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Transformation Groups and Duality in the Analysis of Musical Structure

Janine du Plessis

TRANSFORMATION GROUPS AND DUALITY IN THE ANALYSIS OF MUSICAL STRUCTURE

by

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Under the Direction of Dr. Mariana Montiel

ABSTRACT

One goal of music theory is to describe the resources of a pitch system. Traditionally, the study of pitch intervals was done using frequency ratios of the powers of small integers. Modern mathematical music theory offers an independent way of understanding the pitch system by considering intervals as transformations. This thesis takes advantage of the historical emergence of algebraic structures in musicology and, in the spirit of transformational theory, treats operations that form mathematical groups. Aspects of Neo-Riemannian theory are explored and developed, in particular the T/I and PLR groups as dual. Pitch class spaces, such as $\mathbb{Z}_{12}$, can also be defined as torsors. In addition to surveying the group theoretical tools for music analysis, this thesis provides detailed proofs of many claims that are proposed but seldom supported.

INDEX WORDS: Group theory, Flat torus, Torsors, Mathematical music theory, Transformational theory, Duality, Pitch class set, *Tonnetz*

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1. INTRODUCTION

The mere thought of a connection between mathematics and music has been accepted throughout history, and the scope of that connection has been significantly expanded since it was first exposed by Pythagoras of Samos [4]. This Greek mathematician from the 6th century B.C. defined the "consonant" acoustic relationships between strings of proportional lengths [28]. As the start of the 18^{th} century celebrated the arrival of the first English Dictionary [20] and the first pianoforte [14], it was quivering with a low-bubbling excitement born from a handful of fundamental discoveries made by Leonhard Euler that served as the basic building blocks of Mathematical Music Theory [9]. The musical significance of this early work was sadly overlooked since it was for “musicians too mathematical and for mathematicians too musical” [9]. Since then, scholars of science, mathematics, and music have paved the way for many opportunities to analyze music with math. Of these many pioneers of mathematical music theory, a select few have inspired the subject of this thesis, which is based on the transformational nature of music.

1.1. General Overview and History

In particular, the foundation of this study can be attributed to David Lewin who developed transformational theory [11], and gave rise to a new form of music theory designed to analyze modern music. This new theory is known as neo-Riemannian theory or modern music theory. The spring board for this alternative theory was musical set theory, which gave musicians a way of analyzing music by observing the intervallic

relationships among pitches instead of relating each pitch or musical event to a governing tone of an entire piece (known as *tonality*) [27].

Neo-Riemannian theory was inspired by the German music theorist, Hugo Riemann, who contributed much to the efforts of forming coherent relationships between pitches and intervals in the absence of tonality. The need for this change was born out of the industrial, political, and social changes occurring at the turn of the 19th century. It was inevitable that there would be a substantial effect on the music of the time, and these changes were often expressed via adventurous modulation, innovative chord progressions, atypical dissonance and resolution, and in general, much less preparation for sharp changes in the music. These radical changes gave way, in music, to post-romanticism and eventually to atonality and post-tonality. Music that did not follow all the conformities of tonal music was considered atonal or post-tonal. Naturally, tonal music theory could no longer fulfill its responsibility and new tools had to be constructed in order to analyze and explain this evolving music - hence, the birth of Riemannian theory [18].

While Riemann was primarily interested in substituting the current labeling system of chords and musical events, Lewin saw the potential for these labels to rather describe the motion between these musical events. Lewin's work takes shape in his extensive contribution to the definition of operations that describe musical motion (i.e. transformational theory), and goes further by applying group theory to music [23]. Not only do these sets of transformations form groups, but they are isomorphic to each other and to the dihedral group. Furthermore, they satisfy various properties that allow us to conclude duality (as defined in [13]). In the midst of all these remarkable

deductions, we can also observe relationships with the *torus* (as shown by Riemann [13] and Balzano [5]), and with *torsors* (as seen in Hook [22]). The scope of group theory that is tapped into causes the same problem as with Euler's work, where many scholars of music are not equipped with the training that would enable them to read the group theory aspects with depth and understanding. As a result, much of the work that revolves around neo-Riemannian theory introduces and makes use of group theoretic structures without substantiating them. Herein lies the primary goal of this thesis, which is to provide detailed proofs of these unsupported claims. All the proofs, figures and tables in this thesis are the work of the author with the following exceptions. Theorem 5.1.15 and Theorem 5.3.4 are provided in [13] with some alterations. All figures and tables are original work or have been reconstructed except for figure 6.1 [21]. In addition to offering comprehensive explanations of mathematical concepts, this thesis provides concrete examples that can be added to the mathematical repertoire of group theory.

1.2. Musical Background

The twelve pitches of our modern day, 12-tone music system are labeled using the first 7 letters of the alphabet. Each different letter represents a different frequency, and the letters repeat when the frequency of a pitch or letter is doubled. The range of pitches from the start of one frequency to the double of the same frequency is known as an *octave*. Since the octave is divided into 12 equal frequencies, each pitch is $2^{1/12}$ times the note below it. This is known as *equal tempered tuning*. Prior to equal temperament, musicians used just intonation which is a system with notes having

frequencies that are related by ratios of whole numbers. The difference in frequency between each pitch is called a *semitone* or *half step*. With only 7 letters and twelve notes to label, the symbol ♯ (know as *sharp*) is used to denote a pitch that is a semitone above the current pitch and the symbol ♭ (known as *flat*) is used to denote a pitch that is a semitone below the current pitch. For example, if we are talking about the pitch G, then the note a half step up would be G#, and a half step below would be G♭. The entire set of twelve notes is called the chromatic scale and is musically denoted as follows

C, C#, D, D#, E, F, F#, G, G#, A, A#, B

As mentioned before, each note differs by a semitone so where we previously took a semitone above G to be G#, it is also a semitone below A, which can be denoted as A♭. The property of any note having multiple names is known as *enharmonic equivalence*. This is displayed, along with the frequency values of each pitch, in table 1.1.

Table 1.1 Frequencies corresponding to notes

Note	Frequency (Hz)		Note	Frequency (Hz)
C	261.63		F♯/G♭	369.99
C♯/D♭	277.18		G	392.00
D	293.66		A♭/G♯	415.30
E♭/D♯	311.1		A	440.00
E	329.63		B♭/A♯	466.16
F	349.23		B	493.88

Since all the multiples of a certain frequency are represented by the same letter, it makes it very convenient to mathematically represent this set of 12 notes by the set of equivalence classes modulo 12 ($\mathbb{Z}_{12}$), where each element is a class and represents an infinite set of numbers. In keeping with the literature on Mathematical Music Theory, we will assign the following numbers with the corresponding letters.

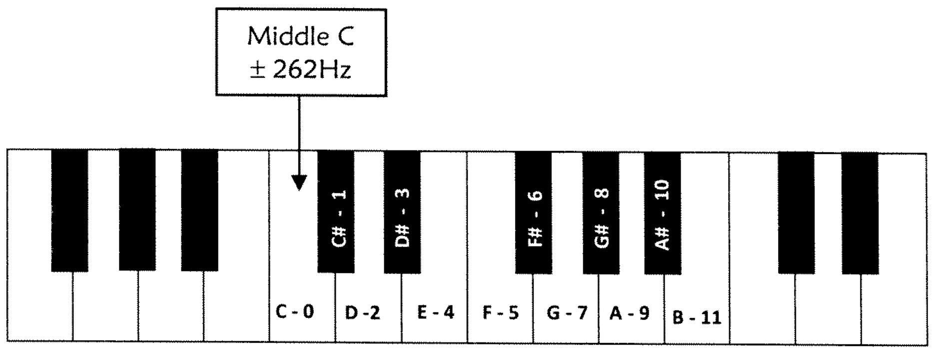


Figure 1.1 Notes assigned to numbers in $\mathbb{Z}_{12}$

This assignment is not set in stone, so it is perfectly alright to let any one of the 12 notes begin with zero. This is one of the attributes that make this set a good candidate for torsors, but more on that later.

1.3. Notation Used in Mathematical Music Theory

For the purpose of this thesis, we are interested in notes that are played simultaneously. These sets of simultaneously played notes are referred to as *chords.* In particular, we will concentrate on the group of chords know as *triads*, which means that only 3 notes are being played simultaneously. Now, there are $\binom{12}{3}$ or 220 combinations of 3 note sets from the entire 12 note set, but we will limit the number of 3

element sets to include only those that are known as *major* and *minor* chords. The three notes in a triad are respectively known as the *root*, the *third* and the *fifth* where each triad is named after its root. We define the major and minor triads mathematically as follows:

Definition 1.3.1 Let <a,b,c> be a **major** chord, then b = a + 4, and c = a + 7 where a, b, c $\in \mathbb{Z}_{12}$.

Major chords in this form are said to be in *root position* and are generally denoted by upper case letters, such as G# = <8,0,3>. Recall that the triad is made of a root, third and fifth. In the case of G#, the root is G# = 8, the third is B# = 0, and the fifth is D# = 3.

Definition 1.3.2 Let <a,b,c> be a **minor** chord, then b = a + 3, and c = a + 7 where a, b, c $\in \mathbb{Z}_{12}$.

Minor chords in this form are said to be in *root position* and are generally denoted by lower case letters, such as f = <5,8,0>.

Now that we have defined the elements, we will refer to them as *pitch class triads* and will denote the set of these 24 major and minor triads to be M. The entire set is displayed in Table 1.2 and can also be seen as

$$M = \{<x,x+3,x+7> \text{ and } <X,X+4,X+7> \mid x, X \in \mathbb{Z}_{12}\}$$

It is important to note that the 3-element sets are not necessarily ordered. In other words, the f-minor chord <5,8,0> is the same set as <0,8,5>. Musically the f-minor

chord is formed by playing the notes F, A♭, and C simultaneously. If we take <0,8,5> we are still indicating that the notes C, A♭, and F be played together, which would indicate that we intend to play the f-minor chord. So the root position of the chord, as in definitions 1.3.1 and 1.3.2 shows the spelling of the chord and is more of an algorithm for obtaining the elements of the set of all major and minor triads. The individual notes may be distributed in multiple ways without changing the identity of the chord.

It is common in the current literature to see the triads displayed in a different order to that of table 1.2. This is relevant only when working with the P, L and R transformations described later. There will be a reminder about this when it becomes important.

Table 1.2 Set of all major and minor triads

Major Chords		Minor Chords	
C	<0,4,7>	C	<0,3,7>
C♯/D♭	<1,5,8>	c♯/d♭	<1,4,8>
D	<2,6,9>	D	<2,5,9>
E♭/D♯	<3,7,10>	e♭/d♯	<3,6,10>
E	<4,8,11>	E	<4,7,11>
F	<5,9,0>	F	<5,8,0>
F♯/G♭	<6,10,1>	f♯/g♭	<6,9,1>
G	<7,11,2>	G	<7,10,2>
A♭/G♯	<8,0,3>	a♭/g♯	<8,11,3>
A	<9,1,4>	A	<9,0,4>
B♭/A♯	<10,2,5>	b♭/a♯	<10,1,5>
B	<11,3,6>	B	<11,2,6>

The term pitch class triad draws our attention to a subtle yet important distinction that must be made. We must be very clear about what elements we are working with, because up to now we have referred to a pitch class triad simply as a triad. An element x in M is a triad, where x = <a,b,c>, and where a, b, c $\in \mathbb{Z}_{12}$. This notation however, is used for convenience and simplicity, but to be more precise we must remember that $\mathbb{Z}_{12}$ is a set of classes.

$$\mathbb{Z}_{12} = \{[0], [1], \ldots, [11]\}, \quad \text{where} \quad \begin{aligned} [0] &= \{\ldots, -24, -12, 0, 12, 24, \ldots\} \\ &\vdots \\ [11] &= \{\ldots, -13, -1, 11, 23, \ldots\} \end{aligned}$$

So, when you read a, b, c $\in \mathbb{Z}_{12}$, it is actually [a], [b], [c] $\in \mathbb{Z}_{12}$. We call these classes, *pitch classes* because (as mentioned before) every note from a to g represents all the pitches which are a multiple of it - much the same way that each class in $\mathbb{Z}_{12}$ represents all the numbers modulo 12 that are multiples. So, when you read x = <a,b,c>, it is actually [x] = <[a],[b],[c]>. Therefore we extend this idea of pitch classes to pitch class triads where all the elements in M are also classes. As an example, let's take the C-major chord x = <0,4,7>. If we look at C-major as a triad class, we should represent it in the following way:

$$C = [x] = \langle[0],[4],[7]\rangle = \{\ldots, \langle -12,-8,-5\rangle, \langle 0,4,7\rangle, \langle 12,16,19\rangle, \ldots\}$$

Now that we have clarified the distinction between basic elements and classes, for simplicity, we will continue to denote [x] simply as x.

2. T/I AND PLR TRANSFORMATIONS

A musical piece is much more than just a combination of pitches and durations of pitches. The entire musical experience is substantiated by rhythm, texture, timbre, dynamics and the other elements with which pitches are bound up. All these properties of music allow for vast amounts of possibilities, but with our 12 tone system the range of choices (albeit large) is still limited. Fortunately, there are ways of manipulating music through transformations which vastly expand the means with which music can entice tone, excite curiosity and surpass expectation. Amongst the multitude of techniques used to create variation in music, *transposition*, *inversion*, *parallel*, *relative*, and *leading tone exchange* transformations are of particular interest.

On an elementary level, we know a transformation to be a mapping from a set of elements to another set of elements. These mappings or functions subject the domain elements to certain rules or conditions, and in altering the original elements the range of the function is formed. Musical actions and tools perform in much the same way. The entire set of pitches and chords are simply manipulated in various ways in order to form more pitches and chords in altered states. The domain and range of our functions then, very conveniently turn out to be one and the same. Following are the definitions of the transposition, inversion, parallel, relative, and leading tone exchange transformations. Our domain and codomain for these functions is the set M, which has been previously defined.

2.1. The T and I Transformations

Transposition in music theory refers to the process of moving a pitch or set of pitches up or down by a constant interval. Whether it is one musician transposing another musician's work to fit within their own vocal range or two musicians performing together at different pitches to create a harmony, transposition is a very common tool in music. Conveniently, the musical definition of this transformation translates directly into mathematical transformation [13].

Definition 2.1.1 Let $x \in M$, where $x = \langle a,b,c \rangle$. A **transposition**, denoted T, is a bijective mapping $T_n: M \to M$, such that

$$T_n(x) = x + n = \langle a+n,b+n,c+n \rangle \qquad \text{for all } n \in \mathbb{Z}$$

There are only 24 elements (triads) in M to apply T_n to, but we have an infinite amount of transpositions of any triad since $n \in \mathbb{Z}$, however after having transposed any triad 12 times, we get the same sequence of triads again. For example

$$T_0(C) = T_0(\langle 0,4,7 \rangle) = \langle 0,4,7 \rangle$$

$$T_1(C) = T_1(\langle 0,4,7 \rangle) = \langle 1,5,8 \rangle$$

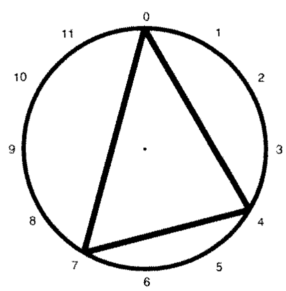
$$\vdots$$

$$T_{12}(C) = T_{12}(\langle 0,4,7 \rangle) = \langle 0,4,7 \rangle = T_0(C)$$

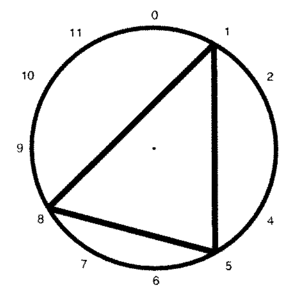
$$T_{13}(C) = T_{13}(\langle 0,4,7 \rangle) = \langle 1,5,8 \rangle = T_1(C)$$

We see then that T_0 behaves like the identity function, and for each triad there are no more than 12 distinct transpositions.

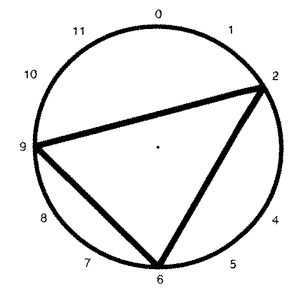
Geometrically we can see transpositions as rotations of a triangle through 12 equally spaced points on a circle. The three vertices of the triangle represents the pitches of that triad and below we see the example of the C-major cord <0,4,7> being rotated one vertex or pitch at a time to the right. In other words, we see an illustration in figure 2.1 of $T_n(<0,4,7>)$, for all $0 \leq n < 12$.



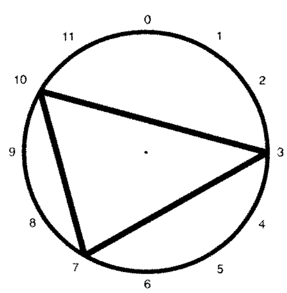
$T_0(<0,4,7>)$



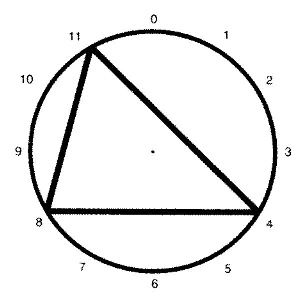
$T_1(<0,4,7>)$



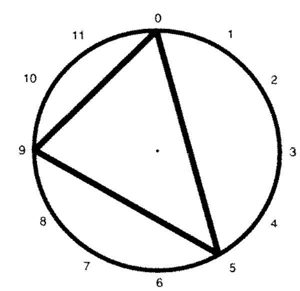
$T_2(<0,4,7>)$



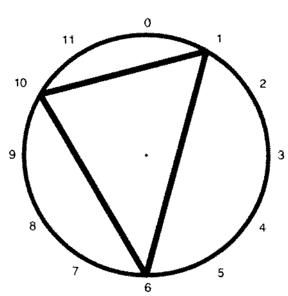
$T_3(<0,4,7>)$



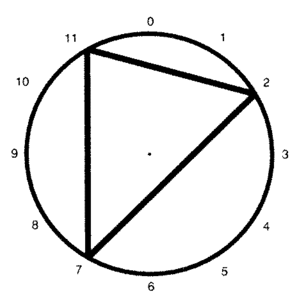
$T_4(<0,4,7>)$



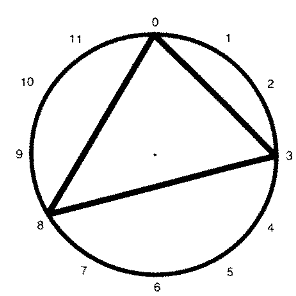
$T_5(<0,4,7>)$



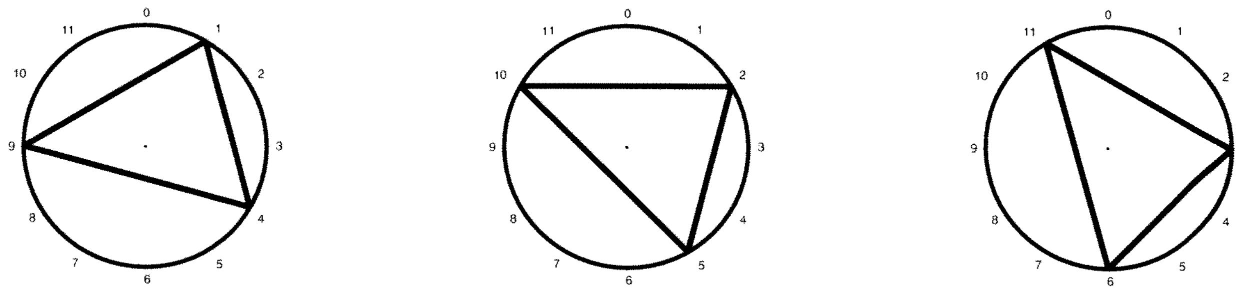
$T_6(<0,4,7>)$



$T_7(<0,4,7>)$



$T_8(<0,4,7>)$



$T_9(<0,4,7>)$ $T_{10}(<0,4,7>)$ $T_{11}(<0,4,7>)$

Figure 2.1 All 12 transpositions of the C-major triad <0,4,7>

Since translations maintain the relation between all three digits in the chord, we see that T maps all major and minor chords to major and minor respectively. Inversions also map the entire set M to itself, yet major chords are mapped to minor chords and vice versa. The musical definition of inversion does not correspond to the mathematical definition quite in the same way that transposition does, however the algebraic definition of inversion is used by the music theorists of mathematical music theory. Musically, an inversion of a chord is a rearrangement of the pitches in a chord. For example, when the root of a triad is moved from the front of a triad to the last position of the triad it is known as *first inversion*. The first inversion of C-major <0,4,7> would then be <4,7,0>. In this study, the term inversion has been used in the algebraic sense, and musicians are cautioned not to confuse this definition of inversion with the musical one.

Definition 2.1.2 Let $x \in M$, where $x = <a,b,c>$. An **inversion**, denoted I, is a bijective mapping $I_n: M \to M$, such that

$$I_n(x) = -x + n = <-A+n,-B+n,-C+n> \qquad \forall\, n \in \mathbb{Z}$$

As with transpositions, there are 24 triads to invert and an infinite number of inversions of each triad, and when we apply I_n, for $n \geq 12$ we generate the same triads that have been generated for $0 \leq n \leq 11$. For example

$$I_0(C) = I_0(<0,4,7>) = <0,8,5>$$

$$I_1(C) = I_1(<0,4,7>) = <1,9,6>$$

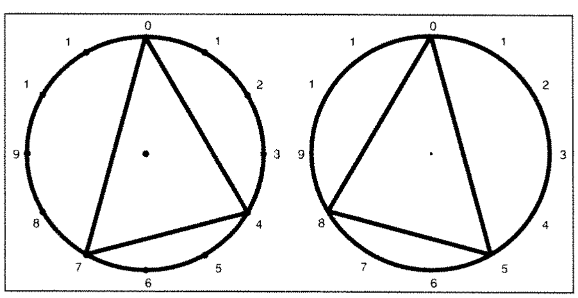
$$\vdots$$

$$I_{12}(C) = I_{12}(<0,4,7>) = <0,8,5> = I_0(C)$$

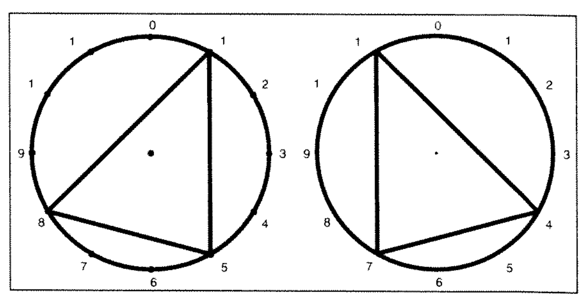
$$I_{13}(C) = I_{13}(<0,4,7>) = <1,9,6> = I_1(C)$$

We see then that for each triad there are no more than 12 distinct inversions.

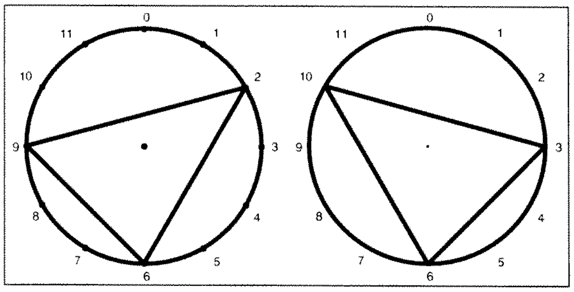
In contrast to the algebraic representation of inversion, the geometric representation is somewhat more intuitive. All inversions can be illustrated as the reflections of triangles about the vertical axis from 0 to 6 on the circle. Unlike the transpositions, Figure 2.2 is not simply the 12 inversions of <0,4,7>. Rather, in order to show the notion of reflection, the illustrations provide the inverted form of each major triad, which is ultimately a transposition of the original triad <0,4,7>.



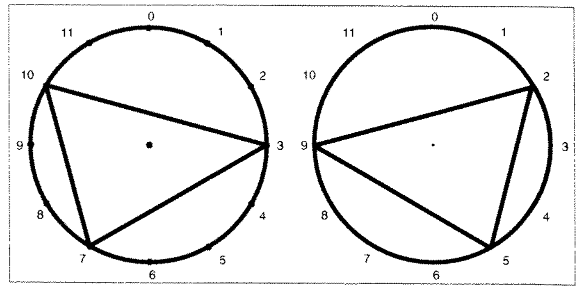
$I_0(<0,4,7>) = <0,8,5>$



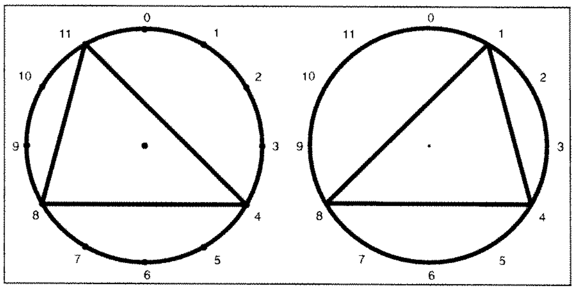
$I_0(<1,5,8>) = <11,7,4>$



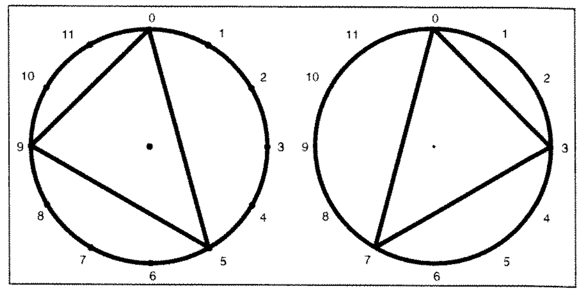
$I_0(<2,6,9>) = <10,6,3>$



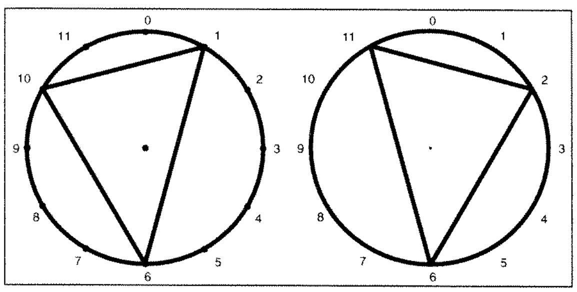
$I_0(<3,7,10>) = <9,5,2>$



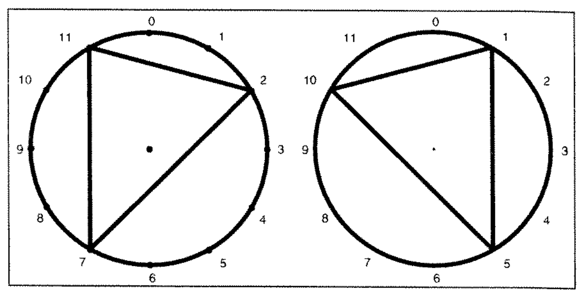
$I_0(<4,8,11>) = <8,4,1>$



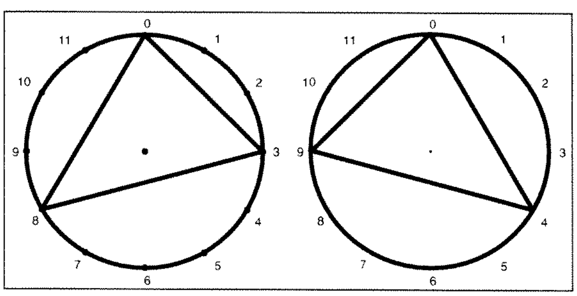
$I_0(<5,9,0>) = <7,3,0>$



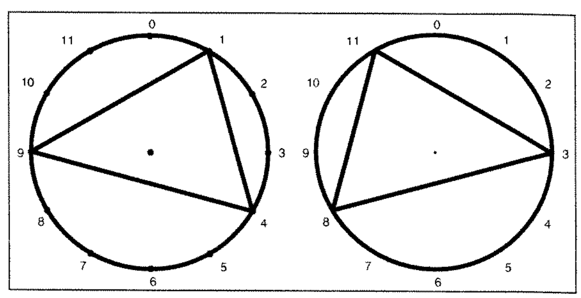
$I_0(<6,10,1>) = <6,2,11>$



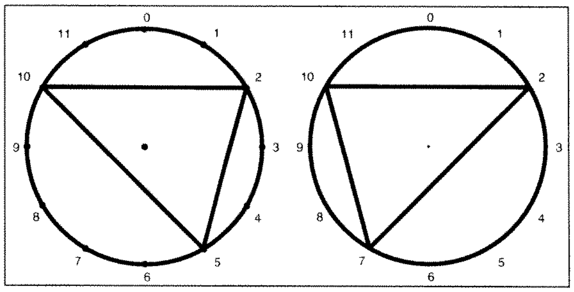
$I_0(<7,11,2>) = <5,1,10>$



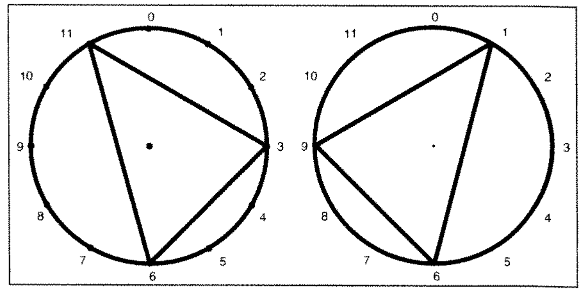
$I_0(<8,0,3>) = <4,0,9>$



$I_0(<9,1,4>) = <3,11,8>$



$I_0(<10,2,5>) = <2,10,7>$



$I_0(<11,3,6>) = <1,9,6>$

Figure 2.2 I_0 of all 12 major triads

By observation we see that there are no more than 12 transpositions and 12 inversions, so while the indices are in the set of integers (which includes negative and positive numbers and is infinite), they are always equivalent to some $k \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. In effect, any triad being acted on changes according to the index and therefore does not change by more than 11. We formalize this in the following Theorem.

Theorem 2.1.3 For all $n, k \in \mathbb{Z}$, such that $n \equiv k \bmod 12$,

$$T_n = T_k \text{ and } I_n = I_k.$$

Proof:

Since $n \equiv k \bmod 12$, then $n = 12q + k$, for some $q \in \mathbb{Z}$

$$T_n = T_{k \bmod 12} = T_{12q+k} = T_{12q} \circ T_k = (T_0)^q \circ T_k = (i)^q \circ T_k = T_k$$

and

$$I_n = I_{k \bmod 12} = I_{12q+k} = T_{12q} \circ I_k = (T_0)^q \circ I_k = (i)^q \circ I_k = I_k$$

■

Both sets of T and I functions constitute the entire set called the T/I-set, which is illustrated in table 2.1. The term T/I is standard notation in mathematical music theory and should not be confused with the notation of quotient groups.

Table 2.1 The T/I-set

T_0	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}	T_{11}
I_0	I_1	I_2	I_3	I_4	I_5	I_6	I_7	I_8	I_9	I_{10}	I_{11}

Definition 2.1.4 The set of all transposition and inversion functions is known as **T/I**, and defined as

$$T/I = \{T_n \text{ and } I_n \mid n = 0, \ldots, 11\}$$

It turns out we can actually represent all these elements in a more compact manner, if we take a look at all 4 possible compositions of T and I functions.

Lemma 2.1.5 The results of the following equations account for all the functions in the T/I-set:

i) $T_m \circ T_n = T_{m+n \bmod 12}$ (1)

ii) $T_m \circ I_n = I_{m+n \bmod 12}$ (2)

iii) $I_m \circ T_n = I_{m-n \bmod 12}$ (3)

iv) $I_m \circ I_n = T_{m-n \bmod 12}$ (4)

Proof:

i) $T_m \circ T_n$
$$\begin{aligned} &= T_m(T_n(\langle a,b,c\rangle)) \\ &= T_m(\langle a+n,b+n,c+n\rangle) \\ &= \langle a+n+m,b+n+m,c+n+m\rangle \\ &= \langle a+(m+n),b+(m+n),c+(m+n)\rangle \\ &= T_{m+n \bmod 12} \end{aligned}$$

ii) $T_m \circ I_n$
$$\begin{aligned} &= T_m(I_n(\langle a,b,c\rangle)) \\ &= T_m(\langle -a+n,-b+n,-c+n\rangle) \\ &= \langle -a+n+m,-b+n+m,-c+n+m\rangle \\ &= \langle -a+(m+n),-b+(m+n),-c+(m+n)\rangle \\ &= I_{m+n \bmod 12} \end{aligned}$$

iii) $I_m \circ T_n$
$$\begin{aligned} &= I_m(T_n(\langle a,b,c\rangle)) \\ &= I_m(\langle a+n,b+n,c+n\rangle) \\ &= \langle -a-n+m,-b-n+m,-c-n+m\rangle \\ &= \langle -a+(m-n),-b+(m-n),-c+(m-n)\rangle \\ &= I_{m-n \bmod 12} \end{aligned}$$

iv) $I_m \circ I_n$
$$\begin{aligned} &= I_m(I_n(\langle a,b,c\rangle)) \\ &= I_m(\langle -a+n,-b+n,-c+n\rangle) \\ &= \langle a-n+m,b-n+m,c-n+m\rangle \\ &= \langle a+(m-n),b+(m-n),c+(m-n)\rangle \\ &= T_{m-n \bmod 12} \end{aligned}$$

∎

When all the functions from table 2.1 are consecutively applied to any triad in M, then the entire set M is reproduced. As an example, table 2.2 demonstrates the result of applying all T and I functions to C = <0,4,7>.

Table 2.2 All transpositions and inversions applied to C-major

Prime Forms	Inverted Forms
T_0(<0,4,7>) = **<0,4,7>**	I_0(<0,4,7>) = **<0,8,5>**
T_1(<0,4,7>) = **<1,5,8>**	I_1(<0,4,7>) = **<1,9,6>**
T_2(<0,4,7>) = **<2,6,9>**	I_2(<0,4,7>) = **<2,10,7>**
T_3(<0,4,7>) = **<3,7,10>**	I_3(<0,4,7>) = **<3,11,8>**
T_4(<0,4,7>) = **<4,8,11>**	I_4(<0,4,7>) = **<4,0,9>**
T_5(<0,4,7>) = **<5,9,0>**	I_5(<0,4,7>) = **<5,1,10>**
T_6(<0,4,7>) = **<6,10,1>**	I_6(<0,4,7>) = **<6,2,11>**
T_7(<0,4,7>) = **<7,11,2>**	I_7(<0,4,7>) = **<7,3,0>**
T_8(<0,4,7>) = **<8,0,3>**	I_8(<0,4,7>) = **<8,4,1>**
T_9(<0,4,7>) = **<9,1,4>**	I_9(<0,4,7>) = **<9,5,2>**
T_{10}(<0,4,7>) = **<10,2,5>**	I_{10}(<0,4,7>) = **<10,6,3>**
T_{11}(<0,4,7>) = **<11,3,6>**	I_{11}(<0,4,7>) = **<11,7,4>**

This same set of triads will be produced regardless of the triad used. Notice that certain triads in this table differ in order from those in table 1.2. As mentioned before, the ordering of each triad is inconsequential, but it will make certain investigations easier if this ordering is preserved. The headings of the two columns are commonly used in the literature on this subject, and when this ordering of a triad is preferred or required, the triad is referred to as a triad in *prime* or *inverted* form. This is will become apparent when working with the P, L, and R operations.

Before we move on to more operations on the set of 24 triads, we will first verify that the results of T_n and I_n do not depend on the element in M they are acting on, and furthermore that it doesn't matter which representative within each pitch class is used.

Theorem 2.1.6 The operations T_n and I_n are well defined and therefore, if [x] is a pitch class triad in M then for all $x_1, x_2 \in [x]$ we have

i) $T_n(x_1) = T_n(x_2)$, i.e. $T_n(<a_1,b_1,c_1>) = T_n(<a_2,b_2,c_2>)$

ii) $I_n(x_1) = I_n(x_2)$, i.e. $I_n(<a_1,b_1,c_1>) = I_n(<a_2,b_2,c_2>)$

Proof:

Let $x_1, x_2 \in [x] \in M$, where $x_1 = <a_1,b_1,c_1>$ and $x_2 = <a_2,b_2,c_2>$. So, x_1, and x_2 are elements in the triad class $[x] = <[a],[b],[c]>$. We see then that $a_1, a_2, \in [a] \in \mathbb{Z}_{12}$, and $b_1, b_2, \in [b] \in \mathbb{Z}_{12}$, and $c_1, c_2, \in [c] \in \mathbb{Z}_{12}$

i) $T_n(<a_1,b_1,c_1>) = <a_1+n,b_1+n,c_1+n>$

and,

$T_n(<a_2,b_2,c_2>) = <a_2+n,b_2+n,c_2+n>$

since,

$a_1 \in [a]$, then $(a_1+n) \in [a+n]$ and $a_2 \in [a]$, then $(a_2+n) \in [a+n]$

Similarly

$(b_1+n), (b_2+n) \in [b+n]$ and $(c_1+n), (c_2+n) \in [c+n]$

which gives us

$\mathbf{T_n(x_1)} = T_n(<a_1,b_1,c_1>) = <a_1+n,b_1+n,c_1+n> =<a_2+n,b_2+n,c_2+n> = T_n(<a_2,b_2,c_2>) = \mathbf{T_n(x_2)}$

Therefore, T_n is well defined.

ii) $I_n(<a_1,b_1,c_1>) = <-a_1+n,-b_1+n,-c_1+n>$

and,

$I_n(\langle a_2, b_2, c_2 \rangle) = \langle -a_2+n, -b_2+n, -c_2+n \rangle$

since,

$-a_1 \in [a]$, then $(-a_1+n) \in [a+n]$ and $-a_2 \in [a]$, then $(-a_2+n) \in [a+n]$

Similarly

$(-b_1+n), (-b_2+n) \in [b+n]$ and $(-c_1+n), (-c_2+n) \in [c+n]$

which gives us

$$\mathbf{I_n(x_1)} = I_n(\langle a_1, b_1, c_1 \rangle) = \langle -a_1+n, -b_1+n, -c_1+n \rangle = \langle -a_2+n, -b_2+n, -c_2+n \rangle = I_n(\langle a_2, b_2, c_2 \rangle) = \mathbf{I_n(x_2)}$$

Therefore, I_n is well defined.

■

2.2. The P, L, and R Transformations

In addition to the T and I transformations that we apply to the set M, we also have the parallel (P), leading tone exchange (L), and relative (R) functions [13]. As with the T and I functions, there are musical, group theoretic and geometric descriptions of the P, L and R functions. The descriptions and definitions of these three will not be separated as with the T and I functions, and extended examples are provided below the formal definition in the event that the following brief examples are not clear.

Two triads are said to be parallel if they have the same letter name but of opposite parity (parity meaning major or minor). For instance, the parallel minor of F-major <5,9,0> is f-minor <5,8,0>. Both triads are named with the letter f, but one is major and the other is minor. Two triads are said to be relative if they are again of opposite parity, and if the root of the minor triad is three semitones below the root of

major triad. To illustrate, we take F-major <5,9,0> and count three semitones below 5, which is 2 and then build a minor chord on 2. This yields the d-minor chord <2,5,9>, and d minor is the relative minor of F-major. Lastly, the leading tone exchange is derived from the fact that a semitone below any pitch is called the *leading tone* of that pitch. Therefore, the leading tone exchange of any triad is also of opposite parity, and the root of the major triad is replaced with its leading tone. We use F-major <5,9,0>once again to illustrate. The root of F is 5, which is replaced with its leading tone, 4. It suffices for now to say that the only minor chord with the pitches 4, 9, and 0 is the a-minor chord, <4,0,9>. Richard Cohn [12] describes it as:

> "P (for Parallel) which relates triads that share a common fifth; L (for Leading-tone exchange), which relates triads that share a common minor third; and R (for Relative), which relates triads that share a common major third."

Below are the mathematical definitions of P, L, and R which are then followed by examples. Recall that the triad ordering in table 2.2 is recommended.

Definition 2.2.1 Let x, Y ∈ M, where x = <a,b,c> a minor triad and Y = <A,B,C> a major triad, then

P(x) = P(<a,b,c>) = <C,B+1,A> (5)

P(Y) = P(<A,B,C>) = <c,b-1,a> (6)

L(x) = L(<a,b,c>) = <A+1,C,B> (7)

L(Y) = L(<A,B,C>) = <a-1,c,b> (8)

R(x) = R(<a,b,c>) = <B,A,C-2> (9)

R(Y) = R(<A,B,C>) = <b,a,c+2> (10)

Notice that each function keeps two notes in common with the original chord, but switches their positions. In addition to this, each function converts the original triad from major to minor or vice versa. These relationships are more musically meaningful and are easier to detect for those that are musically inclined. If one takes a purely numerical view of the results it is not as intuitive, so we once again interject with a reminder to use the triad ordering in table 2.2. Each function has musical relevance, and these transformations are commonly used during modulation (the change of one key to another). The goal is to make the transition as smooth as possible, so the more notes in common between the connecting triads, the less noticeable it is to hear the change of course. The following examples provide concrete implementation of definition 2.2.1.

$P(c) = P(<7,3,0) = <0,4,7> = C$ and $P(F) = P(<5,9,0.>) = <0,8,5> = f$

$L(e) = L(<11,7,4>) = <0,4,7> = C$ and $L(G) = L(<7,11,2>) = <6,2,11> = b$

$R(b) = R(<6,2,11>) = <2,6,9> = D$ and $R(A) = R(<9,1,4>) = <1,9,6> = f\#$

We now explore what the set of P, L, and R functions looks like and we will start with the geometric representation. As with the T and I functions, there is a rather interesting representation of the P, L, and R functions, known as the *Tonnetz*. The word *Tonnetz* is German for "tone network" and was invented by Leonhard Euler [12]. It was Hugo Riemann that explored its capacity to chart harmonic motion or the movement from one pitch or triad to another. For various reasons, the original *Tonnetz* has undergone various alterations [12], but we will use the version shown in figure 2.3. Note that the vertices are pitch classes and the triangles represent major and minor triads.

As mentioned before, the P, L, and R transformations preserve 2 pitches when applied to any triad. Therefore, the rotation of a triangle about any one of its edges yields another triangle which is equivalent to one of the three triads that P, L, or R would produce. Notice that if we expand the diagram with more vertices, we see that they start repeating vertically and horizontally and in effect, the grid wraps around and therefore lies on a torus. This idea will be expanded upon in chapter 6.

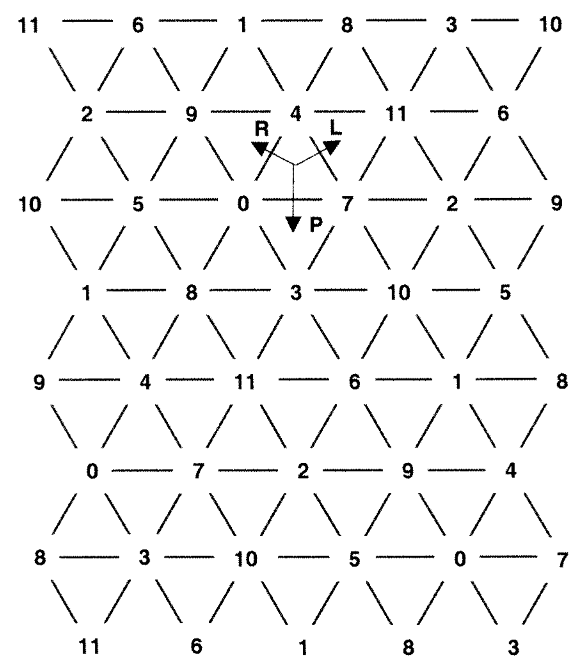


Figure 2.3 The Oettingen/Riemann *Tonnetz*

While we are treating P, L, and R as transformations we will determine what sequence of functions maps each element of M to a distinct image, as with the T and I functions. Although, unlike T and I, we do not have the subscripts of each function and therefore the elements of the set are not as obvious. The *Tonnetz* makes this

exploration visual and in turn much more effortless than having to calculate a series of transformations by hand. We look to figure 2.3 to see that there are multiple ways of tracing any triad to itself; however we are concerned with a path that traverses every element in M. Figure 2.4 displays such a path and we see that a series of R and L functions map the triad <5,9,0> to itself by mapping to every other triad in the process. We are not claiming yet that this is the entire set of P, L, and R functions, but rather that this gives us a sense of what the elements might look like.

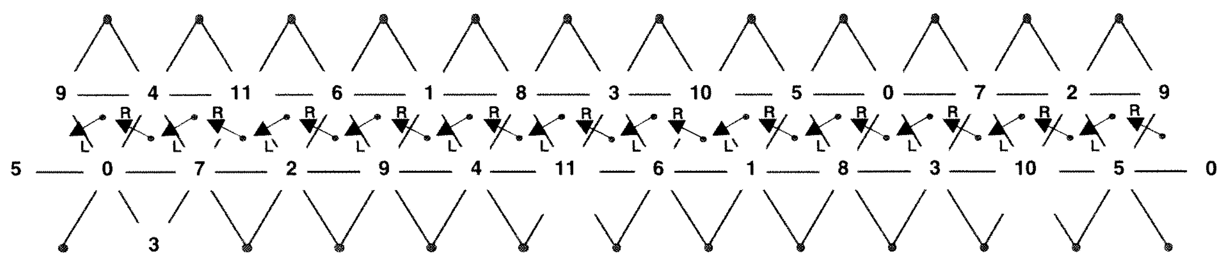


Figure 2.4 A unique path of functions on the *Tonnetz*

From this result, we turn then to the compositions of P, L, and R and the powers of those compositions in order to determine the elements of the set. We notice quickly that P, L, and R are involutive. In other words,

$$P^2 = L^2 = R^2 = i$$

We will show this for the P function for example, and the L and R functions will behave in the same way.

$$P \circ P(<a,b,c>) = P(<C,B+1,A>) = <a,(b+1)-1,c> = <a,b,c>$$

$$= i(<a,b,c>) \text{ (the identity function)}$$

Note: After the first application of P, the resulting triad is a major triad, so care must be taken in using the correct computation from definition 2.2.1.

As demonstrated above, by consecutively applying R to any triad and then L to the result, you will produce the following sequence of triads (again, upper case representing major triads and lower case representing minor triads).

C, a, F, d, B♭, g, E♭, c, A♭, f, D♭, b♭, G♭, e♭, B, g#, E, c#, A, f#, D, b, G, e, C

This sequence so happens to be a famous progression in Beethoven's *Ninth Symphony* [15]. In order to see this sequence unfolding, recall that it is makes the explanation simpler when using the order of the triads found in table 2.2. Initially, for example, we take $C = <0,4,7>$ and start applying the R and L functions to get the following

$$R(<0,4,7>) = <4,0,9> = a$$

So, we see that C is taken to its relative minor, a. Then

$$LR(<0,4,7>) = L(<4,0,9>) = <5,9,0> = F$$

which shows that a is taken to its leading tone, F-major.

Continuing in this way will eventually produce the string of triads displayed above and in table 2.3. In fact, when the R and L functions are applied to any major triad in that order, the same sequence of triads will result. Alternatively, the sequence is produced in reverse order when applied to any minor triad. Furthermore (as seen in figure 2.4), once all the elements in M are produced via the R and L functions, the sequence starts repeating. We illustrate this by the following operation on a triad x in M.

$$(LR)^{12}(x) = (LR)(LR)^{11}(x)$$

$$(LR)(LR)^{11}(x) = (LR)(LR)^{11}(<A,B,C>)$$

$$= (LR)^{11}(L(R(<A,B,C>))) \qquad = (LR)^{11}(L(<b,a,c+2>))$$

$$= (LR)^{11}(\langle B+1,C+2,A\rangle) \qquad = (LR)^{10}(L(R(\langle B+1,C+2,A\rangle)))$$

$$= (LR)^{10}(L(\langle c+2,b+1,a+2\rangle)) \qquad = (LR)^{10}(\langle C+3,A+2,B+1\rangle)$$

$$= (LR)^{9}(L(R(\langle C+3,A+2,B+1\rangle))) \qquad = (LR)^{9}(L(\langle a+2,c+3,b+3\rangle))$$

$$= (LR)^{9}(\langle A+3,B+3,C+3\rangle) \qquad = (LR)^{8}(L(R(\langle A+3,B+3,C+3\rangle)))$$

$$= (LR)^{8}(L(\langle b+3,a+3,c+5\rangle)) \qquad = (LR)^{8}(\langle B+4,C+5,A+3\rangle)$$

$$= (LR)^{7}(L(R(\langle B+4,C+5,A+3\rangle))) \qquad = (LR)^{7}(L(\langle c+5,b+4,a+5))$$

$$= (LR)^{7}(\langle C+6,A+5,B+4\rangle) \qquad = (LR)^{6}(L(R(\langle C+6,A+5,B+4\rangle)))$$

$$= (LR)^{6}(L(\langle a+5,c+6,b+6\rangle)) \qquad = (LR)^{6}(\langle A+6,B+6,C+6\rangle)$$

$$= (LR)^{5}(L(R(\langle A+6,B+6,C+6\rangle))) \qquad = (LR)^{5}(L(\langle b+6,a+6,c+8\rangle))$$

$$= (LR)^{5}(\langle B+7,C+8,A+6\rangle) \qquad = (LR)^{4}(L(R(\langle B+7,C+8,A+6\rangle)))$$

$$= (LR)^{4}(L(\langle c+8,b+7,a+8\rangle)) \qquad = (LR)^{4}(\langle C+9,A+8,B+7\rangle)$$

$$= (LR)^{3}(L(R(\langle C+9,A+8,B+7\rangle))) \qquad = (LR)^{3}(L(\langle a+8,c+9,b+9\rangle))$$

$$= (LR)^{3}(\langle A+9,B+9,C+9\rangle) \qquad = (LR)^{2}(L(R(\langle A+9,B+9,C+9\rangle)))$$

$$= (LR)^{2}(L(\langle b+9,a+9,c+11\rangle)) \qquad = (LR)^{2}(\langle B+10,C+11,A+9\rangle)$$

$$= (LR)(L(R(\langle B+10,C+11,A+9\rangle))) \qquad = (LR)(L(\langle c+11,b+10,a+11\rangle))$$

$$= (LR)(\langle C+12,A+11,B+10\rangle) \qquad = (L(R(\langle C+12,A+11,B+10\rangle)))$$

$$= L(\langle a+11,c+12,b+12\rangle) \qquad = \langle A+12,B+12,C+12\rangle)$$

$$= \langle A,B,C\rangle$$

$$= i(\langle A,B,C\rangle)$$

Similarly, if you apply $(LR)^{12} = (LR)(LR)^{11}$ to a minor triad, you will end up with that same triad. The next iteration is $R(LR)^{12}$, which gives us

$$R(LR)^{12}(\langle A,B,C\rangle)= \langle b,a,c+2\rangle = R(\langle A,B,C\rangle)$$

Similarly, if we compute $(LR)^{13}$ we get

$$(LR)^{13}(\langle A,B,C \rangle) = \langle B+1,C+2,A\rangle = LR(\langle A,B,C \rangle)$$

We can then see that $(LR)^{12}$ behaves like the identity and so we can say that

$$(LR)^{12} = i = 1 = (LR)^{0}$$

Table 2.3 R and L functions applied to C-major

$R(\langle 0,4,7\rangle)$ = **<4,0,9>** = a	$R(LR)^{6}(\langle 0,4,7\rangle)$ = **<10,6,3>** = e♭
$LR(\langle 0,4,7\rangle)$ = **<5,9,0>** = F	$(LR)^{7}(\langle 0,4,7\rangle)$ = **<11,3,6>** = B
$R(LR)(\langle 0,4,7\rangle)$ = **<9,5,2>** = d	$R(LR)^{7}(\langle 0,4,7\rangle)$ = **<3,11,8>** = g#
$(LR)^{2}(\langle 0,4,7\rangle)$ = **<10,2,5>** = B♭	$(LR)^{8}(\langle 0,4,7\rangle)$ = **<4,8,11>** = E
$R(LR)^{2}(\langle 0,4,7\rangle)$ = **<2,10,7>** = g	$R(LR)^{8}(\langle 0,4,7\rangle)$ = **<8,4,1>** = c#
$(LR)^{3}(\langle 0,4,7\rangle)$ = **<3,7,10>** = E♭	$(LR)^{9}(\langle 0,4,7\rangle)$ = **<9,1,4>** = A
$R(LR)^{3}(\langle 0,4,7\rangle)$ = **<7,3,0>** = c	$R(LR)^{9}(\langle 0,4,7\rangle)$ = **<1,9,6>** = f#
$(LR)^{4}(\langle 0,4,7\rangle)$ = **<8,0,3>** = A♭	$(LR)^{10}(\langle 0,4,7\rangle)$ = **<2,6,9>** = D
$R(LR)^{4}(\langle 0,4,7\rangle)$ = **<0,8,5>** = f	$R(LR)^{10}(\langle 0,4,7\rangle)$ = **<6,2,11>** = b
$(LR)^{5}(\langle 0,4,7\rangle)$ = **<1,5,8>** = D♭	$(LR)^{11}(\langle 0,4,7\rangle)$ = **<7,11,2>** = G
$R(LR)^{5}(\langle 0,4,7\rangle)$ = **<5,1,10>** = b♭	$R(LR)^{11}(\langle 0,4,7\rangle)$ = **<11,7,4>** = e
$(LR)^{6}(\langle 0,4,7\rangle)$ = **<6,10,1>** = G♭	$(LR)^{0}(\langle 0,4,7\rangle)$ = **<0,4,7>** = C

Again we see that the powers of the functions will always be in the set {0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11}. Once again we formalize this in a theorem.

Theorem 2.2.2 For all $n, k \in \mathbb{Z}$, such that $n \equiv k \bmod 12$,

$$(LR)^{n} = (LR)^{k} \text{ and } R(LR)^{n} = R(LR)^{k}.$$

Proof:

Since $n \equiv k \bmod 12$, then $n = 12q + k$, for some $q \in \mathbb{Z}$

$$(LR)^n = (LR)^{k \bmod 12} = (LR)^{12q+k} = (LR)^{12q}(LR)^k = ((LR)^0)^q(LR)^k = (i)^q(LR)^k = (LR)^k$$

and

$$R(LR)^n = R(LR)^{k \bmod 12} = R(LR)^{12q+k} = R(LR)^{12q}(LR)^k = (R(LR)^0)^q(LR)^k$$
$$= (R(i)^q)(LR)^k = R(LR)^k$$

■

This yields 12 functions of the form $(LR)^n$ and 12 functions of the form $R(LR)^n$ for a minimum of 24 distinct functions in our set of P, L, and R functions so far. While this says nothing about the maximum amount of functions in the set, we will verify shortly that the entire PLR-set actually consists only of the functions in table 2.4.

Table 2.4 The PLR-set

$R = R(LR)^0$	$R(LR)^2$	$R(LR)^4$	$R(LR)^6$	$R(LR)^8$	$R(LR)^{10}$
$(LR)^1$	$(LR)^3$	$(LR)^5$	$(LR)^7$	$(LR)^9$	$(LR)^{11}$
$R(LR)^1$	$R(LR)^3$	$R(LR)^5$	$R(LR)^7$	$R(LR)^9$	$R(LR)^{11}$
$(LR)^2$	$(LR)^4$	$(LR)^6$	$(LR)^8$	$(LR)^{10}$	$(LR)^0 = i$

Definition 2.2.3 By theorem 2.2.2 we see that the set of all parallel, relative, and leading tone exchange functions is known as **PLR**, and defined as

$$PLR = \{(LR)^n \text{ and } R(LR)^n \mid n = 0, \ldots, 11\}$$

It is curious that the single functions P and L don't explicitly show up in table 2.3 where we are able to generate the entire set M without employing either of them. Further investigation will reveal that both P and L are represented above because

$$P = R(LR)^3 \qquad \text{and} \qquad L = R(LR)^{11}$$

These equalities are irrespective of whether you apply them to a major or minor triads. Take for example a minor triad such as f and apply P to it as stated in definition 2.2.1.

$$P(f) = P(<0,8,5>) = <5,9,0> = F$$

Now if we use the R and L functions as described before, we get

$$R(f) = R(<0,8,5>) = <8,0,3>$$

$$LR(<0,8,5>) = L(<8,0,3>) = <7,3,0>$$

$$R(LR)(<0,8,5>) = R(<7,3,0>) = <3,7,10>$$

$$(LR)^2(<0,8,5>) = L(<3,7,10>) = <2,10,7>$$

$$R(LR)^2 (<0,8,5>) = R(<2,10,7>) = <10,2,5>$$

$$(LR)^3(<0,8,5>) = L(<10,2,5>) = <9,5,2>$$

$$R(LR)^3(<0,8,5>) = R(<9,5,2>) = <5,9,0> = F$$

On the other hand if we take a major triad such as D, we get

$$P(D) = P(<2,6,9>) = <9,5,2> = d$$

Now if we use the R and L functions as described before, we get

$$R(D) = R(<2,6,9>) = <6,2,11>$$

$$LR(<2,6,9>) = L(<6,2,11>) = <7,11,2>$$

$$R(LR)(<2,6,9>) = R(<7,11,2>) = <11,7,4>$$

$$(LR)^2(<2,6,9>) = L(<11,7,4>) = <0,4,7>$$

$$R(LR)^2(<2,6,9>) = R(<0,4,7>) = <4,0,9>$$

$$(LR)^3(<2,6,9>) = L(<4,0,9>) = <5,9,0>$$

$$R(LR)^3(<2,6,9>) = R(<5,9,0>) = <9,5,2> = d$$

This shows $P = R(LR)^3$, irrespective of major or minor triads and the same will be true of $L = R(LR)^{11}$.

Another somewhat obvious concern might be that the functions of the form (RL) may introduce additional distinct functions into the set. Interestingly since the P, L, and R functions are involutive, when you consecutively apply the function L to a triad first and then R, you will also produce the above mentioned sequence of triads in reverse order. Table 2.5 shows the equivalences between the LR and RL functions.

Table 2.5 Equivalences between (LR) and (RL) functions

$R(LR)^0 = R = L(RL)^{11}$	$R(LR)^6 = L(RL)^5$
$LR = (RL)^{11}$	$(LR)^7 = (RL)^5$
$R(LR) = L(RL)^{10}$	$R(LR)^7 = L(RL)^4$
$(LR)^2 = (RL)^{10}$	$(LR)^8 = (RL)^4$
$R(LR)^2 = L(RL)^9$	$R(LR)^8 = L(RL)^3$
$(LR)^3 = (RL)^9$	$(LR)^9 = (RL)^3$
$R(LR)^3 = L(RL)^8$	$R(LR)^9 = L(RL)^2$
$(LR)^4 = (RL)^8$	$(LR)^{10} = (RL)^2$
$R(LR)^4 = L(RL)^7$	$R(LR)^{10} = L(RL)$
$(LR)^5 = (RL)^7$	$(LR)^{11} = RL$
$R(LR)^5 = L(RL)^6$	$R(LR)^{11} = L(RL)^0 = L$
$(LR)^6 = (RL)^6$	$(LR)^0 = (RL)^0$

We can now conclude that the list of all $(LR)^n$ and $R(LR)^m$ functions in table 2.4 is a far more comprehensive list than what it appears to be. It contains a set of compositions that include the P, L and R functions as well as the compositions of (LR) and (RL). Moreover, these 24 distinct functions map any element of M to 24 distinct elements of M. From our results thus far however, we cannot definitively say that this accounts for all possible P, L, and R compositions (for example PLRPRLLR). We can however note that the two basic elements $(LR)^n$ and $R(LR)^m$ already produce all the triads in M, so any other amount of P, L or R functions (before or after them) has nothing else but to map to another triad that has already been mapped to. We see then (informally) that any possible composition of P, L, and R is just another way of writing an element that already exists in table 2.4. We arrive at the solution to this problem more formally through induction, but first we look at the results of all possible compositions of the two basic functions $(LR)^n$ and $R(LR)^m$.

Lemma 2.2.4 The PLR-set is closed under $\circ$.

Proof:

Since we have two main functions in the set ($(LR)^n$ and $R(LR)^m$), we have four possible compositions that must be verified to exist in the PLR-set.

Case 1: $$R(LR)^n \circ R(LR)^m = R\underbrace{(LRLR \ldots LRLR)}_{\text{n times}} \circ R\underbrace{(LRLR \ldots LRLR)}_{\text{m times}}$$

$$= R\underbrace{(LRLR \ldots LR)}_{\text{n-1 times}} (L)\ (RR)\ \underbrace{(LRLR \ldots LRLR)}_{\text{m times}}$$

$$= R\underbrace{(LRLR \ldots LR)}_{\text{n-1 times}}(L)\underbrace{(LRLR \ldots LRLR)}_{\text{m times}} \qquad \text{since } R^2 = i$$

$$= R\underbrace{(LRLR \ldots LR)}_{\text{n-1 times}}(LL)(R)\underbrace{(LR \ldots LRLR)}_{\text{m-1 times}}$$

$$= R\underbrace{(LRLR \ldots LR)}_{\text{n-1 times}}(R)\underbrace{(LR \ldots LRLR)}_{\text{m-1 times}} \qquad \text{since } L^2 = i$$

$$\vdots$$

Continue in this way to get

$$= R(LRLRLR)\,(R)(LRLR) \qquad \text{since } L^2 = i$$

$$= RLRLRL\,(RR)\,LRLR$$

$$= RLRLR\,(LL)\,RLR \qquad \text{since } R^2 = i$$

$$= RLRL\,(RR)\,LR \qquad \text{since } L^2 = i$$

$$= RLR\,(LL)\,R \qquad \text{since } R^2 = i$$

$$= RL\,(RR) \qquad \text{since } L^2 = i$$

$$= RL \qquad \text{since } R^2 = i$$

As in table 2.5, $RL = (LR)^{11} \in PLR$

Therefore, $R(LR)^n \circ R(LR)^m \in PLR$

Note: When $n = m$ in case 1, we get

$$= R\underbrace{(LRLR \ldots LR)}_{\text{n-1 times}}(R)\underbrace{(LR \ldots LRLR)}_{\text{n-1 times}} \qquad \text{since } L^2 = i$$

$$\vdots$$

Continue in this way to get

$= R(LR)\,(R)(LR)$ since $L^2 = i$

$= RL\,(RR)\,LR$

$= R\,(LL)\,R$ since $R^2 = i$

$= RR$ since $L^2 = i$

$= i$ since $L^2 = i$

Case 2: $(LR)^n \circ (LR)^m = (LR)^{n+m} = (LR)^{n+m \bmod 12} \in PLR$

Case 3: $R(LR)^n \circ (LR)^m = R(LR)^{n+m} = R(LR)^{n+m \bmod 12} \in PLR$

Case 4: $(LR)^n \circ R(LR)^m = \underbrace{(LRLR \ldots LRLR)}_{\text{n times}} \circ R\underbrace{(LRLR \ldots LRLR)}_{\text{m times}}$

$= \underbrace{(LRLR \ldots LR)}_{\text{n-1 times}}\,(L)\,(RR)\,\underbrace{(LRLR \ldots LRLR)}_{\text{m times}}$

$= \underbrace{(LRLR \ldots LR)}_{\text{n-1 times}}\,(L)\,\underbrace{(LRLR \ldots LRLR)}_{\text{m times}}$ since $R^2 = i$

$= \underbrace{(LRLR \ldots LR)}_{\text{n-1 times}}\,(LL)\,(R)\,\underbrace{(LR \ldots LRLR)}_{\text{m-1 times}}$

$= \underbrace{(LRLR \ldots LR)}_{\text{n-1 times}}\,(R)\,\underbrace{(LR \ldots LRLR)}_{\text{m-1 times}}$ since $L^2 = i$

$\vdots$

Continue in this way to finally get

$= (LRLR)\,(L)\,(LRLRLR)$ since $L^2 = i$

$= LRLR\ (LL)\ RLRLR$

$= LRL\ (RR)\ LRLR$ since $L^2 = i$

$= LR\ (LL)\ RLR$ since $R^2 = i$

$= L\ (RR)\ LR$ since $L^2 = i$

$= (LL)\ R$ since $R^2 = i$

$= R$ since $L^2 = i$

Where $R \in PLR$

We conclude then, for all $n, m \in \mathbb{Z}$, all compositions of $(LR)^n$ and $R(LR)^m$ are in PLR.

■

Before we present the induction proof, we look at one more lemma that will be utilized in the induction process.

Lemma 2.2.5 Let $x \in \{(LR)^n, R(LR)^n\}$, and let $y = a \circ x$, where $a \in \{P, L, R\}$ then

Case 1: Let $x = (LR)^n$ then $y = a \circ x \in PLR$

Case 2: Let $x = R(LR)^n$ then $y = a \circ x \in PLR$

Proof:

For each case there are 3 sub-cases, namely

i) If $a = P = R(LR)^3$, ii) If $a = L = R(LR)^{11}$, and iii) If $a = R$.

Case 1: Let $x = (LR)^n$

<u>i) If a = P</u>:

$y = a \circ x$

$= P \circ (LR)^n$

$= R(LR)^3\ (LR)^n$ since $P = R(LR)^3$

By case 3 of Lemma 2.2.3, $R(LR)^3\ (LR)^n \in PLR$

ii) If a = L:

$y = a \circ x$

$= L \circ (LR)^n$

$= R(LR)^{11}\ (LR)^n$ since $L = R(LR)^{11}$

By case 3 of Lemma 2.2.3, $R(LR)^{11}\ (LR)^n \in PLR$

iii) If a = R:

$y = a \circ x$

$= R \circ (LR)^n$

$= R(LR)^n$

As in table 2.4, $R(LR)^n \in PLR$

Case 2: Let $x = R(LR)^n$

i) If a = P:

$y = a \circ x$

$= P \circ R(LR)^n$

$= R(LR)^3\ R(LR)^n$ since $P = R(LR)^3$

By case 1 of Lemma 2.2.3, $R(LR)^3\ R(LR)^n \in PLR$

ii) If a = L:

$y = a \circ x$

$= L \circ R(LR)^n$

$= R(LR)^{11}\ R(LR)^n$ since $L = R(LR)^{11}$

By case 1 of Lemma 2.2.3, $R(LR)^{11}\ R(LR)^n \in PLR$

iii) If a = R:

$$\begin{aligned} y &= a \circ x \\ &= R \circ R(LR)^n \\ &= (RR)\ (LR)^n \\ &= (LR)^n \qquad\qquad \text{since } R^2 = \boldsymbol{i} \end{aligned}$$

As in table 2.4, $(LR)^n \in PLR$

In conclusion, all compositions of P, L, and R functions are represented in the set PLR.

■

Now we will show by induction that all possible compositions of P, L, and R are equivalent to some composition of $(LR)^n$ and $R(LR)^m$.

Theorem 2.2.6 All possible compositions of P, L, and R are in the PLR-set.

Proof:

Let x be any composition of P, L, and R functions. We say that x has length at most n if there exists a decomposition for x as a composition of at most n total P, L, and R functions. We will prove by induction that any composition of P, L, and R functions is in the PLR-set.

Base case: Verify that any x of length n = 1, is in the PLR-set.

Case 1: $x = P = R(LR)^3$, then $x \in PLR$

Case 2: If $x = L = R(LR)^{11}$, then $x \in PLR$

Case 3: If $x = R$, then $x \in PLR$

We assume that we have proved that if x has length at most k, for $k \geq 1$ then $x \in$ PLR.

Inductive Step: Verify that if x has length k+1, then $x \in$ PLR.

Let y be of length k+1, so by definition y is a composition of k+1 total P, L, and R functions. Let a be the first function in the composition, therefore $a \in \{P, L, R\}$, and then $y = a \circ x$. Now x has length $\leq k$, and by the base case, we know that $x \in$ PLR. Therefore, there exists an n such that $x = (LR)^n$ or $x = R(LR)^n$

Now apply lemma 2.2.5 and we see that $a \circ x = y \in$ PLR.

■

Note that since $P^{-1} = P$, $L^{-1} = L$, and $R^{-1} = R$, then theorem 2.2.6 shows that the subgroup generated by P, L, and R (which is the group of all the possible compositions including inverses), is the PLR-set. This concludes our investigation of the elements in the PLR-set. We know that there are no less and no more than 24 elements in the set, which are illustrated in table 2.4. As with the T/I functions, we verify that the PLR functions are well defined.

Theorem 2.2.7 The operations P, L, and R are well defined and therefore, if [x] is a pitch class triad in M then for all $x_1, x_2 \in [x]$ we have

i) $P(x_1) = P(x_2)$, i.e. $P(<a_1,b_1,c_1>) = P(<a_2,b_2,c_2>)$

ii) $L(x_1) = L(x_2)$, i.e. $L(<a_1,b_1,c_1>) = L(<a_2,b_2,c_2>)$

ii) $R(x_1) = R(x_2)$, i.e. $R(<a_1,b_1,c_1>) = R(<a_2,b_2,c_2>)$

Proof:

Let $x_1, x_2 \in [x] \in M$, where $x_1 = <a_1,b_1,c_1>$ and $x_2 = <a_2,b_2,c_2>$. So, x_1, and x_2 are elements in the triad class $[x] = <[a],[b],[c]>$. We see then that $a_1, a_2, \in [a] \in \mathbb{Z}_{12}$, and $b_1, b_2, \in [b] \in \mathbb{Z}_{12}$, and $c_1, c_2, \in [c] \in \mathbb{Z}_{12}$

i) $P(<a_1,b_1,c_1>) = <C_1,B_1+1,A_1>$

and,

$P(<a_2,b_2,c_2>) = <C_2,B_2+1,A_2>$

since,

$A_1 \in [A]$ then $(A_1+n) \in [A+n]$ and $A_2 \in [A]$ then $(A_2+n) \in [A+n]$

Similarly

$(B_1+n), (B_2+n) \in [B+n]$ and $(C_1+n), (C_2+n) \in [C+n]$

which gives us

$\mathbf{P(x_1)} = P(<a_1,b_1,c_1>) = <C_1,B_1+1,A_1> = <C_2,B_2+1,A_2> = P(<a_2,b_2,c_2>) = \mathbf{P(x_2)}$

Hence, P is well defined.

ii) $L(<a_1,b_1,c_1>) = <A_1+1,C_1,B_1>$

and,

$L(<a_2,b_2,c_2>) = <A_2+1,C_2,B_2>$

since,

$A_1 \in [A]$ then $(A_1+n) \in [A+n]$ and $A_2 \in [A]$ then $(A_2+n) \in [A+n]$

Similarly

$(B_1+n), (B_2+n) \in [B+n]$ and $(C_1+n), (C_2+n) \in [C+n]$

which gives us

$\mathbf{L(x_1)} = L(<a_1,b_1,c_1>) = <A_1+1,C_1,B_1> = <A_2+1,C_2,B_2> = L(<a_2,b_2,c_2>) = \mathbf{L(x_2)}$

Hence, L is well defined.

iii) $R(<a_1,b_1,c_1>) = <B_1,A_1,C_1-2>$

and,

$R(<a_2,b_2,c_2>) = <B_2,A_2,C_2-2>$

since,

$A_1 \in [A]$ then $(A_1+n) \in [A+n]$ and $A_2 \in [A]$ then $(A_2+n) \in [A+n]$

Similarly

$(B_1+n), (B_2+n) \in [B+n]$ and $(C_1+n), (C_2+n) \in [C+n]$

which gives us

$\mathbf{R(x_1)} = R(<a_1,b_1,c_1>) = <B_1,A_1,C_1-2> = <B_2,A_2,C_2-2> = R(<a_2,b_2,c_2>) = \mathbf{R(x_2)}$

Hence, R is well defined.

■

3. T/I AND PLR GROUPS

After having established the T/I and PLR sets of functions to be well defined, it is safe to move onto the exploration of these sets as *groups* under composition. We have already noticed that the two sets look very different. The elements of the T/I-set are more intuitively derived through their indices whereas the elements of the PLR-set are not quite so intuitive. The group properties of each will follow in a similar fashion. We first recall the definition of a group.

Definition 3.0.1 A nonempty set G with a binary operation $*$ on G is called a **group** if the following axioms hold:

i. For all a, b $\in$ G, $a*b = c$ such that c $\in$ G (G is closed under the operation)

ii. $a*(bc) = (ab)*c$ for all a, b, c $\in$ G

iii. There exists $i \in$ G such that $i*a = a$ for all a $\in$ G

iv. For every a $\in$ G there exists a' $\in$ G such that $a'*a = i$

3.1. The T/I-Group

Using the results of the compositions of all T and I functions, the properties of the T/I-group can be represented in a much more concise manner.

Theorem 3.1.1 T/I forms a group under composition.

Proof:

i. For all f, g $\in$ T/I, $f \circ g = h \in$ T/I, by equations (1), (2), (3), and (4).

   Hence, T/I is closed under composition.

ii. $T_0 \circ T_n = T_{0+n} = T_n$

$T_n \circ T_0 = T_{n+0} = T_n$

$T_0 \circ I_n = I_{0+n} = I_n$

$I_n \circ T_0 = I_{n-o} = I_n$

Therefore, $T_0 = \boldsymbol{i} \in T/I$ (i.e. T_0 is the identity element)

iii. $T_n \circ T_{12-n} = T_{n+12-n} = T_{12} = T_0$

$T_{12-n} \circ T_n = T_{12-n+n} = T_{12} = T_0$

Therefore, $T_n^{-1} = T_{12-n}$

$I_n \circ I_n = T_{n-n} = T_0$

Therefore, $I_n^{-1} = I_n$

iv. By the properties of the composition of functions, the operation $\circ$ is associative.

Hence, T/I is a group under composition

■

3.2. The PLR-Group

With the fine details taken care of in section 2.2, we will use the main outcomes in order to simplify our proof here.

Theorem 3.2.1 PLR forms a group under composition and $(R(LR)^n)^{-1} = R(LR)^n$ and $((LR)^n)^{-1} = (LR)^k$ where $-n = k \bmod 12$.

Proof:

By the note after theorem 2.2.6, PLR is the subgroup generated by all P, L, and R functions and therefore fulfills all the properties of being a group. While we know that

the inverses exist, it is of interest to see what the inverses are of the two main elements in the PLR-group. By lemma 2.2.4, for all functions of the form $R(LR)^n$, then

$$R(LR)^n \circ R(LR)^n = i$$

Therefore, $(R(LR)^n)^{-1} = R(LR)^n$

As for all functions of the form $(LR)^n$

$$((LR)^n)^{-1} = (LR)^{-n} = (LR)^{-n \bmod 12}$$

Then

$$(LR)^n \circ (LR)^{-n} = (LR)^{n-n} = (LR)^0 = i$$

Therefore, $((LR)^n)^{-1} = (LR)^k$ where $-n = k \bmod 12$.

Hence there exists an inverse function for all functions in PLR.

Hence, PLR is a group under composition.

■

4. T/I AND PLR ISOMORPHISMS

To summarize briefly, we now have established two groups of transformations where the elements of each group are functions that map the set M to itself. The next level of exploration stems from the fact that both the T/I and PLR groups have generators with similar properties, and that there exists a homomorphism between them.

4.1. The Dihedral Group

By observing the generators of each group, we see immediately that they are each isomorphic to the *dihedral group* when n = 12. We can also see this geometrically where the transpositions in the T/I-group are represented as rotations of the 12 vertices of the dodecagon and where the inversions are represented as the reflections.

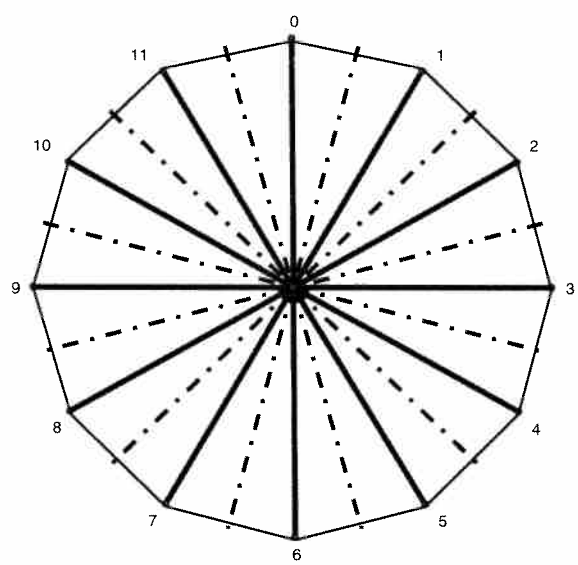


Figure 4.1 The dodecagon showing the axes of rotation and reflection

Theoretically we can demonstrate this by first recalling the definition of the dihedral group of degree n [7].

Definition 4.1.1 The dihedral group $\mathbf{D_n}$ (i.e. group of symmetries), is a group of order 2n generated by two elements f and g satisfying

$$f^n = i, \qquad g^2 = i, \qquad \text{and} \qquad gf = f^{n-1}g$$

where,

$$D_n = \{i, f, f^2, \ldots, f^{n-1}, g, gf, \ldots, gf^{n-1}\}$$

These properties exist in both the T/I-group and PLR-group, as seen in the following theorems.

Theorem 4.1.2 The T/I-group is isomorphic to the dihedral group when n = 12 where,

i) $(T_1)^n = i$, and $(I_0)^2 = i$

ii) $(I_0)(T_1) = (T_1)^{n-1}(I_0)$

iii) $T/I = \{i, T_1, T_2, \ldots, T_{n-1}, I_0, I_1, \ldots, I_{n-1}\}$

Proof:

i. $(T_1)^n = (T_1)^{12}$ since n = 12

$$= \underbrace{(T_1)(T_1) \ldots (T_1)(T_1)}_{\text{12 times}}$$

$$= T_{\underbrace{1+1+\ldots+1+1}_{\text{12 times}}}$$

$$= T_{12}$$

$= T_0$

$= \boldsymbol{i}$

and,

$(I_0)^2 = (I_0)(I_0)$

$= T_{0+0}$

$= T_0$

$= \boldsymbol{i}$

ii. $(I_0)(T_1) = I_{0-1}$

$= I_{-1}$

$= I_{11}$ since $11 = -1 \bmod 12$

$= I_{12-1+0}$

$= (T_{12-1})(I_0)$

$= (T_1)^{12-1}(I_0)$

$= (T_1)^{n-1}(I_0)$

iii. The T/I-group is

$\{\boldsymbol{i}, T_1, (T_1)^2, \ldots, (T_1)^{11}, I_0, (I_0)(T_1), \ldots, (I_0)(T_1)^{11}\}$

$= \{\boldsymbol{i}, T_1, T_2, \ldots, T_{11}, I_0, I_{0-1}, \ldots, I_{0-11}\}$ Since $(T_1)^m = T_m$, and $= (I_n)(T_m) = I_{n-m}$

$= \{\boldsymbol{i}, T_1, T_2, \ldots, T_{11}, I_0, I_1, \ldots, I_{11}\}$ Since $I_k = I_{-n \bmod 12}$

Hence, we see that the T/I-group is isomorphic to the dihedral group of the dodecagon.

■

Theorem 4.1.3 The PLR-group is isomorphic to the dihedral group when n = 12 where,

i) $(LR)^n = i$, and $R^2 = i$

ii) $R(LR) = (LR)^{n-1}R$

iii) $PLR = \{i, (LR), (LR)^2, \ldots, (LR)^{n-1}, R, R(LR), \ldots, R(LR)^{n-1}\}$

Proof:

i. $(LR)^n = (LR)^{12}$ since n = 12

$= (LR)^0$ since 12 = 0 mod 12

$= i$

and,

$(R)^2 = i$ since R is involutive

ii. $R(LR) = L(LR)^{10}$ by table 2.5

$= L(LR)^{10}(RR)$ since $(R)^2 = i$

$= (L(LR)^{10}R)\,R$

$= (LR)^{11}R$

$= (LR)^{12-1}R$

$= (LR)^{n-1}R$

iii. The PLR-group is

$\{i, (LR), (LR)^2, \ldots, (LR)^{n-1}, R, R(LR), \ldots, R(LR)^{11}\}$

Hence, we see that the PLR-group is isomorphic to the dihedral group of the dodecagon.

■

4.2. PLR to T/I

There is an additional isomorphism between the T/I-group and PLR-group, although by transitivity, as each are isomorphic to the dihedral group of the dodecagon, they are isomorphic to each other. This additional mapping between the T/I-group and PLR-group is not (as one might think) a mapping between the same two functions that map any triad x to the same triad y, since this mapping proves not to be a homomorphism. Consider the new mapping denoted by φ as seen in the following table, which in fact is an isomorphism and will be verified by theorem 4.2.2.

Table 4.1 Isomorphism φ: PLR $\rightarrow$ T/I

PLR $\rightarrow$ T/I	PLR $\rightarrow$ T/I	PLR $\rightarrow$ T/I
$R \rightarrow I_0$	$R(LR)^4 \rightarrow I_8$	$R(LR)^8 \rightarrow I_4$
$LR \rightarrow T_1$	$(LR)^5 \rightarrow T_5$	$(LR)^9 \rightarrow T_9$
$RLR \rightarrow I_{11}$	$R(LR)^5 \rightarrow I_7$	$R(LR)^9 \rightarrow I_3$
$(LR)^2 \rightarrow T_2$	$(LR)^6 \rightarrow T_6$	$(LR)^{10} \rightarrow T_{10}$
$R(LR)^2 \rightarrow I_{10}$	$R(LR)^6 \rightarrow I_6$	$R(LR)^{10} \rightarrow I_2$
$(LR)^3 \rightarrow T_3$	$(LR)^7 \rightarrow T_7$	$(LR)^{11} \rightarrow T_{11}$
$R(LR)^3 \rightarrow I_9$	$R(LR)^7 \rightarrow I_5$	$R(LR)^{11} \rightarrow I_1$
$(LR)^4 \rightarrow T_4$	$(LR)^8 \rightarrow T_8$	$(LR)^0 \rightarrow T_0$

We see then, instead that the mapping is established by first pairing the generators and the identities of the T/I-group and the PLR-group with each other. In other words, the generators of T/I are T_1 and I_0, and the generators of the PLR-group are (LR) and R.

Therefore we map T_1 to (LR) and I_0 to R. The identities T_0 and $(LR)^0$ are then also paired with each other. The remaining functions reveal a pattern between the powers of the R and L compositions and the sub indices of the T and I functions. We notice that all functions of the form $(LR)^x$ map to T_n functions where n = x, and all $R(LR)^x$ functions map to I_n functions where $n \equiv -x \bmod 12$. For now, we make nothing more of this, other than it being a consequential pattern which is interesting at the least.

Theorem 4.2.2 There exists a bijective homomorphism φ: PLR $\rightarrow$ T/I, such that $\varphi((LR)^x) = T_n$, where n = x; and $\varphi(R(LR)^x) = I_n$, where $n \equiv -x \bmod 12$.

Proof:

As shown in table 4.1, we see that each element of the domain is mapped to exactly one element in the codomain, and that all the elements in the codomain have a pre-image. Hence we can conclude that the mapping φ is bijective. Following is a *pointwise* evaluation of the homomorphism property, using all possible compositions of the generators of the PLR-group. In other words we will show that for all g, h $\in$ PLR and for all x $\in$ M,

$$\varphi(g \circ h)(x) = \varphi(g)(x) \circ \varphi(h)(x)$$

Case 1: Let x = $\langle a,b,c \rangle \in$ M, and let g = (LR) and h = R, then

The left side of the equation results in

$$\begin{aligned} \varphi(g \circ h)(\langle a,b,c \rangle) &= \varphi((LR) \circ R)(\langle a,b,c \rangle) \\ &= \varphi(LRR)(\langle a,b,c \rangle) \\ &= \varphi(L)(\langle a,b,c \rangle) && \text{since } R^2 = i \\ &= \varphi(R(LR)^{11})(\langle a,b,c \rangle) \end{aligned}$$

$= I_1(\langle a,b,c\rangle)$ by table 4.1

$= \langle -a+1,-b+1,-c+1\rangle$

While the right side results in

$\varphi(g)(\langle a,b,c\rangle) \circ \varphi(h)(\langle a,b,c\rangle) = \varphi((LR))(\langle a,b,c\rangle) \circ \varphi(R)(\langle a,b,c\rangle)$

$= T_1(\langle a,b,c\rangle) \circ I_0(\langle a,b,c\rangle)$ by table 4.1

$= I_{1+0}(\langle a,b,c\rangle)$ by T/I composition

$= I_1(\langle a,b,c\rangle)$

$= \langle -a+1,-b+1,-c+1\rangle$

Case 2: Let $x = \langle a,b,c\rangle \in M$, and let $g = R$ and $h = (LR)$, then

The left side of the equation results in

$\varphi(g \circ h)(\langle a,b,c\rangle) = \varphi(R \circ (LR))(\langle a,b,c\rangle)$

$= \varphi(RLR)(\langle a,b,c\rangle)$

$= I_{11}(\langle a,b,c\rangle)$ by table 4.1

$= \langle -a+11,-b+11,-c+11\rangle$

While the right side results in

$\varphi(g)(\langle a,b,c\rangle) \circ \varphi(h)(\langle a,b,c\rangle) = \varphi(R)(\langle a,b,c\rangle) \circ \varphi((LR))(\langle a,b,c\rangle)$

$= I_0(\langle a,b,c\rangle) \circ T_1(\langle a,b,c\rangle)$ by table 4.1

$= I_{0-1}(\langle a,b,c\rangle)$ by T/I composition

$= I_{-1}(\langle a,b,c\rangle)$

$= I_{11}(\langle a,b,c\rangle)$

$= \langle -a+11,-b+11,-c+11\rangle$

Case 3: Let $x = \langle a,b,c\rangle \in M$, and let $g = (LR)$ and $h = (LR)$, then

The left side of the equation results in

$$\varphi(g \circ h)(\langle a,b,c\rangle) = \varphi((LR) \circ (LR))(\langle a,b,c\rangle)$$

$$= \varphi((LR)^2)(\langle a,b,c\rangle)$$

$$= T_2(\langle a,b,c\rangle) \qquad \text{by table 4.1}$$

$$= \langle a+2,b+2,c+2\rangle$$

While the right side results in

$$\varphi(g)(\langle a,b,c\rangle) \circ \varphi(h)(\langle a,b,c\rangle) = \varphi((LR))(\langle a,b,c\rangle) \circ \varphi((LR))(\langle a,b,c\rangle)$$

$$= T_1(\langle a,b,c\rangle) \circ T_1(\langle a,b,c\rangle) \qquad \text{by table 4.1}$$

$$= T_{1+1}(\langle a,b,c\rangle) \qquad \text{by T/I composition}$$

$$= T_2(\langle a,b,c\rangle)$$

$$= \langle a+2,b+2,c+2\rangle$$

Case 4: Let $x = \langle a,b,c\rangle \in M$, and let $g = R$ and $h = R$, then

The left side of the equation results in

$$\varphi(g \circ h)(\langle a,b,c\rangle) = \varphi(R \circ R)(\langle a,b,c\rangle)$$

$$= \varphi(R^2)(\langle a,b,c\rangle)$$

$$= \varphi(i)(\langle a,b,c\rangle)$$

$$= \varphi((LR)^0)(\langle a,b,c\rangle) \qquad \text{by table 2.3}$$

$$= T_0(\langle a,b,c\rangle) \qquad \text{by table 4.1}$$

$$= \langle a,b,c\rangle$$

While the right side results in

$$\varphi(g)(\langle a,b,c\rangle) \circ \varphi(h)(\langle a,b,c\rangle) = \varphi(R)(\langle a,b,c\rangle) \circ \varphi(R)(\langle a,b,c\rangle)$$

$$= I_0(\langle a,b,c\rangle) \circ I_0(\langle a,b,c\rangle) \qquad \text{by table 4.1}$$

$$= T_{0+0}(\langle a,b,c\rangle) \qquad \text{by T/I composition}$$

$= T_0(\langle a,b,c \rangle)$

$= \langle a,b,c \rangle$

Therefore $\varphi(g \circ h)(x) = \varphi(g)(x) \circ \varphi(h)(x)$ for all $g, h \in PLR$ and for all $x \in M$.

This shows us that there exists a homomorphism φ: $PLR \rightarrow T/I$.

Hence, φ is an isomorphism.

■

5. DUALITY OF THE T/I AND PLR GROUPS

The term duality has been used in various different disciplines and has revealed itself to have highly specialized definitions. There are occurrences of duality in both music and math, but should not be confused with the definition in this study. In group theory we see duality pertaining to abelian groups and projective planes [17], which is very far removed from our use of the term. In music we see harmonic duality, which dates back to the 19th century [18], but later resurfaces in Lewin's research as he notices harmonic duality between the TI and PLR groups [23]. Harmonic duality is informally expressed as "whatever a transformation does to a major triad, its effect on a minor triad is precisely the opposite" [22]. Julian Hook expanded on this and formalized a mathematical definition of duality. It is this mathematical definition as explicitly stated in [13] that will be used when analyzing the properties of the T/I and PLR groups. We formally define duality in section 5.3 when it is pertinent, but in order to get a sense of the task ahead, we define it briefly as two groups being dual if they both act simply and transitively on the same set and each is the centralizer of the other, within some larger group.

5.1. Simple Transitivity

In order to derive simple transitivity we need to verify a few underlying properties of both groups. We take a closer look at these prerequisites with regards to the T/I and PLR groups before concluding simple transitivity. The proofs of how T/I and PLR satisfy these conditions are placed directly after the definitions, which come from [24] and [7].

Definition 5.1.1 Let G be a group and X a set. Then G is said to act on X if there is a mapping $\phi: G \times X \to X$, with $\phi(g,x)$ written as $g*x$, such that for all $a, b \in G$, $x \in X$,

i. $a*(b*x) = (ab)*x$

ii. $i*x = x$

The mapping ϕ is called the **action** of G on X, and X is said to be a G-set.

Lemma 5.1.2 The set M is a T/I-set. In other words the T/I-group acts on M.

Proof:

Let $\phi: T/I \times M \to M$. We define the T and I actions on M as in definition 2.1.1 and 2.2.2, for all $T_m, I_n \in T/I$ and $x, y, z \in M$.

We also recall equations (1), (2), (3), and (4) of the compositions of all T and I functions, because (as mentioned before) they represent all the functions in the T/I-group.

i. For equation (1): $T_m*\mathbf{(}T_n*x\mathbf{)} = T_m \circ (T_n(<a,b,c>))$
$$= (T_m(T_n(x)))$$
$$= \mathbf{(}T_mT_n\mathbf{)}*x$$

For equation (2): $T_m*\mathbf{(}I_n*x\mathbf{)} = T_m \circ (I_n(<a,b,c>))$
$$= (T_m(I_n(x)))$$
$$= \mathbf{(}T_mI_n\mathbf{)}*x$$

For equation (3): $I_m*\mathbf{(}T_n*x\mathbf{)} = I_m \circ (T_n(<a,b,c>))$
$$= (I_m(T_n(x)))$$
$$= \mathbf{(}I_mT_n\mathbf{)}*x$$

For Equation (4): $I_m*\mathbf{(}I_n*x\mathbf{)} = I_m \circ (I_n(<a,b,c>))$
$$= (I_m(I_n(x)))$$

$$= (I_m I_n) * x$$

ii. Since $T_0 = i \in T/I$ then

$$i * x = T_0(x) = x$$

Therefore, for all $T_m, I_n \in T/I$ and $x \in M$, T/I acts on M.

■

Lemma 5.1.3 The set M is a PLR-set. In other words the PLR-group acts on M.

Proof

Let ϕ: PLR $\times$ M $\rightarrow$ M. Recall that we have established PLR to be comprised of two main function compositions, namely $(LR)^n$ and $R(LR)^m$. We will again consider all compositions of these two main types of functions (in Lemma 2.2.4) for the first property of the action.

i. Case 1: $R(LR)^n * (R(LR)^m * x) = R(LR)^n \circ R(LR)^m(x)$

$$= R(LR)^n(R(LR)^m(x))$$

$$= (R(LR)^n R(LR)^m) * x$$

Case 2: $(LR)^n * ((LR)^m * x) = (LR)^n \circ (LR)^m(x)$

$$= (LR)^n((LR)^m(x))$$

$$= ((LR)^n (LR)^m) * x$$

Case 3: $R(LR)^n * ((LR)^m * x) = R(LR)^n \circ (LR)^m(x)$

$$= R(LR)^n((LR)^m(x))$$

$$= (R(LR)^n (LR)^m) * x$$

Case 4: $(LR)^n * (R(LR)^m * x) = (LR)^n \circ R(LR)^m(x)$

$$= (LR)^n(R(LR)^m(x))$$

$$= ((LR)^n R(LR)^m)*x$$

ii. Since $(LR)^0 = i$, then

$$i*x = (LR)^0(x) = x$$

Therefore, for all f, g ∈ PLR and x ∈ M, PLR acts on M.

■

Definition 5.1.4 Let X be a G-set, then for all x ∈ X, Gx = {gx | g ∈ G} is the **orbit** of x or the orbit generated by x.

Lemma 5.1.5 For all x ∈ M, the orbit of x is (T/I)x = M.

Proof:

As illustrated in table 2.2, all the functions applied to a single chord generate the entire set M. Even though the table only shows the functions being applied to the C major triad, it is easily verified that the functions act in the same way on any major or minor triad. Thus, the orbit of the T/I-group is then

$$(T/I)x = \{fx \mid f \in T/I\} = M$$

The entire set M is generated when all functions in T/I are applied to any triad in M.

■

Lemma 5.1.6 For all x ∈ M, the orbit of x is (PLR)x = M.

Proof:

As illustrated in table 2.3, all the functions applied to a single triad, generate the entire set M. The orbit of x in the PLR-group is

$$(PLR)x = \{fx \mid f \in PLR\} = M$$

Again, there is only one orbit for all $x \in M$, since the entire set M results from applying all functions in the PLR-group to any triad in M.

■

Definition 5.1.7 Let $x \in X$, and X be a G-set then $G_x = \{g \in G \mid gx = x\}$ is the **stabilizer** of x.

We are interested in the cardinality of the stabilizers and orbits, and will therefore make use of the following orbit-stabilizer theorem from [1].

Theorem 5.1.8 (*orbit-stabilizer*) Suppose that a group G acts on a set X. Let Gx be the orbit of $x \in X$, and let G_x be the stabilizer of x. Then the size of the orbit is the index of the stabilizer, that is, $|Gx| = [G:G_x]$. Thus if G is finite, then $|Gx| = |G|/|G_x|$.

Lemma 5.1.9 Let $x \in M$, and M be a T/I-set then the stabilizer of x is

$$(T/I)_x = \{f \in T/I \mid fx = x\} = T_0$$

Proof:

We use theorem 5.1.8. to see that

$$|(T/I)_x| = |T/I|/|(T/I)x|$$

where

$|T/I| = 24$ since there are 24 functions in T/I

and

$|(T/I)x| = 24$ since $(T/I)x = M$

This give us

$|(T/I)_x| = 24/24 = 1$

$(T/I)_x = \{T_0\} = \{i\}$ since $T_0(x) = x$

Thus, the stabilizer is trivial.

■

Lemma 5.1.10 Let $x \in M$, and M be a PLR-set then the stabilizer of x is

$$(PLR)_x = \{f \in PLR \mid fx = x\} = (LR)^0$$

Proof:

Again, we use theorem 5.1.8. to see that

$|(PLR)_x| = |PLR|/|(PLR)x|$

where

$|PLR| = 24$ since there are 24 functions in PLR

and

$|(PLR)x| = 24$ since $(PLR)x = M$

This give us

$|(PLR)_x| = 24/24 = 1$

$(PLR)_x = \{(LR)^0\} = \{i\}$ since $(LR)^0(x) = x$

Thus, the stabilizer is trivial.

■

Definition 5.1.11 The action of G on X is said to be **free** if for all $x \in X$, and for all $g \in G$ if $g*x = x$ then $g = i$ (i.e. the stabilizer $G_x = \{i\}$ for all $x \in X$).

Lemma 5.1.12 The T/I-group and PLR-group act freely on M.

Proof:

We have already established above, that for all $x \in M$ the stabilizers are

$$(T/I)_x = \{i\}$$

$$(PLR)_x = \{i\}$$

Thus, the actions of the T/I-group and PLR-group on M satisfy the definition of being free.

■

Definition 5.1.13 The action of G on X is said to be **transitive** if for all $x, y \in X$ there exists a $g \in G$ such that $g*x = y$ (i.e. X consists of a single orbit).

Definition 5.1.14 The action of G on X is said to be **simply transitive** if it is both free and transitive (i.e. for all $x, y \in X$ there exists a unique $g \in G$ such that $g*x = y$).

We now examine how T/I and PLR act simply and transitively on M. For reference, a slightly different form of this proof is provided in [13].

Theorem 5.1.15 The actions of the T/I-group and PLR-group on M are both simply transitive.

Proof:

We can actually deduce simple transitivity from the illustrations in table 2.2 and table 2.3. We first see that all the functions acting on any x produces the entire set M. In other words, there exists a function such that for all x and y in M, $g(x) = y$. Furthermore, this result occurs without a repetition of triads, which means that only one function maps a triad to any other triad. Hence, g is unique.

Alternatively, we can use the findings regarding the above definitions to infer simple transitivity. First we established

$$(T/I)x = M \qquad \text{and} \qquad (PLR)x = M$$

which means that for all $x, y \in M$, there exists an $f \in T/I$ and a $g \in PLR$ such that

$$f(x) = y \qquad \text{and} \qquad g(x) = y$$

This shows that T/I and PLR act transitively on M.

We then observed that T/I and PLR act freely on M, which means that for all $f \in T/I$ and $g \in PLR$ where $fx = x$, and $gx = x$, then

$$f = \boldsymbol{i} = g$$

Now, suppose the functions f and g are not unique. In order to analyze this possibility, we suppose that there exists g_1, g_2 in T/I or in PLR such that $g_1(x) = g_2(x)$ and $g_1 \neq g_2$. Then

$$g_1(x) = g_2(x)$$

$$g_2^{-1}g_1(x) = g_2^{-1}g_2(x)$$

$$g_2^{-1}g_1(x) = x$$

but for both the T/I and PLR groups the stabilizer is trivial and therefore

$$g_2^{-1}g_1 = i$$

$$g_1 = g_2$$

Hence, g is unique and both T/I and PLR act simply and transitively on M.

■

5.2. Commutativity

The concept of centralizers is based on *commutativity*, and we therefore turn to the commutativity between the elements of T/I and PLR. Recall the definition from [7].

Definition 5.2.1 An operation $*$ on a set X is **commutative** if $x*y = y*x$, for all $x, y \in X$.

Lemma 5.2.2 All elements of the PLR-group and of the T/I-group commute. It suffices to show commutativity of the generators of each group, so we have

i) $T_1 \circ (LR) = (LR) \circ T_1$

ii) $T_1 \circ (R) = (R) \circ T_1$

iii) $I_0 \circ (LR) = (LR) \circ I_0$

iv) $I_0 \circ (R) = (R) \circ I_0$

Proof:

i) $T_1 \circ (LR)(<A,B,C>= T_1(L(<b,a,c+2>)) = T_1(<B+1,C+2,A>) = <B+2,C+3,A+1>$

$(LR) \circ T_1(<A,B,C>= (LR)(<A+1,B+1,C+1>) = L(<b+1,a+1,c+3>)= <B+2,C+3,A+1>$

ii) $T_1 \circ (R)(<A,B,C>= T_1(<b,a,c+2>) = <b+1,a+1,c+3>$

$(R) \circ T_1(<A,B,C>= R(<A+1,B+1,C+1>) = <b+1,a+1,c+3>$

iii) $I_0 \circ (LR)(<A,B,C>= I_0(L(<b,a,c+2>)) = I_0(<B+1,C+2,A>) = <-b-1,-c-2,-a>$

$(LR) \circ I_0(<A,B,C>= (LR)(<-a,-b,-c>) = L(<-B,-A,-C-2>) = <-b-1,-c-2,-a>$

iv) $I_0 \circ (R)(<A,B,C>= I_0(<b,a,c+2>) = <-B,-A,-C-2>$

$(R) \circ I_0(<A,B,C>= R(<-a,-b,-c>) = <-B,-A,-C-2>$

Hence, for all $f \in T/I$, and for all $g \in PLR$, $fg = gf$.

■

As a reminder, refer to definitions 2.1.1, 2.1.2, and 2.2.1 in order to ensure the correct transformations of major and minor triads. Figure 5.1 and 5.2 demonstrate musical depictions of commutativity with the use of commutative diagrams (as in category theory).

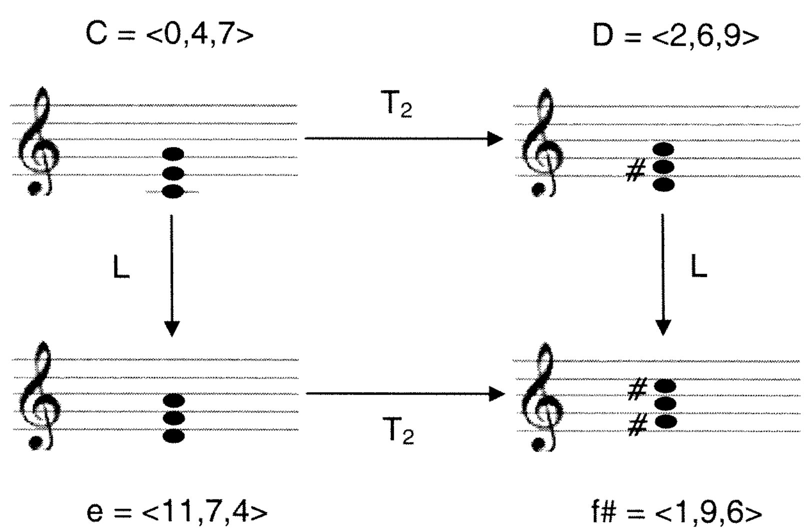


Figure 5.1 Musical illustration of $T_2L(C) = LT_2(C)$

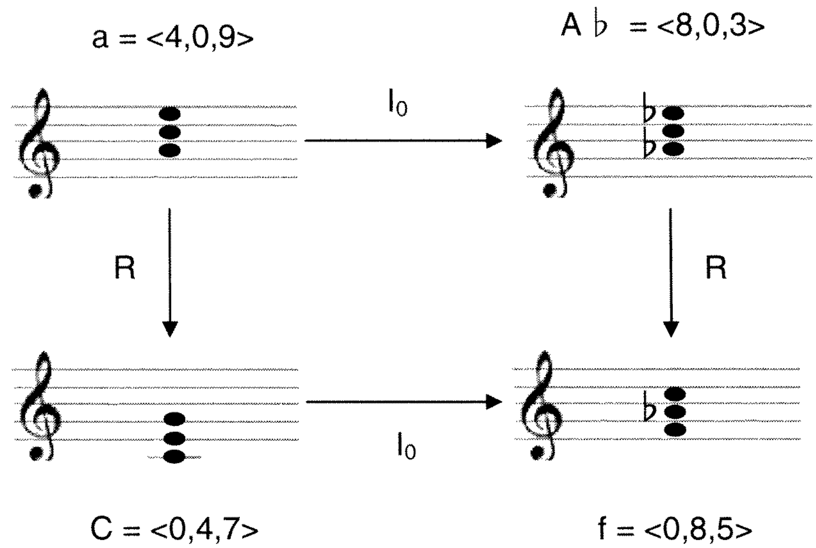


Figure 5.2 Musical illustration of $I_0R(a) = RI_0(a)$

More generally we can assert this relationship for the entire set M, and for all f_n in T/I and for all g in PLR.

$$\begin{tikzcd} M \arrow[r, "f_n"] \arrow[d, "g"'] & M \arrow[d, "g"] \\ M \arrow[r, "f_n"'] & M \end{tikzcd}$$

Figure 5.3 Commutative diagram showing that $f_ng = gf_n$

The commutative nature of the T/I and PLR groups leads us to the last notion needed for duality. While most of the proofs in this study have not been documented, a proof for duality is provided in [13].

5.3. T/I and PLR are Dual

Since both the T/I-group and PLR-group are mappings from M to itself, they are regarded as permutation groups, and therefore subgroups of the Symmetric group on M (i.e. Sym(M)). We will examine the *centralizers* of each group within the larger group Sym(M), with the following definition in mind [7].

Definition 5.3.1 Let X be a nonempty subset of a group G. The **centralizer** of X in G is

$$C(X) = \{g \in G \mid xg = gx, \text{ for all } x \in X\}$$

Theorem 5.3.2 C(T/I) = PLR and C(PLR) = T/I

Proof:

First consider the centralizer of the T/I-group.

$$C(T/I) = \{g \in Sym(M) \mid fg = gf, \text{ for all } f \in T/I\}$$

Lemma 5.2.2 shows that for all $g \in$ PLR, $fg = gf$, for all $f \in$ T/I, therefore PLR is contained in C(T/I). Since C(T/I) is in Sym(M), we must verify that there exist no other functions than those from the PLR-group. We start by investigating the stabilizer of x in C(T/I). Suppose, that $h \in C(T/I)$ fixes $x \in M$ (where $h \neq i$), and let $g \in$ T/I, then

$$h(x) = x$$

$$g(h(x)) = g(x)$$

$$h(g(x)) = g(x) \qquad \text{since } h \in C(T/I)$$

By theorem 5.1.15 we know that T/I acts simply transitively, so for all $y \in M$, $y = g(x)$.

This shows that for all $x, y \in M$

$$h(g(x)) = h(y) = g(x) = y$$

Therefore

$$h(y) = y \qquad \text{for all } y \in M$$

Thus, the only element in C(T/I) that fixes any element in C(T/I) is h. However, since the identity element fixes all triads in M, then h must be the identity element. Hence, $(C(T/I))_x = \{i\}$, the trivial group. We now apply the orbit-stabilizer theorem 5.1.8. to $(C(T/I))_x$, which is the stabilizer of x in C(T/I).

$$|(C(T/I))x| = |(C(T/I)| \,/\, |(C(T/I))_x|$$

We know that the orbit for all $x \in M$ is the entire set M, therefore

$$|(C(T/I))x| = 24$$

$$|(C(T/I))_x| = 1 \qquad \text{as shown above}$$

$$|(C(T/I))x| = |(C(T/I)| \,/\, |(C(T/I))_x| = |(C(T/I)| \,/\, 1 = 24$$

then

$$|(C(T/I)| = 24/1 = 24$$

$$C(T/I) = PLR \qquad \text{since } |PLR| = 24 \text{ and } PLR \subseteq C(T/I)$$

Hence, the centralizer of T/I is the PLR-group.

It remains to show that the centralizer of the PLR-group is the T/I-group, however one need only to reverse the roles of the T/I-group with the PLR-group from the start of the proof in order to show that C(PLR) = T/I.

■

Definition 5.3.3 Let H and K be subgroups of the symmetric group Sym(X) (i.e. H and K are permutation groups on the set X). H and K are said to be **dual** if each acts simply transitively on X and each is the centralizer of the other in Sym(X).

Due to the complexity of this last theorem, the core of it resides in the multiple lemmas and theorems prior to it.

Theorem 5.3.4 The T/I-group and PLR-group are dual.

Proof:

It follows from theorem 5.1.15 and from theorem 5.3.2 that T/I and PLR are dual groups.

■

Duality between the T/I-group and the PLR-group has existed in music before it was formalized mathematically. We see its appearance in both traditional classical music and modern music.

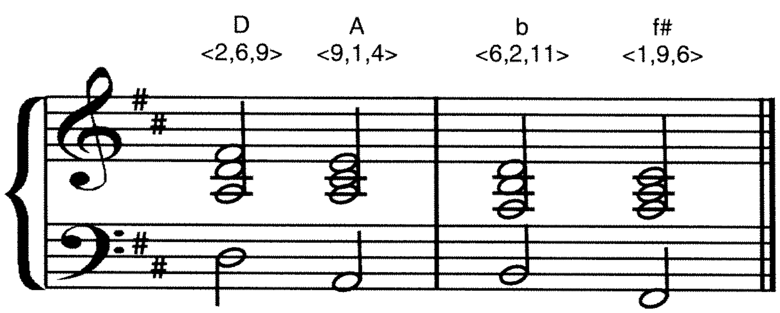


Figure 5.4 Selection from Pachelbel's Canon in D

This particular chord progression appears in 28 variations in Johann Pachelbel's Canon in D. The duality can be seen clearly via the following commutative diagram.

$$\begin{tikzcd} D \arrow[r, "T_7"] \arrow[d, "R"'] & A \arrow[d, "R"] \\ b \arrow[r, "T_7"'] & f\# \end{tikzcd}$$

Figure 5.5 Commutative diagram for selection from Pachelbel's Canon in D

Since we are observing the duality between the T/I-group and the PLR-group, we are interested in compositions of T/I and PLR functions. For this reason the functions running horizontally will always be in a different group to those running vertically. The object is to find the transformations between each triad while keeping in mind the need for both T/I and PLR functions and the changes in parity. In figure 5.5 the parity of the triads that are horizontally across from one another are the same and those triads vertical to each other are of opposite parity. We know that T functions are the simplest way of maintaining the parity of the triad being acted on so the horizontal function will be a transposition. The vertical transformation must change the parity of the triad and must therefore be an odd number of compositions of functions in PLR (since we have already used a function from T/I). The major chord D is seven semitones away from A, so we see that the horizontal transformations are T_7. We now have to find a function or

composition of functions from the PLR-group that maps A-major to f#-minor and D-major to b-minor. Both A and D are three semitones higher than the triad they are being mapped to (which means they are the relative majors of their respective minors). The vertical function is simply R then.

Another musical example, which is this time from the modern music era, is "Religion" written by Charles Ives'.

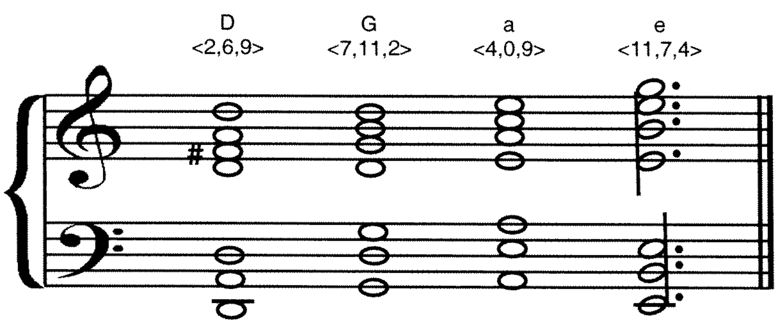


Figure 5.6 Selection from Ives' "Religion"

The commutative diagram for this musical example is then

$$
\begin{tikzcd}
D \arrow[r, "I_6"] \arrow[d, "LR"'] & a \arrow[d, "LR"] \\
G \arrow[r, "I_6"'] & e
\end{tikzcd}
$$

Figure 5.7 Commutative diagram for selection from Ives' "Religion"

For this example, we see that the triads horizontally across from one another change in parity and therefore need to be an I function or an odd number of compositions from PLR. We can do either as long as the vertical transformation comes from the opposite group. The simplest transformation then from D-major to a-minor and G-major to e-minor is through I_6. The vertical transformations must be from the PLR-group, and since the triads across from one another (vertically) are of the same parity, we need an even number of compositions of functions from PLR. The simplest transformation here is the composition LR, which takes D-major to G-major and a-minor to e-minor.

These two musical examples show us two things. Firstly, the mathematical duality between these groups of transformations existed before it was even defined, and secondly, we are able to use duality as a tool to analyze not only modern music, but traditional classical music.

6. FURTHER REPRESENTATIONS OF MUSICAL STRUCTURE

The analysis carried out in this thesis thus far has used some basic components of our music system, the 12 pitches and 24 triads. After equipping these components with mathematical structure (as a group, in the case of the 12 pitches), we proceeded to the operations that are commonly performed on them. These operations both have algebraic and geometric representations. This section proceeds along a similar path, where we first see an alternative representation of the triads as a torsor and secondly we elaborate on the already mentioned claim that the *Tonnetz* lies on the torus.

6.1. Torsors

The triad is essentially a 3-tuple, and therefore the set of 24 triads can be seen as a subset of $\mathbb{Z}_{12}\mathrm{x}\mathbb{Z}_{12}\mathrm{x}\mathbb{Z}_{12}$. Since the numbers applied to the pitches are arbitrary, we don't have a convenient or pre-defined identity element of the set, and thus it cannot form a group. This gives us the freedom to assign any element to be the identity element (if we so choose), which is one of the properties that enable us to view M as a torsor. The mathematical physicist, John Baez describes torsors in the following way [2] and [3].

Definition 6.1.1 Let G be a group acting on X. If, for all x, y $\in$ X gx = y such that g $\in$ G is unique, then X is called a **G-torsor**.

It is here where the previous comment about torsors (in section 1.2) is revealed, but first we recall some definitions.

Definition 6.1.2 A **homogeneous** space for a group G is a set X on which G acts continuously by symmetry in a transitive way.

Definition 6.1.3 A **principal homogeneous** space, or **torsor**, for a group G is a set X on which G acts freely and transitively. That is, X is a homogeneous space for G such that the stabilizer of any point is trivial.

From this perspective, we have already demonstrated that T/I and PLR act freely and transitively on M, and thus M is both a T/I-torsor and a PLR-torsor. We demonstrate then the difference between a group and a torsor in terms of the elements of each structure. There exists a relation between the elements of the group and the elements of a set (under some action) which produces elements of the set (the G-torsor). However, the elements of the set cannot relate to other elements in the set to form more elements of the set. Rather, elements of the set are related to other elements of the set (under some action) which result in elements of the group. In terms of our groups T/I, PLR, and our set M, we can say that when any function (in T/I or PLR) is applied to any triad it should form yet another triad in M. In addition, any two triads that are added or subtracted should not form another triad, but rather a transformation that exists in T/I or PLR. This will become clearer in a moment.

The idea of subtraction is very useful when applying the concept of torsors to other areas such as physics. For example, we don't perceive energies but rather differences in energies (i.e. we don't measure an instance of energy but rather an interval of energy). The same can be applied to some aspects of musical analysis. When we

listen to a section of music, we don't listen to each note isolated from the one before and the one thereafter. Instead, we follow the changes in the music and in some cases, anticipate the next move. For this reason, we are then interested in our set of triads (M) as being a T/I-torsor and a PLR-torsor.

We then test the notions expressed above and first apply a transformation to any triad, which does in fact result in another triad, but when we subtract two triads, we don't explicitly get an element of the T/I-group or PLR-group. For example <1,5,8> - <0,4,7> = <1,1,1>. None of the functions (T, I, P, L, or R) map any triad to this 3-tuple, and therefore the action of subtracting two elements of M is not equivalent to the action of any function in the T/I-group or PLR-group. This directs us to the task of giving some sort of structure to M which will allow us to perform these operations. Such a structure is provided by Julian Hook, with the following definitions [22].

Definition 6.1.4 A **triad** is denoted by $\Delta = (r,\sigma)$ where r is the root or number representing the name of the triad and σ is the sign of the triad (+ for major, and – for minor).

As and example of this, the A-major triad is <9,1,4> and would be expressed as

$$\Delta = (9,+)$$

Definition 6.1.5 Arithmetic performed on the numerical parameter of the triad will follow the rules of **arithmetic** mod 12. The signs will follow the rules of **multiplication** where

$$--=+, \qquad +-=-, \qquad -+=-, \qquad ++=+$$

Therefore, if we have $\Delta_1 = (r_1,\sigma_1)$, and $\Delta_2 = (r_2,\sigma_2)$ then the relationship between them is

$$\Delta_2 - \Delta_1 = (r_2 - r_1,\ \sigma_2\sigma_1)$$

This represents the interval between the triads Δ_2 and Δ_1. Note that this arithmetic gives us once again a result in the form $\Delta_3 = (r_3,\sigma_3)$, which we are interpreting to be an interval, but can also be read as another triad. For example, the interval from g-minor (7,–) to B-major (11,+) is

$$\Delta_2 - \Delta_1 = (11,+) - (7,-) = (11-7,\ +\,-) = (4,-) = \Delta_3$$

We see then, that the interval between g and B is a major third, but the result can also be seen as e-minor. One of the reasons for seeing this as a triad is to verify that all operations don't produce triads outside of our set M (which will become clearer in theorem 6.1.6). Seeing the result of subtraction as a triad is also useful when observing the second property of torsors, which states that after subtracting two elements of M we should be able to represent the result with an element of T/I or PLR. If we take our above example of moving from g-minor (7,–) to B-major (11,+), we can either see it as moving from g-minor through B-major to get e-minor, or we can see that we have moved an interval of (4,–) from g-minor to B-major through some transformation . We can then represent both of these movements in multiples ways using transformations that we have already defined. The first representation of this is transforming g-minor to e-minor by passing through B-major. Consider then,

$$I_1(g) = T_1(I_0(<2,10,7>)) = T_1(<10,2,5>) = <11,3,6> = B$$

then

$$I_{10}(B) = T_{10}(I_0(<11,3,6>)) = T_{10}(<1,9,6>) = <11,7,4> = e$$

Therefore, we have that $I_{10}I_1(g)$ = e-minor.

The second representation involves moving from g-minor to B-major via some transformation, of which five such possibilities are proposed here.

Transformation 1: $T_4(g) = T_4(<2,10,7>) = <6,2,11> = b$

then

$P(g) = P(<6,2,11>) = <11,3,6> = B$

Transformation 2: $I_1(g) = T_1(I_0(<2,10,7>)) = T_1(<10,2,5) = <11,3,6> = B$

Transformation 3: $P(g) = P(<2,10,7>) = <7,11,2> = G$

then

$LP(g) = LP(<2,10,7>) = <6,2,11> = b$

followed by

$PLP(g) = PLP(<2,10,7>) = <11,3,6> = B$

Transformation 4: $L(g) = L(<2,10,7>) = <3,7,10> = D\#$

then

$PL(g) = PL(<2,10,7>) = <10,6,3> = d\#$

followed by

$LPL(g) = LPL(<2,10,7>) = <11,3,6> = B$

Transformation 5: $R(g) = R(<2,10,7>) = <10,2,5> = E$

then

$LR(g) = LR(<2,10,7>) = <9,5,2> = d$

followed by

$PLR(g) = PLR(<2,10,7>) = <2,6,9> = D$

then

RPLR(g) = RPLR(<2,10,7>) = <6,2,11> = b

followed by

PRPLR(g) = PRPLR(<2,10,7>) = <11,3,6> = B

These examples are just some of the few ways in which this transformation can be represented. The object is to show that there exists at least one operation in T/I and PLR that represents the subtraction between any two triads in the torsor, M. We verify both properties of the torsor in the following theorem, which completes our declaration of M being a torsor.

Theorem 6.1.6 M is a T/I-torsor and a PLR-torsor.

Proof:

Let $\Delta_1 = (r_1,\sigma_1)$, and $\Delta_2 = (r_2,\sigma_2)$, for all Δ_1, $\Delta_2 \in$ M. Then for all f $\in$ T/I or PLR,

i) $f(\Delta_1) = \Delta_2$ as shown in table 2.2 and table 2.3.

and

ii) $\Delta_2 - \Delta_1 = (r_2 - r_1, \sigma_2\sigma_1)$

Since $r_2 - r_1, \in \mathbb{Z}_{12}$ and $\sigma_2\sigma_1 \in \{+,-\}$, then

$$\Delta_2 - \Delta_1 = (r_2 - r_1, \sigma_2\sigma_1) = (r_3,\sigma_3) = \Delta_3 \in M$$

This shows us that the subtraction between any two triads result only in triads that are in our set M. When we view Δ_3 as an interval we are returning to the notion of transformations. Therefore, the subtraction between two triads is a transformation which we can view as a mapping of one triad to another via some operation. Since all of the transformations studied here (T, I, P, L, and R), undeniably map any triad to

another triad and no other element outside of M, we see then the definition fulfilled. The difference between two elements in M represents a transformation in T/I or in PLR. Thus M is a T/I-torsor, and M is a PLR-torsor.

■

6.2. The *Tonnetz* and the Torus

We have already seen the torus surface in chapter 3, but here is where we elaborate on its presence in music. We informally define the torus as a topological object that looks like an inner tube filled with air [16]. More formally, we consider circle C_1 (from figure 6.1) and rotate it about the z-axis. The full rotation of circle C_1 forms a second circle, C_2. Point (p_1, p_2) is the result of rotating C_1 about the z-axis until it lies in a vertical plane containing p_2. We see then that all the points of the torus are obtained by the Cartesian product of these two circles.

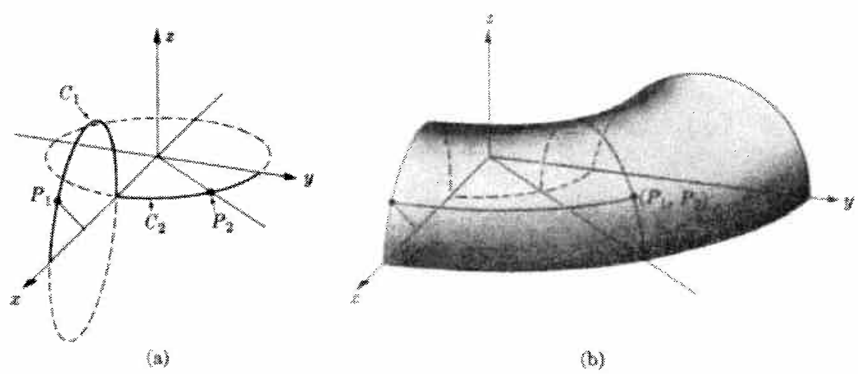


Figure 6.1 The torus generated by C_1xC_2

While each circle in itself is generated by the Cartesian product of two sets of real numbers, we see that the torus is a product of 4 sets of real numbers. Alternatively, we

can say that it is imbedded in Euclidean 4-space, where a Euclidean n-space is defined as n-tuples of n real numbers [21]. The work to follow does not rely on the formal definition of the torus, but it is provided nonetheless.

Definition 6.2.1 The **torus** is a product space or 4-d Euclidean embedding such that $\langle x,y,z,w \rangle = \langle \alpha\cos x, \alpha\sin x, \beta\cos y, \beta\sin y \rangle$ where α and β are constants.

A more practical way of working on the torus is by modeling it on a flat (or two dimensional) surface, where the points on the surface of the torus are repeated continually in all four directions of the plane to form the *flat torus*. The torus and flat torus are topologically the same, since the surface is unaffected by deformation. We can say that the four edges of the flat torus are glued together to form the surface of a doughnut or inner tube. Geometrically, on the other hand, they are different because they have different curvatures (one is flat and one is circular and cylindrical in nature). The flat torus is not to be confused with the actual Cartesian plane, which is infinite. The torus is a finite area with no edges [29], and while this area could be repeated infinitely on a flat surface, every identical point on the plane is actually the same point on the torus. In other words, an infinite path that repeatedly passes over a certain point is not a path that stretches out to infinity. Rather it is a path that wraps around the torus infinitely.

The torus appears in various forms in musical analysis, one of which is the *Tonnetz* which we have seen before. The *Tonnetz* is used as a tool for its ability to trace harmonic motion or chord progressions in music. Of all the descriptions of its

construction, the most eloquent comes from Balzano [5], who sets out to represent n-fold music systems that are analogous to the common 12-tone system.

Balzano proceeds to examine Cartesian products of subgroups of $\mathbb{Z}_{12}$.and the potential isomorphisms with $\mathbb{Z}_{12}$. We have the following theorems at our disposal to help determine the subgroups of $\mathbb{Z}_{12}$ and its isomorphisms [7].

Theorem 6.2.2 (*Lagrange*) Let G be a finite group. Then the order of any subgroup of G divides the order of G.

Theorem 6.2.3 If G is a cyclic group of order mn, where (m,n) = 1, then G $\cong$ H x K, where H is a subgroup of order m, and K is a subgroup of order n.

The following are then subgroups of orders 2, 3, 4, 6 respectively, and while the trivial group and $\mathbb{Z}_{12}$ itself are technically also subgroups, they are of no concern at this moment.

$\mathbb{Z}_{12}/6\mathbb{Z} = \{0,6\} \cong \mathbb{Z}_2 = \{0, 1\}$

$\mathbb{Z}_{12}/4\mathbb{Z} = \{0, 4, 8\} \cong \mathbb{Z}_3 = \{0, 1, 2\}$

$\mathbb{Z}_{12}/3\mathbb{Z} = \{0, 3, 6, 9\} \cong \mathbb{Z}_4 = \{0, 1, 2, 3\}$

$\mathbb{Z}_{12}/2\mathbb{Z} = \{0, 2, 4, 6, 8, 10\} \cong \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$

We see then that $\mathbb{Z}_3$x$\mathbb{Z}_4$, which is the set

$$\{(0,0), (0,1), (0,2), (0,3), (1,0), (1,1), (1,2), (1,3), (2,0), (2,1), (2,2), (2,3)\}$$

is the only possible isomorphism with $\mathbb{Z}_{12}$, and the following mapping is proposed.

Theorem 6.2.4 There exists a bijective homorphism ρ: $\mathbb{Z}_3 \text{x} \mathbb{Z}_4 \to \mathbb{Z}_{12}$, such that

$$\rho(a,b) = (4a + 3b)$$

Proof:

Let the operation $*$ be defined as

$$(a,b)*(c,d) = (a+c,b+d), \text{ for all } (a,b), (c,d) \in \mathbb{Z}_3 \text{x} \mathbb{Z}_4$$

We verify the homomorphism as $\rho((a,b)*(c,d)) = \rho(a,b) * \rho(c,d)$

The left side yields

$$\rho((a,b)*(c,d)) = \rho(a+c,b+d) = 4(a+c) + 3(b+d) = 4a+4c+3b+3d$$

The right side yields

$$\rho(a,b)*\rho(c,d) = (4a + 3b) + (4c + 3d) = 4a+4c+3b+3d$$

Hence ρ is a homomorphism, from $\mathbb{Z}_3 \text{x} \mathbb{Z}_4$ to $\mathbb{Z}_{12}$, where the mapping goes as follows

Table 6.1 Isomorphism ρ: $\mathbb{Z}_3 \text{x} \mathbb{Z}_4 \to \mathbb{Z}_{12}$

$\mathbb{Z}_3 \text{x} \mathbb{Z}_4 \to \mathbb{Z}_{12}$					
$(0,0) \to 0$	$(0,1) \to 3$	$(0,2) \to 6$	$(0,3) \to 9$	$(1,0) \to 4$	$(1,1) \to 7$
$(1,2) \to 10$	$(1,3) \to 1$	$(2,0) \to 8$	$(2,1) \to 11$	$(2,2) \to 2$	$(2,3) \to 5$

We see from the table that it is a bijective mapping, therefore $\mathbb{Z}_3 \text{x} \mathbb{Z}_4 \cong \mathbb{Z}_{12}$.

■

For the geometric interpretation, we introduce the following definitions [6].

Definition 6.2.5 The distance between any two pitches is called the **interval** between them and is measured in half-steps or semitones. A **semitone** is equivalent to 1, and the following are common intervals

i) 3 semitones between two pitches is called a **minor third** (denoted m3)

ii) 4 semitones between two pitches is called a **major third** (denoted M3)

iii) 7 semitones between two pitches is called a **perfect fifth** (denoted P5)

We are able to plot the intervals on the Cartesian plane from the mapping shown in table 6.1. We see then that one axis is generated by intervals of 3 and the other axis is generated by intervals of 4 as shown in figure 6.2.

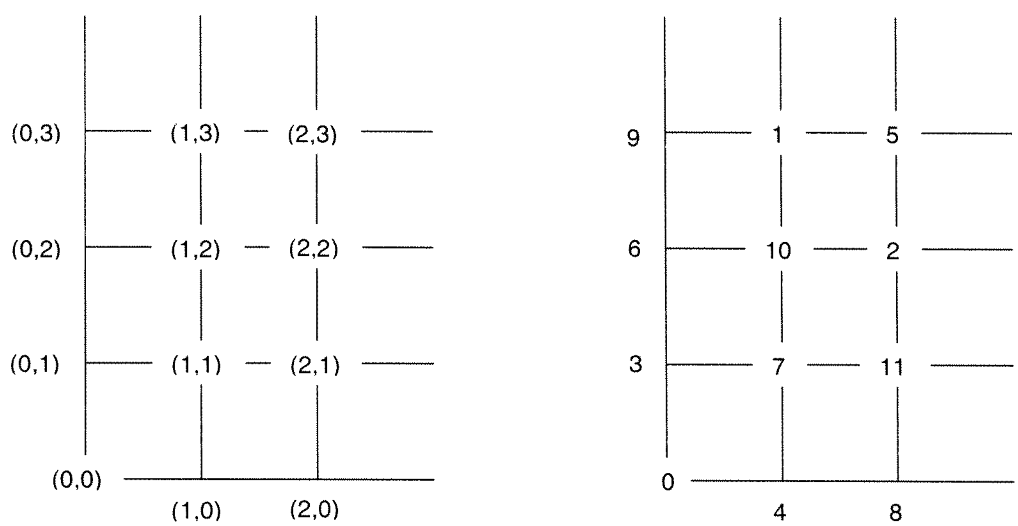


Figure 6.2 The geometric representation of $\mathbb{Z}_3 \mathrm{x} \mathbb{Z}_4 \cong \mathbb{Z}_{12}$

So, we see that each interval can be described in terms of the number of major and minor thirds in the interval. For example, take the point 7 in figure 6.2, on the grid

to the right. The interval 7 is a perfect fifth (P5), where the point 7 on the right corresponds to the point (1,1) on the left, which translates into

$$7 = 4(1) + 3(1) = \text{(1 major third) and (1 minor third)} = (1,1)$$

Another example, would be the point (2,3) on the left grid of figure 6.2. This point indicates that we are moving an interval equivalent to 2 major thirds and 3 minor thirds away from the origin, which gives us

$$(2,3) = \text{(2 major thirds) and (3 minor thirds)} = 4(2) + 3(3) = 17$$

Therefore, the size of our interval from the origin is 17 = 5 mod 12, as shown on the right grid.

Since both groups ($\mathbb{Z}_3 \text{x} \mathbb{Z}_4$ and $\mathbb{Z}_{12}$) are cyclic, the numbers repeat as we extend up and across, giving a grid as in figure 6.3. As described previously, this grid is nothing more than the flat or 2-dimensional representation of the torus that these points lie on. Notice that this diagram looks very similar to figure 2.3 (the *Tonnetz).* In fact, it only differs in historical and minor geometric aspects. Figure 2.3 was constructed for the purpose of analyzing music and ends up forming equilateral triangles between notes to form a lattice-like diagram. Figure 6.3 on the other hand was constructed as a result of the isomorphism between $\mathbb{Z}_3 \text{x} \mathbb{Z}_4$ and $\mathbb{Z}_{12}$. As a consequence, it ends up forming right-angled triangles between vertices. While different, they both embody the same musical tools which will be easier to explain after a few more definitions ([6], [26], and [30]). Recall that the three notes in a triad are called the root, the third and the fifth, where the fifth is 7 semitones (or a P5) away from the root.

Definition 6.2.6 The **circle of fifths** is a sequence of intervals, where all notes are 7 semitones or a P5 apart.

Definition 6.2.7 An **augmented triad** is a triad consisting of two major thirds. In other words, when the fifth of a major triad is raised by a semitone, it is augmented.

Definition 6.2.8 A **diminished triad** is a triad consisting of two minor thirds. In other words, when the fifth of a minor triad is lowered by a semitone, it is diminished.

Definition 6.2.9 A **diatonic scale** is a sequence of whole and half steps as follows

{whole, whole, half, whole, whole, whole, half}

The most common example of this is the diatonic major scale, for instance the C-major scale {0,2,4,5,7,9,11}, which essentially is made of all the white keys on the piano within one octave.

Definition 6.2.10 A topological space X is said to be **compact** if every open covering U of X contains a finite subcollection that also covers X.

Definition 6.2.11 Let X be a topological space. A **separation** of X is a pair U, V of disjoint nonempty open subsets of X whose union is X. The space X is said to be **connected** if there does not exist a separation of X.

Definition 6.2.12 A set S is **convex** in $\mathbb{R}^n$ if for any two points a, b, the straight line segment joining a and b is contained in A.

A note about the way in which we are describing the grid in figure 6.3 deserves some attention. Notice that the grid was generated by $\mathbb{Z}_3 x \mathbb{Z}_4$ as a group isomorphic to $\mathbb{Z}_{12}$, however we can view it topologically (as mentioned in the start of this section). The grid in figure 6.3 is a subset of the flat torus, which we recall, is a subset of the plane ($\mathbb{R}^2$). The flat torus is just a 2-dimensional representation of the torus, which is defined as an embedding in $\mathbb{R}^4$. This clarification enables us to apply the above definitions to our discussion without confusion. Now that we have these terms established, we can reveal the remaining musical representations found on the flat torus - each of which are represented in figure 6.3.

The most basic of observations is the intervals between all the vertices and the musical structures they form. We see that all the maximally compact, connected structures are indeed four basic chords (major, minor, augmented, and diminished 7th chords). The major and minor triads are formed by constructing triangles between the points. The major triad triangles are formed by edges that connect any vertex to the vertex to its right, then to the vertex above that and then back to the original point. The minor triad triangles are just the opposite, where the edge extends from any vertex to the first vertex to its left and then one vertex below that. Recall the ordering that we suggest for the ease of the P, L, and R operations (in table 2.2). Notice that this is the ordering of the triads formed from the above instructions. These triads are illustrated in the top right-hand corner of figure 6.3.

The augmented triads and diminished 7^{th} chords are slightly less involved and are a direct consequence of how the grid is constructed. We know, from our definition that an augmented triad is made of two major thirds and a diminished triad is made of two minor thirds (a diminished 7^{th} chord is then just 3 minor thirds). The columns are constructed using intervals of 3 and therefore the diminished 7^{th} chords are formed simply by grouping any four adjacent notes on any column. The rows are constructed using intervals of 4, and so the augmented triads are then groups of any three adjacent notes on any row. These are displayed at the top of figure 6.3.

By further investigating adjacent tones, we see that the sequence of semitones runs diagonally from top left to bottom right, which is labeled on the middle right side of the grid. In contrast to that, we see the sequence of fifths emerge on the diagonals running from bottom left to top right (shown on the middle left side of the grid). It is to be noted as well, that by adjoining adjacent triads along the fifths intervals a portion of the diatonic major scale is formed (as shown in the middle of figure 6.3). The entire diatonic major scale is made of the following triads (where upper case is major and lower case is minor).

{CEG, dfa, egb, FAC, GBD, ace, bdf}

Mathematically these triads in M are

$$\{\langle 0,4,7\rangle, \langle 9,5,2\rangle, \langle 11,7,4\rangle, \langle 5,9,0\rangle, \langle 7,11,2\rangle, \langle 4,0,9\rangle, \langle 5,2,11\rangle\}$$

As illustrated in figure 6.3, we see six out of the seven triads in the diatonic major scale, which form a convex, compact space. It spans a maximum amount of space along both axes before any one pitch repeats, but does not contain the last triad in the scale. We

notice that <5,2,11> is a diminished triad, which is not an element of our set M and therefore will not appear in this space.

The last musical depiction on the flat torus becomes apparent after connecting the vertices to form the major and minor triads. We have the P, L, and R transformations, as previously represented on the Oettingen/Reimann *Tonnetz*, on the right side of the lower half of figure 6.3. We also have the T and I functions on the lower half of the figure. Notice that the triads under transposition are right-angled triangles facing the same direction (major triads with the hypotenuse facing up toward the left, and minor with the hypotenuse facing down and to the right). The triads under inversion have been marked with symbols inside them to match the triad and its inverted form.

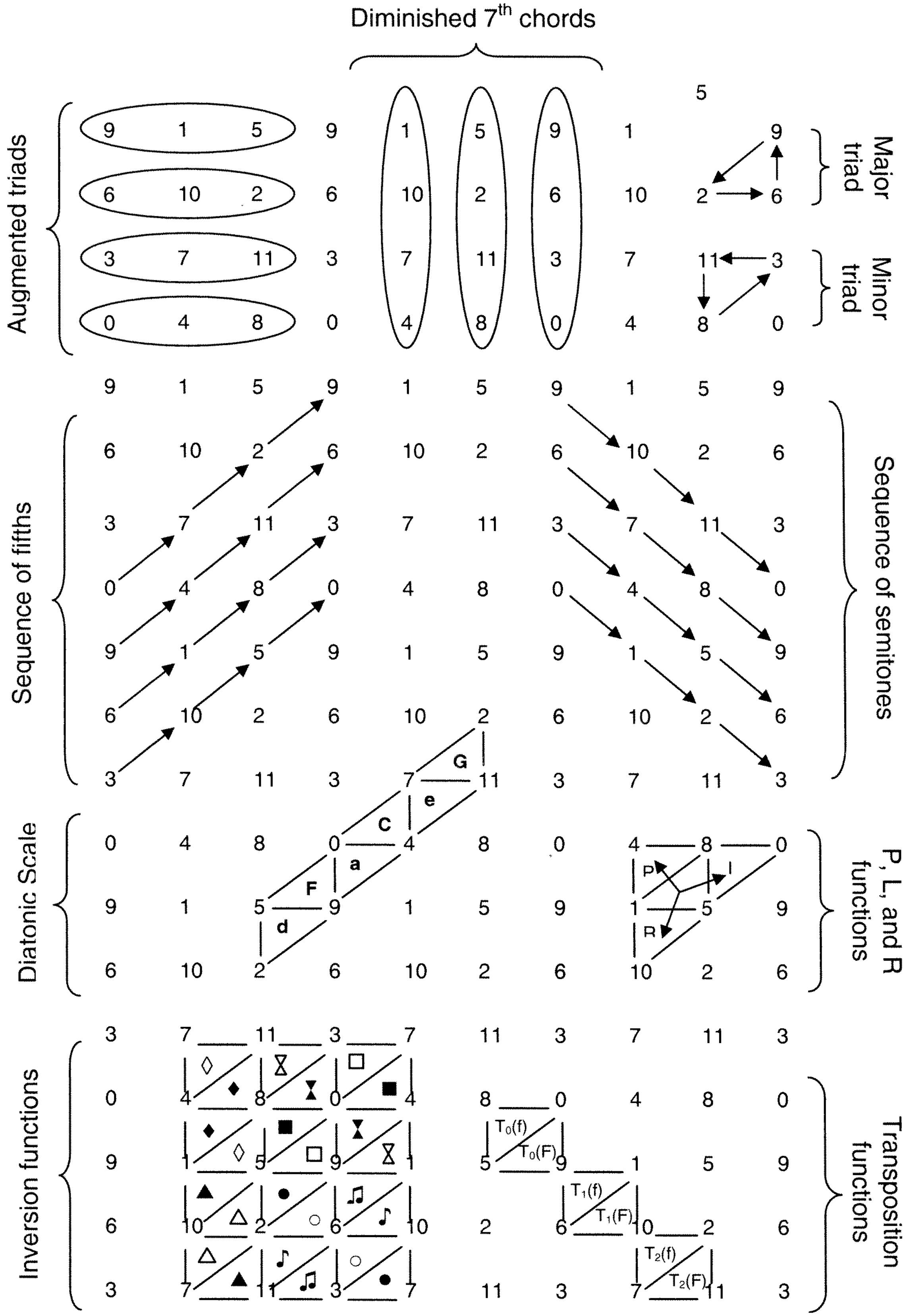


Figure 6.3 Musical structure on the flat torus

CONCLUSION

The evolution of the connection between math and music has proved to be a symbiotic relationship. Most evidently, it presents itself as an aid to musicologists, composers, performers and listeners in their pursuit to attain a comprehensive understanding of music. Although, we see in some instances that mathematical tools designed for music are beneficial to other disciplines such as geography, computer science, and physics [25] and [10]. So, the research done in mathematical music theory has both resulted in new mathematical structures, and as in this study, revealed the presence of already existing structures such as transformational operations, groups, isomorphisms, geometric and topological objects, and torsors.

All of these structures materialize after establishing the fundamental notion of relationships or intervals between pitches. This idea of movement between musical objects eventually translates into mathematical transformations between numbers, which extends into sets of functions and then extends further into groups of functions under composition. We end up with the T/I-group consisting of transpositions and inversions, and the PLR-group consisting of parallel, leading tone exchange and relative functions. Both groups have group theoretic representations which are best expressed through their similarity to the dihedral group of order 24. They both form isomorphisms with D_{12} and are thereby isomorphic to each other. An additional isomorphism exists between the T/I-group and PLR-group, which arises out of construction - since both groups map one triad to another there is manifestly a matching of elements between T/I and PLR.

Aside from these relationships, we define actions of the T/I and PLR groups on M (the set of 24 major and minor triads), which facilitate the investigation of the T/I and PLR orbits and stabilizers. With the use of the orbit-stabilizer theorem, we discover that the actions on M are simply transitive. Furthermore, the commutativity of these two groups reveals that each group is the centralizer of the other within the larger group Sym(M). The property of being one another's centralizers, in conjunction with their actions on M being simply transitive, leads to the notion of duality. Duality in this thesis is specifically defined, and as mentioned previously, is not to be confused with other instances of duality. This notion is best represented with the help of category theory, where we can show duality via commutative diagrams.

Finally we have some alternative representations of music theory, one of which was the set of 24 triads as a torsor (instead of just a set of unordered 3-tuples). We are able to give structure to the set of triads in order to exhibit the way in which it fulfills the conditions of being a T/I-torsor and PLR-torsor. The second observation results in a topological representation of the 12-tone system and the operations performed on it. This topological space is generated by way of the isomorphism between $\mathbb{Z}_3 \times \mathbb{Z}_4$ and $\mathbb{Z}_{12}$, which results in an infinite grid of the twelve pitches. This repeated grid lies on the torus but is presented on the Cartesian plane, because we are able to employ the interchangeable topology of the torus and flat torus. This allows us to graphically demonstrate the various musical structures that are used in both neo-Riemannian and tonal music theory. The flat torus then provides a topological equivalent to the tone network or *Tonnetz*, and is therefore used to model music analysis.

In closing, we see various mathematical structures emerge from the very fabric of music theory. It has been made evident by means of multiple lemmas and theorems that the claims presented in the literature on transformational theory are mathematically sound. These mathematical models not only enhance music theory by serving as tools in the task of analysis, but also augment mathematics by serving as concrete examples that have the potential to broaden the understanding of abstract concepts and lead to the formulation of new mathematical structures.

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