning and problem-solving do so at the K-12 level or the elementary prospective teacher level (Phillip, 2007). Currently, an understanding of the meaningful integration of cognition and affect remains under-conceptualized (Schoenfeld, 1992).

Although limited, recent work on beliefs points to issues of importance that integrate cognition and affect. McLeod (1992) found that "the role of beliefs is central to the development of attitudinal and emotional responses to mathematics" (p. 579). He also found that mathematics-related beliefs that practicing teachers and prospective teachers hold about the nature of mathematics, what it means to do mathematics and their attitudes towards problem-solving can interfere with their ability to learn and understand mathematics deeply. Beliefs can also interfere with a teacher's ability to help his/her students become successful in problem-solving, and in learning and understanding mathematics (McLeod, 1989; 1992). Researchers (Emenaker, 1996; Karp, 1991; McLeod, 1992; Schoenfeld, 1981) have suggested that negative mathematical attitudes do nothing to encourage learners to engage in independent mathematical thinking; whereas, positive attitudes encourage learners to aggressively explore and discover mathematical reasoning and interrelationships in order to gain a deeper understanding of mathematics.

Mathematics education research is incomplete when its focus is only on cognitive aspects of mathematics learning and problem-solving, without considering affective factors, making it difficult for others within or outside our community to relate our research findings to real situations that occur in the classroom (Malmivuori, 2001; McLeod, 1985). McLeod (1992) suggested that when researchers integrate affective factors into studies that address cognitive issues, it strengthens all mathematics education research. Oatley and Nundy (1996) explained, "Neglecting the influence of the emotional realm would distort an understanding of the cognitive process of education in general" (p. 258). Considering the lack of research on the intersection of affect and cognition at the secondary mathematics prospective teacher level along with the possibility that poor

mathematics related beliefs can lead to poor attitudes toward mathematics, mathematics learning, and problem-solving; there is a need to examine how secondary prospective mathematics teachers' affect and cognition interrelate during the problem-solving process.

Prospective teachers' affective dimensions can play a critical role in the formation of their mathematics knowledge, beliefs, and attitudes as well as those of their potential students (Thompson, 1992). Mathematics education researchers can no longer overlook or ignore the influence of affective factors on cognitive and metacognitive processes if, as reported in the literature, prospective teachers' affective dimensions hinder their cognitive mathematical problem-solving process. An increased understanding of the role of affective factors and metacognition in mathematics learning, understanding, and problem-solving will enable mathematics educators to understand how they can advance the learning experiences of prospective secondary mathematics teachers. Mathematics educators and mathematicians can begin to support prospective secondary mathematics teachers in understanding mathematics deeply as they engage in mathematical problem-solving if mathematicians and mathematics educators have a better understanding of the interplay among affective, metacognitive, and cognitive behavior. Teacher education programs and curriculum development can receive new directions and improve based on the insights gained from this study.

Research Questions

This study seeks to answer the main question: What is the characterization of the interplay among prospective teachers' mathematical beliefs, mathematical behavior, and

mathematical knowledge in the context of solving mathematics problems? In answering this main question, I also answer the following questions:

(a) What are the mathematics-related beliefs of prospective secondary mathematics teachers?

(b) What mathematical behaviors do prospective secondary mathematics teachers demonstrate as they engage in mathematical problem-solving?

(c) What mathematics knowledge do prospective secondary mathematics teachers use as they engage in mathematical problem-solving?

Significance of the Study

This study provides knowledge about the mathematical problem-solving process used by prospective secondary mathematics teachers when solving non-routine problems. More specifically, it provides knowledge about the intersection of prospective teachers' mathematics-related beliefs, affective behavior, metacognition, and mathematical cognition during the problem-solving process. The findings in this study will help to extend the current research on mathematical problem-solving processes. Characterizing the interplay among prospective teachers' problem-solving experiences as they engage in the mathematics problem-solving process will help mathematicians and mathematics teacher educators make the necessary changes in curriculum, instruction, and expectations that can better support prospective secondary mathematics teachers' development of deep mathematics knowledge and understanding in a problem-solving environment. Prospective secondary mathematics teachers will gain a better understanding of how their mathematics-related beliefs, attitudes toward mathematics,

how they view mathematics can affect their mathematics learning and instructional practices.

The results of the study may apply to the development of teaching methods and curriculum. The results can improve mathematics and mathematics methods courses and facilitate prospective mathematics teachers in attending to their own affect, while developing deep mathematics knowledge and understanding, and enhancing problem-solving competence. To gain a better understanding of prospective secondary mathematics teachers' knowledge, beliefs, and affect during mathematics problem-solving, the conclusions and recommendations of this study suggest directions for further research.

Definitions

Problem-solving refers to a cognitive process in which the student determines how to solve a problem that he or she does not readily know how to solve (Mayer, 1992).

Problem-solving processes refer to actions and strategies that students employ to solve problems.

Non-routine problems/Mathematical problems are problems that the solver perceives as challenging and unfamiliar, yet not insurmountable (Becker & Shimada, 1997). They demand thinking flexibility and extension of previous knowledge and may involve discovery of connections among mathematical ideas (Schoenfeld, et al., 1999). A mathematical problem is also a task (a) in which the student is interested and engaged and for which he/she wishes to obtain the resolution, and (b) for which the student does not have a readily accessible mathematical means by which to achieve that resolution. (p. 71).

Prospective secondary mathematics teacher is a college student whose goal is to teach mathematics at the middle-grades 4 – 8 or secondary grades 6-12 level. They have acceptance into a National Council for the Accreditation of Teacher Education (NCATE) approved teacher education program.

Mathematics teacher or Practicing teacher is a teacher who is currently teaching secondary school mathematics in a public school setting.

Conceptual Framework

Historically, research in mathematics education and problem-solving has placed a lot of emphasis on cognitive and metacognitive aspects involved in the process of solving mathematics problems (Lester, 1980; Malmivouri, 2001; Pehkonen & Zimmerman, 1990; Schoenfeld, 1992; Silver, 1985). Research on mathematical knowledge and understanding provide several theoretical frameworks that explain either what it means to understand a concept or how an individual makes meaning of mathematics Hiebert & Carpenter, 1992; Hiebert & Lefevre, 1986; Schoenfeld, 1992; Skemp, 1976). However, there has been much less research on the role affect plays in problem-solving and mathematics learning and understanding (Malmivouri, 2001; McLeod, 1992; Schoenfeld, 1992). The lack of theoretical models, accurate definitions, and detailed constructions in consideration of affective characteristics in mathematics education provide incomplete research results on the role of affect in mathematics learning and understanding (Malmivouri, 2001; McLeod, 1988; Schoenfeld, 1992). The framework used to guide this study utilized the theoretical frameworks of several researchers in the field of affective behavior, metacognition, mathematics learning and understanding, and problem-solving.

Mathematical Cognition

In this study, mathematical cognition refers to mathematical thinking, knowledge, and understanding. In the literature, two major domains analyze the nature of mathematical knowledge and understanding: *conceptual knowledge* and *procedural knowledge*. Hiebert and LeFevre (1986) characterize conceptual knowledge as that which is “rich in relationships and thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information” (pp. 3-4). Conceptual knowledge enables one to build relationships between existing pieces of mathematical knowledge and new pieces of mathematical knowledge. Procedural knowledge, on the other hand, consist of two components, one part being “the formal language or symbol representation system, of mathematics” while the other refers to “rules, algorithms, or procedures used to solve mathematical tasks” (Hiebert & LeFevre, 1986, p. 6). Hiebert and LeFevre further emphasized that, while learning without meaning represents procedural knowledge, conceptual knowledge is learning with meaning. Conceptual knowledge allows one to transfer and adapt mathematical procedures to new situations by appropriately relating the mathematical concept to the symbols used to denote mathematics.

In his framework, Skemp (1976) distinguished between two types of mathematical understanding. Similar to Hiebert and LeFevre’s conceptualization of procedural knowledge, Skemp referred to *instrumental understanding* as a type of understanding that focuses primarily on “rules without reason” (p. 9). On the other hand, he described *relational understanding* as “knowing both what to do and why” (p. 9)

which is similar in nature to what Hiebert and LeFevre referred to as conceptual knowledge.

To explore the notion of mathematical knowledge and understanding further, I considered Hiebert & Carpenter's (1992) conceptualization of understanding. Hiebert and Carpenter defined mathematics understanding based on the way an individual structures and represents information. They explained that,

> "A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and strength of connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections" (p. 67).

They found that understanding increased when individuals were able to talk about how they solved a problem or why they proposed specific strategies or approaches as well as connect new knowledge with existing knowledge or existing knowledge is modified, updated, or assimilated with new knowledge. I applied the frameworks of Hiebert and LeFevre, Skemp, and Hiebert and Carpenter when exploring prospective teachers' mathematical thinking, knowledge, and understanding when engaged in mathematics learning and problem-solving.

Affective Dimensions

This study also focused on the affective dimensions of prospective teachers' relationships with mathematics learning, understanding, and problem-solving. Although there is very little literature on the intersection between teachers and affect, several researchers (Mandler, 1989; McLeod, 1992; Hannula, 2002a) have developed frameworks that examine and evaluate student affect. I adopt the frameworks of these researchers to guide this study. Although these frameworks primarily focus on the affect

of K-12 students, I contend that the frameworks will help me in understanding the affect of prospective mathematics teachers engaged in mathematics learning and problem-solving.

McLeod (1989; 1992) developed a framework for studying the affective domain in mathematics education research and for understanding the influences of affect on problem-solving. His framework extends the work of Mandler (1989), a cognitive theorist, who emphasized the role of interruptions in a learner’s planned behavior. Mandler found that when a learner’s behavior is interrupted, the normal pattern of completion could not occur. As a result, the learner experiences a physical arousal such as frustration, anger, disappointment or some other emotion. McLeod (1990) identified three concepts used in the research on affect that can influence mathematics problem-solving performance. They are beliefs, attitudes, and emotions that differ from each other in stability, intensity, and in development.

According to Mandler (1989), researchers must take care in defining and using the term *affect*. In my study of affect, I adopt the view of McLeod (1992) who claimed, “The affective domain refers to a wide range of beliefs, feelings, and moods that are generally regarded as going beyond the domain of cognition” (p. 592). According to McLeod, emotions, attitudes, and beliefs are the three terms that make up the affective domain. I include mathematics-related beliefs as a component of affect because many researchers who have studied affect usually include beliefs as a component of affect (Phillipp, 2007). McLeod (1992) defined emotions as positive and negative feelings that change rapidly during mathematics activities. He described emotions as including feelings such as joy, frustration, pride, satisfaction, disappointment or anger. According

to McLeod (1992), attitudes refer to "affective responses that involve positive and negative feelings of moderate intensity and reasonable stability" (p. 581). Attitudes are cognitive and stable more so than emotions but they are felt less intensely. Beliefs are deep-seated convictions or internal representations that the believer attributes to truth and validity (McLeod, 1988; 1992; Schoenfeld, 1989; 1992). They are also cognitive, stable, and are felt less intensely than emotions. McLeod included four categories of mathematics-related beliefs in his framework: beliefs about mathematics, beliefs about self, beliefs about mathematics teaching, and beliefs about the social context. I utilized McLeod's framework to inform my knowledge on affective behavior.

Hannula (2002a) also extended the work of Mandler by adding the affective influences that are less intensive emotionally, such as learner's reactions during general and specific mathematical thinking. His framework classified the mathematics-related emotions a learner experiences using four evaluative processes. They are (a) expectations of a learner when thinking about doing mathematics, (b) associations a learner makes when thinking about mathematics or asked how they feel about mathematics, (c) emotions exhibited when actually doing mathematics based on mathematics-related goals or expectations, and (d) a cognitive analysis of their progress in achieving their mathematics-related goals. According to Hannula, each is a process that produces an expression of an evaluation or judgment of mathematics.

The frameworks of McLeod and Hannula addressed affective factors experienced by learners while engaged in mathematical problem-solving and understanding. Each framework plays an important role in my investigation of prospective teachers' mathematics-related affect. In this study, I apply McLeod's framework when considering

emotions, attitudes, and beliefs. I also apply the framework of Hannula when considering the associations, expectations, and values a learner holds while engaged in mathematics learning and problem-solving process.

Metacognition

We engage in metacognitive activities every day. Metacognition enables a learner to be successful, and it has been associated with intelligence (Borkowski, Carr, & Pressley, 1987; Sternberg, 1984, 1986a, 1986b). Metacognition, defined as one's knowledge and control of one's cognitive system, is a central component in problem-solving (Brown, 1987; Garofalo & Lester, 1985; Schoenfeld, 1992). Its focus is on one's self-awareness of cognitive knowledge. Furthermore, it guides and regulates cognitive processes and strategies as individuals engage in solving mathematics problems (Tobias & Everson, 2000). It is important to understand metacognitive behavior to determine how learners apply their cognitive resources, because metacognition plays a critical role in successful learning (Brown, 1987; Garofalo & Lester, 1985).

There are a number of theoretical models representing varying viewpoints of metacognition. The theoretical perspective guiding this study extends and combines aspects of the previously established metacognition models of Flavell (1976) and Brown (1987). Both models focus on the metacognitive knowledge and metacognitive experiences of individuals who are engaged in learning. Flavell (1976) proposed that our metacognitive knowledge base consist of what we have learned, through experience, about cognitive activities. He divided metacognitive knowledge into three interactive knowledge variables: task variables, which involve the learner's perceived difficulty of the task; strategy variables, related to the effectiveness of the strategies used; and

personal variables, related to the attitude, motivation, and prior knowledge of the individual. Flavell (1976) suggested that these variables are interrelated and work together to form learning. He also suggested that metacognitive knowledge is critical to successful learning and good learners have meta-cognitive knowledge about themselves as learners, about the nature of the current mathematical task, and about appropriate strategies for reaching their academic goals.

Brown (1987) divided metacognition into two broad components. The first component is related to knowledge of cognition, which involves the reflection of cognitive abilities and activities. This involves the conscious reflection of one's cognitive abilities and the current task. The second component is related to self-regulation, which is often employed during the learning or problem-solving process. According to Brown, the two are closely related. Knowledge about cognition is stable information that individuals have about their own thinking. It requires that learners step back and reflect on their cognitive processes. Regulation of cognition consists of the activities one uses to regulate and keep track of their learning. These processes include planning, which includes choosing a strategy or applying trial and error; monitoring, which includes revising steps or selecting another strategy; and evaluating, which includes checking or reflecting on the solution.

Malmivuori (2001) presented a theoretical analysis that focuses on the interrelationship among affect, beliefs, cognition, metacognition, self-monitoring, self-perceptions, motivation, and the influence of context on these variables. Her theory, based on socio-cognitive and constructivist theories, focuses on the interaction of affect, cognition, and beliefs in specific social environments. I applied parts of her framework

when considering the influence of affective behavior and context on metacognitive knowledge and self-regulation in a problem-solving context. Malmivuori asserted that learners who have high confidence are more decisive and less critical of the decisions they make when engaging in mathematical tasks; whereas, learners who lack mathematical confidence are more likely to hesitate in their decision-making in pursuing their mathematics-related goals.

A sound conceptual or theoretical framework is critical for any study. Garofalo and Lester (1985) recommended that any framework for analyzing mathematical performance should allow for a wide range of possible behaviors, cognitive or otherwise. The applied framework considers this recommendation and includes a range of mathematical beliefs, metacognitive behaviors, and mathematical cognition. The aim of this study is to provide a synthesis, analysis, and rich description of the interplay of prospective teachers' affective behavior, mathematical cognition, and metacognitive behavior as they engage in mathematical problem-solving situations. As such, my goal is to provide a reasonable explanation for the meaningful integration of affective, metacognitive, and mathematical behavior as prospective secondary mathematics teachers engage in problem-solving activities. In order to provide a holistic view of the interactions among these concepts, I used a multi-theoretical approach in the development of the applied conceptual framework (see Figure1).

Summary

A critical aspect to preparing to teach mathematics is the development of a deep conceptually grounded understanding of mathematics (CBMS, 1996; NCTM, 2000). Too often, teacher educators take for granted that teachers' knowledge of the content of

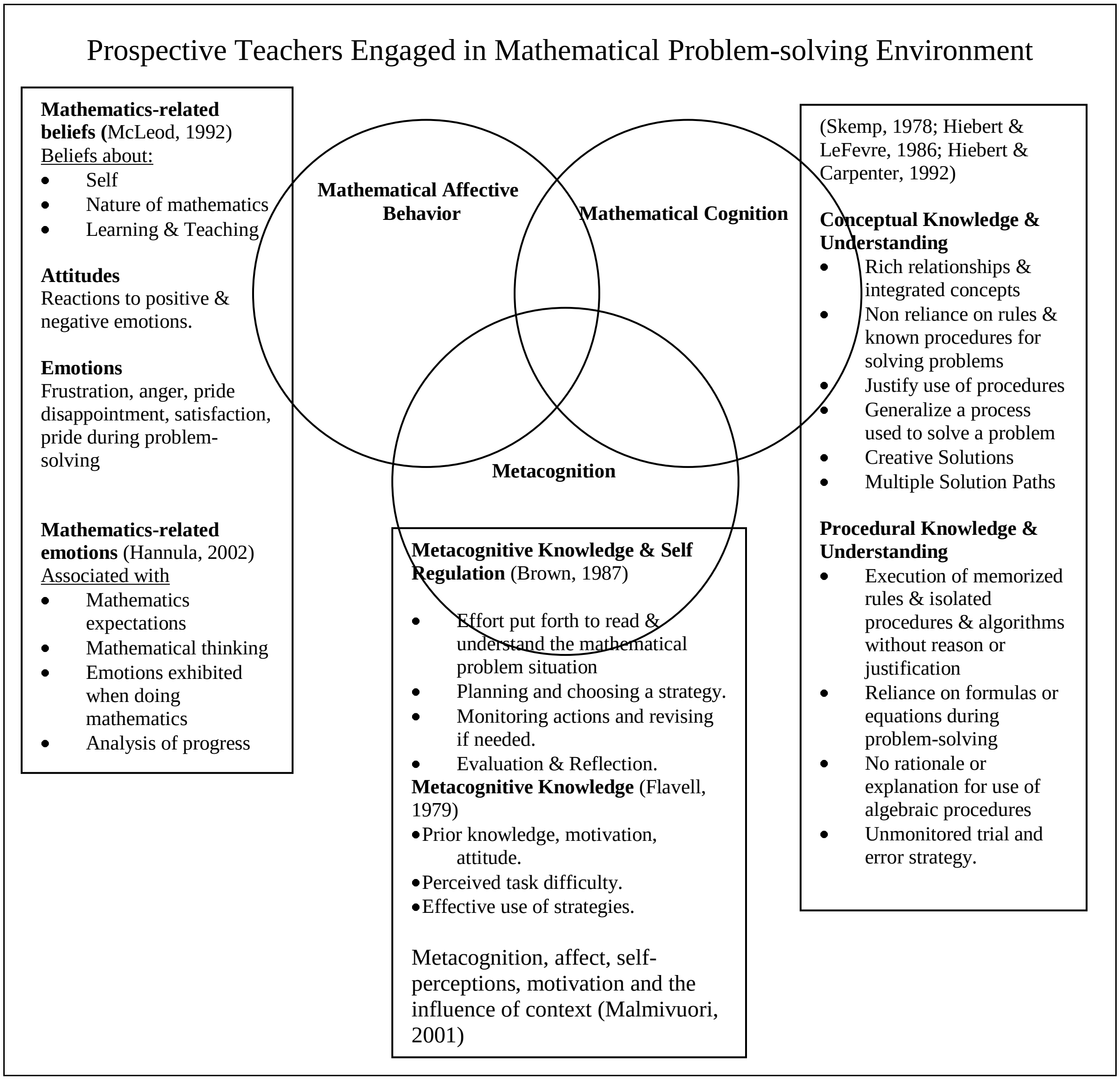
school mathematics is in place by the time they complete their own K-12 learning experiences. Teachers teach mathematics in a way that differs substantially from how they were taught (Schoenfeld, 1992). Researchers (Borko & Putnam, 1996; L. Ma, 1999; Usiskin, 2001; Usiskin, Perissini, Marchisotto, & Stanley, 2003) have recommended that mathematics educators or mathematicians provide prospective teachers with opportunities to revisit school mathematics topics in ways that will allow them to develop deeper understandings of mathematics.

Research has suggested that prospective teachers can develop deep conceptual understanding of mathematics through conjecturing, reasoning, and problem-solving (Francisco & Maher, 2005; Schoenfeld, 1992; Usiskin et al., 2003). However, researchers (DeBellis & Goldin, 1997; Hannula, 2004; Lester, 1994; McLeod, 1992; Schoenfeld, 1992) have also suggested that affective behaviors such as beliefs, emotions, and attitudes are often instrumental in determining how prospective teachers learn, understand, and think about mathematics. Affective behaviors interact with cognition and can either hinder or facilitate the process of learning, understanding, and solving mathematics problems (Carlson & Bloom, 2005).

Belief systems shape cognition and metacognition and they are both instrumental in determining the perspective with which an individual solves mathematics problems, and in turn, influences how an individual learns mathematics (Schoenfeld, 1992). Metacognition is also viewed as being a central component of problem-solving because it focuses on self-awareness of mathematical knowledge, thinking, and understanding. It guides and regulates cognitive processes and strategies during problem-solving (Tobias & Everson, 2000).

Figure 1

Conceptual Framework of Mathematical Affect, Metacognition, and Mathematical Cognition



While there has been an abundance of research in mathematics education emphasizing cognitive aspects of learning and understanding mathematics, there has been less research on affective characteristics in mathematics learning and understanding. Furthermore, research studies have failed to investigate how affective behaviors interact

with cognitive and metacognitive behavior to influence mathematics performance and achievement. This study addresses this gap in the literature by examining the dynamic interplay between and among affective behaviors, mathematical knowledge and understanding, and mathematical metacognition in problem-solving situations.

For conceptual clarity, a framework of affective behaviors combining the works of McLeod (1992) and Hannula (2002a, b) is applied. The framework also includes the combined works of Hiebert and LeFevre (1986), Skemp (1976), and Heibert and Carpenter (1992) to investigate and provide conceptual clarity for mathematical thinking, knowledge, and understanding. Finally, a framework for mathematical metacognitive behavior that combined the works of Flavell (1976), Brown (1987), and Malmivuori (2001) is applied. The research questions, aligned with the goals of the study and the framework, will provide a holistic view of the dynamic interplay among the variables in the study.

In the next chapter, I elaborate on the ideas introduced here. I set the stage for the study with a review of the literature pertaining to affective behavior, mathematics-related beliefs, mathematical knowledge and understanding, and metacognition.

CHAPTER 2

REVIEW OF THE LITERATURE

The literature I review for this study is organized in three main sections: affective behavior, mathematical cognition, and metacognitive behavior associated with problem-solving. I begin with a review of the literature pertaining to the affective domain, specifically emotions, attitudes, and mathematics-related beliefs. The majority of literature reviewed in this area deals primarily with affective behaviors as they relate to performance or competence during mathematics problem-solving situations. Next, I review the literature related to mathematical cognition. In this area, I review what the literature reveals about the nature of mathematical knowledge, mathematical thinking, and mathematics understanding in a learning and problem-solving environment. Finally, I review the literature on the theory of metacognition as it relates to mathematics problem-solving. I end the review of literature with a summary.

Before I begin my discussion of the three main sections of the review of literature, let me briefly explain how problem-solving fits within the study. As discussed in the introduction of the study, conjecturing, reasoning, and problem-solving is seen in the mathematics and mathematics education community as a vehicle for learning mathematics deeply (Lester & Lambdin, 2004; Schoenfeld, 1992; Stein, Boaler & Silver, 2003; Vergnaud, 1982). Problem-solving is a form of inquiry learning where existing knowledge is applied to new or unfamiliar situations in order to gain new knowledge (Killen, 1996; Sternberg, 1995). It is also a vehicle for learners to construct, evaluate, and

refine their own beliefs and theories about mathematics as it relates to the beliefs and theories of others (NCTM, 1989). Engaging in problem-solving involves, not only finding an answer for a particular problem, but also encouraging learners to develop their own ability to think mathematically (Schoenfeld, 1992). The processes involve use of content knowledge, procedures, strategies, language, and reflections (Garofalo & Lester, 1985; Schoenfeld, 1985, 1987).

Drawing from my own experience teaching mathematics and mathematics methods in a problem-solving environment, I have witnessed students' positive and negative affective behavior contributing to and detracting from their problem-solving ability. Some students are able to move beyond their mathematics anxiety, frustration, anger, and disappointment by putting forth an enormous amount of effort to correct their misconceptions and understand the mathematics. In the most difficult and challenging mathematics situations, they buckle down and follow through to the solution process and mathematics understanding. Yet there are other students who, when experiencing mathematics frustration, anxiety, and disappointment, abandon the problem solution process altogether. On the one hand, I have students whose negative affect is cognitively productive and there are other students whose negative affect produces counterproductive cognitive outcomes. I was interested in understanding the interaction among affective, cognitive, and metacognitive behavior as students engage in mathematics-problem solving. In this study, my focus was on gaining new information about the interaction of affect, cognition, and metacognition during problem-solving. My desire to understand this interaction made it appropriate to investigate these phenomena in a problem-solving

context where the phenomena of the study would be most likely revealed (DeBellis & Goldin, 1997; McLeod, 1992; Schoenfeld, 1985).

In this study, I follow Schoenfeld's (1993) definition of what a mathematical problem is. He states:

> For any student, a mathematical problem is a task (a) in which the student is interested and engaged and for which he wishes to obtain the resolution, and (b) for which the student does not have a readily accessible mathematical means by which to achieve that resolution. (p. 71).

I also follow Polya's (1957) conception of problem-solving as learning to grapple with new and unfamiliar mathematics problems when a solution method is not readily available or known to the problem solver. Each section of the literature of review that follows is discussed in the context of mathematics problem-solving.

Schoenfeld (1985) suggests that there are competencies that problem solvers need for becoming successful problem solvers and these competencies should be considered in any analysis of problem-solving. They are *resources*, *heuristics, control*, and *beliefs*. With respect to this study, *resources* refer to the mathematics knowledge processed by an individual during problem-solving and will be reviewed in the mathematical cognition section; *heuristics* or strategies used during problem solving and *control* or monitoring one's solution path will be reviewed with metacognition; and *beliefs* will be reviewed within the literature related to the affective domain.

Affective Dimensions

Research in mathematics education regards cognition and affect as two different fields. Over the past two decades, the majority of research into mathematical thinking focuses primarily on the cognitive aspects of learning mathematics. More recently, researchers (Op t' Eynde 2000; Gomez Chacon, 2000; Sherer, 2000; Malmivuori, 2002;

McLeod, 1989, 1992; Lester, Garofalo & Kroll, 1989; DeBellis & Goldin, 1999) have focused on the affective domain and its interaction with cognition during mathematics learning and problem-solving. McLeod (1989, 1992) describes three components that make up the affective domain: emotions, attitudes, and beliefs.

The majority of research studies that investigate the relationship between affective behavior and learning focus mainly on attitudes or beliefs and less on emotions (McLeod, 1992). Partly because, unlike beliefs and attitudes, emotions arise from immediate situations and are viewed as unstable and fleeting (McLeod, 1992). DeBellis and Goldin (1999) introduced a fourth component known as values, which pertain to an individual's feelings about when or if they should ask for help during a problem-solving situation. For the purpose of this study, I focus on McLeod's (1989, 1992) three affective components of mathematics and use them to inform my study. Consequently, I include an operational definition for each affective component. In the applied framework for the current study, aspects of values are embedded within the subcategories of the three components; therefore, I do not include it as a separate affective component in this study.

Emotions

Although there is no clear definition of *emotion* in the literature, there is agreement among researchers that there are three important components of emotions. There is a subjective component of feelings dealing with personal goals, a physiological or motor component of arousal or expressive gesture, and a functional component, which deals with how we cope and adapt to the mathematical problem-solving situations we find ourselves (Barbalet, 1998; Goldin, 2000; Hannula, 2004; Hannula, Evans, Philippou, & Zan, 2004; Op t' Eynde, DeCorte, & Verschaffel, 2002). Emotions are fleeting, intense

and unstable, negative or positive, and can either facilitate or debilitate an individual's self-esteem or confidence, which can often have a detrimental effect on mathematics learning and performance (DeBellis, 1998; DeBellis & Goldin, 1997; Ma & Kishor, 1997; Malmivouri, 2001; McLeod, 1992; Op t' Eynde et al., 2002; Schoenfeld, 1989). Examples of intensive negative feelings that are related to mathematics problem-solving are fear, anger or even panic when an individual cannot solve a non-routine mathematics problem (Op t' Eynde et al., 2002). A short-term positive emotional reaction might be an "Aha!" moment during problem-solving. On the other hand, when an individual experiences an emotional reaction of satisfaction or joy after solving a challenging mathematics problem it is considered to be a longer-term positive emotion (Malmivouri, 2001).

Emotions are the most visually apparent of the three components of affective behavior. However, they are thought to be the most difficult to analyze and understand primarily because they "may involve little cognitive appraisal and may appear and disappear rather quickly, as when frustration of trying to solve a hard problem is followed by the joy of finding a solution" (McLeod, 1992, p. 579). While McLeod (1989; 1992) views emotions as involving "little cognition" (p. 579), DeBellis and Goldin (1997) disagree and view the level of cognitive activity involved in emotions as being very high, in fact, higher than those of beliefs or attitudes. DeBellis and Goldin (1997) and Goldin (2000) suggest that affective behavior is not auxiliary to cognition, but instead is integrated with cognition. They suggest that curiosity is an emotion that elicits cognition. For example, frustration can evoke anxiety or fear in some; but it can also lead others to cognitive information that suggests the implementation of an effective strategy and

persistence during problem-solving, which could contribute to problem understanding or problem-solving success. DeBellis and Goldin (1997), however, admit that the cognitions interacting with fleeting emotions are often difficult to identify.

Mandler (1989), a cognitive psychologist, is one of the first researchers to examine the role affect played in mathematics problem-solving. He theorizes emotions as developing from the interruption of an individual's planned activity or behavior. According to Mandler (1989), emotions such as joy or frustration are demonstrated when a cognitive interruption of an expected event either occurs or does not occur. If, for example, an individual engages in solving a problem but encounters a block of difficulty and is unable to complete the task or has to apply a new strategy then he or she might exhibit an emotional response such as frustration, disappointment or anger. If the individual is able to proceed successfully through the block of difficulty using a new strategy, then emotions of pride, satisfaction, or joy can occur.

Feelings of frustration or struggle can often lead to an impasse resulting in an ineffective use of strategies, whereas feelings of pride and confidence may serve as a motivating factor leading to the exploration of a variety of strategies during problem-solving (DeBellis & Goldin, 1997). Mandler (1989) suggests that the more alternative strategies one has available immediately following an interruption the more likely the individual will remain engaged and problem-focused, which leads to less anxiety and more problem-solving success. Mandler's finding can play an important role in this study because the mathematical cognition research studies initially reviewed for this study suggests that those who have mathematical knowledge and understanding that is deep,

conceptual, and well-connected are more likely to use alternative strategies to solve mathematics problems.

In a mathematical research study into affective behavior, McLeod, Metzger, and Craviotto (1989) found that experts and novices exhibit similar kinds of emotional reactions as they engage in problem-solving; however, experts are better able to control their emotional reactions and use their knowledge more effectively than novice problem solvers partly because they have more well-connected mathematical knowledge . In a similar empirical study investigating the mathematical behavior of eight research mathematicians and four doctoral mathematics students, Carlson and Bloom (2005) finds that those who had well-connected, conceptual mathematical knowledge were able to cope with their affective behaviors and persist toward finding a problem solution.

Research reveals that emotions are intense and often short-lived but can either facilitate or debilitate a learner's ability to complete mathematical tasks (DeBellis & Goldin, 1997; Goldin, 2000; McLeod, 1992). Although difficult to identify, there is evidence to suggest that emotions can elicit cognition. For example, Goldin (2000) finds that frustration can evoke anxiety in some, but it can lead others to persist in identifying an effective problem-solving strategy that results in successful problem completion. He proposes the construct of *meta-affect* to refer to "the monitoring of affect, and affect itself as monitoring" (p. 62). He explains how a particular experience, such as difficulty solving a mathematics problem, might be interpreted in different ways depending upon the beliefs and values held by the problem solver. A problem solver who encounters problem difficulty might interpret the difficulty as a reflection of their failure; whereas, another problem solver might view the difficulty as a learning opportunity with

anticipation for a feeling of joy or pride in obtaining the correct solution and learning something new.

The studies reviewed in the area of emotions are limited. An individual's emotional response to problem-solving situations depends greatly on their beliefs, values, and interpretation of the situation. Well-connected, conceptual mathematical knowledge and understanding of mathematics can provide an individual with more alternative strategies for solving mathematics problems, which could lead to less anxiety and more problem-solving success. The knowledge gained from my review of the research literature on emotions will inform my investigation of prospective teachers' demonstrated emotions in problem-solving situations.

Attitudes

Emotions are but one aspect of affective behavior that has been found to influence mathematics learning and understanding, and problem-solving. Attitudes, acquired through learning and developed through experience, have also been found to predict mathematical behavior (Morris, 1996). A common theme within the literature suggests that attitudes toward mathematics have three main components: cognitive, emotional and behavioral components. The cognitive component of an attitude consists of thoughts, beliefs, and perceptions relative to mathematics and problem-solving. The emotional component of an attitude involves subjective feelings such as fear, anger, like or dislikes during problem-solving (Avelson, 1979; Hannula, 2002a). The behavioral component determines how an individual expresses their beliefs and subjective feelings about mathematics and problem-solving or the context in which they learn mathematics (Hannula, 2002a).

McLeod (1992) described a person's attitude towards mathematics as "affective responses that involve positive and negative feelings of moderate intensity and reasonable stability" (p. 581). Attitudes related to mathematics include enjoying, liking, and interest in mathematics, or the opposite, and the worst case can be described as mathematics phobia—an overall fear of mathematics (Ernest, 1998). They can be formed from repeated emotional reactions that stabilize into an attitude. For example, if an individual has repeated negative experiences in solving discrete mathematics problems, their reaction to similar tasks can become more automatic. When discrete mathematics problems are encountered, the individual automatically views the experience as negative. Moreover, an attitude can be formed when an already existing attitude is assigned a new and related task. An example of this would include an individual who has an existing attitude toward geometry and who attaches that same attitude to proof or discrete math (McLeod, 1992).

The literature I review for this study includes empirical and theoretical studies investigating attitudes toward mathematics and achievement in mathematics. The majority of studies I review include the use of quantitative methods such as pre- and post-test, surveys, or questionnaires. As a result, much of the literature fails to provide clear empirical findings connecting attitudes and mathematics achievement or problem-solving success (Ma & Kishor, 1997; Zan & DiMartino, 2003). In an action research study, Amato (2004) focuses on the liking dimensions of twenty-four student teachers' attitudes. Based on pre- and post- questionnaires, interviews, diaries, and pre-and post tests, the results of Amato's study indicates that those who dislike mathematics also did not

understand mathematics in school and those who express a liking for mathematics indicate that they understood mathematics well during school.

Stanic and Hart (1995) found attitude towards mathematics and mathematics confidence relates to achievement at the school level, while Maree, Petorius & Eiselen (2003) found similar results to hold at the first year university level. But the correlation in both of these studies is quite small (Op t' Eynde et al., 2002). Bershinsky (1993) conducts a study involving developmental mathematics students at the college level. The study's purpose is to identify attitudinal and achievement variables that were important in predicting student outcomes. Attitudinal variables include feelings about self, school, and mathematics. The findings indicate that outcomes for this group of students represent their feelings about self, school, and mathematics.

Ma and Kishor (1997) conduct a meta-analysis integrating and summarizing the findings for 113 studies concerning the relationship between attitudes toward mathematics and achievement in mathematics. Their findings indicate that there is not a significance difference in attitudes toward mathematics and mathematics achievement. Ma and Kishor (1997) find that the correlations are low suggesting that attitude in mathematics and achievement in mathematics is weak and cannot be considered to be of practical significant in education (Robinson, 1975; Vachon, 1984; Wolf & Blixt, 1981).

There are many different attitudes toward mathematics and the term *attitude* can mean different things for different disciplines (McLeod, 1992). Ma and Kishor (1997) suggest that instead of investigating attitudes toward mathematics in general, researchers should focus on a specific area of mathematics. In this study, I investigate attitude in the context of problem-solving. Hannula (2002a, 2002b) suggests that to establish a clear

picture of attitude, researchers should define attitude and what aspects of attitude are under investigation. I adopt McLeod's (1992) conceptualization of attitude as an affective response or learned tendency or predisposition to respond in a consistently negative or positive manner to some concept, situation, or object. For this study, I refer to attitude as the tendency on the part of a prospective teacher to respond positively or negatively toward a mathematical concept, situation, or person that could possibly affect their disposition toward mathematics. Ma and Kishor (1997) find that individuals who have positive feelings or disposition about mathematics exert more effort, spend more time on mathematics tasks, and are more effective learners than those with poor attitudes. This finding can inform my study in the investigation of prospective teachers' attitudes in mathematics problem-solving situations.

Mathematics-Related Beliefs

As with emotions and attitude, beliefs play an important role in learning and doing mathematics (Furinghetti & Pehkonen, 2000; Kloosterman, 2002; Lester, Garofalo, & Kroll, 1989; Op t' Eynde et al., 2002; Shaughnessy, 1985). There are many difficulties associated with defining beliefs. Some researchers (Furinghetti and Pehkonen, 2002) consider beliefs a part of knowledge. Others consider it a part of attitudes (Grigutsch, 1998). Beliefs, stable and cognitive, are internal representations, which the believer attributes to truth and validity (Thompson, 1992). The relationship between beliefs and knowledge is sometimes fuzzy. However, Furinghetti and Pehkonen (2002) define two parts of knowledge to situate and provide a better understanding of the relationship between beliefs and knowledge. They distinguish between different types of knowledge: objective and subjective. Objective knowledge is defined as one hundred percent

generally accepted truth whereas subjective knowledge is uniquely based on personal experiences and understanding. According to Furinghetti and Pehkonen (2002), beliefs belong to subjective knowledge because they are based primarily on personal experiences.

Lester, Garofalo and Kroll (1989) find that beliefs about the nature of mathematics and mathematics learning shape cognition and determine how an individual approaches a mathematics problem and which problem strategies are used. In a survey of beliefs, Schoenfeld (1992) finds that most students believe that all problems have only one right answer and one correct solution method. He also notes that many students in the study believe that ordinary students should rely on memorizing rules and applying procedures or algorithms because they cannot understand mathematics conceptually.

A survey conducted by the National Assessment of Educational Progress (NAEP, 1983) reveals that fifty-percent of the students who respond agree that learning mathematics consist mostly of memorizing facts. Seventy-five percent agree that doing mathematics requires repeated practice of rules. Ninety percent of those surveyed agree that there is always a rule to apply when solving mathematics problems. Schoenfeld (1985) suggests that beliefs are “important determinants of students’ mathematical behavior” (p. 198).

In his study of secondary students beliefs about mathematics and learning, Kloosterman (2002) finds a connection between belief and effort stating that “students’ belief is something that the student knows or feels that affects effort—in this case effort to learn mathematics” (p. 248). Stage and Kloosterman (1991) examines the relationship between beliefs about mathematics and achievement among college students enrolled in a

developmental mathematics course. The findings from the study indicate that students have a poor conception of the nature of mathematics and their ability to do mathematics.

Most studies investigating beliefs and other affective variables do so in isolation of each other. In other words, research studies focus on investigating the different kinds of beliefs that influence learning, how beliefs develop, and how beliefs motivate students to engage in mathematics. As a result, there remains little known about how various mathematics-related beliefs relate to each other and mathematics learning and understanding in a problem-solving context. Schoenfeld (1985) argued this point during his study of college students' problem-solving behaviors:

> One's beliefs about mathematics can determine how one chooses to approach a problem, which techniques will be used or avoided, how long and how hard one will work on it, and so on. Beliefs establish the context within which resources, heuristics, and control operate. (p. 45)

In this statement, Schoenfeld (1985) proposes the construct of belief systems. That is, an individual's mathematical world view and the perspective with which he/she approaches mathematics or a mathematics problem situation. When you view mathematics-related beliefs in these terms, you begin to understand how beliefs encompasses or represents the whole person—their social life, goals, needs, emotions, attitudes, the context they find themselves in, and their knowledge.

Op t' Eynde et al.'s, (2002) framework and definition of belief, referred to earlier in the introduction, reflects Schoenfeld's (1985) conception of belief and it describes what constitutes a mathematics-related beliefs system. Their frame and definition will inform this study by providing a more comprehensive understanding of the role beliefs play in mathematics learning and problem-solving. According to Op t' Eynde et al. (2002), an individual's mathematics-related belief systems include beliefs about

mathematics teaching and learning, beliefs about oneself, and beliefs about the learning environment.

Op t' Enyde et al. emphasize that each component or category of what constitutes a mathematics-belief system can consist of subcategories. For example in this study, under the category of beliefs about mathematics education, I include the subcategories of (a) beliefs about the nature of mathematics as conceptual or procedural (b) beliefs about problem-solving. The category of beliefs about self includes the subcategories (a) judgment/expectation about one's achievement in mathematics and their ability to do mathematics (b) desire to know and understand mathematics. The category of beliefs about the class context includes the subcategories (a) beliefs about one's functioning role as student (b) beliefs about the instructor's functioning role (c) beliefs about usefulness of assigned tasks. In this section, I briefly review what the literature reveals about each category and subcategory as it pertains to this study.

Prospective mathematics teachers often enter teacher education programs with beliefs about mathematics, the nature of mathematics, and teaching and learning mathematics that they have developed over a lifetime (Cooney, 1994). In many cases, they hold naïve and incorrect beliefs about mathematics (Lampert, 1990). For example, prospective teachers may erroneously believe that mathematics proficiency or mathematical understanding primarily involves the mastery of facts, the rote performance of procedures, and memorizing computational formulas (Lampert, 1990). With respect to beliefs about mathematics, in this study I examine the beliefs of prospective teachers in an effort to determine if they tend to favor beliefs about mathematics from on a conceptual or procedural viewpoint.

Beliefs about mathematics can also influence how a learner engages in problem-solving (Schoenfeld, 1985, 1989). For example, if a learner believes that a mathematics problem can be solved quickly then he/she might not persist to find a solution and instead stop the solution process prematurely (Schoenfeld, 1985). If prospective teachers believe that there is only one way to solve any mathematics problem, then they will not be inclined to seek an original approach, represent the mathematics in multiple ways, or see how the problem could be connected to other mathematical ideas (Thompson, 1992). As a result, their development of conceptual mathematical knowledge and understanding will be limited and they will be unable to develop their students' deep understanding of mathematics (Ma, 1999). Researchers Carlson & Bloom, 2005; Lester, 1994, Schoenfeld, 1985, 1989) find that individuals who are successful problem solvers possess positive mathematical beliefs, more well-connected knowledge and rich schemata, and are persistent in their efforts.

Beliefs about one's ability to do mathematics are related to attitudes and emotions with respect to confidence, security in oneself, and self-efficacy are found to influence mathematics learning (Cooper & Robinson, 1991; Gomez-Chacon, 2000). Social cognitive theorists hypothesize that students' self-efficacy beliefs, that is, their judgment or expectations about their capability to accomplish specific academic tasks, are important determinants of academic motivation, choices, and performance (Bandura, 1986, 1997; Pajares, 1992). In this study, beliefs about oneself refer to judgment/expectations in one's mathematics achievement or ability to do mathematics, desire to know and understand mathematics, and attributions to success or failure in mathematics (Aiken, 1996; Gomez-Chacon, 2000; McLeod, 1992).

Within one's beliefs about their learning environment, one can differentiate between external authority and internal authority. The textbook or instructor represents an external source of knowledge whereas individuals's validation of their own knowledge is internal. Confrey (1994) suggests that a learner's knowledge level matures when they are able to transition from depending on the instructor or textbook as their source of knowledge and understanding to viewing the instructor and textbook as a facilitator of their knowledge and understanding of mathematics. In the literature, autonomy is described as being a belief that one is responsible for his/her own knowledge and answers and that mathematics is valid and acceptable when it makes sense to the them (Confrey, 1994; Fennema & Romberg, 1999; Goodyear, 2000)

Piaget (1973) proposes that one of the goals of education should be to develop autonomous learners. The *Learning Principle* (NCTM, 2000) supports the idea that learning with understanding helps students become autonomous learners. Students become motivated and confident when (a) their instructors provide them with learning support when engaged in independent and cooperative problem-solving tasks, (b) they choose to engage in the mathematical task, and (c) they achieve success in completing the task (Fennema & Romberg, 1999; Fennema, Sowder, & Carpenter, 1999). Classroom interactions enhance the development of prospective teachers who are autonomous learners, as they propose mathematics ideas, conjectures, and learn to evaluate their own thinking and the thinking of others (Ball & Bass, 2003; Lampert, 1990; NCTM, 2000; Yackel & Cobb, 1996). Students learn more when they "take control of their learning by defining their goals and monitoring their progress" (NCTM, 2000, p. 21).

Mathematical Cognition

In this study, mathematical cognition refers to mathematical knowledge, thinking, and understanding (Schoenfeld, 1992). Skemp (1976) uses the terms *relational* and *instrumental* to distinguish two types of mathematical understanding. He describes relational understanding as "knowing both what to do and why" when confronted with a mathematical tasks (p. 9). Instrumental understanding is "rules without reason" (p.9). Hiebert and Lefevre (1986) also propose two types of mathematical knowledge and understanding: conceptual and procedural. Conceptual knowledge refers to knowledge that is well-connected and rich in relationships, whereas procedural knowledge consists mostly of procedures, algorithms, and memorized rules.

There are learners who believe that mathematics consist mostly of procedures, rules, and algorithms. While these are important aspects of mathematics, other competencies that are critical to knowing and understanding mathematics such as reasoning, conceptual understanding, and problem-solving (Hiebert & Lefevre, 1986; Schoenfeld, 1992; Stanic & Kilpatrick, 1989). Knowing and understanding mathematics includes not only having knowledge of facts, rules, algorithms, and procedures; but it includes having knowledge of how and when to use specific mathematical methods, strategies, procedures, and reflecting on the outcome. Knowing how and when to use your mathematical knowledge effectively is control or monitoring, which plays a critical role in achieving problem-solving success (Carlson & Bloom, 2005; Schoenfeld, 1985).

Several studies (i.e., Carlson, 1999; Lester, 1994, 1980; Schoenfeld, 1985) investigating problem-solving behavior suggest that even when individuals have the resources or knowledge to solve a problem, they often do not practice control or access

their knowledge to produce a problem solution. Schoenfeld (1992) said it best when he stated that "It's not just what you know; it's how, when, and whether you use it" (p. 355). Schoenfeld (1985) based this statement on a problem-solving study of undergraduate students who did not notice when their problem-solving efforts were unproductive, due to their practice of poor control during problem-solving. Lester (1994) proposes that effective mathematical problem solvers possess mathematical knowledge that is deep, conceptual, and well-connected and they appear to have knowledge and awareness of their weaknesses and strengths as it relates to the problem.

Problem solvers who possess mathematical knowledge that is deep, conceptual, and well-connected are able to manage their affective behaviors and persist toward a solution to a problem (Carlson & Bloom, 2005). Deep understanding of mathematics implies that mathematics concepts, procedures, and strategies are well-represented and well-connected (Haylock, 1982). Learners who have a deeper understanding of mathematics can monitor their own problem-solving and are faster overall at solving problems because they understand the meaningful relationships between pieces of information, which reduces their need to remember rules, formulas, and procedures (Carpenter, 1988; Hiebert & LeFevre, 1986).

Metacognition

Metacognition encompasses both knowledge and regulation of cognitive activity (Moses and Baird, 1999). Metacognitive knowledge is knowledge an individual has about their cognitive abilities, cognitive strategies, and about mathematical tasks (Flavell, 1979). Metacognitive regulation refers to process that coordinate cognition; for example, monitoring which refers to error detection and control which refers to error correction

(Reder & Schum, 1996). The relationship between developing conceptual knowledge and metacognitive knowledge is established in several studies. The development of knowledge about when, where, and how to apply strategies, understanding the mathematical task, and an awareness of the need to reflect on the content of one's knowledge has been examined by several researchers (Garofalo & Lester, 1985; Kuhn, Garcia-Mila, Zohar, & Anderson, 1995; Schoenfeld, 1985). Schoenfeld found that even when problem solvers have the required mathematical knowledge, they often do not know when or how to use it when solving non-routine problems. Kuhn and his colleagues found that a good use of strategy requires knowledge about when and when not to apply that strategy. They refer to this knowledge as meta-strategic knowledge. Meta-strategic knowledge emerges from conceptual knowledge and metacognition and it has been found to guide further learning (Kuhn et al., 1995; Schoenfeld, 1985, 1987).

Successful completion of mathematical tasks requires more than the application of knowledge, it requires the combination and coordination of both cognitive strategies and processes and metacognitive behavior (Hammouri, 2003; Schoenfeld, 1985). As Schoenfeld (1985) suggested, it is not just, what you know it's how you use what you know in a problem-solving situation where emotions, beliefs, and possibly attitudes are interacting to influence your decisions and performance. While content knowledge is essential for successful problem-solving, metacognitive factors enable a problem solver to monitor and regulate processes and evaluate solutions.

While metacognition and cognition are related, they are quite distinct as well. Lester (1985) states that, "cognition is involved in doing whereas metacognition is involved in choosing and planning what to do and monitoring what is being done" (p.

164). There are a number of similar descriptions of metacognition in the reviewed literature. Baird (1999) describes metacognition as having three components: metacognitive knowledge, metacognitive awareness, and metacognitive-control. Metacognitive knowledge deals with knowledge that relates to the nature of learning, learning techniques and personal learning characteristics. Metacognitive awareness relates to progress during task completion, and metacognitive control describes the making of effective decisions about one's problem-solving progress and outcome. The more problem solvers are able to control and monitor the strategies they use emotional behavior the better their abilities to solve a mathematical problem (Kapa, 1999; McLeod, 1989a, 1992; Schoenfeld, 1985, 1987).

Garofalo and Lester (1985) categorize metacognitive behaviors related to problem-solving. Their cognitive framework is comprised of four categories or activities involved in performing mathematics task: orientation, organization, execution, and verification. Each category is associated with specific behaviors. For example, during the orientation phase, the learner is involved in understanding the problem. During the organization phase, planning and monitoring behavior such as identifying specific goals related to problem completion. Execution involves implementing a strategy to solve the problem. The final phase, verification, involves the evaluation of decisions and the results of an implemented strategy. Garofalo and Lester's (1985) cognitive framework identifies areas where decisions based on metacognitive behaviors are most likely to have an impact on cognitive actions. In this study, their framework is used to analyze the metacognitive behavior of prospective teachers during varying phases of the problem-solving process.

In an effort to understand the relationship between metacognition and problem-solving, several empirical research studies are reviewed. Simon's (1987) investigation of students problem-solving reveals that monitoring, regulation, and orientation processes show up more frequently in the problem-solving protocols of more successful subjects. He also suggests that even though a person might have the knowledge they need to solve a problem in a given situation, they might not access or apply it when needed.

In a study focusing on the metacognitive behaviors of middle school students, Lester (1989) concluded that an individual's orientation to the problem has the most important effect on performance. While Lester's study involved school aged students, Schoenfeld (1985) found similar results with college students in his study on problem-solving. He concludes that the choices problem solvers make at vital points during the problem-solving process are critical to problem-solving success. This link between metacognition and success in mathematics problem-solving is well documented in the literature (see Artz & Armour-Thomas, 1992; Carr & Biddlecomb, 1998; Shelia, 1999; Schoenfeld, 1985, 1989, 1993). Metacognition enables a solver to analyze a new problem, judge how far they are from the goal of obtaining a solution, allocate attention, select a strategy, attempt a solution, monitor the success or failure of the solution process, and decide whether a new strategy is needed to move the process forward (Flavell, 1979).

Summary

This chapter discusses the literature related to affective behaviors such as emotions, attitudes, beliefs, mathematical cognition, and metacognition during problem-solving. Previous research is critical in understanding how these phenomena are identified, and interact to influence mathematics learning and problem-solving. The lack

of a clear definition and theoretical frame used to characterize attitudes and emotions creates difficulty in providing a cohesive body of literature in the affective domain of any real power. Consequently, Op t' Eynde (2002) and McLeod's (1992) frameworks referenced earlier in this paper are especially important tools for interpreting and conducting research in the affective domain. The review on mathematical cognition and metacognition will be useful in informing my study about prospective teachers' mathematical thinking and understanding.

In the next chapter, I provide a thorough description of the methodology including the research paradigm, brief overview of the method, and the intended procedures used for analyzing the data to answer the research questions guiding the study.

CHAPTER 3

METHODOLOGY

This chapter describes the methods I used to investigate prospective teachers' experiences as they engaged in solving mathematics problems. While the research reports that affective and cognitive behaviors influence problem-solving performance, there is little information about the nature of cognitive and metacognitive processes and their interaction with beliefs and other affective behaviors during problem-solving (McLeod, 1992; Phillipp, 2007; Schoenfeld, 1992). The rationale for using qualitative methodology, and more importantly the choice of grounded theory methods is presented along with the procedures used to ensure trustworthiness of the findings.

Brief Overview of the Study

This study was conducted using qualitative methods. Grounded theory methods were used to analyze the data. I utilized qualitative methods of semi-structured face-to-face interviews, observations, a video-based think-aloud interview, memo logs, and a selection of artifacts from each participant to gather as much information as possible with the intent of analyzing, interpreting, and explaining the interaction among specific problem-solving behaviors. I used theoretical and purposeful sampling techniques to select a sample from a group of prospective middle-grades (4-8) and secondary (6-12) prospective mathematics teachers enrolled in a sixteen-week undergraduate mathematics course. The description, comparison, and analysis of several prospective secondary

mathematics teachers' experiences during problem-solving activities provided me with an understanding of their mathematical behavior. Finally, the findings in this study will help to extend the current research on prospective teachers' mathematical problem-solving processes.

The main research question is: What is the characterization of the interplay among prospective teachers' mathematical beliefs, mathematical behavior, and mathematical knowledge in the context of solving mathematics problems? In answering this main question, I also answer the following questions:

(a) What are the mathematics-related beliefs of prospective secondary mathematics teachers?

(b) What mathematical behaviors do prospective secondary mathematics teachers demonstrate as they engage in mathematical problem-solving?

(c) What mathematics knowledge do prospective secondary mathematics teachers use as they engage in mathematical problem-solving?

The Research Paradigm

A paradigm is comprised of the researcher's views about the existence of reality, knowledge, choice of methods used to conduct a research study, the style of research reporting, and the importance of the implications of the research (Ernest, 1998).
There are two major research paradigms, quantitative and qualitative (Creswell, 1994). The quantitative research paradigm is based on numbers used to interpret a phenomenon under study. It involves using statistical analysis and statistical variables for the interpretation of data. The qualitative research paradigm is an inquiry process of understanding a social or human problem by examining the patterns of meaning which

emerge from the data represented by the participants' own words (Creswell, 1994). Direct observation and interaction with the participants enables the researcher to understand not just the words of the participant, but the meanings they give to their words and why they give the words their meaning in a specific context.

In a qualitative or constructivist (Mertens, 2005) research paradigm, for which I am closely aligned, the focus is on exploring interactions with the emphasis on the world as socially constructed reality involving multiple perspectives. The perceptions and the values of all the participants in a situation are needed in order to explore the various possible interpretations (Ernest, 1998; LeCompte & Schensul, 1999b; Mertens, 2005).

Rationale for Conducting Qualitative Research

In mathematics education, research has shifted from the predominantly positivist, quantitative paradigm perspective to that of the naturalistic or qualitative research paradigm (Ernest, 1998). Qualitative methods are useful when the goal is to obtain intricate details about phenomena such as feelings, thought processes, and emotions that are difficult to understand using quantitative methods (Strauss & Corbin, 1998). Moreover, qualitative methods are best used to explore areas of importance for which little is known. Because the integration of affect and cognition is under- conceptualized or little is known about it, a qualitative inquiry seemed appropriate for this study.

Lincoln and Guba (1985) made distinctions between two types of studies. They represent the situation where the researcher "knows what he or she doesn't know" (p. 209) and can therefore explain the means for finding it out, and the situation where the researcher "does not know what he or she knows" (p. 209) in which case the researcher

needs to maintain a more open-ended approach in their search for an explanation. The later is usually a qualitative or naturalistic inquiry (Lincoln & Guba, 1985).

Philosophical Claims of Qualitative Research

Each paradigm inquiry consists of three underlying philosophical claims. The first is ontology or the belief about the nature of reality, the second is an epistemology or the belief about the nature of knowledge and the third is a methodology or the approach to inquiry. Lincoln and Guba (1994) suggests that one's belief about the nature of reality, knowledge, and one's approach to inquiry are interconnected with and constrained by each other.

The qualitative or constructivist research paradigm is based on the assumption that the world is not an objective one, but exists in multiple realities (Mertens, 2005). In qualitative research, the world is a subjective phenomenon that is open to interpretation and not mathematical measurement. The world exists because of human interaction and perceptions, which can only be explored and discovered through meaningful description and interpretation. The research and evidence is achieved through exploration and inductive processes (Merriam, 1998). Lincoln and Guba (1999) explained that in a qualitative research paradigm one denies the existence of an objective reality, "asserting instead that realities are social constructions of the mind, and that there exist as many such constructions as there are individuals, although clearly many constructions will be shared" (p. 431). Reality is relative, local, and socially constructed and every construct carries equal importance (Lincoln & Guba, 1994; Mertens, 2005).

Eisner (1979) suggested that the sources of knowledge are as diverse as the information provided by our senses. Each of our five senses provides a unique experience

that cannot be replicated by the other. An individual's knowledge only has meaning within a given situation or context and each individual's perception can differ, creating many different interpretations. A constructivist epistemological paradigm views all knowledge and meaningful reality as socially constructed through interactions between individuals and the world around them in a social context (Crotty, 1998; Mertens, 2005). For this study, I was interested in understanding the interaction of prospective teachers' affective behavior, problem-solving behavior and experiences during their participation in an undergraduate mathematics course specifically focused on developing deep mathematics understanding in a problem-solving context. This understanding is found in the "realm of the knower" (Smith, 1983, p. 46)—the prospective secondary mathematics teacher. I approached the study from a constructivist epistemological paradigm because mathematics knowledge and understanding is socially constructed (Ernest, 1996).

In a constructivist paradigm, knowledge and understanding is achieved through the interaction between the researcher and the researched. These knowledge and understanding claims are subjective in nature**.** Mertens (2005) explained that the elimination of objectivity and bias is nearly impossible when researching the social world. Values, culture, training, and experience have "pride and place" (Lincoln and Guba, 1994, p. 114) and measures should be taken to maintain the awareness of bias and the role of bias should be addressed in the interpretation of findings.

Ontological and epistemological beliefs influence methodological approach to doing qualitative research. Qualitative research is based on the information gained through listening, watching, and interacting. Prospective teachers' mathematical behavior and experiences during problem-solving can be viewed as a complex, multi-layered

phenomenon. Some behaviors can be observed; however, there are aspects or levels of experience that only an individual who is living or has lived the experience can describe and explain. To explore the affective behavior, mathematical behavior and experiences of prospective secondary mathematics teachers during problem-solving situations would entail many perspectives, their social behaviors, and the context of the class setting. Tests, questionnaires, and surveys can measure some element of behavior, but to understand the nature of prospective teachers' problem-solving and mathematics learning processes, a qualitative study was necessary. The analysis of interviews and observations provided me with insight into prospective secondary mathematics teachers' mathematical behavior, how their cognitive processes operated as they explained their mathematics thinking during problem solution presentations, and how their minds functioned as they explained their mathematics reasoning and justified their problem solutions.

A Grounded Theory Approach

Grounded theory (GT) methodology, a type of qualitative or naturalistic inquiry, focuses on the perceptions, thoughts, and actions of individuals, as well as how individuals define their situations (Denzin, 1989). According to Strauss and Corbin (1990), it is a "qualitative research method that uses a systematic set of procedures to develop an inductively derived grounded theory about a phenomenon" (p. 24). It draws on the strengths of both positivist and interpretivist approaches (Charmaz, 2000). Orlikowski (1993) characterized GT as interpretive because it uses qualitative and unstructured data that represents subjective understanding, it involves subjective sampling (Flick, 1998), and the theory-building process is mostly inductive (Strauss & Corbin, 1990). The method is influenced by positivistic approaches because it provides a

systematic coding procedure designed to eliminate "speculative assumptions not founded on observations" (Schweizer, 1998, p. 44), and deductive verification of concepts and relationships is obtained during the inductive process. For example, during the process of constantly comparing the data, at times, I made assumptions about themes emerging from the data and then collected more data to verify those assumptions.

In this study, I used Charmaz's (2005) coding procedures and included aspects of Strauss and Corbin's (1998) coding techniques to move beyond making statements about the data to making analytic interpretations of the participants lived experiences. There was always the understanding that my background, beliefs, and culture would introduce subjectivity into the process based on how I collected data, interpreted the data provided by my participants, and coded the data. I collected data in an inductive manner (Morse, 2001) and I began the study with no preconceived ideas to either prove or disprove. My desire was to explore and understand the interplay among affective behavior, mathematical behavior, and mathematics knowledge as prospective mathematics teachers engaged in the problem-solving process. Moreover, I wanted to characterize the interaction among these phenomena using a model. The goal of GT is the construction of theory that gives understanding about important issues in people's lives and it allows the researcher to generate explanatory theory about social phenomena rather than generating results to support or test existing theories (Glaser & Strauss, 1967).

Strauss and Corbin (1998) explained that, "analysis is the interplay between the researcher and the data (p. 13). They suggested that GT analyst work to "uncover relationships among categories…by answering the questions of who, what, when, why, how, and with what consequences" (p. 127). Taking a grounded theory approach, I

constructed theory from the data created from the participants' perspectives and voices and my experiences and relationships with the participants (Charmaz, 2005). Charmaz explained that when the researcher starts with the data from the lived experiences of the participants, from the beginning of the study the researcher attends to how they construct their worlds. She explained that "lived experiences shape the researchers' approach to data collection and analysis" (p. 8). Taking a grounded theory approach to conducting this qualitative research study enabled me to construct a theory addressing the relationship among the participants' affect, mathematics learning, and problem-solving behavior based on my interpretation of the meanings participants gave to their realities in the context of problem-solving (Charmaz, 2000).

The constructivist paradigm is familiar in the mathematics education community and my own background and view is closely aligned with this view as well. The constructivist paradigm fits well within the interpretivist philosophy, both of which have the goal understanding "the world of human experience" (Cohen & Manion, 1994, p.36), suggesting, "reality is socially constructed" (Mertens, 2005, p. 12). As such, this study took an interpretivist-constructivist approach to GT. Like interpretivist and constructivist researchers, I relied on the participants' views of the situation being studied (Creswell, 2003) while recognizing the impact my own background and experiences have on the research. The GT resulting from this study is my interpretation of the meaning the participants gave to their mathematics learning and problem-solving experiences--it is not be an exact truth.

Three important features of GT research are theoretical sampling, coding, memo writing, and the constant comparative method of analysis, which involve the continuous

cycle of collecting, labeling, and analyzing data. Theoretical sampling is purposive and refers to the process of data collection where the selection of new participants for the sample is based on the results collected from a previous sample (Glaser & Strauss, 1967). As explained by Strauss and Corbin (1998), "theoretical sampling is cumulative" (p. 203). As an explanation for what is happening in the field emerges and the investigation focuses, so too does the sample of participants. With respect to the constant comparative method, the researcher begins analyzing the data as soon as it is collected and then continues on to compare the analysis of one set of data with another in an effort to develop categories. The basic strategy of the constant comparative method involves constantly comparing terms or phrases the participants use or incidents from interviews, field notes, researcher memos, and other documents with another incident in the same data set or other data set. Each comparison and sorting of field notes and interview transcriptions lead to the construction of categories representing the meanings the participants gave to certain incidents. As the research progresses, the researcher continues to interact with the participants and review the categories as additional new data is collected. The idea is to allow the categories to emerge from my interaction with the participants and with what the participants are saying and doing.

According to Strauss and Corbin (1998), coding is the "building blocks of theory" (p. 101) and it involves the simple act of labeling an event, object or action in the data that represents the participants' voices or a specific incident. Coding is an important component of grounded theory as it enables the researcher to group similar events, happenings and ideas under a common category. I used initial and focused coding (Charmaz, 2005) and axial and selective coding (Strauss & Corbin, 1998) to formulate an

understanding of the interplay among affective, cognitive, and metacognitive behavior during problem-solving. During initial coding, data were compared and I learned what the participants viewed as problematic during the mathematics learning and problem-solving processes. Through focused coding, I compared the participants' experiences, actions, and interpretations, which enabled me to develop categories (Charmaz, 2005). Strauss and Corbin's approach to grounded theory outlined the use of axial coding which provided a frame that enabled me to develop subcategories and to link them to the categories developed during focused coding. I also found that using Strauss and Corbin's axial coding eliminated ambiguity and provided clarity as I sorted and synthesized the large amount of collected data. Selective coding enabled me to specify the relationships among categories and to tell a story, based on the interpretation I gave to participants' statements and actions, that integrates participants' disparate experiences (Charmaz, 2005).

Throughout the coding process and during my memo writing, I spent numerous hours each week formulating and constructing explanations from my interactions with the participants and used this information to analyze my data and to report findings. In that sense, my memo writings represented what Charmaz (2003) referred to as "researcher created and ensuing analysis" (p. 523). Throughout this study, there were instances where I was shaped by the data, but there were also times when I shaped the data (Charmaz, 2003).

The research paradigm associated with a study is an important factor in the research design. The reliability and credibility of research is critical to research findings, and it is only through the quality of the data that meaningful and valid results are

developed. Unlike the rigor and validity applied to quantitative research, the quality and validity of qualitative research is judged by its trustworthiness, credibility, and transferability.

Meeting Trustworthiness Criteria

Credibility, the degree to which the researcher's interpretations are consistent with the meanings intended by the participants, is the foundation on which all other validity is formed (Strauss & Corbin, 1998). If the data are not accurately reported, all else is irrelevant (Glaser & Strauss, 1967; Strauss & Corbin, 1998). Credibility in this study is achieved via prolonged engagement, persistent observation, triangulation, and member checking. I spent ten weeks (30 hours) in the field observing the participants, focusing only on elements of the situation that apply to the study's purpose. I used a number of data collection methods, including interviews, observations, and video-taped think-aloud problem-solving episodes. After each interview and classroom observation, I reflected on what I saw and heard, and recorded my thoughts and hunches in the form of memos. I transcribed each interview verbatim and included features of the participants, such as the appearance of stress and verbal pitch in order to understand thoroughly the interviews. During subsequent interviews, the participants verified the accuracy of specific aspects of their transcribed interviews. Furthermore, I asked a colleague who was familiar with my study, but not directly involved in the study, to code the first few interviews and to provide support or corrective feedback on findings. She also questioned me about my methods, emerging theory, and biases.

Confirmability refers to the extent to which the findings are rooted in the data or the participants' voices and views and are not a reflection of my own ideas or

preconceived notions. Creating and maintaining an audit trail facilitates confirmability. During the interviews, I asked follow-up questions based on previous observations, and I looked for clues within the transcript to assure accurate evaluation of the interviews. I read and re-read line-by-line the transcribed interviews, fieldnotes, and memos to assure accurate evaluation of what the informants say and do. I recorded codes in a codebook along with their meanings. I provided a competent well-trained colleague with a trail of raw data, theoretical notes, and memo notes and elicited her help to conduct the inquiry audit.

Finally, generalizability can be problematic in the sense that the theory resulting from my study is universally applicable. The research study specifically focuses on the characteristics of prospective middle-grades and secondary prospective teachers enrolled in an undergraduate mathematics class; therefore, the findings or theory may or may not apply to similar groups (Strauss & Corbin, 1998). Instead of generalizability, the results could have explanatory power or “predictive ability” (Strauss & Corbin, 1998, p. 267). That is, the results could have the ability to explain what might happen in a given situation. The results of the study are more likely to be internally generalized in the sense that theory specific to this study is developed from the repetitive themes, patterns, and categories and are applicable to the participants in this study or participants in similar situations.

Transferability measures how well the researcher informs the readers of how the data are interpreted and analyzed. Transferability is facilitated by clear descriptions of how the concepts are named and categories developed by the qualitative grounded theory inquirer. Thick description of the process of analysis, phenomena under study, and as

much of the context in which the study took place as possible will contribute significantly to facilitating transferability decisions (Lincoln & Guba, 1989). I explain my worldview, method of sample selection, and include interview protocols and coding procedures in the appendixes so that the reader will have a clear understanding of how I arrive at my findings. I also maintained a reflective memo journal throughout the research process so that I remained aware of my own subjectivity and its potential to influence the research.

Researcher Subjectivity/Sensitivity

The grounded theory approach, according to Strauss and Corbin (1990, p. 24), is a "qualitative research method that uses a systematic set of procedures to develop and inductively derived grounded theory about a phenomenon." GT draws on the strengths of both interpretive and positivistic approaches. Even though I used a systematic approach to coding my data, I viewed the whole process as one that was interpretive, maintaining the view that reality or coming to know is constructed inter-subjectively through meanings, understandings, and interpretations that are developed or constructed socially. Meaning took shape as the data collection proceeded and during the inductive process of the study, I worked from the participants' data to generate categories during the process rather than in advance of data analysis. Themes, concepts, and categories that emerged in this study were filtered through my worldview and colored by my own experiences.

I used Strauss and Corbin's (1998) systematic coding procedures while understanding that my background, beliefs, and culture would introduce subjectivity into the process. I strived for transparency about the approach and interpretations. As such, I acknowledge my bias, sensitivity, and subjectivity. I believe that in doing so, the degree of transferability and the richness of the study are enhanced. I believe that there are

multiple realities. Therefore, perhaps the same study could be done again with different results.

My values, beliefs, background, knowledge, and experience not only provide the means for helping me understand the world in which I live, but they sensitize me to the issues and phenomena I am investigating in this study. Some beliefs are relevant to the study and I share these relevant beliefs, experiences, and values with the reader below.

My Mathematics Education Beliefs

I currently work as a mathematics instructor and as a mathematics teacher educator teaching general education mathematics and secondary mathematics methods courses in a university setting. Previously, I taught middle-grades, secondary mathematics courses, and general mathematics courses at a community college. During my years of teaching, I have encountered students who exhibit a variety of emotional responses, attitudes, and academic abilities when learning mathematics concepts and solving mathematics problems. Therefore, I am very familiar with some of the effects of mathematics-related beliefs, affective behavior, and cognition on problem-solving success. There is a plethora of literature documenting beliefs, affective behavior, cognitive behavior, and problem-solving behaviors. While preparing my prospectus, I familiarized myself with some of that literature. The literature focuses mostly on describing problem-solving behavior, on identifying attributes that contribute to problem-solving success, and on reporting problem-solving success as a function of many factors such as knowledge, control, beliefs, and other affective dimensions. These studies could have possibly played role in my preconceived notions and biases.

The literature, as well as my experience teaching mathematics, has shaped my beliefs about mathematics. I believe that mathematics does not consist primarily of rules and algorithms to be used during problem-solving. Instead, mathematics is rich in relationships that are connected by discrete pieces of conceptual and procedural mathematical knowledge that can be used in problem-solving situations. While my worldview, in some ways, shapes my study, I am open to being shaped by my research experiences and to having my thinking informed by the data.

As the primary instrument for collecting and analyzing data, I was very careful to express the participants' views and perspectives but there were times when I negotiated meaning and defined what was happening through "shared interpretations" (Charmaz, 2002, p. 684). I revisited and reviewed the audio and video recordings of interviews, fieldnotes, and my reflection logs throughout the analysis of data to make meaning of what was happening. Finally, during the course of this study my preconceived notions, assumptions, any emerging theory during data analysis were challenged and debated with a colleague who was familiar with my study.

Context

Selection of Participants

Participants were drawn from a population of prospective middle grades and secondary mathematics teachers enrolled in a third year mathematics course entitled *Advanced Perspectives on Mathematics* (APM). During the semester long course, prospective teachers are revisiting key ideas in school mathematics while using the skills and understandings of college course work in mathematics to solve mathematics problems. In this course, they reason, listen to, respond to, question their instructor and

one another, make conjectures, explore examples, solve problems, present solutions, and justify their reasoning and problem solutions. Since solving problems was a critical aspect of the APM course, I assumed that prospective teachers enrolled either have some experience with problem-solving or are in the process of gaining knowledge of problem-solving.

In GT, the selection of participants is based on the developing nature of the research and cannot be predicted at the start of the study (Glaser, 1978; Strauss & Corbin, 1998). Instead, as the research progresses, data analysis guided the questions for subsequent data collection and sampling. Therefore, the researcher is not expected to specify how large the sample will be before the start of the study. The sampling was done with purpose and was guided by theoretical assumptions emerging from the data and, in some instances, suggested by the literature (Strauss & Corbin, 1998). Events, happenings, or incidents, that represent phenomena pertinent to the study, are sampled and not individuals per se (Strauss & Corbin, 1998). Individuals were the means to obtain pertinent data that were used in the development of concepts and categories essential in developing theory. During the process of data analysis, as concepts and categories begin to emerge, participants who were seen as having experiences and knowledge pertinent to answering the research questions were approached and asked to be a part of the study. I determined the final sample size by theoretical saturation or the failure to obtain new relationships or new information for the categories identified in the study.

After collecting the consent and background information forms, I separated the background information forms into two groups. One for prospective middle-grades teachers and the other contained prospective secondary teachers. I read and labeled the

answers for the two questions on the background information form, “What are your general feelings about mathematics?” and “How would you describe your own ability to do mathematics?” in order to get a sense of the participants self-perceived mathematical ability and their general attitudes about mathematics. I noted any responses that sparked my attention. For example, one potential participant stated that “I struggle with mathematics most of the time and it takes a while for me to ‘get it’ but I really want to teach middle-grades mathematics so that my students can really understand math.” I considered the responses to these questions as provisional data and focused primarily on the responses that pertained to the study. The provisional data provided me with a starting point for areas of observation and interview questions and a point from which to begin my data collection.

I began the study with a classroom observation. Following the first observation, I approached a middle-grades prospective teacher who subsequently agrees to participant in an initial semi-structured interview. After the initial coding of data, the interview from the first participant leads to the selection of an additional middle-grades prospective teacher and two secondary prospective teachers. Each interview and observation provided me with a piece of data on which I could build as I moved back and forth through the data in order to find, compare and verify patterns, concepts, and categories until I reach the point where no new patterns, concepts, or categories emerge. This method of sampling led to the final selection of participants for this study.

Participants

In this study, the participants were chosen from a group of 15 middle-grades and secondary prospective teachers enrolled in the APM course. There were 13 female and

two male students enrolled in the course, seven of whom are prospective secondary (6-12) mathematics teachers and eight of whom are prospective middle-grades (4-8) mathematics teachers. The professor teaching the course provided me with unlimited access to the prospective teachers enrolled in the course. These prospective teachers were engaged in conjecturing, reasoning, and solving non-routine mathematics problems during each class session. The literature reviewed for this study revealed that, as individuals actively engage in problem-solving, a variety of affective behaviors, metacognitive behaviors, and cognition can be identified (Carlson & Bloom, 2005; Garofalo & Lester, 1989; Schoenfeld, 1985, 1992). Because the research questions in this study concern, affective behavior, metacognition, and mathematical cognition in a problem-solving context, choosing participants who are actively engaged in conjecturing, reasoning and problem-solving ensure the possibility of study-relevant sources (Miles & Huberman, 1994). The selection of participants from this course generated the final sample from which the most learning and understanding could occur.

On the first day of class, I explained the focus and intent of the study to all prospective teachers enrolled in the APM course and asked them to complete a background information form indicating whether they were willing to participate, would need more information, or were not willing to participate. To gain initial information about the prospective teachers' mathematics background and self-perceived mathematical ability and confidence level, I also asked them how they viewed their ability to do mathematics. I used three selection criteria. The first is willingness to participate. I wanted to be sure that the potential participants would not mind being video-taped during problem-solving episodes since these episodes are critical in verifying affective,

metacognitive, and mathematical cognitive behavior. The second criterion is for the participants to be secondary mathematics or middle-grades education major, and the third criterion is acceptance into the teacher education program. The purpose of the second and third criteria was to focus on issues related to teaching and learning mathematics in a classroom setting and in the future possibly extend the study into the participants' field experience.

Fifteen prospective teachers were enrolled in the APM course. Ten agreed to participate, two were not willing to participate, two needed further information, and one of the two male students did not meet the second and third criteria. On the second day of class, I revisited the class and explained the study in further detail to those who needed further information. After I answered their questions, the two who needed further information agreed to participate. Thus, the population from which I applied theoretical sampling included twelve potential participants—eleven females and one male. The study eventually included four participants—Tanya and Mandy (pseudonyms), both of whom are prospective middle-grades (4-8) mathematics teachers, and Mark and Cindy (pseudonyms), both of whom are prospective secondary (6-12) mathematics teachers.

Data Collection

The data sources for each participant included four face-to-face semi-structured interviews one of which was a videotaped think-aloud problem-solving interview, ten weeks (30 hours) of classroom observation, and researcher memo logs. A rationale and description for each data source is provided in detail below. A data collection schedule is also listed in appendix K.

Face-to-face Interviews

Face-to-face interviews are important for gaining information on the nature of prospective teachers' mathematics-related beliefs (i.e., beliefs about the nature of mathematics, oneself, and the classroom context) and their attitudes toward mathematics. These behaviors cannot be observed. Patton (1990, p. 196) explained that "we interview people to find out from them those things we cannot directly observe. In my attempt to understand prospective teachers' mathematics-related beliefs, emotions, attitudes toward mathematics, and their metacognitive behavior during problem-solving, face-to-face semi-structured interviews seem appropriate. The interviews were conducted in a private location with adequate lighting, space, acoustics and safeguards against interruptions.

The third interview was a think-aloud problem-solving interview. In mathematics, the think-aloud method is used to examine the processes involved in metacognition, self-regulation or control during problem-solving (Artzt & Amour-Thomas, 1992; Ericsson & Simon, 1984; Schoenfeld, 1985). The think-aloud technique involves the participants verbalizing their thoughts; that is, speaking aloud all that comes to their minds during the problem-solving process. To investigate the processes involved during problem-solving, Ericsson and Simon's (1984) think-aloud protocol and technique were used during the third interview (see Appendix D).

Video-taped Think-Aloud Problem-solving Interviews

Three verbal methods can be used to collect data about the processes an individual experiences while completing a task (Ericsson and Simon, 1980). They are introspective, retrospective, or think-aloud. During the introspective method, the researcher interrupts the participant while he or she performs the task to answer questions

based on what he or she is thinking, feeling, and doing. This method is not considered reliable because not only is the participant's working memory interrupted, but also the task completion process is considered inefficient (Nisbett & Wilson, 1977). The retrospective method requires the participant to complete the task in its entirety and, upon completion, describe the strategies used while completing the task (Rowe, 1985). Nisbett and Wilson (1980) explained that this method is unreliable because, due to the potential prolonged time between task completion and the participant's response, he/she is more likely to forget the process.

According to Ericsson and Simon (1980), the think-aloud method shows the most reliability for reporting the problem-solving process (Ericsson & Simon, 1980). Before the interview, I thoroughly explained the think-aloud method to the participant. The participant was instructed to verbalize everything he or she was feeling or thinking during the solution process. For example, during the videotaped problem-solving interview Mark stated, "Hmmm, what did I do wrong here? Why isn't the answer checking out right…hmmm, I must have done something wrong…okay, let me go back and see where I went wrong here." Ericsson and Simon (1980) explain that think-aloud is natural and automatic because it does not interfere with the participant's working memory, nor does it give the participant time to make his or her own interpretations about their thinking (Ericsson & Simon, 1980). According to Ericsson and Simon (1980), this process provides for reliability and validity when compared to the introspective and retrospective methods. Therefore, for the purpose of this study, think-aloud is used to collect data about the participants' affective behavior, cognition, and metacognition during the problem-solving process.

Follow-up Interview

This fourth and final interview occurred at most two days after the participant was finished solving the problem. During the interview, the participant was given copies of his or her solution to the problem from the think-aloud session. To help the participant recall his/her problem solution process, I played the video and stopped it at specific points to question the participant about his/her thinking and feelings at the time he or she was solving the problems. Questions about my observations of their problem-solving process were asked to clarify the participants' problem-solving thinking and behavior and to clarify my interpretations. The majority of the questions are based on my observations during their problem-solving process; however, some questions are adopted from Randell, Lester, & O'Daffer's (1987) "How to Evaluate Progress in Problem-Solving."

Observations

This research study began with an observation of the APM classroom to decide how the participants may be approached for an initial interview. Face-to-face interviews enable me to investigate prospective teachers' affective behaviors and metacognitive behaviors. Stated affective behaviors, such as beliefs, are often different from the affective behaviors that can be inferred from an individual's actions (DeBellis & Goldin, 1997; Thompson, 1992; Schoenfeld, 1985). To investigate the informants' problem solving experiences, classroom observations are conducted. I observed the participants' as they asked questions, solved mathematics problems at the board and their desks, explained their solutions, and as they engaged in mathematics discourse in the classroom environment.

Observations made it possible to record behavior as it was happening and it provided knowledge of the classroom context. Observations of discourse and interactions with their peers and instructor enabled me to monitor their mathematical thinking and understanding as they engaged in mathematics discourse. Information gained from the mathematics discourse during the observations guided theoretical sampling and in conjunction with interviews crystallized findings.

Observations occurred during each class session for 1 hour 15 minutes, twice per week for the duration of ten weeks from January 15 through March 20, 2008. On January 22, the professor cancelled class due to inclement weather; therefore, no observations occurred on that day. I observed participants on February 28 while completing their midterm exam to identify any affective behavior exhibited during this process. No observations occurred on March 4 and March 6 due to the university's spring break. Field notes were taken during each observation, adding important contextual content to the study.

Memo and Reflection Logs

As a way of monitoring the data collection process and to begin analyzing the information gained from the interviews, immediately following each interview and observation I recorded my ideas, speculations and hunches, feelings, descriptions and perceived moods of the participants (Bogden & Biklen, 1992; Creswell, 2003; Merriam, 1998). Memos are meant to be analytical and conceptual rather than descriptive (Strauss & Corbin, 1998). However, I included descriptions as well. My reflections and memos were used as part of the data collection and analysis process. The following is an excerpt from a memo made after an observation.

> About eight of the 15 prospective teachers (pt) in the class today were consistently participating in the class discussion. This is a very enthusiastic bunch. They seemed very intelligent, knowledgeable, and confident. The instructor posed the question "What is mathematics?" and "How do you learn mathematics?" I had planned to ask this question in a future interview, so I was excited about how the students would respond. After listening to a plethora of responses, I still have questions. What do the students think mathematics is not? What is meant by "doing math?" Does "getting it right" and "explaining why math works" affect how one handles difficulties associated with "doing math?" (Field note 1, January 15, 2008).

This is the first of many, many, memos used to provide direction for the study and to keep the research grounded in the data. The memos provided an opportunity to describe the participants and their moods, the classroom environment, my own personal feelings, questions about phenomena I did not quite understand, and it enabled me to do a great deal of reflective thinking. The memos also enabled me to remain focused and aware of the relationships among emerging categories.

Artifacts

The artifacts included student work and any handouts given during observations. Analysis of these artifacts enabled me to make inferences about the prospective teachers' mathematics knowledge, thinking, and understanding. These artifacts were also used in conjunction with interviews and observations to substantiate findings.

Procedure

The first two interviews were the same for each participant. A complete list of the interview protocol is located in appendix B. In the first interview, I focused on the prospective teachers' beliefs about the nature of mathematics and their self-perceived mathematical ability. The second interview focused on beliefs about the context of the classroom; primarily their learning and their beliefs about whom they perceive as being most responsible for their learning of mathematics. I video-taped the third interview and focused on the participants' specific metacognitive behaviors, emotions, attitudes, and their use of mathematics knowledge during a problem-solving episode. The participants solved a non-routine mathematics problem (see appendix D). The problem chosen follows Schoenfeld's (1993) definition of a mathematical problem; that is, a task in which the student is interested and wishes to obtain a solution, and for which the student does not have a readily accessible mathematics means by which to achieve that resolution. However, the problem chosen was not so difficult that the participant could not provide details about their solution process. To investigate the processes involved during problem-solving, Ericsson and Simon's (1984) think-aloud protocol and technique were used during this interview.

I asked each participant to translate their thoughts into words, recite the translation aloud, and verbalize aloud all the steps that occur as he or she problem solve. When he/she failed to verbalize, for a specified period of time (i.e., 5 seconds), I used prompts such as "keep telling me what you are doing", "keep talking," or "say everything you are thinking and doing" (Montague & Applegate, 1993). As they were solving the problems, I asked the participant to verbalize what they were doing and why

they decided to do what they were doing. I noted all emotional behavior or gestures that the participants demonstrated. I also noted the extent to which each participant used strategies, procedures, conceptual knowledge, and multiple solution paths during the problem-solving process. This third interview was videotaped for response recall during the fourth interview, at which time I asked specific questions about their problem solution.

The fourth and final interview focused on the informants' affective, cognitive, and meta-cognitive responses from the videotaped think-aloud interview. To avoid interruptions during the problem-solving process, I conducted this interview no more than two days after each participant has their think-aloud videotaped interview. Each participant was asked questions as they reviewed their problem solution. I provided this opportunity so that the participants could express what they were thinking and feeling as they solved the mathematics problem. It also gave me an opportunity to verify my interpretations with respect to what I observed during the actual think-aloud interview. In addition, the participants had the opportunity to clarify their think-aloud responses, interpret their use of mathematical knowledge and problem-solving strategies.

Summary

This chapter provides a brief overview of the study, a description of the participants as well as data collection. The rationale for using qualitative methodology, and more importantly the choice of grounded theory methods was presented. I discussed the research procedures used during the study and established the trustworthiness of the research. In the next chapter, I discuss in detail my analysis of the data

CHAPTER FOUR

DATA ANALYSIS

According to Glaser and Strauss (1990), the constant comparison and coding of collected data is at the heart of GT. Creativity of researchers is an essential ingredient in coding collected data. Strauss and Corbin (1998) explained that "analysis procedures were designed not to be followed dogmatically but rather to be used creatively and flexibly by researchers as they deem appropriate" (p. 13) and each person must find a system that works best for him or her during the process of applying GT techniques for analyzing data. Charmaz (1983) holds that every researcher that uses the GT method will tend to develop his or her own variations of techniques. I coded my data using the coding methods suggested by Charmaz (2005) along with Strauss and Corbin's (1998) axial and selective coding. When using Strauss and Corbin's technique, I was not necessarily looking to discover meaning or truth inherent in the data but instead I recognized that both data and analysis is created from shared experiences and relationships with the participants (Charmaz, 2005).

The data represented participants' complex and varied views of the situation (Creswell, 2003). Some views or meanings were common and could be easily grouped or categorized, but others required further interactions, discussions, and negotiation. There were also times when I relied on my work related experiences or used the knowledge I gained from reviewing the literature to negotiate meaning. In this chapter, I explained my

approach to the constant comparative method, open and focused coding, axial, and selective coding. I describe and illustrate open, focused, and axial coding as applied in this study. I describe selective coding in the following chapter.

Merriam explained that, "without ongoing analysis, the data can be unfocused, repetitious, and overwhelming in the sheer volume of material that needs to be processed" (Merriam, 1998, p. 161). She emphasized that, data analysis during the collection process is both "parsimonious and illuminating" (p. 161). In this study, data collection and analysis occurred in alternating sequences on January 10, 2008 and proceeded through July 2008.

Open Coding

In this study, open coding consisted of reading line by line and labeling each line, sentence, or paragraph with a word or phrase that best captured the meaning of what participants were saying and doing during classroom discourse, interactions with their peers, and as they participated in learning and problem-solving. When open coding, I remained open to exploring whatever theoretical possibilities I could discern in the data (Charmaz, 2005). I labeled the participants' actions and interactions by writing the codes in the margins of the transcribed interviews, observations, and other documents. Some labels were given names in the terms used by the participants using "in vivo codes" (Strauss & Corbin, 1998, p. 105) in order to capture insights into their mathematics learning and problem-solving experiences. Other labels given to the concepts, ideas, or categories were negotiated socially and were formed because of my interactions and discussions with and among the participants.

As concepts began to emerge, they were compared for similarities and differences and grouped together by similar code phrases into named categories. The process is repetitive and ensures that the generation of categories and their properties actually emerge from the meanings participants give to an incident or situation (Charmaz, 2003; Creswell, 2005; Strauss & Corbin, 1998). As suggested by Charmaz (2005) during open coding, I asked questions such as "who or what was the participant talking about," "when did what they were talking about happen?", and "how were they saying this?" Asking these questions guided my analysis of interviews, observations, and provided insights about what kinds of data to collect next.

To give you an idea of how I coded during the coding process, I provide an illustration of open coding on an excerpt from two different participants. In Table 1, each line was coded using an open code that represented what the participants said and found problematic. I also coded incidents in my observational data. Open coding helped me identify implicit concerns, actions, and explicit statements from the participants. In doing so, I learned a lot about the participants' mathematics learning and problem-solving worlds. During my observations of the participants in the classroom, I compared what I observed with the knowledge and understanding I gained from initial open coding. In my first observation of Mandy in the classroom, I developed the code of "confidence in doing mathematics" because Mandy demonstrated mathematics confidence as she enthusiastically volunteered to explain and justify her solution to a problem at the board. Later, during an interview she confirmed my development of the code "confidence in doing mathematics" when discussing her mathematics ability she stated, "mathematics is my best subject... I've always been good at it."

Focused Coding

Focused coding (see Table 2) is the second phase of the coding process (Charmaz, 2005). After open coding the interviews and observations, I began to move across interviews and observations and compared the participants' interpretations, experiences,

Table 1

Example of open coding of excerpts from an interview with Mandy and Tanya

Excerpt	Open Code
Excerpt 1 Mandy	
yeah, I love math and it's something	Loves math
I've always been good at, but it's	Good at math
something I have to work at too. Math is	Work at being good at math
everywhere; it is	Math is everywhere
everything in real life.	Math is real-life
What I find difficult about math is functions.	Functions are difficult
Excerpt 2 Tanya	
Math is about discovering how to use	Math is discovering
theories and concepts. It's about being	
creative and seeing patterns of similarities	Math is creative, patterns
and differences. It's about understanding	Math involves understanding
Yeah, I like math but it does not come	Likes math
natural or easy for me. I have to work hard	Math is not easy, hard work

at practicing problems over and over again	Practice math
in order to get it. It's not my best subject.	Not my best subject

Table 2

Example of focused coding of excerpts from an interview with Mandy and Tanya

Open Code	Focused Code
Excerpt 1	Mandy
Loves math	Feelings associated with doing math
Good at math	Math confidence
Have to work at being good at math	Attitude about math
Math is everywhere, real-life	Belief about nature of math
Functions are difficult	Math difficulty
Excerpt 2	Tanya
Math is about discovering, creative, patterns	Belief about nature of math
Math involves understanding	Belief about understanding math
Likes math	Feelings associated with doing math
Have to work at being good at math	Attitude about math
Math is not easy, hard work	Attitude about math
Practice math to get it	Attitudes about math
Not my best subject	Beliefs about math ability

and actions (Charmaz, 2005). I grouped and categorized similar ideas and concepts under a common category. As data collection and analysis progressed, when I discovered another idea or concept that I identified as sharing a common characteristic from an earlier idea or concept, I gave it the same name (Strauss & Corbin, 1998).

During focused coding, I used the most significant and frequent codes from earlier coding to sift through and compare the data (Charmaz, 2005). I made decisions about which of the focused codes were most pertinent in answering the research questions and which ones made the most sense to categorize the data. As a result, some codes were dropped. The focused codes I found most useful in Table 2 were “feelings associated with doing mathematics”, “beliefs about mathematics”, and “attitudes toward mathematics”. I compared incidents in which participants exhibited or expressed emotional behavior during the problem-solving process with those in which they had not exhibited such behavior. I considered the intensity and impact of the participants’ emotional behavior. The focused code of “feelings associated with doing mathematics” was developed into a category as well as “beliefs about mathematics.” I grouped and categorized similar ideas and concepts under a common category. Coding, in general, is not a linear process. I constantly returned to earlier experiences that had been overlooked or were not explicit. Some terms were changed slightly to make them more concise in later stages. The changes were made based on further interactions with the participants during subsequent interviews and observations.

Axial Coding

I applied axial coding to the categories after the development of categories. It relates categories to subcategories and specifies the properties and dimensions of a category (Charmaz, 2005). For example, the category of “feelings associated with doing mathematics” has dimensions: positive emotions associated with doing mathematics and negative emotions associated with doing mathematics. Participants demonstrate positive or negative emotions in different instances. The participants demonstrated negative emotional behavior when they were unable to obtain a problem solution or they were unable to justify a procedure used to obtain a problem solution. The participants were happy, prideful, and confident when they were able to understand challenging mathematics concepts and justify or explain their problem solutions. Other categories had similar dimensions and properties. For example, “beliefs about mathematics” was subcategorized into beliefs about the nature of mathematics, self-mathematical ability, learning mathematics, and teaching mathematics. The process of axial coding enabled me to elaborate on a category through the development of subcategories (Strauss & Corbin, 1998). The process enabled me to link categories to subcategories and to make sense of the data.

When applying axial coding, I re-read the transcripts without stopping to look at the codes, all the while paying close attention to what the participants were talking about and why they described what was happening as they did during the interviews or observations. By re-reading the transcriptions in their entirety, instead of reviewing only the coded sections of data, I was able to learn and understand the participants’ experiences and to developed ideas about the relationships among the categories and

subcategories that represented their experiences. I also began memo-writing to explore in-depth my ideas about the categories and how they represented what the participants were telling me about their problem-solving experiences. The categories developed from the focused codes were further developed using axial coding in the following way (see Table 3).

Table 3

Example of axial coding from the focused coding of excerpts from Mandy and Tanya's interview

Category	Axial Code
Feelings associated with doing math	(+/-) Math Emotions
Belief about the nature of math	Mathematics-related belief
Beliefs about learning/understanding math	Mathematics-related belief
Self-Efficacy math belief	Mathematics-related belief
Attitudes toward math	(+/-) Attitudes toward math

Table 4 illustrates steps in the process of developing a main category using excerpts from Mandy and Tanya's initial interview. Other main categories were developed using the same coding process. Main categories are considered to be of a higher, more abstract order than are open or initial codes (Strauss & Corbin, 1998). I explored possible relationships between categories and subcategories and how they related, influenced, or contradicted each other. This was done mostly through the drawing of diagrams and flowcharts throughout the research process and writing memos that described and discussed the categories and their relationships. Subsequent questioning,

observations, and interactions with the participants were conducted to get a better understanding of the relationships among the categories. Affective behavior such as beliefs, emotional behavior, and attitudes toward mathematics occurred in problem-solving and mathematics learning situations. For example, the participants' beliefs about their own mathematical ability, the nature of mathematics, mathematics teaching, and mathematics learning was manifested in their emotional behavior and attitudes toward mathematics and the problem-solving process.

Table 4

Example of developing a major category from the axial coding of excerpts of Mandy and Tanya's interview

Axial Code	Major Category
(+/-)Math Emotions	Affective Behavior
Mathematics-related beliefs	Affective Behavior
(+/-)Attitudes toward math	Affective Behavior

According to Charmaz (2005) and Strauss and Corbin (1998), the categories are saturated as data becomes redundant when one piece is compared to another and no further categories or dimensions and properties emerge from the data or new information does not add much more to the explanation of phenomena. I was uncertain as to whether saturation had occurred so I returned to collect further data and to make further comparisons. I maintained a reflective journal throughout this process and continued further interaction with the participants, in each case striving to understand the meaning the participants were giving to their experiences.

The constant comparative method of grounded theory means that this process was not linear (Strauss & Corbin, 1998).   I continued to return to the words of the participants previously collected and analyzed data to influence future data collection and analysis until no new insights and no new properties were revealed for the major categories. Memo writing (Charmaz, 2000) throughout this process enabled me to be reflexive about the research process and track how data analysis and interpretation emerged throughout the process. My memos focused on reflecting on the research process such as how I was developing rapport with the participants, any difficulty I as having with interpretations, and how I decided whom to include in the study next, and when saturation might be reached.

Additional participants were selected to explore themes as they emerged in the study and as analytic interpretations focused further data collection. As a way of verifying and clarifying, I sought differentiation among the participants experiences by deliberately seeking out participants whose experiences may not have fit with what I viewed as the emerging theory. The common thread woven among the participants was their goal to become a secondary mathematics teacher. Their interpretations of what they were experiencing during problem-solving and mathematics learning were quite similar. But there were differences in their perceptions of their mathematics learning and problem-solving abilities.

As a way of gaining a better understanding of the participants similarities and in differentiating among the participants’ experiences, a cross comparison of the participants’ experiences was completed. In doing so, major categories emerged from further interaction with the participants and exploration of the categories developed

during axial coding. The major categories represented the participants' mathematics-related beliefs, affective behavior, common heuristics used during problem-solving, their mathematical conceptual and procedural knowledge, and metacognitive behavior in the context of mathematics learning and problem solving. I returned to the data to crystallize my understanding of what the participants were telling me about what problem-solving and mathematics learning experiences were like for them as they participated in a course focused on developing their deep knowledge and understanding of school mathematics. I also returned to the data to crystallize further my understanding of their mathematics-related beliefs, affective behavior, metacognitive behavior, and mathematical knowledge.

I analyzed the data and worked toward understanding their mathematics-related beliefs, affective behavior, and mathematical behavior during problem-solving from the participants' viewpoint. I based my interpretations on what the participants were telling me about their problem-solving experiences. I used this information to define the major categories (Charmaz, 2006) that represented the participants' mathematics-related beliefs, positive and negative attitudes/emotions, use of problem-solving strategies, and mathematical cognition demonstrated during the mathematics learning and problem-solving process. My exploration of the categories indicated that the participants' attitudes and emotional behavior was mostly positive, with some negative attitudes and emotions associated with learning mathematics, the problem-solving process, and past learning experiences.

The data analyzed during open, focused, and axial coding revealed that their beliefs about teaching mathematics contributed to their positive attitude and motivation to, not only learn mathematics, but understand it deeply. Knowledge of strategies,

successful use of strategies, and their access to conceptual knowledge and understanding resulted in positive emotional behavior and often played a role in managing negative emotional behavior. It appeared that their mathematics-related beliefs were linked to all other major categories. Their attitudes appeared to be a manifestation of their beliefs about teaching and learning mathematics. Their emotions represented the high expectations they held for themselves based on their commitment to understanding mathematics deeply. Emotional behavior was linked to their beliefs about mathematics difficulty, their ability to solve math problems successfully, learning, and teaching mathematics.

The participants used a variety of strategies to assist with their efforts to understand mathematics and successfully solve mathematics problems. Strategies, such as relating the problem to a similar or familiar problem, solving an easier problem, breaking the problem into parts, and using a diagram or picture during problem-solving were linked to beliefs about what it means to learn and understand mathematics, to conceptual and procedural knowledge, and to their emotional behavior. Knowing when and how to use a specific strategy, identified in the literature as metacognition, appeared to keep emotional behavior at a minimum.

After identifying the major categories and examining and identifying initial relationships, I returned to the raw data to re-read several interviews, observations, and artifacts to help stimulate my thinking and to search for examples of data not matching the defined relationships. I also began to further integrate the major categories and refine my understanding of what the participants were telling me about their experiences learning mathematics and what problem-solving is like for them. I searched for clues in

the data that could explain how the six major categories might further relate to each other by "moving from description to conceptualization" (Strauss and Corbin, 1998, p.149). This process was not an objective process. The preliminary results or initial theory began to emerge during the memo writing process as I described the relationships among the categories for the participants.

My goal of returning to the original data was to determine what aspects of their problem-solving and mathematics learning experiences were most important to them. I asked myself how I could best represent what they shared with me about their experiences in a way that could help others understand what problem-solving and learning mathematics was like for them and how their experiences interrelated. Further conceptualization of the data resulted in the integration of their mathematics-related beliefs, emotional/attitudinal behavior, use of problem-solving strategies, metacognition, and mathematical knowledge and understanding, which lead to the emergence of four final four major categories associated with the participant's mathematics learning, understanding, and problem-solving processes, are:

1. Affective Behavior
2. Heuristics
3. Metacognition
4. Mathematical Cognition

These categories will be discussed in detail along with an explanation of how they interact during mathematical learning, understanding, and problem-solving in Chapter 5.

Summary

To assist with the data analysis process, I utilized GT methodology to analyze the

data by applying open and axial coding on the participants' individual interviews, observations, and artifacts, which enabled me to develop categories and reach theoretical saturation. Open and axial coding are not sequential acts, but instead are conducted together. I constantly moved between labeling data, constructing categories that describe the experiences of the participants in the study to relating those categories in order to form a more precise explanation about what is happening.

Coding of the data is a dynamic and fluid process. Initially, five categories were developed from the participants' interviews, observations, and artifacts describing their problem-solving behavior. Then through axial coding several categories were integrated into four main categories. Categories can be developed on several levels, and insights about how categories relate occur throughout the axial coding process. I described and illustrated how the categories at each level were developed and how they related to each other. I eventually moved away from describing the categories into conceptualization/interpretation, which lead to the construction of the final three main categories. From the participants' point of view and based on the meanings they gave their problem-solving experiences these categories described their mathematical behavior as they learned mathematics in a problem-solving context. In chapter five, I discussed the results of the study and described in detail the final major categories and the role they play in constructing a model representing the interrelationships among the participants' affective behavior, heuristics, metacognition, and mathematical cognition.

CHAPTER 5

RESULTS

The main purpose of this study is understand the interaction of prospective secondary mathematics teachers' mathematical behavior and experiences as they participated in an undergraduate mathematics course that focused on developing their deep understanding of school mathematics. In doing so, the study also explored the mathematics-related beliefs of prospective secondary mathematics teachers enrolled in a problem-solving mathematics course. I also determined what affective and metacognitive behaviors were demonstrated by prospective teachers as they engaged in mathematics learning and problem-solving. Finally, the study explored what mathematics knowledge and understanding was accessed by prospective teachers as they engaged in mathematics problem-solving.

In this chapter, I describe the results of the analysis of the interviews, observations, and participant artifacts. The results of the analysis are presented using the participants' direct voices from the interview transcripts, discussion of the observations and participant artifacts, and drawing on the literature related to mathematics-related beliefs, affective behaviors, metacognition, and mathematics knowledge. These major categories contained several subcategories that emerged from my interaction with the participants during the course of the study. Each major category, along with its subcategories, are discussed and related to existing literature in order to elucidate or place each major category within a broader context.

Affective Dimensions

Within the mathematics education literature, *affective dimension* is a term used to represent all of the feelings that individuals have about mathematics learning (McLeod, 1988). Researchers (Lester, et al., 1989; McLeod, 1992; Schoenfeld, 1992) suggest that affective dimensions include beliefs, attitudes, and emotions and they all have a powerful influence on the behavior of the problem solver. Beliefs are cognitive, more so than attitudes or emotions (McLeod, 1985). Beliefs are considered to be deep-seated convictions or internal representations that the believer attributes to truth and validity (McLeod, 1985; 1992; Schoenfeld, 1989; 1992). In this study, I examined the participants' beliefs about mathematics, mathematics learning and teaching, self-efficacy, and about the learning environment. I also noted their emotions and attitudes as they engaged in mathematics learning and problem-solving.

Schoenfeld (1992) suggested that although emotions are more evident than beliefs during problem-solving, beliefs play an important role in shaping cognition and the decisions that are made during the problem-solving process. McLeod (1992) explained that "the role of beliefs is central in the development of attitudinal and emotional responses to mathematics" (p. 579). In this study, within affective dimensions, there were several subcategories, namely mathematics-related beliefs, emotions, and attitudes. In an effort to tell the participants' story about the role affective dimensions play in their problem-solving and mathematics learning experiences, I discussed each of the subcategories using the participants' individual voices, a synthesis of their combined voices, with particular references to literature related to the major and subcategories.

Mathematics-Related Beliefs

Teachers' beliefs about the nature of mathematics are conscious or subconscious beliefs, concepts, meanings, rules, mental images, and preferences concerning mathematics as a discipline that affects their behavior (Ernest, 1989; Schoenfeld, 1992). Thompson (1992) explained that it is important to consider at least two kinds of beliefs. One is beliefs about mathematics and the other is beliefs about teaching and learning mathematics. Identifying and understanding the beliefs about the nature of mathematics is important because as Thompson (1992) suggested, these beliefs often have a crucial role in influencing learning experiences and teaching practices.

Beliefs about the nature of mathematics

Several beliefs about the nature of mathematics surfaced through analysis. The participants' beliefs about the nature of mathematics referred to aspects of mathematics creativity, content, problem-solving, and its usefulness in everyday life. The following are examples of their responses indicating their conception of mathematics.

> Math is about discovering how to use theories and concepts. It's about being creative and seeing patterns of similarities and differences. It's about understanding how to solve math problems (comment by Tanya during interview 1, January 18).
>
> Mathematics is the study of relationships among numbers, quantities, shapes, measurements and it plays a role in almost all aspects of my life (comment by Mandy during interview 1, February 5).
>
> Mathematics is a lifestyle. It's all around us; a part of our real-life. It's using numbers, shapes, variables, problem-solving, building concepts in our everyday lives, trying different things to see if you can come up with solutions to problems (comment by Cindy during interview 1, January 28).
>
> Mathematics is the study of numbers and their operations (comment by Mark during interview 1, January 30).

In many ways, the participants' views on the nature of mathematics parallel the views represented in the NCTM *Standards* document. The participants used terms such as discovering, understanding, creative, patterns, relationships, experimenting, and real-life to express their conceptions of mathematics. I was somewhat surprised by their responses because some of the findings reported in the literature suggest that some teachers and prospective teachers see mathematics as a fixed body of knowledge, a collection of procedures, rules, formulas, and theorems that are disconnected (Ball, 1990; 2001; 2003; Nyaumwe, 2004; Thompson, 1992). So, it was heartening to see that the participants' conceptions of mathematics parallels the views of the NCTM *Standards, which* are based on best practices, research on the teaching and learning of mathematics, and are well respected within the mathematics education community as a solid curriculum program for learning mathematics.

Beliefs about learning mathematics

Participants' views on learning mathematics appear to be a manifestation of their beliefs about the nature of mathematics. They viewed mathematics as is a lifestyle, the study of patterns and relationships, numbers, operations, and problem-solving. Because the participants held beliefs about mathematics that do not include the memorization of rules, formulas, and procedures, they approach learning mathematics from the perspective of understanding mathematics.

Schoenfeld (1985a; 1989a, b) found that students typically believe that there is only one way to solve any math problem, math problems can be solved quickly and in isolation, and solutions do not have to make sense. This was not the case for these participants. For them, learning mathematics for understanding has little to do with

remembering rules, formulas, or finding a single solution path. Instead, learning mathematics includes understanding mathematics. Tanya described her mathematics learning experiences as challenging and difficult. In fact, this was her second time enrolling in the course. For her learning mathematics was a struggle partly because she never had an understanding of mathematics that would enable her to explain her mathematical reasoning or the methods behind the procedures used in problem-solving.

Cindy, a secondary prospective teacher, can be described as one who is very confident in her ability to learn and do mathematics. She boasted about her ability to memorize formulas and procedures say, “I was always good at memorizing, that’s why I did so good in my statistics courses where all you had to do was remember the formulas or use your formula sheet.” Factoring, completing the square, applying the quadratic formula were mathematical procedures she performed with little or no difficulty. When it came time to explain the reasoning underlying these mathematical procedures she was unable to do so. Cindy, along with other participants, attributed her inability to explain why a procedure worked to a lack of mathematics understanding. For these participants, understanding mathematics involved knowing how to explain one’s solution process, solving problems in more than one way, and explaining why mathematical formulas and procedures work the way they do. Cindy explained that,

> Math is learned when you are solving problems and you understand why something works the way it does and you can explain it to other people (comment by Cindy during interview 1, January 28).

This mathematics learning belief was not unique to Cindy. Mark and Mandy expressed similar beliefs when asked about their mathematics beliefs.

> Learning math and understanding math involves explaining why; not just how it's done, but why it's done. In this class, I'm always trying to remember the rules or the steps for how to do certain things when now I realize how important it is for me to understand. It's [trying to remember the rules] holding me back. I didn't ever really understand it [quadratic formula] now I'm being forced to understand how it came about. I'm finding out why (comment by Mark during interview 1, January 30).

> The goal of learning math is to understand why something is done instead of just memorizing formulas (comment by Mandy during interview 1, February 5).

Their beliefs about learning mathematics parallel Hiebert's (1997) view that mathematics is not learned when rules are memorized for the purpose of applying a paper and pencil procedure to solve a problem. According to Hiebert et al. (1997), actively engaging in solving mathematics problems contributes to mathematics learning and mathematics understanding.

The participants referred to mathematics as something you "do." They did not view mathematics as necessarily carrying out procedures or calculations during the process of mathematics learning or problem-solving. For them, "doing" mathematics included "playing" with mathematical ideas that could eventually lead to problem solutions, explaining or justifying the reasoning underlying problem solutions, examining the methods their peers used to solve problems, and receiving support and encouragement from the instructor were elements that played a crucial role in their mathematics learning and understanding. Mark and Cindy explained that,

> Mathematics is a player's sport. In order to learn mathematics you have to play the game, you have to do it (comment by Mark during interview 2, February 6).

> I need hands on learning, a chance to kinda play around with the math and try a lot of different things; a chance to see how my classmates are working the same problem (comment by Cindy during interview 2, February 8).

For them *playing* with the mathematics included trying several methods, strategies, or solution paths during the problem-solving process without fear of “getting the problem wrong.” Playing with the mathematics created a less stressful, less tense learning environment. The participants seemed to view working through a problem and getting it wrong as a normal part of the process of understanding mathematics. They wanted opportunities to work through the mathematics without being evaluated for a grade. Tanya and Mandy talk about getting the wrong solution during the solution process,

> I don’t always get it the first time around and when this happens I go back over the problem to see where I went wrong and I see how other students have worked the problem…I mean, just going back over seeing where you went wrong can help you understand (comment by Tanya during interview 2, February 13).

> Understanding mathematics is not about getting the right answer. Getting an answer is not learning (comment by Cindy during interview 2, February 8).

> Understanding the why is an important part of learning mathematics. Knowing formulas is not going to help you understand (comment by Mandy during an observation, January 17).

Brownell (1946, p. 121) explained that “a problem is not necessarily solved because the correct answer has been made. A problem is not truly solved unless the learner understands what he has done and knows why his actions were appropriate” The participants’ espoused beliefs that learning and understanding mathematics included “getting it wrong,” looking back through the process, and rethinking the solution. In the literature DeBellis and Goldin (1999) identified a learner’s desire to want to understand or justify their mathematical reasoning as opposed to getting the right answer, as mathematics integrity.

Beliefs about teaching mathematics

Overall, the participants' beliefs about teaching could be explained by their own experiences as students along with their mathematics ability and achievement. They look on the instruction they received during their own education as either the correct way or incorrect way that mathematics should be taught. The participants expressed an awareness of the inadequacy of their past learning experiences and did not want to teach in the manner they were taught during middle school, high school, and in some college courses. It appeared that if the method of teaching was successful for them but was based primarily on their memorizing theorems, rules, formulas, and procedures and less on understanding the underlying mathematics, the participants rejected the method as being good teaching.

Mark, a prospective secondary teacher, is a sports fanatic whose goal is to both coach football and teach mathematics. His best high school mathematics learning experience occurred in Coach Carter's (a pseudonym) class. Mark's dreamed of becoming an inspiring teacher like Coach Carter, except for one thing—he wants his students to understand why mathematics procedures work the way they do. Mark explained that,

> Mathematics is a player's sport, not something to be watched. Mathematics has to be taught so that the students not only learn how to do the steps to solve the problems, but why those steps were done the way they were. I'll probably end up taking what good teachers I've had in math and end up teaching it the way they did it. I would make it as comfortable as I can and still make sure they are really learning and understanding the math (comment by Mark during interview 2, February 6).

Mandy, a prospective middle-grades mathematics teacher and the mother of two school-aged children, expressed her beliefs about teaching with respect to her own learning. She explained that,

> I believe that if you are going to be a good math teacher you have to be able to explain the concepts you are teaching in more than one way. People learn in different ways and when a teacher only explains it one way, I don't always understand that one way. I'm going to use manipulatives when I teach some of the math concepts to my students. This helped me to understand a lot of things I already knew but didn't really understand before. It [using manipulatives] gave me a way to think about explaining things in a different way (comment by Mandy during interview 2, February 15).

Cindy expressed that "mathematics is not easy for me" emphasizing that she puts forth a great deal of effort to understand mathematics but she believes that the teacher plays a critical role in her mathematics learning. In an interview, she stated

> I believe that a good teacher explains the math concepts in more than one way. They have to know more than one way to do a problem. I have a hard time understanding math when the teacher explains the math in just one way and I don't get that way. I also need hands-on learning experiences, like working with manipulatives or using a picture or diagram to solve a problem. This has always worked for me. This is my idea of a good teacher and this is the kind of teacher I want to be (comment by Cindy, February 8).

The participants' beliefs about mathematics do not associate teaching with telling. Instead, for them, teaching mathematics involves explaining mathematical concepts and procedures in multiple ways. Their views about what constitutes good teaching is a reflection of what Thompson (1999) refers to as conceptual orientations. Thompson et al. (1994) used the term "orientation" to refer to teacher's views about mathematics and mathematics teaching. Teachers with a conceptual orientation are interested in directing students to understand relationships among mathematical concepts and to explain and justify the reasoning behind their solutions. These participants espouse beliefs that are aligned with the conceptual orientation Thompsons described.

Beliefs about the learning environment

The participants' beliefs about their learning environment are demonstrated in their comments about teaching and learning. For example, Cindy and Mark's comments reflected their beliefs about the mathematics classroom environment.

> In order to learn mathematics you have to play the game, you have to do it (comment by Mark during interview 2, February 6).
>
> Learning math and understanding math involves explaining (comment by Mark, interview 1, January 30).
>
> I need hands on learning, a chance to kinda play around with the math and try a lot of different things; a chance to see how my classmates are working the same problem. I need for them [the instructor] to give me a chance to work some of the problems and kinda be there to answer questions. I don't need for them to work all the problems while I watch them (comment by Cindy during interview 2, February 8).
>
> Math is learned when you are solving problems (comment by Cindy during interview 1, January 28).

Mandy and Tanya made similar comments that reflected their views about the learning environment.

> I'm going to use manipulatives when I teach some of the math concepts to my students (comment by Mandy, interview 2, February 15).
>
> I don't always get it the first time around and when this happens I go back over the problem to see where I went wrong and I see how other students have worked the problem (comment by Tanya, interview 2, February 13).

Participants envisioned a mathematics learning environment that engaged students in "doing" mathematics. I asked the participants to define what "doing" mathematics would be like for them. A pattern emerged representing that for them, "doing" mathematics included "playing" with mathematical ideas that could lead to problem solutions, using manipulatives to assist in the understanding of mathematical concepts, examining the solution methods used by others enrolled in the course, and receiving support and

encouragement from the instructor. It seemed as though this kind of learning environment played a critical role in their mathematics learning and understanding and problem-solving success.

Self-Efficacy Beliefs

Bandura (1986) described self-efficacy as an individual's beliefs and judgments about one's self and their ability to execute an action that will lead to a desired result or outcome. According to Bandura, an individual's self-efficacy greatly influenced one's behavior and the choices they make and where they direct their actions. In this study, self-efficacy is described as the participants' beliefs about their ability to do mathematics, learn mathematics, and understand mathematics. I used interview protocols and observations to learn more about the participants and gain an understanding of their perceived self-efficacy and the impact it has on their mathematics learning and the problem-solving process.

All but one of the participants held the belief that mathematics is their best subject. Unlike the other participants, Tanya discussed the difficulty and challenges she encounters when learning mathematics and she confessed that mathematics is not her best subject. She was very emotional when recalling some of the difficulties she has encountered in other mathematics courses. I know of no one who has shown Tanya's level of determination in mathematics learning, often spending numerous hours practicing or re-working mathematics problems. She finds mathematics challenging, but explains,

> Yeah, I like math but it does not come natural or easy for me. I have to work hard at practicing problems over and over again in order to get it (interview 1, January 18).

I asked her to explain and she replied,

> I don't usually get it the first time around. I mean, I have to go back over what we do in class; I spend a lot of time with it. (Interview 1, January 18)

Unlike Tanya, Mandy is confident in her mathematics ability. Teaching mathematics is a second career for Mandy. She holds a bachelors degree in business and has worked in the business industry for 15 years. She explained that mathematics is her "absolute" best subject. During a classroom observation, Mandy demonstrated her mathematical confidence during an in-class mathematics task assignment. The instructor asked the entire class if division by zero is possible and to justify their answer. At their desk, some of the students worked independently on the problem and others worked in pairs. As the students were working at their desk, I quietly circulated the room to listen to their dialogue and monitor their progress. I noticed that Mandy worked through the solution correctly, but was hesitant in presenting her solution at the board. I was surprised by her lack of confidence to present based on her belief that mathematics is her "absolute" best subject. She eventually volunteered to work through the solution process at the board. At each step she explained her solution process, but near the end of her explanation she appeared a bit fuzzy. In an interview that took place a few days later, she explained what it was like for her to work the problem at the board,

> I knew the answer—you can't divide by 0, I've heard it a million times. I was a little unsure about how to go about explaining why division by zero can't be. I mean, I know that you can't divide by zero, but I had to be sure I could show why (Interview 1, February, 12).

According to DeBellis and Goldin (1999), Mandy's response reflects her mathematics integrity. She was not only concerned about working through the problem and getting the correct solution, Mandy's desire was justifying her solution.

Mark expressed his confidence in his mathematics ability when he talked about the mathematics difficulty associated with solving problems.

> Math is absolutely my best subject and I'm good at it. I believe there are no hard math problems, just long ones. If I spend enough time working on a problem, I'm going to eventually get it (Interview 1, January 30).

During this interview, Mark also talked about the importance of not only knowing how to apply a mathematics procedure when solving a problem, but understanding why the procedure works. His mathematics integrity is reflected when he talked about deriving the quadratic formula.

> I've always known how to use the quadratic formula but I never knew why or exactly where it came from. It was just a formula. It wasn't until we were asked to derive it in this class that I finally got in (interview 2, February 6).

I must admit, I was quite surprised by Mark's comment. Deriving the quadratic formula is an activity in which I engage my college algebra students each semester. So it was a bit surprising that at Mark's level of upper undergraduate mathematics study, he had never derived the quadratic formula. Cindy and Mandy openly discussed their lack of confidence as it relates to geometry, proofs, and functions.

> For me, the most challenging math is geometry. I have a really had time writing geometry proofs for theorems. I'm just not good at it (comment by Cindy during interview 2, February 8).

> I'm not good at functions for some reason. It's something that I've always had a problem with comment by Mandy during Interview 2, February 15).

The participants' beliefs about their mathematical ability are a reflection of their beliefs about mathematics learning and teaching. They demonstrate persistence and motivation in mathematics learning and problem solving situations. I asked each participant to describe what it was like for him or her to fail to find a satisfactory problem solution or

understand a mathematical concept either during or after a lesson. Overwhelmingly, the participants' responses indicated persistence.

> I don't give up easily (comment by Mandy during interview 2, February 6).

> I don't always get it the first time around and when this happens I go back over the problem (comment by Tanya during interview 2, February 13).

> When I get stuck on a problem or I can't figure it out right then, I'll take a break from it and go back to it later. But, one thing I don't usually do is give up (comment by Tanya during Interview 2, February 13).

The participants' espoused self-efficacy beliefs are represented by their effort, persistence, and motivation in problem-solving and mathematics learning situations. For these participants, failure is due to the lack of effort and not the lack of ability. Therefore, they have a willingness to increase their efforts and persist in an attempt to learn and understand mathematics and to reach their problem-solving goals.

Emotions

As the participants talked about what it means to understand mathematics, they refer to "getting it wrong" as being an important part of the learning process. Getting the wrong solution, however, is not without its consequences. When the participants get the wrong solution or are unable to obtain a solution during the problem-solving process they experienced disappointment, frustration, and embarrassment. The participants talked about this extensively during their interviews and I witnessed both positive and negative emotions during my observations of them during class and the think-aloud problem-solving interview. Mark and Cindy talked about the anxiety and frustration they often experienced when solving problems and learning mathematics.

> There are times when the problems we do in this class frustrate me; like, only seeing one way to work through a problem and getting stuck and not being able to finish. I'm frustrated because I can't find a way to solve the problem using a

> different way when I get stuck (comment by Cindy during interview 2, February 8).

Also during this interview, Cindy explained that her frustration positively impacts her learning and mathematics understanding. Again, I was very surprised by Cindy's comment because frustration is not generally associated with positive thinking (Mandler, 1989). She explained that,

> Some of the math problems we do frustrate me, but at the same time I'm challenged and when I'm challenged I learn. I wouldn't learn much if every problem I came across I already knew how to do. I mean, that wouldn't be learning anyway.

Mark had a similar comment during an interview,

> I like to problem-solve; but, not understanding why something works, like knowing how but not why, aggravates and frustrates me; … it makes me want to work harder to understand it (Interview 2, February 6).

An interruption in a problem-solver's solution plan has been reported to cause frustration, anger, and occasionally problem abandonment (Mandler, 1989). The solver may reduce their frustration by giving up on the problem or finding a new plan that might lead to success (McLeod, 1989). Interested in knowing how the participants handled themselves when unable to complete the solution process, I asked the participants to discuss their experiences with unsuccessful attempts at problem-solving. Tanya and Mandy responded with the following,

> When I get stuck on a problem or I can't figure it out right then, I'll take a break from it and go back to it later. But, one thing I don't usually do is give up. Like the time I could not get the polygons problem. Call me a nerd, but I thought about the problem when I was driving home in traffic trying to think about where I went wrong. I got home, and later went back over it again and finally figured out what I did wrong and reworked the problem (comment by Tanya during interview 2, February 13).

> I don't give up easily. I feel like I have to know what I'm doing because one day I'm going to be teaching math. Sure I get upset, but even if I get it wrong or I

> don't know what I'm doing, I still need to know what I did wrong. If I can't solve a problem I don't mind going to someone else in the class or the professor to get help so that I can get through it; but, no I don't usually give up (comment by Mandy during interview 2, February 6).

Their concern and commitment to understanding mathematics can be thought of as a reflection of their mathematics intimacy. DeBellis and Goldin (1999) defined mathematics intimacy as the relationship between the learner and mathematics that connects with their sense of and value of self. A learner who is committed to working on a problem until there is a sense of satisfaction that a solution or understanding is achieved is considered to have intimacy with the mathematics. As prospective teachers, these participants have a desire to understand mathematics in a way that would enable them to justify their mathematical reasoning and explain their solutions because they believe it is a crucial part of becoming an effective teacher.

During the think-aloud follow-up interview, the participants also demonstrated emotional behavior when unable to find a correct solution to a mathematics problem or they are unable to determine the error in their mathematical solution process. Their emotional reactions include nail biting, finger and pencil tapping, sighing, and in some cases problem abandonment. In my observation of Tanya during her problem-solving interview, after working on the problem for forty-five minutes and using a number of different solution methods she eventually abandoned the problem saying,

> Okay, I've tried everything I know but there is something I'm missing; not sure what. This is really upsetting because I've seen this problem before or one like it and for some reason it's not working out the way I remember. I know there's probably something small I'm doing wrong, but I just can't see it right now. (comment during think-aloud, March 12)

With respect to this specific mathematics problem, Tanya experienced mathematics intimacy. The time and effort she spent working toward a solution was a reflection of her

concern about, and commitment to solving the problem. Tanya demonstrated disappointment in her inability to successfully solve the problem when she explained,

> When I first read the problem, I thought I knew I could do it because I had done a similar problem on the board earlier in the semester and got it wrong. I spent time going over that problem to make sure I understood it. So, it's sort of disappointing that I couldn't get it this time either. (follow-up interview 4, March 13)

During my observation of Mark's problem-solving episode, there were moments when he demonstrated confidence, frustration, and satisfaction. When Mark read the problem he smiled and said, "Okay, I can do this." This was an expression of his confidence and motivation. As he continued to work through the problem he reached a point where he recognized a pattern but was unable to translate it into a formula. It is at this point that he demonstrated frustration. He expressed his frustration by resting his chin in his hands, taking a long pause, sighing, and eventually scratching through all previous work. He carefully reworks the problem, looks over what he has done, frowns, clinches his lips between his teeth, takes a long pause, taps his pencil on the desk and says, "Okay, I see where I went wrong." I sensed a moment of relief and satisfaction. He confirmed my sense with a smile and said, "I think I've got it now." In the follow-up interview, I asked him to explain his feelings, mathematical thinking, and actions during the think-aloud episode. He explained,

> I was a little frustrated because I knew how to do the problem, but for some reason it wasn't working out. I mean, I could clearly see a pattern, but the numbers weren't working out. It was something simple I was doing wrong. So, I went back over it again and realized that I had made one little mistake with the number of sides. After, finding the formula and trying it out I knew I had it right and that's when I started to feel better. I'm really glad I was able to figure it out (March 21).

During several classroom observations, I observed very little negative emotional reactions. Instead, there were numerous moments of excitement, enthusiasm, and pride

during their classroom participation. Although some of the participants were unable to complete assigned homework problems successfully, they did not demonstrate sadness or disappointment. I asked the participants to describe what homework completion, class work, and the classroom environment was like for them. The general emerging theme centered around the following comment provided by Mark during an interview, “the mathematics is sometimes challenging, but we have opportunities to (a) ask questions, (b) compare our solutions to others in the class, (c) make mistakes without being judged or graded, and (d) we work problems both independently and cooperatively with others.” The participants described the learning environment in the following ways:

> I feel like the professor is one of us; I really do. We ask questions and she answers with a question and we have to try to figure things out on our own or with our classmates (comment by Cindy during interview 2, February 8).

> I know the professor knows the answers to all of the problems we work on, but she doesn’t give us answers. Like the time we had the question about the pick up sticks game and instead of giving us the answer we actually played the game to see if we could discover a winning strategy (comment by Mandy during interview 2, February 15).

> I am a perfectionist and I like to know that I’m right before I go to the board to explain, but I don’t necessarily feel like I have to do that in here because there’s always someone who can kinda help you figure out where you went wrong. So I’m a lot better about going up and explaining even when I’m unsure about whether it’s right (comment by Tanya during interview 2, February 15).

> At first I thought this class was a push over. But then I realized I could do the math but couldn’t explain why the math worked. So that was embarrassing at first. But now I’m beginning to think that it’s not such a push over after all and I’m finally beginning to understand the why’s behind what I’ve been doing all these years and I guess I do feel pretty good about that (comment by Mark during interview 2, February 6).

During the second interview and two days before the first exam I asked the participants to talk about their experiences with test anxiety. Tanya wrote about her

struggles with test anxiety on the background information form and discussed it in detail during the interview.

> I have test anxiety or math…something like that; anyway, I know this. But I like math, it takes a while for me to get it, I have to think things through a lot longer than most of the people in this class, and it's sometimes painful. I have to put a lot of effort into it, but I know I can do it; eventually I do get it, I just have to work harder to get it.

For Tanya, math anxiety does not appear to hinder her goal to become a middle-grades mathematics teacher. As the study progressed, I became inspired by her commitment, effort, and persistence to stick with the program in light of the mathematics challenges and difficulty she has encountered during her program of study. Tanya has repeated numerous mathematics courses. During our first interview, Tanya shared an experience she had with a counselor who questioned her decision to pursue mathematics in light of her mathematics anxiety. Her response seemed to demonstrate a level of mathematics intimacy as she explained,

> I have a love/hate relationship with math. Math challenges me, I struggle with it; but it gives me a new way of thinking. My number one thing in all this is to learn it…understand it. I like the challenge. When I'm challenged, I learn. When I'm not, I'm bored. (interview 1, January 18)

Mandy did not discuss test anxiety specifically, instead she talked about the anxiety she often experiences due to the expectations she has as a prospective teacher. She explained that,

> My biggest fear is that a student will ask me a question and I can't answer it. I feel bad about thinking this way. I mean, I think since I'm going to teach middle school I should at least know the answer to the questions any middle school student might ask me; but, then again I might not and that scares me sometimes. So I'm really trying hard to understand all that I can. (interview 2, February 15)

Mandy's anxiety is a reflection of her primary belief that mathematics be presented in a way that students can understand. Thompson (1999) identified some beliefs as primary

and others as derivative. She asserts that some beliefs serve as a foundation of others. In this same study, Thompson described a primary belief as a teacher's belief that presenting mathematics is important while a derivative belief might be that teachers should be able to answer any question their students ask. Mandy's belief about answering students' questions appears to be a derivative belief.

Cindy associates her experience with anxiety with her teachers. She explained:

> It's not the math that makes me nervous and upset. Most of the time, for me, it's the professor who makes me nervous. Especially geometry, because I have a hard time with proofs. But like I said, I learn better if the professor can explain the concepts to me in different ways in case I don't get it the first time. I like to know the professor cares and really wants me to understand what it is I'm doing (Interview 2, February 8).

The participants' emotions can be associated with several constructs discussed in the literature including mathematical intimacy and mathematical integrity. Each is a manifestation of the participants' beliefs. Goldin (2000) asserts that beliefs are a stabilizing factor in affective behavior. If a learner believes that when mathematics is challenging they will learn and understand it deeply, then there is an anticipation of satisfaction and joy for the success that will occur. It is the anticipation of satisfaction that stabilizes affect behavior (Goldin, 2000). Mathematics intimacy and integrity appeared to be a crucial factor in the participants' mathematics perseverance. Mathematics intimacy increases mathematics integrity. That is, when students are engaged in mathematics they will become more interested in understanding the mathematics and less in getting the correct answer, which tends to increase mathematics integrity. When the participants demonstrated mathematics intimacy and mathematics integrity in learning situations they expressed confidence and persistence when faced with difficult and challenging mathematics learning and problem-solving situations.

Attitudes

Attitudes are often the manifestation of beliefs (Liljedahl, 2005). That is, the participants' attitudes are the responses that they have to their beliefs. The participants' beliefs about the nature of mathematics, learning and teaching mathematics results in attitudes of concern for understanding why rules, formulas, and procedures for solving problems work the way they do. Mandy, Tanya, and Mark express attitudes such as,

> The goal is for students to understand why something is done instead of just memorizing formulas (comment by Mandy during interview 2, February 15).
>
> I use to think of math as just working problems; but now I can see how important it is for me to actually solve the problems without worrying about getting the right answer. If I really don't understand what's going on, I ask for help so that when I get into the classroom I'm able to help my students (comment by Tanya during interview 1, January 18).
>
> My attitude is that there are no hard math problems; some just take a long time to solve with many different steps that require the solver to have an overall understanding of concepts (comment by Mark during interview 2, February 6).
>
> For a long time I believed I couldn't do this stuff because math is challenging for. Watching the teacher work the problems and copying everything down does not work for me. For me, I have to have a chance to work the problems or see how other people have worked the problems during class so that I can get things figured out before I have to work the problems on my own (comment by Cindy during interview 2, February 8).

Themes emerging from the participants comments associated with their attitudes were themes regarding the role of solving mathematics problems without feeling anxious about getting the correct answer, asking for help, having an expectation that mathematics is challenging, perseverance, and the need for time during the process of mathematics learning and understanding and problem-solving.

Heuristics

I describe the strategies participants use during the problem-solving process as heuristics. The diagonal problem (Appendix D) is the main vehicle for generating data on the heuristics used by each participant during the task-based think-aloud problem-solving interview. Fieldnotes and interview transcripts were scrutinized for incidents of the participants using strategies or heuristics to assist the learning or problem-solving process.

Heuristics are the systematic search for and utilization of strategies in problem analysis, representation, and transformation that help the learner to make sense of a problem and to make progress toward a solution (Verschaffel, De Corte, & Borghart, 1997). Similarly, Schoenfeld (1987) defined heuristics as a general proposal, which helps a learner to understand and use known sources effectively to solve a problem. Heuristics play an important role in the creative thinking involved in problem-solving (Carlson & Bloom, 2005; Montague & Applegate, 1993; Schoenfeld, 1992). They are non-algorithmic tools and techniques used during problem-solving to find a conceptual solution.

On their journey to solving the diagonal problem, the participants move through several problem-solving phases. I label the phases as orientation, exploration, implementation, and evaluation. During the orientation phase each participant reads the problem to obtain an initial understanding. During the exploration phase each participant considers whether they have completed a similar problem. Based on their considerations, a strategy believed to be useful in solving the problem is selected. During the implementation phase the participants implement their choice of strategy. Finally, they

evaluate the reasonableness of their choice of strategy and solution. I focus primarily on the heuristics the participants use as they progress through the solution process. I am interested in exploring the heuristics participants use during problem-solving and not so much the phases through which they move during the process.

Common uses of heuristics in this study include draw a pictorial diagram, search for a pattern, formulate an equivalent problem, and make a table. These heuristics have been empirically found in the literature as playing an important role in effective problem-solving performance. Based on what I observed during the think-aloud problem-solving interview, I explained how the participants employ each heuristic to help with their understanding and problem-solving. During their initial engagement with the problem, each participant thought about whether or not they had solved a similar problem at an earlier time. All participants used a pictorial description of the problem to reformulate or describe it in a simpler but equivalent form. They each began with a diagram of a triangle then moved to a pictorial diagram of a square, pentagon, and hexagon while drawing and counting the diagonals for each figure. After using a pictorial description, each participant created a table to help organize the number of vertices, sides, and diagonals in each polygon. The participants searched for patterns within the table of results in their attempt to find an equation to describe the relationship between the vertices or sides of the polygon and the number of diagonals.

Mandy, Mark, Tanya, and Cindy discussed how their use of a pictorial diagram and table was helpful in simplifying the problem, recognizing a pattern, and finding an equation to describe the relationship between the vertices, sides, and diagonals of the polygon.

> Okay, now I'm gonna draw a seven sided polygon. I'm seeing if there is a pattern to the number of points and trying to see if I can get it into a formula or equation (comment by Mark, problem-solving interview March 20).
>
> I'll start by drawing a polygon for each one to make it easier to count the diagonals (comment by Mandy March 18).
>
> I'll start by drawing a triangle. I'm going to use colored pencils to keep me from losing track of the number of diagonals I draw. Okay, I'm going to draw a pentagon and its diagonals (comment by Tanya, Appendix I, lines 3, 5, 6).
>
> Okay, draw a picture of each polygon as far as I can go and put that info in a table to help me see if there is some kind of pattern (comment by Cindy, Appendix J, lines 2- 3).

Two participants employed a familiar strategy used to solve a similar problem. Mandy, Cindy, and Mark recalled the problem-solving success in an earlier experience when using the current strategy to solve a previous problem. They remembered using a pictorial diagram, table, and pattern recognition, which lead to the successful representation of an equation for finding the sum of the interior angles of a convex polygon.

> I remember solving a problem like this one, the interior angles problem, a using a picture and a table really helped me see the pattern and come up with the formula (comment by Mark during the retrospective interview 4).
>
> Okay, this problem was like the interior angles problem. So I figured it would be a lot easier if I started by using a picture to actually do it (comment by Mandy during the retrospective interview 4).
>
> This problem is kinda like the one we did when we had to find a formula for determining the sum of the interior angles of a convex polygon) comment by Cindy during the retrospective interview 4).

These comments reflected the participants' ability to evaluate whether it is useful to use a specific strategy before employing it to solve a problem. They remembered using the strategy of drawing a pictorial diagram, creating a table, and pattern recognition to solve a previous problem and based that information on whether it would be useful to use these strategies to solve the diagonal problem. They also evaluated if, after applying the

heuristic, it was worthwhile. This process included checking their work and backing up to a previous step when recognizing that the current strategy is not working. The use of this heuristic determined the success of the path they followed as they proceeded to solve the problem. There was a certain amount of confidence demonstrated by these participants in their use of familiar strategies.

> The picture and table because I'm beginning to see a pattern. Now all I have to do is figure out how to write what's happening in the table into a formula (comment by Mark, Appendix H, line 16).

> The first thing I do when I read a problem is draw a picture. It's a habit for me; it's just something I always do because I have to see what the problem represents. I am a visual learner (comment by Cindy during follow up interview 4, March 11).

Unlike the other participants, Tanya used colored pencils to keep track of the number of diagonals contained in each polygon. She viewed this as a worthwhile strategy because it helped her stay focused and organized. In the follow-up interview she explains that,

> I used colored pencils to draw my diagonals and I'm glad I did, because there is no way I would have made the diagonal and vertices connection.

Tanya did not use a table initially, but her use of different pencil colors enabled her to quickly count the number of diagonals connected to each vertex. As a result, she was able to see that there is a pattern that explains the relationship between the vertex and diagonals even though at this point she could not represent the pattern using a formula. Unlike the other participants, Tanya did not make reference to using the strategy of drawing a pictorial diagram to assist her in recognizing a pattern or finding a solution. The observed heuristics or strategies the participants use indicate that using heuristics, as suggested by Schoenfeld (1985), requires a certain amount of sophistication. First the learner has to choose a familiar strategy she thinks might lead to success, she must be

able to break the problem down and relate its parts to familiar problems, and apply the solutions of previously solved problems to the task at hand. The use of problem-solving strategies can contribute significantly to problem-solving success if the solver has had experience using the strategy and is able to make judgments about or monitor whether the strategy is worthwhile.

Metacognition

In this section, I describe the range and patterns of metacognitive knowledge and monitoring or self-regulatory processes employed by each participant when completing the diagonal problem.

The diagonal problem (Appendix D) is the main vehicle for generating data on metacognitive processes by each participant during their problem solution process. The focus of the study is not to identify the problem-solving phases in the problem-solving process but instead examine the uses of metacognitive processes during each participant's solution process. Activities such as planning how to approach a given learning task, monitoring comprehension, and evaluating progress toward the completion of a task are metacognitive in nature. This information was used to gain a better understanding of how the participants' metacognition interacted with their beliefs, affective behavior, use of heuristics, and mathematical cognition in a problem-solving context. The task-based interview transcripts (Appendixes G – J) are coded to identify common uses of the metacognition during the problem-solving process.

In the literature, metacognition is divided into two broad categories: metacognitive knowledge and metacognitive control or self-regulation. Metacognitive knowledge includes knowledge of strategies that can be used to solve mathematical

problems, knowledge of the conditions under which certain strategies can be used, and knowledge of the extent to which certain strategies are effective, and knowledge of self (Flavell, 1979). Metacognitive control or self-regulatory processes are cognitive processes that learners use to monitor, control, and regulate their thinking and learning. They include activities such as planning, checking, and evaluating. Metacognitive knowledge refers to knowledge of cognitive strategies and not the use of those strategies.

The solution processes for the diagonal problem used by each participant are unique in many ways. Although each of the participants demonstrates varying levels of understanding, monitoring and control, it is evident that there are recurring patterns in their solution processes. I frequently observed all participants monitoring and reflecting on the effectiveness and efficiency of their use of heuristic in the solution process.

Mandy (Appendix G) perceived that she could solve the problem. She read the problem and began to make sense of it. After she read the problem, she began planning what strategy she would use to solve problem. After implementing her chosen strategy, monitoring her progress, she stopped to check if her use of the strategy would lead to a possible solution.

> Humm, you can’t come out with an uneven number. Humm, what am I doing wrong? Okay, let me see, so I know when I have a triangle [a triangle] doesn’t have any [diagonals] so that would be…so that should have been n=3. (interview 3, lines 13, 16-20)

During a classroom observation, Mandy also demonstrated metacognitive behavior as she worked a problem on the board. She monitored her thinking and engaged in internal dialogue as she explained why divisibility by zero is impossible and reflected on her explanation.

> Okay that's right. So, that makes sense to everybody? Does everyone see that? (transcribed fieldnotes # 2)

After questioning the possibility that $\frac{0}{0}$ has a solution, she monitors her thinking and responds by saying,

> Anything divided by itself is 1, but then again couldn't it also be 0 because 0 x 0 is really 0. Now I'm confused, I remember talking about this before, but I don't know what it is called (transcribed fieldnotes #2).

During the task-based interview and classroom observation, Mandy's metacognition represents her strategic knowledge, evaluation of problem difficulty, monitoring, and reflection. Her metacognitive knowledge enabled her to reflect on the usefulness of a familiar strategy and its implementation during the problem-solving process. She used a specific strategy because it enabled her to better understand and explain a problem, organize the information within a problem in order to find a solution. Mandy exhibited self-knowledge as she evaluated the difficulty of the diagonal problem and her ability to solve it. Her self-regulatory processes included interpretation, planning, self-questioning, checking, reflecting, and recalculating.

Mark (Appendix H) initially perceived the diagonal problem as one he can solve with very little difficulty. In the follow-up interview he explained that,

> I remembered doing a similar problem so I thought I could be solved this one in the same way. I did not think it would be as difficult as it was for me.

After reading the problem, Mark immediately began to draw a pictorial diagram of the first five polygons along with their diagonals. I noticed he approached the problem as though he has an immediate plan and strategy that will lead to a successful solution; he appears confident. After implementing his strategy and recognizing a pattern he reflected on his work before proceeding to the next step. On numerous occasions during the think-

aloud interview, I observed Mark looking back over his steps (lines 9, 13, 15, 16, 24, 28), monitoring his work (lines 6, 8, 17), and reflecting on and evaluating his solutions (19, 28, 30, 31). After obtaining his first solution (line 26) he checks his work and says,

> Okay, that doesn't work (line 28).
> Alright, let's see…No maybe…I think that works because you have n-3 (line 29).
> I'm still trying to figure out the pattern. I'm trying to figure out how I can represent the relationship between the diagonals and sides. I'm seeing that however many points the figure has, if you take away two from it that's how many points have diagonals it has coming from it. No, no I'm doing sides not vertices; never mind. Okay, well it still works the same though. (lines 16, 17, 19, 20).

Mark demonstrated his metacognitive knowledge and control as he evaluated the problem difficulty, monitored his problem-solving steps, and reflected on his solution. As he worked through the diagonal problem, he realized that the problem was more difficult than he had thought initially. Each time he obtained a possible solution he checked to see if his solution made sense. He revisited his steps along the solution path numerous times and started over when things did not appear to be going well. Although Mark did not manage to solve the problem successfully, his metacognition provided him the opportunity to persevere in his effort to find a successful solution. During Mark's task-based follow-up interview he explained that with more time he would have taken a break and returned to the problem later and perhaps been more successful in obtaining a solution.

Tanya (Appendix I) did not express the difficulty level associated with the problem. After reading the problem, she used a pictorial diagram to organize her thinking. Of all the participants, Tanya decided not to organize her thinking using a table. She used colored pencils in order to keep track of the number of diagonals drawn in each polygon. The colored pencils enabled her to organize her thinking and recognize a pattern

related to the number of diagonals and vertices. In order to represent this relationship she chose to use the method of finite differences. Tanya pursued this approach for the next 15 minutes making very little progress. After evaluating her progress, she realized things were not proceeding well and decided to consider another option.

> That [referring to her use of the method of finite difference] would not help my pattern. Can I get 4 another way? No, that won't work. Hum, that's not working (lines 13, 14, 15).

She created a table to help organize the name of the polygon, number of polygons, number of vertices, and number of diagonals. Again, she returned to the method of finite differences in an effort to find the relationship among the variables in the problem. Tanya seemed convinced that she can find an equation using this method and continued to pursue this approach throughout the process. However, she continued to monitor and check her work and responded accordingly,

> Hum, that's not working. Okay let's go back and take a look, the square had 4 starting points and they all went to one place and you ended up with 4. I'll need to go back and clarify if you have less than 4 sides 'cause that would make a difference in the function or equation I'm looking for. (lines 19, 20, 27)

Tanya explained that she chose to apply the method of finite differences to the diagonals problem because of her recent successful use of it in the classroom setting. She explained:

> I would never have thought about finite differences, but since we've been working with them in this class I'm beginning to like the idea (line 10)

It appears that Tanya approached the problem using this method because of her recent experience applying it and not because she possessed the meta-strategic knowledge needed to recognize the appropriate situation in which this method is best used. She appears to understand how the method is applied, but her limited experience in applying

the method prevents her from knowing if it would be useful in this specific problem situation. Even though she was unable to make progress using this strategy, she remained optimistic about her solution process and confident that the strategy would be useful in obtaining a solution. She continued to apply this method for the remainder of the problem with no success in obtaining a solution.

Tanya's metacognitive knowledge and regulation was a reflection of her persistence, motivation, monitoring, and reflection. She drew a pictorial diagram, created a table, and applied the method of finite differences during the solution process. She was persistent in her efforts and frequently revisited her steps to assess her progress in reaching her goal. Her mathematical intimacy became evident in the effort she put forth to obtaining a solution. Unlike the others, it appeared as though she and the mathematics became one. Although she was not making progress in obtaining a solution, she proceeded with finding a recursive formula by applying the method of finite differences. She moved between examining her pictorial diagram, creating a table, searching for a pattern, applying the method of finite differences, guessing a formula, and reflecting on her progress. She appeared confused and indecisive about choosing a strategy or direction that would lead her down a path of success. She confirmed my thinking in a comment as she abandoned her 45 minute solution process.

> Now it's like too many ways I can look at the problem. It's like which way should I do it, they're all running together line 29)

During her problem-solving processes, Tanya demonstrated that her knowledge of procedures and strategies was not enough to lead to a successful solution path. Having knowledge about a strategy and experience using it in a variety of problem situations is a more useful kind of knowledge (Star, 1999). Knowledge of a procedure or strategy

enables one to identify a strategy or procedure that can be applied in a problem-solving situation. However, knowledge about a procedure or strategy goes several steps further and includes having (a) an understanding the goals of the procedure or strategy, (b) knowledge of the type of situation in which the strategy or procedure is best used, and (c) knowledge about what using a procedure or strategy will accomplish. This knowledge is similar to what Davis (1983) described as planning knowledge; but it deals strictly with knowledge about the procedures used to solve mathematical tasks.

Cindy (Appendix J) perceived the diagonal problem as difficult but demonstrated confidence in solving it. She began the solution process by reading and interpreting the meaning of the problem. She restated the problem using her on words and thought carefully about the method or approach that would be most helpful in leading to a solution. She explained,

> Okay, so you're asking me to explain how I would go about finding the equation that I can use to tell how many diagonals there are in any given polygon, right? (line 3)

> Like I said before, drawing a picture or diagram is something I always try to do because it always helps me visualize the problem. (lines 5, 6, 7)

Cindy's plan included drawing a pictorial diagram and organizing her thinking in a table. She chose this approach because "it makes the problem easier to deal with and it lets me see what the problem is actually asking" (follow-up interview, March 11). Cindy has used this heuristic strategy successfully in similar problem-solving situations.

At each step she looked back over her work, engaged in self-talk, and checked her solution to ensure that it made sense.

> Okay, I'm doing that right. Okay, I think the vertices and the diagonals are related, not the sides. Well… vertices and sides are the same. So, the only thing I

> need to do now is to find the equation that can represent this relationship between the vertices and the diagonal. (line 14)

After examining the table of values, Cindy recognized that there was a relationship between the number of diagonals and the number of vertices. She used this information to write an expression to represent the relationship between the vertices and the diagonals of a polygon. She monitored and reflected on her solution process.

> Okay, I think its N – 2. So let me check to see if this works. The square…4 vertices so 4 – 2 = 2; so that works. The pentagon…5 -2 = 3; No that doesn't work. The hexagon, 6 – 2 = 4. No that won't work. Okay go back. (line, 23)

In general, Cindy appeared to be aware of her strengths, weaknesses, and learning style. This is reflected in her comments made during an interview when asked to discuss how she learns.

> I need hands on learning, a chance to play around with the math. I also want to be able to solve a problem using several approaches. Like drawing a picture, making a table, using a graph, working the problem backward (comment by Cindy during interview 2, February 8).

Cindy's awareness of her intellectual strengths and weaknesses is a reflection of her metacognitive knowledge and self-regulatory process. She has an awareness of the strategies that she relies on to help her learn mathematics, interpret, and understand the mathematics problems she encounters.

> I have no idea how I got that. I mean, I can show you, but I basically used guess and check. I would definitely need to go back and see if I can figure out why you would divide by two. Because to be honest, I basically guessed and then went back and checked. (Appendix I, interview 3, lines 28, 29)

Cindy's metacognitive knowledge enabled her to know what, when, and how to use specific strategies. Her ability to select, combine, coordinate chosen strategies, monitor, and regulate her progress during problem-solving and mathematics learning appears to be an important component of her metacognitive control or self-regulation.

However, her comment also reflected her lack of confidence and understanding of the solution process. She was unable to make the connection between the total number of diagonals and the fact that each is drawn twice, so the expression must be divided by 2.

Metacognition in mathematics problem-solving and learning involves the processes of planning, monitoring, evaluating specific problems, and selecting appropriate strategies (Flavell, 1992). Learners who take time to understand and make sense of the facts in the problem, check their work for accuracy, break complex problems into simpler steps, and engage in self-questioning and answering are likely to perform better during problem-solving (Artzt & Armour-Thomas, 1992). Knowledge about strategies and their use influences the problem-solving process, however knowing when and why it is appropriate to use is a specific strategy along with understanding what using the strategy will bring to the problem-solving effort is important for overcoming obstacles and achieving goals.

Mathematical Cognition

While metacognitive knowledge and processes have been found to help problem solvers become more efficient at handling mathematics problems (Flavell, 1992; Schoenfeld, 1992), several researchers (i.e., Ambrose, 2004; Hiebert, 1999; Rittle-Johnson & Kroedinger, 2002) have found that procedural and conceptual knowledge is important in studying problem-solving and knowledge for teaching and learning mathematics. This study seeks to characterize the interplay among prospective teachers' affective behavior, meta-cognition, and mathematical cognition in the context of problem-solving. In doing so, I defined and characterized the mathematics knowledge structure prospective teachers used as they engaged in mathematical problem-solving as

conceptual or procedural. Skemp (1992) and Hiebert and LeFevre (1992) proposed that a learner's mathematical knowledge structure can be characterized as either conceptual or procedural. Conceptual knowledge refers to knowledge or understanding that is integrated. Skemp described conceptual knowledge as "knowing what to do and why" (p. 1953). Procedural knowledge refers to the knowledge and understanding based on the execution of rules, procedures, and formulas without reference to their rationale or underlying meaning or origin. To gauge the depth, richness, and to characterize the knowledge demonstrated by the participants as procedural, conceptual, or otherwise, I relied on fieldnotes from classroom observations, participants' written work, and interview transcripts. Transcripts and fieldnotes are reviewed for statements or actions suggestive of a view of mathematics as procedural or conceptual, inferences from their written work and solution process are coded according to the criteria in appendix K.

During the task-based interview Tanya, Mandy, Mark, and Cindy appear to have some knowledge that an initial drawing of a pictorial diagram and creating a table of values would be helpful in simplifying the problem, recognizing a pattern, and representing a relationship between the diagonals of the polygon and its vertices or sides. These participants use computational procedures during the problem-solving process; but they approach the problem through exploration or trial and error and not by recalling a formula they used in the past. Overall, the participants seemed to have a plan for approaching the problem using the strategies "draw a pictorial diagram" and "create a table," because they each had some awareness of what using those strategies would accomplish. Although the problem is unfamiliar, having knowledge of the outcome of using a specific technique or strategy is familiar to all participants. I observed the

application of their strategic knowledge in action during a classroom observation. The students were asked to solve the following problem: John can paint the house in 3 hours. Mark can paint the house in 4 hours, and Zach can paint the house in 5 hours. How long does it take them to complete the house if they all work together? I circulated the room to observe the participants solving the problem and examined their written work. All participants used a pictorial diagram, trial and error, conjecturing, and reasoning during the process of trying to obtaining a solution to this problem. No one began the process by attempting to recall a formula; instead they began with a strategy and used their reasoning.

The participants' views about mathematics and their approach to solving mathematics problems appeared to be mostly conceptual. During the interviews the participants described mathematics as real-life, creative, pattern exploration and discovery, a life-style, and the relationships among quantities. For them, mathematics is not a fixed body of knowledge, a collection of procedures, rules, formulas, and theorems to be memorized. Mark, Mandy, Cindy, and Tanya repeatedly asserted the importance of leaning mathematics for understanding. They expressed beliefs that suggested memorizing mathematics is not the same as learning or understanding mathematics. This view was reflected in the following comments.

> Math is learned when you are solving problems and you understand why something works the way it does and you can explain it to other people (comment by Cindy, interview1, January 28).

> Learning math and understanding math involves explaining why; not just how it's done, but why it's done. In this class, I'm always trying to remember the rules or the steps for how to do certain things when now I realize how important it is for me to understand. I didn't ever really understand it [quadratic formula] now I understand how it came about. I'm finding out why (comment by Mark, interview1, January 30).

> The goal of learning math is to understand why something is done instead of just memorizing formulas (comment by Mandy, interview 1, February 5).

These comments represented the participants' assertions that it is important to understand the rationale in mathematics. Hiebert and LeFerve (1986) and Skemp (1976) would agree that their comments could be identified as someone who has a conceptual view of mathematics. Mathematics procedures are an important part of mathematics and problem-solving; however, for these participants, the rationale for procedures is equally important when explaining concepts to others. For them, learning mathematics is synonymous to understanding mathematics. They view understanding mathematics as eliminating their need to memorize formulas and rules.

Throughout the task-based interview, the participants monitored their progress by checking that the procedures and strategies they used to solve the diagonals problem were executed correctly and the solution process made sense. Some participants felt that mathematics problems generally have one solution, but multiple solution paths should lead to the solution. During an interview, when discussing what it means to understand mathematics Mandy and Cindy explained that,

> I think one of the most important things in learning and teaching math and solving problems is you have different ways to approach, work; you have to have more than one way to explain a problem (comment by Mandy, interview 4, March 18)

> If you're going to be a good math teacher you have to be able to explain the concepts you are teaching in more than one way (comment by Cindy, interview 4, March 10).

The participants perceived understanding mathematics as being able to justify their reasoning and solutions, solving problems using more than one approach, and explaining a problem in multiple ways. During a classroom observation, Mandy and

Mark appeared to value their ability to explain and justify their reasoning when solving problems, especially at the board. While Mandy knows that division by zero is undefined, she was unsure about how to explain why this is true. In other words, she was unable to justify her solution. Consequently, her hesitation to explain her solution to the class at the board demonstrated a lack of confidence. In a follow-up interview she explained,

> I need to understand not just memorize, I have to not only learning it, I have to understand it if I'm going to teach this stuff. (Interview 4, March 19)

Mark explained that while he knew how to use the quadratic formula as a way of finding solutions to a quadratic equation, he did not fully understand where or how it originated. He explained that,

> Learning math and understanding math involves explaining why; not just how it's done, but why it's done. I didn't ever really understand it [quadratic formula] now…I'm finding out why. (Interview 2, February 6)

During the task-based interview, each participant demonstrated flexibility as they moved from one problem-solving approach to another upon evaluating the effectiveness of a current approach. The participants evaluated their problem-solving and mathematics learning success based on their understanding of the mathematics process and less on obtaining the correct solution. This was evident during their task-based interview. Failure to obtain a solution did not seem to be an indication that their process was incorrect. For them, the process was valid even if the solution was not correct. In an earlier interview when asked in general how she feels when she fails to get the correct solution, Cindy's reply was, "Understanding mathematics is not about getting the right answer; getting the right answer is not learning" (Interview 2, February 8).

The mathematical behavior of the participants during the problem-solving process indicated that their mathematics knowledge appears to be rooted in both procedural and

conceptual mathematics knowledge and understanding. They did not initially resort to using an algebraic manipulation to solve the diagonal problem. While they each searched for an equation to describe how they would find the number of diagonals in a polygon, they began the search with an initial exploration, trial and error based on conjecturing and deduction. While they each had the expectation that an equation or algebraic expression would result, the participants used the integration of a pictorial diagram, pattern recognition, table of quantities representing the sides, vertices, and diagonals of the polygons in the process of finding the solution to the diagonals problem. The participants are able to move between these representations to validate and justify their work.

Summary

I used a synthesis of Charmaz's (2003) constructivist grounded theory and Strauss and Corbin's (1998) grounded theory procedures of coding fieldnotes, interview transcriptions, and participants' written work to define or construct categories. My interest was in understanding and explaining what was happening based on what the participants were telling me about what their mathematics learning and problem-solving experiences were like for them. The collection and analysis of data was shaped by me and my participants (Charmaz, 2003). The categories represented the participants' affective behavior, use of problem-solving strategies or heuristics, metacognition, and mathematical cognition during problem-solving and mathematics learning.

The first category represented affective behavior. In this study, affective behavior refers to the participants' mathematics-related beliefs, emotions, and attitudes related to mathematics learning and problem-solving. Their beliefs about the nature of mathematics represent aspects of mathematics creativity, content, problem-solving, and its usefulness

in everyday life. For them mathematics is more than a collection of rules, formulas, and theorems to be memorized. The participants' views on learning and teaching mathematics appeared to be a manifestation of their beliefs. Because they are prospective teachers they believe that it is important to learn mathematics with understanding.

Their beliefs about teaching appeared to be a result of their own experiences as students. As prospective teachers, they made distinctions between effective methods of teaching and ineffective methods of teaching based on their past learning experiences. If a method of teaching was successful for them but was based on memorizing mathematics, they rejected the methods as good teaching. If, on the other hand, the method of teaching was successful and their understanding of the mathematics enabled them to explain justify their reasoning for a specific solution, the participants accepted the method as good teaching. The participant's perceived understanding mathematics as being able to justify their reasoning and solutions, solve problems using more than one approach, and explain a problem in multiple ways.

Learning mathematics in a problem-solving environment plays an important role in understanding mathematics. For them, learning and understanding mathematics cannot be achieved without doing mathematics in an environment that offers support, encouragement, and opportunities for "playing" with mathematics without fear of "getting it wrong." Solving problems is an important part of learning mathematics; but, for them, getting the correct answer during problem-solving does not constitute learning. The participants view "getting it [the answer] wrong" as a normal part of understanding mathematics.

Getting the wrong solution, however, was not without consequence. When the participants get an incorrect solution or experience an interruption during the process of obtaining a solution they experience disappointment, frustration, and often embarrassment. Their affective behaviors appear to be related to their beliefs about teaching and learning. While they do not espouse beliefs that a teacher has all the answers, they believe that a good mathematics teacher is someone who understands mathematics. For them, understanding mathematics deeply is associated with being able to explain your solution process, solve problems using multiple solution paths, and explain why mathematical procedures work and how they connect to mathematical concepts. The participants believe that mathematics should be presented in a way that students can understand. They believe this can be achieved if the teacher understands mathematics.

The participants' emotions appeared to be a manifestation of their beliefs. Goldin (1995) proposes that beliefs are a stabilizing factor in affective behavior. The participants asserted that they learn more mathematics when they are challenged mathematically. As a result, when confronted with mathematics difficulty the participants were able to control their emotional behavior and persevere through difficulty because they believed that it comes with the territory. They see it as being an important factor in their mathematics learning and understanding.

The second category represented the participants' use of heuristics in problem-solving. In this study, heuristics are strategies, methods, or approaches participants use during the mathematics learning or problem-solving process. They range anywhere from trial and error to draw a pictorial diagram to make a generalization. The participants' use

of heuristics enables flexibility in planning and is a reflection of their ability to reason mathematically.

In the third category, I described the participants' metacognitive knowledge and metacognitive control. The participants frequently monitor and reflect on the effectiveness and efficiency of their solution process. Knowledge about strategies and their use influence the problem-solving process, however knowing when and why it is appropriate to use is a specific strategy appears to play an important role in overcoming obstacles and achieving problem solution goals.

The fourth and final category represented the mathematical knowledge participants' access during problem-solving. The participants' comments and actions during the interviews, observations, and in examples of their written work appear to convey a view about mathematics that is mostly conceptual. However, there are instances when they implement procedures to solve unfamiliar problems; but they do so with initial planning, exploration, reasoning or deduction. Their ability to plan how they will approach an unfamiliar mathematical problem is a reflection of their knowledge of problem-solving strategies, their ability to evaluate whether a specific strategy will be useful in obtaining a problem solution and to monitor and control their thinking during problem-solving. The participants' knowledge of the (a) structure of the problem or task, (b) use of a strategy or procedure (b) procedural steps and goals of the procedural steps, and (c) situation in which a procedure or strategy is most effectively used is knowledge that appears to play a role in the problem-solving process (Star, 1999).

In the next chapter, I further explore the principles underlying major categories in an effort to construct a core category or a central theme or story line of the data

(Charmaz, 2003) that will explain how the participants' affective behavior, metacognition, and mathematical cognition interact in the context of problem-solving.

CHAPTER 6

CONCLUSION AND DISCUSSION

Introduction

This chapter begins with a brief summary of the overall study. A discussion of the selective coding technique used to develop the main category and the model is presented, along with the conclusions as they relate to the research questions. A description of the model depicting my interpretation and understanding of the interplay among affective, metacognitive, and mathematical behavior, its development, and evaluation follows. Next, the implications of the study for future research, mathematics teacher educators and mathematicians, and curriculum development are outlined, and then limitations of the study are discussed. Finally, closing statements about the study as a whole are provided.

Summary of the Study

The recent NCTM (2000) reform movement called for mathematics teachers to provide students with experiences and opportunities in problem-solving throughout the secondary school mathematics curricula. If prospective secondary mathematics teachers are expected to meet the problem-solving goals set by the NCTM, they too must be provided with experiences and opportunities to develop substantial deep mathematics understanding for teaching in a problem-solving context (Ball, Bass, & Hill, 2005; CBMS, 2001; Even, 1993; Ma, 2004; Usiskin, et al., 2003). However, when helping teachers and prospective teachers learn ways to improve their problem-solving competence and enhance their mathematics thinking as well as that of their students,

Thompson (1992) found that they often encountered a number of hindrances such as beliefs and attitudes toward mathematics and problem-solving.

To get a better understanding of how affective behavior interacts with metacognition and mathematics cognition, this study investigated prospective teachers' mathematical problem-solving experiences as they participated in an undergraduate course focused on deepening their understanding of school mathematics. The main purpose of this study was to explore and explain the interplay of prospective teachers' affective behaviors, metacognition, and mathematical knowledge. The aim was to capture what these prospective teachers were thinking, saying, and doing while learning mathematics and solving mathematics problems and gain a better understanding of what the experience meant for them. I explored their mathematics-related beliefs, emotions, and attitudes along with their metacognitive and mathematics knowledge and understanding in the context of mathematics learning and problem-solving. The goal of the study was achieved by analyzing prospective teachers' perspectives of their mathematics-related beliefs, emotions, attitudes, metacognition, and mathematical knowledge as they engaged in the mathematics learning and problem-solving process.

The main question guiding this study was: What is the characterization of the interplay among prospective teachers' mathematical beliefs, mathematical behavior, and mathematical knowledge of prospective in the context of solving mathematics problems? In answering this main question, the following questions will also be answered:

(a) What are the mathematics-related beliefs of prospective secondary mathematics teachers?

(b) What mathematical behaviors are demonstrated by prospective secondary mathematics teachers as they engage in mathematical problem-solving?

(c) What mathematics knowledge is used by prospective secondary mathematics teachers as they engage in mathematical problem-solving?

Theoretical or purposeful sampling of participants was used to identify four participants whom I perceived would provide the maximum amount of information for offering the best potential to add variation, depth, and breadth to the themes emerging from the data (Strauss & Corbin, 1998). Throughout the research process, grounded theorists develop analytic interpretations of their data to focus further data collection, which they use to inform and refine *their* developing theoretical analysis (Charmaz, 200, p. 509). A multistep data analysis and flexible coding technique was used to analyze the data (Strauss & Corbin, 1998). The coded and categorized data reflected emerging ideas that were used to assist in the construction of an analysis of the data rather than a description (Charmaz, 2003). Throughout the entire process of data analysis, memos were written which explored ideas about the data, codes, categories, or themes (Charmaz, 1983).

Through the process of open and axial coding, individual and group data were analyzed to initially form four main categories. The main categories were grounded in the comparison of data from each participant; therefore, they have relevance for and are applicable to all participants in the study. I used an interpretive process of selective coding which enabled me to identify patterns and relationships between these patterns, which I then presented as four interrelated main categories: affective behavior, heuristics, metacognition, and mathematical-cognition. Further review of patterns, themes,

literature, and the reexamination of the four interrelated main categories lead to the discovery that heuristics were central to metacognition. During the problem-solving process, when the participants utilized a strategy or completed a series of procedures, they made an evaluation of its usefulness. Flavell (1976) suggested that thoughtful planning, and the decision-making, or evaluation of a heuristic is metacognitive in nature. He explained that in order to apply a heuristic, an individual must engage in the process of selecting a heuristic. In this context, it is necessary to monitor the progress continually and make revisions when necessary. In their study of students' use of problem-solving strategies, Artz and Armour-Thomas (1992) found that this monitoring process required metacognitive knowledge and self-regulation. Based on this information and considering the context of the study, I merged heuristics and metacognition to form the single category of metacognitive heuristics that represents the participants' reflection on their use of problem-solving strategies.

Selective Coding

During open coding the researcher is concerned with labeling phenomena based on the information provided by interviewees and observations. The researcher collects participant meaning and makes interpretations of the data (Creswell, 1998). Focused coding consists of using the most significant codes to categorize large segments of data (Charmaz, 2005). During axial coding statements of relationships among the categories are made. By listening to the participants and interacting with them during the interviews and in the field, I was able to make statements about how and what the participants were saying and doing, as well as why they were saying and doing it. Each of the participants had their own story to tell and much of what they were saying had common themes.

These categories and common themes enabled me to make assertions about the relationships among the four main categories.

During selective coding, a central theme was selected around which I could represent my overall understanding of what the problem-solving and mathematics learning process was like for the participants (Strauss & Corbin, 1998). I related all the major categories and selected a major theme by determining what was most striking and interesting about the participants' experiences. Through shared interpretations, the core category or central theme was linked to the main categories by telling a story about what was happening (Charmaz, 2002).

During selective coding I began to examine the main categories and any patterns to emerge during the process of the study and to explain in a few words, or using a diagram, my interpretation of what the research was all about and what appeared to be the issues or problems important to the participants. In many cases, I used the words of the participants to tell the story about what they experienced as they solved mathematics problems. Each of the main categories told a story of its own. For example, the main category affective behavior told a story about the participants' beliefs about mathematics, their emotions, and attitudes demonstrated during mathematics learning and problem-solving. It also told the story of how the participants felt during the process of mathematics learning and problem-solving. In combination, the categories represent what was important and relevant to the participants as they engaged in mathematics learning and problem-solving. What follows is a rendering or a story of my interpretation of what I saw and learned about the participants in the context of problem-solving.

The Storyline

My analysis of the participants' interviews, observations, and written work indicated that understanding the underlying mathematics concepts associated with a problem solution, justifying one's reasoning, and solving a problem using multiple approaches is more important than applying a procedure to a problem or obtaining a correct answer. For the participants, *knowing how and knowing why* was indicative of mathematics understanding. Collectively, the participants agreed that understanding mathematics deeply included both the acquisition of knowledge of procedures as well as the mathematics concepts underpinning the procedures.

Tanya, a prospective middle grades mathematics teacher, demonstrated optimism in the most difficult of mathematics circumstances. Tanya explained that mathematics has not been her best subject, but she loves the challenges it presents. She described her relationship with mathematics as one of "love and hate". Mathematics is sometimes problematic for her, but she finds joy in deeply understanding mathematics. This course is a second time around for Tanya and this time she believes she can "understand the mathematics". Procedures, algorithms, and formulas are easily memorized, but understanding the mathematics underpinning the procedures presents a challenge for Tanya. In an interview she stated, "learning mathematics is understanding mathematics, and understanding mathematics is knowing when, how, and why you use procedures".

During her problem-solving think-aloud interview, Tanya struggled and was unsuccessful in obtaining a problem solution. She exhibited emotional behavior such as frustration and disappointment but her frustration did not appear to be counterproductive.

Instead of abandoning the problem after using an initial problem-solving strategy unsuccessfully, she demonstrated persistence as she worked, reworked, and reflected on her problem-solving steps. Despite past negative experiences with mathematics learning and problem-solving, she believed that with increased effort she could eventually find success during problem-solving and in mathematics understanding. For Tanya, failure to understand a particular mathematics concept was not a reflection of her lack of ability. Instead, she viewed failure to understand as a lack of effort. She consistently spent a great deal of time practicing mathematics problems or redoing problems she worked incorrectly on a test or homework assignment. Tanya believed that an increased effort on her part would improve her mathematics ability. She demonstrated positive deposition about mathematics learning and understanding, persistence during difficult and challenging mathematics situations, and she valued understanding mathematics over obtaining a correct answer.

Mandy, also a prospective middle grades mathematics teacher, currently holds a business administration degree and is the mother of two school aged children. During the study, she demonstrated powerful mathematics knowledge and understanding in mathematics learning and problem-solving situations. She consistently submitted class work, homework, and problem solutions that reflected her conceptual knowledge and understanding of mathematics. For Mandy, understanding why a procedure works is as important as implementing a procedure to obtain a correct solution. She demonstrated persistence in her efforts to understand the mathematics underlying the procedures she implemented to solve problems. She is a self professed perfectionist and mathematics has always been her best subject.

Failure to learn mathematics with understanding has not been an option for Mandy. She explains that, “I have to understand if I’m going to teach”. She specifically chose middle-grades mathematics because she believed she could learn fourth through eighth grade mathematics deeply and in doing so enable her to answer all of her students’ mathematics questions. While she believed that getting the correct answer was not important for successful mathematics learning, Mandy was convinced that middle school mathematics teachers should be able to answer middle school students’ mathematics questions. I related this belief to the lack of autonomy she occasionally demonstrated during the study. Before volunteering to explain her problem solutions at the board, she needed assurance that her solution was correct from either the instructor or at least one other student. She valued having the ability to explain and justify her solution process as opposed to *only* implementing a procedure or strategy to obtain the correct solution.

Mark demonstrated knowledge and understanding that extended beyond knowing how to find a solution to a problem. For him, “understanding how, but not why a procedure works the way it does” is both frustrating and motivating to him. He explained that he had recently changed his attitude about what it means to understand mathematics. For Mark, it is not about getting the right answer. More than anyone else in this study, Mark consistently explained that mathematics learning occurs when one understands mathematics. He demonstrated persistence and autonomy in mathematics learning and problem-solving situations. Deriving the quadratic formula was an “aha” moment for him. He demonstrated excitement when sharing his derivation process with me. “It finally makes sense to me”, he said. His comment represents the pride he felt during this process.

Mark holds the belief that “there are no hard problems, just long ones”. For him, spending time with a problem would eventually lead him to a solution. He was confident in his ability to “do” mathematics. It appeared to me that Marks’ persistence, intimacy and integrity with mathematics was a reflection of his mathematics-related beliefs and values. It was important for him to not only arrive at a successful solution, but understand the mathematics underlying the procedure as well. Mark exhibited a consistent and ongoing use of self-reflection and self-regulation. He does not put a problem aside before he has made progress toward finding a solution. His intimate engagement with mathematics and his focus on making sense of mathematics was empowering for him. Empowering because Mark believes that when you understand mathematics you can explain mathematics, and when you are able to explain mathematics you can effectively teach mathematics.

Cindy demonstrated an unyielding commitment to understanding mathematics. She demonstrated persistence and autonomy in her mathematics learning and problem-solving. Mathematics is her favorite subject, but she explains that as she advances in her mathematics study it has become more difficult and challenging. Her ability to take a difficult and challenging problem-solving situation and use it as a learning opportunity was a direct reflection of her “don’t give up” attitude. She stated that the “mathematics we do in here is challenging, but when I’m challenged I learn”. Cindy values reasoning and explaining her problem solutions and she believes “all problems have multiple solution paths”. Very rarely did she abandon her problem solution for a peer or instructor solution process even when she had obtained an incorrect solution. Instead she made it a point to understand where she went wrong during her problem-solving process. Obtaining

a correct solution as not as important as understanding the mathematical solution process. Cindy demonstrated mathematics knowledge, understanding, persistence, and autonomy during the problem-solving process.

All participants appeared to demonstrate knowledge of procedures and strategies. However, all but one of the participants demonstrated planning knowledge (Davis, 1983). Planning knowledge includes understanding the goals of the procedure or strategy, the type of situation in which the strategy or procedure is best used, and knowledge about what using the procedure or strategy will accomplish. Each participant demonstrated a different level of mathematics knowledge and understanding during the problem-solving interview, but they all appeared to value exploration, understanding, and self-monitoring of progress as opposed to recalling a formula and using it to solve the problem. This was true for homework problems and in problem-solving situations in the classroom as well. Participants' frustration, anger, and disappointment was demonstrated not so much when they obtained an incorrect solution, but when they were unable to explain their reasoning or justify their correct solutions.

The participants' mathematics-related beliefs, values, and attitudes played an important role in their interpretation of mathematics learning and problem-solving experiences. The effective use of metacognitive knowledge, conceptual knowledge, and procedural knowledge in problem-solving and mathematics learning situations are important in stabilizing and controlling their affective behavior. This conclusion was reached by using coding techniques, memo writing, and diagramming throughout the research process.

Conclusions

The Central Theme

The central theme represented the participants' synthesized voices and experiences. Lincoln and Guba (1985) referred to the process of constructing the core category or central theme as developing "pattern theories" that represent interconnected thoughts or parts linked to a whole. I was interested in understanding what the whole experience was like for the participants from the beginning of the problem-solving process through the end.

In my application of selective coding, I related the three main categories (affective behavior, heuristics/metacognition, and mathematical cognition) to each other in an effort to understand the interaction among the participants' affective, metacognitive, and mathematical cognitive behavior during problem-solving. The overall common theme that appeared to interconnect the main categories pertained to the participants' belief about mathematics learning and understanding. A goal shared by all participants was understanding mathematics deeply which for them meant both knowing how to apply a procedure and why the procedure works—"knowing how and knowing why." For them, understanding mathematics in this manner was the first step to effective mathematics teaching. Some participants considered mathematics their best subject and encountered little difficulty or conflict during the mathematics learning and problem-solving process. Others struggled to understand mathematics and often found themselves spending numerous hours working on problem solutions.

Using GT methods, the central theme emerging from the participants' understanding of themselves as mathematics problem-solvers, prospective teachers, and

mathematics learners was *Mathematics Understanding: Knowing How and Knowing Why.* The study revealed that for the participants involved in this study, learning mathematics with understanding was a process that involved having both knowledge and understanding of mathematics concepts, procedures, and problem-solving strategies. Their beliefs appeared to center around the theme, "learning math and understanding math involves explaining why; not just knowing how it's done." For them, mathematics knowledge and understanding is validated when they are able to justify the procedures they use, explain their reasoning, and explain (in multiple ways) their problem-solving processes. This is a reflection of their "mathematics integrity," associated with a learner's desire to want to understand and justify one's reasoning (DeBellis and Goldin, 1999).

The Emerging Interpretive Model in Relation to the Research Questions

The purpose of this grounded theory study was to understand the process of prospective teachers' mathematics learning and problem-solving and how their affective behavior, metacognition, and mathematical cognition interacted as they participated in a course focused on developing their understanding of school mathematics. The intended outcome of this study was an interpretive model representing the interaction among prospective teachers' affective behavior, metacognition, and mathematical cognition. By answering the research questions below, I gained a better understanding of the interrelationship among the participants' demonstrated affective behavior, metacognitive heuristic behavior, and mathematical knowledge. The interrelationship represents the participants' potential use of conceptual knowledge and understanding, procedural knowledge, procedural fluency, and procedural understanding, which develops into powerful productive mathematics learning and problem-solving.

What are the mathematics-related beliefs of prospective secondary mathematics teachers?

The prospective teachers' views on the nature of mathematics paralleled the views put forth in the NCTM Standards document. They used terms such as discovering, making connections, understanding patterns, relationships, and real-life to express their conceptions of mathematics. With respect to learning and understanding mathematics, the participants did not believe that failure to understand mathematics was an option for them as prospective teachers. While they did not hold the belief that the teacher has all the answers, each believed that teachers should present mathematics in a way that engages students' mathematical thinking and supports their mathematical efforts. The participants held beliefs that teachers should be able to explain mathematics in more than way. Each participant believed that a good teacher is one who directs their students to understand relationships among mathematical concepts and to explain and justify the reasoning behind their solutions.

With respect to the learning environment, the participants perceived that the ideal learning environment would provide opportunities for them to "play with the mathematics." For them, this involved having the opportunity to experiment with a variety of methods, strategies, and solution paths during the mathematics learning and problem-solving process without the pressure associated with it "counting against" them. According to the participants, "playing with the mathematics" reduced their anxiety level and increased their potential for learning and understanding. The participants perceived that while getting a correct answer was desirable, getting a correct answer was not an indication that learning or understanding had occurred. Their attitudes were a

manifestation of their belief that "getting it wrong" was a normal part of learning and understanding mathematics.

What mathematical behaviors are demonstrated by prospective secondary mathematics teachers as they engage in mathematical problem-solving?

The prospective teachers' mathematics-related beliefs and values were a reflection of their attitudes and disposition, which played a role in how they interpreted the learning and problem-solving environment. For them, solving challenging and difficult mathematics problems often created moments of frustration, anger, and disappointment; but they interpreted these difficult situations as opportunities for deep mathematics understanding to occur. The prospective teachers demonstrated a "don't give up" disposition in mathematics learning and problem-solving situations. As prospective teachers, failure was not an option. For them, understanding mathematics was a critical aspect of teaching mathematics. They associated failure with a lack of effort and were therefore willing to increase effort and persistence in order to achieve their mathematics learning and problem-solving goals. "One thing I don't usually do is give up" and "I don't give up easily" were common responses that demonstrated the participants' persistence and self-determination. When describing what it was like engaging in challenging mathematics learning situations and problem-solving difficulty, they used phrases such as "hard work", "persistence", and "keep trying".

The participants demonstrated "mathematics intimacy" (DeBellis & Golding, 1999) in their persistence and determination to develop deeper understandings of the mathematics they plan to teach. They were willing to spend time and effort toward understanding mathematics and finding solutions to challenging problems. For these prospective teachers, difficulty and unsuccessful problem-solving or mathematics

learning situations were not seen as moments of failure. Instead, they viewed these challenging moments with the anticipation of pride and joy in an expected success.

Use of heuristics or problem-solving strategies was central to metacognition. During the problem-solving process, when the participants utilized a strategy or completed a series of procedures, they made an evaluation of its usefulness. They also demonstrated meta-cognitive behavior. Their thoughtful planning, decision-making, and evaluation of a selected strategy was identified as metacognitive in nature. When the participants applied a heuristic or problem-solving strategy, they engaged in self-questioning, reflection, and monitoring. The participants frequently monitored and reflected on the effectiveness and efficiency of their solutions. Self-talk, mathematics discourse, self-reflections, and self-knowledge were applied frequently during the problem-solving process.

What mathematics knowledge is demonstrated by prospective secondary mathematics teachers as they engage in mathematical problem-solving?

The prospective teachers' mathematics understanding was rooted in both procedural and conceptual knowledge. For them, mathematics was not a collection of procedures, rules, or formulas to be memorized. They perceived having knowledge of the concepts which underpin procedures, rules, and formulas as vital to their mathematics understanding. They demonstrated a non-reliance on rules, formulas, and procedures in problem-solving situations. For them, "getting a right answer" had very little meaning if the right answer could not be justified. Understanding mathematics included having the ability to explain reasoning and justify solutions. For them, learning mathematics included understanding mathematics; and, understanding mathematics eliminated their need to memorize formulas, rules, and procedures. The participants believed that

knowing why a procedure worked was as important as knowing how to use a procedure. The participants' desire to explain their reasoning and justify their solutions appeared to be a reflection of their persistence and autonomy.

What is the characterization of the interplay among prospective teachers' mathematical beliefs, mathematical behavior, and mathematical knowledge of prospective in the context of solving mathematics problems?

I analyzed what these prospective secondary mathematics teachers said and did while learning mathematics and solving mathematics problems. In doing so, I found six key principles that appeared to emerge from the core category of *Mathematics Understanding: Knowing How and Knowing Why.* The key principles were (a) "getting it [the solution] wrong" is a part of the mathematics learning process, (b) don't give up easily, keep trying (c) " playing" with the mathematics decreases anxiety and increases learning and understanding, (d) mathematics difficulty and challenges provide learning opportunities, (e) understanding mathematics involves explaining and justifying solutions in multiple ways, and (f) learning in a supportive environment.

These key principles represented the participants' mathematics-related beliefs, emotions, values, persistence, autonomy, and their views on mathematics learning and understanding and provided insights into how they viewed themselves as mathematics learners and prospective mathematics teachers. The main categories, central theme, and the six key principles lead to the development of a substantive theoretical model, *Knowing How and Knowing Why: Mathematics Knowledge and Understanding that Empowers.* Mathematics knowledge and understanding appeared to empower the participants in challenging and difficult problem-solving and mathematics learning situations. The model is a rendering of my interpretation of the meaning the participants

gave to their mathematics learning and problem-solving experiences while enrolled in a mathematics course designed to develop their deep understanding of school mathematics.

Development of the Model

The model*, Knowing How and Knowing Why: Mathematics Knowledge and Understanding That Empowers (*see figure 2) represents patterns and relationships between these patterns in the context of mathematics learning and mathematics problem-solving. It was developed by bringing together the key principles and key elements that emerged from the participants' insights around their mathematics learning, understanding, and problem-solving experiences and the process of making meaning of the participants' experience. The three main categories leading to the central theme were: affective behavior, meta-cognitive heuristics, and mathematical cognition. Each main category is represented by a circle. Diagramming was used as a way of capturing the relationships among the participants' affective behavior, mathematics cognition, and meta-cognition. Each pair of overlapping circles represented an interrelationship to emerge from the study. For example, the intersection of affective behavior and metacognitive heuristics was characterized as autonomy and persistence in a mathematics learning and problem-solving context. These interrelationships synthesized the process of making sense of the participants' mathematics knowledge and understanding, their mathematics-related beliefs, emotions, and attitudes, what mathematics learning and problem-solving was like for them, and what I observed in the classroom.

The intersection of the three overlapping circles characterizes the interplay of the three major categories (affect, metacognitive heuristics, and mathematical cognition) as *Empowering Mathematics Understanding* that represents mathematics knowledge and

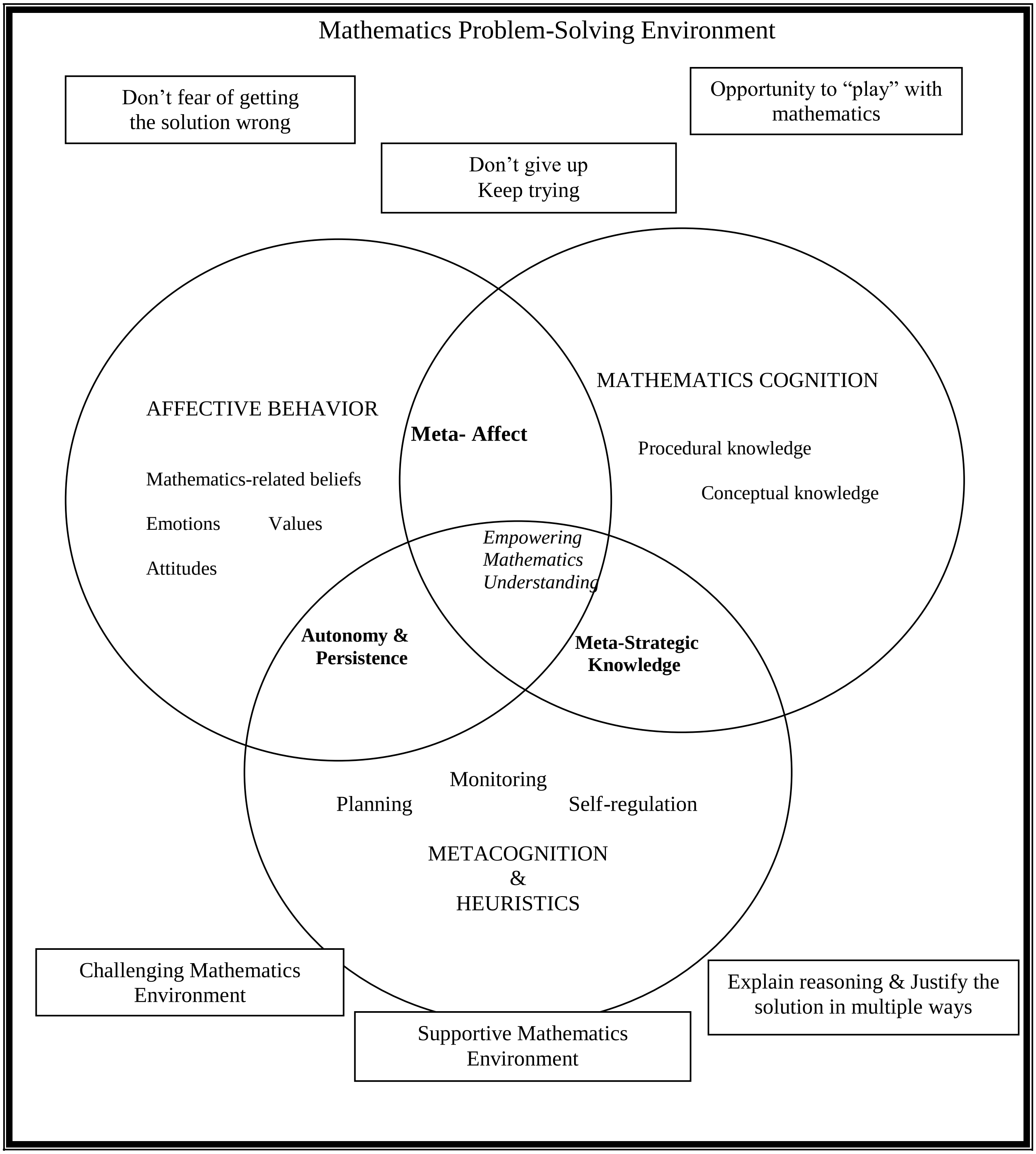


Figure 2:

Knowing How and Knowing Why: Mathematics Knowledge and Understanding That Empower

understanding that can lead to productive problem-solving experiences. This intersection represents mathematics intimacy, mathematics integrity, conceptual knowledge and understanding, procedural knowledge, and procedural fluency. The interactions occurred in a problem-solving and mathematics learning context; therefore, the overlapping circles are enclosed in a surrounding square that represents the mathematics learning and problem-solving environment and the culture in the classroom.

The key principles that emerged from the core category were vital to the participants' problem-solving competence, deep mathematics understanding, and metacognitive knowledge; and thus, were placed on the outside of the three circles but inside the square. For the prospective secondary mathematics teachers in this study, affective behavior, a metacognitive use of heuristics, and mathematical cognition interact and react to represent: *Knowing How and Knowing Why: Knowledge and Understanding That Empowers* (see figure 2). Key elements of the model are presented and discussed below.

Key Elements in the Model

Meta-strategic Knowledge

I characterize the interplay among metacognition and heuristics, and mathematical cognition as *meta-strategic knowledge*. This characterization is in keeping with the relationship between developing conceptual knowledge and metacognitive knowledge found in the study of Kuhn, Garcia-Mila, Zohar, and Anderson (1995). Meta-strategic knowledge is knowledge about where, when, and how to apply strategies and an understanding of the structure of the current mathematics task. It also includes

information about the strengths and weaknesses of each strategy and the effort involved in implementing the strategy (see figure 2 on page 150). None of the participants were successful in finding a specific equation for determining the solution to the diagonals problem. Obtaining the correct answer was not the objective. I was interested in exploring the processes they used to solve the problem.

The participants demonstrated conceptual understanding and procedural fluency and they used prior knowledge during the problem-solving process. They demonstrated their ability to use strategies and reasoning by investigating and selecting appropriate problem-solving strategies and using a process that could possibly lead to a correct solution. Understanding how strategies are related to each other and the current mathematics problem includes knowing when and when not to apply a specific strategy. Thus, meta-strategic knowledge appears to be an important factor to mathematics understanding and problem-solving competence. I conceptualize that the interaction between metacognitive/heuristics and mathematical cognition can be characterized meta-strategic knowledge.

Persistence and Autonomy

The participants' desire to explain their reasoning and justify their solutions was expressed during the interviews and observations. For these prospective secondary mathematics teachers, explaining the underlying mathematics of an applied procedure validates their mathematics understanding. In the literature, autonomy is described as having a belief that one is responsible for his/her own knowledge and answers and that mathematics is valid and acceptable when it makes sense to the prospective mathematics teacher (Confrey, 1994; Fennema & Romberg, 1999; Goodyear, 2000).The participants

believed that knowing why a procedure worked was as important as knowing how to use a procedure. The participants' desire to explain their reasoning and justify their solutions appeared to be a reflection of their autonomy. Throughout the problem-solving protocols and observation fieldnotes the participants demonstrated that failure to explain or justify reasoning is unacceptable. They have a willingness to increase efforts and persist in their attempt to understand mathematics and to reach their problem-solving goals.

Covington (1985) and Borkowski, Carr, and Rellinger (1990) proposed the theory of self-worth, which equates human value with ability. Those who believe failure is due to lack of effort are willing to work hard to achieve their goals. Those who believe that failure is due to a lack of ability will not put forth great effort because they hold the belief that they do not have the ability to succeed under any circumstance. The participants' beliefs and values related to mathematics and mathematics learning and understanding were in keeping with DeBellis and Goldin's (1999) construct of mathematics intimacy and integrity. The extent to which the participants' concern about and involvement in learning and understanding mathematics was a reflection of their intimacy with mathematics. Their focus on explaining why a procedure worked and not only on how to apply a procedure was a reflection of their mathematics integrity. In this study, mathematical intimacy and mathematics integrity were dimensions of the affective domain that provide important information about how the participants approached mathematics, mathematics learning, and problem-solving. I propose that the interplay between metacognition and affective behavior can be characterized as *persistence* and *autonomy.*

Meta-Affect

The prospective teachers' affective behavior is demonstrated throughout the study in varying situations. They experienced frustration, disappointment, and embarrassment when they are unable to explain or justify their mathematical thinking or problem solutions, but not so much when encountering mathematics difficulty. The participants' self-efficacy was demonstrated in the expectations they held about their mathematics abilities. They are willing to accept mathematics challenges because they believe they will be efficacious in meeting the challenge and in successfully performing the mathematics task. Strong self-efficacy beliefs were demonstrated in the effort and persistence they exerted during problem-solving and mathematics learning situations.

Their beliefs and values appeared to play a major role in how they interpreted the difficulties and challenges they experienced during mathematics learning and problem-solving. For them, learning and understanding mathematics deeply occurred when they were challenged by the mathematics, and not so much when they solved routine problems or the instructor worked through the solution process as they watched. When they were unable to successfully solve a problem, while they might experience frustration, they viewed that difficulty and frustration with the anticipation that learning and understanding the mathematics would result in a feeling of pride and satisfaction at the expected success. Goldin (2000a) proposed that this behavior represents the participants' meta-affect or how they feel about what they feel, value, or believe, which he suggested is a type of monitoring. Monitoring and regulating one's affect, metacognition, and cognition has been found to assist with successful problem-solving (Carlson & Bloom, 2005; DeBellis & Goldin, 1999; Goldin, 2000a). In this study, the interplay between

affective behavior and mathematics knowledge can be characterized as *meta-affect* (see figure 2).

Validating the Model

According to Strauss and Corbin (1998), in order to ensure that a theory or model is represented in the data, the adequacy of the study's research process and the grounding of the findings must be established. In this study, the substantive theoretical model, *Knowing How and Knowing Why: Knowledge and Understanding That Empowers*, conceptualized the interaction among affective behavior, metacognition, mathematical cognition. The model emerged through the open, axial, selective coding of solid, rich data. The core category, *Mathematics Understanding: Knowing How and Knowing Why*, was developed by identifying patterns and relationships through the shared meanings and interpretations with the participants. The main categories (affective behavior, metacognitive heuristics, and mathematical cognition) merged to represent these patterns and relationships. At the heart of the model is mathematics knowledge and understanding that is empowering for these prospective secondary mathematics teachers.

Discussion

I believe this study will make a valuable contribution to the body of knowledge on affect and cognition, because very little is known about the role affect plays in mathematics learning, understanding, and problem-solving. This study highlights the problem-solving and mathematics learning experiences of prospective secondary mathematics teachers who are enrolled in an upper level mathematics course. There is very little research addressing the knowledge and understanding associated with the interaction among meta-affect, meta-strategic knowledge, and persistence and autonomy.

This interaction is represented at the intersection of the affective behavior, metacognition, and mathematical cognition. The knowledge at this intersection appears to extend the application of conceptual, procedural, and meta-strategic knowledge in problem-solving situations.

In this study, it appears that having a deep understanding of procedures—knowledge of what, when, and how a specific procedure might work in conjunction with a specific strategy being used in a given problem-solving situation—can be as helpful as having conceptual or meta-strategic knowledge. Star (1999) suggests that this type of procedural understanding is deep and includes "knowledge of such things as the order of steps, the goals and subgoals of steps, the environment or type of situation in which the procedure is used, constraints imposed on the procedure, and any heuristics or common sense knowledge which are inherent in the environment" (p. 84). I propose that deep procedural understanding and knowledge together with metacognition, affect, and mathematical cognition lies at the intersection of the model and represents a kind of deep mathematics knowledge and understanding that is powerful in facilitating mathematics success.

In this study, affect appears to play an influential role in mathematics learning and problem-solving success; however, unlike in some research studies on affect and cognition (Harper & Daane, 1998; Mapolelo, 1998; Thompson, 1992) affective behavior does not necessarily impede learning. The participants in this study experience emotional behaviors such as anger, frustration, disappointment, and struggle; however, these emotional behaviors did not negatively influence their efforts or they way they interpreted the situation. A problem-solving environment that is challenging while at the

same time supportive, where mistakes and incorrect answers are viewed as learning opportunities, and one can experiment or "play" with mathematics, appears to positively influence mathematics learning and understanding.

Although these prospective teachers often encountered difficulty solving mathematics problems and often found that the mathematics they learned was challenging, their mathematics-related beliefs and values enabled them to interpret these situations as learning opportunities. While they often experienced frustration, disappointment, anger, and struggle when confronting mathematics challenges and difficult problems, they seemed to anticipate the feeling of satisfaction at an expected success. The participants demonstrated positive attitudes, in part, because of their desire to become effective mathematics teachers. Failure to learn the mathematics with understanding was not an option for these prospective teachers.

Implications

This research study investigated the interplay of prospective teachers' mathematics-related beliefs, attitudes, emotions, metacognition, and mathematics cognition as they participated in an undergraduate mathematics course focused on deepening their understanding of school mathematics in a problem-solving context. The findings in this study have implications for mathematics education researchers, mathematics teacher educators, prospective mathematics teachers, and curriculum developers. These insights come from the substantive theoretical model that emerged from the study but also from the process coming to the model through the interviews, observations, participant artifacts, and the process of making meaning of the participants' experiences.

Implications for Mathematics Education Researchers

Traditionally, mathematics education research has focused on cognitive or metacognitive aspects of mathematics learning and problem-solving. There are very few studies in mathematics education integrating cognitive and affective factors (McLeod, 1992) and even fewer studies investigating the intersection between prospective teachers' affect, metacognition, and cognition in mathematics learning and problem-solving contexts. Emotional responses demonstrated by those learning school mathematics and associated attitudes have been found to linger into college-level mathematics. Some studies (Harper & Daane, 1998; Mapolelo, 1998) have suggested that affect impedes mathematics learning. This study suggests that this might not be entirely true in the case of prospective secondary mathematics students. Affect does not appear to necessarily impede their mathematics learning, thinking, or problem-solving goals. It appears that their beliefs and the meaning they attach to what it means to be an effective teacher plays a significant role in mathematics learning.

The prospective teachers in this study experience frustration, anger, and disappointment in difficult, challenging mathematics problem-solving situations, but they appeared to interpret these challenges, not as a failure, but instead as opportunities to gain deep mathematics understanding. There seemed to be an expectation that the acquisition of deep mathematics understanding will be accompanied by frustration, anger, and disappointment. In other words, they demonstrated the attitude of "it comes with the territory."

This study sheds light on the interrelationships among affect, metacognition, and mathematics cognition calling into question the idea that affective behaviors negatively

influences academic achievement. Affective behavior in prospective mathematics teachers appears to be associated with the high expectations they hold for themselves as prospective teachers. Their passion, frustration, disappointment, and anger can be viewed as the driving force behind their persistence and autonomy and their desire to learn mathematics with understanding. For them understanding mathematics is a critical part of effective teaching. Perhaps, prospective teachers' affective behavior can be beneficial in motivating them to take the view that mathematics frustration, disappointment, and even anger should not be associated with failure. Previous research studies have failed to look closely at this relationship, which creates the challenge of building a robust theoretical knowledge base for this area of research. For researchers, the findings in this study suggest that future studies should consider the relationship among affective behavior and cognition.

Implications for Mathematics Educators and Mathematicians

The participants preferred not to be told how to apply a procedure to obtain a solution. They identify this as ineffective teaching. For them, getting a right answer is not indicative of mathematics understanding. This is demonstrated throughout the study in their interviews and observations. When unable to obtain a correct solution, they demonstrate emotional behavior such as disappointment, frustration, and anger; however, they view these emotional behaviors as an often necessary part of understanding mathematics deeply.

As prospective teachers, they have the desire to explain and justify their reasoning and solutions. Their interpretation of a learning environment or problem situation is based on their mathematics-related beliefs and values. Where others might interpret

failure to obtain a solution as their lack of knowledge, these prospective teachers viewed the difficulty as an opportunity to ask questions and obtain a better understanding of the underlying mathematics. An environment that is mathematically challenging, engaging, and collaborative is one in which learning and understanding will occur. This finding contributes to the body of literature that suggests that an environment that engages students in challenging mathematics problem-solving situations positively influences mathematics learning and understanding.

Because prospective teachers have a desire to deeply understand mathematics, teacher educators might consider providing challenging and robust mathematics problem-solving opportunities no matter how frustrating it might be for students. Students can develop their mathematics integrity when teacher educators provide robust mathematics problem-solving opportunities for students to become intimate with the mathematics. Problem situations that challenge students to spend quality time interacting with mathematics. A prospective teacher in this study stated that “when I am challenged, I learn” and another stated “even though the mathematics we learn makes me frustrated and angry, I know that I really understand the mathematics for the first time”. They view a difficult, challenging mathematics problem situation or an incorrect solution, not as a failure, but with anticipation for a feeling of satisfaction at an expected successful outcome. As teacher educators we can help students attend to their frustrations and let them know that frustration, disappointment, and struggle are critical aspects of learning and understanding mathematics. With that said, we can also provide a practicing environment where students can act as practicing mathematicians working on problem solutions in an environment that is supportive and nonthreatening.

Implications for Curriculum Developers

Curriculum developers can develop mathematics and mathematics methods courses that have problem-solving as its foundation. This can be a powerful vehicle to facilitate prospective teachers' deep mathematics understanding. Mathematics courses offered to prospective secondary mathematics teachers address conceptual understanding, procedural fluency, strategic knowledge, and reasoning. If the courses mathematics educators and mathematicians develop and offer to prospective secondary mathematics teachers fail to address disposition or affective factors, it is possible that teacher education programs will continue to produce teachers who lack the positive disposition associated with creating a positive mathematics learning environment and experiences for their students. According to Phillips (2007), teachers' affect is critically important. Therefore, if prospective or practicing mathematics teachers are to develop deeper mathematics knowledge and understanding, and productive mathematics-related beliefs then affect has to be addressed in the mathematics and mathematics methods courses offered to prospective mathematics teachers.

Summary

In summary, this study has implications for researchers, teacher educators, and curriculum developers. The study contributes to the knowledge base of mathematics and mathematics education by identifying prospective secondary mathematics teachers' mathematics-related beliefs and dispositions with respect to mathematics learning and problem-solving. It emphasizes the important need for research in the affective domain related to mathematics learning and understanding. It exemplifies the need for a theoretical framework for considering practicing and prospective teacher affect.

Limitations

While similarities may be drawn with the experiences of prospective secondary mathematics teachers, the experiences, mathematics-related beliefs, attitudes, emotions, metacognition, and mathematical cognition expressed or demonstrated by those who are interviewed and observed are individual and therefore, unique. In order to consider if the views of these prospective teachers are representative of others, the model needs to be validated through interviews and observations of prospective teachers at other colleges enrolled in similar mathematics course.

Fifteen participants agreed to participate in the study and after a theoretical or purposive sampling of four participants, no new information emerged from the data. The focus of the study is on the prospective teachers. The professor was not interviewed; however, during the participants' interviews it became apparent that she plays an important role in their perceptions of the learning environment and their motivation and perhaps even their beliefs. The professor is not asked to comment on the content of the prospective teacher interviews or my observation fieldnotes. Only the perspectives of the prospective teachers are reported.

The effects of the professor on the students' beliefs, knowledge, metacognition, and use of heuristics are not explicitly studied. The participants discuss their past experiences, their professor's role in learning mathematics, and the strategies they use to learn and understand mathematics, which indicates that the learning environment is closely connected to their mathematics-related beliefs and values. The teacher is a part of the learning environment; therefore, her mathematics-related beliefs, expectations, and the manner in which she presents the content might possibly send an unspoken message

to the students about the nature of mathematics, mathematics learning and teaching. The participants overall conceptual view of mathematics and their positive mathematics-related beliefs, values, and attitudes could be heavily influenced by what the professor says or does. Further study can be done to include the professor mathematics-related beliefs, values, and expectations to get a better understanding of the level of influence a professor has on their students' affective, metacognitive, and cognitive behavior.

Recommendations

Recommendations for Practice

Mathematics teacher educators and mathematicians can work with prospective secondary mathematics teachers as they develop beliefs about and a deep understanding of mathematics. They can support prospective secondary mathematics teachers in the need to reason, make conjectures, justify their solutions, and communicate mathematically. The value prospective secondary mathematics teachers place on justifying their reasoning and problem solutions can contribute to the construction of mathematics integrity, referring to their desire to understand the underlying mathematics associated with procedures. Mathematicians and mathematics teacher educators should be explicit about the behavior that could result in working difficult, challenging, non-routine mathematics problems. Instructors should let prospective mathematics teachers know that frustration and satisfaction comes with solving problems. An important affective goal in mathematics should not be to eliminate frustration or to make all mathematical activity easy and fun. Rather, instructors should encourage prospective secondary mathematics teachers in their development of meta-affect where their feelings or emotions associated with difficulty or impasse have a positive or productive impact on

their mathematics learning (Hannula, 2002). The feeling of frustration or anger with a mathematical problem solution process could indicate that the problem is challenging, non-routine or interesting. These feelings could be experienced with the anticipation of problem-solving success and understanding new mathematics deeply.

In this study, prospective secondary mathematics teachers' mathematics- related beliefs and values often influenced their emotions. They value reaching challenging mathematical goals related to understanding mathematics and justifying mathematical reasoning. Prospective secondary mathematics teachers who hold such values often hold beliefs that are productive. Beliefs that, although mathematics is sometimes difficult and challenging success is in fact likely to occur if accompanied by persistence and effort. As mathematics educators and mathematicians, we must help prospective secondary mathematics teachers to consider not only what mathematics they are teaching but also the learning experiences they create for their students. Teachers make important and critical decisions about how they present mathematics and if they hold mathematics-related beliefs, emotions, and dispositions that are counterproductive to mathematics learning and understanding or if they do not give explicit attention to their affect it could possibly negatively influence the mathematics-learning experiences they create for their students.

Recommendations for Research

Based on the limitations discussed above, future studies should examine the interplay of affect, metacognition, and mathematical cognition in a similar problem-solving context at other colleges. Also, because phenomenon being considered is in the context of a classroom and the professor is a vital aspect in the classroom, he or she

should be included in the study. I find it impossible to accept that the participants are in no way influenced by the professor's mathematics-related beliefs, values, and expectations. In fact, the participants often referenced her teaching methods. However, the question that remains for me is, "To what extent are the participants influenced by the professor's own affect, metacognition, and cognition?" The participants in this study are prospective secondary teachers, who generally have more positive affect than prospective elementary teachers toward mathematics. Do mathematics majors generally have more positive affect than prospective secondary mathematics teachers? Can the model be applied to mathematics majors?

I would like to follow the participants into their field experience to examine whether or not their teaching practices are in conflict with the mathematics-related beliefs and attitudes demonstrated and observed in this study. It would be interesting to discover whether the participants present the mathematics to their students from a conceptual orientation or a procedural orientation. Will the participant present mathematics with a focus primarily on procedural fluency? Will the participant encourage the students to focus on understanding the mathematics underpinning the procedures? There are studies to suggest that teacher-beliefs are known to influence their students' beliefs. It would also be interesting to learn if the mathematics integrity, mathematics intimacy, autonomy and persistence demonstrated by the participants during the study are active in their classroom. Are the students influenced by the mathematics-related beliefs, values, and attitudes of their teacher? How might the teacher's affective behavior influence the students?

Finally, this study examined the interplay of affect, metacognition, and cognition. The affective factors within the study only consider a subset of the participants' mathematics-related beliefs. The mathematics-related beliefs in this study focus primarily on teaching and learning, the nature of mathematics, and self-efficacy. There a numerous mathematics-related beliefs held by prospective teachers which when related to metacognition, heuristics, and cognition could produce very different results. Additional research can incorporate other beliefs such as beliefs about technology, beliefs about gender, and beliefs about reform. The participants in this study are prospective secondary teachers, who generally have more positive affect toward mathematics than prospective elementary teachers.

Closing Statement

The study offers new insights into the relationship among prospective secondary mathematics teachers' beliefs, affective behavior, metacognition, and mathematical cognition in a context of mathematics learning and problem solving. Few studies focus on the intersection between prospective teachers' cognition and their affective behavior during problem-solving (McLeod, 1992). As a result, the meaningful integration of affect, metacognition, and cognition is under-conceptualized and in need of new explanatory models (Schoenfeld, 1992). This study explains what prospective secondary mathematics teachers believe, think, feel, and do during mathematics learning and problem-solving situations and how what they believe, think, and feel interact to influence what they do in a mathematics learning and problem-solving context.

These prospective teachers' overall beliefs and dispositions are productive. Mathematics makes sense to them; it is useful, worthwhile, challenging, and with effort and

persistence, they perceive that they have the capability to learn and understand mathematics deeply. The prospective teachers in this study want to understand mathematics deeply; and for them, understanding mathematics is integrally related to understanding the mathematics underpinning the procedures they apply to solve problems, solving problems using multiple solution paths, and explaining their reasoning and justifying their solutions.

As mathematics teacher educators and researchers, we should not only support prospective teachers in their development of conceptual understanding, procedural fluency and understanding, and strategic competence; we must also provide opportunities for prospective teachers to become more aware of their affect and the role it plays in their mathematics learning. We can support prospective teachers in developing positive affect and mathematics dispositions by creating positive challenging mathematics learning and problem-solving experiences. Then, as a result, prospective teachers will gain experience in not only monitoring and controlling their cognition but also their affect toward mathematics, mathematics learning, and problem-solving. This will perhaps enable prospective teachers to think about both the mathematics they teach and the mathematics learning and problem-solving experiences they create for their students.

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APPENDIXES

APPENDIX A

CONSENT FORM

Georgia State University

Department of Middle, Secondary, and Instructional Technology

Title: Conceptualizing Prospective Teachers' Affective Behavior, Metacognitive Behavior, and Mathematics Cognition During Problem Solving.

Principal Investigator: Dr. Christine D. Thomas, MSIT

Student Investigator: Ms. Belinda P. Edwards, MSIT

Sponsor: None

I. Purpose:

You are invited to participate in a research study. The purpose of this study is to determine how prospective middle-grades and secondary mathematics teachers' affective behaviors, metacognitive behaviors, and mathematical cognition interplay as you conjecture, reason and communicate mathematically, and solve mathematics problems.

You are invited to participate because you are currently a prospective middle-grades or secondary mathematics teacher enrolled in the course entitled Advanced Perspectives on Mathematics. At least 3 participants will be recruited for the study.

II. Procedures:

If you decide to participate, you will be observed during your Advanced Perspectives on Mathematics course twice per week as you interact with mathematics, your peers who are also enrolled in the course, and your instructor. You should understand that all students in the mathematics course will solve mathematics problems during class; therefore, participation in the study does not involve any extra assignments for you. You should also understand that you will be interviewed about your previous mathematics experiences and your mathematics-related beliefs. One of the interviews will be videotaped as you explain your thinking as you solve a mathematics problem. You will be interviewed five times over a period of eight weeks (January 2008 – February 2008). The interviews will take between 30 to 60 minutes and will be audio-taped (one will be videotaped) and transcribed by the investigator. The investigator will use a pseudonym rather than your name on all records. All transcripts of the interviews, your inscriptions and work, and a copy of the videotape will be kept under lock and key in a private office and destroyed by fire at the end of the study.

III. Risks:

There are no known risks or discomfort to you from participation in the study. However, there might be times when you feel uncomfortable working or discussing mathematics problems while being observed or you might feel embarrassed when you are not successful when solving a mathematics problem. You might also reveal something in your mathematical background or a belief about mathematics that you might later

regret. You can be assured that none of the information you provide will be traced back to you personally or used against you in any way. There will be no judgments made about your ability or inability to perform mathematical tasks.

IV. Benefits:

Participation in this study will not likely be any direct benefit to you, but knowledge gained from this study may contribute to a better understanding of how affective behavior, mathematical behavior, and use of mathematical understanding and knowledge interplay as you learn mathematics. The information gain from this study will inform and assist mathematicians and mathematics educators in their efforts to improve prospective teacher education.

V. Voluntary Participation and Withdrawal:

Participation in research is voluntary. You have the right not to be in this study. If you decide to be in the study and change your mind, you have the right to drop out at any time. You may skip questions or stop participating at any time. Whatever you decide, you will not lose any benefits to which you are otherwise entitled. Whether you choose to participate at all, or decide not to continue at a later time, will have no effect on the grade you receive in your Advance Perspectives on Mathematics course.

VI. Confidentiality:

We will keep your records private to the extent allowed by law. We will use a pseudonym rather than your name on all records. The key that connects you to the pseudonym will be kept in a locked file cabinet in my private office. All transcripts of the interviews, the videotape, and your inscriptions and work will be kept under lock and key in a private office. Only the principal investigator (Dr. Christine Thomas) and principal student investigator (Belinda Edwards) will have access to the information you provide. It will be stored in a file cabinet under lock and key and on a password- and firewall-protected computer located in the student investigator's private office. The key to the file cabinet will be stored in a separate location from the data to protect your privacy. Your name and other facts that might point to you will not appear when we present this study or publish its results. The findings will be summarized and reported in group form. You will not be identified personally. The audiotapes, videotapes, transcribed interviews, your inscriptions and work will be destroyed by fire at the end of the study.

VII. Contact Persons:

You may call or email Dr. Christine Thomas at 404 - 413- 8065, cthomas11@gsu.edu or Belinda Edwards at 770-420-4727, bedwards@kennesaw.edu if you have questions about this study. If you have questions or concerns about your rights as a participant in this research study, you may contact Susan Vogtner in the Office of Research Integrity at 404-413-3513 or svogtner1@gsu.edu.

VIII. Copy of Consent Form to Subject:

We will give you a copy of this consent form to keep.
If you are willing to volunteer for this research and to have your interviews audio taped and video taped, please sign below.

____ _________________

Participant Date

____ _________________

Belinda P. Edwards (Co-investigator) Date

APPENDIX B

INTERVIEW 1

At the beginning of the interview, the informant will be reminded that they are not obligated to answer questions that they wish not to answer.

1. Tell me about some of your experiences learning mathematics?
2. Do you like to solve mathematical problems? If so, why?
3. What kind of mathematical problems do you like to solve?
4. Do you think you are a good at solving mathematics problems? Why? Why not?
5. What do you think makes someone a good problem solver?
6. What do you think is most descriptive of mathematics?
7. What do you think is least descriptive of mathematics?
8. What do you like most about mathematics?
9. What do you like least about mathematics?
10. Describe what learning mathematics is like for you?
11. What was learning like you in elementary school? Middle school? High school?
12. What is the hardest thing about learning mathematics?
13. How do you feel when you are asked to solve unexpected mathematics problems?
14. What do you think it means to learn mathematics?
15. Are there times when it is more important to learn mathematics through memory as opposed to understanding? Explain.
16. How do you relate to mathematical ideas you have learned?

17. What does it mean to be creative in mathematics?
18. What do you think are the important components of a mathematics problem solution?
19. Describe your view of mathematics?
20. Why do you think you have this view?
21. What do you think is the difference between a conceptual understanding and a procedural understanding?
22. Do you think one [conceptual versus procedural] is more important to have than the other is?
23. What is the most important thing you can tell me about your beliefs about mathematics?
24. For you, what does it mean to understand mathematics?
25. Do you think that in order to learn and understand mathematics you have to enjoy it?

APPENDIX C

INTERVIEW 2

At the beginning of the interview, the informant will be reminded that they are not obligated to answer questions that they wish not to answer.

1. Describe the best environment for mathematics learning. *[Possible follow up questions: What would you be doing in this environment? What would the teacher be doing?]*
2. Describe the characteristics of a good instructor.
3. Describe the characteristics of a bad instructor.
4. Why do you want to teach mathematics?
5. When did you decide you wanted to be a mathematics teacher? *[At what point in your life; college, elementary, middle, high school, etc.]*
6. What is the most important thing your instructors can do to help you become, what you consider, an excellent teacher?
7. As a prospective teacher, what do you think is the most frightening aspect of being a teacher?
8. What are some things other people do to help you learn mathematics?
9. What role should your mathematics instructor play in your learning?
10. What are some strategies you use to help you learn mathematics, understand mathematics, or do mathematics?
11. How do you learn mathematics?
12. What do you do after you have solved a problem?

13. How long would you work on a problem to find a solution? Explain.
14. What do you do if you are unable to solve a problem or you are having difficulty with your homework?
15. What do you do when you do not understand a mathematics concept explained by your instructor during class?
16. How do you feel when you don't understand the mathematics being explained?
17. How do you feel about being assigned homework problems that the instructor has not previously reviewed in class? Explain.
18. How do you feel about problems being placed on the test that the instructor has not reviewed in class? Explain.
19. If you could create the perfect learning environment, what characteristics would it have?
20. How important do you think it is to get the right answer when solving a problem?

APPENDIX D

INTERVIEW 3

VIDEO TAPED TASK-BASED THINK-ALOUD INTERVIEW

Instructions:

The title of this study is "Affective Behaviors, Metacognition, and Cognition in a Problem Solving Context." I am interested in the processes you use when solving problems. I cannot read your mind; however, the think-aloud method will help me understand your ideas while solving problems. There is no time limit in which you are expected to complete this problem.

Please always speak aloud while you are working on these problems and describe how you are solving them. Your participation will be videotaped. The videotape will be erased when the study is completed and a pseudonym will be used anytime the videotape is referenced.

Here are paper and pencils for you to use. You can use your calculator if you wish. When you use it, just let me know why you chose to use it. Remember to speak aloud as you work on the problem. Remember, to solve this problem, you can use as much time as you need. Do you have any questions before we start?

THE DIAGONAL PROBLEM

How do you determine the number of diagonals possible in an n-sided polygon?

APPENDIX E

INTERVIEW 4

VIDEO TAPED THINK-ALOUD FOLLOW-UP INTERVIEW

Instructions:
Before starting the interview, the informant will be given copies of the problem and the participant's solution paper from the think-aloud session to be reviewed with the interviewer. I will also play the video and stop it at specific points to question the participant about their thinking and feelings as they solved the problems. The following questions below will be used as a guideline for the interview. Other questions may be added to further prompt informants' thought processes when they answer in varying ways. If the participant does not understand what they are being asked, the interviewer may clarify the question.

Understanding the Participant's Problem Solving Processes

Understanding the Problem

Tell me about this problem?
Have you ever seen a similar problem like this before?
If yes, did it affect how you solved this problem?
Did you have difficulty understanding the given information in the problem?
If yes, explain what parts confused you? How did this make you feel?
What did you do after reading the problem?
Before you started working, did you think the problem was difficult to solve?

Planning

How did you plan to solve this problem at the beginning? Explain.
What did you do to overcome any difficulties?
Please explain your solution plan to me?

Executing

How did you decide to carry out your solution plan?
What mathematical content did you use? Explain why and how.
What mathematical strategies and procedures did you consider as potentially useful for solving the problem?
Did you follow your solution plan? If no, explain why not.
How did you know you solved the problem correctly?
What did you do when you got stuck on the problem?

Verifying

How can you be sure that your solution is correct?
Did you check that your solution with your plan and the given conditions of the problem?

Is it possible to get the correct answer and still not understand the problem? Explain.
Do you have any other comments about your work and thoughts while working on this problem?

APPENDIX F

CODE NOTES

Category	Code Note
(+) Math Emotions (+AE)	Excitement, enthusiasm, confidence, smiles, laughter, love, satisfaction, pride, like, joy.
(-) Math Emotions (-AB)	Frustration, dislike, hate, intimidation, confusion, insecurity, anxiety, hesitation, struggle, pain staking effort, aggravating.
Emotional Actions (AEB)	Nail or Pencil Biting, Frowning, Long Pause, Sighing, Pencil Tapping,
Beliefs About Math (BAM)	Nature, Difficulty, Learning, Understanding, Teaching
Positive Self Belief (+BS)	Ego, math certainty, ability to explain, understand, and do math
Negative Self Belief (-ABS)	Inability to explain or do math
(+) Math Attitudes (+MA)	Enjoyment, Interest, Engagement
(-) Math Attitudes (-MA)	Dislike for a specific math topic/area
Problem Solving Strategies (PSS)	Strategic methods to assist with problem solving: draw a picture, write down or organize given information; take your time,

	play with it, file back through, endure/persist, ask for help
Metacognition (MC)	Plan, Reflect, Rethink, Restate, Recognize what works, Verify, Conjecture, Talk it through, Detect Errors, Correct Errors
Procedural Math Engagement (PME)	Procedural Thinking/Understanding: Explores problem and Executes algorithms, expresses desire to justify procedures, monitored trial and error, feels the need to recall and use a procedure
Conceptual Math Engagement (CMU)	Conceptual Thinking: Justifies procedures, Uses multiple approaches to solving and explaining, Uses number sense or approximate, Checks for reasonableness of answer, Looks back and provides summary.

APPENDIX G

Mandy's transcribed and coded task-based think-aloud interview.

Mandy's Transcribed Interview	Open code located in the margins of the transcribed interview.
(1) Participant Reads the Problem	Initial Engagement/Motivation
(2) *I'll start by drawing a polygon for each one to make it easier to count the diagonals.*	Strategy
(3) *Say I had a 5 sided, one, two, three, four,*	Organizes Knowledge
(4) *five sided* (draws a pentagon) *so you have*	Strategy
(5) *to have n times n minus two divided by 2.*	Procedural Knowledge
(6) (She writes on the paper $\frac{n(n-2)}{2}$).	Organizes Math Knowledge
(7) *Because you can draw a diagonal from every vertex to another vertex in the polygon except the one that's right next to it because then it wouldn't be a diagonal, because if you did that it would be a straight line so that means if you have a 5-sided, you can only have*	Explains/Justifies a Math Procedure
(8) *uhmmm 1, 2, 3, you should have 3, okay*	Conjecture

(9) *1, 2, 3 and the reason you're dividing it by 2 is because you only want to use one, each vertex one time, otherwise you would have uhmmm, you'd end up with 6 of them but some of them are already used again.*	Explains/Justifies a Math Procedure
(10) *So, if you had an n-sided polygon, say for instance you had a 5-sided it would be*	Explains/Justifies a Math Procedure
5 times (11) 2 divided by 2 (she writes $\frac{5(5-2)}{2}$).	Organizes Math Knowledge/Procedure
(12) *Which is* $\frac{5(3)}{2} = \frac{15}{2}$.	Executes a Procedure
(13) *Hummm, you can't come out with an uneven number, humm, what am I doing*	Monitoring, Reflecting on Solution
wrong.	Exhibits Emotional Behavior
(14) (long pause)	
(15) *Hummm n times n minus 2.*	Monitoring
(16) *I know that's right… I remember this…..*	Confidence, Recall
(17) *Okay, let me see, so I know when I*	Monitoring
(18) *have a triangle* (she draws a triangle)	Math Knowledge, Strategy
doesn't have any so that would be …..	Organizes Knowledge
(19) (hand to chin, twist lips/mouth to left side)	Exhibits Emotional Behavior
(20) *So that would have been n= 3, so 3 times 3 minus 2 over 2. So that's 3*	Monitoring/Checking/Reflecting

times 3 minus 2 is 1 over 2	
(21) (She writes $\frac{3(3-2)}{2}=\frac{3}{2}$) *that's 3 halves.*	Organizes Math Knowledge
(22) *Okay, what am I doing wrong? Its n times n minus 2* (she writes $n(n-2)$.	Monitoring/Reflecting on Solution
(23) *So let's draw a square. That's an easy one, so 1, 2, 3, 4 sides* (draws a square and its diagonals), *so you get 1, 2 diagonals so I'll find that will be*	Strategy
(24) *2 times 2 minus 2* (she writes $2(2-2)=2(0)=0$,	Executes Procedure
(25) *so no that's wrong.*	Monitoring/Reflecting on Solution
(26) *Because you'll end up with 0 and 2 times,*	Conjectures
(27) *that's 0 and you do have 2 diagonals here*	
in the square I drew, not 0.	Monitoring
(28) *Okay, let's see….humm…. 1, 2, is it that minus?*	Monitoring
(Long pause, hand to chin, taps pencil to desk).	Exhibits Emotional Behavior
(29) *Okay, that's what's wrong… you have 4 sides, okay 1, 2, 3, 4. So, n is 4.*	Monitoring/Reflection
(30) *4 times 4 minus 2 over 2, right?*	Organizes Math Knowledge
(31) (She writes $\frac{4(4-2)}{2}=\frac{4(2)}{2}=\frac{8}{2}=4$). *4 minus 2 is 2 and 4 times 2 is 8 divided*	Executes a Procedure

by 2 is 4 and there is only 2 diagonals.	
(32) *Is it, could it be 2 times n minus 2 over 2?*	Conjecture
(33) (She writes $\frac{2(n-2)}{2}$). *Let's try that one. which would be 4 minus 2 is 2 and 2 times 2 is 4 and that's over 2, that is 2.*	Organizes Math Knowledge Executes Procedure
(34) *Let's see if it works for the other one. Let's see,*	Monitoring/Reflecting/Checking
(35) *5 sided would be 2 times 5 minus 2 which is 3 times 2 is 6. That is divided by 2 is 3.*	Executes Procedure
(36) (She writes $\frac{2(5-2)}{2} = \frac{3(2)}{2} = \frac{6}{2} = 3$).	Organizes knowledge
(37) *Okay, that's what I did wrong; so it's 2 times n-2 over 2.*	Reflecting on process
(38) (She writes, $\frac{2(n-2)}{2}$).	Organizes Knowledge
(39) *This is how you do it for the nth polygon; n = the number of sides of a polygon. n is the number of sides. I think that works.*	Explains/Justifies a Procedure
(40) Wow (shakes her head left to right). *Harder than I thought.*	Exhibits Emotional Behavior Reflection on difficulty of the problem

APPENDIX H

Mark's transcribed and coded task-based think-aloud interview.

Mark's Transcribed Interview	Open cod**e located in the margins of the** transcribed interview.
(1) Reads the problem	Initial Engagement/Motivation
(2) *What I'm gonna do is, I'm gonna draw one shape.*	Strategy
(3) *I'm gonna make a table, okay I'm gonna start with a shape that has 4 sides.*	Strategy
(4) *So, how many diagonals does it have, it has 2 diagonals and it has one point, two points, three, four.*	Mathematical Cognition
(5) *Okay, now I'm gonna draw the next shape with five sides, it should have 5 points.*	Strategy Conjecture
(6) *Okay, that's right it has one, two, three, four, five diagonals*	Monitoring
(7) *A six sided polygon* (He draws the polygon) *is probably gonna have 7*	Mathematical Cognition Conjecture

diagonals. (He draws the diagonals for the six sided figure)	Strategy
(8) *Yeah, it does, that's right.*	Monitoring
(9) (He goes back over everything he has already done)	Reflects on process
(10) *Okay, now I'm gonna draw a seven sided polygon.*	Strategy
(He attempts to draw a seven sided figure, but then erases it--struggles)	Affect
(11) *Well I'm seeing that the first point that I go to has four diagonals coming from it. The second one has one, two, three, four. The third one has one, two, three.*	Conjecture Organizes thinking
(12) *I'm seeing if there is a pattern to the number of points and trying to see if I can get it into a formula or rule.*	Organizing
(13) (Silence, He goes back to the square and redraws the diagonals and again to the pentagon; struggles)	Monitoring, Reflecting on process Affect
(14) *I see that there is a pattern. There is a pattern with the number of points.*	Conjecture
(15) (He scratches through the hexagon	Affect

and redraws it along with all its diagonals--struggles)	
(16) *The picture and table helps because I'm beginning to see a pattern. Now all I have to do is figure out how to write what's happening in the table into a formula.* (Rethinks process)	Monitoring, Reflecting on process
(17) *I'm seeing that however many points the figure has, if you take away two from it that's how many points have diagonals it has coming from it.*	Monitoring Mathematical Cognition
(18) *So it looks like whatever you want to use for points—say its n minus 2.*	Conjecture
(19) *No, no I'm doing sides, never mind.*	Monitoring, Reflecting on process
(20) *Okay, well it still works the same.*	Monitoring, Reflecting on process
(22) *Sides minus 2 gives you how many…*	Executes a procedure
(23) *(long pause—10 seconds)*	
	Affect
(24) (He looks back over what he has written)	Reflects on process
(25) *Okay, sides minus 3 factorial gives you the number of diagonals.*	Executes

(26) *Okay, I think this will work. Let me try a few numbers to see if it works.*	Conjecture
(27) *Okay there are 4 sides, a square so that's 4(6-3) 4(3!) which is 4(6) which equals 24.*	Mathematical Cognition
(28) Okay that doesn't work. *(He goes back to the beginning of the problem and reconsiders his steps)*	Reflects on process
(29) *Alright, let's see. (4 – 3)! + 1 No, maybe… N(N-3)!*	Executes
I think that works because you have N – 3. You're gonna go 2, 2!	Conjecture
(30) (He rethinks his steps, taps his pencil, sits silently for about 20 seconds)	Reflects on process
(31) *There has to be something like the number of sides times something. I think there is a relationship between the number of sides and the diagonals. The number of sides, factorial.*	Reflects on problem and his solution
N! + 3 (Sits in silence)	Affect
(32) *I'm not sure*…….(disappointment)	Affect

APPENDIX I

Tanya's transcribed and coded task-based think-aloud interview.

	Tanya's Transcribed Interview	Open code located in the margins of the transcribed interview.
(1)	Reads the problem	Initial Engagement/Motivation
(2)	*Convex being a really cool word because it means it's on the inside.*	Mathematical Cognition
(3)	*If we did, you can't have a one sided or two sided shape so you have to start with three sides. So the number of diagonals inside will actually be the wall.*	Mathematical Cognition
(4)	*Do a triangle it has 0 diagonals. So if we do a square, then that would be two diagonals.*	Mathematical Cognition
(5)	*I'm going to use colored pencils to keep me from losing track of the number of diagonals I'm drawing.*	Strategy
(6)	*Okay, I'm going to draw a pentagon and draw it's diagonals* (She draws a	Strategy

triangle, square, and pentagon along with their diagonals)	
(7) *I'm going to predict now that I know that I have enough …I have a finite difference.*	Conjecture
(8) *The first difference is 0, second one being 4, the next one being 9.*	Mathematical Cognition
(9) *Okay, so we have 0^2, 2^2, 3^2; just for fun let's do the next one which will be six sides.*	Organizes thinking
(10) *I would have never thought about finite differences, but since we've been working with them in this class I'm beginning to like that idea.*	Knowledge of Self
(laughter)	Affect
(11) *Okay, I'm going to use colored pencils because that makes it easier to double check my diagonals.*	Strategy
(12) (She goes back to the previously drawn polygons and recounts the diagonals for the triangle, square, and pentagon)	Monitoring, Reflecting on process
(13) *5, 10, 15, 18, that would not be four*	Organizes thinking

	cubed. That would not help my pattern at all.	
(14)	*Can I get 4 another way?*	Conjectures
(15)	*So the factors of 18 are 2, 9, 3, 6; factors of 9 are 9, 3; factors of 4 are 2. So none of those are going to work. No, that won't work.*	Mathematical Cognition
(15)	*That would be linear, that would be quadratic, that would be cubic, tougher but possible.* (trying a number of different solution methods, self talk)	Mathematical Cognition
(16)	*Okay, let me think. I vaguely remember a concept something about the number of sides related to the*	Organizes thinking
	number of diagonals. So, the square has 4 sides and 2 diagonals; so, 2^2 equals 4.	Mathematical Cognition
(17)	*5 sides to 9 diagonals and 6 sides to 18 diagonals.*	Mathematical Cognition
	Hum, that's not working.	Monitoring, Reflecting on process
(18)	*Okay. Let's play with it.* (She draws a table)	Strategy

(19) *Okay let's go back and take a look, the square had 4 starting points and they all went to one place and you ended up with 4.*	Monitoring, Reflecting on process
(20) *The pentagon had 5 starting points and you ended up with 9. That one* (points to the hexagon) *had 6 and it went to 18. Okay, now I have a different thought.*	Rethinking process
(21) *Let's see; that's what it was; one* [diagonal] *out of each point, two out of each point, three out of each point…so this one will probably have 4 out of each place.*	Conjecture
(22) *The only reason I noticed that, is because I used colored pencils. There is no way I would have made that correction had I not used colored pencils because the numbers would not have jumped out at me like the colors did.*	Reflecting on process
(23) *Looking at the table, I think there might be a pattern, but now how do I*	Conjecture

write it so I can explain it to someone else.	
(24) *This is where talking it through out loud really helps because I can process it better.*	Monitoring
(25) (long pause, taps pencil, looks over previous work)	Affect Monitoring
(26) *All those* (referring to the table values) *have a difference of one, so it would be one plus the previous one.*	Mathematical Cognition
(27) *I'll need to go back and clarify if you have less than 4 sides 'cause that would make a difference in the function.*	Monitoring
(28) *After that you can have an infinite number of sides and that would follow the pattern and* (struggles)….*Hum…*	Mathematical Cognition Affect
(29) *Now, it's like I too many ways I can look at this problem. It's like which way should I do it, they're all running together.*	Monitoring, Reflecting on process
(30) *Well, at least I'll be able to explain it*	Reflecting

	in several different ways.	
(31)	(long pause, 3 minutes)	Affect
(32)	*Okay, I'm thinking about how do I do this. I know in my head, okay think back*	Self talk

APPENDIX J

Cindy's transcribed and coded task-based think-aloud interview.

Cindy's Transcribed Interview	Open code located in the margins of the transcribed interview.
(1) *Reads the problem*	Initial Engagement/Motivation
(2) *Hmmmm, let's see*	Mathematical Cognition
(3) *Okay, so you're asking me to explain how I would go about finding the equation that I can use to tell how many diagonals there are in any given polygon, right?*	Restate the problem, Make sense of problem
(4) *Well, let's see the first thing ….*	Planning
(5) *Well, the first thing I'm gonna do is draw a picture because I'm a visual person.*	Planning, Strategy Metacognition
(6) *Okay, a triangle first then a square, a pentagon, a hexagon. Okay that's probably far enough.*	Organizing thinking
(7) *Like I said before, drawing a picture or diagram is something I always try to do because it always helps me visualize the problem. (lines 5, 6, 7)*	Knowledge of
(6) *(She draws a triangle, square, and pentagon along with their diagonals)*	Strategy
(7) *Now that I have them drawn, I can draw in the diagonals for each one*	Planning
(8) *Okay the triangle doesn't have any [diagonals]*	Mathematical Cognition
(9) *The square has two, pentagon 5, hexagon, hmmmm, 1,2,3,4,5,6,7,8, Okay the hexagon has 9.*	Mathematical Cognition' Organizes thinking
(10) *Okay, so 2, 5, 9. (she writes these numbers in a vertical column) see if there might be some kind of pattern (she taps pencil, smiles)*	Reflects Organizes thinking Affect
(11) *Okay for the square that's 4 sides and 2 diagonals; Well the triangle has 3 sides but no diagonals, so that's 3 and 0. Hmmmmm; the pentagon has*	Mathematical Cognition

	5 sides and 5 diagonals; the hexagon has 6 sides and 9 (she is now pairing the sides with the diagonals—sides in one column, diagonals in another)	Strategy Organizes thinking
(12)	*Hmmmmm, let's see (she looks back over what she has done)*	Monitoring, Reflecting on process
(13)	*So, 3 for 0; 4 for 2; 5 for 5; and 6 for 9. There has to be some kind of pattern in this.*	Organizes thinking and Reflects Conjecture
(14)	*Okay, I'm doing that right. Okay, I think the vertices and the diagonals are related, not the sides. Well… vertices and sides are the same. So, the only thing I need to do now is to find the equation that can represent this relationship between the vertices and the diagonal.* (silence for 4 minutes)	Conjecture Monitoring, Reflecting
(15)	*Well… okay, let's see*	Monitoring
(16)	*Okay it's probably something related to the diagonals and the vertices, since I use the vertices to draw the diagonals.*	Conjecture
(17)	*Let me try, let me try a guess first and then plug in some numbers…hmmmm*	Strategy Organizes thinking
(18)	*Let's see… 4 and 2; 4 – 2 is 2; 5 and 5; 5-2… N-2 maybe…no that's not right. Well, let's see (N-2) could be …. N is the number of vertices*	Mathematical Cognition Monitoring, Reflecting on process
(19)	*Okay let me go back and look at the picture….(silence for 3 minutes)*	Monitoring
(20)	*Alright..the square. I can draw 2 diagonals but I only use 2 vertices, the other 2 I don't use because if would just be a duplicate.*	Monitoring, Reflecting on process
(21)	*Now, the pentagon. I can draw the diagonals from 3 of the vertices. If I use the other 2, I would be duplicating again. But the third vertex has one has 1 less*	Rethinking process
(22)	*Okay, I going to guess again. Let's see. (N-2) times 2. (she writes the expression)*	Conjecture
(23)	*So let me check to see if this works. The square…4 vertices so 4 – 2 = 2;*	Reflecting on process

so that works. The pentagon…5 -2 = 3; No that doesn't work. The hexagon, 6 – 2 = 4. No that won't work. Okay go back…	Monitoring
(24) *Hmmmm. Maybe N-3 will work. 4 – 3 = 1; 5-3=2; 6-3=3* *Square is 4 – 3 = 1* *Pentagon is 5 – 3 = 2* *Hexagon is 6 – 3 = 3*	Conjecture
(25) *Okay, I have to be close. I see by subtracting I get 1, 2, 3. There is a pattern here.*	Monitoring
(26) Okay multiply by 2 I get 2 for the square; *Multiply by 2 I get 10 for the pentagon;* *Multiply by 3 I get 18 for the hexagon.*	Mathematics Cognition
(27) *Okay, that's it I divide by 2 and that gives it ... So (N – 3)3 divided by 2*	Monitoring
(28) *I have no idea how I got that. I mean, I can show you, but I basically used guess and check.*	Self-questioning, awareness Reflection, Awareness
(29) *I would definitely need to go back and see if I can figure out why you would divide by two. Because to be honest, I basically guessed and then when back and checked.*	Reflection, Awareness

APPENDIX K

Data Collection Schedule

Monday	Tuesday	Wednesday	Thursday	Friday
January 7 Semester begins	January 8 Explain Study Issue Consent & Background Info.	January 9	January 10 Collect Consent & Background Info. Form	January 11 Note Taking & Coding Background Info.
January 14	January 15 Classroom Observation	January 16 Note Taking & Coding	January 17 Classroom Observation	January 18 Tanya Interview1
January 21 Note Taking & Coding	January 22 Snow Day * Memo writing	January 23 Memo writing Coding	January 24 Classroom Observation	January 25 Note Taking Coding
January 28 Cindy Interview1 Coding	January 29 Classroom Observation Note Taking	January 30 Mark Interview1 Coding	January 31 Classroom Observation Note Taking	February 1 Memo writing Coding
February 4 Review the Literature	February 5 Mandy Interview1 Classroom Observation	February 6 Note Taking Mark Interview2 Artifact Coding	February 7 Classroom Observation Coding	February 8 Cindy Interview2 Coding
February 11 Memo Writing	February 12 Classroom Observation Tanya Interview2	February 13 Mandy Interview2 Coding	February 14 Classroom Observation Cindy Interview3	February 15 Coding Notes
February 18 Review the Literature	February 19 Classroom Observation Artifact Coding	February 20 Classroom Observation Coding	February 21 Classroom Observation	February 22 Review the Literature
February 25 Review the Literature	February 26 Classroom Observation	February 27 Coding Memo Writing	February 28 Test Classroom Observation	February 29 Coding/Memo writing

March 10 Cindy Video-taped Interview 3	March 11 Cindy Retrospective Interview 4 Classroom Observation	March 12 Tanya Video-taped Interview 3 Coding	March 13 Tanya Retrospective Interview4 Classroom Observation	March 14 Coding Sorting
March 17 Coding Sorting	March 18 Mandy Video-taped Interview3 Classroom Observation	March 19 Mandy Follow-up Interview5	March 20 Mark Video-taped Interview 3 Classroom Observation	March 21 Mark Follow-up Interview4 Sorting /Coding