

8-13-2019

DENSITY AND CHROMATIC INDEX, AND MINIMUM RANKS OF SIGN PATTERN MATRICES

Guangming Jing

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DENSITY AND CHROMATIC INDEX, AND MINIMUM RANKS OF SIGN PATTERN
MATRICES

by

GUANGMING JING

Under the Direction of Zhongshan Li, PhD, and Guantao Chen, PhD

ABSTRACT

Given a (multi)graph, the density is defined by

$$\Gamma(G) = \max \left\{ \frac{2|E(U)|}{|U| - 1} : U \subseteq V, |U| \geq 3 \text{ and odd} \right\}.$$

The **chromatic index** $\chi'(G)$ of a graph G is the minimum number of colors that required to color the edges of G such that two adjacent edges receive different colors. It is known

that $\chi'(G) \geq \Gamma(G)$. The **cover index** $\xi(G)$ of G is the greatest integer k for which there is a coloring of E with k colors such that each vertex of G is incident with at least one edge of each color. A sign pattern is a matrix whose entries are from the set $\{+, -, 0\}$.

In part 1, we will generally discuss the connections between the density and the chromatic index. In particular, the Goldberg-Seymour conjecture states that $\chi'(G) = \lceil \Gamma(G) \rceil$ if $\chi'(G) > \Delta + 1$, where Δ is the maximum degree of G . Some open problems are mentioned at the end of part 1. In particular, a dual conjecture to the Goldberg-Seymour conjecture on the cover index is discussed. A proof of the Goldberg-Seymour conjecture is given In part 2.

In part 3, we will present a connection between the minimum ranks of sign pattern matrices and point-line configurations.

INDEX WORDS: Density, Chromatic index, Goldberg-Seymour conjecture, Sign pattern, Minimum rank, Point-hyperplane configuration.

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GUANGMING JING

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy
in the College of Arts and Sciences
Georgia State University

2019

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Guangming Jing
2019

DENSITY AND CHROMATIC INDEX, AND MINIMUM RANKS OF SIGN PATTERN
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by

GUANGMING JING

Committee Chair: Zhongshan Li

Committee: Marina Arav

Guantao Chen

Frank Hall

Hendricus van der Holst

Michael Stewart

Electronic Version Approved:

Office of Graduate Studies

College of Arts and Sciences

Georgia State University

April 2019

DEDICATION

This dissertation is dedicated to my parents and my wife.

ACKNOWLEDGEMENTS

The completion of this research could not have been possible without the assistance of so many people whose names may not all be enumerated. Nonetheless, I would like to express my special appreciation to the following:

Prof. Zhongshan Li and Prof. Guantao Chen, my mentors, for the support, guidance, valuable comments that benefited me much not only in the completion of this study but also in my life.

Prof. Frank J. Hall, Prof. Hendricus van der Holst, Prof. Marina Arav, and Prof. Michael Stewart, for serving on my dissertation committee and for their time and effort in checking this manuscript.

My parents, Mr. Xisheng Jing and Mrs. Xiuli Tan, for their love, caring, and support in every aspect throughout my graduate program.

Dr. Wei Gao, Dr. Jie Han, Dr. Songling Shan, Dr. Shuenn Siang Ng, Yan Cao, and Shushan He, for their support and caring as friends.

Dr. Wenyi Wang, my beloved wife, for her love, caring, understanding, and help in life. I would not have achieved this much without her.

At last, I would like to thank everyone who helped me in Atlanta.

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PART 1

DENSITY AND CHROMATIC INDEX

1.1 Introduction

Graphs in this Chapter are finite, undirected, and without loops. We denote the vertex set and edge set of G by V and E , respectively. Graphs can be used to model many types of relations and processes in physical, biological, social and information systems. **Graph edge coloring** is a well established subject in the field of graph theory, it is one of the basic combinatorial optimization problems: color the edges of a graph G with as few colors as possible such that no two adjacent edges receive the same color. The minimum number of colors needed for such a coloring of G is called the **chromatic index** of G , written $\chi'(G)$. The **Density** of a graph G is defined by

$$\Gamma(G) = \max \left\{ \frac{2|E(U)|}{|U| - 1} : U \subseteq V, |U| \geq 3 \text{ and odd} \right\}.$$

Let $\Delta := \Delta(G)$ be the maximum degree of G . The multiplicity of two distinct vertices $x, y \in V(G)$ is $\mu_G(x, y) = |E_G(x, y)|$. Let $\mu := \mu(G)$ be the maximum multiplicity of G . It is clear that $\Delta(G) \leq \chi'(G)$, as we require that no two adjacent edges can receive the same color. Another lower bound for $\chi'(G)$ is $\Gamma(G)$. Since the edges of G with the same color form a matching, we have $|E(H)| \leq \chi'(G) \lfloor |V(H)|/2 \rfloor$ for any $H \subseteq G$. Thus $\Gamma(G) \leq \chi'(G)$.

How about the upper bound? Shannon [37] in 1949 showed that $\chi'(G) \leq \frac{3}{2}\Delta$. Later in 1965 Vizing [41] showed that $\chi'(G) \leq \Delta + \mu$. G is called simple if $\mu = 1$. So $\chi'(G)$ is either Δ or $\Delta + 1$ for a simple graph G . By a result of Holyer [26], the determination of the chromatic index is NP-Complete. Therefore it is unlikely to find a fast algorithm to determine the chromatic index of an arbitrary graph. However, in the 1970s, Goldberg [18] and Seymour [36] independently made the following conjecture, which is known as the Goldberg-

Seymour conjecture.

Conjecture 1.1.1. *Let G be a graph. If $\chi'(G) > \Delta + 1$, then $\chi'(G) = \lceil \Gamma(G) \rceil$.*

The Goldberg-Seymour conjecture is one of the most important conjectures in edge chromatic graph theory. If it is true, it would provide the best possible approximation to the chromatic index within one color of the optimum in polynomial time. In fact, this conjecture implies that there exists a polynomial-time algorithm to determine $\chi'(G)$ unless $\chi'(G)$ lies in the set $\{\Delta, \Delta + 1\}$ by computing the **fractional chromatic index**. A *fractional edge coloring* of G is a non-negative weighting $w(\cdot)$ of the set $\mathcal{M}(G)$ of matchings in G such that, for every edge $e \in E(G)$, $\sum_{M \in \mathcal{M}: e \in M} w(M) = 1$. Clearly, such a weighting $w(\cdot)$ exists. The *fractional chromatic index* $\chi'_f := \chi'_f(G)$ is the minimum total weight $\sum_{M \in \mathcal{M}} w(M)$ over all fractional edge colorings of G . By definitions, we have $\chi' \geq \chi'_f \geq \Delta$. It follows from Edmonds' characterization of the matching polytope [11] that χ'_f can be computed in polynomial time and

$$\chi'_f = \max \left\{ \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} : H \subseteq G \text{ with } |V(H)| \geq 3 \right\} \text{ if } \chi'_f > \Delta.$$

In this sense, determining $\chi'(G)$ is considered one of the “easiest” NP-Complete problems, since a lot of other NP-complete problems such as the traveling salesman problem do not have good approximations. Over the past four decades, the study of this conjecture has stimulated a significant amount of work; see McDonald [30] for a survey on this conjecture and Stiebitz et al. [13] for a comprehensive account of edge-colorings. In particular, the book [13] written by Stiebitz et al. used the title “Graph Edge Colouring: Vizing’s Theorem and Goldberg’s Conjecture”, indicating the great significance of this conjecture. Besides, several weaker conjectures implied by the Goldberg-Seymour conjecture, such as the Seymour’s r -graph conjecture, have been proposed. In part 2, we will present a proof of the Goldberg-Seymour conjecture. The proof relies heavily on the Tashkinov tree method which is proposed by Tashkinov in [40].

1.2 A dual to the Goldberg-Seymour Conjecture

Let $\delta(G)$ be the minimum degree of G and let $\Phi(G)$ be the **co-density** of G , defined by

$$\Phi(G) = \min \left\{ \frac{2|E^+(U)|}{|U|+1} : U \subseteq V, |U| \geq 3 \text{ and odd} \right\},$$

where $E^+(U)$ is the set of all edges of G with at least one end in U . Clearly, $\xi(G) \leq \min\{\delta(G), \Phi(G)\}$. In 1978 Gupta proposed the following co-density conjecture: Every multigraph G satisfies $\xi(G) \geq \min\{\delta(G) - 1, \lfloor \Phi(G) \rfloor\}$, which is the dual version of the Goldberg-Seymour conjecture on edge-colorings of multigraphs. In 1978 Gupta [23] proposed the following co-density conjecture, which is the counterpart of the Goldberg-Seymour conjecture.

Conjecture 1.2.1. *Let G be a multigraph. If $\Phi(G) < \delta(G)$, then $\xi(G) = \lfloor \Phi(G) \rfloor$.*

The reader is referred to Stiebitz *et al.* [13] for more information about this conjecture. To our knowledge, the bound $\xi(G) \geq \min\{\lfloor \frac{7\delta(G)+1}{8} \rfloor, \lfloor \Phi(G) \rfloor\}$ established by Gupta [23] in 1978 remains to be the best approximate version of Conjecture 1.2.1.

As is well known, the inequality $\chi'(G) \leq \Delta(G) + \mu(G)$ holds for any multigraph G , where $\mu(G)$ is the maximum multiplicity of an edge in G . This result has been successfully dualized by Gupta [22] to packing edge covers: $\xi(G) \geq \delta(G) - \mu(G)$. Observe that this dual version actually follows from Conjecture 1.2.1 as a corollary, because $\Phi(G) \geq \delta(G) - \mu(G)$. To see this, let U be a subset of V with $|U| \geq 3$ and odd, let $F(U)$ be the set of all edges of G with precisely one end in U , and let $G[U]$ be the subgraph of G induced by U . Since each vertex in U is adjacent to at most $(|U| - 1)\mu(G)$ edges in $G[U]$ and at most $|F(U)|$ edges outside $G[U]$, we have $\delta(G) \leq (|U| - 1)\mu(G) + |F(U)|$, which implies that $\delta(G)|U| + |F(U)| \geq (\delta(G) - \mu(G))(|U| + 1)$. As $2|E^+(U)| = 2|E(U)| + 2|F(U)| \geq \delta(G)|U| + |F(U)|$, we obtain $2|E^+(U)| \geq (\delta(G) - \mu(G))(|U| + 1)$ and hence $\Phi(G) \geq \delta(G) - \mu(G)$, as desired.

Gupta [22] discovered that the lower bound $\delta(G) - \mu(G)$ for $\xi(G)$ is sharp when $\mu(G) \geq 1$ and $\delta(G) = 2p\mu(G) - q$, where p and q are two integers satisfying $q \geq 0$ and $p > \mu(G) +$

$\lfloor (q-1)/2 \rfloor$. This led Gupta [22] to suggest the following conjecture, which aims to give a complete characterization of all values of $\delta(G)$ and $\mu(G)$ for which no multigraph G with $\xi(G) = \delta(G) - \mu(G)$ exists.

Conjecture 1.2.2. *Let G be a multigraph such that $\delta(G)$ cannot be expressed in the form $2p\mu(G) - q$, where p and q are two integers satisfying $q \geq 0$ and $p > \mu(G) + \lfloor (q-1)/2 \rfloor$. Then $\xi(G) \geq \delta(G) - \mu(G) + 1$.*

As edge covers are more difficult to manipulate than matchings, it is no surprise that a direct proof of conjecture 1.2.1 would be more complicated and sophisticated than that of Conjecture 1.1.1. One purpose of this note is to establish a slightly weaker version of conjecture 1.2.1 by using Conjecture 1.1.1.

Theorem 1.2.3. *(Assuming Conjecture 1.1.1) Let G be a multigraph and $\Phi(G) < \delta(G)$. If $\Phi(G)$ is not an integer, then $\xi(G) = \lfloor \Phi(G) \rfloor$. If $\Phi(G)$ is an integer, then $\Phi(G) \geq \xi(G) \geq \Phi(G) - 1$. In particular, if $\Phi(G)$ is an integer and $\Phi(G)$ is archived by an unique odd subset U , then $\xi(G) = \lfloor \Phi(G) \rfloor$.*

We shall also demonstrate that Conjecture 1.2.2 is contained in Conjecture 1.2.1 as a special case.

Theorem 1.2.4. *Conjecture 1.2.1 implies Conjecture 1.2.2.*

Throughout we shall repeatedly use the following terminology and notations. Let $G = (V, E)$ be a multigraph. A subset U of V is called an **odd set** if $|U|$ is odd and $|U| \geq 3$. For each $v \in V$, let $d_G(v)$ be the degree of v in G . For each $U \subseteq V$, let $E_G(U)$ be the set of all edges of G with both ends in U , let $E_G^+(U)$ be the set of all edges of G with at least one end in U , and let $F_G(U)$ be the set of all edges of G with exactly one end in U . For any two subsets X and Y of V , let $E_G(X, Y)$ be the set of all edges of G with one end in X and the other end in Y . We write $E_G(x, y)$ for $E_G(X, Y)$ if $X = \{x\}$ and $Y = \{y\}$. We shall drop the subscript G if there is no danger of confusion.

The proofs of the above two theorems will take up the entire remainder of this note.

1.2.1 Weaker Version

We present a proof of Theorem 1.2.3 in this section. Let $G = (V, E)$ be a multigraph and let $Z \subseteq V$. A set $C \subseteq E$ is called a Z -cover if every vertex of Z is incident with at least one edge of C . Note that if $Z = V$, then Z -covers are precisely edge covers of G . To prove the theorem, we shall actually establish the following variant.

Theorem 1.2.5. *Let $G = (V, E)$ be a multigraph, let $Z \subseteq V$, and let k be a positive integer.*

* *If $d(z) \geq k + 1$ for all $z \in Z$, and $|E^+(U)| \geq \frac{|U|+1}{2}k$ for all odd sets $U \subseteq Z$, then G contains $k - 1$ disjoint Z -covers. In particular, if $|E^+(U)| = \frac{|U|+1}{2}k$ is achieved by an unique odd subset U , then G contains k disjoint Z -covers.*

** *If $d(z) \geq k + 1$ for all $z \in Z$, and $|E^+(U)| > \frac{|U|+1}{2}k$ for all odd sets $U \subseteq Z$, then G contains k disjoint Z -covers.*

Note that by **, we have the first “if” part of Theorem 1.2.3, and * gives the second “if” part and “In particular” part.

Proof. Let’s first prove *. Clearly, we may assume that all vertices outside Z have degree one. Suppose for a contradiction that Theorem 1.2.5 is false. We reserve the triple (G, Z, k) for a counterexample with the minimum $\sum_{z \in Z} d(z)$, and break the proof into some simple observations. By the hypothesis of this theorem, $d(z) \geq k + 1$ for all $z \in Z$. For convenience, we call an odd set $U \subseteq Z$ **optimal** if $|E^+(U)| = \frac{|U|+1}{2}k$.

Claim 1. $d(z) = k + 1$ for all $z \in Z$.

Otherwise, $d(z) \geq k + 2$ for some $z \in Z$. If z is contained in no optimal odd set $U \subseteq Z$, letting H be obtained from G by splitting an one edge $e \in E(y, z)$ from z , then (H, Z, k) would be a smaller counterexample than (G, Z, k) , a contradiction. Hence

(1) there exists an optimal odd set $U_1 \subseteq Z$ containing z ; subject to this, $|U_1|$ is minimum.

Since $(|U_1| + 1)k = 2|E^+(U_1)| = 2|E(U_1)| + 2|F(U_1)| \geq (k + 1)|U_1| + |F(U_1)|$, we have $|F(U_1)| \leq k - |U_1| < d(z)$. So z is adjacent to some vertex $y \in U_1$. Let H be arising from G by splitting off one edge $e \in E(y, z)$ from z . We propose to show that

(2) (H, Z, k) is a smaller counterexample than (G, Z, k) .

Assume the contrary. Then $|E_H^+(U_2)| < \frac{|U_2|+1}{2}k$ for some odd set $U_2 \subseteq Z$ by the hypothesis of this theorem. Thus

$$(3) \ z \in U_2, \ y \notin U_2, \text{ and } |E^+(U_2)| = \frac{|U_2|+1}{2}k.$$

Let $T_1 = U_1 \setminus U_2$ and $T_2 = U_2 \setminus U_1$. By (3), we have $y \in U_1 \setminus U_2$, so $T_1 \neq \emptyset$. By the minimality assumption on $|U_1|$ (see (1)), U_2 is not a proper subset of U_1 , which implies $T_2 \neq \emptyset$. Since $z \in U_1 \cap U_2$, we obtain $|U_1 \cap U_2| \geq 1$. Let us consider two cases, according to the parity of $|U_1 \cap U_2|$.

Case 1. $|U_1 \cap U_2|$ is odd.

It is a routine matter to check that

$$(4) \ |E^+(U_1 \cup U_2)| + |E^+(U_1 \cap U_2)| = |E^+(U_1)| + |E^+(U_2)| - |E(T_1, T_2)|.$$

In this case, $U_1 \cup U_2$ is an odd set. So $|E^+(U_1 \cup U_2)| \geq \frac{|U_1 \cup U_2|+1}{2}k$ by the hypothesis of this theorem.

$$(5) \ |E^+(U_1 \cap U_2)| \geq \frac{|U_1 \cap U_2|+1}{2}k + 1.$$

To justify this, note that if $|U_1 \cap U_2| = 1$, then $|E^+(U_1 \cap U_2)| = d(z) \geq k + 2$. So (5) holds. If $|U_1 \cap U_2| \geq 3$, then $U_1 \cap U_2$ is not an optimal odd set by the minimality assumption on $|U_1|$ (see (1)). Thus we also get (5).

From (4) and (5) we deduce that $\frac{|U_1 \cup U_2|+1}{2}k \leq |E^+(U_1 \cup U_2)| \leq |E^+(U_1)| + |E^+(U_2)| - |E^+(U_1 \cap U_2)| \leq \frac{|U_1|+1}{2}k + \frac{|U_2|+1}{2}k - \frac{|U_1 \cap U_2|+1}{2}k - 1 = \frac{|U_1 \cup U_2|+1}{2}k - 1$, a contradiction.

Case 2. $|U_1 \cap U_2|$ is even.

It is easy to see that $|E^+(U_1)| + |E^+(U_2)| = |E^+(T_1)| + |E^+(T_2)| + 2|E(U_1 \cap U_2)| + |E(U_1 \cap U_2, T_1 \cup T_2)| + 2|E(U_1 \cap U_2, \overline{U_1 \cup U_2})|$, where $\overline{U_1 \cup U_2} = V - (U_1 \cup U_2)$. Thus

$$(6) \ |E^+(U_1)| + |E^+(U_2)| \geq |E^+(T_1)| + |E^+(T_2)| + 2|E(U_1 \cap U_2)| + |F(U_1 \cap U_2)|.$$

In this case, $|T_i|$ is odd, so $|E^+(T_i)| \geq \frac{|T_i|+1}{2}k$ for $i = 1, 2$ by the hypothesis of this theorem. It follows from (3) and (6) that $\frac{|U_1|+1}{2}k + \frac{|U_2|+1}{2}k \geq \frac{|T_1|+1}{2}k + \frac{|T_2|+1}{2}k + 2|E(U_1 \cap U_2)| + |F(U_1 \cap U_2)| \geq \frac{|T_1|+1}{2}k + \frac{|T_2|+1}{2}k + |U_1 \cap U_2|(k + 1) = \frac{|U_1|+1}{2}k + \frac{|U_2|+1}{2}k + |U_1 \cap U_2|$, a contradiction.

Combining the above two cases, we obtain (2). This contradiction justifies Claim 1.

By Claim 1, $d(z) = k + 1$ for all $z \in Z$. Thus we have $|U|(k + 1) = 2|E(U)| + |F(U)| = |E(U)| + |E^+(U)| \geq |E(U)| + \frac{|U|+1}{2}k$. Thus $|E(U)| \leq \frac{|U|-1}{2}(k + 2) + 1 \leq \frac{|U|-1}{2}(k + 3)$ for each odd set $U \subseteq Z$. By Conjecture 1.1.1, the chromatic index of $G[S]$ is at most $k + 3$. Since all vertices outside Z have degree one, we further obtain $\chi'(G) \leq k + 3$. So E can be partitioned into $k + 3$ matchings M_1, M_2, \dots, M_{k+3} . Note that

(7) each vertex $z \in Z$ is disjoint from precisely two of M_1, M_2, \dots, M_{k+3} (as $d(z) = k+1$).

Let H be the subgraph of G induced by edges in $M_k \uplus M_{k+1} \uplus M_{k+2} \uplus M_{k+3}$, where \uplus is the multiset sum, and let N be an orientation of H such that $|d_N^+(v) - d_N^-(v)| \leq 1$ for each vertex v . (It is well known that such an orientation exists.) From (7) and this orientation we see that

(8) if a vertex $z \in Z$ is disjoint from precisely one of M_1, M_2, \dots, M_{k-1} , then $d_H(z) = 3$ and $d_N^-(z) \geq 1$; if z is disjoint from precisely two of M_1, M_2, \dots, M_{k-1} , then $d_H(z) = 4$ and $d_N^-(z) = 2$.

For each $i = 1, 2, \dots, k - 1$, let C_i be obtained from M_i as follows: for each $z \in Z$, if z not covered by M_i , add an edge from N that is directed to z and has not yet been used in $C_1 \uplus C_2 \uplus \dots \uplus C_{i-1}$, where $C_0 = \emptyset$. From this construction and (8) we deduce that C_1, C_2, \dots, C_{k-1} are disjoint and each of them is a Z -cover in G .

Now we assume the optimum odd set U_1 of Z is unique. Let H be obtained from G by splitting off an edge $e \in E(y, z)$ from z , where $y, z \in U_1$. Thus we have $\frac{|U|-1}{2}(k + 2)$ for all odd sets $U \subseteq V(H)$, and Conjecture 1.1.1 yields $\chi'(H) \leq k + 2$. So the edge set of H can be partitioned into $k + 2$ matchings M_1, M_2, \dots, M_{k+2} . Since z has degree k , precisely two of M_1, M_2, \dots, M_{k+2} are disjoint from z , say M_{k+1} and M_{k+2} (rename subscripts if necessary). Let N be the subgraph of G induced by edges in $M_{k+1} \uplus M_{k+2}$, where \uplus is the multiset sum. Note that each component of N is either a path or an even cycle. We direct edges of N such that each component is either a directed path or a directed cycle. For each $i = 1, 2, \dots, k$, let C_i be obtained from M_i as follows: for each $z \in Z$ not covered by M_i , add the edge from N that is directed to z . Then C_1, C_2, \dots, C_k are disjoint and each of them is a Z -cover in G .

The proof of ** is almost incidental as *, for which we consider the sets U such that

$|E^+(U)| - 1 = \frac{|U|+1}{2}k$ optimal. Here we omit the details

□

1.2.2 Implication

The purpose of this section is to show that Conjecture 1.2.2 can be deduced from Conjecture 1.2.1.

Proof of Theorem 1.2.4. We may assume that

(1) G is connected, otherwise consider its components separately.

By hypothesis, $\delta(G)$ cannot be expressed in the form $2p\mu(G) - q$, where p and q are two integers satisfying $q \geq 0$ and $p > \mu(G) + \lfloor (q-1)/2 \rfloor$. Since $0 \leq q \leq 2p - 2\mu(G)$, setting $q = 0, 1, \dots, 2p - 2\mu(G)$ respectively, we see that $\delta(G)$ does not belong to the set

$$\Omega_p = \{2(p+1)\mu(G) - 2p, 2(p+1)\mu(G) - 2p + 1, \dots, 2p\mu(G)\},$$

where $p \geq \mu(G)$. Note that $2\mu(G)^2$ is the only member of $\Omega_{\mu(G)}$ and that the gap between Ω_p and Ω_{p+1} consists of all integers i with $2p\mu(G) + 1 \leq i \leq 2(p+2)\mu(G) - (2p+3)$. So

(2) either $\delta(G) \leq 2\mu(G)^2 - 1$ or $2p\mu(G) + 1 \leq \delta(G) \leq 2(p+2)\mu(G) - (2p+3)$ for some $p \geq \mu(G)$.

To prove the theorem, it suffices to show that for any odd set U of G , we have $\frac{2|E^+(U)|}{|U|+1} \geq \delta(G) - \mu(G) + 1$, or equivalently,

$$(3) \quad 2|E(U)| + |F(U)| \geq (|U| + 1)(\delta(G) - \mu(G) + 1).$$

Set $k = \mu(G)$ if $\delta(G) \leq 2\mu(G)^2 - 1$ and set $k = p + 1$ if $2p\mu(G) + 1 \leq \delta(G) \leq 2(p+2)\mu(G) - (2p+3)$ for some $p \geq \mu(G)$. We consider two cases according to the size of U .

Case 1. $|U| \geq 2k + 1$.

We divide the present case into two subcases.

Subcase 1.1. $U \subsetneq V$ or $U = V$ and $\delta(G)$ is odd. In this subcase,

$$(4) \quad 2|E(U)| + |F(U)| \geq |U|\delta(G) + 1.$$

Indeed, if $U \subsetneq V$, then $|F(U)| \geq 1$ by (1). If $U = V$ and $\delta(G)$ is odd, then G contains at least one vertex of degree at least $\delta(G) + 1$, because $|V|$ is odd and the total number of vertices with odd degree is even. Hence (4) is true.

$$(5) \quad |U|\delta(G) + 1 \geq (|U| + 1)(\delta(G) - \mu(G) + 1).$$

Note that (5) amounts to saying that $\delta(G) \leq (|U|+1)(\mu(G)-1)+1$. If $\delta(G) \leq 2\mu(G)^2 - 1$, then $\delta(G) \leq (2\mu(G) + 2)(\mu(G) - 1) + 1 = (2k + 2)(\mu(G) - 1) + 1 \leq (|U| + 1)(\mu(G) - 1) + 1$. If $\delta(G) \leq 2(p+2)\mu(G) - (2p+3)$, then $\delta(G) \leq 2(k+1)\mu(G) - (2k+1) = (2k+2)(\mu(G) - 1) + 1 \leq (|U| + 1)(\mu(G) - 1) + 1$. So (5) is established.

The desired statement (3) follows instantly from (4) and (5).

Subcase 1.2. $U = V$ and $\delta(G)$ is even. In this subcase, we have $\delta(G) \leq 2\mu(G)^2 - 2$ if $\delta(G) \leq 2\mu(G)^2 - 1$ and $\delta(G) \leq 2(p+2)\mu(G) - (2p+4)$ if $\delta(G) \leq 2(p+2)\mu(G) - (2p+3)$. So $\delta(G) \leq (2k+2)(\mu(G) - 1)$ and hence

$$(6) \quad \delta(G) \leq (|U| + 1)(\mu(G) - 1).$$

From (6) we deduce that $|U|\delta(G) \geq (|U| + 1)(\delta(G) - \mu(G) + 1)$. Therefore (3) holds, because $2|E(U)| + |F(U)| \geq |U|\delta(G)$.

Case 2. $|U| \leq 2k - 1$. (So $k \geq 2$ as $|U| \geq 3$.)

By the Pigeonhole Principle, there is a vertex $v \in U$ which is incident with at most $\frac{|F(U)|}{|U|}$ edges in $F(U)$. Note that v is incident with at most $(|U| - 1)\mu(G)$ edges in $G[U]$, so $d(v) \leq (|U| - 1)\mu(G) + \frac{|F(U)|}{|U|}$. Hence

$$(7) \quad \delta(G) \leq (|U| - 1)\mu(G) + \frac{|F(U)|}{|U|}.$$

We proceed by considering two subcases.

Subcase 2.1. $2p\mu(G) + 1 \leq \delta(G) \leq 2(p+2)\mu(G) - (2p+3)$, where $p \geq \mu(G)$.

From (7) and the hypothesis of the present subcase, we deduce that $2p\mu(G) + 1 \leq (|U| - 1)\mu(G) + \frac{|F(U)|}{|U|}$. Thus $|F(U)| \geq |U|(2p + 1 - |U|)\mu(G) + |U|$. So

$$(8) \quad |U|\delta(G) + |F(U)| \geq |U|\delta(G) + |U|(2p + 1 - |U|)\mu(G) + |U|.$$

Let us show that

$$(9) \quad |U|\delta(G) + |U|(2p + 1 - |U|)\mu(G) + |U| \geq (|U| + 1)(\delta(G) - \mu(G) + 1).$$

To justify this, note that (9) is equivalent to

$$(10) \quad \delta(G) \leq \{|U|(2p+2-|U|)+1\}\mu(G) - 1.$$

By the hypothesis of the present subcase, $\delta(G) \leq 2(p+2)\mu(G) - (2p+3)$. To establish (10), we turn to proving that $2(p+2)\mu(G) - (2p+3) \leq \{|U|(2p+2-|U|)+1\}\mu(G) - 1$, or equivalently

$$(11) \quad \{-|U|^2 + 2(p+1)|U| - (2p+3)\}\mu(G) \geq -(2p+2).$$

Let $f(x) = -x^2 + 2(p+1)x - (2p+3)$. Then $f(x)$ is a concave function on \mathbb{R} . So on any interval $[a, b]$, $f(x)$ achieves the minimum at a or b . By the hypothesis of the present case, $|U| \leq 2k-1 = 2p+1$, so $3 \leq |U| \leq 2p+1$. By direct computation, we obtain $f(3) = 4p-6 \geq -2$ and $f(2p+1) = -2$. Thus $f(|U|) \geq -2$ for $3 \leq |U| \leq 2p+1$, which implies that the LHS of (11) $\geq -2\mu(G) \geq -(2p+2) = \text{RHS of (11)}$, because $p \geq \mu(G)$. This proves (11) and hence (10) and (9).

Since $2|E(U)| + |F(U)| \geq |U|\delta(G) + |F(U)|$, the desired statement (3) follows instantly from (8) and (9).

$$\textbf{Subcase 2.2.} \quad \delta(G) \leq 2\mu(G)^2 - 1.$$

We may assume that

$$(12) \quad \delta(G) \geq (|U|+1)(\mu(G)-1)+1, \text{ for otherwise, } |U|\delta(G) \geq (|U|+1)(\delta(G)-\mu(G)+1).$$

So (3) holds.

By (12) and the hypothesis of the present subcase, either $2t(\mu(G)-1)+1 \leq \delta(G) \leq 2(t+1)(\mu(G)-1)$ for some t with $\frac{|U|+1}{2} \leq t \leq \mu(G)-1$ or $\delta(G) = 2t(\mu(G)-1)+1$ for $t = \mu(G)$.

By (7), we have $2t(\mu(G)-1)+1 \leq (|U|-1)\mu(G) + \frac{|F(U)|}{|U|}$. So $\frac{|F(U)|}{|U|} \geq (2t-|U|+1)\mu(G) - 2t+1$, and hence

$$(13) \quad |U|\delta(G) + |F(U)| \geq |U|\{\delta(G) + (2t-|U|+1)\mu(G) - 2t+1\}.$$

We propose to show that

$$(14) \quad |U|\{\delta(G) + (2t-|U|+1)\mu(G) - 2t+1\} \geq (|U|+1)(\delta(G) - \mu(G) + 1).$$

To justify this, note that (14) is equivalent to that

$$(15) \quad \delta(G) \leq \{|U|(2t+2-|U|)+1\}\mu(G) - |U|2t-1.$$

Suppose $\delta(G) = 2\mu(G)^2 - 1$. Then $t = \mu(G)$. So (15) says that $2\mu(G)^2 - 1 \leq$

$\{|U|(2\mu(G) + 2 - |U|) + 1\}\mu(G) - |U|2\mu(G) - 1$, equivalently $(|U| - 1)(|U| - 2\mu(G) + 1) \leq 0$, which holds trivially because $|U| \leq 2\mu(G) - 1$ by the hypothesis of the present case.

So we assume that $\delta(G) \leq 2(t + 1)(\mu(G) - 1)$ for some t with $\frac{|U|+1}{2} \leq t \leq \mu(G) - 1$. We prove (15) by showing that $2(t + 1)(\mu(G) - 1) \leq \{|U|(2t + 2 - |U|) + 1\}\mu(G) - |U|2t - 1$, or equivalently, $\{|U|(2t + 2 - |U|) - 2t - 1\}\mu(G) - |U|2t \geq -2t - 1$. Let $g(x) = \{x(2t + 2 - x) - 2t - 1\}\mu(G) - 2tx$. Then $g(x)$ is a concave function on \mathbb{R} . So on any interval $[a, b]$, $g(x)$ achieves the minimum at a or b . By direct computation, we obtain $g(3) = 4(t - 1)\mu(G) - 6t$ and $g(2t - 1) = 4(t - 1)\mu(G) - 2t(2t - 1)$. It is easy to see that $\min\{g(3), g(2t - 1)\} \geq -2t - 1$, because $\mu(G) \geq t + 1 \geq 3$. This proves (15) and hence (14) and (13).

Since $2|E(U)| + |F(U)| \geq |U|\delta(G) + |F(U)|$, the desired statement (3) follows instantly from (13) and (14), completing the proof of Theorem 1.2.4. \square

1.3 Open Problems

Beside the Goldberg-Seymour conjecture, it is suspected that there are more connections between $\chi'(G)$ and $\Gamma(G)$. For instance, what if $\lceil \Gamma \rceil < \Delta$? Goldberg [19] has the following conjecture.

Conjecture 1.3.1. *Let G be a graph. If $\lceil \Gamma(G) \rceil < \Delta$, then $\chi'(G) = \Delta$.*

This conjecture implies that the only difficulty of determining the chromatic index lies in determining the chromatic index of a graph G with $\lceil \Gamma(G) \rceil = \Delta$. Moreover, the following Seymour's exact conjecture proposed by Seymour in [36] suggests that it is "easy" to determine the chromatic index of a planar graph by computing the density of G .

Conjecture 1.3.2. *Let G be a planar graph. Then $\chi'(G) = \max\{\Delta, \lceil \Gamma(G) \rceil\}$.*

By observing the examples of graphs such that $\chi'(G) = \Delta + 1$ and $\lceil \Gamma(G) \rceil \leq \Delta$, I'm suspecting the following.

Question 1.3.3. *Let G be a graph with $\chi'(G) = \Delta(G) + 1$. Then either $\lceil \Gamma(G) \rceil = \Delta(G) + 1$, or the underlying simple graph H of G satisfies $\chi'(H) = \Delta(H) + 1$ and $\lceil \Gamma(H) \rceil = \Delta(H)$.*

Note that Question 1.3.3 and the following Vizing's planar graph conjecture implies Seymour's exact conjecture, as there is no simple planar graph H with $\chi'(H) = \Delta(H) + 1$ and $[\Gamma(H)] = \Delta(H)$.

Conjecture 1.3.4. *Let G be a simple planar graph with $\Delta \geq 6$. Then $\chi'(G) = \Delta$.*

Vizing in [41] verified Conjecture 1.3.4 when $\Delta \geq 8$. The case $\Delta = 7$ was independently proved by Grünewald [21], by Sanders and Zhao [34], and by Zhang [42].

Note that all the above conjectures can also be asked naturally for the cover index in a dual fashion, here we omit these problems.

PART 2

A PROOF OF THE GOLDBERG-SEYMOUR CONJECTURE

2.1 Notation and terminology

We will generally follow the book [39] for notation and terminology. Let $G = (V, E)$ be a graph. An edge- k -coloring of a graph G is a map $\varphi: E(G) \rightarrow \{1, 2, \dots, k\}$ that assigns to every edge e of G a color $\varphi(e) \in \{1, 2, \dots, k\}$ such that no two adjacent edges of G receive the same color. Denote by $\mathcal{C}^k(G)$ the set of all edge- k -colorings of G . The chromatic index $\chi'(G)$ is the least integer $k \geq 0$ such that $\mathcal{C}^k(G) \neq \emptyset$. A graph G is called k -critical edge chromatic (simply k -critical) for an integer $k \geq \Delta(G)$ if $\chi'(G) = k + 1$ and $\chi'(H) \leq k$ for any proper subgraph H of G . A vertex-edge alternating sequence $T = (y_0, e_1, y_1, e_2, \dots, y_{p-1}, e_p, y_p)$ of distinct vertices y_i and edges e_i of G is called a tree-sequence if the endvertices of each e_i are y_{i+1} and y_r for some $r \in \{1, 2, \dots, i\}$. Clearly, the edges of a tree-sequence indeed induce a tree, which is also denoted by T , and its vertex set and edge set are denoted by $V(T)$ and $E(T)$, respectively. Denote by \prec_ℓ the linear order naturally generated by the tree-sequence. For every element $x \in T$, let T_x or $T(x)$ be the sequence generated by elements $\prec_\ell x$ and x , and call it an x -segment. For a color α , denote by $T_{v(\alpha)}$ or $T(v(\alpha))$ the segment of T ending at $v(\alpha)$ where $v(\alpha) = v(T, \alpha)$ is defined to be the first vertex missing color α in \prec_ℓ of T if $\alpha \in \overline{\varphi}(T)$, and $T_{v(\alpha)} = T$ where $v(\alpha) = v(T, \alpha)$ is defined to be the last vertex of T if $\alpha \notin \overline{\varphi}(T)$. Since we will basically work on special tree-sequences, our notation and terminology will be based on tree-sequences although they could be defined more generally.

Let T be a tree-sequence. An edge f is *incident* with T if at least one of its endvertices is in $V(T)$, and is a *boundary edge* of T if exact one of its endvertices is in T . Denote by $\partial(T)$ the set of all boundary edges of T and $\partial_{\varphi, \delta}(T) = \{f \in \partial(T), \varphi(f) = \delta\}$. Let f be an edge of G . If both endvertices of f are in $V(T)$, let $a(f)$ and $b(f)$ denote two endvertices of f with $a(f) \prec_\ell b(f)$; If $f \in \partial(T)$, let $a(f)$ be the endvertex of f in $V(T)$ and

$b(f)$ be the other endvertex of f . More generally, for a set F of edges incident with T , let $a(F) = \{a(f) : f \in F\}$ and $b(F) = \{b(f) : f \in F\}$.

For the rest of this part, we will consider an edge- k -critical graph, an edge $e \in E(G)$ and a coloring $\mathcal{C}^k(G - e)$. For short, we call them a k -triple and denote it by (G, e, φ) .

We state the main Theorem as the following.

Theorem 2.1.1. *If G is a k -critical graph with $k \geq \Delta + 1$, then $\chi' = \lceil \frac{|E(G)|}{|V(G)|} \rceil$.*

Let (G, e, φ) be a k -triple and T be a tree-sequence. For any vertex v , let $\varphi(v)$ denote the set of colors assigned to the edges incident to v and $\bar{\varphi}(v)$ denote the set of colors not in $\varphi(v)$. Clearly, $|\varphi(v)| + |\bar{\varphi}(v)| = k$. We call $\varphi(v)$ the set of colors seen by v and $\bar{\varphi}(v)$ the set of colors missing at v . Let $\bar{\varphi}(T) = \cup_{v \in V(T)} \bar{\varphi}(v)$ and $\varphi(T) = \cup_{f \in E(T)} \varphi(f)$. Let H be a subgraph of G . We call H *elementary* if $\bar{\varphi}(u) \cap \bar{\varphi}(v) = \emptyset$ for any two distinct vertices $u, v \in V(H)$. We call H *closed* if no missing color in $\varphi(\partial(H))$, and call H *strongly closed* if additionally all colors on its boundary edges are distinct. Since each color class is a matching in $E(G)$, it is fairly easy to check that to prove Theorem 2.1.1 we only need to show that G is elementary or, more generally, there is a tree-sequence T with $E(T) \subset E(G)$, and is both elementary and strongly closed. Given a color set B , we say T is *B -closed* if $\varphi(\partial(T)) \cap B = \emptyset$ and T is *B^- -closed* if T is $(\bar{\varphi}(T) - B)$ -closed. We also say a color α is *closed in T* and T is *closed for α* if $\alpha \notin \varphi(\partial(T))$.

For a k -triple (G, e, φ) and two colors α, β , Let E_α be the set of α edges and let $E_{\alpha, \beta} = E_\alpha \cup E_\beta$. Clearly, E_α is matching and $G[E_{\alpha, \beta}]$ is a union of disjoint even cycles or paths, which are called (α, β) -chains. We call an (α, β) -chain an (α, β) -path if it is indeed a path. For each vertex $v \in V(G)$, denote by $P_v(\alpha, \beta, \varphi)$ the unique (α, β) -chain containing v . Clearly, for any two vertices u and v , $P_u(\alpha, \beta, \varphi)$ and $P_v(\alpha, \beta, \varphi)$ are either the same or vertex-disjoint. When $P_u(\alpha, \beta, \varphi)$ is a path and u is an endvertex of it, it generates a linear order $\preceq_{P_u(\alpha, \beta, \varphi)}$ such that $v \preceq_{P_u(\alpha, \beta, \varphi)} w$ if and only if v is between u and w in $P_u(\alpha, \beta, \varphi)$. We define $\varphi/P_v(\alpha, \beta, \varphi)$ to be a new coloring obtained by switching colors α and β on the path $P_v(\alpha, \beta, \varphi)$. Clearly $\varphi/P_v(\alpha, \beta, \varphi)$ is still an edge- k -coloring of G . If $V(T)$ is an elementary set of (G, e, φ) and $\alpha, \beta \in \bar{\varphi}(T)$ where $\alpha, \beta \notin \varphi(\partial(T))$, the coloring obtained by switching α with β among all

edges colored α and β in $E(G) - E(T)$ is also an edge- k -coloring of $G - e$. We denote such a coloring by $\varphi/(G - T, \alpha, \beta)$. Let T be a tree sequence and P be a nonempty sub-chain of an (α, β) -chain. If $V(P) \cap V(T) \neq \emptyset$, we say P intersects T .

Definition 1. Let (G, e, φ) be a k -triple and T be a tree-sequence.

- A color δ is called a **defective** color of T if it appears more than once in $\partial(T)$, i.e. $|\partial_\delta(T)| \geq 2$. The corresponding edges with color δ are called *defective edges*.
- Colors α and β are **T -interchangeable** if there are at most one (α, β) -path intersecting T . We also say α is interchangeable with β in T if α and β are T -interchangeable.
- For any color set C , an edge- k -coloring φ^* of $G - e$ is **(T, C, φ) -stable** if the following two properties hold.

$$(1) \quad \varphi^*(f) = \varphi(f) \text{ for every edge } f \text{ incident to } T \text{ with } \varphi(f) \in \overline{\varphi}(T) \cup C.$$

$$(2) \quad \overline{\varphi}^*(v) = \overline{\varphi}(v) \text{ for any } v \in V(T), \text{ which gives } \overline{\varphi}^*(T) = \overline{\varphi}(T).$$

We say a coloring φ^* is $(\emptyset, \emptyset, \varphi)$ -stable if φ^* is an edge- k -coloring φ^* of $G - e$.

- An edge- k -coloring φ^* of $G - e$ is **(T, φ) -wstable** if $\overline{\varphi}^*(v) = \overline{\varphi}(v)$ for every $v \in V(T)$ and $\varphi^*(f) = \varphi(f)$ for every $f \in E(T)$. Note that a (T, C, φ) -stable coloring is also (T, φ) -wstable if $\overline{\varphi}(T) \cup C = \varphi(T)$.
- A subpath P' of an (α, β) -path P is called an **T -exit path** or **exit path for T** if one endvertex v of P' is in T with $V(P') \cap V(T) = \{v\}$, and either α or β is missing at the other endvertex of P' outside T . The vertex v in this case is called a **T -exit of P** or **exit of P for T** . A vertex u is called a **T -exit of (α, β)** or **(α, β) exit for T** if there exist an (α, β) -path P such that u is a T -exit of P .
- Given a color set C , an edge $f \in \partial(T)$ is **$T \vee C$ -nonextendable** if there exists a $(T, \{\varphi(f)\} \cup C, \varphi)$ -stable coloring φ^* and a color $\gamma \in \overline{\varphi}^*(a(f))$ such $a(f)$ is a T -exit with $(\gamma, \varphi(f))$. Otherwise, f is called an extendable edge of T .

Clearly colorings being wstable to each other is an equivalence relation. The next lemma shows that (T, C, \cdot) -stable colorings for a given color set C and a given tree sequence T also form an equivalence relation.

Lemma 2.1.1. *For any given color set C and tree sequence T , (T, C, \cdot) -stable colorings form an equivalence relation.*

Proof. Let C be a color set. Clearly, every coloring φ itself is (T, C, φ) -stable. Suppose φ^* is (T, C, φ) -stable. Then $\overline{\varphi^*}(v) = \overline{\varphi}(v)$ for any $v \in V(T)$. Let f be an edge incident to T such that $\varphi^*(f) \in \overline{\varphi^*}(T) \cup C = \overline{\varphi}(T) \cup C$. We claim $\varphi(f) = \varphi^*(f)$. Suppose the contrary. Then $\varphi(f) \notin \varphi(T) \cup C$ since φ^* is (T, C, φ) -stable. Let $v \in V(T)$ be incident to f . Since $\overline{\varphi^*}(v) = \overline{\varphi}(v)$, there is another incident edge f_1 such that $\varphi(f_1) = \varphi^*(f) \in \overline{\varphi}(T) \cup C$. Since φ^* is (T, C, φ) -stable, we must have $\varphi(f_1) = \varphi^*(f_1)$. Thus $\varphi^*(f_1) \neq \varphi^*(f)$, which gives a contradiction since φ^* is a proper coloring. Therefore φ is (T, C, φ^*) -stable.

Suppose φ^* is (T, C, φ) -stable and φ^{**} is (T, C, φ^*) -stable. Let f be an edge incident to T with $\varphi(f) \in \overline{\varphi}(T) \cup C$. Since φ^* is (T, C, φ) -stable, $\varphi^*(f) = \varphi(f)$ holds. Since $\overline{\varphi^*}(T) \cup C = \overline{\varphi}(T) \cup C$ and φ^{**} is (T, C, φ^*) -stable, we have $\varphi^{**}(f) = \varphi^*(f)$. So, $\varphi^{**}(f) = \varphi(f)$. Moreover, $\overline{\varphi}(v) = \overline{\varphi^*}(v)$ and $\overline{\varphi^*}(v) = \overline{\varphi^{**}}(v)$ imply $\overline{\varphi}(v) = \overline{\varphi^{**}}(v)$. Therefore φ^{**} is (T, C, φ) -stable. \square

Moreover, follow by definition, if a coloring is (T, C, φ) -stable then it is (T', C, φ) -stable for any tree sequence T' where T' is a segment of T . The followings are two lemmas involve some connections between stable colorings and nonextendable edges.

Lemma 2.1.2. *Let T be a closed tree-sequence associated with a k -triple (G, e, φ) and let C be color set. If edge f is $T \vee C$ -nonextendable, then for any $\gamma \in \overline{\varphi}(a(f))$ there is a $(T, \{\varphi(f)\} \cup C, \varphi)$ -stable coloring φ^* such that $a(f)$ is a T -exit of $(\gamma, \varphi(f))$.*

Proof. Since f is non-extendable, there exists a $(T, \{\varphi(f)\} \cup C, \varphi)$ -stable coloring φ' and a color $\beta \in \overline{\varphi}(a(f))$ such that there is an T exit path $P(\beta, \varphi(f))$. Let $\varphi^* = \varphi' / (G - T, \beta, \gamma)$. Since T is closed, φ^* is a proper coloring. Then, $P_{a(f)}(\gamma, \varphi(f), \varphi^*) = P(\gamma, \varphi(f))$ is the desired T -exit path. \square

Lemma 2.1.3. *Let T be a closed tree-sequence associated with a k -triple (G, e, φ) and let C be color set. If there exists a $(T, \{\varphi(f)\} \cup C, \varphi)$ -stable coloring φ^* such that $\overline{\varphi^*}(b(f)) \cap \overline{\varphi^*}(T) \neq \emptyset$, then f is $T \vee C$ -nonextendable.*

Proof. Let $\alpha \in \overline{\varphi^*}(T) \cap \overline{\varphi^*}(b(f))$. Let $\gamma \in \overline{\varphi^*}(a(f))$. Since T is closed, $\partial(T) \cap E_{\alpha, \gamma} = \emptyset$. So, the coloring $\varphi^{**} = \varphi^*/(G - T_n, \alpha, \gamma)$ is a proper coloring. Clearly, it is $(T, \{\varphi(f)\} \cup C, \varphi^*)$ -stable, so is $(T, \{\varphi(f)\} \cup C, \varphi)$ -stable. Then, $P_{a(f)}(\alpha, \gamma, \varphi^{**}) = (a(f), f, b(f))$ is an (α, γ) -exit path, so it is $T \vee C$ -nonextendable. \square

Definition 2. *Let (G, e, φ) be a k -triple and T be tree-sequence. If T is not closed, the algorithm of adding an edge $f \in \partial(T)$ and corresponded vertex $b(f)$ with $\varphi(f) \in \overline{\varphi}(T)$ to T is called Tashkinov Augmenting Algorithm (TAA). The closure \overline{T} is a tree-sequence obtained from T by applying TAA until T becomes closed.*

Given a T and coloring φ , we note that \overline{T} may not be unique although $V(T)$ is.

2.2 Extended Tashkinov Tree (ETT)

Given a tree-sequence $T = (y_0, e_1, y_1, e_2, \dots, y_{p-1}, e_p, y_p)$ and index $i \leq p$, let T_{v_i} denote the subsequence of T from the beginning vertex y_0 to vertex v_i , and call it a *segment* of T ending at v_i . Given a k -triple (G, e, φ) , a tree-sequence T is called a Tashkinov tree if $e_1 = e$ and for each $i \geq 2$, $\varphi(e_i) \in \overline{\varphi}(T_{v_{i-1}})$.

Theorem 2.2.1 (Tashkinov [40]). *All Tashkinov trees are elementary for any k -triple (G, e, φ) with $k \geq \Delta + 1$.*

Theorem 2.2.2. [Scheide [35]] *Let G be a k -critical graph with $k \geq \Delta + 1$. If $|T| < 11$ for all Tashkinov trees T , then G is elementary.*

A Tashkinov tree T is called a *maximum Tashkinov tree* if $|V(T)|$ is maximum among all k -triples of G . In this paper, we assume all maximum Tashkinov trees have at least 11 vertices due to Theorem 2.2.2. Tashkinov trees have been extended recently in [8, 9, 7, 35, 39] in many various formats. But, they all are defined under a fixed coloring. Our proof of

Goldberg's conjecture is based on the following complicated extension involving a coloring sequence.

Definition 3. Let (G, e, φ) be a k -triple. A series Tashkinov Tree (STT) \mathcal{T} is a series of tuples $(T_n, \varphi_{n-1}, S_{n-1}, F_{n-1}, \Theta_{n-1})$, where T_n is a closed tree-sequence under φ_{n-1} containing T_{n-1} as a segment, $\varphi_{n-1} \in \mathcal{C}^k(G - e)$, S_{n-1} is a color set with $|S_{n-1}| \leq 2$, F_{n-1} is an edge set with $|F_{n-1}| \leq 2$, and $\Theta_{n-1} = RE, SE, PE$ or \emptyset .

For $n = 1$, let T_1 be a closed Tashkinov tree and φ_0 be the corresponding coloring, $S_0 = F_0 = \emptyset$, $\Theta_0 = \emptyset$ and $T_0 = \emptyset$. Suppose for each i with $1 \leq i \leq n$, $(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1})$ is defined, and let $D_i = \cup_{h \leq i} S_h - \overline{\varphi}_i(T_i)$. If T_n is strongly closed under a $(T_n, D_{n-1}, \varphi_{n-1})$ -stable coloring, we stop. Otherwise, we define $(T_{n+1}, \varphi_n, S_n, F_n, \Theta_n)$ according the following three conditions below.

(R): Under φ_{n-1} , there exists an $h < n$ with $\Theta_h = PE$ and $S_h = \{\delta_h, \gamma_h\}$ such that there is a (γ_h, δ_h) -cycle Q containing a subpath P intersecting T_h and an edge $f_n \in \partial_{\varphi_{n-1}, \gamma_n}(T_n)$ with $V(P) \subset V(T_n)$ and $a(f_n) \in V(P)$.

If (R) is satisfied, we do the following RE extension. If condition R is not satisfied, we proceed with the following. For each $(T_n, D_{n-1}, \varphi_{n-1})$ -stable coloring φ_{n-1}^* , let $v_{\varphi_{n-1}^*}$ be the maximum $a(f)$ along \prec_ℓ over all $f \in \partial(T_n)$ such that $\varphi_{n-1}^*(f)$ is a defective color of T_n , and let v_n be the maximum vertex along \prec_ℓ over all $v_{\varphi_{n-1}^*}$. Denote by φ'_{n-1} and f_n the corresponding coloring and boundary edge. Let $\delta_n = \varphi'_{n-1}(f_n)$. If the following (S) is satisfied, we do SE extension. Otherwise, (P) is satisfied by Lemma 2.1.3 and we do PE extension.

(S): Under every $(T_n, D_{n-1} \cup \{\delta_n\}, \varphi'_{n-1})$ -stable coloring φ''_{n-1} , $\overline{\varphi}''_{n-1}(T_n) \cap \overline{\varphi}''_{n-1}(b(f_n)) = \emptyset$.

(P): For any missing color $\gamma \in \overline{\varphi}_{n-1}(v_n)$, there exists a $(T_n, D_{n-1} \cup \{\delta_n\}, \varphi'_{n-1})$ -stable coloring φ''_{n-1} such that $P_{v_n}(\gamma, \delta_n, \varphi''_{n-1})$ is a (δ_n, γ) -exit path.

Revisiting Extension (RE): Let $\varphi_n = \varphi_{n-1}$, $T_{n+1} = \overline{T_n \cup (f_n, b(f_n))}$ under φ_n which is a closure of $(T_n, f_n, b(f_n))$ under φ_n , $\delta_n = \delta_h$, $\gamma_n = \gamma_h$, $S_n = \{\delta_n, \gamma_n\}$, $F_n = \{f_n\}$, and $\Theta_n = RE$. In this case, we call f_n a RE connecting edge.

Series Extension (SE): Let $\varphi_n = \varphi'_{n-1}$, $T_{n+1} = \overline{T_n \cup (f_n, b(f_n))}$ under φ_n , $S_n = \{\delta_n\}$, $D_n = \cup_{h \leq n} S_h - \overline{\varphi_n}(T_n)$, $F_n = \{f_n\}$ and $\Theta_n = SE$. In this case, we call the maximum defective vertex v_n (f_n , resp.) extension vertex (SE connecting edge, resp.).

Parallel Extension (PE): We call the vertex v_n a supporting vertex in this case. Let $\gamma_n \in \overline{\varphi''}(v_n)$ with preference for colors in $\overline{\varphi''}_{n-1}(v_n) \cap (\cup_h S_h)$, where the index h runs for all supporting vertices v_h with $h < n$ and $v_h = v_n$. Let $\varphi_n = \varphi''_{n-1}/P_{v_n}(\gamma_n, \delta_n, \varphi''_{n-1})$, $S_n = \{\delta_n, \gamma_n\}$, $D_n = \cup_{h \leq n} S_h - \overline{\varphi_n}(T_n)$, $F_n = \emptyset$, and $\Theta_n = PE$. Let T_{n+1} be a closure $\overline{T_n}$ under φ_n .

The long and tedious definition above contains a few facts and freedom of choices below.

- (1) By Lemma 2.1.3, if condition (S) does not hold, then condition (P) holds. But, we add condition (R) and extension RE because of its important role in our proofs. To emphasize the importance of priority of the extension types, we call it **Extension Rule (ER)**: After each PE extension, we do a sequence RE's until condition (R) does not hold.
- (2) When $\Theta_i = PE$, there are a few choices for $\gamma_i \in \overline{\varphi''}_{i-1}(v_i)$. However, we always pick $\gamma_i \in \cup_h S_h$ if possible where the index h runs for all supporting vertices v_h with $h < i$ and $v_h = v_i$. With this restriction, we claim that $|\overline{\varphi''}_{i-1}(v_i) \cap (\cup_h S_h)| \leq 1$. Actually we can see that for the smallest i' where $v_{i'} = v_i$ is used as a supporting vertex, the claim holds since $|\overline{\varphi''}_{i'-1}(v_{i'}) \cap (\cup_h S_h)| = \emptyset$ where h runs for all supporting vertices v_h with $h < i'$ and $v_h = v_{i'}$. Hence $\delta_{i'} \in \overline{\varphi_{i'}}(v_n)$. Then, for the smallest j such that v_j is a supporting vertex with $v_j = v_{i'}$ and $j > i'$, we have that $|\overline{\varphi''}_{j-1}(v_j) \cap (S_{i'})| = \delta_{i'}$. Thus we have to pick $\gamma_j = \delta_{i'}$. Continue in this fashion, we see that each time we pick a γ_h , the previous possible color $\delta_{h'}$ in some $S_{h'}$ is swapping from missing color to be seen at v_i and we have as claimed. In fact we have a few choices for γ_i in the first time v_i is used as a supporting vertex, and each time $v_k = v_i$ is used as a supporting vertex again, we only have one choice for the color γ_k . We call this property **the Uniqueness at supporting vertices**.

- (3) The color set D_n , a subset set of $\cup_{i \leq n} S_n$, will play a key role in our proofs. So we single it out in the definition.

We also notice the following three facts. (a) For every $i \geq 1$, δ_i is a defective color of T_i under φ_i for PE or SE . However, γ_n is a defective color of T_i under φ_i when $\Theta_i = RE$ because f_i is contained in a (δ_i, γ_i) cycle, $\delta_i \in \bar{\varphi}_i(T_i)$ and T_i is closed for colors in $\bar{\varphi}_i(T_i)$ under φ_i for RE extensions. (b) When $\Theta_n = PE$, v_n is the only vertex in $a(\partial_{\varphi_n, \gamma_n}(T_n))$ since $\gamma_n \in \bar{\varphi}_{n-1}(T_n)$ and T_n is closed for γ_n under φ_{n-1} . (c) We have $\bar{\varphi}_{i-1}(T_i) \cup D_{i-1} \subset \bar{\varphi}_{j-1}(T_j) \cup D_{j-1}$ for $i \leq j$ because $\bar{\varphi}_{i-1}(T_i) \cup D_{i-1} \subset \bar{\varphi}_i(T_i) \cup D_i$ for each $i \geq 1$.

For convenience, we call RE connecting edges and SE connecting edges *connecting edges* and the color of the edge in F_n under φ_n a *connecting color*.

Definition 4. A tree sequence T is called an *Extended Tashkinov Tree (ETT)* if there exists an STT \mathcal{T} such that $T \subseteq \mathcal{T}$. The unique nonnegative integer n such that $T_n \subsetneq T \subseteq T_{n+1}$ is called the *rung number of T* and denoted by $n(T)$. The sequence $T_0 \subset T_1 \subset \dots \subset T_n \subset T$ is called the *ladder of T* . The corresponding coloring sequence $(\varphi_0, \varphi_1, \dots, \varphi_n)$ is called the *coloring-sequence of T* and φ_n is called the *last coloring of T* .

Clearly every STT is also an ETT. Note that if T is an ETT then any segment T' of T is also an ETT. Let φ_n^* be a (T_n, D_n, φ_n) -stable coloring. Although φ_n^* is (T_n, D_n, φ_n) -stable, φ_n^* might not be a valid last coloring for an ETT. One can see that the tree sequence T' obtained from $T_n \cup F_n \cup b(F_n)$ using TAA under φ_n^* might not satisfy the definition of ETT under last coloring φ_n^* , as we may not be able to find all colorings $\varphi_0^*, \dots, \varphi_{n-1}^*$ corresponding to each T_i where $0 < i \leq n$. Moreover, we have the following remark:

- If $\Theta_n = SE$, since v_n is still “maximum” under φ^* and φ^* is (T_n, D_n, φ_n) -stable, T_n satisfies condition S under φ_n^* and can be extend to T' as a PE extension.
- If $\Theta_n = PE$, we may have $P_{v_n}(\delta_n, \gamma_n, \varphi_n^*) \cap T_n \neq \{v_n\}$ under φ_n^* . In this case, T' may not be an ETT under last coloring φ_n^* , as we might not find a corresponding coloring φ_{n-1}'' such that $\varphi_n^* = \varphi_{n-1}''/P_{v_n}(\delta_n, \gamma_n, \varphi_{n-1}'')$ with $P_{v_n}(\delta_n, \gamma_n, \varphi_{n-1}'') \cap T_n = \{v_n\}$.

- If $\Theta_n = \text{RE}$, then instead of being an edge contained in a (δ_h, γ_h) cycle intersecting T_h where h is the largest index such that $\Theta_h = \text{PE}$, f_n could be contained in a (δ_h, γ_h) path. In this case, T' is not an ETT under last coloring φ_n^* .

Therefore, we introduce the following concept.

Definition 5. Let T be an ETT with ladder $T_0 \subset T_1 \subset \dots \subset T_n \subset T$ which is contained in an STT $\mathcal{T} = \{(T_i, \varphi_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$. A coloring θ_n is called $\varphi_n \bmod T_n$ if there is an ETT $T^* \subset \mathcal{T}^*$ with ladder $T_0 \subset T_1 \subset \dots \subset T_n \subset T^*$ with coloring-sequence $(\theta_0, \theta_1, \dots, \theta_n)$ such that

- (1) $\mathcal{T}^* = \{(T_i^*, \theta_{i-1}, S_{i-1}, F_{i-1}, \Theta_{i-1}) : 1 \leq i \leq n+1\}$ is also an STT and $T_i^* = T_i$ for $1 \leq i \leq n$;
- (2) θ_i is (T_i, D_i, φ_i) -stable for each $i = 0, 1, 2, \dots, n$.

We call the ETT T^* above a corresponding ETT of θ_n .

Note that a $\varphi_n \bmod T_n$ coloring θ_n have many choices for corresponding ETTs. From the definition above, we immediately see that if θ_n is a $\varphi_n \bmod T_n$ coloring, then every ETT obtained from $T_n \cup F_n \cup b(F_n)$ under θ_n by TAA is actually a corresponding ETT of θ_n . In this paper, we will deal with a lot of last colorings which are $\varphi_n \bmod T_n$.

Definition 6. Let G be an edge- k -critical graph and T be an ETT with ladder $T_0 \subset T_1 \subset T_2 \subset \dots \subset T_n \subset T$ and coloring sequence $(\varphi_0, \varphi_1, \dots, \varphi_n)$. We call T satisfy Maximum Property (MP) if T_1 is the Tashkinov tree with maximum number of vertices over all k -triples (G, e, φ) , and for each i with $1 \leq i \leq n-1$, $|T_{i+1}|$ is maximum over all (T_i, D_i, φ_i) -stable colorings, i.e. $|\overline{T_i \cup F_i \cup b(F_i)}| \leq |T_{i+1}|$ under all (T_i, D_i, φ_i) -stable colorings.

Given an ETT with ladder $T_0 \subset T_1 \subset T_2 \subset \dots \subset T_n \subset T$ and coloring sequence $(\varphi_0, \varphi_1, \dots, \varphi_n)$, MP only requires $|V(T_i)|$ to be maximal among all $(T_{i-1}, D_{i-1}, \varphi_{i-1})$ -stable colorings for $i = 1, 2, \dots, n$. Therefore if T satisfies MP, T_i under φ_{i-1} also satisfies MP because it is an ETT with ladder $T_0 \subset T_1 \subset T_2 \subset \dots \subset T_{i-1} \subset T_i$. Moreover, any Tashkinov tree satisfies MP because MP does not require anything on T_0 . However, the existence

of an non-trivial ETT satisfying MP is not clear, as choosing $|T_i|$ being maximum over all $(T_{i-1}, D_{i-1}, \varphi_{i-1})$ -stable colorings may conflict the definition of ETT. Despite of such a problem, the existence of an ETT satisfying MP will be showed in Corollary 2.2.1. In fact, Corollary 2.2.1 shows under MP, any (T_n, D_n, φ_n) -stable coloring is a $\varphi_n \bmod T_n$ coloring and it can be used to extend T_n by the same extension type as in φ_n . Moreover, we have the following result.

Lemma 2.2.1. *Let T be an ETT satisfying MP with ladder $T_1 \subset T_2 \subset \dots \subset T_n \subset T$ and coloring sequence $(\varphi_0, \varphi_1, \dots, \varphi_n)$. If θ_n is a $\varphi_n \bmod T$ coloring and T^* is the corresponding ETT, then T^* satisfies MP with ladder $T_1 \subset T_2 \subset \dots \subset T_n \subset T^*$ and coloring sequence $(\theta_0, \theta_1, \dots, \theta_n)$.*

Proof. By (1) in the definition of $\varphi_n \bmod T$ coloring, we see that T_1 is also a maximum Tashkinov tree under θ_0 . Since (2) in the definition of $\varphi_n \bmod T$ coloring requests θ_i being (T_i, D_i, φ_i) -stable for each $i = 0, 1, 2, \dots, n$ and colorings being (T_i, D_i, φ_i) -stable is an equivalent relation, we have that $|\overline{T_i \cup F_i \cup b(F_i)}| \leq |T_{i+1}|$ under all (T_i, D_i, θ_i) -stable colorings for $0 < i < n$. Therefore, T^* also satisfies MP as desired. \square

Theorem 2.2.3. *Let n be a nonnegative integer and (G, e, φ) be a k -triple with $k \geq \Delta + 1$. If T is an ETT satisfying MP with ladder $T_1 \subset T_2 \subset \dots \subset T_n \subset T$ and coloring sequence $(\varphi_0, \varphi_1, \dots, \varphi_n)$, then T is elementary under the coloring φ_n .*

Based on Tashkinov's theorem, for any k -triple (G, e, φ) , all Tashkinov trees are elementary. Our proof is divided into two steps: (a) Adding a connecting edge and a vertex (a edge vertex alternating sequence .resp) to a closed ETT for SE and RE (PE .resp) while keeping the elementary properties and (b) Adding edges through TAA keeps the elementary property.

Proof of Theorem 2.1.1 Let T be an STT. Then T is also an ETT by definition. Note that by the definition of STT, we can always extend T unless T is strongly closed under a final coloring. Moreover, by Lemma 2.2.1, we can always assume T satisfies MP, and therefore T is always elementary. \square

Let T be an ETT starting with a k -triple (G, e, φ) with ladder $T_1 \subset T_2 \subset \dots \subset T_n \subset T$ and $v \in V(T)$. For any $v \in V(T)$, let $m(v)$ be the smallest index m such that $v \in V(T_m)$, where we denote T by T_{n+1} for convenience.

Definition 7. A closed ETT T with $n(T) = n$ has closed interchangeability for missing colors (CIMC) if under any (T, D_n, φ_n) -stable coloring, any two colors are T -interchangeable provided one of them is in $\bar{\varphi}_n(T)$.

Note that in our definition of ETT, we have a coloring series while G might be colored differently for each coloring. However, the next lemma shows that the color of the edges colored by $\bar{\varphi}_{s-1}(T_s) \cup D_s$ contained in $E(G[T_s])$ will stay the same in later colorings for $s \leq n$.

Lemma 2.2.2. Let T be an ETT with ladder $T_0 \subset T_1 \subset \dots \subset T_n \subset T$. For any $s \leq n$ and any edge f incident to T_s , if $\varphi_s(f) \in \bar{\varphi}_{s-1}(T_s) \cup D_s$, then $\varphi_t(f) = \varphi_{s-1}(f)$ for any t with $s \leq t \leq n$ unless $f = f_{t'} \in F_{t'}$ where $\Theta_{t'} = PE$ and $s \leq t' \leq t$. In particular, if $f \in E(G[V(T_s)])$ and $\varphi_s(f) \in \bar{\varphi}_{s-1}(T_s) \cup D_s$, then $\varphi_t(f) = \varphi_{s-1}(f)$ for any t with $s \leq t \leq n$.

Proof. Note that if $f \in E(G[V(T_s)])$, then $f \notin \partial(T_{t'})$ for any t' with $s \leq t' \leq t$ and consequently, $f \neq f_{t'} \in F_{t'}$ for any t' where $\Theta_{t'} = PE$ and $s \leq t' \leq t$. Thus the ‘‘In particular’’ part holds. Because $\bar{\varphi}_{s-1}(T_s) \cup D_s \subset \bar{\varphi}_s(T_{s+1}) \cup D_{s+1}$, we only need to show Lemma 2.2.2 holds for $t = s$ and apply it repeatedly to get to all $t \leq n$. Let f be any edge incident to T_s with $\varphi_s(f) \in \bar{\varphi}_{s-1}(T_s) \cup D_s$. Since $\varphi_{s-1}(f) \in \bar{\varphi}_{s-1}(T_s) \cup D_s$, $\varphi_{s-1}^*(f) = \varphi_{s-1}(f)$ for any $(T_s, D_s, \varphi_{s-1})$ -stable coloring φ_{s-1}^* . Since φ_s is $(T_s, D_s, \varphi_{s-1})$ -stable if $T_s \rightarrow T_{s+1}$ is an SE or RE, we have $\varphi_{s-1}(f) = \varphi_s(f)$. If $T_s \rightarrow T_{s+1}$ is a PE, then $\varphi_{s-1}''(f) = \varphi_{s-1}(f)$ for the same reason. Since $\varphi_s = \varphi_{s-1}''/P_{v_s}(\delta_s, \gamma_s, \varphi_{s-1})$ and $P_{v_s}(\delta_s, \gamma_s, \varphi_{s-1})$ only contains an edge f_s incident to T_s , we have $\varphi_s(f) = \varphi_{s-1}(f)$ unless $f = f_s$ as desired. \square

With the above preparation, we will state our main result, which is slightly stronger than Theorem 2.2.3.

Theorem 2.2.4. Let n be a nonnegative integer and (G, e, φ) be a k -triple with $k \geq \Delta + 1$. Then for every ETT T satisfying MP with ladder $T_0 \subset T_1 \subset \dots \subset T_n \subset T$ and coloring

sequence $(\varphi_0, \varphi_1, \dots, \varphi_n)$, the following five statements hold.

A1: (1) T is elementary under φ_n and (2) T has CIMC property if T is closed.

A2: If $\Theta_n = PE$, then under any (T_n, D_n, φ_n) -stable coloring φ_n^* , we have $P_{v_n}(\gamma_n, \delta_n, \varphi_n^*) \cap T_n = \{v_n\}$ where $S_n = \{\delta_n, \gamma_n\}$.

A3: For any (T_n, D_n, φ_n) -stable coloring φ_n^* , if δ is a defective color of T_n under φ_n^* and $v \in a(\partial_{\varphi_n^*, \delta}(T_n))$ where v is not the smallest vertex along \prec_ℓ in $a(\partial_{\varphi_n^*, \delta}(T_n))$, then $v \prec_\ell v_i$ for any supporting or extension vertex v_i with $i \geq m(v)$.

A4: For any positive integer l with $l \leq n$, if v_l is a supporting vertex and $m(v_l) = j$, then every (T_l, D_l, φ_l) -stable coloring φ_l^* is $(T_{v_l} - \{v_l\}, D_{j-1}, \varphi_{j-1})$ -stable, particularly, φ_l^* is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$ -stable. For any two supporting vertices v_s and v_t with $s, t \leq n$, if $m(v_s) = m(v_t)$ but $v_s \neq v_t$, then $S_s \cap S_t = \emptyset$.

A5: Every (T_n, D_n, φ_n) -stable coloring φ_n^* is a $\varphi_n \bmod T$ coloring and every corresponding ETT T^* obtained from T_n under φ_n^* using the same extension type as $T_n \rightarrow T$ also satisfies MP.

As immediate consequences of (A5), we have the following two corollaries.

Corollary 2.2.1. Let T be an ETT with ladder $T_0 \subset T_1 \subset T_2 \subset \dots \subset T_n \subset T$. If T satisfies MP, there exists a closed T_{n+1} with ladder $T_0 \subset T_1 \subset \dots \subset T_n \subset T_{n+1}$ under a $\varphi_n \bmod T$ coloring such that $|T_{n+1}|$ is maximum over all (T_n, D_n, φ_n) -stable colorings. Furthermore, if T_{n+1} is not strongly closed then T_{n+1} can be extended further to get an ETT T' satisfying MP.

Proof. Suppose the statement A5 of Theorem 2.2.4 holds for any ETT satisfying MP. Then there exists an closed ETT T_{n+1} satisfying MP with the same extensions and ladder as T where $|T_{n+1}|$ is maximum under a (T_n, D_n, φ_n) -stable coloring φ_n^* among all (T_n, D_n, φ_n) -stable colorings. In fact, this implies that we can assume $|T_i|$ is maximal for each i and hence MP condition is well defined. Furthermore, if T_{n+1} is not strongly closed then T_{n+1} can be extended further to get an ETT T' satisfying MP. \square

2.3 Proof of Theorem 2.2.4

We will prove Theorem 2.2.4 by induction on $n = n(T)$, the number of rungs. Note that when $n = 0$, we have (A2), (A3), (A4) and (A5) hold trivially and (A1) (1) holds by Theorem 2.2.1. (A1) (2) for $n = 0$ will be proved in Lemma 7. For the inductive step we assume Theorem 2.2.4 holds for rungs smaller than n and prove it for n . The inductive step will be divided into a few Lemmas and propositions.

Proposition 1. *Let n be a positive integer. Suppose that (A1), (A2), (A3), (A4) and (A5) hold for all ETT T' with $n(T') \leq n - 1$ satisfying MP. Let T be an ETT with ladder $T_0 \subset T_1 \subset \dots \subset T_n \subset T$ satisfying MP with coloring sequence $(\varphi_0, \varphi_1, \dots, \varphi_n)$. Then, (A4) holds for T , i.e.,*

- (1) *For any positive integer l with $l \leq n$, if v_l is a supporting vertex and $m(v_l) = j$, then every (T_l, D_l, φ_l) -stable coloring φ_l^* is $(T_{v_l} - \{v_l\}, D_{j-1}, \varphi_{j-1})$ -stable, particularly, φ_l^* is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$ -stable.*
- (2) *For any two supporting vertices v_s and v_t with $s, t \leq n$, if $m(v_s) = m(v_t)$ but $v_s \neq v_t$, then $S_s \cap S_t = \emptyset$.*

Proof. We first prove (1). Assume on the contrary that there exists a (T_l, D_l, φ_l) -stable coloring φ_l^* which is not $(T(v_l) - \{v_l\}, D_{j-1}, \varphi_{j-1})$ -stable with $m(v_l) = j$. Since stable coloring is an equivalence relation, we see that all edges incident to T_l colored by $\varphi_l(T_l) \cup D_l$ under φ_l are colored the same under φ_l^* . Note that $j \leq l$, we have $\overline{\varphi}_{j-1}(T(v_l) - \{v_l\}) \cup D_{j-1} \subset \overline{\varphi}_{j-1}(T_j) \cup D_{j-1} \subset \overline{\varphi}_l(T_l) \cup D_l$. Therefore φ_l is not $(T(v_l) - \{v_l\}, D_{j-1}, \varphi_{j-1})$ -stable. Let $V_l^- = V(T_{v_l}) - \{v_l\}$. Then, either there exists an edge f incident to V_l^- with $\varphi_{j-1}(f) \in \overline{\varphi}_{j-1}(V_l^-) \cup D_{j-1}$ such that $\varphi_{j-1}(f) \neq \varphi_l^*(f)$ or there exist a vertex $v \in V_l^-$ such that $\overline{\varphi}_{j-1}(v) \neq \overline{\varphi}_j(v)$. By Lemma 2.2.2 and the definition of $\varphi_j, \varphi_{j+1}, \dots, \varphi_l$, we have that there exist a supporting vertex $v_k \in V_l^-$ with $j \leq k < l$, so $j < l$. Moreover, we have $v_k \prec_\ell v_l$ because $v_k \in V_l^-$. Since $v_l \in T_j$, we have $v_l \in T_{l-1}$. Since v_l is the maximal vertex in $X_{\varphi'_{l-1}, \varphi'_{l-1}(f_l)}(T_l)$, we see that $\varphi'_{l-1}(f_l)$ is a defective color of $\partial(T_{l-1})$ under φ'_{l-1} and v_l

is not the smallest vertex in $X_{\varphi'_{l-1}, \varphi'_{l-1}(f_l)}(T_{l-1})$. Since φ'_{l-1} is $(T_l, D_{l-1}, \varphi_{l-1})$ -stable by the definition of STT, it is also $(T_{l-1}, D_{l-1}, \varphi_{l-1})$ -stable. Applying (A3) with $v = v_l$ and $i = k$ for φ'_{l-1} , we get $v_l \prec_\ell v_k$, a contradiction.

Now we prove (2). Suppose on the contrary there exist two distinct supporting vertices v_s and v_t with $m(v_s) = m(v_t) = r$ and $S_s \cap S_t \neq \emptyset$. Additionally, we assume $s < t$, and f_s and f_t are the corresponding edge with $\varphi'_{s-1}(f_s) = \delta_s$ and $\varphi'_{t-1}(f_t) = \delta_t$, respectively. Since $r \leq s - 1 < t$ and $v_t \in T_r$, $v_t \in T_{t-1}$. Note that δ_t is a defective color in $\partial(T_{t-1})$ under φ'_{t-1} and v_t is not the smallest in $X_{\varphi'_{t-1}, \delta_t}(T_{t-1})$. Since φ'_{t-1} is $(T_t, D_{t-1}, \varphi_{t-1})$ -stable by the definition of STT, it is also $(T_{t-1}, D_{t-1}, \varphi_{t-1})$ -stable. Applying (A3) for φ'_{t-1} on T_{t-1} , we have $v_t \prec_\ell v_s$. The remaining proof is divided into two cases: $\gamma_s \in S_t$ and $\delta_s \in S_t$, respectively.

First we assume $\gamma_s \in S_t$. Note that T_s is an ETT under φ_{s-1} with $n(T_s) = s - 1$ and $s - 1 < t \leq n$, it is elementary under φ_{s-1} by (A1). Thus we have $\gamma_s \notin \bar{\varphi}_{s-1}(v_t)$. Let f be the edge incident to v_t with $\varphi_{s-1}(f) = \gamma_s$. Since T_s is closed under φ_{s-1} , $f \in E(G[T_s])$. By Lemma 2.2.2, we have $\varphi_{t-1}(f) = \varphi_{s-1}(f)$. So, we have $\gamma_s \notin \bar{\varphi}_{t-1}(v_t)$ and $f \notin \partial(T_{t-1})$. Because φ''_{t-1} is $(T_t, D_{t-1}, \varphi_{t-1})$ -stable and $\gamma_s \in \bar{\varphi}_{s-1}(T_s) \cup D_{s-1} \subset \bar{\varphi}_{t-1}(T_t) \cup D_{t-1}$, we have that $\gamma_s \notin \bar{\varphi}''_{t-1}(v_t)$ and $f \notin \partial(T_{t-1})$ under φ''_{t-1} . Thus we have $\gamma_s \notin S_t$, a contradiction.

We now assume $\delta_s \in S_t$. Note that T_{s+1} is an ETT under φ_s with $n(T_{s+1}) = s$ and $s < t \leq n$, it is elementary under φ_s by (A1). Because T_{s+1} is elementary under φ_s and $\delta_s \in \bar{\varphi}_s(v_s)$, we have $\delta_s \notin \bar{\varphi}_s(v_t)$. Let f be the edge incident to v_t with $\varphi_s(f) = \delta_s$. Since T_{s+1} is closed under φ_s , $f \in E(V(T_{s+1}))$. Since $t > s$, we have $\varphi_{t-1}(f) = \varphi_s(f)$ by Lemma 2.2.2. Because φ''_{t-1} is $(T_t, D_{t-1}, \varphi_{t-1})$ -stable and $\delta_s \in \bar{\varphi}_s(T_s) \cup D_s \subset \bar{\varphi}_s(T_{s+1}) \cup D_s \subset \bar{\varphi}_{t-1}(T_t) \cup D_{t-1}$, we have that $\delta_s \notin \bar{\varphi}''_{t-1}(v_t)$ and $f \notin \partial(T_{t-1})$ under φ''_{t-1} . Thus $\delta_s \notin S_t$, a contradiction. \square

Proposition 2. *If (A1), (A2), (A3) and (A5) hold for any ETT T' satisfying MP with $n(T') < n$ and (A4) holds for any ETT T' satisfying MP with $n(T') \leq n$, then (A2) holds for any ETT T with $n(T) = n$ satisfying MP with $\Theta_n = PE$.*

Proof. Assume on the contrary that there exists an ETT T satisfying MP with ladder $T_0 \subset T_1 \subset \dots \subset T_n \subset T$ and a (T_n, D_n, φ_n) -stable coloring φ^* such that $P_{v_n}(\gamma_n, \delta_n, \varphi_n^*) \cap T_n \neq \{v_n\}$.

Let $j := m(v_n)$. We will lead a contradiction to the maximality of T_j by finding a coloring φ^* agreeing with φ_{j-1} on edges in $E(T_j)$ and missing color sets for vertices in $V(T_j)$ except for v_n where we have a missing color δ_n instead. In the case which we can not find such a coloring, we show that there is a contradiction to the elementariness in (A1) for ETTs with less than n rungs. Before proceeding with the proof, we introduce some notation and properties.

Denote by L the set of indices i with $i \geq j$ such that $\Theta_i = \text{PE}$ and $m(v_i) = j$ where v_i is a supporting vertex. We further divide L into disjoint subsets L_1, L_2, \dots, L_k such that two indices $s, t \in L$ are in the same set if and only if $v_s = v_t$. For each L_i , let w_i denote the common supporting vertex and assume, without loss of generality, that $w_1 \prec_\ell w_2 \prec_\ell \dots \prec_\ell w_k$. For each L_i , let P_i be the graph with $V(P_i) = \cup_{t \in L_i} \{\delta_t, \gamma_t\}$ and $E(P_i) = \{\delta_t \gamma_t : t \in L_i\}$.

Claim 2.3.1. P_1, P_2, \dots, P_k are vertex-disjoint paths.

Proof. By proposition 1 (2), $S_s \cap S_t = \emptyset$ if s and t are in different index sets. So, P_1, P_2, \dots, P_k are mutually vertex-disjoint graphs. We only need to show that P_i is a path for each $0 \leq i \leq k$.

Let $L_i = \{i_1, i_2, \dots, i_{s_i}\}$. Following the rule of **Uniqueness at supporting vertices** (Page 10) in the definition of PE, we have $\delta_{i_t} = \gamma_{i_{t+1}}$ for $1 \leq t < s_i$. So, P_i is a walk from γ_{i_1} to $\delta_{i_{s_i}}$. To show that it is a path, we need to prove $\gamma_{i_s} \neq \delta_{i_t}$ whenever $s_i \geq t \geq s \geq 1$. Assume on the contrary that $\gamma_{i_s} = \delta_{i_t}$ for a pair of indices s and t with $s \leq t$. Let $v \in T$ be an arbitrary vertex with $v \prec_\ell w_i$. Since $\gamma_{i_s} \in \overline{\varphi}_{i_s-1}(w_i)$, $\gamma_{i_s} \in \overline{\varphi}_{i_s-1}(v)$. Let f be the edge incident with v with $\varphi_{i_s-1}(f) = \gamma_{i_s}$. Since T_{i_s} is closed under coloring φ_{i_s-1} , both ends of f are in $V(T_{i_s})$. Since $\gamma_{i_s} \in S_{i_s}$, for any $m \geq i_s$ the color $\varphi_{i_s-1}(f)$ on f stays the same under φ_m by Lemma 2.2.2. Moreover, it stays the same in φ'_m and φ''_m (if they exist). Thus $v \notin a(\partial_{\varphi_{i_t-1}, \gamma_{i_s}}(T_{i_t}))$, which in turn shows that w_i can not be the supporting vertex of T_{i_t} , a contradiction. \square

By (A4), φ_n is $(T_n(v_n) - v_n, D_{j-1}, \varphi_{j-1})$ -stable. We claim that $w_1 = v_n$. Assume $w_1 \neq v_n$, then $w_1 \prec_\ell v_n$. By claim 2.3.1, we have $\varphi_{j-1}(w_1) \neq \varphi_n(w_1)$, a contradiction with φ_n being

$(T_n(v_n) - v_n, D_{j-1}, \varphi_{j-1})$ -stable. Thus $w_1 = v_n$.

Claim 2.3.2. *Let v_s be a supporting vertex with $s < n$, φ_s^* be a (T_s, D_s, φ_s) -stable coloring and $s' > s$ be the smallest index such that $\Theta_{s'} \neq RE$. Then the followings are true:*

(1) $P_v(\delta_s, \gamma_s, \varphi_s^*)$ is a cycle for any $v \in T_s - v_s$.

(2) For any $v \in T_s - v_s$, $P_v(\delta_s, \gamma_s, \varphi_{s'-1})$ is a cycle and it is contained in $G[V(T_{s'})]$.

Moreover, we have $\varphi_{s'-1}(f) = \varphi_{s'}(f) = \dots = \varphi_n(f) = \varphi_n^*(f)$ for any edge $f \in P_v(\delta_s, \gamma_s, \varphi_{s'-1})$.

We first prove (1). Suppose there exist a vertex $v \in T_s - v_s$ which is contained in a path P under a (T_s, D_s, φ_s) -stable coloring φ_s^* . By (A2), $P_{v_s}(\delta_s, \gamma_s, \varphi_s^*) \cap T_s = \{v_s\}$ and therefore, $v \notin P_{v_s}(\delta_s, \gamma_s, \varphi_s^*)$. Thus P is disjoint with $P_{v_s}(\delta_s, \gamma_s, \varphi_s^*)$. Let $\varphi_{s-1}^* = \varphi_s^*/P_{v_s}(\delta_s, \gamma_s, \varphi_s^*)$. Since $\varphi_s = \varphi_{s-1}^*/P_{v_s}(\delta_s, \gamma_s, \varphi_{s-1}^*)$, φ_s^* is (T_s, D_s, φ_s) -stable and φ_{s-1}^* is $(T_s, D_{s-1}, \varphi_{s-1})$ -stable, P_{s-1}^* is $(T_s, D_{s-1}, \varphi_{s-1})$ -stable. By A1, T_s has ICMC property under φ_{s-1}^* , i.e., there are at most one (δ_s, γ_s) -path intersecting T_s . However we have two disjoint paths $P_{v_s}(\delta_s, \gamma_s, \varphi_{s-1}^*)$ and P intersecting T_s , a contradiction.

Now we prove (2). We have that all (δ_s, γ_s) -cycles intersecting T_s are contained in $G[V(T_{s'})]$ under $\varphi_{s'-1}$ by the **extension rules** within the definition of ETT. In fact, we have $\Theta_{s+1} = \Theta_{s+2} = \dots = \Theta_{s'-1} = RE$ when $s' - 1 \geq s + 1$ and therefore $\varphi_{s'-1}$ is (T_s, D_s, φ_s) -stable. Thus by (1), we have that $P_v(\delta_s, \gamma_s, \varphi_{s'-1})$ is a cycle under $\varphi_{s'-1}$ with $v \in T_s - v_s$. Therefore $P_v(\delta_s, \gamma_s, \varphi_{s'-1})$ is contained in $G[V(T_{s'})]$ under $\varphi_{s'-1}$. Let f be an edge contained in $P_v(\delta_s, \gamma_s, \varphi_{s'-1})$. Then, $f \in E(G[V(T_{s'})])$. Since $\varphi_{s'-1}(f) \in S_s \subseteq D_{s'} \cup \bar{\varphi}_{s'-1}(T_{s'})$, we have $\varphi_{s'-1}(f) = \varphi_{s'}(f) = \dots = \varphi_n(f)$ by Lemma 2.2.2. \square

Note that by Claim 2.3.2, a (δ_s, γ_s) cycle Q intersects $T_s - v_s$ under $\varphi_{s'-1}$ if and only if Q intersects $T_s - v_s$ under φ_n . Since φ_n^* is (T_n, D_n, φ_n) -stable, a (δ_s, γ_s) cycle Q intersects $T_s - v_s$ under $\varphi_{s'-1}$ if and only if Q intersects $T_s - v_s$ under φ_n^* . Now our first step is to retrieve all colors originally missing in each representative of V_i other than v_n . By originally missing we mean the missing colors of the representative of V_i under φ_{j-1} .

Claim 2.3.3. *There exist a coloring φ^* satisfying the followings:*

- (1) φ^* is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$ -stable.
- (2) $\bar{\varphi}^*(v) = \bar{\varphi}_{j-1}(v)$ for all $v \in T_j - v_n$, $\bar{\varphi}^*(v_n) = \bar{\varphi}_n(v_n)$ and $\varphi^*(f) = \varphi_n(f) = \varphi_{j-1}(f)$ for all $f \in T_j - T_j(v_n)$.
- (3) $\varphi^*(f) = \varphi_n^*(f)$ for all $f \in G$ with $\varphi_n^*(f) \in \cup_{i \in L_1} S_i$
- (4) For all $i \in L_1 - \{n\}$, we have $\varphi^*(f) = \varphi_n(f)$ for $f \in Q$ where Q is any (δ_i, γ_i) -cycle intersecting $T_i - v_i$ under φ_n .

By Claim 2.3.2, every (δ_i, γ_i) -cycle Q intersecting $T_i - v_i$ under φ_n is contained in $G[V(T_n)]$ for all $i \in V_1 - \{n\}$. Since φ_n^* is (T_n, D_n, φ_n) -stable, we have that (3) implies (4). So we will not check (4) during the proof. Moreover, by Lemma 2.2.2 we have $\varphi_n(f) = \varphi_{j-1}(f)$ for all $f \in T_j - T_j(v_n)$. Therefore in (2) for the part $\varphi^*(f) = \varphi_n(f) = \varphi_{j-1}(f)$ for all $f \in T_j - T_j(v_n)$, we only need to show $\varphi^*(f) = \varphi_n(f)$ for all $f \in T_j - T_j(v_n)$. We will now find the coloring φ^* by the following procedure. Denote $L_i = \{i_1, \dots, i_{p_i}\}$ where $p_i = |L_i|$ and $i_s < i_t$ if $s < t$. We define a linear ordering \prec_v for $\cup_{1 < i \leq k} L_i$ where $i_t \prec_v h_s$ if $h < i$ or $h = i$ but $s < t$. Then we will inductively define a series of colorings along the linear ordering \prec_v starting from k_{p_k} . We let $\varphi_{k,p_k} = \varphi_n^*$ and $\varphi_{h,s} = \varphi_{i,t}/P_{v_{i_t}}(\delta_{i_t}, \gamma_{i_t}, \varphi_{i,t})$, where $h_s \neq i_t$ is the smallest index bigger than i_t along \prec_v . Finally we let $\varphi_{2,0} = \varphi_{2,1}/P_{v_{2,1}}(\delta_{2,1}, \gamma_{2,1}, \varphi_{2,1})$. Denote $U_i = \{v_s | s \in L_h, 1 \leq h \leq i\}$. We claim that $\varphi_{h,s}$ satisfies the followings where $h_s \neq i_t$ is the smallest index bigger than i_t along \prec_v :

- 1* $\varphi_{h,s}$ is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$ -stable.
- 2* $\bar{\varphi}_{h,s}(v) = \bar{\varphi}_{j-1}(v)$ for all $v \in T_j - U_i$, $\bar{\varphi}_{h,s}(v) = \bar{\varphi}_n(v)$ for $v \in U_i - v_{i_t}$, $\bar{\varphi}_{h,s}(v_{i_t}) = \bar{\varphi}_{h_s}(v_{i_t})$ and $\varphi_{h_s}(f) = \varphi_n(f)$ for all $f \in T_j - T_j(v_n)$.
- 3* $\varphi_{h,s}(f) = \varphi_n^*(f)$ for all $f \in G$ with $\varphi_n^*(f) \in \cup_{i \in U_{1 \leq g < h} L_g} S_i$.
- 4* $\varphi_{h,s}(f) = \varphi_n^*(f)$ for all $f \in G$ with $\varphi_n^*(f) \in \cup_{h_r, 0 < r < s} S_{h_r}$.
- 5* $\varphi_{h,s}(f) = \varphi_n(f)$ for $f \in Q$ where Q is any $(\delta_{h_s}, \gamma_{h_s})$ -cycle intersecting $T_{h_s} - v_{h_s}$ under φ_n .

Note that φ_n^* satisfies statements 1*–4* above because φ_n^* is (T_n, D_n, φ_n) -stable and φ_n is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$ -stable by (A4). By Claim 2.3.2, every $(\delta_{h_s}, \gamma_{h_s})$ -cycle Q intersecting $T_{h_s} - v_{h_s}$ under φ_n is contained in $G[V(T_n)]$. Since φ_n^* is (T_n, D_n, φ_n) -stable, 5* also holds for φ_n^* . Moreover, the above claim immediately implies $\varphi_{2,0}$ is the coloring φ^* in Claim 2.3.3. Thus we only need to prove the claim above. We will do an induction along \prec_v starting from $\varphi_{k,k(i)} = \varphi_n^*$, where the induction base is clear for φ_n^* . Now we prove the claim holds for $\varphi_{h,s}$. By Claim 2.3.2, all vertices in $T_{i_t} - v_{i_t}$ are contained in $(\delta_{i_t}, \gamma_{i_t})$ -cycles under φ_n . By 5* for $\varphi_{i,t}$, we see that all vertices in $T_{i_t} - v_{i_t}$ are contained in $(\delta_{i_t}, \gamma_{i_t})$ -cycles under $\varphi_{i,t}$. Therefore $P_{v_{i_t}}(\delta_{i_t}, \gamma_{i_t}, \varphi_{i,t}) \cap T_{i_t} = \{v_{i_t}\}$. Hence 1* holds by 1* of $\varphi_{i,t}$. Additionally, since $T_j \subset T_{i_t}$, the holding of 2* relies on 2* for $\varphi_{i,t}$, $P_{v_{i_t}}(\delta_{i_t}, \gamma_{i_t}, \varphi_{i,t}) \cap T_{i_t} = \{v_{i_t}\}$ and the uniqueness at supporting vertices in the definition of ETT. By Proposition 1, $S_s \cap S_t = \emptyset$ if $v_s \neq v_t$ where v_s and v_t are supporting vertices. Moreover, Claim 2.3.1 implies $S_{i,t} \cap \cup_{0 < r < s} S_{h_r} = \emptyset$. Therefore the holding of 3* and 4* for $\varphi_{i,t}$ imply 3* and 4* hold for $\varphi_{h,s}$. Finally we will prove 5*. If $i \neq h$, then $S_{h,s} \cap S_{i,t} = \emptyset$ by Proposition 1. In this case 5* holds by 3* of $\varphi_{i,t}$ and Claim 2.3.2 because φ_n^* is (T_n, D_n, φ_n) -stable. Therefore we assume $i = h$. We then have $\gamma_{i_t} = \delta_{h_s}$ by Claim 2.3.1. By Claim 2.3.2, all $(\delta_{h_s}, \gamma_{h_s})$ -cycles intersecting $T_{h_s} - v_{h_s}$ under φ_n are contained in $G[V(T_{i_t})]$. Since φ_n^* is (T_n, D_n, φ_n) -stable, 4* for $\varphi_{i,t}$ implies all $(\delta_{h_s}, \gamma_{h_s})$ -cycles intersecting $T_{h_s} - v_{h_s}$ under φ_n is colored the same way in $\varphi_{s,t}$. Because $P_{v_{i_t}}(\delta_{i_t}, \gamma_{i_t}, \varphi_{i,t}) \cap T_{i_t} = \{v_{i_t}\}$ and $\varphi_{h,s} = \varphi_{i,t} / P_{v_{i_t}}(\delta_{i_t}, \gamma_{i_t}, \varphi_{i,t})$, all $(\delta_{h_s}, \gamma_{h_s})$ -cycles intersecting $T_{h_s} - v_{h_s}$ under φ_n is colored the same in $\varphi_{s,t}$ and $\varphi_{h,s}$. Thus 5* holds for $\varphi_{h,s}$. \square

Now we consider the coloring φ^* in Claim 2.3.3. Let $T_n^{d^*} = \overline{T_j(v_n)}$ under φ^* . Since δ_n is defective in T_n under φ_{n-1} and all boundary edges colored by δ_n are not incident to vertices after v_n in \prec_ℓ by our choice of v_n , there are two edges $f, g \in \partial_{\varphi_n, \delta_n}(T_n)$ and incident to vertices before v_n in \prec_ℓ . Therefore $\varphi^*(f) = \varphi^*(g) = \delta_n$ by (3) of Claim 2.3.3 and the fact that φ_n is (T_n, D_n, φ_n) -stable. Since $\delta_n \in \overline{\varphi^*(v_n)}$ by (2) of Claim 2.3.3, $b(f)$ and $b(g)$ are contained in $T_n^{d^*}$. Thus $T_n^{d^*} - T_n \neq \emptyset$. By (1) of Claim 2.3.3, φ^* is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$ -stable and hence it is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$ -stable. Thus $T_n^{d^*}$ is an ETT satisfying MP under last coloring φ^* with $j - 1 < n$ rungs by (A5). Hence it is elementary by (A1). We then

consider the following two cases:

Case 1. $\overline{\varphi}^*(T_n^{d^*} - T_j(v_n)) \cap (\cup_{i \in L_1} S_i) \neq \emptyset$.

Since $T_n^{d^*}$ is elementary and $\delta_n \in \overline{\varphi}^*(v_n)$, $\delta_n \notin \overline{\varphi}^*(T_n^{d^*} - T_j(v_n))$. Since $\delta_{1_s} = \gamma_{1_{s+1}}$ for any $1_s \in L_1$ with $1_s < n$, we may assume that there is a smallest index $m \in L_1$ such that $\gamma_m \in \varphi^*(T_n^{d^*})$. Then $m \geq j$. Let $w \in T_n^{d^*}$ such that $\gamma_m \in \overline{\varphi}^*(w)$. By the uniqueness at supporting vertices, $\overline{\varphi}^*(v_n) \cap (\cup_{m \in L_1} S_m) = \{\delta_n\}$, which in turn shows $w \neq v_n$. We claim that $w \in T_n^{d^*} - T_m$. Assume that on the contrary, $w \in T_m$. By Claim 2.3.3 (3), $\gamma_m \in \overline{\varphi}_n^*(w) = \overline{\varphi}_n(w)$ because φ_n is (T_n, D_n, φ_n) -stable. Note that T_m is elementary under φ_{m-1} by (A1) because it is an ETT satisfying MP with $m - 1$ rungs. Recall that $w \neq v_n$, we have the edge f colored by γ_m incident to w is contained in $E(G[V(T_m)])$ under φ_{m-1} because $\gamma_m \in \overline{\varphi}_{m-1}(v_n)$ and T_m is elementary under φ_{m-1} . Thus $\varphi_n(f) = \gamma_m$ by Claim 2.2.2, which is a contradiction to $\gamma_m \in \overline{\varphi}_n(w)$. Hence we have $w \in T_n^{d^*} - T_m$. Let $m = 1_p$ for some $1 \leq p \leq 1_{p_1}$ where $L_1 = \{1_1, \dots, 1_{p_1}\}$. Then $\gamma_m = \delta_{1_{p-1}}$ if $p > 1$. Our idea for this part of the proof is to make γ_{1_1} a missing color of w , where we have a bigger ETT. Similarly as Claim 2.3.3, we make the following Claim.

Claim 2.3.4. *There exist a coloring φ^2 satisfying the followings:*

- (1) φ^2 is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$ -stable.
- (2) $\overline{\varphi}^2(v) = \overline{\varphi}^*(v)$ for all $v \in T_j \cup T_n^{d^*}(w) - w$ and $\varphi^2(f) = \varphi^*(f)$ for all $f \in T_j \cup T_n^{d^*}(w)$.
- (3) $\gamma_{1_1} \in \overline{\varphi}^2(w)$.

Similarly as before, we define $\varphi_{1,i}$ with $0 < i \leq p - 1$ inductively from the largest index where $\varphi_{1,p} = \varphi^*$ and $\varphi_{1,i} = \varphi_{1,i+1}/P_w(\delta_{1_i}, \gamma_{1_i}, \varphi_{1,i+1})$. We claim the following holds for each $\varphi_{1,i}$ with $0 < i \leq p$:

- 1* $\varphi_{1,i}$ is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$ -stable.
- 2* $\overline{\varphi}_{1,i}(v) = \overline{\varphi}^*(v)$ for all $v \in T_j \cup T_n^{d^*}(w) - w$ and $\varphi_{1,i}(f) = \varphi^*(f)$ for all $f \in T_j \cup T_n^{d^*}(w)$.
- 3* $\gamma_{1_i} \in \overline{\varphi}_{1,i}(w)$.

4* $\varphi_{1,i}(f) = \varphi_n(f)$ for $f \in Q$ where Q is any $(\delta_{1_s}, \gamma_{1_s})$ -cycle intersecting $T_{1_s} - v_{1_s}$ under φ_n with $0 < s < i$.

5* $\overline{\varphi}_{1,i}(T_n^{d^*}(w) - w) \cap (\cup_{0 < t < p} S_{1_t}) = \emptyset$ and $\varphi_{1,i}(T_n^{d^*}(w) - T_j(v_n)) \cap (\cup_{0 < t < p} S_{1_t}) = \emptyset$.

We will prove all five statements by induction. We first prove all statements for $\varphi_{1,p}$. Note that 2*, 3* are trivial and 1*, 4* follow from Claim 2.3.3. Therefore we only need to show 5* for $\varphi_{1,p}$. We first show $\overline{\varphi}_{1,p}(T_n^{d^*}(w) - \{w\}) \cap (\cup_{0 < t < p} S_{1_t}) = \emptyset$. Let $S = \cup_{0 < t < p} S_{1_t}$. The choice of w implies no color in S is in $\overline{\varphi}^*(T_n^{d^*}(w) - T_j(v_n) - \{w\})$. Note that φ^* is $(T_j(v_n) - \{v_n\}, D_{j-1}, \varphi_{j-1})$ -stable by Claim 2.3.3 (1). Suppose $\gamma_{1_t} \in \overline{\varphi}^*(T_j(v_n) - \{v_n\})$ with $\gamma_{1_t} \in \overline{\varphi}^*(v)$. Then $\gamma_{1_t} \in \overline{\varphi}_n(v)$ by Claim 2.3.3 (2) and the assumption of $v \neq v_n$. On the other hand, the edge f colored by γ_{1_t} incident to v is contained in $E(V(T_{1_t}))$ under $\varphi_{1_{t-1}}$ because $\gamma_{1_t} \in \overline{\varphi}_{1_{t-1}}(v_n)$ and T_{1_t} is an elementary ETT by (A1). Thus $\varphi_n(f) = \gamma_{1_t}$ by Lemma 2.2.2, where we have a contradiction with $\gamma_{1_t} \in \overline{\varphi}_n(v)$. Finally by the uniqueness at supporting vertex, no color in S appears in $\overline{\varphi}^*(v_n) = \overline{\varphi}_n^*(v_n)$ because we have $\delta_n \in \overline{\varphi}_n^*(v_n)$. Since $T_n^{d^*}$ is obtained by TAA from $T_j(v_n)$ under φ^* , $\overline{\varphi}_{1,p}(T_n^{d^*}(w) - \{w\}) \cap S = \emptyset$ implies $\varphi_{1,p}(T_n^{d^*}(w) - T_j(v_n)) \cap S = \emptyset$.

We now verify the claim inductively for general $\varphi_{1,i}$ by assume it holds for $\varphi_{1,i+1}$. Clearly 5* holds because $\varphi_{1,i} = \varphi_{1,i+1}/P_w(\delta_{1_i}, \gamma_{1_i}, \varphi_{1,i+1})$, $S_{1_i} \subset S$ and 5* holds for $\varphi_{1,i+1}$. By 4* for $\varphi_{1,i+1}$ and Claim 2.3.2, all vertices in $T_{1_i} - v_{1_i}$ are contained in $(\delta_{1_i}, \gamma_{1_i})$ cycles under $\varphi_{1,i+1}$. Therefore $P_w(\delta_{1_i}, \gamma_{1_i}, \varphi_{1,i+1}) \cap T_{1_i} = \{v_{1_i}\}$ or \emptyset . Hence 1* holds by 1* for $\varphi_{1,i+1}$. Since $T_j \subset T_{1_i}$ and $P_w(\delta_{1_i}, \gamma_{1_i}, \varphi_{1,i+1}) \cap T_{1_i} = \{v_{1_i}\}$ or \emptyset , $\overline{\varphi}_{1,i}(v) = \overline{\varphi}^*(v)$ for all $v \in T_j$ and $\varphi_{1,i}(f) = \varphi^*(f)$ for all $f \in T_j$. Since 5* holds for $\varphi_{1,i+1}$ and $P_w(\delta_{1_i}, \gamma_{1_i}, \varphi_{1,i+1}) \cap T_{1_i} = \{v_{1_i}\}$ or \emptyset , the other end of $P_w(\delta_{1_i}, \gamma_{1_i}, \varphi_{1,i+1})$ is not in $T_n^{d^*}(w)$ and no edge in $T_n^{d^*}(w)$ is contained in $P_w(\delta_{1_i}, \gamma_{1_i}, \varphi_{1,i+1})$. Thus by 2* for $\varphi_{1,i+1}$, $\overline{\varphi}_{1,i}(v) = \overline{\varphi}^*(v)$ for all $v \in T_n^{d^*}(w) - w$ and $\varphi_{1,i}(f) = \varphi^*(f)$ for all $f \in T_n^{d^*}(w)$. Therefore, 2* holds for $\varphi_{1,i}$. Additionally 3* for $\varphi_{1,i}$ implies 3* for $\varphi_{1,i}$ because $\varphi_{1,i} = \varphi_{1,i+1}/P_w(\delta_{1_i}, \gamma_{1_i}, \varphi_{1,i+1})$. Finally by Claim 2.3.2, every $(\delta_{1_s}, \gamma_{1_s})$ -cycle intersecting $T_{1_s} - v_{1_s}$ with $0 < s < i$ under φ_n is contained in $G[V(T_{1_{s+1}})] \subset G[V(T_{1_i})]$. Since $P_w(\delta_{1_i}, \gamma_{1_i}, \varphi_{1,i+1}) \cap T_{1_i} = \{v_{1_i}\}$ or \emptyset and 4* for $\varphi_{1,i+1}$, every $(\delta_{1_s}, \gamma_{1_s})$ -cycle intersecting $T_{1_s} - v_{1_s}$ under φ_n is colored the same in φ_{1_i} and $\varphi_{1,i+1}$. Thus 4* holds for $\varphi_{1,i}$.

By letting $\varphi_{1,1} = \varphi^2$, we have as desired. \square

Now under $\overline{\varphi}^2$, we consider $T^2 = \overline{T_j(v_n)}$. By Claim 2.3.4 (1), φ^2 is $(T_{v_n} - v_n, D_{j-1}, \varphi_{j-1})$ -stable and therefore, it is also $(T_{j-1}, D_{j-1}, \varphi_{j-1})$ -stable. By (A5), T^2 is an ETT satisfying MP with $j-1 < n$ rungs and ladder $T_0 \subset T_1 \subset \dots \subset T_{j-1} \subset T^2$. Therefore it is elementary by (A1). Note that we have $\gamma_{1_1} \in \overline{\varphi}^2(w)$ in Claim 2.3.4 (3). By Claim 2.3.3 (2) and the uniqueness at supporting vertices, $\overline{\varphi}^*(v_n) = \overline{\varphi}_n(v_n)$ and $\overline{\varphi}_{j-1}(v_n) \subset \overline{\varphi}_n(v_n) \cup \{\gamma_{1_1}\}$. Since $V(T_j) = V(\overline{T_j(v_n)})$ under φ_{j-1} , Claim 2.3.3 (2) and Claim 2.3.4 (2) ensure that $V(T_j) \cup V(T_n^{d^*}(w)) \subset V(T^2)$. However, this contradicts the maximality of $|V(T_j)|$ in MP because $w \notin T_j$.

Case 2. $\overline{\varphi}^*(T_n^{d^*} - T_j(v_n)) \cap (\cup_{i \in L_1} S_i) = \emptyset$.

Recall that $L_1 = \{1_1, 1_2, \dots, 1_{p_1}\}$. For convenience, we let $p_1 = l$. Then $1_l = n$. Denote $\cup_{i \in L_1} S_i$ by S' . We first prove the following claim.

Claim 2.3.5. $\overline{\varphi}^*(T_n^{d^*}) \cap S' = \{\delta_n\}$, $\varphi^*(T_n^{d^*} - T_j(v_n)) \cap S' = \{\delta_n\}$, $\delta_n \notin \overline{\varphi}^*(T_j - T_j(v_n))$ and $\delta_n \notin \varphi^*(T_j - T_j(v_n))$.

Since $T_n^{d^*}$ is obtained from $T_j(v_n)$ by TAA under φ^* , we have $\varphi^*(T_n^{d^*} - T_j(v_n)) \cap S' = \{\delta_n\}$ if $\overline{\varphi}^*(T_n^{d^*}) \cap S' = \{\delta_n\}$. Note that $T_n^{d^*}$ is elementary, we have that $\delta_n \notin \overline{\varphi}^*(T_j(v_n) - v_n)$. By the uniqueness at supporting vertices and Claim 2.3.1, we see that $\delta_n \notin \overline{\varphi}_{j-1}(v_n)$. Therefore if $\delta_n \notin \overline{\varphi}^*(T_j - T_j(v_n))$, then (2) in Claim 2.3.3, $\delta_n \notin \overline{\varphi}_{j-1}(v_n)$ and the fact $\delta_n \notin \overline{\varphi}^*(T_j(v_n) - v_n)$ imply $\delta_n \notin \overline{\varphi}_{j-1}(T_j)$, which in turn gives $\delta_n \notin \varphi^*(T_j - T_j(v_n))$ because T_j is obtained from $T_j(v_n)$ by TAA and edges in $E(T_j)$ are colored the same in φ^* and φ_{j-1} . Hence we only need to prove $\overline{\varphi}^*(T_n^{d^*}) \cap S' = \{\delta_n\}$ and $\delta_n \notin \overline{\varphi}^*(T_j - T_j(v_n))$. Since $T_n^{d^*}$ is elementary and by the case assumption in Case 2, we have $\overline{\varphi}^*(T_n^{d^*} - T_j(v_n)) \cap S' = \emptyset$ and $\delta_n \notin \overline{\varphi}^*(T_j(v_n) - v_n)$. Suppose there exist $\gamma_{1_s} \in \overline{\varphi}^*(T_j(v_n) - v_n)$ with $\gamma_{1_s} \in \overline{\varphi}^*(v)$ and $0 < s \leq l$. Since φ_n is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$ -stable by (A4), $\gamma_{1_s} \in \overline{\varphi}_n(v)$ by Claim 2.3.3 (1) and the assumption of $v \neq v_n$. On the other hand, since $\gamma_{1_s} \in \overline{\varphi}_{1_s-1}(v_n)$ and T_{1_s} is elementary under φ_{1_s-1} by (A1), $\gamma_{1_s} \notin \overline{\varphi}_{1_s-1}(v)$ and the edge f colored by γ_{1_s} incident to v is contained in $E(G[V(T_{1_s})])$ under φ_{1_s-1} . Thus $\varphi_n(f) = \gamma_{1_s}$ by Claim 2.2.2, where we have a contradiction with $\gamma_{1_s} \in$

$\bar{\varphi}_n(v)$. Finally by the uniqueness at supporting vertex, no color in S' except δ_n appears in $\bar{\varphi}^*(v_n) = \bar{\varphi}_n^*(v_n)$. Thus we have $\bar{\varphi}^*(T_n^{d*}) \cap S' = \{\delta_n\}$. Now we suppose $\delta_n \in \bar{\varphi}^*(T_j - T_j(v_n))$. Then by Claim 2.3.3 (2), $\delta_n \in \bar{\varphi}_{j-1}(T_j - T_j(v_n))$. Note that T_j is an ETT satisfying MP under φ_{j-1} , it is elementary under φ_{j-1} by (A1). Then $\delta_n \notin \bar{\varphi}_{j-1}(v_n)$ and the edge f incident to v_n colored by δ_n under φ_{j-1} is contained in $E(G[V(T_j)])$. By Claim 2.2.2, $\varphi_n(f) = \varphi_{j-1}(f) = \delta_n$, a contradiction with $\delta_n \in \bar{\varphi}_n(v_n)$. Thus we have $\delta_n \notin \bar{\varphi}^*(T_j - T_j(v_n))$ as desired. \square

Note that no boundary edge colored by δ_n is incident to vertices after v_n along \prec_ℓ in T_n under φ_n and $X_{\gamma_n, \varphi_n}(T_n) = \{v_n\}$. Thus no boundary edge colored by δ_n is incident to vertices after v_n along \prec_ℓ in T_n under φ_n^* and $X_{\gamma_n, \varphi_n^*}(T_n) = \{v_n\}$, because φ_n^* is (T_n, D_n, φ_n) -stable. Then by Claim 2.3.3 (3), no boundary edge colored by δ_n is incident to vertices after v_n along \prec_ℓ in T_n under φ^* and $X_{\gamma_n, \varphi^*}(T_n) = \{v_n\}$. Because the vertices in T_n before v_n along \prec_ℓ are all contained in T_n^{d*} , $\delta_n \in \bar{\varphi}^*(v_n)$ and T_n^{d*} is closed under φ^* , δ_n is closed in $T_n \vee T_n^{d*}$. Note that Claim 2.3.3 (3) implies $P_{v_n}(\gamma_n, \delta_n, \varphi^*) = P_{v_n}(\gamma_n, \delta_n, \varphi_n^*)$. Recall that we assumed $P_{v_n}(\gamma_n, \delta_n, \varphi_n^*) \cap T_n \neq \{v_n\}$, so v_n can not be the $T_n \vee T_n^{d*}$ -exit of $P_{v_n}(\gamma_n, \delta_n, \varphi_n^*)$. Since $X_{\gamma_n, \varphi_n^*}(T_n) = \{v_n\}$ and δ_n is closed in $T_n \vee T_n^{d*}$ under φ^* , the $T_n \vee T_n^{d*}$ -exit of $P_{v_n}(\gamma_n, \delta_n, \varphi_n^*) = P_{v_n}(\gamma_n, \delta_n, \varphi_n^*)$ is in $T_n^{d*} - T_n$. Let its exit be w and $\beta \in \bar{\varphi}^*(w)$. Recall that T_n^{d*} is elementary under φ^* , $\beta \neq \delta_n$ because $\delta_n \in \bar{\varphi}^*(v_n)$. Then $\beta \notin S'$ by Claim 2.3.5. We consider the coloring $\varphi^{2*} = \varphi^*/(G - T_n^{d*}, \beta, \delta_n)$. Note that both β and δ_n are closed in T_n^{d*} . Then φ^{2*} is $(T_n^{d*}, D_{j-1}, \varphi^*)$ -stable and, therefore it is $(T_j(v_n) - v_n, D_{j-1}, \varphi^*)$ -stable. Since φ^* is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$ -stable, φ^{2*} is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$ -stable and therefore it is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$ -stable. Moreover, $\beta \notin \bar{\varphi}^{2*}(T_j(v_n)) = \bar{\varphi}^*(T_j(v_n))$ because $\beta \in \bar{\varphi}^*(w)$ and T_n^{d*} is elementary under φ^* . Note that $\beta \in \bar{\varphi}^*(T_j - T_{v_n})$ might happen. Our goal is to make γ_{1_1} missing in w if $\beta \notin \bar{\varphi}^*(T_j)$ and to make δ_n a missing color in $\bar{\varphi}^*(T_j - T_{v_n})$ otherwise. In the first possibility we have a contradiction with the maximality of $|V(T_j)|$ and the second one contradicts (A1) for ETTs with less than n rungs. Similarly as earlier, we have the following claim.

Claim 2.3.6. *There exist a coloring φ^3 satisfying the followings:*

- (1) φ^3 is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$ -stable.

(2) If $\beta \notin \overline{\varphi}^*(T_j)$, we have $\overline{\varphi}^3(v) = \overline{\varphi}^*(v)$ for all $v \in T_j \cup T_n^{d^*}(w) - w$ and $\varphi^3(f) = \varphi^*(f)$ for all $f \in T_j \cup T_n^{d^*}(w)$. If $\beta \in \overline{\varphi}^*(T_j - T_j(v_n))$, then $\overline{\varphi}^3(v) = \overline{\varphi}^*(v)$ for all $v \in T_j(v(\beta)) \cup T_n^{d^*}(w) - w - v(\beta)$ and $\varphi^3(f) = \varphi^*(f)$ for all $f \in T_j(v(\beta)) \cup T_n^{d^*}(w)$.

(3) $\gamma_{1_i} \in \overline{\varphi}^3(w)$. If $\beta \in \overline{\varphi}^*(T_j - T_j(v_n))$, then $\delta_n \in \overline{\varphi}^3(v(\beta))$.

We define $\varphi_{1,i}$ with $0 < i < l$ inductively from the largest index where $\varphi_{1,l} = \varphi^{2^*}/P_w(\beta, \gamma_{1_l}, \varphi^{2^*})$ and $\varphi_{1,i} = \varphi_{1,i+1}/P_w(\delta_{1_i}, \gamma_{1_i}, \varphi_{1,i+1})$. Note that $\gamma_{1_l} = \delta_{1_{l-1}}$. We claim the following holds for each $\varphi_{1,i}$ with $0 < i \leq l$:

1* $\varphi_{1,i}$ is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$ -stable.

2* If $\beta \notin \overline{\varphi}^*(T_j - T_j(v_n))$, we have $\overline{\varphi}_{1,i}(v) = \overline{\varphi}^*(v)$ for all $v \in T_j \cup T_n^{d^*}(w) - w$ and $\varphi_{1,i}(f) = \varphi^*(f)$ for all $f \in T_j \cup T_n^{d^*}(w)$. If $\beta \in \overline{\varphi}^*(T_j - T_j(v_n))$, then $\overline{\varphi}_{1,i}(v) = \overline{\varphi}^*(v)$ for all $v \in T_j(v(\beta)) \cup T_n^{d^*}(w) - w - v(\beta)$ and $\varphi_{1,i}(f) = \varphi^*(f)$ for all $f \in T_j(v(\beta)) \cup T_n^{d^*}(w)$.

3* $\gamma_{1_i} \in \overline{\varphi}_{1,i}(w)$. If $\beta \in \overline{\varphi}^*(T_j - T_j(v_n))$, then $\delta_n \in \overline{\varphi}_{1,i}(v(\beta))$.

4* $\varphi_{1,i}(f) = \varphi_n(f)$ for $f \in Q$ where Q is any $(\delta_{1_s}, \gamma_{1_s})$ -cycle intersecting $T_{1_s} - v_{1_s}$ under φ_n with $0 < s < i$.

5* We have $\overline{\varphi}_{1,i}(T_n^{d^*} - w) \cap S' = \{\delta_n\}$ and $\varphi_{1,i}(T_n^{d^*} - T_j(v_n)) \cap S' = \{\delta_n\}$.

Again we will prove all five statements by induction. We first prove all statements for $\varphi_{1,l}$. Since $w \in T_n^{d^*} - T_n$ is the $T_n \vee T_n^{d^*}$ -exit of $P_{v_n}(\delta_n, \gamma_n, \varphi^*)$ under φ^* , $P_w(\beta, \gamma_{1_l}, \varphi^{2^*}) \cap (T_n \cup T_n^{d^*}) = \{w\}$. Recall that φ^{2^*} is $(T_j(v_n) - v_n, D_{j-1}, \varphi_{j-1})$ -stable, we have 1* holds for $\varphi_{1,l} = \varphi^{2^*}/P_w(\beta, \gamma_{1_l}, \varphi^{2^*})$. Let $v(\beta, T_j) = v$ if $\beta \in \overline{\varphi}^*(T_j - v_n)$. Since $T_n^{d^*}$ is elementary under φ^* and $\beta \in \overline{\varphi}^*(w)$, $v \in T_j - T_n^{d^*}$. Therefore 3* holds because $\varphi^{2^*} = \varphi^*/(G - T_n^{d^*}, \beta, \delta)$ and $\varphi_{1,l} = \varphi^{2^*}/P_w(\beta, \gamma_{1_l}, \varphi^{2^*})$. By Claim 2.3.3 (4), $\beta \notin S'$ and By Claim 2.3.2, all $(\gamma_{1_s}, \delta_{1_s})$ cycles intersecting $T_{1_s} - v_{1_s}$ under φ_n are contained in $G[V(T_n)]$ for $0 < s < l$. Since $\beta \notin S'$ and $\delta_n \notin S_{1_s}$ for $0 < s < l$, all $(\gamma_{1_s}, \delta_{1_s})$ cycles intersecting $T_{1_s} - v_{1_s}$ under φ_n are colored the same under φ_n and φ^{2^*} , so they are still contained in $G[V(T_n)]$. Since $P_w(\beta, \gamma_{1_l}, \varphi^{2^*}) \cap (T_n \cup T_n^{d^*}) = \{w\}$ and $w \notin T_n$, 4* holds for $\varphi_{1,l}$. Because $\varphi^{2^*} = \varphi^*/(G - T_n^{d^*}, \beta, \delta)$, we have $\overline{\varphi}^{2^*}(T_n^{d^*} - w) \cap S' =$

$\{\delta_n\}$ and $\varphi^{2*}(T_n^{d*} - T_j(v_n)) \cap S' = \{\delta_n\}$ by Claim ???. Since $P_w(\beta, \gamma_{1_i}, \varphi^{2*}) \cap (T_n \cup T_n^{d*}) = \{w\}$, we have $\bar{\varphi}^{1_i}(T_n^{d*} - w) \cap S' = \{\delta_n\}$ and $\varphi^{1_i}(T_n^{d*} - T_j(v_n)) \cap S' = \{\delta_n\}$. Thus 5* holds. Finally we will prove 2*. Recall that $\beta \notin \bar{\varphi}^*(T_j(v_n))$ because $\beta \in \bar{\varphi}^*(w)$ and T_n^{d*} is elementary under φ^* . Thus $\beta \notin \bar{\varphi}_n(v_n)$ by Claim 2.3.3 (2). By the uniqueness at supporting vertices and the fact $\beta \notin S'$, $\beta \notin \bar{\varphi}_{j-1}(v_n)$. Thus by applying Claim 2.3.3 (2) again, we have $\beta \notin \bar{\varphi}_{j-1}(T_j(v_n))$. We first assume $\beta \notin \bar{\varphi}^*(T_j - T_j(v_n))$. Then $\beta \notin \bar{\varphi}_{j-1}(T_j)$ by Claim 2.3.3 (2) and the fact $\beta \notin \bar{\varphi}_{j-1}(v_n)$. Thus in this case $\beta \notin \varphi_{j-1}(T_j - T_j(v_n))$ because T_j is obtained from $T_j(v_n)$ by TAA under φ_{j-1} . By Claim 2.3.3 (2), $\beta \notin \varphi^*(T_j - T_j(v_n))$. Moreover, $\delta_n \notin \bar{\varphi}^*(T_j - T_j(v_n))$ and $\delta_n \notin \varphi^*(T_j - T_j(v_n))$ by Claim 2.3.5. Since $\varphi^{2*} = \varphi^*/(G - T_n^{d*}, \beta, \delta_n)$, $\beta \notin \varphi^*(T_j - T_j(v_n))$, $\beta \notin \bar{\varphi}^*(T_j - T_j(v_n))$, $\delta_n \notin \bar{\varphi}^*(T_j - T_j(v_n))$ and $\delta_n \notin \varphi^*(T_j - T_j(v_n))$, we have $\bar{\varphi}^{2*}(v) = \bar{\varphi}^*(v)$ for all $v \in T_j \cup T_n^{d*}(w) - w$ and $\varphi^{2*}(f) = \varphi^*(f)$ for all $f \in T_j \cup T_n^{d*}(w)$. Since $\varphi_{1,l} = \varphi^{2*}/P_w(\beta, \gamma_{1_i}, \varphi^{2*})$ and $P_w(\beta, \gamma_{1_i}, \varphi^{2*}) \cap (T_n \cup T_n^{d*}) = \{w\}$, we have $\bar{\varphi}_{1,l}(v) = \bar{\varphi}^*(v)$ for all $v \in T_j \cup T_n^{d*}(w) - w$ and $\varphi_{1,l}(f) = \varphi^*(f)$ for all $f \in T_j \cup T_n^{d*}(w)$. We then assume $\beta \in \bar{\varphi}^*(v)$ where $v \in T_j - T_j(v_n)$. Recall that $\beta \notin \bar{\varphi}_{j-1}(T_j(v_n))$. By Claim 2.3.3, $\beta \in \bar{\varphi}_{j-1}(v)$. Since T_j is elementary under φ_{j-1} by (A1), $\beta \notin \bar{\varphi}_{j-1}(T_j(v) - v)$. Since T_j is obtained from $T_j(v_n)$ by TAA under φ_{j-1} , we have $\beta \notin \varphi_{j-1}(T_j(v))$, and therefore $\beta \notin \varphi^*(T_j(v)) - T_j(v_n)$. Similarly as in the case $\beta \notin \bar{\varphi}^*(T_j - T_j(v_n))$, we have $\delta_n \notin \bar{\varphi}^*(T_j - T_j(v_n))$ and $\delta_n \notin \varphi^*(T_j - T_j(v_n))$ by Claim 2.3.5. Again since $\varphi^{2*} = \varphi^*/(G - T_n^{d*}, \beta, \delta_n)$, we have $\bar{\varphi}^{2*}(v) = \bar{\varphi}^*(v)$ for all $v \in T_j(v) \cup T_n^{d*}(w) - w - v$ and $\varphi^{2*}(f) = \varphi^*(f)$ for all $f \in T_j(v) \cup T_n^{d*}(w)$. Finally since $\varphi_{1,l} = \varphi^{2*}/P_w(\beta, \gamma_{1_i}, \varphi^{2*})$ and $P_w(\beta, \gamma_{1_i}, \varphi^{2*}) \cap (T_n \cup T_n^{d*}) = \{w\}$, we have $\bar{\varphi}_{1,l}(v) = \bar{\varphi}^*(v)$ for all $v \in T_j(v) \cup T_n^{d*}(w) - w - v$ and $\varphi_{1,l}(f) = \varphi^*(f)$ for all $f \in T_j(v) \cup T_n^{d*}(w)$, and, 3* holds for both cases.

We now verify the claim inductively for general $\varphi_{1,i}$ by assuming it holds for $\varphi_{1,i+1}$. By 4* for $\varphi_{1,i+1}$ and Claim 2.3.2, all vertices in $T_{1_i} - v_{1_i}$ are contained in $(\delta_{1_i}, \gamma_{1_i})$ -cycles under $\varphi_{1,i+1}$. Therefore $P_w(\delta_{1_i}, \gamma_{1_i}, \varphi_{1,i+1}) \cap T_{1_i} = \{v_n\}$ or \emptyset . Hence 1* holds by 1* of $\varphi_{1,i+1}$. Since $T_j \subset T_{1_i}$, $P_w(\delta_{1_i}, \gamma_{1_i}, \varphi_{1,i+1}) \cap T_{1_i} = \{v_n\}$ or \emptyset and $\delta_{1_i}, \gamma_{1_i} \in S' - \{\delta_n\}$, we have 2* holds for $\varphi_{1,i}$ by 2* and 5* for $\varphi_{1,i+1}$. Because 5* holds for $\varphi_{1,i+1}$ and $P_w(\delta_{1_i}, \gamma_{1_i}, \varphi_{1,i+1}) \cap T_{1_i} = \{v_n\}$ or \emptyset , the other end of $P_w(\delta_{1_i}, \gamma_{1_i}, \varphi_{1,i+1})$ is not in T_n^{d*} and no edge in T_n^{d*} is in $E(T_n^{d*})$ under $\varphi_{1,i+1}$.

Thus 5* holds for $\varphi_{1,i}$. Additionally 3* for $\varphi_{1,i+1}$ implies 3* for $\varphi_{l,i}$. Finally by Claim 2.3.2, all $(\delta_{1_s}, \gamma_{1_s})$ -cycles intersecting $T_{1_s} - v_{1_s}$ with $0 < s < i$ under φ_n are contained in $G[V(T_{1_{s+1}})]$ and therefore, they are contained in $G[V(T_{1_i})]$. Since $P_w(\delta_{1_i}, \gamma_{1_i}, \varphi_{1,i+1}) \cap T_{1_i} = \{v_n\}$ or \emptyset , Claim 2.3.2 and 4* for $\varphi_{1,i+1}$ imply any $(\delta_{1_s}, \gamma_{1_s})$ -cycle intersecting $T_{1_s} - v_{1_s}$ under φ_n is colored the same in φ_{1_i} and $\varphi_{1,i+1}$. Thus 4* holds for $\varphi_{1,i}$. By letting $\varphi_{l,1} = \varphi^3$, we have as desired. \square

Now we consider φ^3 . By (1) of Claim 2.3.6 and (A5), $T^3 = \overline{T_j(v_n)}$ under φ^3 is still and ETT satisfying MP. Moreover, (A1) implies T^3 is elementary. Note that $\gamma_{1_1} \in \overline{\varphi^3}(w)$ by Claim 2.3.6 (3) and $\overline{\varphi}_{j-1}(v_n) \subset \overline{\varphi}_n(v_n) \cup \{\gamma_{1_1}\}$ by the uniqueness at supporting vertices. Now if $\beta \notin \overline{\varphi^*}(T_j)$, we have $V(T_j \cup T_n^{d^*}(w)) \subset V(T^3)$ by Claim 2.3.6 (2) and Claim 2.3.3 (2). However, $w \notin T_j$, a contradiction to MP. If $\beta \in \overline{\varphi^*}(T_j)$, we have $V(T_j(v) \cup T_n^{d^*}(w)) \subset V(T^3)$ by Claim 2.3.6 (2) and Claim 2.3.3 (2), and we have $\delta_n \in \overline{\varphi^3}(v_n) \cap \overline{\varphi^3}(v)$ with v being the vertex missing β in T_j under φ^* by Claim 2.3.6 (3). Thus we reach a contradiction to the elementariness of T^3 under φ^3 . \square

Next Proposition is an inductive proof of (A3).

Proposition 3. *Let n be a positive integer. If (A1), (A3) and (A5) hold for all ETTs satisfying MP with at most $n - 1$ rungs and (A2), (A4) hold for all ETTs with at most n rungs, then (A3) holds for all ETTs satisfying MP with n rungs.*

Proof. Let T be an ETT with ladder $T_0 \subset T_1 \subset \dots \subset T_n \subset T$ satisfying MP with coloring sequence $(\varphi_0, \varphi_1, \dots, \varphi_n)$. Assume on the contrary that there is a vertex $v \in T_n$ and a defective color δ for T_n under a (T_n, D_n, φ_n) -stable coloring φ_n^* where v is not the smallest vertex under \prec_ℓ inside $X_{\varphi_n^*, \delta}(T_n)$ and there exist a supporting or extension vertex v_i such that $i \geq m(v)$ and $v_i \prec_\ell v$ with $v \neq v_i$.

We first consider the case $\Theta_n = \text{PE}$. Note that $\gamma_n \in S_n$. Since γ_n is not defective for T_n under φ_n , it can not be defective for T_n under the (T_n, D_n, φ_n) -stable coloring. Thus $\delta \neq \gamma_n$. Note that if $\delta = \delta_n$, then $v \prec_\ell v_n$. By Proposition 2, we have $P_{v_n}(\gamma_n, \delta_n, \varphi_n^*) \cap T_n = \{v_n\}$. Recall that $\varphi_{n-1}'' = \varphi_n / P_{v_n}(\gamma_n, \delta_n, \varphi_n)$, we have that

$\varphi_{n-1}^* = \varphi_n^*/P_{v_n}(\gamma_n, \delta_n, \varphi_n^*)$ is $(T_n, D_{n-1}, \varphi_{n-1}'')$ -stable and therefore it is $(T_n, D_{n-1}, \varphi_{n-1})$ -stable. Thus φ_{n-1}^* is also $(T_{n-1}, D_{n-1}, \varphi_{n-1})$ -stable. If $i < n$, we have $v \in T_{n-1}$ because $m(v) \leq i < n$. Since v is not the smallest vertex under \prec_ℓ inside $X_{\varphi_n^*, \delta}(T_n)$, δ is still defective for T_{n-1} under φ_{n-1}^* . Therefore we have a contradiction with (A3) in T_{n-1} with $n(T_n) = n - 1 < n$ by the same δ , v and v_i . If $i = n$, then we must have $\delta \neq \delta_n$. Recall that v_n is “maximum” by our choice of v_n in the definition of STT. However, since $v_n \prec_\ell v$ and φ_{n-1}^* is $(T_n, D_{n-1}, \varphi_{n-1})$ -stable, we have a contradiction to the choice of v_n .

Now we suppose $\Theta_n = \text{RE}$ or SE . Since we either have a SE or RE extension from T_n to T , φ_n^* is also $(T_n, D_{n-1}, \varphi_{n-1})$ -stable and hence is $(T_{n-1}, D_{n-1}, \varphi_{n-1})$ -stable. If $i < n$, then we must have $m(v) < n$. Therefore δ is defective for T_{n-1} under φ_n^* because $v \in T_{n-1}$. However, this is a contradiction to (A3) with $n - 1 < n$ rungs. If $i = n$, then we must have $\delta \neq \delta_n$. Again we have a contradiction to the choice of v_n since $v_n \prec_\ell v$ and φ_n^* is $(T_n, D_{n-1}, \varphi_{n-1})$ -stable, .

□

Proposition 4. *If (A2) and (A4) hold for all ETTs satisfying MP with at most n rungs and (A1), (A3), (A5) hold for all ETTs satisfying MP with at most $n - 1$ rungs, then (A5) holds for every ETT T satisfying MP with n rungs.*

Proof. Let T be an ETT with ladder $T_0 \subset T_1 \subset \dots \subset T_n \subset T$ satisfying MP with coloring sequence $(\varphi_0, \varphi_1, \dots, \varphi_n)$. Recall that (A5) claims every (T_n, D_n, φ_n) -stable coloring φ_n^* is a $\varphi_n \bmod T$ coloring and every corresponding ETT T^* obtained from T_n under φ_n^* also satisfies MP.

First we consider the case $\Theta_n = \text{PE}$. Since φ_n^* is (T_n, D_n, φ_n) -stable, $P_{v_n}(\gamma_n, \delta_n, \varphi_n^*) \cap T_n = \{v_n\}$ by (A2). Let $\varphi_{n-1}^* = \varphi_n^*/P_{v_n}(\gamma_n, \delta_n, \varphi_n^*)$. Recall that in the definition of the PE extension, there is a $(T_n, D_{n-1}, \varphi_{n-1})$ -stabling coloring φ_{n-1}'' such that $\varphi_n = \varphi_{n-1}''/P_{v_n}(\delta_n, \gamma_n, \varphi_{n-1}'')$. Then, φ_{n-1}^* is $(T_n, D_{n-1}, \varphi_{n-1}'')$ -stable and therefore it is $(T_n, D_{n-1}, \varphi_{n-1})$ -stable and $(T_{n-1}, D_{n-1}, \varphi_{n-1})$ -stable. Since T_n is an ETT satisfying MP under φ_n with $n(T_n) = n - 1$, by (A5), there is an ETT T' satisfying MP with lad-

der $T_0 \subset T_1 \subset \dots \subset T_{n-1} \subset T'$ and coloring sequence $(\varphi_0^*, \varphi_1^*, \dots, \varphi_{n-1}^*)$ under the $\varphi_{n-1} \bmod T_n$ coloring φ_{n-1}^* . Since $(T_n, D_{n-1}, \varphi_{n-1})$ -stable, we may assume $T' = T_n$. Since $P_{v_n}(\delta_n, \gamma_n, \varphi_{n-1}^*) = P_{v_n}(\delta_n, \gamma_n, \varphi_n^*)$, it is an exit path at v_n . Moreover, φ_n^* is (T_n, D_n, φ_n) -stable. Therefore by the choice of v_n along \prec_ℓ , condition P is satisfied under coloring φ_n^* . So, we can perform a PE extension on T_n with the same exit v_n and same S_n to get an ETT T^* under φ_n^* . Moreover, since φ_n^* is (T_n, D_n, φ_n) -stable, φ_n^* is a $\varphi_n \bmod T$ coloring. By Lemma 2.2.1, T^* also satisfies MP.

We then consider the case $\Theta_n = \text{SE}$. In this case, $\varphi_n = \varphi'_{n-1}$, where φ' is $(T_n, D_{n-1}, \varphi_{n-1})$ -stable. Thus, φ_n^* is both $(T_n, D_{n-1}, \varphi_{n-1})$ -stable and $(T_{n-1}, D_{n-1}, \varphi_{n-1})$ -stable. In this case we let $\varphi_{n-1}^* = \varphi_n^*$. Applying (A5) to φ_{n-1}^* and T_n , we see that T_n is an ETT satisfying MP under the $\varphi_{n-1} \bmod T_n$ coloring φ_{n-1}^* . Since φ_n^* is (T_n, D_n, φ_n) -stable, δ_n is still defective color of T_n under φ_n^* and condition S is satisfied. Thus by the choice of v_n , f_n can be added to T_n as part of an SE extension under φ_n^* to get an ETT T^* . Similarly as before, we have that φ_n^* is a $\varphi_n \bmod T$ coloring. By Lemma 2.2.1, T^* also satisfies MP.

Finally we consider the case $\Theta_n = \text{RE}$. Again φ_n^* must be both $(T_n, D_{n-1}, \varphi_{n-1})$ -stable and $(T_{n-1}, D_{n-1}, \varphi_{n-1})$ -stable because $\varphi_{n-1} = \varphi_n$. Let $\varphi_{n-1}^* = \varphi_n^*$. Similarly by (A5), T_n is an ETT satisfying MP under the $\varphi_{n-1} \bmod T_n$ coloring φ_{n-1}^* because φ_n^* is both $(T_n, D_{n-1}, \varphi_{n-1})$ -stable and $(T_{n-1}, D_{n-1}, \varphi_{n-1})$ -stable. Now we check condition R. Let f_n be the connecting edge, Q be the (δ_h, γ_h) -cycle and P be the subpath of Q in $V(T_n)$ from T_h to f_n . Since φ_n^* is (T_n, D_n, φ_n) -stable, P is still a subpath of a (δ_h, γ_h) -chain. We only need to show that, under the coloring φ_n^* , P is still contained in a (δ_h, γ_h) -cycle. Let Q^* be the (δ_h, γ_h) -chain containing P . Since $S_h \subset \overline{\varphi}_n(T_n) \cup D_n$, φ_n^* is also (T_h, D_h, φ_h) -stable by Extension Rules. If Q^* is a cycle, we are done. So, we assume Q^* is a path. Note that P is contained in a (δ_h, γ_h) -cycle under $\varphi_n = \varphi_h$ and $P_{v_h}(\delta_h, \gamma_h)$ is a path, $v_h \notin V(P)$. Thus P contains a vertex v in T_h other than v_h . If $v_h \in Q^*$, then under φ_n^* , Q^* intersects T_h at v and v_h , giving a contradiction to (A2) because φ_n^* is (T_n, D_n, φ_n) -stable and (T_h, D_h, φ_h) -stable. Thus, $v_h \notin Q^*$. By (A2) again, $P_{v_h}(\delta_h, \gamma_h, \varphi_n^*)$ is an T_h -exit path of (δ_h, γ_h) . Since Q^* is a path and $P \subset Q^*$, Q^* and $P_{v_h}(\delta_h, \gamma_h, \varphi_n^*)$ are two (δ_h, γ_h) paths intersecting T_{h+1} . So, colors

α_h and γ_h are not interchangeable in T_{h+1} under the $(T_{h+1}, D_h, \varphi_h)$ -stable coloring φ_n^* , a contradiction to (A1) (2). Thus condition R is satisfied and we can extend T_n using f_n by RE extension to get a ETT T^* . Thus φ_n^* is indeed a $\varphi_n \bmod T$ coloring. By Lemma 2.2.1, T^* also satisfies MP. \square

Divided by extension types, we prove (A1) (1) in the next two sections.

2.3.1 PE extension:

Let $T_n^d = \overline{T_n(v_n)}$ under φ_n and let $m(v_n) = j$.

Lemma 2.3.1. *Let n be a positive integer. Suppose (A2), (A3), (A4) and (A5) hold for ETTs satisfying MP with at most n rungs and (A1) holds for ETTs satisfying MP with at most $n-1$ rungs. Let φ_n^* be a (T_n, D_n, φ_n) -stable coloring and $T_n^{d*} = \overline{T_n(v_n)}$ under φ_n^* . Then $T_n \vee T_n^{d*}$ is elementary.*

We call the sequence $T_n \vee T_n^{d*}$ in the Lemma above a **half closure** of T_n under φ_n^* . Thus $T_n \vee T_n^d$ is a half closure of T_n under φ_n .

Proof. Since T_n is an ETT with ladder $T_0 \subset T_1 \subset \dots \subset T_{n-1} \subset T_n$ under last coloring φ_{n-1} , it satisfies MP and it is elementary by (A1) because $n(T_n) = n-1 < n$. Since $\varphi_n = \varphi_{n-1}''/P_{v_n}(\delta_n, \gamma_n, \varphi_{n-1}'')$ and $\delta_n \notin \overline{\varphi_{n-1}''}(T_n)$, T_n is elementary under φ_n . Hence it is also elementary under φ_n^* . By (A4) and the fact that φ_n^* is (T_n, D_n, φ_n) -stable, we see that φ_n^* is $(T_j(v_n), D_{j-1}, \varphi_{j-1})$ -stable. Therefore T_n^{d*} is an ETT satisfying MP with ladder $T_0 \subset T_1 \subset \dots \subset T_{j-1} \subset T_n^{d*}$ and $j-1 < n$ rungs by (A5). By (A1), T_n^{d*} is elementary under φ_n^* . Suppose on the contrary that $T_n \vee T_n^{d*}$ is not elementary under φ_n^* . Now there must exist $\alpha \in \overline{\varphi_n^*}(u) \cap \overline{\varphi_n^*}(v)$ such that $u \in T_n - T_n^{d*}$ and $v \in T_n^{d*} - T_n$. Note that $\alpha \neq \delta_n$ because $\delta_n \notin \overline{\varphi_n^*}(v_n)$. Moreover, because $\gamma_n \in \overline{\varphi_{n-1}}(v_n)$ and T_n is elementary under φ_{n-1} , we have $\gamma_n \notin \overline{\varphi_n}(T_n)$ and $\gamma_n \notin \overline{\varphi_n^*}(T_n)$. Therefore $\alpha \neq \gamma_n$. As a consequence, $\alpha \notin S_n$. Since A defective color must appear on the boundary for at least three times and by our choice of v_n , v_n must be added to T_n after both end vertices of e . Therefore $|\varphi_n^*(T_n \cap T_n^{d*})| \geq |\overline{\varphi_n^*}(T_j) - 1| \geq 4$. Hence we have a color θ such that $\theta \in \overline{\varphi_n^*}(T_n \cap T_n^{d*}) - \delta_n$.

Recall that $\gamma_n \notin \overline{\varphi}_n^*(T_n)$. Hence $\theta \neq \gamma_n$. Say $\theta \in \overline{\varphi}_n^*(w)$ for some $w \in T_n \cap T_n^{d*}$. Since φ_n^* is (T_n, D_n, φ_n) -stable and $\alpha, \theta \notin S_n$, edges colored by α and θ incident to $V(T_n)$ stayed with the same color between φ_n^* and φ_{n-1}'' . Because T_n is elementary and closed under φ_{n-1} , it is also closed and elementary under φ_{n-1}'' . Thus we must have $P_u(\alpha, \theta, \varphi_n^*) = P_u(\alpha, \theta, \varphi_{n-1}'') = P_w(\alpha, \theta, \varphi_{n-1}'') = P_w(\alpha, \theta, \varphi_n^*)$. However, since T_n^{d*} is also an elementary and closed ETT under φ_n^* , we must have $P_v(\alpha, \theta, \varphi_n^*) = P_w(\alpha, \theta, \varphi_n^*)$, a contradiction. \square

As an immediate consequence of Lemma 2.3.1, $T_n \vee T_n^d$ is elementary under φ_n . Clearly $T_n \vee T_n^d$ is an ETT satisfying MP under last coloring φ_n because it can be obtained from T_n by TAA under φ_n . By our choice of v_n , $b(\partial_{\varphi_n, \delta_n})(T_n) \subset T_n^d$. Since T_n^d is closed for δ_n under φ_n , $T_n \vee T_n^d$ is closed for δ_n under φ_n . Moreover, since T_n is a closed ETT under φ_{n-1} and T_n^{d*} is a closed ETT under φ_n , $T_n \vee T_n^d$ is closed for colors in $\overline{\varphi}_n(T_n \cap T_n^d)$ under φ_n .

Definition 8. We call a coloring φ_n^* is $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable if φ_n^* is both (T_n, D_n, φ_n) -stable and $(T_n^d, \emptyset, \varphi_n)$ -stable, i.e., the following holds:

- (1) $\varphi_n^*(f) = \varphi_n(f)$ for any edge f incident to $V(T_n)$ with $\varphi_n(f) \in \overline{\varphi}_n(T_n) \cup D_n$.
- (2) $\varphi_n^*(f) = \varphi_n(f)$ for any edge f incident to $V(T_n^d)$ with $\varphi_n(f) \in \overline{\varphi}_n(T_n^d)$.
- (3) $\overline{\varphi}_n^*(v) = \overline{\varphi}_n(v)$ for any $v \in T_n \cup T_n^d$.

Since colorings being (T, C, φ) -stable is an equivalent relation, colorings being $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable is also an equivalent relation. Moreover, we can immediately see a coloring φ_n^* being $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable implies φ_n^* is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$ -stable by (A4). Applying (A5), T_n^d is still an ETT satisfying MP under the $\varphi_{j-1} \bmod T_j$ coloring φ_n^* . Note that edges incident to $V(T_n^d - T_n)$ colored by colors in $D_n - \overline{\varphi}_n(T_n^d)$ may have their colors changed from φ_n to φ_n^* . Thus, every $(T_n \vee T_n^d, D_n, \varphi_n)$ -stable coloring is $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable but the converse is not true. In addition, $T_n \vee T_n^d$ is closed for colors in $\overline{\varphi}_n^*(T_n \cap T_n^d)$ under φ_n^* which $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable, because $T_n \vee T_n^d$ is closed for colors in $\overline{\varphi}_n(T_n \cap T_n^d)$ under φ_n .

Lemma 2.3.2. Suppose (A2), (A3), (A4) and (A5) hold for ETTs satisfying MP with at most n rungs and (A1) holds for ETTs satisfying MP with at most $n - 1$ rungs. Let T, T_n

and T_n^d be defined as above, and φ_n^* be a $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable coloring. Then the followings hold:

- (1) α and β are T_n^d -interchangeable under φ_n^* if $\alpha \in \overline{\varphi_n^*}(T_n^d)$.
- (2) α and β are T_n -interchangeable under φ_n^* if $\alpha \in \overline{\varphi_n^*}(T_n)$.
- (3) α and β are $T_n \vee T_n^d$ -interchangeable if α is closed in $T_n \vee T_n^d$ under φ_n^* .
- (4) α and β are $T_n \vee T_n^d$ -interchangeable under φ_n^* if $\alpha \in \overline{\varphi_n^*}(T_n)$ and $\beta \in \overline{\varphi_n^*}(T_n^d)$.

Proof. We first prove (1). Because φ_n^* is $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable, it is also (T_n, D_n, φ_n) -stable. Hence φ_n^* is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$ -stable by (A4). Note that (2) and (3) in the definition of $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable imply edges in $E(T_n^d)$ are colored the same under both φ_n and φ_n^* and $\overline{\varphi_n^*}(v) = \overline{\varphi_n}(v)$ for any $v \in T_n^d$. Thus T_n^d can be obtained from $T_n(v_n)$ by TAA under φ_n^* . Therefore by (A5), T_n^d must be an ETT satisfying MP with ladder $T_0 \subset T_1 \subset \dots \subset T_{j-1} \subset T_n^d$ under the $\varphi_{j-1} \bmod T_j$ coloring φ_n^* with $j-1 < n$ rungs. Recall that $T_n^d = \overline{T_n(v_n)}$ under φ_n , it is closed under φ_n . Since φ_n^* is $(T_n^d, \emptyset, \varphi_n)$ -stable, T_n^d is still closed under φ_n^* . Since φ_n^* is clearly $(T_n^d, D_{j-1}, \varphi_n^*)$ -stable, we can apply (A1) (2) for T_n^d under last coloring φ_n^* , where we have as desired from CIMC.

Now we prove (2). Let $\alpha \in \overline{\varphi_n^*}(T_n)$ and β be an arbitrary color. Since $\gamma_n \notin \overline{\varphi_n}(T_n) = \overline{\varphi_n^*}(T_n)$, $\alpha \neq \gamma_n$. We first consider the case $\alpha, \beta \notin S_n = \{\gamma_n, \delta_n\}$. By (A2), $P_{v_n}(\gamma_n, \delta_n, \varphi_n^*) \cap T_n = \{v_n\}$ because φ_n^* is also (T_n, D_n, φ_n) -stable. If α is not interchangeable with β in T_n under φ_n^* , they are not interchangeable in T_n under $\varphi^{**} = \varphi_n^*/P_{v_n}(\gamma_n, \delta_n, \varphi_n^*)$ because $\alpha, \beta \notin S_n$. Note that φ^{**} is $(T_n, D_n, \varphi_{n-1}')$ -stable, it is $(T_n, D_{n-1}, \varphi_{n-1})$ -stable. Hence by (A1) (2) for T_n under φ_{n-1} , α is interchangeable with β in T_n under φ^{**} , a contradiction. We now assume $\beta = \gamma_n$ and $\alpha \neq \delta_n$. In this case $|X_{\varphi_n^*, \gamma_n}(T_n)| = 1$ and T_n is closed for α . Since T_n is elementary under φ_n^* , there is at most one (α, γ_n) -path intersecting T_n . Hence α is interchangeable with β in T_n under φ_n^* . The case $\alpha = \delta_n$ or $\beta = \delta_n$ will be proved in the proof of (3).

We then prove (3). Suppose on the contrary there exist two colors α and β such that α is closed in $T_n \vee T_n^d$ under φ^* and β is not interchangeable with α in $T_n \vee T_n^d$.

Note that all colors in $\overline{\varphi}^*(T_n \cap T_n^d)$ are closed in $T_n \vee T_n^d$. We claim that we may assume $\alpha \in \overline{\varphi}^*(T_n \cap T_n^d)$. Let $\alpha^* \in \overline{\varphi}^*(T_n \cap T_n^d)$. Because α, β are not $T_n \vee T_n^d$ -interchangeable and $T_n \vee T_n^d$ is elementary under φ_n^* , two (α, β) -paths intersecting $T_n \vee T_n^d$ have at least 2 ends not contained in $T_n \vee T_n^d$, which create at least two (α, β) exit paths for $T_n \vee T_n^d$ under φ_n^* . Note that both α and α^* are closed in $T_n \vee T_n^d$. Therefore under the $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable coloring $\varphi^* = \varphi_n^*/(G - T_n \vee T_n^d, \alpha, \alpha^*)$, there are two (α^*, β) exit paths for $T_n \vee T_n^d$. Hence α^*, β are not $T_n \vee T_n^d$ -interchangeable because $P_{v(\alpha^*)}(\alpha^*, \beta, \varphi^*)$ only contains at most one (α^*, β) exit path. By letting φ^* be φ_n^* and α^* be α , we have as claimed. Say $\alpha \in \overline{\varphi}_n^*(w)$ where $w \in T_n \cap T_n^d$. Since $P_w(\alpha, \beta, \varphi_n^*)$ is an (α, β) path intersecting T_n^d , $P_w(\alpha, \beta, \varphi_n^*)$ is the only (α, β) path intersecting T_n^d by (1). Since α is not interchangeable with β in $T_n \vee T_n^d$, there must exist an (α, β) -path P which intersects T_n but does not intersect T_n^d . Therefore α and β are not interchangeable in T_n under φ_n^* because $P_w(\alpha, \beta, \varphi_n^*)$ also intersects T_n . Note that $\alpha \neq \gamma_n$. First we consider the case $\alpha \neq \delta_n$. Then T_n is closed for α under φ_n^* . Since $|X_{\varphi_n^*, \gamma_n}(T_n)| = 1$, we have $\beta \neq \gamma_n$. Since T_n is closed for α and all δ_n edges on the boundary of T_n are incident to $V(T_n \cap T_n^d)$ by the choice of v_n , all (α, δ_n) paths intersecting T_n intersect T_n^d . Thus we have that $\beta \neq \delta_n$. Therefore we proceed the same as proving (2), where we consider the coloring $\varphi^{**} = \varphi_n^*/P_{v_n}(\gamma_n, \delta_n, \varphi_n^*)$. Since $\alpha, \beta \notin S_n$, α and β are not T_n -interchangeable under φ^{**} . Since φ^{**} is $(T_n, D_{n-1}, \varphi_{n-1})$ -stable, we have a contradiction with (A1) (2) for T_n under φ_{n-1} because T_n has $n-1$ rungs. Now we assume $\alpha = \delta_n$. By our choice of v_n , we have that δ_n is closed for $T_n \vee T_n^d$. Let $\theta \in \varphi_n^*(T_n \cap T_n^d) - \delta_n$. Then $\varphi^{**} = \varphi_n^*/(G - T_n \vee T_n^d, \delta_n, \theta)$ is also $(T_n \vee T_n^d, D_n, \varphi_n)$ -stable. However, $\theta \neq \delta_n$ is not interchangeable with β in $T_n \vee T_n^d$ under φ^{**} , where we reach the previous case $\alpha \neq \delta_n$.

Finally we prove (4) by contradiction. By (3), we may assume $\alpha \in \overline{\varphi}_n^*(u)$ and $\beta \in \overline{\varphi}_n^*(v)$ with $u \in T_n$ and $v \in T_n^d$ such that α, β are not interchangeable in $T_n \vee T_n^d$. By (1) and (2), we see that $P_u(\alpha, \beta, \varphi_n^*)$ is the only (α, β) -path intersecting T_n and $P_v(\alpha, \beta, \varphi_n^*)$ is the only (α, β) -path intersecting T_n^d . Therefore we have $P_u(\alpha, \beta, \varphi_n^*) \neq P_v(\alpha, \beta, \varphi_n^*)$. Moreover, by (2) we have that $P_v(\alpha, \beta, \varphi_n^*)$ does not intersect T_n . Let $\varphi^{**} = \varphi_n^*/P_v(\alpha, \beta, \varphi_n^*)$. Hence φ^{**} is (T_n, D_n, φ_n) -stable. Since $\beta \in \overline{\varphi}_n^*(v)$, we have $\beta \notin \overline{\varphi}^*(T_n^d)$ because $T_n \vee T_n^d$ is elementary under

φ_n^* by Lemma 2.3.1. Since $\alpha \in \overline{\varphi_n^*}(u) \subset \overline{\varphi_n^*}(T_n - T_n^d)$, $\alpha \notin \overline{\varphi_n^*}(T_n(v_n)) = \overline{\varphi_n}(T_n(v_n))$. Since T_n^d is obtained from $T_n(v_n)$ by TAA under φ_n , we must have both $\alpha, \beta \notin \varphi_n(T_n^d(v) - T_n(v_n)) = \varphi_n^*(T_n^d(v) - T_n(v_n))$. Hence $\alpha, \beta \notin \varphi^{**}(T_n^d(v) - T_n(v_n))$. Moreover, the edges and vertices of $T_n^d(v) - T_n(v_n) - \{v\}$ are colored the same as in φ_n^* and φ^{**} . Let $T_n^{d*} = \overline{T_n(v_n)}$ under φ^{**} . Since φ^{**} is (T_n, D_n, φ_n) -stable, $T_n \vee T_n^{d*}$ is elementary by Lemma 2.3.1. However, because the edges and vertices of $T_n^d(v) - T_n(v_n) - \{v\}$ are colored the same as in φ_n^* and φ^{**} , we have $v \in V(T_n^{d*})$ under φ^{**} after applying TAA. Therefore we have a contradiction to $\alpha \in \overline{\varphi^{**}}(u) \cap \overline{\varphi^{**}}(v)$ because $u, v \in T_n \vee T_n^{d*}$. \square

Lemma 2.3.3. *Let T, T_n, φ_n and T_n^d be defined as above. We have $\varphi_n(T_n^d - T(v_n)) \cap \overline{\varphi_n}(V(T_n) - V(T_n^d)) = \emptyset$. Moreover, we have $\varphi_n(T_n) - \overline{\varphi_n}(T_n) \subset D_n$ and therefore $(\varphi_n(T_n) - \overline{\varphi_n}(T_n)) \cap \overline{\varphi_n}(T_n^d) \subset D_n \cap \overline{\varphi_n}(T_n^d - T_n)$.*

Proof. Since T_n^d is obtained from $T_n(v_n)$ by TAA under φ_n , no edge in $E(T_n^d - T(v_n))$ is colored by colors in $\overline{\varphi_n}(V(T_n) - V(T_n^d))$. By the construction of T_n , edges in $E(T_n)$ colored by colors not in $\overline{\varphi_n}(T_n)$ are colored by colors in D_n . Thus we see that edges in $E(T_n)$ colored by colors not missing in $\overline{\varphi}(T_n)$ but colors missing in $\overline{\varphi}(T_n^d)$ are actually colored by colors in $\overline{\varphi}(T_n^d - T_n) \cap D_n$. \square

In the remainder of this section, we will prove the elementariness of T if T is an ETT satisfying MP with ladder $T_0 \subset T_1 \subset \dots \subset T_n \subset T$, coloring sequence $(\varphi_0, \varphi_1, \dots, \varphi_n)$, $\Theta_n = \text{PE}$ and $T_n \subset T_n \vee T_n^d \subset T$. In the proof we will divide the sequence of $T - T_n$ into a number of subsequence. We call the nested sequence $T_n := T_{n,0} \subset T_n \vee T_n^d := T_{n,1} \subset T_{n,2} \subset \dots \subset T_{n,q} \subset T := T_{n,q+1}$ a *PE split tail* with q *splitters* for T or simply a *split tail* of T , where $T_{n,i} = T_{v_i}$ for $1 < i \leq q$ and $v_1 \prec_\ell v_2 \prec_\ell \dots \prec_\ell v_q$ are vertices called *splitters* in $V(T - T_n \cup T_n^d)$. The ETT T with the form $T_1 \subset T_2 \subset \dots \subset T_n \subset T_{n,1} \subset T_{n,2} \subset \dots \subset T_{n,q} \subset T := T_{n,q+1}$ is called a *PE refinery* of T with q *PE splitters*, or simply a *refinery* of T . During this section, we omit the word PE when we talk about split tails for convenience. Since we will work on the last coloring φ , we let $\varphi := \varphi_n$ if there is no confusion arise. Note that $T_n \vee T_n^d$ is

elementary by Lemma 2.3.1, we may assume that $T_n \vee T_n^d$ is not closed and $q \geq 1$ when we mention the number of splitters.

Recall $D_n = \cup_{h \leq n} S_h - \bar{\varphi}(T_n)$, where $S_h = \{\delta_h\}$ if $\Theta_h = \text{SE}$ and $S_h = \{\delta_h, \gamma_h\}$ otherwise. Moreover, when $\Theta_h = \text{PE}$ we have $\delta_h \in \bar{\varphi}_h(T_h)$ and $\gamma_h \in \bar{\varphi}_h(T_h)$ when $\Theta_h = \text{SE}$. Thus $|D_n| \leq n$. Denote $D_n = \{\eta_1, \dots, \eta_{n'}\}$ with $n' \leq n$.

Definition 9. *A split tail with q splitters of an ETT T with n -rungs satisfies condition R2 if it satisfies the following conditions for $0 < j \leq q$:*

- (1) *When $j \geq 2$, there exists a two color set $\Gamma_h^j = \{\gamma_{h_1}^j, \gamma_{h_2}^j\} \subseteq \bar{\varphi}(T_{n,j}) - \varphi(T_{n,j+1}(v(\eta_h)) - T_{n,j})$ for each $\eta_h \in D_{n,j} = D_n - \bar{\varphi}(T_{n,j})$. When $j = 1$, for each $\eta_h \in D_{n,1} = D_n - \bar{\varphi}(T_{n,1})$, there exists $\Gamma_h^1 = \{\gamma_{h_1}^1, \gamma_{h_2}^1\} \subseteq \bar{\varphi}(T_n) - \varphi(T_{n,2}(v(\eta_h)) - T_{n,1})$ such that $|\bar{\varphi}(T_n \cap T_n^d) - \Gamma^1|$ is maximum under the restriction of $T_{n,1}$ not being $(\Gamma^1)^-$ -closed, where $\Gamma^j = \cup_{\eta_h \in D_{n,j}} \Gamma_h^j$. Moreover, the followings hold for each j :*

(a) $\Gamma_h^j \cap \Gamma_g^j = \emptyset$ if $\eta_h \neq \eta_g \in D_{n,j}$.

(b) $\Gamma^{j+1} - \Gamma^j \subseteq \bar{\varphi}(T_{n,j+1} - T_{n,j})$.

- (2) $T_{n,j}$ is $(\cup_{\eta_h \in D_{n,j}} \Gamma_h^{j-1})^-$ -closed when $j \geq 2$.

We say an ETT T satisfies condition R2 under φ for convenience if there is a split tail of T satisfies condition R2. We call each Γ_h^j in the above definition a Γ set. Note that $D_n - D_{n,1} \subset \bar{\varphi}(T_n^d - T_n)$. Moreover, condition R2 (1) requests that colors in Γ_h^j could only be used in $T_{n,j+1}$ after $v(\eta_h)$. Thus if $\eta_h \notin \bar{\varphi}(T_{n,j+1})$, then $\Gamma_h^j \cap \varphi(T_{n,j+1} - T_{n,j}) = \emptyset$. Note that we updated Γ set from $T_{n,j}$ to $T_{n,j+1}$, $T_{n,j+1} \neq T_{n,j}$ for $j \geq 2$ by our construction. In R2 (1) the case $j = 1$, we require that $T_{n,1}$ is not $(\Gamma^1)^-$ -closed. We have such requirement just to make sure that $T_{n,2} \neq T_{n,1}$. To prove the elementariness, we need to perform a lot of kempe changes. However switching colors in $D_{n,j}$ usually destroys the stable coloring, we may use colors in Γ_h^j as stepping stones to switch with colors in D_n while keeping the coloring stable in later proofs. Thus we may consider the set Γ_h^j as a color set reserved for η_h . Condition R2 (1.a) requires different colors in D_n to reserve different color sets. Condition R2 (1.b) requests we replace the colors in the reserved set within the missing colors in $\bar{\varphi}(T_{n,j+1} - T_{n,j})$.

Condition R2 (2) implies that we always extend the ETT $T_{n,j}$ until we have to use colors in Γ_h^{j-1} before η_h missing in $T_{n,j}$. If $T_{n,j}$ gets to maximal and we cannot add edges without using colors in Γ_h^{j-1} before η_h missing in $T_{n,j}$, we replace colors in Γ_h^{j-1} to make Γ_h^j , where we continue to build $T_{n,j+1}$. Moreover, R2 (1) actually involves $T_{n,q+1}$ for $j = q$ while R2 (2) only involves $T_{n,q}$. Thus during the proof of an ETT satisfying R2, we usually prove that $T_{n,q}$ satisfies R2 with R2 (2) holding for $T_{n,q}$ itself first, and then we prove $T - T_{n,q}$ satisfies R2 without checking (2). Let φ^* be a (T_n, D_n, φ_n) -stable coloring. By (A5), φ^* is a $\varphi_n \bmod T$ coloring. However, we note that a split tail under φ_n may not be a split tail under φ^* even when T is still a corresponding ETT of the $\varphi_n \bmod T$ coloring φ^* , because $T_n^{d*} = \overline{T_n(v_n)}$ under φ^* may not contain the same vertices as $T_n^d = \overline{T_n(v_n)}$ under φ and as a consequence, $T_n \vee T_n^d$ may not be a half closure of T_n under φ^* . Nonetheless, we have the following Lemma.

Lemma 2.3.4. *Let T be an ETT satisfying MP under last coloring φ_n with ladder $T_0 \subset T_1 \subset \dots \subset T_n \subset T$ and split tail $T_{n,0} \subset T_n \vee T_n^d = T_{n,1} \subset T_{n,2} \subset \dots \subset T_{n,q} \subset T := T_{n,q+1}$. Let φ^* be a $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable coloring and $m(v_n) = j$. Moreover we assume that T can still be obtained by TAA from $T_{n,1}$ under φ^* . Then T is still an ETT satisfying MP under the $\varphi_n \bmod T$ coloring and $T_{n,0} \subset T_n \vee T_n^d \subset T_{n,2} \subset \dots \subset T_{n,q} \subset T := T_{n,q+1}$ is still a split tail of T .*

Proof. Since φ^* is $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable, it is (T_n, D_n, φ_n) -stable and $(T_n^d, \emptyset, \varphi_n)$ -stable. Hence φ^* is $(T_{j-1}, D_{j-1}, \varphi_{j-1})$ -stable by (A4). Note that (2) and (3) in the definition of $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable imply edges in $E(T_n^d)$ are colored the same under both φ_n and φ^* and $\overline{\varphi^*}(v) = \overline{\varphi_n}(v)$ for any $v \in T_n^d$. Thus T_n^d can be obtained from $T_n(v_n)$ by TAA under φ^* . Therefore by (A5), T_n^d must be an ETT satisfying MP with ladder $T_0 \subset T_1 \subset \dots \subset T_{j-1} \subset T_n^d$ under the $\varphi_{j-1} \bmod T_j$ coloring φ^* with $j - 1 < n$ rungs. Recall that $T_n^d = \overline{T_n(v_n)}$ under φ_n , it is closed under φ_n . Since φ^* is $(T_n^d, \emptyset, \varphi_n)$ -stable, T_n^d is still closed under φ^* . Thus $T_n^d = \overline{T_n(v_n)}$ under φ^* , and therefore $T_n \vee T_n^d$ is still a half closure of T_n under φ^* . Moreover, since T can still be obtained by TAA from $T_{n,1}$ under φ^* , T is a corresponding ETT of φ^* , and

therefore it satisfies MP by (A5). Therefore $T_{n,0} \subset T_n \vee T_n^d \subset T_{n,2} \subset \cdots \subset T_{n,q} \subset T := T_{n,q+1}$ is still a split tail of T under φ^* . \square

Lemma 2.3.5. *Suppose (A2),(A3),(A4) and (A5) hold for ETTs satisfying MP with at most n rungs and (A1) holds for ETTs satisfying MP with at most $n - 1$ rungs. Let T be an ETT with n rungs satisfying MP under last coloring φ with $\Theta_n = PE$. Let φ^* be obtained from φ by switching α and β edges of some (α, β) -chains in $G - V(T)$. Then φ^* is (T, D_n, φ_n) -stable and T is an ETT satisfies MP under the $T \bmod \varphi$ coloring φ^* . Moreover, if T satisfies R2 under last coloring φ , then the same split tail satisfies R2 with the same Γ sets under last coloring φ^* and φ^* is both $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable and (T, φ_n) -wstable, and additionally if $T = T_{n,q+1}$ itself satisfies R2 (2), i.e. $T_{n,q+1}$ is $(\cup_{\eta_h \in D_{n,q+1}} \Gamma_h^q)^-$ -closed where $D_{n,q+1} = D_n - \bar{\varphi}(T_{n,q+1})$, then $T_{n,q}$ itself also satisfies R2 (2) under φ^* .*

Proof. Since $\varphi(f) = \varphi^*(f)$ for every edge f incident to $V(T)$, φ^* is (T, D_n, φ_n) -stable and therefore (T_n, D_n, φ_n) -stable. Applying (A5), we see that T is an ETT satisfying MP under the $\varphi \bmod T$ coloring φ^* . Suppose T satisfies R2 with a split tail $T_n = T_{n,0} \subset T_n \vee T_n^d = T_{n,1} \subset T_{n,2} \subset \cdots \subset T_{n,q} \subset T = T_{n,q+1}$. Since φ^* is (T, D_n, φ_n) -stable, it is clearly $(T_n \vee T_n^d, D_n, \varphi_n)$ -stable and (T, φ_n) -wstable. Thus φ^* is both $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable and (T, φ_n) -wstable. Note T is still obtained from T_n by TAA under φ^* and it is an corresponding ETT of φ^* . By Lemma 2.3.4, $T_n \subset T_n \vee T_n^d \subset T_{n,2} \subset \cdots \subset T_{n,q} \subset T = T_{n,q+1}$ is still a split tail of T under φ^* . Since the colors who are $T_{n,i}$ -closed under φ stay $T_{n,i}$ -closed under φ^* for each $0 \leq i \leq q$, $\varphi(f) = \varphi^*(f)$ for every edge f incident to $V(T)$ and $\bar{\varphi}(v) = \bar{\varphi}^*(v)$ for every $v \in V(T)$, R2 is also satisfied for the same split tail under φ^* . Now we assume $T_{n,q+1}$ itself satisfies R2 (2) under φ . If $q = 0$, $T_{n,1}$ itself satisfies R2 (2) under φ^* because R2 (2) starts for $T_{n,2}$. If $q \geq 1$, then $T = T_{n,q+1}$ being $(\cup_{\eta_h \in D_{n,q+1}} \Gamma_h^q)^-$ -closed under φ implies that it is also $(\cup_{\eta_h \in D_{n,q+1}} \Gamma_h^q)^-$ -closed under φ^* . \square

We prove the following proposition which is an inductive proof of (A1) (1) when $\Theta_n = PE$.

Proposition 5. *Let n be a positive integer. Suppose (A2),(A3),(A4) and (A5) hold for*

ETT's satisfying MP with n rungs and (A1) holds for ETT's satisfying MP with at most $n-1$ rungs. If T is an ETT satisfying MP with ladder $T_1 \subset T_2 \subset \dots \subset T_n \subset T_n \vee T_n^d \subset T$ with coloring sequence $(\varphi_0, \varphi_1, \dots, \varphi_n)$ and $\Theta_n = PE$, then the following statements A and B hold, which imply T is elementary.

A. If T satisfies condition R2, then T is elementary.

B. If A holds, then there exists a closed ETT T' with $V(T) \subset V(T')$ and ladder $T_1 \subset T_2 \subset \dots \subset T_n \subset T_n \vee T_n^d \subset T'$ satisfying MP and R2.

Note that MP is satisfied for corresponding ETT's of all (T_n, D_n, φ_n) -stable colorings by (A5). We place the proof of Statement B first since it is much shorter than the proof of Statement A. We assume A holds in the proof of B.

2.3.1.1 Proof of Statement B in Proposition 5 Let T be an ETT satisfying MP with ladder $T_1 \subset T_2 \subset \dots \subset T_n \subset T$ under last coloring $\varphi_n = \varphi$. We will construct an ETT T' with ladder $T_1 \subset T_2 \subset \dots \subset T_n \subset T_n \vee T_n^d \subset T'$ under φ_n such that $V(T) \subset V(T')$ with a split tail $T_n =: T_{n,0} \subset T_{n,1} := T_n \vee T_n^d \dots \subset T_{n,q} \subset T' := T_{n,q+1}$ satisfying condition R2. Recall that we assume $T_n \vee T_n^d = T_{n,1}$ is not closed under φ_n . We first build $T_{n,2}$ after $T_{n,1}$ using the following algorithm.

- (1) Pick distinct $\Gamma_h^1 = \{\gamma_{h1}^1, \gamma_{h2}^1\} \subset \overline{\varphi}(T_n)$ for each $\eta_h \in D_{n,1}$ such that $|\overline{\varphi}(T_n \cap T_n^d) - \Gamma^1|$ is maximum under the restriction of $T_{n,1}$ not being $(\Gamma^1)^-$ -closed. Let $T_{n,2} = T_{n,1} \cup \{f, b(f)\}$, where $\varphi(f) \in \overline{\varphi}(T_{n,1})$ and $f \in \partial(T_{n,1})$.
- (2) If there exists $f \in \partial(T_{n,2})$ with $\varphi(f) \in \overline{\varphi}(T_{n,2})$, we augment $T_{n,2}$ by letting $T_{n,2} := T_{n,2} \cup \{f, b(f)\}$ under the restriction $\Gamma_h^1 \cap \varphi(T_{n,2}(v(\eta_h)) - T_{n,1}) = \emptyset$ for all $\eta_h \in D_{n,1}$ until we can not add any new edge.

Since T_i is elementary under φ_{i-1} by (A1) (1) for $0 < i \leq n$, each $|T_i|$ has odd number of vertices. Thus $|T_i - T_{i-1}| \geq 2$ for each $0 < i \leq n$. Since T satisfies MP, T_1 is a maximal Tashkinov tree and $|T_1| \geq 11$. Recall that T_n is elementary under φ_n . Thus we have

$|\overline{\varphi}(T_n)| \geq 11 + 2n$. Since $|D_n| \leq 2n$, $|D_{n,1}| \leq 2n$. Since $T_{n,1}$ is not closed under φ , there must exist a an edge $f \in \partial(T_{n,1})$ with $\varphi(f) \in \overline{\varphi}(T_{n,1})$. Note that $\varphi(f) \notin \overline{\varphi}(T_n \cap T_n^d)$ because T_1 is closed for colors in $\overline{\varphi}(T_n \cap T_n^d)$. Thus we have enough colors to pick each Γ_h^1 distinctly for distinct $\eta_h \in D_{n,1}$ from $\overline{\varphi}(T_n) - \varphi(f)$ such that $|\overline{\varphi}(T_n \cap T_n^d) - \Gamma^1|$ is maximum under the restriction of $T_{n,1}$ not being $(\Gamma^1)^-$ -closed, and therefore step (1) is feasible. Note that $T_{n,2}$ obtained from the algorithm above clearly satisfies MP since $T_{n,2}$ shares the same ladder with T and we did not change the coloring. Moreover, R2 is also satisfied by our construction. Suppose $T_{n,j-1}$ is defined for some $j \geq 3$. If $T_{n,j-1}$ is closed, then $V(T) \subset V(T_{n,j-1})$ and we let $T_{n,j-1} = T'$. If $T_{n,j-1}$ is not closed, we will continue to build $T_{n,j}$ from $T_{n,j-1}$ inductively as follows:

- (1) Let $T_{n,j} = T_{n,j-1} \cup \{f, b(f)\}$ with $\varphi(f) \in \Gamma_h^{i,j-2}$ for some $\eta_h \in D_{n,j-1}$ and $f \in \partial(T_{n,j-1})$. Let $\Gamma_h^{j-1} \subset \overline{\varphi}(T_{n,j-1} - T_{n,j-2})$ with $|\Gamma_h^{j-1}| = 2$ for η_h and $\Gamma_i^{j-1} = \Gamma_i^{j-2}$ for any other η_i with $\eta_i \in D_{n,j-1}$.
- (2) If there exists $f \in \partial(T_{n,j-1})$ where $\varphi(f) \in \overline{\varphi}(T_{n,j-1})$, we let $T_{n,j} := T_{n,j-1} \cup \{f, b(f)\}$ under the restriction $\Gamma_h^{j-1} \cap \varphi(T_{n,j-1}(v(\eta_h)) - T_{n,j-1}) = \emptyset$ for all $\eta_h \in D_{n,j-1}$ until we can not add any new edge.

Since $T_{i,j-1}$ is not closed but is $(\cup_{\eta_h \in D_{n,j-1}} \Gamma_h^{j-2})^-$ closed, such a η_h in (1) exists. By Statement A, $T_{n,j-1}$ is elementary. Therefore, $|\overline{\varphi}(T_{n,j-1} - T_{n,j-2})| \geq 2$. Thus step (1) is feasible. Note that $T_{n,j}$ obtained from the algorithm above satisfies R2. $T_{n,j}$ also satisfies MP since $T_{n,j}$ shares the same ladder with T and we did not change the coloring. Now if $T_{n,j}$ is closed, then $V(T) \subset V(T_{n,j})$ and we let $T_{n,j} = T'$. If $T_{n,j}$ is not closed, we will continue to build $T_{n,j+1}$. Finally we will obtain a closed T' as desired. \square

2.3.1.2 Proof of Statement A in Proposition 5 We prove statement A by induction on q which is the number of splitters. Denote the ETT T by $T_1 \subset T_2 \subset \dots \subset T_n := T_{n,0} \subset T_{n,1} := T_n \vee T_n^d \subset \dots \subset T_{n,q} \subset T = T_{n,q+1}$, where the split tail satisfies R2. With $q = 1$, $T_{n,q} = T_{n,1} = T_n \vee T_n^d$. By Lemma 2.3.1, $T_{n,1}$ is elementary. Suppose statement A

holds for all ETTs satisfying MP and R2 with at most $q-1$ splitters. Thus $T_{n,q}$ is elementary because it satisfies R2 with $q-1$ splitters. We will show $T_{n,q+1} = T$ is elementary. Denote T by $T_{n,q} \cup \{e_0, y_0, e_1, \dots, e_p, y_p\}$ following the order \prec_ℓ . We define the path number $p(T)$ of T as the smallest index $i \in 0, 1, \dots, p$ such that the sequence $y_i T := (y_i, e_{i+1}, \dots, e_p, y_p)$ is a path in G . Suppose on the contrary that T is a counterexample to the theorem, i.e., conditions MP holds for T under last coloring φ and R2 holds for a split tail of q splitters, but $V(T)$ is not elementary. Furthermore, we assume that among all counterexamples under colorings both $(T_n \cup T_n^d, D_n, \varphi)$ -stable and $(T_{n,q}, \varphi)$ -wstable where R2 is satisfied for the same splitters, the following two conditions hold:

- (1) $p(T)$ is minimum,
- (2) $|T - T_{n,q}|$ is minimum subject to (1), i.e. p is minimum subject to (1).

We can indeed seek counterexamples through all colorings which are both $(T_n \cup T_n^d, D_n, \varphi)$ -stable and $(T_{n,q}, \varphi)$ -wstable, because a coloring being $(T_n \cup T_n^d, D_n, \varphi)$ -stable implies we have an ETT satisfying MP by (A5), and the coloring is also $(T_{n,q}, \varphi)$ -wstable implies we can indeed grow to $T_{n,q}$ by TAA. Note that we also require R2 being satisfied for the same splitters. Such requirement is valid, because by Lemma 2.3.4 we can have the same splitters under a coloring which is both $(T_n \cup T_n^d, D_n, \varphi)$ -stable and $(T_{n,q}, \varphi)$ -wstable. We call such counter-examples “minimum” counter-examples for convenience. During this proof, we may say a coloring is $T_{n,q}$ -stable (resp. T -stable) for convenience instead of saying it’s both $(T_n \cup T_n^d, D_n, \varphi)$ -stable and $(T_{n,q}, \varphi)$ -wstable (resp. both $(T_n \cup T_n^d, D_n, \varphi)$ -stable and (T, φ) -wstable). Note that a coloring being $T_{n,q}$ -stable is an equivalence relation, because both $(T_n \cup T_n^d, D_n, \varphi)$ -stable and $(T_{n,q}, \varphi)$ -wstable are equivalence relations. We may often mention that an ETT satisfies R2 in the proof, and if the splitters and Γ sets are not specified, we always mean that R2 is satisfied for the same splitters and Γ sets in T . Thus a coloring being $T_{n,1}$ -stable means it is actually $(T_n \cup T_n^d, D_n, \varphi)$ -stable, because $(T_n \cup T_n^d, D_n, \varphi)$ -stable implies $(T_n \vee T_n^d)$ -wstable. By our choice, $V(Ty_{p-1})$ is elementary, where $Ty_{p-1} = T_{n,q}$ when $p = 0$. Since $V(T)$ is not elementary, there exist a color $\alpha \in \overline{\varphi}(y_p) \cap \overline{\varphi}(v)$ for some $v \in V(Ty_{p-1})$.

For simplification of notations, we let $\Gamma_h^q = \{\gamma_{h1}, \gamma_{h2}\}$ for $\eta_h \in D_{n,q}$ during the proof.

2.3.1.2.1 A few basic properties

Let T be a “minimum” counter-example as mentioned earlier for Proposition 5 with splitters v_1, v_2, \dots, v_q .

Claim 2.3.7. *For every $T_{n,j}$ with $1 < j \leq q$ and two colors α, β , if $\alpha \in \overline{\varphi}(T_{n,j})$ and is closed in $T_{n,j}$, then α and β are $T_{n,j}$ -interchangeable under φ .*

Instead of proving Claim 2.3.7, we will prove the next Claim which implies Claim 2.3.7 by letting $\varphi' = \varphi$ and $T'_{n,j} = T_{n,j}$.

Claim 2.3.8. *Let φ' be a $T_{n,1}$ -stable coloring and $T'_{n,j}$ be an ETT with ladder $T_0 \subset T_1 \subset \dots \subset T_n \subset T'_{n,j}$ and split tail $T_n \subset T_{n,1} \subset T'_{n,2} \subset \dots \subset T'_{n,j}$ under last coloring φ' with $1 < j \leq q$. Suppose under φ' the split tail satisfies R2 and $T'_{n,j}$ itself satisfies R2 (2), i.e. $T'_{n,j}$ is $(\cup_{\eta_h \in D_{n,j}} \Gamma_h^{j-1})^-$ -closed. For any two colors α and β , if $\alpha \in \overline{\varphi'}(T_{n,j})$ and is closed in $T_{n,j}$ under φ' , then α and β are $T_{n,j}$ -interchangeable under φ' .*

Recall if there are q splitters for a split tail, then R2 (1) are required up to index $q + 1$ and R2 (2) stops at index q . Thus a split tail satisfies R2 does not imply that the ETT itself ($T_{n,q+1}$) satisfies R2 (2). For the situation in Claim 2.3.8 where $T'_{n,j}$ satisfies R2 by a split tail and R2 (2) is also satisfied for $T'_{n,j}$ itself, later in the proof we may simply say R2 is satisfied for $T'_{n,j}$ **up to itself** if the content is clear.

Proof. Let j be the minimum index such that Claim 2.3.8 fails. Then there are two (α, β) -paths intersecting $T'_{n,j}$ under φ' . For simplification of notations, we denote φ' still by φ and $T'_{n,i}$ by $T_{n,i}$ for each $1 < i \leq j$. We then consider the following two cases.

Case I: $\beta \in \overline{\varphi}(T_{n,j})$.

Note that $T_{n,j}$ is an ETT satisfying MP under φ by (A5), because φ is $T_{n,1}$ -stable. Since $T_{n,j}$ satisfies R2 with $j - 1 < q$ splitters, $T_{n,j}$ is elementary by induction hypothesis. Since α is closed in $T_{n,j}$, $|V(T_{n,j})|$ is odd. Therefore $|\partial_\beta(T_{n,j})|$ is even. Thus there are even number of (α, β) exits by the elementariness of $T_{n,j}$. Since $T_{n,j}$ is elementary and there exist two

(α, β) -paths intersecting $T_{n,j}$, we assume that there exist two exit vertices $u, v \in T_{n,j}$ which belong to exit paths $P_u^{ex}(\alpha, \beta, \varphi)$ and $P_v^{ex}(\alpha, \beta, \varphi)$, respectively. We may assume $u \preceq_l v$.

Case I.a: $v \in T_{n,j} - T_{n,j-1}$.

By R2 (2), $T_{n,j}$ is $(\cup_{\eta_h \in D_{n,j}} \Gamma_h^{j-1})^-$ -closed. Since $\bar{\varphi}(T_{n,j} - T_{n,j-1}) \cap \Gamma^{j-1} = \emptyset$, $T_{n,j}$ is closed for colors in $\bar{\varphi}(T_{n,j} - T_{n,j-1})$. Since $\beta \in \varphi(\partial(T_{n,j}))$ and $\beta \in \bar{\varphi}(T_{n,j})$, β is not closed in $T_{n,j}$ under φ . Thus we have $v(\beta) \in V(T_{n,j-1})$. Let $\gamma \in \bar{\varphi}(v)$. Then $\gamma \notin \Gamma^{j-1}$ and therefore γ is closed in $T_{n,j}$ by R2. Thus $T_{n,j}$ is closed for both α and γ . Hence $\varphi^* = \varphi/(G - T_{n,j}, \alpha, \gamma)$ is $T_{n,j}$ -stable, and MP, R2 are still satisfied for $T_{n,j}$ up to itself under φ^* by Lemma 2.3.5. Note that the paths $P_v^{ex}(\alpha, \beta, \varphi) = P_v^{ex}(\gamma, \beta, \varphi^*) = P_v(\gamma, \beta, \varphi^*)$ and $P_u^{ex}(\alpha, \beta, \varphi) = P_u^{ex}(\gamma, \beta, \varphi^*)$ are two (γ, β) exit paths under φ^* . Let $\varphi^2 = \varphi^*/P_v^{ex}(\gamma, \beta, \varphi^*)$. Because $P_v^{ex}(\gamma, \beta, \varphi^*) \cap T_{n,j} = \{v\}$, all edges incident to $V(T_{n,j}(v) - v)$ are colored the same under φ^* and φ^2 . By Lemma 2.3.5, $T_{n,j-1}$ satisfies MP and R2 up to itself under φ^2 by the same splitters and Γ sets. Moreover, we have $\varphi^*(f) = \varphi^2(f)$ for $f \in T_{n,j}(v)$ and $\bar{\varphi}^*(v') = \bar{\varphi}^2(v')$ for $v' \in T_{n,j}(v) - v$. Thus φ^* is $T_{n,1}$ -stable and $T_{n,j}(v)$ still satisfies MP by (A5). In addition, $T_{n,j}(v)$ itself still satisfies R2 (1) because $T_{n,j}$ satisfies R2 (1) under φ^* by the same splitters and same Γ^i set for $1 \leq i \leq j-1$. Thus $T_{n,j}(v)$ is an ETT satisfying MP and R2 with $j-1$ splitters. Hence it is elementary because it has $j-1 < q$ splitters. However, we have $\beta \in \bar{\varphi}^2(T_{n,j-1})$ and $\beta \in \bar{\varphi}^2(v)$, where we reach a contradiction.

Case I.b: $v \in T_{n,j-1}$.

We claim that there exists $\alpha^* \in \bar{\varphi}(T_{n,j-1})$ such that α^* is closed in both $T_{n,j-1}$ and $T_{n,j}$ under φ . First we consider the case when $j = 2$. Note that by condition R2 (1a), $|\Gamma^1| = 2|D_{n,1}| \leq 2n$. Since $|\bar{\varphi}(T_1)| \geq 13$ and T_n is elementary under φ with $|T_i| = \text{odd}$ for $i \leq n$, we have $|\bar{\varphi}(T_n)| \geq 11 + 2n > |\Gamma^1| + 1$. Note that we have a color $\theta \in \bar{\varphi}(T_{n,1})$ where θ is not closed in $T_{n,1}$ under φ . Thus if we pick colors for Γ^1 from $\bar{\varphi}(T_n) - \theta$ with priority from colors in $\bar{\varphi}(T_n) - \bar{\varphi}(T_n \cap T_n^d)$, then $\bar{\varphi}(T_n \cap T_n^d) - \Gamma^1$ is not empty and $T_{n,1}$ is not $(\Gamma^1)^-$ -closed. Recall that R2 requires us to pick colors for Γ^1 such that $|\bar{\varphi}(T_n \cap T_n^d) - \Gamma^1|$ is maximum with the restriction of $T_{n,1}$ not being $(\Gamma^1)^-$ -closed, there must exist a color $\alpha^* \in \bar{\varphi}(T_n \cap T_n^d) - \Gamma^1$. Since $T_{n,2}$ is $(\cup_{\eta_h \in D_{n,2}} \Gamma_h^1)^-$ -closed by condition R2(2) and $T_{n,1}$ is

closed for colors in $\overline{\varphi}(T_n \cap T_n^d)$ because φ is $T_{n,1}$ -stable, we have that α^* is closed in both $T_{n,1}$ and T_n . Now we assume $j \geq 3$. By condition R2(2), T_{j-1} is $(\cup_{\eta_h \in D_{n,j-1}} \Gamma_h^{j-2})^-$ -closed. Similarly as the case $j = 2$, we have $|\overline{\varphi}(T_{n,j-2})| \geq 11 + 2n \geq |\cup_{\eta_h \in D_{n,j-1}} \Gamma_h^{j-2}|$ because $|\cup_{\eta_h \in D_{n,j-1}} \Gamma_h^{j-2}| \leq 2|D_{n,j-1}| \leq 2n$, and there exists $\alpha^* \in \overline{\varphi}(T_{n,j-2}) - (\cup_{\eta_h \in D_{n,j-1}} \Gamma_h^{j-2})$. Hence α^* is closed in $T_{n,j-1}$. By condition R2, we have that $\Gamma^j - \Gamma^{j-1} \subset \overline{\varphi}(T_{n,j} - T_{n,j-1})$, and therefore $\alpha^* \notin \Gamma^j$. Thus $\alpha^* \notin (\cup_{\eta_h \in D_{n,j}} \Gamma_h^{j-1}) \subset \Gamma^j$. Now by condition R2(2), α^* is also closed in $T_{n,j}$, where we have the color α^* as claimed.

Since α and α^* are closed in $T_{n,j}$, $\varphi^* = \varphi/(G - T_{n,j}, \alpha, \alpha^*)$ is $T_{n,1}$ -stable and $T_{n,j}$ satisfies MP and R2 with the same splitters under φ^* by Lemma 2.3.5. Thus $T_{n,j-1}$ satisfies MP and R2 by the same splitters while R2 (2) is satisfied for $T_{n,j-1}$. Note that $T_{n,j-1}$ is elementary under φ^* , $\alpha^* \in \overline{\varphi^*}(T_{n,j-1})$ and α^* is still closed in $T_{n,j-1}$ under φ^* . However, because $P_u^{ex}(\alpha^*, \beta, \varphi^*) = P_u^{ex}(\alpha, \beta, \varphi)$ and $P_v^{ex}(\alpha^*, \beta, \varphi^*) = P_v^{ex}(\alpha, \beta, \varphi)$ are two (α^*, β) exit paths of $T_{n,j-1}$ under φ^* , we see that α^* and β are not interchangeable in $T_{n,j-1}$ under φ^* . Thus we reach a contradiction to the minimality of j when $j > 2$ and reach a contradiction to Lemma 2.3.2 (3) when $j = 2$ because of α^* being closed in $T_{n,j-1}$ under φ^* .

Case II: $\beta \notin \overline{\varphi}(T_{n,j})$.

In this case $|\partial_\beta(T_{n,j})| = \text{odd}$. Hence $T_{n,j}$ has at least three (α, β) exit paths since $T_{n,j}$ is elementary under φ . Let u, v, w be exits from three (α, β) exit paths for $T_{n,j}$ and assume $u \prec_l v \prec_l w$.

Case II.a: $w \in T_{n,j} - T_{n,j-1}$.

In this case, we pick the counter-example of Claim 2.3.8 with above w, u, v and $w \in T_{n,j} - T_{n,j-1}$ such that $L = |V(P_w^{ex}(\alpha, \beta, \varphi))| + |P_u^{ex}(\alpha, \beta, \varphi)| + |P_v^{ex}(\alpha, \beta, \varphi)|$ is minimum. Let $\gamma \in \overline{\varphi}(w)$. By definition, $\gamma \notin \Gamma^{j-1}$, and hence $T_{n,j}$ is closed for γ by condition R2(2). Note that γ may be η_h for some $h \leq n'$. By Lemma 2.3.5, $\varphi^* = \varphi/(G - T_{n,j}, \alpha, \gamma)$ is $T_{n,j}$ -stable, and MP, R2 are still satisfied under φ^* for $T_{n,j}$ by the same splitters and Γ sets as φ . Moreover, under φ^* , we have $P_w^{ex}(\gamma, \beta, \varphi^*) = P_w(\gamma, \beta, \varphi^*) = P_w^{ex}(\alpha, \beta, \varphi)$, $P_u^{ex}(\gamma, \beta, \varphi^*) = P_u^{ex}(\alpha, \beta, \varphi)$ and $P_v^{ex}(\gamma, \beta, \varphi^*) = P_v^{ex}(\alpha, \beta, \varphi)$ are three (γ, β) -exit paths for $T_{n,j}$. Let the three other end vertices of $P_w^{ex}(\gamma, \beta, \varphi^*)$, $P_u^{ex}(\gamma, \beta, \varphi^*)$ and $P_v^{ex}(\gamma, \beta, \varphi^*)$ not in $T_{n,j}$ be w_2 , u_2 and v_2

respectively. Let u' be the vertex in $P_u^{ex}(\gamma, \beta, \varphi^*)$ next to u , and the edge connecting u and u' be f_u ; and v' be the vertex in $P_v^{ex}(\gamma, \beta, \varphi^*)$ next to v , and the edge connecting v and v' be f_v . Note that f_v and f_u are colored β under φ^* . Let $\varphi^2 = \varphi^*/P_w^{ex}(\gamma, \beta, \varphi^*)$. Since $w \in T_{n,j} - T_{n,j-1}$ and $P_w(\gamma, \beta, \varphi^*) \cap T_{n,j} = w$, all edges incident to $V(T_{n,j}(w) - w)$ are colored the same under φ^* and φ^2 . Moreover, we have $\varphi^*(f) = \varphi^2(f)$ for $f \in T_{n,j}(w)$ and $\overline{\varphi^*}(v') = \overline{\varphi^2}(v')$ for $v' \in T_{n,j}(w) - w$. Thus φ^2 is $T_{n,1}$ -stable and $T_{n,j}(w)$ is an ETT satisfies MP under φ^2 by (A5). Moreover, Lemma 2.3.4 allows us to have the same splitters. In addition, $T_{n,j}(w)$ still satisfies R2 under φ^2 by the same splitters and same Γ set as φ^* and φ . Note that in φ^2 , $\beta \in \overline{\varphi^2}(w)$. Since $\beta \notin \Gamma^{j-1}$, we have $\{T_{n,j}(w), f_u, u', f_v, v'\}$ satisfies R2 by the same splitters and Γ sets. Note that by (A5), φ^2 is a φ mod $T_{n,j}$ coloring and any tree sequence obtained from $T_{n,j}(w)$ by TAA is a corresponding ETT of φ^2 , and therefore it satisfies MP. Thus we can keep condition MP by keeping extending $\{T_w, f_u, u', f_v, v'\}$ using TAA under R2 (1) with the same Γ_h^{j-1} set for each $\eta_h \in D_{j-1}$ until maximal, i.e. adding any edge and corresponded vertex to it by TAA will violate R2 (1). Let the resulting ETT be $T_{n,j}^2$. Thus $T_{n,j}^2$ is $(\cup_{\eta_h \in D'_{n,j}} \Gamma_h^{j-1})^-$ -closed, where $D'_{n,j} = D_n - \overline{\varphi^2}(T_{n,j}^2)$. Thus $T_{n,j}^2$ satisfies MP and R2 by the same $j-1$ splitters and same Γ set as φ and φ^* . Moreover, $T_{n,j}^2$ itself satisfies R2 (2), because it is $(\cup_{\eta_h \in D'_{n,j}} \Gamma_h^{j-1})^-$ -closed. Since $j-1 < q$, $T_{n,j}^2$ is elementary. If one of w_2, u_2, v_2 is in $T_{n,j}^2$, then γ must be missing at that vertex since $\beta \in \overline{\varphi^2}(T_{n,j}^2)$. Since both $\gamma, \beta \notin \Gamma^{j-1}$, and both $\gamma, \beta \in \overline{\varphi^2}(T_{n,j}^2)$, we must have all three vertices w_2, u_2, v_2 are in $T_{n,j}^2$. However, all of them miss either γ or β in φ^2 , which gives a contradiction to the elementary property because $T_{n,j}^2$ is an ETT satisfying MP and R2 by $j-1 < q$ splitters. Thus none of the vertices above are in $T_{n,j}^2$. Hence each of $P_u^{ex}(\gamma, \beta, \varphi^*)$, $P_v^{ex}(\gamma, \beta, \varphi^*)$ and $P_w^{ex}(\gamma, \beta, \varphi^*)$ contains a (γ, β) exit path of $T_{n,j}^2$. Let these three correspondence exits be u'', v'' and w'' . Then all of them are in $T_{n,j}^2 - T_{n,j-1}$. Moreover, $|V(P_{w''}^{ex}(\gamma, \beta, \varphi))| + |P_{u''}^{ex}(\gamma, \beta, \varphi)| + |P_{v''}^{ex}(\gamma, \beta, \varphi)| < L$. Recall that $T_{n,j}^2$ satisfies R2 with $j-1$ splitters and itself satisfies R2 (2). Moreover, φ^2 is $T_{n,1}$ -stable. Thus we have a contradiction to the minimality of L , where we treat $T_{n,j}^2$ as $T_{n,j}$ and use γ, β as the two colors.

Case II.b: $w \notin T_{n,j} - T_{n,j-1}$.

The proof of this case is essentially the same as in Case I.b. We first show there exists a color α^* which closed in both $T_{n,j-1}$ and $T_{n,j}$, where we also consider the case $j = 2$ and $j > 2$ differently. Again by Lemma 2.3.5, there is a $T_{n,j}$ -stable coloring $\varphi^* = \varphi/(G - T_{n,j}, \alpha, \alpha^*)$ in which $T_{n,j}$ satisfies conditions MP and R2 by the same splitters. Therefore $T_{n,j-1}$ satisfies R2 up to itself under φ^* and it also satisfies MP. However under φ^* , α^* and β are not interchangeable in $T_{n,j-1}$, giving a contradiction to the minimality of j if $j > 2$ or to Lemma 2.3.2 (3) if $j = 2$. Here we omit the details. \square

Claim 2.3.9. *For any $y \in V(T_{y_{p-1}}) - V(T_{n,q})$, $|\bar{\varphi}(T_y) - \varphi(T_y - T_{n,q}) - \bar{\varphi}(T_n^d - T_n)| \geq 11 + 2n$. Furthermore, if $|\bar{\varphi}(T_y) - \bar{\varphi}(T_n^d - T_n) - \Gamma^q \cup D_{n,q} \cup \varphi(T_y - T_{n,q})| \leq 4$, then there exist 7 distinct colors $\eta_i \in D_{n,q} \cap \bar{\varphi}(T_y)$ such that all colors $\eta_i, \gamma_{i1}, \gamma_{i2} \notin \varphi(T_y - T_{n,q})$.*

Proof. Since $|\bar{\varphi}(T_y - T_{n,q})| \geq |\varphi(T_y - T_{n,q})|$, $|V(T_{n,q})| \geq 11 + 2(n - 1)$ and $|\bar{\varphi}(T_y) - \varphi(T_y - T_{n,q}) - \bar{\varphi}(T_n^d - T_n)| \geq |\bar{\varphi}(T_{n,q}) - \bar{\varphi}(T_n^d - T_n)| \geq |V(T_n)| + 2 \geq 11 + 2n$. Now assume $|\bar{\varphi}(T_y) - \bar{\varphi}(T_n^d - T_n) - \Gamma^q \cup D_{n,q} \cup \varphi(T_y - T_{n,q})| \leq 4$. Since $\bar{\varphi}(T_y) - \bar{\varphi}(T_n^d - T_n) = (\bar{\varphi}(T_y) - \bar{\varphi}(T_n^d - T_n) - \Gamma^q \cup D_{n,q} \cup \varphi(T_y - T_{n,q})) \cup ((\Gamma^q \cup D_{n,q}) \cap \bar{\varphi}(T_y) - \varphi(T_y - T_{n,q})) \cup (\varphi(T_y - T_{n,q}) \cap (\bar{\varphi}(T_y) - \bar{\varphi}(T_n^d - T_n)))$, we have

$$\begin{aligned} & |(\Gamma^q \cup D_{n,q}) \cap \bar{\varphi}(T_y) - \varphi(T_y - T_{n,q})| \\ & \geq |\bar{\varphi}(T_y)| - |\bar{\varphi}(T_n^d - T_n)| - 4 - |\varphi(T_y - T_{n,q}) \cap (\bar{\varphi}(T_y) - \bar{\varphi}(T_n^d - T_n))| \\ & \geq |\bar{\varphi}(T_y)| - |\bar{\varphi}(T_n^d - T_n)| - 4 - |\varphi(T_y - T_{n,q})| \\ & \geq \bar{\varphi}(T_{n,q}) - \bar{\varphi}(T_n^d - T_n) - 4 \geq \bar{\varphi}(T_n) - 4 \geq 2n + 7. \end{aligned}$$

By the Pigeonhole Principle, there are 7 distinguished i such that $\eta_i, \gamma_{i1}, \gamma_{i2} \notin \varphi(T_y - T_{n,q})$ with $\eta_i \in D_{n,q} \cap \bar{\varphi}(T_y)$, so the result holds. \square

Claim 2.3.10. *Let $\alpha, \beta \in \bar{\varphi}(T_{y_{p-1}})$ with $v(\alpha) \prec_\ell v(\beta)$ and $\alpha \notin \varphi(T_{v(\beta)} - T_{n,q})$. Then $P_{v(\alpha)}(\alpha, \beta, \varphi) = P_{v(\beta)}(\alpha, \beta, \varphi)$ if one of the following holds:*

- (1) $q \geq 2$, and $\alpha \in \bar{\varphi}(T_{n,q})$ or $\alpha, \beta \notin D_{n,q}$.
- (2) $q = 1$, and $\alpha \in \bar{\varphi}(T_n)$ or $\alpha, \beta \notin D_n$ or $T_{n,q} \prec_\ell v(\alpha)$ with $\alpha, \beta \notin D_{n,q}$.

Moreover, if $\alpha \in \overline{\varphi}(T_{n,q})$ and α is $T_{n,q}$ -closed, then $P_{v(\alpha)}(\alpha, \beta, \varphi) = P_{v(\beta)}(\alpha, \beta, \varphi)$ and it is the only (α, β) -path intersecting $T_{n,q}$.

Note that $T_{v(\beta)} - T_{n,q} = \emptyset$ if $v(\beta) \in T_{n,q}$. Moreover, α satisfies either (1) or (2) when $\alpha \in \Gamma^q$, because $\Gamma^q \subset \overline{\varphi}(T_n)$ when $q = 1$ and $\Gamma^q \subset \overline{\varphi}(T_{n,q})$ when $q > 1$. Recall that $T_{y_{p-1}}$ is elementary under φ , $T_{n,1} \prec_\ell v(\alpha)$ with $\alpha, \beta \notin D_{n,1}$ actually implies $\alpha, \beta \notin D_n$ in (2). Thus we will not check the case $T_{n,q} \prec_\ell v(\alpha)$ with $\alpha, \beta \notin D_{n,q}$ when $q = 1$ during the proof. Moreover, in Claim 2.3.10, (α, β) -path can not be replaced by (α, β) -chain because there may be (α, β) -cycles intersecting $\partial(T_m)$.

Proof. Let $u = v(\alpha)$ and $w = v(\beta)$. We consider the following few cases according to the locations of u and w .

Case I: $u, w \in T_{n,q}$.

Since $T_{n,q}$ is elementary under φ , u and w are the unique vertices missing α and β , respectively. Thus if $T_{n,q}$ is closed for both α, β , then $E_{\alpha, \beta} \cap \partial(T_{n,q}) = \emptyset$, and therefore $P_u(\alpha, \beta, \varphi) = P_w(\alpha, \beta, \varphi)$ is the only path intersecting $T_{n,q}$, by which Claim 2.3.10 holds. Now we suppose $T_{n,q}$ is closed for α or β but not for both. If $q > 1$, then by Claim 2.3.7 we have that α and β are $T_{n,q}$ -interchangeable, and therefore $P_u(\alpha, \beta, \varphi) = P_w(\alpha, \beta, \varphi)$ is the only (α, β) -path intersecting $T_{n,q}$ by which Claim 2.3.10 holds. If $q = 1$, then by Lemma 2.3.2 (3) we also conclude that α and β are $T_{n,1}$ -interchangeable, and therefore $P_u(\alpha, \beta, \varphi) = P_w(\alpha, \beta, \varphi)$ is the only (α, β) -path intersecting $T_{n,1}$ by which Claim 2.3.10 holds. We now assume neither α nor β is $T_{n,q}$ -closed. Under this assumption, Claim 2.3.10 only requires $P_u(\alpha, \beta, \varphi) = P_w(\alpha, \beta, \varphi)$. We may assume $\beta \in \overline{\varphi}(T_{n,j'} - T_{n,j'-1})$ for some $0 \leq j' < q$ where $T_{n,-1} = \emptyset$ for convenience. Note that $u \prec_\ell w$, so $u \in T_{n,j'}$. By condition R2, β is closed in $T_{n,j'}$ if $j' \neq 1$. We first consider the case $j' > 1$. In the same fashion as we did the case in which $T_{n,q}$ is closed for either α or β , we have $P_u(\alpha, \beta, \varphi) = P_w(\alpha, \beta, \varphi)$ in $T_{n,j'}$ by Claim 2.3.7 because we have $u, v \in T_{n,j'}$ if $j' > 1$. If $j' = 0$, then we immediately have $P_u(\alpha, \beta, \varphi) = P_w(\alpha, \beta, \varphi)$ by Lemma 2.3.2 (2). If $j' = 1$, then $P_u(\alpha, \beta, \varphi) = P_w(\alpha, \beta, \varphi)$ follows from Lemma 2.3.2 (1) because $w \in T_n^d - T_n$ and $u \in T_n^d$.

Case II: $w \notin T_{n,q}$ and $u \in T_{n,q}$.

In this case $\alpha, \beta \notin \varphi(T_w - T_{n,q})$. We first consider the case α is closed in $T_{n,q}$. By Claim 2.3.7 if $q > 1$ and by Lemma 2.3.2 (3) if $q = 1$, we have that α and β are $T_{n,q}$ -interchangeable. Thus $P_u(\alpha, \beta, \varphi)$ is the only (α, β) path intersecting $T_{n,q}$ under φ . Now we assume $P_u(\alpha, \beta, \varphi) \neq P_w(\alpha, \beta, \varphi)$. Thus $P_w(\alpha, \beta, \varphi)$ does not intersect $T_{n,q}$. Therefore by Lemma 2.3.5, $\varphi^* = \varphi/P_w(\alpha, \beta, \varphi)$ is $T_{n,q}$ -stable and $T_{n,q}$ satisfies MP and R2 by the same splitters and Γ sets. Moreover, since $T_{n,q}$ satisfies R2 (2) under φ , $T_{n,q}$ itself satisfies R2 (2) under φ^* by Lemma 2.3.5. In addition, since $\alpha, \beta \notin \varphi(T_w - T_{n,q})$, φ^* is $(T_w - w)$ -stable and therefore T_w is still an ETT obtained from $T_{n,q}$ by TAA under φ^* . Thus it satisfies MP because it has the same ladder as $T_{n,q}$ under φ^* . Moreover, since $\varphi(f) = \varphi^*(f)$ for each $f \in T_w$, $\bar{\varphi}(v') = \bar{\varphi}^*(v')$ for each $v' \in T_w - w$ and we have the same Γ set for T_w under φ^* as under φ , T_w still satisfies R2 (1), and therefore it satisfies R2 because R2 is satisfied for $T_{n,q}$ up to itself under φ^* . However, T_w is not elementary, giving a contradiction to the minimality of $|T - T_{n,q}|$.

Now we assume that α is not closed in $T_{n,q}$. In this case we only need to prove $P_u(\alpha, \beta, \varphi) = P_w(\alpha, \beta, \varphi)$. Assume on the contrary that $P_u(\alpha, \beta, \varphi) \neq P_w(\alpha, \beta, \varphi)$. We first consider case $q = 1$. In this case, the condition of Claim 2.3.10 says that $\alpha \in \bar{\varphi}(T_n)$ or $\alpha, \beta \notin D_n$ or $T_{n,q} \prec_\ell v(\alpha)$ with $\alpha, \beta \notin D_{n,q}$. Thus we have $\alpha \notin D_n$. We claim that $\varphi^* = \varphi/P_w(\alpha, \beta, \varphi)$ is $T_{n,1}$ -stable, i.e. $(T_n \cup T_n^d, D_n, \varphi)$ -stable. If $u \in T_n - T_n^d$, by Lemma 2.3.2 (2) we must have that $P_w(\alpha, \beta, \varphi) \cap T_n = \emptyset$. Hence φ^* is (T_n, D_n, φ) -stable. By TAA, we see that no edge in $E(T_n^d - T(v_n))$ is colored by either α or β under φ . Moreover, $\alpha, \beta \notin \bar{\varphi}(T_n^d)$. Hence φ^* is both (T_n, D_n, φ) -stable and $(T_n^d, \emptyset, \varphi)$ -stable, and therefore it is $T_{n,1}$ -stable. If $u \in T_n^d - T_n$, then the condition of Claim 2.3.10 ensures $\alpha, \beta \notin D_n$. By Lemma 2.3.2 (1) we must have that $P_w(\alpha, \beta, \varphi) \cap T_n^d = \emptyset$. Thus φ^* is $(T_n^d, \emptyset, \varphi)$ -stable. By Lemma 2.3.3, we also have that no edge in $E(T_n)$ is colored by α or β . Moreover, $\alpha, \beta \notin \bar{\varphi}(T_n)$. Hence φ^* is both (T_n, D_n, φ) -stable and $(T_n^d, \emptyset, \varphi)$ -stable, and therefore it is $T_{n,1}$ -stable. If $u \in T_n \cap T_n^d$, then α is closed in $T_{n,1}$, which contradicts our assumption that α is not closed in $T_{n,1}$. Therefore under all three possibilities we have that φ^* is $T_{n,1}$ -stable. Thus $T_{n,1}$ satisfies MP by (A5).

Moreover, since $\alpha, \beta \notin \varphi(T_w - T_{n,q})$, φ^* is $(T_w - w)$ -stable and T_w is an ETT obtained from $T_{n,1}$ by TAA under φ^* . Therefore T_w satisfies MP because it shares the same ladder with $T_{n,1}$. Recall φ^* is $T_{n,1}$ -stable, Lemma 2.3.4 allows us to have the same splitters. Moreover, T_w satisfies R2 (1) by the same splitters for the same Γ_h^1 with $\eta_h \in D_n$ because $\varphi(f) = \varphi^*(f)$ for each $f \in T_w$ and $\bar{\varphi}(v') = \bar{\varphi}^*(v')$ for each $v' \in T_w - w$. Since R2 (2) requires $j \geq 2$, T_w satisfies R2. However, T_w is not elementary, giving a contradiction to the minimality of $|T - T_{n,q}|$.

We then consider the case $q \geq 2$. Note that T_n is closed for colors in $\bar{\varphi}(T_n) - \delta_n$ and $T_{n,1}$ is closed for δ_n . Moreover, R2 implies colors in $\bar{\varphi}(T_{n,j} - T_{n,j-1})$ are closed in $T_{n,j}$ for $2 \leq j \leq q$ because they are not in Γ^{j-1} , and colors in $\bar{\varphi}(T_n^d - T_n)$ are closed in $T_{n,2}$ because they are not in γ^1 . Therefore there exist the largest $q' \geq 0$ such that α is closed in $T_{n,q'}$, because $\alpha \in \bar{\varphi}(T_{n,q})$ with $q \geq 2$. Moreover, $q' \leq 1$ implies $\alpha \in \bar{\varphi}(T_n)$. We claim that $\alpha \notin \varphi(T_w - T_{n,q'})$. Suppose on the contrary $\alpha \in \varphi(T_w - T_{n,q'})$. Recall that we assume $\alpha \notin \varphi(T_w - T_{n,q})$ in Claim 2.3.10. Thus we can assume $\alpha \in \varphi(T_{n,r} - T_{n,r-1})$ for some $q' < r \leq q$. Note that $r \geq 1$ because $r > q' \geq 0$. First we consider the case $r \geq 2$. If $\alpha \in \Gamma_h^{r-1}$ for some $\eta_h \in D_{n,r}$, then α being used in $\varphi(T_{n,r} - T_{n,r-1})$ violates R1 (1). Therefore $\alpha \notin \cup_{\eta_h \in D_{n,r}} \Gamma_h^{r-1}$, and hence α is closed in $T_{n,r}$ by R2 (2), which contradicts the maximality of q' . Thus we assume $r = 1$. In this case we have $q' = 0$, which implies $\alpha \in \bar{\varphi}(T_n - T_n^d)$, because colors in $\bar{\varphi}(T_n^d - T_n)$ are closed in $T_{n,2}$ by R2. However, since $\alpha \in \bar{\varphi}(T_n - T_n^d)$, by TAA no edge is colored by α in $T_n^d - T_n(v_n)$. Note that we have $\varphi(T_{n,1} - T_n) \subset \varphi(T_n^d - T_n(v_n))$ by the definition of $T_n \vee T_n^d$. Thus $\alpha \notin \varphi(T_{n,1} - T_n)$, a contradiction. Hence we indeed have $\alpha \notin \varphi(T_w - T_{n,q'})$. Now we first consider the case $q' \geq 1$. By Claim 2.3.7 when $q' > 1$ and by Claim 2.3.2 (3) when $q' = 1$, α and β are $T_{n,q'}$ -interchangeable under φ , and therefore $P_u(\alpha, \beta, \varphi)$ is the only (α, β) -path intersecting $T_{n,q'}$. Then $P_w(\alpha, \beta, \varphi)$ is disjoint with $T_{n,q'}$. Hence under $\varphi^* = \varphi / P_w(\alpha, \beta, \varphi)$, $T_{n,q'}$ satisfies MP and R2 up to itself with the same splitters and Γ sets by Lemma 2.3.5. Since $\beta \in \bar{\varphi}(w)$, $\alpha, \beta \notin \varphi(T_w - T_{n,q'})$. Thus $\varphi^*(f) = \varphi(f)$ for each $f \in T_w$ and $\bar{\varphi}(v') = \bar{\varphi}^*(v')$ for each $v' \in T_w - w$. Therefore T_w is still an ETT under φ^* , and it satisfies MP because it shares the same ladder as $T_{n,q'}$, and in addition, T_w satisfies R2 (1)

for the same splitters with the same Γ sets because T satisfies R2 under φ . Since $\beta \in \overline{\varphi}(w)$, $\beta \notin \overline{\varphi}(T_w - w) = \overline{\varphi}^*(T_w - w)$. Since α is not closed in $T_{n,s}$ under φ with $1 \leq q' < s \leq q$, $\alpha \in \cup_{\delta_h \in D_{n,s}} \Gamma_h^{s-1}$ by R2 (2). Because from φ to φ^* we only changed colors along $P_w(\alpha, \beta, \varphi)$, all $T_{n,s}$ for $q' < s \leq q$ stay $(\cup_{\delta_h \in D_{n,s}} \Gamma_h^{s-1})^-$ -closed under φ^* , which implies that $T_{n,s}$ satisfies condition R2(2) by the same splitters and Γ sets as under φ . Therefore T_w satisfies MP and R2 under φ^* as an ETT. However, $\alpha \in \varphi^*(w) \cap \varphi^*(T_{n,q})$, giving a contradiction to the minimality of $|T - T_{n,q}|$.

Finally we consider the case $q' = 0$. Recall that in this case we have $\alpha \in \overline{\varphi}(T_n - T_n^d)$. By Lemma 2.3.2 (2), α and β are T_n -interchangeable under φ , and therefore $P_u(\alpha, \beta, \varphi)$ is the only (α, β) -path intersecting T_n . Thus $P_w(\alpha, \beta, \varphi)$ is disjoint with T_n . Since $\alpha \notin \overline{\varphi}(T_n^d)$, $\alpha \notin \varphi(T_n^d - T_n(v_n))$ by TAA. Moreover, $\beta \notin \overline{\varphi}(T_n^d)$ because $\beta \in \overline{\varphi}(w)$. Thus $\beta \notin \varphi(T_n^d - T_n(v_n))$ by TAA. Note $\beta \in D_n$ may happen in this case. But still, φ^* in this case is (T_n, D_n, φ) -stable because $P_w(\alpha, \beta, \varphi)$ is disjoint with T_n . Moreover, φ^* is $(T_n^d, \emptyset, \varphi)$ -stable because $\alpha, \beta \notin \overline{\varphi}(T_n^d)$ and $\alpha, \beta \notin \varphi(T_n^d - T_n(v_n))$. Thus φ^* is $T_{n,1}$ -stable, and $T_{n,1}$ satisfies MP under φ^* by (A5). Recall that $\alpha \notin \varphi(T_w - T_{n,q'})$. Thus we have $\alpha, \beta \notin \varphi(T_w - T_{n,q'})$ because $\beta \in \overline{\varphi}(w)$. Thus $\varphi^*(f) = \varphi(f)$ for each $f \in T_w$ and $\overline{\varphi}(v') = \overline{\varphi}^*(v')$ for each $v' \in T_w - w$. Therefore T_w is still an ETT under φ^* and Lemma 2.3.4 allows us to have the same splitters. Moreover, T_w satisfies MP because it shares the same ladder as $T_{n,1}$, and in addition, T_w satisfies R2 (1) for the same splitters with the same Γ sets because T satisfies R2 under φ . Recall that R2 (2) starts from $j = 2$. Since $\beta \in \overline{\varphi}(w)$, $\beta \notin \overline{\varphi}(T_w - w) = \overline{\varphi}^*(T_w - w)$. Similarly as before, since α is not closed in $T_{n,s}$ under φ with $2 \leq s \leq q$, $\alpha \in \cup_{\delta_h \in D_{n,s}} \Gamma_h^{s-1}$ by R2 (2). Because from φ to φ^* we only changed colors along $P_w(\alpha, \beta, \varphi)$, all $T_{n,s}$ for $1 < s \leq q$ stay $(\cup_{\delta_h \in D_{n,s}} \Gamma_h^{s-1})^-$ -closed under φ^* , which implies that $T_{n,s}$ satisfies condition R2(2) by the same splitters and Γ sets as under φ . Therefore T_w satisfies MP and R2 under φ^* as an ETT. However, $\alpha \in \varphi^*(w) \cap \varphi^*(T_{n,q})$, giving a contradiction to the minimality of $|T - T_{n,q}|$.

Case III: $u, w \notin T_{n,q}$.

In this case, condition of Lemma 2.3.10 requires $\alpha \notin \varphi(T_w - T_{n,q})$. Moreover, condition of Lemma 2.3.10 also requires $\alpha, \beta \notin D_{n,q}$ when $q > 1$ and $\alpha, \beta \notin D_n$ when $q = 1$, which

in turn give $\alpha, \beta \notin D_{n,q} \cup \overline{\varphi}(T_{n,q}) = D_n \cup \overline{\varphi}(T_{n,q})$. Thus by Lemma 2.3.3, $\alpha, \beta \notin \overline{\varphi}(T_n)$. Moreover, $\beta \notin \varphi(T_w - T_n)$. Therefore we have $\alpha, \beta \notin \varphi(T_w)$. Note that $\Gamma^i \subset \overline{\varphi}(T_{n,q})$ for $1 \leq i \leq q$. Suppose on the contrary that $P_u(\alpha, \beta, \varphi) \neq P_w(\alpha, \beta, \varphi)$. Now consider the coloring $\varphi^* = \varphi/P_w(\alpha, \beta, \varphi)$. Since $\alpha, \beta \notin D_n \cup \overline{\varphi}(T_{n,q})$ and $\alpha, \beta \notin \overline{\varphi}(T_w)$, φ^* is (T_n, D_n, φ) -stable, $(T_n^d, \emptyset, \varphi)$ -stable and $(T_w - w, \varphi)$ -wstable. Moreover, $\varphi^*(f) = \varphi(f)$ for each $f \in T_w$ and $\overline{\varphi}(v') = \overline{\varphi}^*(v')$ for each $v' \in T_w - w$. Thus φ^* is $T_{n,q}$ -stable and T_w is also an ETT under last coloring φ^* satisfying MP by (A5). Applying Lemma 2.3.4 for T_w under φ and φ^* , the splitter of T is still splitters of T_w . Since $\varphi^*(f) = \varphi(f)$ for each $f \in T_w$ and $\overline{\varphi}(v') = \overline{\varphi}^*(v')$ for each $v' \in T_w - w$, R2 (1) is satisfied for T_w under φ^* by the same splitters and Γ sets. Since $\Gamma^i \subset \overline{\varphi}(T_{n,q})$ for $1 \leq i \leq q$ and $\alpha, \beta \notin D_n \cup \overline{\varphi}(T_{n,q})$, R2 (2) is still satisfied for T_w under φ^* by the same splitters and Γ sets. Thus T_w is an ETT satisfying MP and R2 under φ^* which is $T_{n,q}$ -stable. However, now $\alpha \in \overline{\varphi}^*(u) \cap \overline{\varphi}^*(w)$, which gives a contradiction to the minimality of $|T - T_{n,q}|$. \square

Claim 2.3.11. *Let $\alpha, \beta \in \overline{\varphi}(T_{y_{p-1}})$. If one of the following holds, then $\varphi^* = \varphi/P$ is $T_{n,q}$ -stable and $T_{n,q}$ satisfies MP and R2 **up to itself** under the $T \bmod \varphi$ coloring φ^* by the same refinery $T_0 \subset T_1 \subset \dots \subset T_n \subset T_{n,1} \subset T_{n,2} \subset \dots \subset T_{n,q}$ and same Γ sets, and consequently any tree sequence obtain from $T_{n,q}$ by TAA under φ^* is an ETT satisfying MP.*

- (1) $q \geq 2$, and $\alpha \in \overline{\varphi}(T_{n,q})$. In this case P is an (α, β) -path disjoint from $P_{v(\alpha)}(\alpha, \beta, \varphi)$.
- (2) $q = 1$, and $\alpha \in \overline{\varphi}(T_n)$ or $\alpha \in \overline{\varphi}(T_{n,1})$ with $\alpha, \beta \notin D_n$. In this case P is an (α, β) -path disjoint from $P_{v(\alpha)}(\alpha, \beta, \varphi)$.
- (3) $T_{n,q} \prec_\ell v(\alpha) \prec_\ell v(\beta)$, $\alpha, \beta \notin D_{n,q}$ and $\alpha \notin \varphi(T_{v(\beta)} - T_{v(\alpha)})$. In this case P is an arbitrary (α, β) -chain.

Moreover if (3) holds, T is an ETT satisfying MP and R2 under φ^* by the same splitters and Γ sets as under φ .

During the remainder of this section, we usually apply Claim 2.3.11 to show that $T_{n,q}$ satisfies MP and R2 up to itself, and if a tree sequence T' is obtained from $T_{n,q}$ by TAA it

then it satisfies MP too. Moreover, since we already shown that $T_{n,q}$ satisfies R2 up to itself, to confirm that condition R2 is indeed satisfied for T' we only need to show that $T' = T_{n,q+1}$ satisfies R2 (1), i.e. no colors from Γ_h^q are used for edges in $T_{n,q+1} - T_{n,q}$ before $v(\eta, T_{n,q+1})$ along \prec_ℓ with $\eta_h \in D_{n,q}$. In addition, it is easy to see that $\Gamma^q \subset \bar{\varphi}(T_{n,q})$ when $q > 1$ and $\Gamma^q \subset \bar{\varphi}(T_n)$ when $q = 1$, so in later proofs we always use Claim 2.3.11 to make Kempe changes on paths involving a color in Γ^q with a color in $D_{n,q}$.

Proof. Note that if φ^* is $T_{n,q}$ -stable, then it is (T_n, D_n, φ) -stable and by (A5), φ^* is a $T \bmod \varphi$ coloring and all corresponding ETTs satisfy MP. Therefore, every tree sequence obtain from $T_{n,q}$ by TAA under φ^* is an ETT satisfying MP because it is an ETT obtained from T_n by TAA under φ^* . Thus we only need to show that φ^* is $T_{n,q}$ -stable during the proof.

We first prove (1) and (2) together, because they share the same P . If one of α and β is closed in $T_{n,q}$, then $P_{v(\alpha)}(\alpha, \beta, \varphi) = P_{v(\beta)}(\alpha, \beta, \varphi)$ is the only (α, β) -path intersecting $T_{n,q}$ by Claim 2.3.10. Thus P is disjoint with $T_{n,q}$. By Lemma 2.3.5, $\varphi^* = \varphi/P$ is $T_{n,q}$ stable and $T_{n,q}$ satisfies MP and R2 up to itself by the same splitters and Γ sets under φ^* , and therefore Claim 2.3.11 holds. We now suppose that neither α nor β is closed in $T_{n,q}$. We first consider the case $q \geq 1$. Note that T_n is closed for colors in $\bar{\varphi}(T_n) - \delta_n$ and $T_{n,1}$ is closed for δ_n . Moreover, R2 implies colors in $\bar{\varphi}(T_{n,j} - T_{n,j-1})$ are closed in $T_{n,j}$ for $2 \leq j \leq q$ because they are not in Γ^{j-1} , and colors in $\bar{\varphi}(T_n^d - T_n)$ are closed in $T_{n,2}$ because they are not in γ^1 . Then similarly to the proof of Claim 2.3.10, there exist the largest q' such that one of α and β is closed in $T_{n,q'}$. We claim that $\alpha, \beta \notin \varphi(T_{n,q} - T_{n,q'})$. Suppose $\alpha \in \varphi(T_{n,q} - T_{n,q'})$. We can assume $\alpha \in \varphi(T_{n,r} - T_{n,r-1})$ for some $r > q'$. Note that $r \geq 1$ because $r > q' \geq 0$. First we consider the case $r \geq 2$. If $\alpha \in \Gamma_h^{r-1}$ for some $\eta_h \in D_{n,r}$, then α being used in $\varphi(T_{n,r} - T_{n,r-1})$ violets R1 (1). Therefore $\alpha \notin \cup_{\eta_h \in D_{n,r}} \Gamma_h^{r-1}$, and hence α is closed in $T_{n,r}$ by R2 (2), which contradicts the maximality of q' . Thus we assume $r = 1$. In this case we have $q' = 0$, which implies $\alpha \in \bar{\varphi}(T_n - T_n^d)$, because colors in $\bar{\varphi}(T_n^d - T_n)$ are closed in $T_{n,2}$ by R2. However, since $\alpha \in \bar{\varphi}(T_n - T_n^d)$, by TAA no edge is colored by α in $T_n^d - T_n(v_n)$. Note that we have $\varphi(T_{n,1} - T_n) \subset \varphi(T_n^d - T_n(v_n))$ by the definition of $T_n \vee T_n^d$. Thus $\alpha \notin \varphi(T_{n,1} - T_n)$, a contradiction. Hence we indeed have $\alpha \notin \varphi(T_{n,q} - T_{n,q'})$.

Now we prove $\beta \notin \varphi(T_{n,q} - T_{n,q'})$. If $\beta \in \overline{\varphi}(T_{n,q})$, we argue the same as in the case when α is not closed in $T_{n,q}$. If $\beta \notin \overline{\varphi}(T_{n,q})$, $\beta \notin \varphi(T_{n,q} - T_{n,q'})$. Hence $\alpha, \beta \notin \varphi(T_{n,q} - T_{n,q'})$. Moreover by our choice of q' , $\alpha, \beta \notin \overline{\varphi}(T_{n,q} - T_{n,q'})$. Suppose not, say $\alpha \in \overline{\varphi}(T_{n,m} - T_{n,m-1})$ where $m > q' \geq 0$. Recall that by R2 we have α being closed in $T_{n,m}$ when $m > 1$ and α being closed in $T_{n,2}$ when $m = 1$, and that is a contradiction to the choice of q' . Thus $v(\alpha) \in T_{n,q'}$ because we have $\alpha \in \overline{\varphi}(T_{n,q})$ in (1) and (2). Let $v(\alpha) = u$. Now we first consider the case $q' \geq 1$. By Claim 2.3.7 when $q' > 1$ and by Claim 2.3.2 (3) when $q' = 1$, α and β are $T_{n,q'}$ -interchangeable under φ , and therefore $P_u(\alpha, \beta, \varphi)$ is the only (α, β) -path intersecting $T_{n,q'}$. Then P is disjoint with $T_{n,q'}$. Hence $\varphi^* = \varphi/P$ is $T_{n,q'}$ -stable and $T_{n,q'}$ satisfies MP and R2 up to itself under φ^* with the same splitters and Γ sets by Lemma 2.3.5. Since $\alpha, \beta \notin \overline{\varphi}(T_{n,q} - T_{n,q'})$ and $\alpha, \beta \notin \varphi(T_{n,q} - T_{n,q'})$, $\varphi^*(f) = \varphi(f)$ for each $f \in T_{n,q}$ and $\overline{\varphi}(v') = \overline{\varphi}^*(v')$ for each $v' \in T_{n,q}$. Therefore φ^* is $T_{n,q}$ -stable and $T_{n,q}$ is still an ETT under φ^* . Thus it satisfies MP because it shares the same ladder as $T_{n,q'}$, and in addition, $T_{n,q}$ satisfies R2 (1) for the same splitters with the same Γ sets because $T_{n,q}$ satisfies R2 under φ . Since α and β are not closed in $T_{n,s}$ under φ with $1 \leq q' < s \leq q$ by our choice of q' , $\alpha, \beta \in \cup_{\delta_h \in D_{n,s}} \Gamma_h^{s-1}$ by R2 (2). Because from φ to φ^* we only changed colors along P , all $T_{n,s}$ for $q' < s \leq q$ stay $(\cup_{\delta_h \in D_{n,s}} \Gamma_h^{s-1})^-$ -closed under φ^* , which implies that $T_{n,s}$ satisfies condition R2(2) by the same splitters and Γ sets as under φ . Therefore $T_{n,q}$ satisfies MP and R2 up to itself by the same splitters and Γ sets under φ^* as an ETT.

We then consider the case $q' = 0$. Recall that in this case we have $\alpha \in \overline{\varphi}(T_{n,0})$ because $u \in T_{n,q'}$. Moreover, $\alpha \notin \overline{\varphi}(T_n^d \cap T_n)$ because colors in $\overline{\varphi}(T_n^d \cap T_n)$ are closed in $T_{n,1}$. Thus $\alpha \in \overline{\varphi}(T_n - T_n^d)$. By Lemma 2.3.2 (2), α and β are T_n -interchangeable under φ , and therefore $P_u(\alpha, \beta, \varphi)$ is the only (α, β) -path intersecting T_n . Thus P is disjoint with T_n . Since $\alpha \notin \overline{\varphi}(T_n^d)$, $\alpha \notin \varphi(T_n^d - T_n(v_n))$ by TAA. Recall that colors in $\overline{\varphi}(T_n^d - T_n)$ are closed in $T_{n,2}$ by R2 and colors in $\overline{\varphi}(T_n^d \cap T_n)$ are closed in $T_{n,1}$, we have $\beta \notin \overline{\varphi}(T_n^d)$. Thus $\beta \notin \varphi(T_n^d - T_n(v_n))$ by TAA. Note $\beta \in D_n$ may happen in this case. But still, φ^* in this case is (T_n, D_n, φ) -stable because P is disjoint with T_n . Moreover, φ^* is $(T_n^d, \emptyset, \varphi)$ -stable because $\alpha, \beta \notin \overline{\varphi}(T_n^d)$ and $\alpha, \beta \notin \varphi(T_n^d - T_n(v_n))$. Thus φ^* is $T_{n,1}$ -stable, and $T_{n,1}$ satisfies MP under φ^* by (A5). Recall

that $\alpha, \beta \notin \varphi(T_{n,q} - T_{n,q'})$ and $\alpha, \beta \notin \overline{\varphi}(T_{n,q} - T_{n,q'})$. Thus $\varphi^*(f) = \varphi(f)$ for each $f \in T_{n,q}$ and $\overline{\varphi}(v') = \overline{\varphi}^*(v')$ for each $v' \in T_{n,q}$. Therefore φ^* is $T_{n,q}$ -stable and $T_{n,q}$ is still an ETT under φ^* . Note that Lemma 2.3.4 allows us to have the same splitters for $T_{n,q}$ under φ^* as under φ . Moreover, $T_{n,q}$ satisfies MP because it shares the same ladder as $T_{n,1}$, and in addition, $T_{n,q}$ satisfies R2 (1) for the same splitters with the same Γ sets because T satisfies R2 under φ . Recall that R2 (2) starts from $j = 2$. Recall that $\alpha, \beta \notin \overline{\varphi}(T_{n,q} - T_{n,q'})$. Similarly as before, since α and β are not closed in $T_{n,s}$ under φ with $2 \leq s \leq q$, $\alpha \in \cup_{\delta_h \in D_{n,s}} \Gamma_h^{s-1}$ by R2 (2). Because from φ to φ^* we only changed colors along P , all $T_{n,s}$ for $1 < s \leq q$ stay $(\cup_{\delta_h \in D_{n,s}} \Gamma_h^{s-1})^-$ -closed under φ^* , which implies that $T_{n,s}$ satisfies condition R2(2) by the same splitters and Γ sets as under φ . Therefore $T_{n,q}$ satisfies MP and R2 up to itself with the same split tail and Γ sets under φ^* as an ETT.

Now we assume that $q = 1$. Let $v(\alpha) = u$. In this case, we have that $\alpha \notin D_n$. If either α or β is closed in $T_{n,1}$, P must be disjoint with $T_{n,1}$ by Claim 2.3.10 and φ^* must be $T_{n,1}$ -stable under which $T_{n,1}$ satisfies MP by (A5). Note that by applying Lemma 2.3.4 for $q = 0$, $T_{n,0} \subset T_{n,1}$ is still a valid split tail under φ^* , and therefore it clearly satisfies R2 up to itself by the same split tail and Γ sets because R2 is empty for $T_{n,1}$. Thus we assume $\alpha, \beta \notin \overline{\varphi}(T_n \cap T_n^d)$ since $T_{n,1}$ is closed for colors in $\overline{\varphi}(T_n \cap T_n^d)$. We first consider the case $u \in T_n - T_n^d$. By Lemma 2.3.2 (2), α and β are T_n -interchangeable. Therefore $P_u(\alpha, \beta, \varphi)$ is the only (α, β) -path intersecting T_n and therefore $P \cap T_n = \emptyset$. If $\beta \in \overline{\varphi}(T_n^d - T_n)$, P must be disjoint with $T_{n,1}$ by Lemma 2.3.2 (4) and φ^* must be $T_{n,1}$ -stable where $T_{n,1}$ satisfies MP. Argue the same as the case either α or β is closed in $T_{n,1}$, we see that $T_{n,0} \subset T_{n,1}$ is still a valid split tail and $T_{n,1}$ still satisfies R2 up to itself by the same splitters and Γ sets. Hence we assume $\beta \notin \overline{\varphi}(T_n^d)$. Therefore by TAA, we see that no edge in $E(T_n^d - T(v_n))$ is colored by either α or β . φ^* in this case is (T_n, D_n, φ) -stable because P is disjoint with T_n . Moreover, φ^* is $(T_n^d, \emptyset, \varphi)$ -stable because $\alpha, \beta \notin \overline{\varphi}(T_n^d)$ and $\alpha, \beta \notin \varphi(T_n^d - T_n(v_n))$. Thus φ^* is $T_{n,1}$ -stable. Argue similarly as before, under φ^* we have that $T_{n,1}$ satisfies MP and $T_{n,0} \subset T_{n,1}$ is still a valid split tail satisfying R2 up to itself with the same splitters and Γ sets. Now we consider the case $u \in T_n^d - T_n$. Then $\beta \notin D_n$. By Lemma 2.3.2 (1) we must

have that $P \cap T_n^d = \emptyset$. Thus φ^* is $(T_n^d, \emptyset, \varphi)$ -stable. If $\beta \in \overline{\varphi}(T_n - T_n^d)$, P must be disjoint with $T_{n,1}$ by Lemma 2.3.2 (3) and φ^* must be $T_{n,1}$ -stable where we argue the same as the case $u \in T_n - T_n^d$ and $\beta \in \overline{\varphi}(T_n^d - T_n)$. Hence we assume $\beta \notin \overline{\varphi}(T_n)$ because $\beta \notin \overline{\varphi}(T_n \cap T_n^d)$. By Lemma 2.3.3, we have that no edge in $E(T_n)$ is colored by α or β since $\alpha, \beta \notin D_n$ and $\alpha, \beta \notin \overline{\varphi}(T_n)$. Hence φ^* is (T_n, D_n, φ) -stable because $\alpha, \beta \notin D_n$ and $\alpha, \beta \notin \overline{\varphi}(T_n)$. Thus φ^* is $T_{n,1}$ -stable. Argue similarly as before, under φ^* we have that $T_{n,1}$ satisfies MP and $T_{n,0} \subset T_{n,1}$ is still a valid split tail satisfying R2 up to itself with the same splitters and Γ sets.

Finally we prove part (3). By Claim 2.3.10, $P_{v(\alpha)}(\alpha, \beta, \varphi) = P_{v(\beta)}(\alpha, \beta, \varphi)$. In this case we have $\alpha, \beta \notin \overline{\varphi}(T_{n,q}) \cup D_{n,q}$, and therefore $\alpha, \beta \notin \overline{\varphi}(T_n) \cup D_n$ and $\alpha, \beta \notin \overline{\varphi}(T_n^d)$. Thus $\alpha, \beta \notin \varphi(T_n)$ by Lemma 2.3.3, and $\varphi^* = \varphi/P$ is (T_n, D_n, φ) -stable because P is an (α, β) chain. Moreover, φ^* is $(T_n^d, \emptyset, \varphi)$ -stable because $\alpha, \beta \notin \overline{\varphi}(T_n^d)$, and therefore it is $T_{n,1}$ -stable. Thus $T_{n,1}$ is an ETT satisfying MP under the $\varphi \bmod T$ coloring φ^* by (A5). Note that by applying Lemma 2.3.4 for $q = 0$, $T_{n,0} \subset T_{n,1}$ is still a valid split tail under φ^* , and therefore it clearly satisfies R2 up to itself by the same split tail and Γ sets because R2 is empty for $T_{n,1}$. Since $\alpha \notin \varphi(T_{v(\beta)} - T_{v(\alpha)})$ and T is obtained from T_n by TAA, we have $\alpha, \beta \notin \varphi(T_{v(\beta)})$. Thus $\varphi(f) = \varphi^*(f)$ for each $f \in T_{v(\beta)}$ and $\alpha, \beta \notin \varphi^*(T_{v(\beta)})$. Moreover, we have $\overline{\varphi}(v') = \overline{\varphi^*}(v')$ for each $v' \in T_{n,q}$ because $\alpha, \beta \notin \overline{\varphi}(T_{n,q})$. Thus φ^* is $T_{n,q}$ -stable and $T_{n,q}$ can be obtained from $T_{n,1}$ by TAA under φ^* . So $T_{n,q}$ is an ETT satisfying MP under φ^* because it shares the same ladders as $T_{n,1}$. Now we will prove that T is an ETT satisfying MP under the $\varphi \bmod T$ coloring φ^* . First we assume $P = P_{v(\alpha)}(\alpha, \beta, \varphi)$. Then $T_{v(\beta)}$ can still be obtained from $T_{n,1}$ by TAA under φ^* because $\alpha, \beta \notin \varphi^*(T_{v(\beta)})$ and T can be obtained from $T_{v(\beta)}$ by TAA because $\alpha, \beta \in \overline{\varphi^*}(T_{v(\beta)})$. Thus T is an ETT under φ^* . It satisfies MP because it shares the same ladder as $T_{n,1}$. We then assume $P \neq P_{v(\alpha)}(\alpha, \beta, \varphi)$. In this case, we still have $\alpha, \beta \notin \varphi^*(T_{v(\beta)})$ and $\alpha, \beta \in \overline{\varphi^*}(T_{v(\beta)})$, and therefore T is an ETT under φ^* , and it satisfies MP because it shares the same ladder as $T_{n,1}$. Finally we claim T still satisfies R2 with the same splitters and Γ sets. Since $\overline{\varphi}(v') = \overline{\varphi^*}(v')$ for each $v' \in T_{n,q}$ and $\varphi(f) = \varphi^*(f)$ for each $f \in T_{v(\beta)}$, $T_{n,s}$ still satisfies R2 (1) for each $1 < s \leq q$ under φ^* by the same Γ^{s-1}

as under φ . Because $T_{n,s}$ satisfies R2 (2) for $1 < s \leq q$, $T_{n,s}$ is closed under φ for colors in $\overline{\varphi}(T_s) - \cup_{\eta_h \in D_{n,s}} \Gamma_h^{s-1}$. Since $\alpha, \beta \notin \overline{\varphi}(T_{n,q}) \cup D_{n,q}$, $T_{n,s}$ is still closed under φ^* for colors in $\overline{\varphi}^*(T_s) - \cup_{\eta_h \in D_{n,s}} \Gamma_h^{s-1} = \overline{\varphi}(T_s) - \cup_{\eta_h \in D_{n,s}} \Gamma_h^{s-1}$ with $1 < s \leq q$, and therefore it still satisfies R2 (2) under φ^* by the same Γ^{s-1} as under φ . Thus $T_{n,q}$ satisfies R2 up to itself under φ^* by the same splitters and Γ sets. Moreover, since $\alpha, \beta \notin \overline{\varphi}(T_{n,q}) \cup D_{n,q}$, $\alpha, \beta \notin \Gamma^q \cup D_{n,q}$. Therefore edges in T colored by Γ^q are colored the same under φ^* as under φ , and the vertex missing η under φ still misses η under φ^* for each $\eta \in D_{n,q} \cap T$. Thus $T = T_{n,q+1}$ satisfies R2 (1) under φ^* . So T is an ETT satisfying MP and R2 by the same split tail and Γ sets. \square

2.3.1.2.2 Case verification

Claim 2.3.12. $p > 0$

Proof. Suppose on the contrary $p = 0$, that is, $T = T_{n,q} \cup \{e_0, y_0\}$. We consider two cases.

Case I: $q = 1$.

Let $\alpha \in \overline{\varphi}(T_{n,q}) \cap \overline{\varphi}(y_0)$. By the construction of T , we have $\varphi(e_0) = \beta \in \overline{\varphi}(T_{n,1})$. Let $\theta \in \overline{\varphi}(T_n \cap T_n^d)$. By Lemma 2.3.2, $P_{v(\theta)}(\alpha, \theta, \varphi)$ is the unique (α, θ) -path intersecting $T_{n,1}$. Then $P_{y_0}(\alpha, \theta, \varphi) \cap T_{n,1} = \emptyset$. Let $\varphi^* = \varphi / P_{y_0}(\alpha, \theta, \varphi)$. Since $\theta \in \overline{\varphi}(T_n)$, by Claim 2.3.11 (2) we have that φ^* is $T_{n,1}$ -stable. Note that θ is closed in φ^* . By Lemma 2.3.2, β and θ are $T_{n,1}$ -interchangeable under φ^* . Thus $P_{v(\theta)}(\theta, \beta, \varphi^*) = P_{v(\beta, T_{n,1})}(\theta, \beta, \varphi^*)$. However, we have $P_{y_0}(\theta, \beta, \varphi^*) \cap T_{n,1} \neq \emptyset$ and $\theta, \beta \in \overline{\varphi}^*(T_{n,1})$, which implies that there are at least two (θ, β) -paths intersecting $T_{n,1}$, a contradiction to β and θ being $T_{n,1}$ -interchangeable under φ^* .

Case II: $q > 1$. In this case $T_{n,q}$ is $(\cup_{\eta_h \in D_{n,q}} \Gamma_h^{q-1})^-$ closed by R2 (2).

Assume without loss of generality that e_0 is colored by $\gamma_0 \in \Gamma^{q-1}$. Moreover, $\varphi(e_0) \notin \Gamma^q$ because T satisfies R2 (1). Let $\alpha \in \overline{\varphi}(T_{n,q}) \cap \overline{\varphi}(y_0)$. Let $\theta \in \overline{\varphi}(T_{n,q} - \Gamma^{q-1})$. Then $\theta \in \overline{\varphi}(T_{n,q})$ is closed in $T_{n,q}$ and $\theta \neq \gamma_0$. By Claim 2.3.10, $P_{v(\theta)}(\alpha, \theta, \varphi) = P_{v(\alpha)}(\alpha, \theta, \varphi)$ is the unique

(α, θ) -path intersecting $T_{n,q}$. Therefore $P_{y_0}(\alpha, \theta, \varphi) \cap T_{n,q} = \emptyset$. Let $\varphi^* = \varphi / P_{y_0}(\alpha, \theta, \varphi)$. Then by Claim 2.3.11, φ^* is $T_{n,q}$ -stable and $T_{n,q}$ satisfies MP and R2 up to itself with the same splitters and Γ sets. Since e_0 is still colored by $\gamma_0 \in \Gamma^{q-1}$, T can be obtained from $T_{n,q}$ by TAA and thus it is an ETT satisfying MP by Claim 2.3.11. Moreover, $T = T_{n,q+1}$ clearly satisfies R2 (1) because $\varphi^*(e_0) \notin \Gamma^q$. Thus T is still a “minimum” counter-example ETT satisfying MP and R2 under φ^* . However, we have $P_{y_0}(\theta, \gamma_0, \varphi^*) \cap T_{n,q} \neq \emptyset$ and $\theta, \gamma_0 \in \overline{\varphi^*}(T_{n,q})$, which implies that there are at least two (θ, γ_0) -paths intersecting $T_{n,q}$, a contradiction to Claim 2.3.10. \square

We now consider three cases according to $P(T)$.

Case 1. $p(T) = 0$. In this case $T - T_{n,q}$ is a path obtained by TAA under φ , so we call T a *Generalized Kierstead path*.

Claim 2.3.13. *We may assume $\alpha \in \overline{\varphi}(y_i) \cap \overline{\varphi}(y_p)$ for some $p > i \geq 0$.*

Proof. Suppose $\alpha \in \overline{\varphi}(y_p) \cap \overline{\varphi}(v)$ for some $v \in V(T_{n,q})$. We will show that we may assume $\alpha \in \overline{\varphi}(T_n)$ when $q = 1$. Assume $\alpha \in \overline{\varphi}(T_n^d - T_n)$. Then $\alpha \notin \Gamma^1$. Since $|\overline{\varphi}(T_n)| \geq 11 + 2n$ by Claim 2.3.9 and $|\Gamma^1| \leq 2n$, there exist $\beta \in \overline{\varphi}(T_n) - \Gamma^1$. By Claim 2.3.10, $P_{v(\alpha)}(\alpha, \beta, \varphi) = P_{v(\beta)}(\alpha, \beta, \varphi)$ and $P_{y_p}(\alpha, \beta, \varphi)$ is disjoint with $T_{n,1}$. By Claim 2.3.11, $\varphi^* = \varphi / P_{y_p}(\alpha, \beta, \varphi)$ is $T_{n,1}$ -stable and $T_{n,1}$ satisfies MP and R2 up to itself with the same splitters and Γ sets. Moreover, T can still be obtained from $T_{n,1}$ by TAA since $\alpha, \beta \in \overline{\varphi^*}(T_{n,1})$, and therefore T satisfies MP. Since both $\alpha, \beta \notin \Gamma^1$ and $\alpha, \beta \notin \overline{\varphi}(T_{y_{p-1}} - T_{n,1})$, T satisfies R2 (1). Under φ^* , $\beta \in \overline{\varphi^*}(T_n) \cap \overline{\varphi^*}(y_p)$. Thus T is a “minimum” counter-example ETT satisfying MP and R2 with q splitters. Hence we may assume $\alpha \in \overline{\varphi}(T_n)$.

We first consider the case $\alpha \notin \varphi(T - T_{n,q})$. Let $\beta \in \overline{\varphi}(y_{p-1})$. Then $\beta \notin \varphi(T - T_{n,q})$. Since $\alpha \in \overline{\varphi}(T_{n,q})$ and $\alpha \in \overline{\varphi}(T_n)$ when $q = 1$, by Claim 2.3.10 we have $P_v(\alpha, \beta, \varphi) = P_{y_{p-1}}(\alpha, \beta, \varphi)$, and therefore $P_{y_p}(\alpha, \beta, \varphi)$ is a difference path from the path above. Let $\varphi^* = \varphi / P_{y_p}(\alpha, \beta, \varphi)$. By Claim 2.3.11, φ^* is $T_{n,q}$ -stable and $T_{n,q}$ is an ETT satisfying MP and R2 up to itself with the same splitters and Γ sets under φ^* . Note that we will apply Claim 2.3.11 lots of times later, so as mentioned in the very beginning of the proof of Statement A, we may

simply say R2 is satisfied for $T_{n,q}$ up to it self instead of mentioning that R2 is satisfied for $T_{n,q}$ with the same splitters and Γ sets over and over again. Since $\alpha, \beta \notin \varphi(T_{y_p} - T_{n,q})$ and $\alpha, \beta \notin \overline{\varphi}(T_{y_{p-1}} - T_{n,q})$, T can still be obtained from $T_{n,q}$ and therefore it is an ETT satisfying MP. Moreover, we have $\varphi^*(f) = \varphi(f)$ for each $f \in T$ and $\overline{\varphi}(v') = \overline{\varphi}^*(v')$ for each $v' \in T_{y_{p-1}}$. Thus R2 (1) is still satisfies for T under φ^* . Therefore T satisfies MP and R2 under φ^* . Note that we have $\beta \in \overline{\varphi}^*(y_{p-1}) \cap \overline{\varphi}^*(y_p)$, Claim 2.3.13 holds.

We now consider the case $\alpha \in \varphi(T - T_{n,q})$. Following order \prec_ℓ , let e_j be the first edge in $T - T_{n,q}$ such that $\alpha = \varphi(e_j)$. We first assume $j \geq 1$. Let $\beta \in \overline{\varphi}(y_{j-1})$. Then $\alpha, \beta \notin \varphi(T_{y_{j-1}} - T_{n,q})$. Since $\alpha \in \overline{\varphi}(T_{n,q})$ and $\alpha \in \overline{\varphi}(T_n)$ when $q = 1$. by Claim 2.3.10, $P_v(\alpha, \beta, \varphi) = P_{y_{j-1}}(\alpha, \beta, \varphi)$ and therefore $P_{y_p}(\alpha, \beta, \varphi)$ is a different (α, β) -path. Therefore by Claim 2.3.11, $\varphi^* = \varphi/P_{y_p}(\alpha, \beta, \varphi)$ is a $T_{n,q}$ -stable coloring and $T_{n,q}$ satisfies MP and R2 up to itself. In addition, $\overline{\varphi}^*(v') = \overline{\varphi}(v')$ for $v' \in T_{y_{p-1}}$. Moreover, since $\alpha, \beta \notin \varphi(T_{y_{j-1}} - T_{n,q})$, T can still be obtained from $T_{n,q}$ by TAA, and therefore T satisfies MP by Claim 2.3.11. Note $\beta \notin \Gamma^q$. If $\alpha \notin \Gamma^q$, then T satisfies R2 (1) because $\overline{\varphi}(v') = \overline{\varphi}(v')$ for $v' \in T_{y_{p-1}}$. If $\alpha \in \Gamma^q$, say $\alpha = \gamma_{i1}$ for some $0 < i \leq n'$, by R2 (1) we have $\eta_i \in \overline{\varphi}(w)$ for some $w \preceq_\ell y_{j-1}$. Since $\alpha, \beta \notin \varphi(T_{y_{j-1}} - T_{n,q})$ and only edges after w in the ordering \prec_ℓ may change colors between α and β , R2 (1) still holds for T under φ^* . Therefore T satisfies MP and R2 under φ^* by the same splitters. Since $\beta \in \overline{\varphi}^*(y_{j-1}) \cap \overline{\varphi}^*(y_p)$, Claim 2.3.13 holds by simply denoting φ^* as φ .

Now we assume that $j = 0$. Therefore we have $\alpha = \varphi(e_0)$. Note that $\alpha \notin \Gamma^q$ by condition R2 (1). We claim that there exists $\gamma \in \overline{\varphi}(T_{n,q}) - \Gamma^q$ when $q > 1$ and $\gamma \in \overline{\varphi}(T_n) - \Gamma^1$ when $q = 1$ such that γ is closed in $T_{n,q}$ under φ . We first assume $q \geq 2$. By condition R2(2), $T_{n,q}$ is $(\cup_{\eta_h \in D_{n,q}} \Gamma_h^{q-1})^-$ closed. Therefore, $T_{n,q}$ is closed for colors in $\overline{\varphi}(T_{n,q}) - \Gamma^{q-1}$ because $\cup_{\eta_h \in D_{n,q}} \Gamma_h^{q-1} \subseteq \Gamma^{q-1}$. Hence we need to show that there exists $\gamma \in \overline{\varphi}(T_{n,q}) - \Gamma^q \cup \Gamma^{q-1}$. Since $\Gamma^q - \Gamma^{q-1} \subseteq \overline{\varphi}(T_{n,q} - T_{n,q-1})$ by condition R2 and the assumption that $T_{n,q}$ is elementary, we have $|(\Gamma^q \cup \Gamma^{q-1}) \cap \overline{\varphi}(T_{n,q-1})| = |\Gamma^{q-1} \cap \overline{\varphi}(T_{n,q-1})| \leq 2n$ and $|\overline{\varphi}(T_{n,q-1})| \geq |\overline{\varphi}(T_1)| + 2(n-1) = 2n+11$. Therefore $|\overline{\varphi}(T_{n,q}) - \Gamma^q \cup \Gamma^{q-1}| = |\overline{\varphi}(T_{n,q-1}) - \Gamma^{q-1}| + |\overline{\varphi}(T_{n,q} - T_{n,q-1}) - (\Gamma^q - \Gamma^{q-1})| \geq |\overline{\varphi}(T_{n,q-1}) - \Gamma^{q-1}| \geq (2n+11) - 2n \geq 11$, where we have γ as desired. We then consider

the case $q = 1$. Recall that we have $|\overline{\varphi}(T_n)| \geq 11 + 2n > |\Gamma^1| + 1$. Note that we have a color $\theta \in \overline{\varphi}(T_{n,1})$ where θ is not closed in $T_{n,1}$ under φ . Thus if we pick colors for Γ^1 from $\overline{\varphi}(T_n) - \theta$ with priority from colors in $\overline{\varphi}(T_n) - \overline{\varphi}(T_n \cap T_n^d)$, then $\overline{\varphi}(T_n \cap T_n^d) - \Gamma^1$ is not empty and $T_{n,1}$ is not $(\Gamma^1)^-$ -closed. Recall that R2 requires us to pick colors for Γ^1 such that $|\overline{\varphi}(T_n \cap T_n^d) - \Gamma^1|$ is maximum with the restriction of $T_{n,1}$ not being $(\Gamma^1)^-$ -closed, there must exist a color $\gamma \in \overline{\varphi}(T_n \cap T_n^d) - \Gamma^1$. Note that γ is closed in $T_{n,1}$ under φ , we have as claimed.

By Claim 2.3.10, $P_{v(\alpha)}(\alpha, \gamma, \varphi) = P_{v(\gamma)}(\alpha, \gamma, \varphi)$, and therefore $P_{y_p}(\alpha, \beta, \varphi)$ is disjoint with $T_{n,q}$. Thus by Claim 2.3.11, $\varphi^* = \varphi/P_{y_p}(\alpha, \gamma, \varphi)$ is $T_{n,q}$ -stable and $T_{n,q}$ satisfies MP and R2 up to itself under φ^* . Moreover, we have $\overline{\varphi}^*(v') = \overline{\varphi}(v')$ for each $v' \in T_{y_{p-1}}$. Since $\alpha, \gamma \in \overline{\varphi}(T_{n,q})$, we have $\alpha, \gamma \in \overline{\varphi}^*(T_{n,q})$, and therefore T can still be obtained from $T_{n,q}$ by TAA. Thus it satisfies MP by Claim 2.3.11. Since $\alpha, \gamma \notin \Gamma^q$ and $\overline{\varphi}^*(v') = \overline{\varphi}(v')$ for each $v' \in T_{y_{p-1}}$, T satisfies R2 (2) under φ^* . Thus T is still a “minimum” counter-example under φ^* . Moreover, since $e_0 \in \partial(T_{n,q})$, $e_0 \notin P_{y_p}(\alpha, \gamma, \varphi)$. Thus $\varphi^*(e_0) = \alpha$. Now $\gamma \in \overline{\varphi}^*(y_p) \cap \overline{\varphi}^*(v)$ for some $v \in V(T_{n,q})$ and $\alpha \neq \gamma$, which either returns to the case $\gamma \notin \overline{\varphi}(T - T_{n,q})$ or the case $\gamma \in \overline{\varphi}(T - T_{n,q})$ where e_j is the first edge in $T - T_{n,q}$ colored by γ with $j \geq 1$. \square

Among all “minimum” counter-examples satisfying MP and R2, we assume that i is the maximum index such that $\alpha \in \overline{\varphi}(y_p) \cap \overline{\varphi}(y_i)$.

Claim 2.3.14. $i = p - 1$.

Proof. Suppose on the contrary $i < p - 1$. We first consider the case $\alpha \notin D_{n,q}$. Note that $\alpha \notin \varphi(T_{y_{i+1}} - T_n)$ because T is obtained from T_n by TAA under φ and $T_{y_{i+1}} - T_n$ is a path. Let $\theta \in \overline{\varphi}(y_{i+1})$. Then $\theta \notin \varphi(T_{y_{i+1}} - T_n)$. If $\theta \notin D_{n,q}$, then $\{\alpha, \theta\} \cap D_{n,q} = \emptyset$. By Claim 2.3.10, $P_{y_i}(\alpha, \theta, \varphi) = P_{y_{i+1}}(\alpha, \theta, \varphi)$. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \beta, \varphi)$. Since both $y_i, y_{i+1} \in T - T_{n,q}$ and $\alpha \notin \varphi(T_{y_{i+1}} - T_n)$, φ^* is $T_{n,q}$ -stable and T is still an ETT satisfying MP and R2 with the same splitters under φ^* by Claim 2.3.11 (3). But $\theta \in \overline{\varphi}^*(y_p) \cap \overline{\varphi}^*(y_{i+1})$, which contradicts the maximality of i . We now consider the case $\theta = \eta_k \in D_{n,q}$ for some $k \leq n$. Then by R2 (1), $\gamma_{k1} \notin \varphi(T_{y_{p+1}} - T_{n,q})$. Because $\gamma_{k1} \in \overline{\varphi}(T_{n,q})$ when $q \geq 2$ and $\gamma_{k1} \in \overline{\varphi}(T_n)$ when $q = 1$, we have that $P_{v(\gamma_{k1})}(\alpha, \gamma_{k1}, \varphi) = P_{y_i}(\alpha, \gamma_{k1}, \varphi)$ and $P_{y_p}(\alpha, \gamma_{k1}, \varphi)$ is a different (α, γ_{k1}) -path

by Claim 2.3.10. Moreover, $\overline{\varphi}^*(v') = \overline{\varphi}(v')$ for each $v' \in T_{y_{p-1}}$. Note that $P_{y_p}(\alpha, \gamma_{k1}, \varphi) = y_p$ can occur if $\gamma_{k1} \in \overline{\varphi}(y_p)$. By Claim 2.3.11, $\varphi^* = \varphi/P_{y_p}(\alpha, \gamma_{k1}, \varphi)$ is a $T_{n,q}$ -stable coloring and $T_{n,q}$ satisfies MP and R2 up to itself under φ^* . Recall that $\gamma_{k1} \notin \varphi(T_{y_{i+1}} - T_{n,q})$ by R2 (1). Moreover, since $\alpha \notin \varphi(T_{y_{i+1}} - T_n)$, T can still be obtained by TAA from $T_{n,q}$ under φ^* , and therefore it still satisfies MP by Lemma 2.3.11. Note that $\eta_k \in \overline{\varphi}^*(y_{i+1})$, $\alpha \notin \Gamma^q$ and $\alpha, \gamma_{k1} \notin \varphi(T_{y_{i+1}} - T_{n,q})$, T still satisfies R2 (2) under φ^* . Thus T is still a “minimum” counter-example under φ^* with $\gamma_{k1} \in \overline{\varphi}^*(y_p) \cup \overline{\varphi}^*(T_{n,q})$. Note that $\gamma_{k1}, \eta_k \notin \varphi^*(T_{y_{i+1}} - T_{n,q})$. Then, by Claim 2.3.10 again, $P_{v(\gamma_{k1})}(\eta_k, \gamma_{k1}, \varphi^*) = P_{y_i}(\eta_k, \gamma_{k1}, \varphi^*)$ and $P_{y_p}(\eta_k, \gamma_{k1}, \varphi^*)$ is a different (η_k, γ_{k1}) -path. Let $\varphi^{**} = \varphi^*/P_{y_p}(\eta_k, \gamma_{k1}, \varphi^*)$. By Claim 2.3.11, φ^{**} is $T_{n,q}$ -stable and $T_{n,q}$ satisfies MP and R2 up to itself under φ^{**} . Moreover, $\overline{\varphi}^{**}(v') = \overline{\varphi}^*(v')$ for each $v' \in T_{y_{p-1}}$. Recall that we have both $\gamma_{k1}, \eta_k \notin \varphi^*(T_{y_{i+1}} - T_{n,q})$. So $\gamma_{k1}, \eta_k \notin \varphi^{**}(T_{y_{i+1}} - T_{n,q})$. Therefore we see that T still can be obtained by TAA under φ^{**} , and therefore it is still an ETT satisfying MP by Claim 2.3.11. Moreover, since we have $\gamma_{k1}, \eta_k \notin \varphi^{**}(T_{y_{i+1}} - T_{n,q})$ and $\overline{\varphi}^{**}(v') = \overline{\varphi}^*(v')$ for each $v' \in T_{y_{p-1}}$, T still satisfies R2 (1) under φ^{**} . Thus T is still a “minimum” counter-example under φ^{**} . However under φ^{**} , $\eta_k \in \overline{\varphi}^{**}(y_p) \cap \overline{\varphi}^{**}(y_{i+1})$, giving a contradiction to the maximality of i .

We now consider the case $\alpha \in D_{n,q}$, say $\alpha = \eta_k \in D_{n,q}$ for some $k \leq n$. Since $\varphi(e_{i+1})$ can not be both γ_{k1} and γ_{k2} , we assume without loss of generality $\varphi(e_{i+1}) \neq \gamma_{k1}$. Then $\eta_k, \gamma_{k1} \notin \varphi(T_{y_{i+1}} - T_{n,q})$. By Claim 2.3.10, $P_{v(\gamma_{k1})}(\eta_k, \gamma_{k1}, \varphi) = P_{y_i}(\eta_k, \gamma_{k1}, \varphi)$, and therefore $P_{y_p}(\eta_k, \gamma_{k1}, \varphi)$ is a different path. Let $\varphi^* = \varphi/P_{y_p}(\eta_k, \gamma_{k1}, \varphi)$. By Claim 2.3.11, φ^* is $T_{n,q}$ -stable and $T_{n,q}$ satisfies MP and R2 up to itself. Moreover, $\overline{\varphi}^*(v') = \overline{\varphi}(v')$ for each $v' \in T_{y_{p-1}}$. Now $\gamma_{k1} \in \overline{\varphi}^*(y_p)$. Moreover, since $\gamma_{k1}, \eta_k \notin \varphi(T_{y_{i+1}} - T_{n,q})$, T can still be obtained from $T_{n,q}$ by TAA, and therefore it satisfies MP. Since $\overline{\varphi}^*(v') = \overline{\varphi}(v')$ for each $v' \in T_{y_{p-1}}$ and $\gamma_{k1}, \eta_k \notin \varphi(T_{y_{i+1}} - T_{n,q})$, we have $\gamma_{k1} \notin \varphi^*(T_{y_{i+1}} - T_{n,q})$ and therefore T still satisfies R2 (1) under φ^* . Thus T still serves as a “minimum” counter-example under φ^* . Let $\theta \in \overline{\varphi}^*(y_{i+1})$. Then $\theta, \gamma_{k1} \notin \varphi^*(T_{y_{i+1}} - T_{n,q})$. By the minimality of $|T - T_{n,q}|$, $\theta \neq \gamma_{k1}$. By Claim 2.3.10, $P_{v(\gamma_{k1})}(\theta, \gamma_{k1}, \varphi^*) = P_{y_{i+1}}(\theta, \gamma_{k1}, \varphi^*)$, and $P_{y_p}(\theta, \gamma_{k1}, \varphi^*)$ is a different path. By Claim 2.3.11, $\varphi^{**} = \varphi^*/P_{y_p}(\theta, \gamma_{k1}, \varphi^*)$ is $T_{n,q}$ -stable and $T_{n,q}$ satisfies MP and R2 up to itself under φ^{**} .

Moreover, $\bar{\varphi}^{**}(v') = \bar{\varphi}^*(v')$ for each $v' \in T_{y_{p-1}}$. Since $\theta, \gamma_{k1} \notin \varphi^*(T_{y_{i+1}} - T_{n,q})$, T can still be obtained by TAA under φ^{**} , and therefore it is still an ETT satisfying MP. Since $\theta, \gamma_{k1} \notin \varphi^*(T_{y_{i+1}} - T_{n,q})$, $\bar{\varphi}^{**}(v') = \bar{\varphi}^*(v')$ for each $v' \in T_{y_{p-1}}$, $\eta_k \in \bar{\varphi}^{**}(y_i)$ and $\theta \notin \Gamma^q$, T still satisfies R2 (1) under φ^{**} . However, $\theta \in \bar{\varphi}^{**}(y_p) \cap \bar{\varphi}^{**}(y_{i+1})$, giving a contradiction to the maximality of i . \square

Now we have $i = p - 1$. Let $\varphi(e_p) = \theta$. Since $\alpha \in \bar{\varphi}(y_p) \cap \bar{\varphi}(y_{p-1})$, we can recolor e_p by α . Denote the new coloring by φ^* . Denote $T_{y_{p-2}} = T_{n,q}$ if $p = 1$. By Lemma 2.3.5, $T_{y_{p-2}}$ still satisfies MP and R2 under φ^* and if $p = 1$, $T_{n,q}$ satisfies R2 up to itself. Moreover we clearly have $\theta \in \bar{\varphi}^*(y_{p-1})$, and φ^* is $T_{n,q}$ -stable. Since $\varphi(f) = \varphi^*(f)$ for each $f \in T_{y_{p-1}}$ and $\bar{\varphi}(v') = \bar{\varphi}^*(v')$ for each $v' \in T_{y_{p-2}}, T_{y_{p-1}}$ still satisfies R2 (1) under φ^* . Note that $v(\theta) \prec_\ell y_{p-1}$, we have a counterexample which has one less vertex than T , giving a contradiction. \square

Case 2. $p(T) = p \geq 1$. In this case, y_{p-1} is not incident to e_p . Let $\theta = \varphi(e_p)$.

We divide this case into a number of sub-cases according whether $v = y_{p-1}$ and $\alpha \in D_{n,q}$. We will prove Case 2.1.1 independently. Case 2.1.2 may redirect to Case 2.1.1 and Case 2.2.1., Case 2.2.1 may redirect to Case 2.1.1., Case 2.2.2 may redirect to Case 2.2.1., which may redirect to Case 2.1.1. Case 2.3.1 may redirect to Case 2.1., Case 2.3.2 may redirect to Case 2.1.1. and Case 2.2.. Therefore, in the end, there is no loophole in our proof.

Case 2.1. $\alpha \in \bar{\varphi}(y_p) \cap \bar{\varphi}(y_{p-1})$ and $\alpha = \eta_m \in D_{n,q}$.

Since $\eta_m \in \bar{\varphi}(y_p)$, we have $\theta \neq \eta_m$. Note that $\theta \in D_{n,q}$ may occur. Moreover, $\alpha \in D_n - D_{n,1}$ will not happen when $q = 1$ since we assumed $\alpha = \eta_m \in D_{n,q}$. Moreover, we have $\alpha \notin \varphi(T - T_{n,q})$ because $\alpha \in \bar{\varphi}(y_p)$.

Case 2.1.1. $\theta \notin \bar{\varphi}(y_{p-1})$.

During this part of the proof, we will consider the tree sequence $T^* = (T_{n,q}, e_0, y_0, e_1, \dots, e_{p-2}, y_{p-2}, e_p, y_p)$ and $T_{y_p, y_{p-1}} = (T_{n,q}, e_0, y_0, e_1, \dots, y_{p-2}, e_p, y_p, e_{p-1}, y_{p-1})$ for different situations. Note that both T^* and $T_{y_p, y_{p-1}}$ can be obtained from $T_{n,q}$ by TAA under φ , thus

they are both ETTs and they satisfy MP. Note that $T_{n,q}$ satisfies R2 up to itself under φ by our assumption.

We first consider the case $\theta \notin \Gamma^q$. Since $\eta_m \in \overline{\varphi}(y_{p-1})$, $\gamma_{m1}, \gamma_{m2} \notin \varphi(T_{y_{p-1}} - T_{n,q})$ by R2 (1). Thus $\gamma_{m1}, \gamma_{m2} \notin \varphi(T - T_{n,q})$. Since $\theta \notin \Gamma^q$, T^* satisfies R2 (1) under φ . Thus T^* satisfies R2 because $T_{n,q}$ satisfies R2 up to itself under φ . We will show that $\gamma_{m1} \notin \overline{\varphi}(y_p)$. Otherwise, $\gamma_{m1} \in \overline{\varphi}(y_p)$ and γ_{m1} is missing twice in the ETT T^* when $p \geq 2$. Here T^* gives a counterexample smaller than T because it satisfies R2. If $p = 1$, we consider $(T_{n,q}, e_1, y_1)$ as a smaller counter-example. Note that $(T_{n,q}, e_1, y_1)$ still satisfies MP because it can be obtained from $T_{n,q}$ by TAA under φ and $T_{n,q}$ satisfies MP under φ . Moreover, $(T_{n,q}, e_1, y_1)$ still satisfies R2 (1) while dropping e_0 , as $\varphi(e_1) = \theta \notin \Gamma^q$. Thus we indeed have $\gamma_{m1} \notin \overline{\varphi}(y_p)$. Note that we have $P_{v(\gamma_{m1})}(\eta_m, \gamma_{m1}, \varphi) = P_{y_{p-1}}(\eta_m, \gamma_{m1}, \varphi)$ by Claim 2.3.10 because $\gamma_{m1} \notin \varphi(T_{y_{p-1}} - T_{n,q})$. Now we consider the ETT $T_{y_p, y_{p-1}}$. Since $\theta \notin \overline{\varphi}(y_{p-1})$, $\theta \notin \Gamma^q$ and T satisfies R2 under φ , $T_{y_p, y_{p-1}}$ also satisfies R2 under φ . Moreover, $\gamma_{m1} \notin \varphi(T_{y_p, y_{p-1}}(y_p) - T_{n,q})$. Applying Claim 2.3.10 again, we have $P_{v(\gamma_{m1})}(\eta_m, \gamma_{m1}, \varphi) = P_{y_p}(\eta_m, \gamma_{m1}, \varphi)$, giving a contradiction to $P_{v(\gamma_{m1})}(\eta_m, \gamma_{m1}, \varphi) = P_{y_{p-1}}(\eta_m, \gamma_{m1}, \varphi)$.

Now we assume $\theta \in \Gamma^q$. Without loss of generality we say $\theta = \gamma_{k1}$ for some $k \leq n'$ with $\eta_k \in D_{n,q}$. Since $\eta_m \in \overline{\varphi}(y_{p-1})$, $\gamma_{m1}, \gamma_{m2} \notin \varphi(T_{y_{p-1}} - T_{n,q})$ by R2 (1). By Claim 2.3.10, $P_{v(\gamma_{m1})}(\eta_m, \gamma_{m1}, \varphi) = P_{y_{p-1}}(\eta_m, \gamma_{m1}, \varphi)$. If $\eta_k \notin \overline{\varphi}(y_{p-1})$, then $\eta_k \in \overline{\varphi}(T_{y_{p-1}} - y_{p-1})$. Thus we are still able to show $T_{y_p, y_{p-1}}$ and T^* satisfying R2. So we can argue the same in the previous case by considering $T_{y_p, y_{p-1}}$ and find a contradiction by showing $P_{v(\gamma_{m1})}(\eta_m, \gamma_{m1}, \varphi) = P_{y_p}(\eta_m, \gamma_{m1}, \varphi)$ and $P_{v(\gamma_{m1})}(\eta_m, \gamma_{m1}, \varphi) = P_{y_{p-1}}(\eta_m, \gamma_{m1}, \varphi)$. Hence we assume $\eta_k \in \overline{\varphi}(y_{p-1})$. Since γ_{k1} can not be both γ_{m1} and γ_{m2} , we assume $\gamma_{m2} \neq \gamma_{k1}$. Then we have $\gamma_{m2} \notin \varphi(T - T_{n,q})$ by R2 (1). Moreover, $\eta_m \notin \varphi(T - T_{n,q})$. By Claim 2.3.10, $P_{v(\gamma_{m2})}(\eta_m, \gamma_{m2}, \varphi) = P_{y_{p-1}}(\eta_m, \gamma_{m2}, \varphi)$, and therefore $P_{y_p}(\eta_m, \gamma_{m2}, \varphi)$ is a different path. Note that $\gamma_{m2} \in \overline{\varphi}(T_{n,q})$ if $q > 1$ and $\gamma_{m2} \in \overline{\varphi}(T_n)$ when $q = 1$, by Claim 2.3.11, $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring $\varphi^* = \varphi/P_{y_p}(\eta_m, \gamma_{m2}, \varphi)$. Because $\gamma_{m1}, \eta_m \notin \varphi(T - T_{n,q})$, T can still be obtained from $T_{n,q}$ by TAA under φ^* and therefore it still satisfies MP. Moreover, it satisfies R2 (1) because $\gamma_{m2}, \gamma_{m1} \notin \varphi(T - T_{n,q})$.

By Claim 2.3.9, $|\overline{\varphi}^*(T_{y_{p-1}}) - \varphi^*(T_{y_{p-1}} - T_{n,q}) - \overline{\varphi}(T_n^d - T_n)| \geq 11 + 2n$. Hence there exists $\beta \in \overline{\varphi}^*(T_{y_{p-2}}) - \Gamma^q - \overline{\varphi}(T_n^d - T_n)$ such that $\beta \notin \varphi^*(T - T_{n,q})$. By Claim 2.3.10, $P_{v(\gamma_{m2})}(\beta, \gamma_{m2}, \varphi^*) = P_{v(\beta)}(\beta, \gamma_{m2}, \varphi^*)$, and $P_{y_p}(\beta, \gamma_{m2}, \varphi^*)$ is a different path than above. Let $\varphi^{**} = \varphi^*/P_{y_p}(\beta, \gamma_{m2}, \varphi^*)$. Applying Claim 2.3.11, we see that the coloring φ^{**} is $T_{n,q}$ -stable and $T_{n,q}$ satisfies MP and R2 up to itself. Moreover, since $\gamma_{m2}, \beta \notin \varphi^*(T - T_{n,q})$, $\gamma_{m2}, \beta \notin \varphi^{**}(T - T_{n,q})$. Therefore $\varphi^{**}(f) = \varphi^*(f)$ for each $f \in T$ and $\overline{\varphi}^{**}(v') = \varphi^*(v')$ for each $v' \in T_{y_{p-1}}$. Hence T satisfies MP because it can be obtained from $T_{n,q}$ by TAA under φ^{**} and moreover, it satisfies R2 (1) under φ^{**} too. Now $\beta \in \overline{\varphi}^{**}(y_p)$. Since $\gamma_{k1}, \beta \notin \varphi^{**}(T_{y_{p-1}} - T_{n,q})$, by Claim 2.3.10, $P_{v(\gamma_{k1})}(\beta, \gamma_{k1}, \varphi^{**}) = P_{v(\beta)}(\beta, \gamma_{k1}, \varphi^{**})$, and therefore $P_{y_p}(\beta, \gamma_{k1}, \varphi^{**})$ is a different path. Finally, we let $\varphi^{***} = \varphi^{**}/P_{y_p}(\beta, \gamma_{k1}, \varphi^{**})$. Note that $\beta \notin \varphi^{**}(T - T_{n,q})$ and $\gamma_{k1} \notin \varphi^{**}(T_{y_{p-1}} - T_{n,q})$, we can see that $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{***} by Claim 2.3.11. Moreover, T still can be obtained from $T_{n,q}$ by TAA under φ^{***} , and therefore it is an ETT satisfying MP. Note that we have $\varphi^{***}(f) = \varphi^{**}(f)$ for each $f \in T_{y_{p-1}}$, $\varphi^{***}(e_p) = \beta$, $\beta \notin \Gamma^q$ and $\overline{\varphi}^{***}(v') = \varphi^{**}(v')$ for each $v' \in T_{y_{p-1}}$, T satisfies R2 (1) under φ^{***} . Since $\varphi^{***}(e_p) = \beta$, $\beta \notin \Gamma^q$, $\gamma_{k1} \in \overline{\varphi}^{***}(y_p)$ and $v(\beta) \prec_\ell y_{p-1}$, we have T^* satisfies MP and R2 (1) under φ^{***} . Moreover, we see that T^* satisfies R2 because $T_{n,q}$ satisfies R2 up to itself under φ^{***} . Thus T^* is a counter-example smaller than T under φ^{***} , giving a contradiction.

Case 2.1.2. $\theta \in \overline{\varphi}(y_{p-1})$.

In this case $\theta \neq \eta_m$ since $\eta_m \in \overline{\varphi}(y_p)$ and $\theta \notin \overline{\varphi}(T_{n,q})$ since $\theta \in \overline{\varphi}(y_{p-1})$. Because T is obtained from $T_{n,q}$ by TAA under φ , $\eta_m \notin \varphi(T_{y_{p-1}} - T_{n,q})$. Thus $\eta_m \notin \varphi(T - T_{n,q})$ because $\theta \neq \eta_m$. Since $\theta \notin \overline{\varphi}(T_{n,q})$, $\theta \notin \Gamma^q$ and therefore $\theta \neq \gamma_{m1}$. By condition R2 (1), $\gamma_{m1} \notin \varphi(T_{y_{p-1}} - T_{n,q})$ because $\eta_m \in \overline{\varphi}(y_{p-1})$. Therefore, $\gamma_{m1} \notin \varphi(T - T_{n,q})$ because $\theta \neq \gamma_{m1}$. By Claim 2.3.10, $P_{v(\gamma_{m1})}(\eta_m, \gamma_{m1}, \varphi) = P_{y_{p-1}}(\eta_m, \gamma_{m1}, \varphi)$ and $P_{y_p}(\eta_m, \gamma_{m1}, \varphi)$ is a different path from above. Let $\varphi^* = \varphi/P_{y_p}(\eta_m, \gamma_{m1}, \varphi)$. By Claim 2.3.11, $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Moreover, $\gamma_{m1}, \eta_m \notin \varphi(T - T_{n,q})$ implies $\varphi^*(f) = \varphi(f)$ for each $f \in T$ and $\overline{\varphi}^*(v') = \varphi(v')$ for each $v' \in T_{y_{p-1}}$. Thus T is clearly an ETT satisfying MP and R2 (1) under φ^* . Therefore T is still a “minimum”

counter-example under φ^* . Since $\gamma_{m1} \in \varphi^*(T - T_{n,q})$, by applying Claim 2.3.10 again, we have $P_{v(\gamma_{m1})}(\theta, \gamma_{m1}, \varphi^*) = P_{y_{p-1}}(\theta, \gamma_{m1}, \varphi^*)$, and therefore $P_{y_p}(\theta, \gamma_{m1}, \varphi^*)$ is a different path. Let $\varphi^{**} = \varphi^*/P_{y_p}(\theta, \gamma_{m1}, \varphi^*)$. Note that $\varphi_{e_p}^{**} = \gamma_{m1}$ because $\varphi_{e_p}^* = \theta$. Now since $\gamma_{m1} \notin \overline{\varphi}^*(T - T_{n,q})$, by applying Claim 2.3.11, we see that $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{**} . Moreover, since $\gamma_{m1}, \theta \notin \varphi(T_{y_{p-1}} - T_{n,q}) = \varphi^*(T_{y_{p-1}} - T_{n,q})$, T can still be obtained from $T_{n,q}$ by TAA under φ^{**} , and therefore it satisfies MP under φ^{**} . Since $\varphi^{**}(e_p) = \gamma_{m1}$ and $\eta_m \in \overline{\varphi}^{**}(y_{p-1}) = \overline{\varphi}^*(y_{p-1})$, T satisfies R2 (1) under φ^{**} . Note that under φ^{**} , $\theta \in \overline{\varphi}^{**}(y_p) \cap \overline{\varphi}^{**}(y_{p-1})$ and $\varphi^{**}(e_p) = \gamma_{m1}$. Thus if $\theta \in D_{n,q}$, then in φ^{**} we have Case 2.1.1. So we may assume $\theta \notin D_{n,q}$, which will be handled in case a of Case 2.2.1 below.

Case 2.2. $\alpha \in \overline{\varphi}(y_p) \cap \overline{\varphi}(y_{p-1})$ and $\alpha \notin D_{n,q}$. In this case we have $\alpha \notin D_n$ when $q = 1$ because $\alpha \notin \overline{\varphi}(T_{n,q})$. Moreover, we have $\alpha \notin \varphi(T - T_{n,q})$ because $\alpha \in \overline{\varphi}(y_p)$.

Case 2.2.1. $\theta \notin \overline{\varphi}(y_{p-1})$.

In this case, $T_{y_p, y_{p-1}}$ is also an ETT under φ satisfying MP and R2 with the same splitters and Γ sets as T except for the case where $\theta \in \Gamma_m^q$ with $\eta_m \in D_{n,q}$ and $\eta_m \in \overline{\varphi}(y_{p-1})$. We first assume $T_{y_p, y_{p-1}}$ satisfies R2, i.e. there does not exist $\eta_m \in D_{n,q}$ such that $\theta \in \Gamma_m^q$ and $\eta_m \in \overline{\varphi}(y_{p-1})$. Thus in this case $T^* = (T_{n,q}, y_0, e_1, \dots, e_{p-2}, y_{p-2}, e_p, y_p)$ satisfies MP and R2 under φ^* . By Claim 2.3.9, we have $|\overline{\varphi}(Ty_{p-1}) - \varphi(Ty_{p-1} - T_{n,q}) - \overline{\varphi}(T_n^d - T_n)| \geq 2n + 11$. So there exists a color $\beta \in \overline{\varphi}(Ty_{p-2}) - D_{n,q}$ when $q > 1$ and $\beta \in \overline{\varphi}(Ty_{p-2}) - D_n$ when $q = 1$ such that $\beta \notin \varphi(T - T_{n,q})$. We claim that $\beta \notin \overline{\varphi}(y_p)$. Otherwise, T^* is a counterexample smaller than T , giving a contradiction. Since $\alpha, \beta \notin D_{n,q}$ when $q > 1$, $\alpha, \beta \notin D_n$ when $q = 1$ and $\alpha, \beta \notin \varphi(T - T_{n,q})$, by Claim 2.3.10 we have $P_{v(\beta)}(\alpha, \beta, \varphi) = P_{y_{p-1}}(\alpha, \beta, \varphi)$. Applying Claim 2.3.10 to $T_{y_p, y_{p-1}}$, we see that $P_{v(\beta)}(\alpha, \beta, \varphi) = P_{y_p}(\alpha, \beta, \varphi)$. So, $P_{v(\beta)}(\alpha, \beta, \varphi)$ has three endvertices $v(\beta)$, y_{p-1} and y_p , a contradiction. Hence, we may assume there exists $\eta_m \in D_{n,q}$ such that $\eta_m \in \overline{\varphi}(y_{p-1})$ and $\theta = \gamma_{m1}$. By R2, $\gamma_{m2}, \alpha \notin \varphi(T_{y_{p-1}} - T_{n,q})$. So $\gamma_{m2}, \alpha \notin \varphi(T - T_{n,q})$ and $\eta_m \notin \varphi(T - T_{n,q})$.

By Claim 2.3.10, $P_{v(\gamma_{m2})}(\alpha, \gamma_{m2}, \varphi) = P_{y_{p-1}}(\alpha, \gamma_{m2}, \varphi)$ and $P_{y_p}(\alpha, \gamma_{m2}, \varphi)$ is a different

path from above. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \gamma_{m_2}, \varphi)$. By applying Claim 2.3.11, we see that $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Moreover, because $\gamma_{m_2}, \alpha \notin \varphi(T - T_{n,q})$, T can still be obtained from $T_{n,q}$ by TAA under φ^* and therefore it satisfies MP under φ^* . Because $\gamma_{m_2}, \alpha \notin \varphi(T - T_{n,q})$, $\varphi^*(f) = \varphi(f)$ for each $f \in T$ and $\bar{\varphi}(v') = \bar{\varphi}^*(v')$ for each $v' \in T_{y_{p-1}}$. Thus T still satisfies R2 (2) under φ^* . If $\eta_m \in \bar{\varphi}^*(y_p)$, then with $\eta_m \in \bar{\varphi}^*(y_p) \cap \bar{\varphi}^*(y_{p-1})$ and $\varphi^*(e_p) = \gamma_{m_1} \notin \bar{\varphi}^*(y_{p-1})$ we have reached Case 2.1.1. Hence we assume $\eta_m \notin \bar{\varphi}^*(y_p)$. Because $\eta_m, \gamma_{m_2}, \alpha \notin \varphi(T - T_{n,q})$, $\eta_m, \gamma_{m_2} \notin \varphi^*(T - T_{n,q})$. Now by Claim 2.3.10, $P_{v(\gamma_{m_2})}(\eta_m, \gamma_{m_2}, \varphi^*) = P_{y_{p-1}}(\eta_m, \gamma_{m_2}, \varphi^*)$, and $P_{y_p}(\eta_m, \gamma_{m_2}, \varphi^*)$ is different from the path above. Let $\varphi^{**} = \varphi^*/P_{y_p}(\eta_m, \gamma_{m_2}, \varphi^*)$. By applying Claim 2.3.11, we see that $T_{n,q}$ is an ETT satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{**} . Since $\gamma_{m_2}, \eta_m \notin \varphi^*(T - T_{n,q})$, $\varphi^{**}(f) = \varphi^*(f)$ for each $f \in T$ and $\bar{\varphi}^*(v') = \bar{\varphi}^{**}(v')$ for each $v' \in T_{y_{p-1}}$. Therefore T satisfies MP and R2 (1) under φ^{**} . Note that under φ^{**} , we have $\eta_m \in \bar{\varphi}^{**}(y_p) \cap \bar{\varphi}^{**}(y_{p-1})$, $\gamma_{m_1} = \varphi^{**}(e_p) \notin \bar{\varphi}^{**}(y_{p-1})$, which also leads us back to Case 2.1.1.

Case 2.2.2. $\theta \in \bar{\varphi}(y_{p-1})$.

We first assume $\theta = \eta_m$ for some $\eta_m \in D_{n,q}$. By R2 (1), $\gamma_{m_1} \notin \varphi(T - T_{n,q})$. By Claim 2.3.10, $P_{v(\gamma_{m_1})}(\alpha, \gamma_{m_1}, \varphi) = P_{y_{p-1}}(\alpha, \gamma_{m_1}, \varphi)$, and $P_{y_p}(\alpha, \gamma_{m_1}, \varphi)$ is a different path. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \gamma_{m_1}, \varphi)$. By Claim 2.3.11, we see that φ^* is $T_{n,q}$ -stable and $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Moreover since $\alpha, \gamma_{m_1} \notin \varphi(T - T_{n,q})$, $\varphi^*(f) = \varphi(f)$ for each $f \in T$ and $\bar{\varphi}(v') = \bar{\varphi}^*(v')$ for each $v' \in T_{y_{p-1}}$. Thus T still satisfies MP and R2 (2) under φ^* . Therefore T serves as a “minimum” counter-example under φ^* . Note that $\gamma_{m_1} \notin \varphi^*(T - T_{n,q})$. By Claim 2.3.10 again, $P_{v(\gamma_{m_1})}(\eta_{m_1}, \gamma_{m_1}, \varphi^*) = P_{y_{p-1}}(\eta_{m_1}, \gamma_{m_1}, \varphi^*)$, and therefore $P_{y_p}(\eta_{m_1}, \gamma_{m_1}, \varphi^*)$ is a different path from above. Let $\varphi^{**} = \varphi^*/P_{y_p}(\eta_{m_1}, \gamma_{m_1}, \varphi^*)$. By applying Claim 2.3.11, $T_{n,q}$ is still an ETT satisfying MP and R2 under the $T_{n,q}$ -stable coloring φ^{**} . Since $\gamma_{m_1} \notin \varphi^*(T - T_{n,q})$, $\eta_m \notin \varphi^*(T_{y_{p-1}} - T_{n,q})$ and $\varphi^*(e_p) = \eta_m$, we have $\varphi^{**}(f) = \varphi^*(f)$ for each $f \in T_{y_{p-1}}$, $\bar{\varphi}^*(v') = \bar{\varphi}^{**}(v')$ for each $v' \in T_{y_{p-1}}$ and $\varphi^{**}(e_p) = \gamma_{m_1}$. Thus T satisfies MP and R2 (2) under φ^{**} because $\eta_m \in \bar{\varphi}^{**}(y_{p-1})$. Note that we have $\eta_m \in \bar{\varphi}^{**}(y_p) \cap \bar{\varphi}^{**}(y_{p-1})$ and $\gamma_{m_1} = \varphi^{**}(e_p) \notin \bar{\varphi}^{**}(y_{p-1})$, which leads to Case 2.1.1.

We then assume $\theta \notin D_{n,q}$. Denote y_{-1} by the last vertex of $T_{n,q}$. We claim that there exists a color $\beta \in \overline{\varphi}(T_{y_{p-2}}) - \overline{\varphi}(T_n^d - T_n) - D_{n,q}$ with $\beta \notin \varphi(T - T_{n,q})$ such that either $\beta \notin \Gamma^q$ or $\beta = \gamma_{r1} \in \Gamma^q$ and $\eta_r \in \overline{\varphi}(T_{y_{p-2}})$. By Claim 2.3.9, if $|\overline{\varphi}(T_{y_{p-2}}) - \overline{\varphi}(T_n^d - T_n) - \Gamma^q \cup D_{n,q} \cup \varphi(T_{y_{p-2}} - T_{n,q})| \leq 4$, then there exist 7 distinct colors $\eta_i \in D_{n,q} \cap \overline{\varphi}(T_{y_{p-2}})$ such that all colors $\eta_i, \gamma_{i1}, \gamma_{i2} \notin \varphi(T_{y_{p-2}} - T_{n,q})$. Therefore, if $|\overline{\varphi}(T_{y_{p-2}}) - \overline{\varphi}(T_n^d - T_n) - \Gamma^q \cup D_{n,q} \cup \varphi(T_{y_{p-2}} - T_{n,q})| > 4$, we have $\beta \in \overline{\varphi}(T_{y_{p-2}}) - \overline{\varphi}(T_n^d - T_n) - D_{n,q}$ with $\beta \notin \varphi(T - T_{n,q})$ such that $\beta \notin \Gamma^q$ because $|E(T - T_{y_{p-2}})| = 2$. Otherwise, there exist 7 distinct colors $\eta_i \in D_{n,q} \cap \overline{\varphi}(T_{y_{p-2}})$ such that all colors $\eta_i, \gamma_{i1}, \gamma_{i2} \notin \varphi(T_{y_{p-2}} - T_{n,q})$. Thus we have $\beta = \gamma_{r1} \in \Gamma^q$ and $\eta_r \in \overline{\varphi}(T_{y_{p-2}})$ as desired since $|E(T - T_{y_{p-2}})| = 2$. Therefore $\alpha, \beta \notin \varphi(T - T_{n,q})$. Moreover, $\alpha, \beta \notin D_n$ since $\beta \notin \overline{\varphi}(T_{n,1} - T_n) \cup D_{n,q}$, $\alpha \in \overline{\varphi}(y_{p-1})$ and $\alpha \notin D_{n,q}$. Thus by Claim 2.3.10, $P_{v(\beta)}(\alpha, \beta, \varphi) = P_{y_{p-1}}(\alpha, \beta, \varphi)$ and therefore $P_{y_p}(\alpha, \beta, \varphi)$ is a different path. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \beta, \varphi)$. Applying Claim 2.3.11, we see that $T_{n,q}$ is an ETT satisfying MP and R2 under the $T_{n,q}$ -stable coloring φ^* . Since $\alpha, \beta \notin \varphi(T - T_{n,q})$, we have $\varphi^*(f) = \varphi(f)$ for each $f \in T$ and $\overline{\varphi}(v') = \overline{\varphi}^*(v')$ for each $v' \in T_{y_{p-1}}$. Thus T satisfies MP and R2 (2) under φ^* . Hence T serves as a “minimum” counter-example under φ^* . Under φ^* , we have $\beta \in \overline{\varphi}^*(y_p) \cap \overline{\varphi}^*(v(\beta))$ and $v(\beta) \neq y_{p-1}$. Note that $\theta \notin \varphi^*(T_{y_{p-1}} - T_{n,q})$ and $\beta \notin \varphi^*(T - T_{n,q})$. Since $\theta \notin D_{n,q}$ and $\theta \in \overline{\varphi}(y_{p-1})$, $\theta \notin D_n$. Since $\beta, \theta \notin D_n$ and $\beta \notin \overline{\varphi}^*(T_{n,1} - T_n)$, by Claim 2.3.10, $P_{v(\beta)}(\theta, \beta, \varphi^*) = P_{y_{p-1}}(\theta, \beta, \varphi^*)$ and therefore $P_{y_p}(\theta, \beta, \varphi^*)$ is different path. Let $\varphi^{**} = \varphi/P_{y_{p-1}}(\theta, \beta, \varphi^*)$. Applying Claim 2.3.11 again, we have that $T_{n,q}$ is an ETT satisfies MP and R2 under the $T_{n,q}$ -stable coloring φ^{**} . Note that we have $\varphi^*(f) = \varphi^{**}(f)$ for each $f \in T_{y_{p-1}}$, $\overline{\varphi}(v')^* = \overline{\varphi}^{**}(v')$ for each $v' \in T_{y_{p-1}}$ and $\varphi^{**}(e_p) = \beta$. Thus T satisfies MP under φ^{**} because it can be obtained from $T_{n,q}$ by TAA under φ^{**} . Now we check R2 for T . Since $\beta \notin \varphi^{**}(T_{y_{p-1}} - T_{n,q})$ and $\theta \notin \varphi^*(T - T_{n,q})$, T satisfies R2 (1) if $\beta \notin \Gamma^q$. For the case when $\beta = \gamma_{r1} \in \Gamma^q$, we have $\eta_r \in \overline{\varphi}(T_{y_{p-2}})$, which in turn gives R2 (1). Under φ^{**} , we have $\theta \in \overline{\varphi}^{**}(y_p) \cap \overline{\varphi}^{**}(y_{p-1})$, $\theta \notin D_{n,q}$ and $\varphi^{**}(e_p) = \beta \notin \overline{\varphi}^{**}(y_{p-1})$, which goes back to Case 2.2.1.

Case 2.3. $\alpha \in \overline{\varphi}(y_p) \cap \overline{\varphi}(v)$ for a vertex $v \prec_\ell y_{p-1}$.

Let $T_{y_{p-2}}$ be $T_{n,q}$ when $p = 1$. We then have the following claim.

Claim 2.3.15. *We may assume $\alpha \in \overline{\varphi}(T_{y_{p-2}})$ with $\alpha \notin \varphi(T - T_{n,q})$ such that either $\alpha \notin D_{n,q} \cup \Gamma^q$ when $q > 1$ and $\alpha \notin D_n \cup \Gamma^1$ when $q = 1$, or $\alpha = \eta_k \in \overline{\varphi}(T)$ with $\eta_k \in D_{n,q}$ and $\eta_k, \gamma_{k1}, \gamma_{k2} \notin \varphi(T - T_{n,q})$.*

Proof. By Claim 2.3.9, we have $|\overline{\varphi}(T_{y_{p-2}}) - \overline{\varphi}(T_n^d - T_n) - D_{n,q} \cup \Gamma^q \cup \varphi(T_{y_{p-2}} - T_{n,q})| \geq 4$ or there exists seven $\eta_i \in D_{n,q}$ such that all η_i, γ_{i1} and $\gamma_{i2} \in \overline{\varphi}(T_{y_{p-2}}) - \varphi(T - T_{n,q})$. The first inequality implies that there exists a color $\beta \in \overline{\varphi}(T_{p-2}) - \overline{\varphi}(T_n^d - T_n) - D_{n,q} \cup \Gamma^q \cup \varphi(T - T_{n,q})$. Hence we have $\beta \notin D_{n,q} \cup \Gamma^q$ when $q > 1$ and $\beta \notin D_n \cup \Gamma^1$ when $q = 1$, because $D_n - D_{n,1} \subset \overline{\varphi}(T_n^d - T_n)$. If the second case happens, there exist $\beta = \eta_k \in \overline{\varphi}(T)$ with $\eta_k \in D_{n,q}$ and $\eta_k, \gamma_{k1}, \gamma_{k2} \notin \varphi(T - T_{n,q})$ because we have $|E(T - T_{y_{p-2}})| = 2$. If $\beta \in \overline{\varphi}(y_p)$, we are done. Hence we assume $\beta \notin \overline{\varphi}(y_p)$. Let $P := P_{y_p}(\alpha, \beta, \varphi)$. We will show one of the following two statement holds.

- a: $\varphi^* = \varphi/P$ is $T_{n,q}$ -stable and T is an ETT satisfying MP and R2 with the same refinery and Γ sets under φ^* where the requirement of Claim 2.3.15 is satisfied by T .
- b: Under φ , there exist a non-elementary ETT T' with refinery $T_0 \subset T_1 \subset \dots \subset T_n \subset T_{n,1} \subset \dots \subset T_{n,q} \subset T'$ such that MP and R2 are satisfied, but $p(T') < p(T)$.

Note that Statement a. gives Claim 2.3.15 while Statement b. gives a contradiction to the minimality of p . We have that $\beta \notin \Gamma^q$ by the choice of β . We proceed with the proof by considering three cases: $\alpha \notin \Gamma^q$, $\alpha \in \Gamma^q - \varphi(T - T_{n,q})$ and $\alpha \in \Gamma^q \cap \varphi(T - T_{n,q})$.

If $V(P) \cap V(T_{y_{p-1}}) = \emptyset$, by Lemma 2.3.5, we have that $T_{y_{p-1}}$ is an ETT satisfying MP and R2 with R2 (2) satisfies up to $T_{n,q}$ under the $T_{n,q}$ -stable coloring $\varphi^* = \varphi/P$. Moreover, T is still an ETT satisfying MP and R2 (1) because $\varphi^*(e_p) = \varphi(e_p)$. Thus statement a. holds. Hence we assume $V(P) \cap V(T_{y_{p-1}}) \neq \emptyset$. Along the order of P from y_p , let u be the first vertex in $V(T_{y_{p-1}} \cap P)$ and P' be the subpath joining u and y_p . Let

$$\begin{aligned} T' &= T_{y_{p-2}} \cup P' && \text{if } u \neq y_{p-1}, \text{ and} \\ T' &= T_{y_{p-1}} \cup P' && \text{if } u = y_{p-1}. \end{aligned}$$

Note that $e_0 \notin T'$ may happen when $q = 1$, but it is easy to see that T' is still an ETT with the same ladder as T and q splitters where $T_{n,q} \subseteq T'$. Moreover, $T_{y_{p-2}}$ is an ETT satisfying MP and R2 with R2 (2) being satisfied for $T_{n,q}$ under φ , because T satisfies MP and R2 under φ . In addition, since $\alpha, \beta \in \overline{\varphi}(T_{y_{p-2}})$, T' can be obtained from $T_{y_{p-2}}$ by TAA, and therefore it satisfies MP under φ .

Case I: $\alpha \notin \Gamma^q$. Since $\alpha, \beta \notin \Gamma^q$, T' satisfies R2 (2). Hence statement b. holds and gives a contradiction to the minimality of $p(T)$.

Case II: $\alpha \in \Gamma^q \cap \varphi(T - T_{n,q})$. Assume $\alpha = \gamma_{m_1}$ for some $m \leq n$ where $\eta_m \in D_{n,q}$. Since $\varphi(e_p) \neq \alpha$, $\alpha \in \varphi(T_{y_{p-1}} - T_{n,q})$. Therefore we must have $\eta_m \in \overline{\varphi}(T_{y_{p-2}})$ by R2 (1). Furthermore, $\beta \in \overline{\varphi}(T_{y_{p-2}})$ with $\beta \notin \Gamma^q$. Therefore T' satisfies R2 (2). Hence statement b. holds.

Case III: $\alpha \in \Gamma^q - \varphi(T - T_{n,q})$. Let $\varphi^* = \varphi/P$. By Claim 2.3.10, $P_{v(\alpha)}(\alpha, \beta, \varphi) = P_{v(\beta)}(\alpha, \beta, \varphi)$, and therefore P is a different (α, β) -path. By Claim 2.3.11, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Note that in this case $\varphi(f) = \varphi^*(f)$ for each edge $f \in (T - T_{n,q})$ and $\overline{\varphi}(v') = \overline{\varphi^*}(v')$ for each $v' \in T_{y_{p-1}}$ because $\alpha, \beta \notin \varphi(T - T_{n,q})$. Therefore T is an ETT satisfying MP and R2 (2) under φ^* . Moreover, $\beta \notin \varphi^*(T - T_{n,q})$. Hence Statement a. holds. \square

Now let φ and α be the coloring and color in Claim 2.3.15. We then consider two cases.

Case 2.3.1. $\theta = \varphi(e_p) \notin \overline{\varphi}(y_{p-1})$.

Recall that $T^* = (T_{n,q}, e_0, y_0, e_1, y_1, \dots, e_{p-2}, y_{p-2}, e_p, y_p)$. In this case, T^* is an ETT satisfies MP under φ because it can be obtained from $T_{n,q}$ by TAA. Note that T' also satisfies R2 with the exception $\theta = \gamma_{m_1}$ with $\eta_m \in D_{n,q}$ and $\eta_m \in \overline{\varphi}(y_{p-1})$, which gives a contradiction because $p(T^*) < p(T)$. Hence we may assume $\theta = \gamma_{m_1}$ with $\eta_m \in D_{n,q}$ and $\eta_m \in \overline{\varphi}(y_{p-1})$. Since $\alpha \in \overline{\varphi}(T_{y_{p-2}})$, $\alpha \neq \eta_m$. By R2 (1), we have $\gamma_{m_1} \notin \varphi(T_{y_{p-1}} - T_{n,q})$. By Claim 2.3.10, $P_v(\alpha, \gamma_{m_1}, \varphi) = P_{v(\gamma_{m_1})}(\alpha, \gamma_{m_1}, \varphi)$, and therefore $P_{y_p}(\alpha, \gamma_{m_1}, \varphi)$ is a different path. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \gamma_{m_1}, \varphi)$. By Claim 2.3.11, $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Since $\alpha \notin \varphi(T - T_{n,q})$ and $\gamma_{m_1} \notin \varphi(T_{y_{p-1}} - T_{n,q})$,

we have $\varphi^*(f) = \varphi(f)$ for each $f \in T_{y_{p-1}}$ and $\bar{\varphi}(v') = \bar{\varphi}^*(v')$ for each $v' \in T_{y_{p-1}}$. Moreover, $\varphi^*(e_p) = \alpha \notin \Gamma^q$. Therefore T is an ETT satisfying MP and R2 (1) under φ^* . Note $\gamma_{m1} \in \bar{\varphi}^*(y_p) \cap \bar{\varphi}^*(v(\gamma_{m1}))$ and $\gamma_{m1} \notin \varphi^*(T - T_{n,q})$. Since $\eta_m \in \bar{\varphi}(y_{p-1})$ and $\alpha \neq \eta_m$, we have $\eta_m \notin \varphi(T - T_{n,q})$. Let $\varphi^{**} = \varphi^*/P_{y_p}(\eta_m, \gamma_{m1}, \varphi^*)$. After applying Claim 2.3.10 and Claim 2.3.11, we can show as before that under the $T_{n,q}$ -stable coloring φ^{**} , T is an ETT satisfying MP and R2, and $\eta_m \in \bar{\varphi}^{**}(y_p) \cap \bar{\varphi}^{**}(y_{p-1})$. So, under φ^{**} we go back to Case 2.1.

Case 2.3.2. $\theta = \varphi(e_p) \in \bar{\varphi}(y_{p-1})$.

We first assume $\theta = \varphi(e_p) = \eta_m$ with $\eta_m \in D_{n,q}$. Thus by R2 (1), $\gamma_{m1} \notin \varphi(T - T_{n,q})$. By Claim 2.3.10, $P_{v(\gamma_{m1})}(\alpha, \gamma_{m1}, \varphi) = P_v(\alpha, \gamma_{m1}, \varphi)$ and $P_{y_p}(\alpha, \gamma_{m1}, \varphi)$ is a different path. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \gamma_{m1}, \varphi)$. By Claim 2.3.11, $T_{n,q}$ is an ETT and satisfying MP and R2 under the $T_{n,q}$ -stable coloring φ^* . Since $\alpha, \gamma_{m1} \notin \varphi(T - T_{n,q})$, under coloring φ^* , $T_{n,q}$ can be extended to T as an ETT in which MP, R2 are satisfied. Now $\gamma_{m1} \in \bar{\varphi}^*(y_p)$ and $\eta_m, \gamma_{m1} \notin \varphi^*(T - T_{n,q})$. Similarly, after applying Claim 2.3.10 and Claim 2.3.11, we can show that under the coloring $\varphi^{**} = \varphi^*/P_{y_p}(\eta_m, \gamma_{m1}, \varphi^*)$, T is also an ETT satisfying MP and R2. Now $\eta_m \in \bar{\varphi}^{**}(y_p) \cap \bar{\varphi}^{**}(y_{p-1})$, which is dealt in Case 2.1.1.

We now consider the case $\theta = \varphi(e_p) \notin D_{n,q}$. Since $\theta \in \bar{\varphi}(y_{p-1})$ and $T_{y_{p-1}}$ is elementary, we have $\theta \notin \Gamma^q$, so $\theta \notin D_{n,q} \cup \Gamma^q$ when $q > 1$ and $\theta \notin D_n \cup \Gamma^q$ when $q = 1$. Suppose $\alpha \neq D_{n,q}$. Then, $\alpha \notin D_{n,q} \cup \Gamma^q$ when $q > 1$ and $\alpha \notin D_n \cup \Gamma^q$ when $q = 1$ by Claim 2.3.15. By Claim 2.3.10, $P_{v(\alpha)}(\alpha, \theta, \varphi) = P_{y_{p-1}}(\alpha, \theta, \varphi)$ and therefore $P_{y_p}(\alpha, \theta, \varphi)$ is a different path. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \theta, \varphi)$. By Claim 2.3.11, $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Since $\theta, \alpha \notin \varphi(T_{y_{p-1}} - T_{n,q})$ and $\alpha, \theta \notin \Gamma^q$, T is an ETT in φ and MP, R2 hold. Now $\theta \in \bar{\varphi}^*(y_p) \cap \bar{\varphi}^*(y_{p-1})$, which is dealt in Case 2.1. Hence we may assume $\alpha = \eta_m$ for some $m \leq n$ where $\eta_m \in D_{n,q}$. By Claim 2.3.15, we have $\gamma_{m1}, \gamma_{m2}, \eta_m \notin \varphi(T - T_{n,q})$. By Claim 2.3.10, $P_{v(\gamma_{m1})}(\alpha, \gamma_{m1}, \varphi) = P_v(\alpha, \gamma_{m1}, \varphi)$ and $P_{y_p}(\alpha, \gamma_{m1}, \varphi)$ is different from the path above. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \gamma_{m1}, \varphi)$. Since $\gamma_{m1}, \eta_m \notin \varphi(T - T_{n,q})$, by Claim 2.3.11, it is easy to check that $T_{n,q}$ can be extended to T as an ETT under φ^* for which MP and R2 (1) hold. Note $\gamma_{m1} \notin \varphi^*(T - T_{n,q})$. Hence by Claim 2.3.10, $P_{v(\gamma_{m1})}(\theta, \gamma_{m1}, \varphi^*) = P_{y_{p-1}}(\theta, \gamma_{m1}, \varphi^*)$ and $P_{y_p}(\theta, \gamma_{m1}, \varphi^*)$ is a different path.

Let $\varphi^{**} = \varphi^*/P_{y_p}(\theta, \gamma_{m1}, \varphi^*)$. Again by Claim 2.3.11, $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Note that from coloring φ^* to coloring φ^{**} , e_p is the only edge changed color from θ to γ_{m1} in $E(T - T_{n,q})$. Since $\eta_m = \alpha \in \varphi^{**}(v)$, $T_{n,q}$ can be extended to T as an ETT under φ^* for which MP and R2 (1) hold. Now $\theta \in \overline{\varphi^{**}}(y_p) \cap \overline{\varphi^{**}}(y_{p-1})$, which is dealt in Case 2.2. This completes Case 2. \square

In the remainder of the proof, let $I_\varphi = \{i \geq 0 : \overline{\varphi}(y_p) \cap \overline{\varphi}(y_i) \neq \emptyset\}$ and let $j = p(T)$. Clearly $I_\varphi = \emptyset$ when $\{v : \overline{\varphi}(y_p) \cap \overline{\varphi}(v) \neq \emptyset\} \subset V(T_{n,q})$. For convention, we denote $\max I_\varphi = -1$ when $I_\varphi = \emptyset$. By the assumption of $p(T)$, we have $j \geq 1$ and y_{j-1} is not incident to e_j .

Case 3. $1 \leq p(T) \leq p - 1$ and $\max(I_\varphi) \geq p(T)$.

This case is similar to Case 1 and can be handled in the same fashion: We first show that $\max(I_\varphi) = p - 1$ and replace color $\varphi(e_p)$ by α to get a smaller counterexample.

Case 4. $1 \leq p(T) \leq p - 1$ and $\max(I_\varphi) < p(T)$.

Let $j = p(T)$. Then $j \geq 1$ and $e_j \notin E_G(y_{j-1}, y_j)$. Let $\min(I_\varphi) = i$ if $I_\varphi \neq \emptyset$. Let y_{j-2} be the last vertex in $T_{n,q}$ when $j = 1$.

Claim 2.3.16. *We may assume there exist $\alpha \in \overline{\varphi}(y_p) \cap \overline{\varphi}(T_{y_{j-2}})$ such that either $\alpha \notin \Gamma^q \cup \overline{\varphi}(T_n^d - T_n)$, or $\alpha = \gamma_{m1} \in \Gamma^q$ with $\eta_m \in D_{n,q}$ and $v(\eta_m) \preceq_\ell y_{j-2}$.*

Proof. We first consider the case when $I_\varphi \neq \emptyset$. Since we assume $\max(I_\varphi) < j$, $i \leq j - 1$. Let $\alpha \in \overline{\varphi}(y_p) \cap \overline{\varphi}(y_i)$. Since $\alpha \in \overline{\varphi}(y_i) \cap \overline{\varphi}(y_p)$, $\alpha \notin \overline{\varphi}(T_{n,q})$. Thus $\alpha \notin \Gamma^q \cup \overline{\varphi}(T_n^d - T_n)$. If $i < j - 1$, then we have $\alpha \in \overline{\varphi}(T_{y_{j-2}})$ as desired. Thus we assume $i = j - 1$. By Claim 2.3.9, we either have $|\overline{\varphi}(T_{y_{j-2}}) - \overline{\varphi}(T_n^d - T_n) - \Gamma^q \cup D_{n,q} \cup \varphi(T_{y_{j-2}} - T_{n,q})| > 4$, or there exist 7 distinct colors $\eta_i \in D_{n,q} \cap \overline{\varphi}(T_{y_{j-2}})$ such that all colors $\eta_i, \gamma_{i1}, \gamma_{i2} \notin \varphi(T_{y_{j-2}} - T_{n,q})$. Because $|E(T_{y_{j-1}-T_{y_{j-1}}})| = 1$, for the first possibility, we have a color $\beta \in \overline{\varphi}(T_{y_{j-2}})$ such that $\beta \notin \Gamma^q \cup \overline{\varphi}(T_n^d - T_n) \cup D_{n,q}$ with $\beta \notin \varphi(T_{y_{j-1}} - T_{n,q})$, and for the second possibility, we have $\beta = \gamma_{m1} \in \Gamma^q$ with $\eta_m \in D_{n,q}$, $v(\eta_m) \preceq_\ell y_{j-2}$ and $\eta_m, \gamma_{m1}, \gamma_{m2} \notin \varphi(T_{y_{j-1}} - T_{n,q})$. Note that in both cases $\beta \notin \varphi(T_{y_{j-1}} - T_{n,q})$ and $\beta \notin D_n$.

We now consider the case $\alpha = \eta_l \in D_{n,q}$. By R2 (1), $\gamma_{l1} \notin \varphi(Ty_i - T_{n,q})$ because $\alpha \in \overline{\varphi}(y_{j-1})$. Thus $\gamma_{l1}, \eta_l \notin \varphi(Ty_i - T_{n,q})$. By Claim 2.3.10, $P_{v(\gamma_{l1})}(\eta_l, \gamma_{l1}, \varphi) = P_{y_i}(\eta_l, \gamma_{l1}, \varphi)$ and $P_{y_p}(\eta_l, \gamma_{l1}, \varphi)$ is a different path. Let $\varphi^* = \varphi/P_{y_p}(\eta_l, \gamma_{l1}, \varphi)$. By Claim 2.3.11, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Since $\gamma_{l1}, \eta_l \notin \varphi(Ty_i - T_{n,q})$ and $\eta_l \in \overline{\varphi}(y_j)$, $\gamma_{l1}, \eta_l \notin \varphi^*(Ty_i - T_{n,q})$ and $\eta_l \in \overline{\varphi^*}(y_j)$. Thus $T_{n,q}$ can still be extended to T as an ETT under φ^* for which MP and R2 (1) hold. Because $\beta \notin D_{n,q}$, $\beta \neq \eta_l$. Moreover, $\beta \neq \gamma_{l1}$ because $\eta_l \in \overline{\varphi}(y_{j-1})$. Thus $\beta, \gamma_{l1} \notin \varphi^*(Ty_i - T_{n,q})$ because $\beta \notin \varphi(Ty_{j-1} - T_{n,q})$. Therefore by Claim 2.3.10, $P_{v(\gamma_{l1})}(\beta, \gamma_{l1}, \varphi^*) = P_{v(\beta)}(\beta, \gamma_{l1}, \varphi^*)$ and $P_{y_p}(\beta, \gamma_{l1}, \varphi^*)$ is different from the path above. Let $\varphi^{**} = \varphi^*/P_{y_p}(\gamma_{l1}, \beta, \varphi^*)$. By Claim 2.3.11, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{**} . Note that $\eta_l \in \overline{\varphi^{**}}(y_{j-1})$ because $\eta_l \neq \beta$ and $\eta_l \in \overline{\varphi^*}(y_{j-1})$. From $\beta, \gamma_{l1} \notin \varphi^*(Ty_{j-1} - T_{n,q})$, we get $\beta, \gamma_{l1} \notin \varphi^{**}(Ty_{j-1} - T_{n,q})$. If $\beta \notin \Gamma^q$, because $\eta_l \in \overline{\varphi^{**}}(y_{j-1})$, $T_{n,q}$ can be extended to T under φ^{**} for which MP and R2 (1) hold. If $\beta = \gamma_{m1}$, we have $\eta_m \in \overline{\varphi}(Ty_{j-2}) = \overline{\varphi^{**}}(Ty_{j-2})$ because $\eta_l \neq \eta_m \in \overline{\varphi}(y_{j-1})$, and therefore $T_{n,q}$ can be extended to T under φ^{**} for which MP and R2 (1) hold. Now $\beta \in \overline{\varphi^{**}}(y_p) \cap \varphi^{**}(v(\beta))$, thus Claim 2.3.16 holds.

We now consider the case $\alpha \notin D_{n,q}$. Then $\alpha \notin D_n \cup \Gamma^q$ because $\alpha \notin \overline{\varphi}(T_{n,q})$. We have $\alpha \notin \varphi(Ty_{j-1} - T_{n,q})$ because $\alpha \in \overline{\varphi}(Ty_{j-1})$. We first consider the case that $\beta \notin \Gamma^q \cup \overline{\varphi}(T_n^d - T_n) \cup D_{n,q}$. Then $\beta \notin \Gamma^q \cup D_n$. Since $\beta \notin \varphi(Ty_{j-1} - T_{n,q})$, by Claim 2.3.10, $P_{v(\beta)}(\alpha, \beta, \varphi) = P_{y_i}(\alpha, \beta, \varphi)$ and $P_{y_p}(\alpha, \beta, \varphi)$ is a different path. By Claim 2.3.11, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring $\varphi^* = \varphi/P_{y_p}(\beta, \alpha, \varphi)$. Since $\alpha, \beta \notin \varphi(Ty_i - T_{n,q})$, we have $\varphi(f) = \varphi^*(f)$ for each $f \in (Ty_i)$ and $\overline{\varphi}(v') = \overline{\varphi^*}(v')$ for each $v' \in Ty_i$. Thus, $T_{n,q}$ can be extended to T as an ETT under φ^* satisfying MP. By our choice of β , we have $\beta \notin \Gamma^q$. Since both $\alpha, \beta \notin \Gamma^q$, T also satisfies R2 (2) under φ^* . It is seen that, Claim 2.3.16 holds under φ^* by considering $\beta \in \overline{\varphi^*}(Ty_{j-2}) \cap \overline{\varphi^*}(y_p)$.

We now assume $\beta = \gamma_{m1}$ with $\eta_m \in D_{n,q}$. By our choice of β , we have $\eta_m, \gamma_{m1}, \gamma_{m2} \notin \varphi(Ty_i - T_{n,q})$ and $\eta_m \in \overline{\varphi}(Ty_{j-2})$. Note that $\alpha \neq \eta_m$ because $\alpha \in \overline{\varphi}(y_i)$. By Claim 2.3.10, $P_{v(\gamma_{m1})}(\alpha, \gamma_{m1}, \varphi) = P_{y_i}(\alpha, \gamma_{m1}, \varphi)$ and therefore $P_{y_p}(\alpha, \gamma_{m1}, \varphi)$ a different path. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \gamma_{m1}, \varphi)$. By Claim 2.3.11, φ^* is $T_{n,q}$ -stable and $T_{n,q}$ satisfies MP and R2 up to

itself under φ^* . Since $\alpha, \gamma_{m1} \notin \varphi(T_{y_i} - T_{n,q})$, we have $\alpha, \gamma_{m1} \notin \varphi^*(T_{y_i} - T_{n,q})$. Thus $T_{n,q}$ can be extended to T as an ETT in φ^* and MP holds. Moreover, since $\eta_m \in \overline{\varphi}(T_{y_{j-2}})$ and $\alpha \neq \eta_m$, $\eta_m \in \overline{\varphi}^*(T_{y_{j-2}})$. Thus T still satisfies R2 (1) because $\alpha \notin \Gamma^q$, $\eta_m \in \overline{\varphi}^*(T_{y_{j-2}})$ and $\gamma_{m1} \notin \varphi^*(T_{y_i} - T_{n,q})$. Thus we have as desired under φ^* .

We then consider the case $I_\varphi = \emptyset$. Then $\alpha \in \overline{\varphi}(T_{n,q})$. Note that if $\alpha \notin \Gamma^q \cup \overline{\varphi}(T_n^d - T_n)$, then we have as desired. Thus we first consider the case $\alpha \in \overline{\varphi}(T_n^d - T_n) - \Gamma^q$. We claim that there exist a color $\beta \in \overline{\varphi}(T_{n,q}) - \overline{\varphi}(T_n^d - T_n) - \Gamma^q$. Since $q \geq 1$ and $T_{n,q}$ is elementary under φ , $|\overline{\varphi}(T_{n,q}) - \overline{\varphi}(T_n^d - T_n)| \geq |\overline{\varphi}(T_n)| \geq 2n + 11$. Recall that $|\Gamma^q| \leq 2n$, we have $\beta \in \overline{\varphi}(T_{n,q}) - \overline{\varphi}(T_n^d - T_n) - \Gamma^q$ as desired. Since $\beta \in \overline{\varphi}(T_{n,q}) - \overline{\varphi}(T_n^d - T_n)$, we have $\beta \in \overline{\varphi}(T_n)$ when $q = 1$ and $\beta \in \overline{\varphi}(T_{n,q})$ when $q > 1$. Thus by Claim 2.3.10, $P_{v(\alpha)}(\alpha, \beta, \varphi) = P_{v(\beta)}(\alpha, \beta, \varphi)$ and $P_{y_p}(\alpha, \beta, \varphi)$ is a different path. By Claim 2.3.11, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring $\varphi^* = \varphi/P_{y_p}(\alpha, \beta, \varphi)$. Mover, T can still be obtained from $T_{n,q}$ by TAA under φ^* because $\alpha, \beta \in \overline{\varphi}(T_{n,q})$. Thus T satisfies MP under φ^* . Moreover, since both $\alpha, \beta \notin \Gamma^q$ by our assumption on α and choice of β , T still satisfies R2 (1) under φ^* . By considering β under φ^* , we have Claim 2.3.16.

Hence we assume $\alpha = \gamma_{m1} \in \Gamma^q$ with $\eta_m \in D_{n,q}$. We first assume that $\eta_m \notin \overline{\varphi}(T_{y_{p-1}})$. Then $\gamma_{m1} \notin \varphi(T - T_{n,q})$. By Claim 2.3.9, we either have $|\overline{\varphi}(T_{y_{p-2}}) - \overline{\varphi}(T_n^d - T_n) - \Gamma^q \cup D_{n,q} \cup \varphi(T_{y_{p-2}} - T_{n,q})| > 4$, or there exist 7 distinct colors $\eta_i \in D_{n,q} \cap \overline{\varphi}(T_{y_{p-2}})$ such that all colors $\eta_i, \gamma_{i1}, \gamma_{i2} \notin \varphi(T_{y_{p-2}} - T_{n,q})$. Thus there exists a color $\beta \in \overline{\varphi}(T_{y_{p-2}})$ with $\beta \notin \varphi(T - T_{n,q})$ such that either $\beta \notin D_{n,q} \cup \Gamma^q \cup \overline{\varphi}(T_n^d - T_n)$, or $\beta = \eta_k$ with $\eta_k \in D_{n,q}$ and additionally $\gamma_{k1}, \gamma_{k2}, \eta_k \notin \varphi(T_{y_p} - T_{n,q})$. Since $\gamma_{m1} \notin \varphi(T - T_{n,q})$, by Claim 2.3.10, $P_{v(\gamma_{m1})}(\beta, \gamma_{m1}, \varphi) = P_{v(\beta)}(\beta, \gamma_{m1}, \varphi)$ and therefore $P_{y_p}(\beta, \gamma_{m1}, \varphi)$ a different path. By Claim 2.3.11, we have that $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring $\varphi^* = \varphi/P_{y_p}(\beta, \gamma_{m1}, \varphi)$. Since $\beta, \gamma_{m1} \notin \varphi(T_{y_p} - T_{n,q})$, we have $\beta, \gamma_{m1} \notin \varphi^*(T_{y_p} - T_{n,q})$. So, as an ETT in φ^* , $T_{n,q}$ can be extended to T for which MP and R2 (1) hold. Note that in φ^* we have Claim 2.3.16 if $v(\beta) \preceq_\ell y_{j-2}$ or Case 3 if $y_j \preceq_\ell v(\beta)$, or the case $I_\varphi \neq \emptyset$ if $v(\beta) = y_{j-1}$, where we can proceed as before.

Now we assume that $\eta_m \in \overline{\varphi}(T_{y_{p-1}})$. Note that $\eta_m \notin \overline{\varphi}(T_{n,q})$ since $\eta_m \in D_{n,q}$. Without

loss of generality, we can assume that $\eta_m \in \overline{\varphi}(y_k)$ for some $0 \leq k \leq p-1$. If $k < j-1$, we have Claim 2.3.16, hence we assume $k \geq j-1$. Since $\eta_m \in \overline{\varphi}(y_k)$, $\eta_m \notin \varphi(T_{y_k} - T_{n,q})$. By R2 (1), $\gamma_{m1} \notin \varphi(T_{y_k} - T_{n,q})$. Thus by Claim 2.3.10, $P_{v(\gamma_{m1})}(\eta_m, \gamma_{m1}, \varphi) = P_{y_k}(\eta_m, \gamma_{m1}, \varphi)$ and therefore $P_{y_p}(\eta_m, \gamma_{m1}, \varphi)$ a different path. By Claim 2.3.11, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring $\varphi^* := \varphi/P_{y_p}(\eta_m, \gamma_{m1}, \varphi)$. Since $\gamma_{m1}, \eta_m \notin \varphi(T_{y_k} - T_{n,q})$, the edges of T colored different in φ^* and φ is in $T_{y_p} - T_{y_k}$, and they are colored by γ_{m1} or η_m in both colorings φ and φ^* , and consequently T satisfies MP and R2 (1) under φ^* because $\eta_m, \gamma_{m1} \in \overline{\varphi^*}(T_{y_k})$. If $k > j-1$, we have Case 3. If $k = j-1$, we have the case $I_\varphi \neq \emptyset$, where we can proceed as before. \square

We then consider the color φ satisfying Claim 2.3.16. By Claim 2.3.9 again, there exists a color $\beta \in \overline{\varphi}(T_{y_{j-2}})$ with $\beta \notin \varphi(T_{y_j} - T_{n,q})$ such that either $\beta \notin D_{n,q} \cup \Gamma^q \cup \overline{\varphi}(T_n^d - T_n)$ or $\beta = \eta_k \in \overline{\varphi}(T_{y_{j-2}})$ with $\eta_k, \gamma_{k1}, \gamma_{k2} \notin \varphi(E(T_{y_j} - T_{n,q}))$ and $\eta_k \in D_{n,q}$. Note that $\beta \notin \Gamma^q$. Now we consider the path $P := P_{y_p}(\alpha, \beta, \varphi)$. First we consider the case $V(P) \cap V(T_{y_{j-1}}) \neq \emptyset$. Along the order of P from y_p , let u be the first vertex in $V(T_{y_{j-1}})$ and P' be the subpath joining u and y_p . Let

$$\begin{aligned} T' &= T_{y_{j-2}} \cup P' && \text{if } u \neq y_{j-1}, \text{ and} \\ T' &= T_{y_{j-1}} \cup P' && \text{if } u = y_{j-1}. \end{aligned}$$

Note that $e_0 \notin T'$ may happen when $q = 1$, but it is easy to see that T' is still an ETT with the same ladder as T and q splitters where $T_{n,q} \subseteq T'$. Moreover, $T_{y_{j-2}}$ is an ETT satisfying MP and R2 with R2 (2) being satisfied for $T_{n,q}$ under φ , because T satisfies MP and R2 under φ . In addition, since $\alpha, \beta \in \overline{\varphi}(T_{y_{j-2}})$, T' can be obtained from $T_{y_{j-2}}$ by TAA, and therefore it satisfies MP under φ .

Case I: $\alpha \notin \Gamma^q$. Since $\alpha, \beta \in \overline{\varphi}(T_{y_{j-2}})$ and $\alpha, \beta \notin \Gamma^q$, T' satisfies R2 (1), giving a contradiction to the minimality of $p(T)$.

Case II: $\alpha \in \Gamma^q$. Then by Claim 2.3.16, $\alpha = \gamma_{m1} \in \Gamma^q$ with $\eta_m \in D_{n,q}$ and $v(\eta_m) \prec_\ell y_{j-2}$. Then $\eta_m \in \overline{\varphi}(T_{y_{j-2}})$. Therefore T' still satisfies R2 (1), giving a contradiction to the

minimality of $p(T)$.

□

Therefore we have $V(P) \cap V(T_{y_{j-1}}) = \emptyset$. Let $\varphi^* = \varphi/P$. Then φ^* is $T_{y_{j-1}}$ -stable and $T_{y_{j-1}}$ satisfies MP and R2 with R2 (2) satisfied for $T_{n,q}$ by Lemma 2.3.5. Moreover, since $\alpha, \beta \in \overline{\varphi}(T_{y_{j-1}})$, T can still be obtained from $T_{n,q}$ by TAA under φ^* , and therefore it is an ETT satisfying MP. If $\alpha \notin \Gamma^q$, T satisfies R2 (1) under φ^* since $\beta \notin \Gamma^q$. If $\alpha \in \Gamma^q$, by Claim 2.3.16, $\alpha = \gamma_{m1} \in \Gamma^q$ with $v(\eta_m) \prec_\ell y_{j-2}$. Therefore T still satisfies R2 (1) because $\eta_m \in \overline{\varphi}(T_{y_{j-2}})$. Because $V(P) \cap V(T_{y_{j-1}}) = \emptyset$, e_j is adjacent to a vertex in $V(T_{y_{j-1}})$ and $\beta \notin \varphi(T_{y_j} - T_{n,q})$, $\beta \notin \varphi^*(T_{y_j} - T_{n,q})$. Moreover, when $\beta = \eta_k \in D_{n,q}$, we also have $\gamma_{k1}, \gamma_{k2} \notin \varphi^*(T_{y_j} - T_{n,q})$ because $\gamma_{k1}, \gamma_{k2} \notin \varphi(T_{y_j} - T_{n,q})$, $V(P) \cap V(T_{y_{j-1}}) = \emptyset$ and e_j is adjacent to a vertex in $V(T_{y_{j-1}})$. Note that $\beta \in \overline{\varphi^*}(y_p) \cap \overline{\varphi^*}(v(\beta))$, where $v(\beta) \prec_l y_{j-2}$. Denote $v = v(\beta)$ for convenience. Let $\gamma \in \overline{\varphi}(y_j)$. Then $\gamma \notin \Gamma^q$. We then denote $\varphi^* = \varphi$ and consider the following two cases.

Case 4.1. $\gamma \notin D_{n,q}$. $\gamma \notin D_n$ when $q = 1$, because $\gamma \in \overline{\varphi}(y_i)$. For the same reason $\gamma \notin \Gamma^q \subset \overline{\varphi}(T_{n,q})$.

By our choice on β , there are two cases.

Case 4.1.1. $\beta \notin D_{n,q} \cup \Gamma^q \cup \overline{\varphi}(T_n^d - T_n)$.

Then $\beta \notin D_{n,q}$ when $q > 1$ and $\beta \notin D_n$ when $q = 1$. By Claim 2.3.10, $P_{v(\beta)}(\beta, \gamma, \varphi) = P_{y_j}(\beta, \gamma, \varphi)$ and $P_{y_p}(\beta, \gamma, \varphi)$ a different path than above. let $\varphi^* = \varphi/P_{y_p}(\beta, \gamma, \varphi)$. Then by Claim 2.3.11, φ^* is $T_{n,q}$ -stable and $T_{n,q}$ satisfies MP and R2 up to itself under φ^* . Since $\gamma, \beta \notin \varphi(T_{y_j} - T_{n,q})$ and $v \prec_l y_j$, and moreover $\gamma, \beta \notin \Gamma^q$, we have that T satisfies MP and R2 (1) under φ^* . Now $\gamma \in \overline{\varphi^*}(y_p) \cap \overline{\varphi^*}(y_j)$, by which we reach Case 3.

Case 4.1.2. $\beta = \eta_k \in D_{n,q}$.

Recall that in this case we proved $\gamma_{k1}, \gamma_{k2} \notin \varphi((T_{y_j} - T_{n,q}))$ before starting Case 4.1. Since $\beta \notin \varphi(T_{y_j} - T_{n,q})$, by Claim 2.3.10, $P_{v(\beta)}(\beta, \gamma_{k1}, \varphi) = P_{v(\gamma_{k1})}(\beta, \gamma_{k1}, \varphi)$ and therefore $P_{y_p}(\beta, \gamma_{k1}, \varphi)$ a different path. Let $\varphi^* = \varphi/P_{y_p}(\gamma_{k1}, \beta, \varphi)$. Then by Claim 2.3.11, $T_{n,q}$ satisfies

MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Moreover, T satisfies MP and R2 under φ^* because $\eta_k = \beta, \gamma_{k1} \in \overline{\varphi}(Ty_{j-2})$ and $\beta, \gamma_{k1} \notin \varphi(Ty_j - T_{n,q})$. Note that we still have $\gamma_{k1}, \gamma_{k2} \notin \varphi^*(Ty_j - T_{n,q})$ because $\gamma_{k1}, \gamma_{k2}, \beta \notin \varphi(Ty_j - T_{n,q})$. Similarly by Claim 2.3.10 again, $P_{v(\beta)}(\gamma, \gamma_{k1}, \varphi^*) = P_{v(\gamma_{k1})}(\gamma, \gamma_{k1}, \varphi^*)$ and $P_{y_p}(\gamma, \gamma_{k1}, \varphi^*)$ is a different path. Let $\varphi^{**} = \varphi^*/P_{y_p}(\gamma_{k1}, \gamma, \varphi^*)$. By Claim 2.3.11, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{**} . Since $\eta_k \in \overline{\varphi^*}(Ty_{j-2}) = \overline{\varphi^{**}}(Ty_{j-2})$, $\gamma \in \overline{\varphi^{**}}(y_j)$ and $\gamma, \gamma_{k1} \notin \varphi(Ty_j - T_{n,q})$, T satisfies MP and R2 (1) under φ^{**} . However, we have $\gamma \in \overline{\varphi^{**}}(y_p) \cap \overline{\varphi^{**}}(y_j)$, where we reach Case 3. \square

Case 4.2. $\gamma = \eta_l \in D_{n,q}$.

Recall that $\beta \notin \Gamma^q$. Then $\gamma_{l1}, \gamma_{l2} \notin \varphi(Ty_j - T_{n,q})$ by R2 (1). Moreover, recall that we proved $\beta \notin \varphi(Ty_j - T_{n,q})$ before starting Case 4.1. By Claim 2.3.10, $P_{v(\beta)}(\beta, \gamma_{l1}, \varphi) = P_{v(\gamma_{l1})}(\beta, \gamma_{l1}, \varphi)$ and therefore $P_{y_p}(\beta, \gamma_{l1}, \varphi)$ is a different path than above. Let $\varphi^* = \varphi/P_{y_p}(\gamma_{l1}, \beta, \varphi)$. Then $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* by Claim 2.3.11. Moreover, since $\gamma_{l1}, \beta \in \overline{\varphi}(Ty_{j-2})$ and $\gamma_{l1}, \beta \notin \varphi(Ty_j - T_{n,q})$, T can be extended from $T_{n,q}$ under φ^* , and therefore it satisfies MP. Because $\eta_l = \gamma \in \overline{\varphi}(Ty_j)$, $\beta \notin \Gamma^q$ and $\beta, \gamma_{m1} \notin \varphi(Ty_j - T_{n,q})$, T still satisfies R2 (1) under φ^* . Because $\beta \in \overline{\varphi}(Ty_{j-2})$ and $\gamma \in \overline{\varphi}(y_j)$, $\beta \neq \gamma$. Moreover, we have $\gamma_{l1}, \gamma \notin \varphi^*(Ty_j - T_{n,q})$ because $\gamma_{l1}, \gamma \notin \varphi(Ty_j - T_{n,q})$ and $\beta \neq \gamma$. Similarly by Claim 2.3.10, $P_{v(\beta)}(\gamma, \gamma_{l1}, \varphi^*) = P_{v(\gamma_{l1})}(\gamma, \gamma_{l1}, \varphi^*)$ and therefore $P_{y_p}(\gamma, \gamma_{l1}, \varphi^*)$ is a different path. Let $\varphi^{**} = \varphi^*/P_{y_p}(\gamma_{l1}, \gamma, \varphi^*)$. By Claim 2.3.11, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{**} . Since $\eta_l = \gamma, \gamma_{l1} \in \overline{\varphi^*}(Ty_j)$ and $\gamma, \gamma_{l1} \notin \varphi(Ty_j - T_{n,q})$, T satisfies MP and R2 (1) under φ^{**} . Now we have $\eta_l \in \overline{\varphi^{**}}(y_p) \cap \overline{\varphi^{**}}(y_j)$, where we reach Case 3.

This completes the proof of Case 4. Now for all cases we arrive at a contradiction, which proved statement A.

2.3.2 RE and SE extension

In this section, we consider the case where we use RE or SE extension to get from T_n to T by a connecting edge f_n and its color δ_n .

Let T be an ETT with ladder $T_1 \subset T_2 \subset \dots \subset T_n \subset T$ of the k -triple (G, e, φ) where $k \geq \Delta + 1$. We call the subsequence $T - T_n$ the *other tail* of T and the sequence of subsets $T_n := T_{n,0} \subset T_{n,1} \subset \dots \subset T_{n,q} \subset T := T_{n,q+1}$ a *other split tail* for T , where $T_{n,i} = T_{v_i}$ for $0 < i \leq q$ and $v_1 \prec_\ell v_2 \prec_\ell \dots \prec_\ell v_q$ are vertices in $V(T - T_n)$. The ETT T with the form $T_1 \subset T_2 \subset \dots \subset T_n := T_{n,0} \subset T_{n,1} \subset \dots \subset T_{n,q} \subset T := T_{n,q+1}$ is called a *other refinery* of T with q *other splitters*, or simply a *refinery* of T . For convenience, we omit the word “other” when we talk about tails in this section. Note that these concepts are essentially the same as their analogies in PE section except for the requirements on $T_{n,1}$.

Let $D_n = \{\eta_1, \dots, \eta_{n'}\}$, where $n' \leq n$. Let $\eta'_n = \gamma_n$ when $\Theta_n = \text{RE}$ and $\eta'_n = \delta_n$ if $\Theta_n = \text{SE}$

Definition 10. *We say an other split tail with q splitters of an ETT T with n -rungs satisfies condition R2 if it satisfies the following conditions for $0 \leq j \leq q$:*

(1) *For each $0 \leq j \leq q$ and each $\eta_h \in D_{n,j} = D_n - \overline{\varphi}(T_{n,j})$, there exists a two color set*

$\Gamma_h^j = \{\gamma_{h_1}^j, \gamma_{h_2}^j\} \subseteq \overline{\varphi}(T_{n,j}) - \varphi(T_{n,j+1}(v(\eta_h))) - T_{n,j}$ *satisfying the following conditions:*

(a) $\Gamma_h^j \cap \Gamma_g^j = \emptyset$ *if $\eta_h \neq \eta_g \in D_{n,j}$.*

(b) $\Gamma^{j+1} - \Gamma^j \subseteq \overline{\varphi}(T_{n,j+1} - T_{n,j})$ *where $\Gamma^j = \cup_{\eta_h \in D_{n,j}} \Gamma_h^j$.*

(2) $T_{n,j}$ *is $(\cup_{\eta_h \in D_{n,j}} \Gamma_h^{j-1})^-$ -closed.*

We say an ETT T satisfies condition R2 in φ for convenience if there is a split tail of T satisfies condition R2.

Lemma 2.3.6. *Suppose (A2),(A3),(A4) and (A5) hold for ETTs satisfying MP with at most n rungs and (A1) holds for ETTs satisfying MP with at most $n - 1$ rungs. Let T be an ETT with n rungs satisfying MP under last coloring φ with $\Theta_n = \text{SE}$ or RE . Let φ^* be obtained from φ by switching α and β edges of some (α, β) -chains in $G - V(T)$. Then φ^* is (T, D_n, φ_n) -stable and T is an ETT satisfies MP under the $T \bmod \varphi$ coloring φ^* . Moreover, if T satisfies R2 under last coloring φ , then the same split tail satisfies R2 with the same Γ sets under last coloring φ^* and φ^* is both (T_n, D_n, φ_n) -stable and (T, φ_n) -wstable, and additionally if $T = T_{n,q+1}$ itself satisfies R2 (2), i.e. $T_{n,q+1}$ is $(\cup_{\eta_h \in D_{n,q+1}} \Gamma_h^q)^-$ -closed where $D_{n,q+1} = D_n - \overline{\varphi}(T_{n,q+1})$, then $T_{n,q}$ itself also satisfies R2 (2) under φ^* .*

Proof. Since $\varphi(f) = \varphi^*(f)$ for every edge f incident to $V(T)$, φ^* is (T, D_n, φ_n) -stable and therefore (T_n, D_n, φ_n) -stable. Applying (A5), we see that T is an ETT satisfying MP under the $\varphi_n \bmod T$ coloring φ^* because it can be obtained from $T_n \cup F_n \cup b(F_n)$ by TAA. Suppose T satisfies R2 with a split tail $T_n = T_{n,0} \subset T_{n,1} \subset T_{n,2} \subset \cdots \subset T_{n,q} \subset T = T_{n,q+1}$. Since φ^* is (T, D_n, φ_n) -stable, it is clearly (T, φ_n) -wstable. Thus φ^* is both (T_n, D_n, φ_n) -stable and (T, φ_n) -wstable. Thus $T_n \subset T_{n,1} \subset T_{n,2} \subset \cdots \subset T_{n,q} \subset T = T_{n,q+1}$ is still a split tail of T under φ^* . Since the colors who are $T_{n,i}$ -closed under φ stay $T_{n,i}$ -closed under φ^* for each $0 \leq i \leq q$, $\varphi(f) = \varphi^*(f)$ for every edge f incident to $V(T)$ and $\bar{\varphi}(v) = \bar{\varphi}^*(v)$ for every $v \in V(T)$, R2 is also satisfied for the same split tail under φ^* . Now we assume $T_{n,q+1}$ itself satisfies R2 (2) under φ . If $q = 0$, T satisfies R2 (2) under φ^* because R2 (2) starts for $q = 1$. If $q \geq 1$, then $T = T_{n,q+1}$ being $(\cup_{\eta_h \in D_{n,q+1}} \Gamma_h^q)^-$ -closed under φ implies that it is also $(\cup_{\eta_h \in D_{n,q+1}} \Gamma_h^q)^-$ -closed under φ^* . \square

Similarly as in PE section, we prove the following Proposition which is a proof of (A1) (1) when $\Theta_n = \text{SE}$ or RE .

Proposition 6. *Suppose (A2), (A3), (A4), (A5) hold for ETTs satisfying MP with at most n rungs and (A1) holds for ETTs satisfying MP with at most $n - 1$ rungs. Let T an ETT satisfying MP with ladder $T_1 \subset T_2 \subset \cdots \subset T_n \subset T$. Then the following statements A and B hold, which imply T is elementary.*

A. *If T satisfies condition R2, then T is elementary.*

B. *If A holds, then there exists a closed ETT T' with $V(T) \subset V(T')$ and ladder $T_1 \subset T_2 \subset \cdots \subset T_n \subset T_n \vee T_n^d \subset T'$ satisfying MP and R2.*

We place the proof of Statement B first since it is much shorter than the proof of Statement A.

2.3.2.1 Proof of Statement B in Proposition 6 Let T be an ETT with ladder $T_1 \subset T_2 \subset \cdots \subset T_n \subset T$ satisfying MP. We will construct an ETT T' with ladder $T_1 \subset T_2 \subset \cdots \subset T_n \subset T_n \subset T'$ and the same extension type under the same last coloring such that

$V(T) \subset V(T')$ where there is a split tail $T_n =: T_{n,0} \subset \dots \subset T_{n,q} \subset T' := T_{n,q+1}$ satisfying condition R2. We first build $T_{n,1}$ after $T_{n,0}$ using the following algorithm.

- (1) Fix distinct $\Gamma_h^0 = \{\gamma_{h1}^0, \gamma_{h2}^0\} \subset \overline{\varphi}(T_{n,0})$ for distinct $\eta_h \in D_{n,0}$. Let $T_{n,1} = T_{n,0} \cup \{f_n, b(f_n)\}$, where f_n is the connecting edge of T after T_n .
- (2) If there exists $f \in \partial(T_{n,1})$ with $\varphi(f) \in \overline{\varphi}(T_{n,1})$, we augment $T_{n,1}$ by letting $T_{n,1} := T_{n,1} \cup \{f, b(f)\}$ under the restriction $\Gamma_h^0 \cap \varphi(T_{n,1}(v(\eta_h)) - T_{n,0}) = \emptyset$ for all $\eta_h \in D_{n,0}$ until we can not add any new edge.

Because T_n is elementary by induction hypothesis, we have $|T_n| \geq 11 + 2n$ and $|D_{n,0}| \leq 2n$. Thus we have enough colors to pick each Γ_h^0 distinctly for distinct $\eta_h \in D_{n,0}$ and therefore step (1) is feasible. Note that $T_{n,1}$ obtained from the algorithm above clearly satisfies MP and R2. Suppose $T_{n,j-1}$ is defined for some $j \geq 2$. If $T_{n,j-1}$ is closed, then $V(T) \subset V(T_{n,j-1})$ and we let $T_{n,j-1} = T'$. If $T_{n,j-1}$ is not closed, we will continue to build $T_{n,j}$ from $T_{n,j-1}$ inductively as follows:

- (1) Let $T_{n,j} = T_{n,j-1} \cup \{f, b(f)\}$ where $\varphi(f) \in \Gamma_h^{j-2}$ for some $\eta_h \in D_{n,j-1}$ and $f \in \partial(T_{n,j-1})$. Let $\Gamma_h^{j-1} \subset \overline{\varphi}(T_{n,j-1} - T_{n,j-2})$ with $|\Gamma_h^{j-1}| = 2$ for η_h and $\Gamma_i^{j-1} = \Gamma_i^{j-2}$ for any other η_i with $\eta_i \in D_{n,j-1}$.
- (2) If there exists $f \in \partial(T_{n,j-1})$ where $\varphi(f) \in \overline{\varphi}(T_{n,j-1})$, we let $T_{n,j} := T_{n,j-1} \cup \{f, b(f)\}$ under the restriction $\Gamma_h^{j-1} \cap \varphi(T_{n,j-1}(v(\eta_h)) - T_{n,j-1}) = \emptyset$ for all $\eta_h \in D_{n,j-1}$ until we can not add any new edge.

Since $T_{i,j-1}$ is not closed but is $(\cup_{\eta_h \in D_{n,j-1}} \Gamma_h^{j-2})^-$ closed, such a η_h in (1) exists. By Statement A, $T_{n,j-1}$ is elementary. Therefore, $|\overline{\varphi}(T_{n,j-1} - T_{n,j-2})| \geq 2$. Thus step (1) is feasible. Note that $T_{n,j}$ obtained from the algorithm above satisfies MP and R2. Now if $T_{n,j}$ is closed, then $V(T) \subset V(T_{n,j})$ and we let $T_{n,j} = T'$. If $T_{n,j}$ is not closed, we will continue to build $T_{n,j+1}$. Finally we will obtain a closed T' as desired. \square

2.3.2.2 Proof of Statement A in Proposition 6

Proof. We prove statement A by induction on q which is the number of splitters. Denote T by $T_1 \subset T_2 \subset \cdots \subset T_n := T_{n,0} \subset T_{n,1} \subset \cdots \subset T_{n,q} \subset T := T_{n,q+1}$. When $q = 0$, we have $T_{n,q} = T_{n,0} = T_n$. Since T_n is an ETT under φ_{n-1} with $n(T_n) = n - 1$, it is elementary by (A1). Moreover, since $\Theta_n = \text{SE}$ or RE , φ_n is $(T_n, D_{n-1}, \varphi_n)$ -stable. Therefore T_n is elementary under φ_n , which serves as our induction base. Now we assume $T_{n,q}$ is elementary and show $T_{n,q+1} = T$ is. Denote T by $\{T_{n,q}, e_0, y_0, e_1, \dots, e_p, y_p\}$ following the order \prec_ℓ . We define the path number $p(T)$ of T as the smallest index $i \in 0, 1, \dots, p$ such that the sequence $y_i T := (y_i, e_{i+1}, \dots, e_p, y_p)$ is a path in G . Suppose on the contrary that T is a counterexample to the theorem, i.e., MP and R2 hold for T under last coloring φ , but $V(T)$ is not elementary. Furthermore, we assume that among all counterexamples under coloring that is (T_n, D_n, φ) -stable and $(T_{n,q}, \varphi)$ -wstable where R2 and MP are satisfied, the following two conditions hold:

- (1) $p(T)$ is minimum,
- (2) $|T - T_{n,q}|$ is minimum subject to (1), i.e. p is minimum subject to (1).

By (A5), we can indeed seek for counterexamples under coloring that is both (T_n, D_n, φ) -stable and $(T_{n,q}, D_n, \varphi)$ -wstable, because they are still ETTs. Later in the proof, we may simply use this fact without mentioning (A5). For convenience, by saying $T_{n,q}$ -stable, we mean a coloring that is both (T_n, D_n, φ) -stable and $(T_{n,q}, \varphi)$ -wstable. Moreover, we assume our counter-example is under φ . By our choice, $V(Ty_{p-1})$ is elementary, where $Ty_{p-1} = T_{n,q}$ when $p = 0$. Since $V(T)$ is not elementary, there exist a color $\alpha \in \bar{\varphi}(y_p) \cap \bar{\varphi}(v)$ for some $v \in V(Ty_{p-1})$. For simplification of notations, we let $\Gamma_h^q = \{\gamma_{h1}, \gamma_{h2}\}$ for $\eta_h \in D_{n,q}$. **We may often mention that an ETT satisfies R2 in the proof, and if the splitters and Γ sets are not specified, we always mean that R2 is satisfied for the same splitters and Γ sets in T .**

2.3.2.2.1 A few basic properties

Claim 2.3.17. *For every $T_{n,j}$ with $0 \leq j \leq q$ and two colors α, β , if $\alpha \in \overline{\varphi}(T_{n,j})$ and is closed in $T_{n,j}$, then α and β are $T_{n,j}$ -interchangeable under φ .*

Note that when $j = 0$, Claim 2.3.17 holds by ICMC property on T_n . Instead of proving Claim 2.3.17, we will prove the next Claim which implies Claim 2.3.17 by letting $\varphi' = \varphi$ and $T'_{n,j} = T_{n,j}$.

Claim 2.3.18. *Let φ' be a T_n -stable coloring (i.e. (T_n, D_n, φ_n) -stable) and $T_{n,j}$ be an ETT with ladder $T_0 \subset T_1 \subset \dots \subset T_n \subset T'_{n,j}$ and split tail $T_n \subset T_{n,1} \subset T'_{n,2} \subset \dots \subset T'_{n,j}$ under last coloring φ' with $0 \leq j \leq q$. Suppose $T'_{n,j}$ satisfies R2 up to itself under φ' . For any two colors α and β , if $\alpha \in \overline{\varphi'}(T_{n,j})$ and is closed in $T_{n,j}$ under φ' , then α and β are $T_{n,j}$ -interchangeable under φ' .*

For any $T_{n,j}$ with $0 \leq j \leq q$ and two colors α, β , if $\alpha \in \overline{\varphi}(T_{n,j})$ and is closed in $T_{n,j}$, then α and β are interchangeable in $T_{n,j}$ under any $T_{n,j}$ -stable coloring where R2 is satisfied for $T_{n,j}$.

Proof. We prove Claim 2.3.17 by induction on j . For convenience, we still denote φ' by φ and $T'_{n,j}$ by $T_{n,j}$. Let the corresponding $T_{n,j}$ -stable coloring where R2 is satisfied for $T_{n,j}$ be φ . The case when $j = 0$ follows from (A1) (2). Now we suppose $j > 0$ and consider the following two cases.

Case I: $\beta \in \overline{\varphi}(T_{n,j})$.

Since $T_{n,j}$ is elementary because $T_{n,j}$ has $j - 1 < q$ splitters, $|V(T_{n,j})|$ is odd by the fact $T_{n,j}$ being α -closed. Therefore $|\partial_\beta(T_{n,j})|$ is even and there are even number of (α, β) exit paths. If there are none, then we have interchangeability for α and β because $T_{n,j}$ is elementary. Hence we assume that there exist two exit vertices $u, v \in T_{n,j}$, and they belong to exit paths $P_u^{ex}(\alpha, \beta, \varphi)$ and $P_v^{ex}(\alpha, \beta, \varphi)$, respectively. We may assume $u \preceq_l v$.

Case I.a: $v \in T_{n,j} - T_{n,j-1}$.

Since $\beta \in \varphi(\partial(T_{n,j}))$ and $\beta \in \overline{\varphi}(T_{n,j})$, i.e $T_{n,j}$ is not closed for β , we have $v(\beta) \in V(T_{n,j-1})$ by R2 (2). Let $\gamma \in \overline{\varphi}(v)$. Then $\gamma \notin \Gamma^{j-1}$ hence γ is closed in $T_{n,j}$ by R2 (2). Therefore $T_{n,j}$ is closed for both α and γ . Hence $\varphi^* = \varphi/(\alpha, \gamma, G - T_{n,j})$ is $T_{n,j}$ -stable, and MP, R2 are still satisfied under φ^* by Lemma 2.3.6. However under φ^* , $P_v^{ex}(\alpha, \beta, \varphi) = P_v^{ex}(\gamma, \beta, \varphi^*) = P_v(\gamma, \beta, \varphi^*)$ and $P_u^{ex}(\alpha, \beta, \varphi) = P_u^{ex}(\gamma, \beta, \varphi^*)$ are two (γ, β) exit paths. Let $\varphi^2 = \varphi^*/P_v^{ex}(\gamma, \beta, \varphi^*)$. Since $P_v^{ex}(\gamma, \beta, \varphi^*) \cap T_v = \{v\}$, φ^2 is $T_{n,j-1}$ -stable and $T_{n,j-1}$ satisfies MP and R2 up to itself by Lemma 2.3.6. Moreover, because $P_v^{ex}(\gamma, \beta, \varphi^*) \cap T_v = \{v\}$, T_v still satisfies MP under φ^2 because it can still be obtained by TAA from $T_n \cup F_n \cup b(F_n)$ under φ^2 . In additional, we have $\varphi^2(f) = \varphi^*(f)$ for every f incident to $V(T_v)$ and $\overline{\varphi}^2(v') = \overline{\varphi}^*(v')$ for every $v' \in T_v - v$. Thus T_v still satisfies R2 (1) under φ^2 , and therefore it is elementary because it has $j - 1 < q$ splitters. However, we have $\beta \in \overline{\varphi}^2(T_{n,j-1})$ and $\beta \in \overline{\varphi}^2(v)$, where we reach a contradiction.

Case I.b: $v \in T_{n,j-1}$.

We claim that there exists $\alpha^* \in \overline{\varphi}(T_{n,j-1})$ such that α^* is closed in both $T_{n,j-1}$ and $T_{n,j}$. First we consider the case when $j = 1$. Note that by condition R2(1a) $|\cup_{\eta_h \in D_{n,1}} \Gamma_h^0| = 2|D_{n,1}| \leq 2n$. Since $|\overline{\varphi}(T_1)| \geq 13$ and T_n is elementary with $|T_i| = \text{odd}$ for $i \leq n$, we have $|\overline{\varphi}(T_n)| \geq 11 + 2n \geq |\cup_{\eta_h \in D_{n,1}} \Gamma_h^0|$. By R2, $\Gamma_h^0 \subset \overline{\varphi}(T_n)$ for each $\eta_h \in D_{n,0}$. Hence we have $\Gamma_h^0 \subset \overline{\varphi}(T_n)$ for each $\eta_h \in D_{n,1}$ because $D_{n,1} \subset D_{n,0}$. Therefore there exists $\alpha^* \in \overline{\varphi}(T_n) - (\cup_{\eta_h \in D_{n,1}} \Gamma_h^0)$. Since $T_{n,1}$ is $(\cup_{\eta_h \in D_{n,1}} \Gamma_h^0)^-$ -closed by R2(2) and T_n is closed, α^* is closed in both $T_{n,1}$ and T_n . Now we assume $j > 1$. By R2(2), T_{j-1} is $(\cup_{\eta_h \in D_{n,j-1}} \Gamma_h^{j-2})^-$ -closed. Similarly as the case $j = 1$, we have $|\overline{\varphi}(T_{n,j-2})| \geq 11 + 2n \geq |\cup_{\eta_h \in D_{n,j-1}} \Gamma_h^{j-2}|$, and there exists $\alpha^* \in \overline{\varphi}(T_{n,j-2}) - (\cup_{\eta_h \in D_{n,j-1}} \Gamma_h^{j-2})$. Hence α^* is closed in $T_{n,j-1}$. By R2, $\Gamma^j - \Gamma^{j-1} \subset \overline{\varphi}(T_{n,j} - T_{n,j-1})$, $\alpha^* \notin \Gamma^j$. Therefore $\alpha^* \notin (\cup_{\eta_h \in D_{n,j}} \Gamma_h^{j-1}) \subset \Gamma^j$. Now by R2(2), α^* is also closed in $T_{n,j}$, where we have the color α^* as claimed.

Since α is closed in $T_{n,j}$, $\varphi^* = \varphi/(\alpha, \alpha^*, G - T_{n,j})$ is $T_{n,j}$ -stable and $T_{n,j}$ satisfies MP and R2 up to itself by the same splitters and Γ sets under φ^* by Lemma 2.3.6. Note that $\alpha^* \in \overline{\varphi}^*(T_{n,j-1})$ and α^* is still closed in $T_{n,j-1}$ under φ^* . Thus $T_{n,j-1}$ satisfies MP and R2 up to itself under φ^* . However $P_u^{ex}(\alpha^*, \beta, \varphi^*) = P_u^{ex}(\alpha, \beta, \varphi)$ and $P_v^{ex}(\alpha^*, \beta, \varphi^*) = P_v^{ex}(\alpha, \beta, \varphi)$ are

two (α^*, β) exit paths of $T_{n,j-1}$ under φ^* , giving a contradiction to the induction hypothesis of the minimality of j .

Case II: $\beta \notin \overline{\varphi}(T_{n,j})$.

In this case $|\partial_\beta(T_{n,j})| = \text{odd}$. Hence $T_{n,j}$ has odd number of (α, β) exit paths. Let u, v, w be exits from three (α, β) exit paths for $T_{n,j}$ with $u \prec_l v \prec_l w$.

Case II.a: $w \in T_{n,j} - T_{n,j-1}$. Here we assume that for all counter-examples of Case II.a in our Claim (i.e $\beta \notin \overline{\varphi}(T_{n,j})$ and $w \in T_{n,j} - T_{n,j-1}$), $L = |P_w^{ex}(\alpha, \beta, \varphi)| + |P_u^{ex}(\alpha, \beta, \varphi)| + |P_v^{ex}(\alpha, \beta, \varphi)|$ is minimum. Let $\gamma \in \overline{\varphi}(w)$. By definition, $\gamma \notin \Gamma^{j-1}$, and hence $T_{n,j}$ is closed for γ by condition R2(2). Note that γ may be η_h for some $h \leq i$. By Lemma 2.3.6, $\varphi^* = \varphi / (\alpha, \gamma, G - T_{n,j})$ is $T_{n,j}$ -stable, and $T_{n,j}$ satisfies MP and R2 up to itself under φ^* . Moreover, in φ^* , we have $P_w^{ex}(\gamma, \beta, \varphi^*) = P_w(\gamma, \beta, \varphi^*) = P_w^{ex}(\alpha, \beta, \varphi)$, $P_u^{ex}(\gamma, \beta, \varphi^*) = P_u^{ex}(\alpha, \beta, \varphi)$ and $P_v^{ex}(\gamma, \beta, \varphi^*) = P_v^{ex}(\alpha, \beta, \varphi)$ are three (γ, β) exit paths for $T_{n,j}$. Let the three other end vertices of $P_w^{ex}(\gamma, \beta, \varphi^*)$, $P_u^{ex}(\gamma, \beta, \varphi^*)$ and $P_v^{ex}(\gamma, \beta, \varphi^*)$ not in $T_{n,j}$ be w_2 , u_2 and v_2 respectively. Let u' be the vertex in $P_u^{ex}(\gamma, \beta, \varphi^*)$ next to u , and the edge connecting u and u' be f_u ; and v' be the vertex in $P_v^{ex}(\gamma, \beta, \varphi^*)$ next to v , and the edge connecting v and v' be f_v . Note that f_v and f_u are colored β in φ^* . Let $\varphi^2 = \varphi^* / P_w^{ex}(\gamma, \beta, \varphi^*)$. Since $w \in T_{n,j} - T_{n,j-1}$ and $P_w(\gamma, \beta, \varphi^*) \cap T_{n,j} = w$, $T_{n,j-1}$ satisfies MP under φ^2 by Lemma 2.3.6. Moreover, T_w satisfies MP and R2 because $\varphi^2(f) = \varphi^*(f)$ for every f incident to $V(T_w)$ and $\overline{\varphi}^2(v') = \overline{\varphi}^*(v')$ for every $v' \in T_w - w$. Note that under φ^2 , $\beta \in \overline{\varphi}^2(w)$. Since $\beta \notin \Gamma^{j-1}$, we have $\{T_w, f_u, u', f_v, v'\}$ satisfies MP and R2. Note that by (A5) we can keep condition MP by keeping extending $\{T_w, f_u, u', f_v, v'\}$ using TAA under condition R2 until it is $(\cup_{\eta_h \in D_{n,j}} \Gamma_h^{j-1})^-$ -closed. Let the resulting ETT be $T_{n,j}^2$. Clearly $T_{n,j}^2$ satisfies MP and R2. By induction hypothesis, $T_{n,j}^2$ is elementary. If one of w_2, u_2, v_2 is in $T_{n,j}^2$, then γ must be missing at that vertex since $\beta \in \overline{\varphi}^2(T_{n,j}^2)$. Since both $\gamma, \beta \notin \Gamma^{j-1}$, and both $\gamma, \beta \in \overline{\varphi}^2(T_{n,j}^2)$, we must have all three vertices w_2, u_2, v_2 are in $T_{n,j}^2$. However, all of them miss either γ or β in φ^2 , which gives a contradiction to the elementary property. Thus none of the vertices above are in $T_{n,j}^2$. Hence each of $P_u^{ex}(\gamma, \beta, \varphi^*)$, $P_v^{ex}(\gamma, \beta, \varphi^*)$ and $P_w^{ex}(\gamma, \beta, \varphi^*)$ contains a (γ, β) exit path of $T_{n,j}^2$. Let u_1, v_1, w_1 be the exits for the (γ, β) exit paths contained in the three paths above respec-

tively. We without loss of generality assume $u_1 \prec_f v_1 \prec_f w_1$. Note that $w_1 \neq w$ since we already have $w \prec_f u' \prec_f v'$ in $T_{n,j}^2$. Note that $P_{u_1}^{ex}(\gamma, \beta, \varphi^2)$ and $P_{v_1}^{ex}(\gamma, \beta, \varphi^2)$ are sub-paths of $P_u^{ex}(\alpha, \beta, \varphi)$ and $P_v^{ex}(\alpha, \beta, \varphi)$ and are shorter than those two. Thus we have a contradiction with the minimality of L .

Case II.b: $w \notin T_{n,j} - T_{n,j-1}$.

The proof of this case is essentially the same as in Case I.b. We first show there exists a color which closed in both $T_{n,j-1}$ and $T_{n,j}$. So there is a $T_{n,j}$ -stable coloring φ^* in which $T_{n,j}$ satisfies all MP and R2 up to itself. However in φ^* , α^* and β are not interchangeable in $T_{n,j-1}$, giving a contradiction to the minimality of j . Here we omit the proof. \square

Claim 2.3.19. *For any $y \in V(T_{y_{p-1}}) - V(T_{n,q})$, $|\bar{\varphi}(T_y) - \varphi(T_y - T_{n,q})| \geq 11 + 2n$. Furthermore, if $|\bar{\varphi}(T_y) - \Gamma^q \cup D_{n,q} \cup \varphi(T_y - T_{n,q})| \leq 4$, then there exist 7 distinct η_i with $\eta_i \in D_{n,q}$ and $\eta_i \in \bar{\varphi}(T_y)$ such that all colors $\eta_i, \gamma_{i1}, \gamma_{i2} \notin \varphi(T_y - T_{n,q})$.*

Proof. Since $|\bar{\varphi}(T_y - T_{n,q})| \geq |\varphi(T_y - T_{n,q})|$, $|V(T_{n,q})| \geq 11 + 2(n - 1)$ and $|\bar{\varphi}(T_y) - \varphi(T_y - T_{n,q})| \geq |\bar{\varphi}(T_{n,q})| \geq |V(T_{n,q})| + 2 \geq 11 + 2n$. Now assume $|\bar{\varphi}(T_y) - \Gamma^q \cup D_{n,q} \cup \varphi(T_y - T_{n,q})| \leq 4$. Since $\bar{\varphi}(T_y) = (\bar{\varphi}(T_y) - \Gamma^q \cup D_{n,q} \cup \varphi(T_y - T_{n,q})) \cup ((\Gamma^q \cup D_{n,q}) \cap \bar{\varphi}(T_y) - \varphi(T_y - T_{n,q}) \cap \bar{\varphi}(T_y)) \cup (\varphi(T_y - T_{n,q}) \cap \bar{\varphi}(T_y))$, we have

$$|(\Gamma^q \cup D_{n,q}) \cap \bar{\varphi}(T_y) - \varphi(T_y - T_{n,q}) \cap \bar{\varphi}(T_y)| \geq |\bar{\varphi}(T_y)| - 4 - |\varphi(T_y - T_{n,q})| \geq 2n + 7.$$

Thus we have

$$|(\Gamma^q \cup D_{n,q}) \cap \bar{\varphi}(T_y) - \varphi(T_y - T_{n,q})| \geq 2n + 7.$$

By the Pigeonhole Principle, there are 7 distinguished i such that $\eta_i, \gamma_{i1}, \gamma_{i2} \notin \varphi(T_y - T_{n,q})$, so the result holds. \square

Claim 2.3.20. *Let α and β be two missing colors of $V(T_{y_{p-1}})$ with $v(\alpha) \prec_\ell v(\beta)$ and $\alpha \notin \varphi(T_{v(\beta)} - T_{n,q})$. If $\alpha \in \bar{\varphi}(T_{n,q})$ or $\alpha, \beta \notin D_{n,q}$, then $P_{v(\alpha)}(\alpha, \beta, \varphi) = P_{v(\beta)}(\alpha, \beta, \varphi)$. Additionally, if $\alpha \in \bar{\varphi}(T_{n,q})$ and α is $T_{n,q}$ -closed, then $P_{v(\alpha)}(\alpha, \beta, \varphi)$ is the only (α, β) -path intersecting $\partial(T_{n,q})$.*

Note that $T_{v(\beta)} - T_{n,q} = \emptyset$ if $v(\beta) \in T_{n,q}$ and in Claim 2.3.20, (α, β) -path can not be replaced by (α, β) -chain because there may be (α, β) -cycles intersecting $\partial(T_m)$.

Proof. Let $v(\alpha) = u$ and $v(\beta) = w$. We consider the following few cases.

Case I: $u, w \in T_{n,q}$.

If $T_{n,q}$ is closed for both α, β , then $E_{\alpha, \beta} \cap \partial(T_{n,q}) = \emptyset$ and $P_u(\alpha, \beta, \varphi) = P_w(\alpha, \beta, \varphi)$ since $T_{n,q}$ is elementary. So Claim 2.3.20 holds. Suppose $T_{n,q}$ is closed for α or β but not for both, by Claim 2.3.17 there is at most one (α, β) path intersecting $T_{n,q}$. If $P_u(\alpha, \beta, \varphi) \neq P_w(\alpha, \beta, \varphi)$, then there are two (α, β) paths intersecting $T_{n,q}$, giving a contradiction to Claim 2.3.17. If $P_u(\alpha, \beta, \varphi)$ is not the unique (α, β) -path intersecting $\partial(T_{n,q})$, we also have a contradiction. Hence $P_u(\alpha, \beta, \varphi)$ is the unique (α, β) -path intersecting $\partial(T_{n,q})$ and $P_u(\alpha, \beta, \varphi) = P_w(\alpha, \beta, \varphi)$. We now assume neither α nor β is $T_{n,q}$ -closed. Under this assumption, we only need to show that $P_u(\alpha, \beta, \varphi) = P_w(\alpha, \beta, \varphi)$. We may assume $\beta \in \overline{\varphi}(T_{n,j'} - T_{n,j'-1})$ for some $0 \leq j' < q$ where $T_{n,-1} = \emptyset$ for convenience. By condition R2, β is closed in $T_{n,j'}$. In the same fashion as we did for the case in which $T_{n,q}$ is closed for either α or β , we have $P_u(\alpha, \beta, \varphi) = P_w(\alpha, \beta, \varphi)$ in $T_{n,j'}$ because we have $u, v \in T_{n,j'}$.

Case II: $w \notin T_{n,q}$ and $u \in T_{n,q}$.

In this case $\alpha \notin \varphi(T_w - T_{n,q})$. We first consider the case α is closed in $T_{n,q}$. By Claim 2.3.17, there is at most one (α, β) path intersecting $T_{n,q}$, which is $P_u(\alpha, \beta, \varphi)$. If $P_u(\alpha, \beta, \varphi) \neq P_w(\alpha, \beta, \varphi)$, then $P_w(\alpha, \beta, \varphi)$ does not intersect $T_{n,q}$. Therefore by Lemma 2.3.6, $T_{n,q}$ satisfies MP and R2 up to itself under $\varphi^* = \varphi/P_w(\alpha, \beta, \varphi)$ which is $T_{n,q}$ -stable. Note that $\beta \notin \varphi(T_w - T_{n,q} - f_n)$ where f_n is the connecting edge in $T - T_n$ colored by δ_n (SE) or γ_n (RE). However, in this case $f_n \notin P_w(\alpha, \beta, \varphi)$ because f_n is incident to $V(T_n)$. Thus we have $\varphi(f) = \varphi^*(f)$ for each $f \in T_w$ and $\overline{\varphi}(v') = \overline{\varphi^*}(v')$ for each $v' \in T_w - w$ because $\alpha \notin \varphi(T_w - T_{n,q})$. Thus φ^* is $(T_w - w)$ -stable and T_w satisfies MP and R2 (1). However, T_w is not elementary, giving a contradiction to the minimality of p .

Now we assume that α is not closed in $T_{n,q}$. In this case we only need to prove $P_u(\alpha, \beta, \varphi) = P_w(\alpha, \beta, \varphi)$. Assume not. Since $\alpha \in \overline{\varphi}(T_{n,q})$, by condition R2, there exist

the largest q' such that α is closed in $T_{n,q'}$. Since the only edge in $T - T_n$ with color not missing before is the connecting edge with color δ_n or γ_n , we have $\beta \notin \varphi(T_w - T_n - f_n)$. We claim that $\alpha \notin \varphi(T_w - T_{n,q'})$. Recall that $\alpha \notin T_w - T_{n,q}$. Suppose $\alpha \in \varphi(T_w - T_{n,q'})$. We can assume $\alpha \in \varphi(T_{n,r} - T_{n,r-1})$ for some $q \geq r > q' \geq 0$. Then $\alpha \notin \cup_{\eta_h \in D_{n,r}} \Gamma_h^{r-1}$, and hence α is closed in $T_{n,r}$ by condition R2 if $\alpha \neq \varphi(f_n)$, which contradicts the maximality of q' . If $\alpha = \varphi(f_n)$, then $r = 1$. Let r' be the index such that $\alpha \in \overline{\varphi}(T_{r'} - T_{r'-1})$. Then $r' \geq 1 = r$. In this case we have $\alpha \notin \Gamma^{q-1}$, hence it is still closed in $T_{n,r'}$ by condition R2, which contradicts the maximality of q' . Hence we indeed have $\alpha \in \varphi(T_w - T_{n,q'})$. By Claim 2.3.17 there is at most one (α, β) path intersecting $T_{n,q'}$, which is $P_u(\alpha, \beta, \varphi)$. Then $P_w(\alpha, \beta, \varphi)$ is disjoint with $T_{n,q'}$. Hence $T_{n,q'}$ satisfies MP and R2 up to itself under the $T_{n,q'}$ -stable coloring $\varphi^* = \varphi/P_w(\alpha, \beta, \varphi)$ by Lemma 2.3.6. Recall that $\eta'_n = \gamma_n$ when $\Theta_n = \text{RE}$ and $\eta'_n = \delta_n$ if $\Theta_n = \text{SE}$. Now we first consider the case $\beta \neq \eta_n$. In this case $\alpha, \beta \notin \varphi(T_w - T_{n,q'})$. Therefore, $\alpha, \beta \notin \varphi^*(T_w - T_{n,q'})$, and φ^* is $T_{n,q}$ -stable, which implies all $T_{n,s}$ for $q' \leq s \leq q$ satisfies R2(1) under φ^* . Note that neither α nor β is closed in $T_{n,s}$ for $q' \leq s \leq q$ for φ , we have all $T_{n,s}$ for $q' \leq s \leq q$ satisfies condition R2(2) because none of the closed colors become non-closed in $T_{n,s}$ for $q' \leq s \leq q$ under φ^* . Moreover, we have $\varphi(f) = \varphi^*(f)$ for each $f \in T_w$ and $\overline{\varphi}(v') = \overline{\varphi^*}(v')$ for each $v' \in T_w - w$ because $\alpha \notin \varphi(T_w - T_{n,q'})$. Thus T_w satisfies R2 (1) itself. The fact T_w satisfies R2 follows from the fact that T_w is still obtained by TAA under φ^* . However, $\alpha \in \varphi^*(w) \cap \varphi^*(T_{n,q})$, giving a contradiction to the minimality of p . For the case $\beta = \delta_n$, we have only f_n in $T_{n,q}$ is colored by β by the construction of $T_{n,q}$ and the assumption $\beta \in \overline{\varphi}(w)$ and $w \notin T_{n,q}$. Moreover, by Claim 2.3.17, α is interchangeable with β in $T_{n,0}$, hence there is only one (α, β) path intersecting $T_{n,0}$. Therefore $P_w(\alpha, \beta, \varphi)$ is disjoint with $T_{n,0}$, and hence $\varphi^*(f_e) = \beta$. Note that we can conclude φ^* is $T_{n,q}$ stable similarly as before, and similarly we have T_w satisfying MP and R2 under φ^* . We then reach a contradiction since $\alpha \in \varphi^*(w) \cap \varphi^*(T_{n,q})$.

Case III: $u, w \notin T_{n,q}$.

In this case, we have $\alpha \notin \varphi(T_w - T_{n,q})$ and $\alpha, \beta \notin D_{n,q}$, which in turn give $\alpha, \beta \notin D_{n,q} \cup \Gamma(T) \cup \Gamma^q$. Suppose on the contrary that $P_u(\alpha, \beta, \varphi) \neq P_w(\alpha, \beta, \varphi)$. Now consider

the proper coloring $\varphi^* = \varphi/P_w(\alpha, \beta, \varphi)$. Since $\alpha, \beta \notin D_{n,q} \cup \Gamma(T) \cup \Gamma^q$, we have $\alpha, \beta \notin \varphi(T_w - T_n)$. Thus φ^* is $T_{n,q}$ -stable. Since $\alpha \notin \varphi(T_w - T_{n,q})$, T is still an ETT under φ^* . Therefore T satisfies condition MP and R2 under φ^* because R2 is not related to colors in $\overline{\varphi}(T - T_{n,q}) - D_{n,q}$. However, now $\alpha \in \overline{\varphi}^*(u) \cap \overline{\varphi}^*(w)$, which gives a contradiction to the minimality of $|V(T - T_{n,q})|$. \square

Claim 2.3.21. *For any two colors $\alpha, \beta \in \overline{\varphi}(T_{y_{p-1}})$, the following two statements hold.*

- (1) *If $\alpha \in \overline{\varphi}(T_{n,q})$ and P is an (α, β) path other than $P_{v(\alpha)}(\alpha, \beta, \varphi)$, then $\varphi^* = \varphi/P$ is $T_{n,q}$ -stable and $T_{n,q}$ satisfies MP, R2 up to itself under φ^* . Consequently by (A5), any tree sequence obtained from $T_{n,q}$ when $q \geq 1$ or from $T_n \cup F_n \cup b(F_n)$ when $q = 0$ by TAA is an ETT satisfying MP under φ^* .*
- (2) *If $T_{n,q} \prec_\ell v(\alpha) \prec_\ell v(\beta) \prec_\ell y_{p-1}$, $\alpha \notin \varphi(T_{v(\beta)})$ and $\alpha, \beta \notin D_{n,q}$, then $\varphi^* = \varphi/P$ is T -stable with the same set of $D_{n,q}, \Gamma(T)$, and Γ^q for any (α, β) -chain P . Moreover, T satisfies MP and R2 under φ^* .*

We can see we actually proved any tree sequence obtained from $T_{n,q}$ when $q \geq 1$ or from $T_n \cup F_n \cup b(F_n)$ when $q = 0$ by TAA is an ETT satisfying MP under φ^* by (A5). **Hence in later proofs, each time after using Claim 2.3.21, we claim MP and R2 are satisfied for T without mentioning why MP is satisfied if T is trivial obtained by TAA.**

Proof. We first prove part (1). If one of α and β is closed in $T_{n,q}$, we have that P is disjoint with $T_{n,q}$ by Claim 2.3.20. Then $\varphi^* = \varphi/P$ is $T_{n,q}$ -stable and $T_{n,q}$ satisfies MP, R2 up to itself under φ^* by Lemma 2.3.6. We now suppose that neither α nor β is closed in $T_{n,q}$. Then similarly to the proof of Claim 2.3.20, by Condition R2(2), there exist the largest q' such that either α or β is closed in $T_{n,q'}$. We claim that $\alpha \notin \varphi(T_{n,q} - T_{n,q'})$ and $\beta \notin \varphi(T_{n,q} - T_{n,q'} - f_n)$. Since $\alpha \in \overline{\varphi}(T_{n,q})$, by the same proof of Claim 2.3.20 Case II where we assume α is not closed in $T_{n,q}$, we see that $\alpha \notin \varphi(T_{n,q} - T_{n,q''})$ where q'' is the largest index such that α is closed in $T_{n,q''}$. By our choice of q' , $q' \geq q''$ and therefore we have $\alpha \notin \varphi(T_{n,q} - T_{n,q'})$. Now we prove

$\beta \notin \varphi(T_{n,q} - T_{n,q'} - f_n)$. If $\beta \in \overline{\varphi}(T_{n,q})$, we argue just as in the case when α is not closed in $T_{n,q}$. If $\beta \notin \overline{\varphi}(T_{n,q})$, the only possibility that $\beta \in \varphi(T_{n,q} - T_{n,q'})$ is $\beta = \eta'_n = \varphi(f_n)$ and $q' = 0$ because T is obtained from $T_n \cup F_n \cup b(F_n)$ by TAA. Thus we have as claimed. Because one of α, β is closed in $T_{n,q'}$, by Claim 2.3.17 there is at most one (α, β) path intersecting $T_{n,q'}$, which is $P_{v(\alpha)}(\alpha, \beta, \varphi)$. Then P is disjoint with $T_{n,q'}$. By Lemma 2.3.6, $T_{n,q'}$ satisfies MP and R2 up to itself under the $T_{n,q'}$ -stable coloring φ^* . Since f_n is incident to $V(T_n)$ and P is disjoint with $T_{n,q'}$ for $q' \geq 0$, we see that $\varphi(f_n) = \varphi^*(f_n) = \eta'_n$. By our choice of q' , $\alpha, \beta \notin \overline{\varphi}(T_{n,q} - T_{n,q'})$ because colors in $\overline{\varphi}(T_{n,j} - T_{n,j-1})$ are closed in T_j under φ for $0 \leq j \leq q$. Because $\alpha, \beta \notin \overline{\varphi}(T_{n,q} - T_{n,q'})$, $\varphi(f_n) = \varphi^*(f_n) = \eta'_n$ and $\alpha, \beta \notin \varphi(T_{n,q} - T_{n,q'} - f_n)$, we see that $\varphi(f) = \varphi^*(f)$ for each $f \in T_{n,q}$ and $\overline{\varphi}(v') = \overline{\varphi}^*(v')$ for each $v' \in T_{n,q}$. Thus φ^* is $T_{n,q}$ -stable, and therefore $T_{n,s}$ satisfies MP and R2 (1) for each $q' < s \leq q$. Since both β, α are not closed in $T_{n,s}$ for $p' < s \leq q$, $T_{n,s}$ satisfies condition R2(2) because none of the closed colors become non-closed in $T_{n,s}$. Thus $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* .

Now we prove part (2). By Claim 2.3.20, $P_{v(\alpha)}(\alpha, \beta, \varphi) = P_{v(\beta)}(\alpha, \beta, \varphi)$. In this case we have $\alpha, \beta \notin \overline{\varphi}(T_{n,q}) \cup D_{n,q}$ and $\alpha, \beta \notin \varphi(T_{v(\beta)})$. If $P = P_{v(\alpha)}(\alpha, \beta, \varphi)$, then T is an ETT in $\varphi^* = \varphi/P$ since $\alpha, \beta \notin \varphi^*(T_{v(\beta)})$. Since $\alpha, \beta \notin \overline{\varphi}(T_{n,q}) \cup D_{n,q}$, T satisfies MP and R2 under φ^* . If $P \neq P_{v(\alpha)}(\alpha, \beta, \varphi)$, then T is an ETT satisfying MP and R2 under φ^* because we still have $\alpha, \beta \notin \varphi^*(T_{v(\beta)})$. \square

2.3.2.2.2 Case verification

Claim 2.3.22. $p > 0$

Proof. Suppose on the contrary $p = 0$, that is, $T = T_{n,q} \cup \{e_0, y_0\}$. We consider two cases.

Case I: $q = 0$. In this case $T_{n,q} = T_n$ is closed and e_0 is a connecting edge.

The case for SE extension follows from the definition, because $T = T_{n,q} \cup \{e_0, y_0\}$ is elementary for all T_n -stable colorings. Now we consider the case for RE extension. Assume

that there exist $\alpha \in \overline{\varphi}(T_n) \cap \overline{\varphi}(y_0)$. Since $\alpha, \delta_n \in \overline{\varphi}(T_n)$, T_n is closed for α and δ_n , and therefore $P_{y_0}(\alpha, \delta_n, \varphi)$ is disjoint with T_n . Therefore $\varphi^* = \varphi/P_{y_0}(\alpha, \delta_n, \varphi)$ is also T_n -stable and $\delta_n \in \overline{\varphi}^*(T_n) \cap \overline{\varphi}^*(y_0)$. By (A5), f_n is still a RE connecting edge under φ^* because φ^* is T_n -stable, i.e. (T_n, D_n, φ) -stable. Thus f_n satisfies condition R under φ^* , and therefore f_n belongs to a (δ_n, γ_n) -cycle under φ^* . However, f_n belongs to $P_{y_0}(\delta_n, \gamma_n, \varphi^*)$, a contradiction.

Case II: $q > 0$. In this case $T_{n,q}$ is not closed although it is $(\cup_{\eta_h \in D_{n,q}} \Gamma_h^{q-1})^-$ closed.

Assume without loss of generality that e_0 is colored by $\gamma_0 \in \Gamma^{q-1}$. Let $\alpha \in \overline{\varphi}(T_{n,q}) \cap \overline{\varphi}(y_0)$. Let $u = a(e_0)$ be the vertex adjacent to e_0 belonging to $T_{n,q}$. We further assume that $u \in T_{n,q'} - T_{n,q'-1}$ for some $q' \leq q$ where $T_{n,-1} = \emptyset$ for convenience. We claim that $v(\gamma_0) \prec_l u$. Otherwise we can assume $v(\gamma_0) \in T_{n,s} - T_{n,s-1}$ for some $q' \leq s \leq q$, and then γ_0 is closed in $T_{n,s}$ by condition R2(2). Combining with the assumption $u \prec_f v(\gamma_0)$, we get $y_0 \in T_{n,s}$, a contradiction. Hence we have as claimed. Let $\gamma \in \overline{\varphi}(u)$. Clearly $\alpha \neq \gamma_0$ and $\gamma \neq \gamma_0$ because $\varphi(e_0) = \gamma_0$ and α, γ are missing at the endpoints of e_0 . Since both $v(\alpha)$ and u are in $T_{n,q}$, we have $P_{v(\alpha)}(\alpha, \gamma, \varphi) = P_u(\alpha, \gamma, \varphi)$ by Claim 2.3.20, and $P_{y_0}(\alpha, \beta, \varphi)$ is different from the path above. Moreover by Claim 2.3.21, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring $\varphi^* = \varphi/P_{y_0}(\alpha, \gamma, \varphi)$. Since $\alpha \neq \gamma_0$ and $\gamma \neq \gamma_0$, $\varphi^*(e_0) = \gamma_0$. Note that $\gamma \in \overline{\varphi}^*(u) \cap \overline{\varphi}^*(y_0)$. Let $u \in T_s - T_{s-1}$ where $0 \leq s \leq q$. Then $P_{v(\gamma_0)}(\alpha, \gamma_0, \varphi^*)$ and $P_u(\alpha, \gamma_0, \varphi^*)$ are two (α, γ_0) -paths intersecting $T_{n,s}$. Since φ^* is $T_{n,q}$ -stable, it is also $T_{n,s}$ -stable. Thus we have a contradiction with Claim 2.3.18. \square

Case 1. $p(T) = 0$.

Claim 2.3.23. *We may assume $\alpha \in \overline{\varphi}(y_i) \cap \overline{\varphi}(y_p)$ for some $p > i \geq 0$.*

Proof. Suppose $\alpha \in \overline{\varphi}(y_p) \cap \overline{\varphi}(v)$ for some $v \in V(T_{n,q})$. We first consider the case $\alpha \notin \varphi(T - T_{n,q})$. Let $\beta \in \overline{\varphi}(y_{p-1})$. By Claim 2.3.20 $P_v(\alpha, \beta, \varphi) = P_{y_{p-1}}(\alpha, \beta, \varphi)$ and $P_{y_p}(\alpha, \beta, \varphi)$ is different from the path above. Let $\varphi^* := \varphi/P_{y_p}(\alpha, \beta, \varphi)$. By Claim 2.3.21, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Since $\alpha, \beta \notin \varphi(T_{y_p} - T_{n,q})$, we have $\varphi(f) = \varphi^*(f)$ for every $f \in T$ and $\overline{\varphi}(v') = \overline{\varphi}^*(v')$ for every $v' \in T_{y_{p-1}}$, and therefore T

satisfies MP and R2 (1) under φ^* . Note that we have $\beta \in \overline{\varphi^*}(y_{p-1}) \cap \overline{\varphi^*}(y_p)$, Claim 2.3.23 holds.

We now consider the case $\alpha \in \varphi(T - T_{n,q})$. Following order \prec_ℓ , let e_j be the first edge in $T - T_{n,q}$ such that $\alpha = \varphi(e_j)$. We first consider the case $j \geq 1$. Let $\beta \in \overline{\varphi}(y_{j-1})$. Then $\alpha, \beta \notin \varphi(T_{y_{j-1}} - T_{n,q})$. By Claim 2.3.20, $P_v(\alpha, \beta, \varphi) = P_{y_{j-1}}(\alpha, \beta, \varphi)$ and $P_{y_p}(\alpha, \beta, \varphi)$ is different from the path above. Moreover, by Claim 2.3.21 $T_{n,q}$ satisfies MP and R2 under $\varphi^* = \varphi/P_{y_p}(\alpha, \beta, \varphi)$ which is a $T_{n,q}$ -stable coloring. Now we check MP and R2 (1) on $T - T_{n,q}$. Since both $\beta, \alpha \notin \varphi(T_{y_{j-1}} - T_{n,q})$, T can still be obtained by TAA and therefore it satisfies MP. Moreover, under φ^* , condition R2 (1) holds for T if $\alpha \notin \Gamma^q$, because $\beta \notin \Gamma^q$. If $\alpha \in \Gamma^q$, say $\alpha = \gamma_{i1}$ for some $0 < i \leq n$, by condition R2 (1) we have $\eta_i \in \overline{\varphi}(w)$ for some $w \preceq_\ell y_{j-1}$. Since only edges after w in the ordering \prec_ℓ may change colors between α and β , condition R2 (1) also holds in φ^* . Since $\beta \in \overline{\varphi^*}(y_{j-1}) \cap \overline{\varphi^*}(y_p)$, Claim 2.3.23 holds by simply denoting φ^* as φ .

Now we assume that $j = 0$. In this case $q > 0$, since $q = 0$ implies $\alpha = \eta'_n$ which is a contradiction to $\alpha \in \overline{\varphi}(T_{n,q})$. Therefore, $\alpha = \varphi(e_0)$ where $\alpha \in \Gamma^{q-1}$. Note that $\alpha \notin \Gamma^q$ by condition R2 (1). We will show that there exists $\gamma \in \overline{\varphi}(T_{n,q}) - \Gamma^q$ such that γ is closed in $T_{n,q}$. By condition R2(2), $T_{n,q}$ is $(\cup_{\eta_h \in D_{n,q}} \Gamma_h^{q-1})^-$ closed. Therefore, $T_{n,q}$ is closed for colors in $\overline{\varphi}(T_{n,q}) - \Gamma^{q-1}$ because $\cup_{\eta_h \in D_{n,q}} \Gamma_h^{q-1} \subseteq \Gamma^{q-1}$. Hence we need to show that there exists $\gamma \in \overline{\varphi}(T_{n,q}) - \Gamma^q \cup \Gamma^{q-1}$. Since $\Gamma^q - \Gamma^{q-1} \subseteq \overline{\varphi}(T_{n,q} - T_{n,q-1})$ by condition R2 and the assumption that $T_{n,q-1}$ is elementary, we have $|(\Gamma^q \cup \Gamma^{q-1}) \cap \overline{\varphi}(T_{n,q-1})| = |\Gamma^{q-1} \cap \overline{\varphi}(T_{n,q-1})| \leq 2n$ and $|\overline{\varphi}(T_{n,q-1})| \geq |\overline{\varphi}(T_1)| + 2(n-1) = 2n + 11$. Therefore $|\overline{\varphi}(T_{n,q}) - \Gamma^q \cup \Gamma^{q-1}| = |\overline{\varphi}(T_{n,q-1}) - \Gamma^{q-1}| + |\overline{\varphi}(T_{n,q} - T_{n,q-1}) - (\Gamma^q - \Gamma^{q-1})| \geq |\overline{\varphi}(T_{n,q-1}) - \Gamma^{q-1}| \geq (2n + 11) - 2n \geq 11$, where we have γ as desired. Now by Claim 2.3.20, $P_{v(\alpha)}(\alpha, \gamma, \varphi) = P_{v(\gamma)}(\alpha, \gamma, \varphi)$, and $P_{y_p}(\alpha, \beta, \varphi)$ is disjoint with $T_{n,q}$. Therefore by Claim 2.3.21, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring $\varphi^* = \varphi/P_{y_p}(\alpha, \gamma, \varphi)$. Moreover, since $e_0 \in \partial(T_{n,q})$, $e_0 \notin P_{y_p}(\alpha, \gamma, \varphi)$. Since $\alpha, \gamma \notin \Gamma^q$ and $\alpha, \gamma \in \overline{\varphi}(T_{n,q})$, T_{y_p} satisfies MP and R2 (1) under φ^* . Now $\gamma \in \overline{\varphi^*}(y_p) \cap \overline{\varphi^*}(v)$ for some $v \in V(T_{n,q})$ and $\alpha \neq \gamma$, which returns to the case either $\gamma \notin \overline{\varphi}(T - T_{n,q})$ or $j \geq 1$. \square

Among all T -stable colorings satisfying MP and R2, we assume that i is the maximum index such that $\alpha \in \overline{\varphi}(y_p) \cap \overline{\varphi}(y_i)$.

Claim 2.3.24. $i = p - 1$.

Proof. Suppose on the contrary $i < p - 1$. We first consider the case $\alpha \notin D_{n,q}$. Let $\theta \in \overline{\varphi}(y_{i+1})$. If $\theta \notin D_{n,q}$, then $\{\alpha, \theta\} \cap D_{n,q} = \emptyset$. Thus $\alpha, \theta \notin \varphi(T_{y_{i+1}} - T_{n,q})$. By Claim 2.3.20, $P_{y_i}(\alpha, \theta, \varphi) = P_{y_{i+1}}(\alpha, \theta, \varphi)$, and therefore $P_{y_p}(\alpha, \beta, \varphi)$ is a different path. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \beta, \varphi)$. Since both $\alpha, \beta \notin D_{n,q}$ and $\alpha \notin \varphi(T_{y_{i+1}} - T_{n,q})$, by Claim 2.3.21, T is also an ETT satisfying MP and R2 under the $T_{n,q}$ -stable coloring φ^* . But $\theta \in \overline{\varphi^*}(y_p) \cap \overline{\varphi^*}(y_{i+1})$, which contradicts the maximality of i . We now consider the case $\theta = \eta_k$ with $\eta_k \in D_{n,q}$. By Claim 2.3.20, $P_{v(\gamma_{k1})}(\alpha, \gamma_{k1}, \varphi) = P_{y_i}(\alpha, \gamma_{k1}, \varphi)$ and $P_{y_p}(\alpha, \gamma_{k1}, \varphi)$ is different from path above. Note that $P_{y_p}(\alpha, \gamma_{k1}, \varphi) = y_p$ can occur if $\gamma_{k1} \in \overline{\varphi}(y_p)$. By Claim 2.3.21, $\varphi^* = \varphi/P_{y_p}(\alpha, \gamma_{k1}, \varphi)$ is a $T_{n,q}$ -stable coloring and $T_{n,q}$ satisfies MP and R2 up to itself under φ^* . Since f_n is incident to $V(T_{n,q})$, $\varphi(f) = \varphi^*(f) = \eta'_n$. Moreover, since $\alpha \notin \varphi(T_{y_{i+1}} - T_{n,q})$, $\beta \notin \varphi(T_{i+1} - T_{n,q} - f_n)$ and $\eta_k \in \overline{\varphi}(y_{i+1})$, T still satisfies MP and R2 under φ^* . Then, by Claim 2.3.20 again, $P_{v(\gamma_{k1})}(\eta_k, \gamma_{k1}, \varphi^*) = P_{y_i}(\eta_k, \gamma_{k1}, \varphi^*)$ and $P_{y_p}(\eta_k, \gamma_{k1}, \varphi^*)$ is different from the path above. Let $\varphi^{**} = \varphi^*/P_{y_p}(\eta_k, \gamma_{k1}, \varphi^*)$. By Claim 2.3.21, φ^{**} is $T_{n,q}$ -stable and $T_{n,q}$ satisfies MP and R2 under φ^{**} . Since f_n is incident to $V(T_{n,q})$, we still have $\varphi^*(f) = \varphi^{**}(f) = \eta'_n$. Note that $\gamma_{k1} \notin \overline{\varphi^{**}}(T_{y_{i+1}} - T_{n,q})$, and η_k is only possibly used by $f_n \in T_{y_{i+1}} - T_n$, we see that T satisfies condition R2 (1) and MP because $\eta_k, \gamma_{k1} \in \overline{\varphi^{**}}(T_{y_{i+1}})$. However under φ^{**} , $\eta_k \in \overline{\varphi^{**}}(y_p) \cap \overline{\varphi^{**}}(y_{i+1})$, giving a contradiction to the maximality of i .

We now consider the case $\alpha = \eta_k \in D_{n,q}$. Since $\varphi(e_{i+1})$ can not be both γ_{k1} and γ_{k2} , we assume without loss of generality $\varphi(e_{i+1}) \neq \gamma_{k1}$. By Claim 2.3.20, $P_{v(\gamma_{k1})}(\eta_k, \gamma_{k1}, \varphi) = P_{y_i}(\eta_k, \gamma_{k1}, \varphi)$ and $P_{y_p}(\eta_k, \gamma_{k1}, \varphi)$ is a different path. Let $\varphi^* = \varphi/P_{y_p}(\eta_k, \gamma_{k1}, \varphi)$. By Claim 2.3.21, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Now $\gamma_{k1} \in \overline{\varphi^*}(y_p)$. Moreover, since $\gamma_{k1} \notin \overline{\varphi}(T_{y_i} - T_{n,q})$ and η_k is only possibly used by connecting edge in $T_{y_{i+1}} - T_n$ where they are colored the same in φ^* because φ^* is $T_{n,q}$ -stable, T satisfies MP and R2 (1), and $\varphi^*(e_{i+1}) \neq \gamma_{k1}$. Hence $\gamma_{k1} \notin \varphi^*(T_{y_{i+1}} - T_{n,q})$. Let $\theta \in \overline{\varphi^*}(y_{i+1})$. By the

minimality of $|V(T - T_{n,q})|$, $\theta \neq \gamma_{k1}$. By Claim 2.3.20, $P_{v(\gamma_{k1})}(\theta, \gamma_{k1}, \varphi^*) = P_{y_{i+1}}(\theta, \gamma_{k1}, \varphi^*)$, and $P_{y_p}(\theta, \gamma_{k1}, \varphi^*)$ is a different path. By Claim 2.3.21, $\varphi^{**} = \varphi^*/P_{y_p}(\theta, \gamma_{k1}, \varphi^*)$ is $T_{n,q}$ -stable and $T_{n,q}$ satisfies MP and R2 up to itself under φ^{**} . Moreover, since $\theta \in \varphi^*(T_{y_{i+1}} - T_{n,q})$ is only possible when $\theta = \eta'_n$ which is used by a f_n and $\gamma_{k1} \notin \varphi^*(T_{y_{i+1}} - T_{n,q})$, φ^{**} being $T_{n,q}$ -stable ensures $\varphi^{**}(f_n) = \eta'_n$. Because $\theta \in \varphi^*(T_{y_{i+1}} - T_{n,q} - f_n)$, $\eta_k \in \overline{\varphi}(T_{y_i})$, $\varphi^{**}(f_n) = \eta'_n$ and $\gamma_{k1} \notin \varphi^*(T_{y_{i+1}} - T_{n,q})$, T satisfies MP and R2 (1) under φ^{**} . However, $\theta \in \overline{\varphi}^{**}(y_p) \cap \overline{\varphi}^{**}(y_{i+1})$, giving a contradiction to the maximality of i . \square

Now we have $i = p-1$. Let $\varphi(e_p) = \theta$. Since $\alpha \in \overline{\varphi}(y_p) \cap \overline{\varphi}(y_{p-1})$, we can recolor e_p by α . Denote the new coloring by φ^* . Then $\theta \in \overline{\varphi}^*(y_{p-1})$, and φ^* is $T_{n,q}$ -stable. $T_{n,q}$ clearly satisfies MP and R2 up to itself under φ^* by Lemma 2.3.6 because $y_{p-1}, y_p \notin T_{n,q}$. Moreover, $T_{y_{p-1}}$ satisfies MP and R2 (1) because we have $\varphi(f) = \varphi^*(f)$ for every $f \in T_{y_{p-1}}$ and $\overline{\varphi}(v') = \overline{\varphi}^*(v')$ for every $v' \in T_{y_{p-1}} - y_{p-1}$. Note that $v(\theta) \prec_\ell y_{p-1}$, we have a counterexample which has one less vertex than T , giving a contradiction. \square

Case 2. $p(T) = p \geq 1$. In this case, y_{p-1} is not incident to e_p . Let $\theta = \varphi(e_p)$. Recall that $p \geq 1$. We define y_{p-2} to be the last vertex of $T_{n,q}$ when $q = 1$ and therefore $T_{y_{-1}} = T_{n,q}$.

We have similar case strategy here as in the PE section, hence there is no loophole in our proof.

Case 2.1. $\alpha \in \overline{\varphi}(y_p) \cap \overline{\varphi}(y_{p-1})$ and $\alpha = \eta_m \in D_{n,q}$. Since $\eta_m \in \overline{\varphi}(y_p)$, we have $\theta \neq \eta_m$. Note that $\theta \in D_{n,q}$ may occur.

Case 2.1.1. $\theta \notin \overline{\varphi}(y_{p-1})$.

We first consider the case $\theta \notin \Gamma^q$. By R2 (1), $\gamma_{m1}, \gamma_{m2} \notin \varphi(T_{y_{p-1}} - T_{n,q})$. Thus $\gamma_{m1}, \gamma_{m2} \notin \varphi(T - T_{n,q})$. If $\gamma_{m1} \in \overline{\varphi}(y_p)$, then γ_{m1} is missing twice in the ETT $T^* = (T_{n,q}, e_0, y_0, e_1, \dots, e_{p-2}, y_{p-2}, e_p, y_p)$ when $p \geq 2$. Because $\theta \notin \Gamma^q$, $\gamma_{m1}, \gamma_{m2} \notin \varphi(T - T_{n,q})$ and $T_{n,q}$ satisfies MP and R2 up to itself under φ , T^* satisfies MP and R2, and therefore T^* gives a counterexample with smaller p and $p(T^*) \leq p(T)$. If $p = 1$, we must have $q > 0$. Otherwise since $T_{n,0}$ is closed for colors in $\overline{\varphi}(T_{n,0})$, we have $\theta \in \overline{\varphi}(y_{p-1})$, giving a contradiction. We then consider $(T_{n,q}, e_1, y_1)$ as a smaller counter-example. Note

that $(T_{n,q}, e_1, y_1)$ still satisfies MP and R2 while dropping e_0 , as $\varphi(e_1) \in \cup_{\eta_h \in D_{n,q}} \Gamma_h^{q-1}$ and $\varphi(e_1) = \theta \notin \Gamma^q$. By Claim 2.3.20, $P_{v(\gamma_{m1})}(\eta_m, \gamma_{m1}, \varphi) = P_{y_{p-1}}(\eta_m, \gamma_{m1}, \varphi)$. Now we consider $T_{y_p, y_{p-1}} = (T_{n,q}, y_0, e_1, \dots, y_{p-2}, e_p, y_p, e_{p-1}, y_{p-1})$ obtained from T by switching the order of joining vertices y_p and y_{p-1} . We can see $T_{y_p, y_{p-1}}$ is also an ETT of (G, e, φ) since $\theta \notin \bar{\varphi}(y_{p-1})$ and $\theta \notin \Gamma^q$ and MP, R2 are satisfied. Applying Claim 2.3.20 again, we have $P_{v(\gamma_{m1})}(\eta_m, \gamma_{m1}, \varphi) = P_{y_{p-1}}(\eta_m, \gamma_{m1}, \varphi)$, giving a contradiction.

Now we assume $\theta \in \Gamma^q$. Without loss of generality we say $\theta = \gamma_{k1}$ with $\eta_k \in D_{n,q}$. By Claim 2.3.20, $P_{v(\gamma_{m1})}(\eta_m, \gamma_{m1}, \varphi) = P_{y_{p-1}}(\eta_m, \gamma_{m1}, \varphi)$. If $\eta_k \notin \bar{\varphi}(y_{p-1})$, $T_{y_p, y_{p-1}}$ also satisfies MP and R2, where we proceed with argument in the previous case and consider $T_{y_p, y_{p-1}}$. Thus we assume $\eta_k \in \bar{\varphi}(y_{p-1})$. Since γ_{k1} can not be both γ_{m1} and γ_{m2} , we assume $\eta_k \in \bar{\varphi}(y_{p-1})$ and $\gamma_{m2} \neq \gamma_{k1}$. By R2 (1), we have $\gamma_{m2} \notin \varphi(T - T_{n,q})$. By Claim 2.3.20, $P_{v(\gamma_{m2})}(\eta_m, \gamma_{m2}, \varphi) = P_{y_{p-1}}(\eta_m, \gamma_{m2}, \varphi)$ and that $P_{y_p}(\eta_m, \gamma_{m2}, \varphi)$ is different from the path above. Note that $\gamma_{m2} \in \bar{\varphi}(T_{n,q})$, by Claim 2.3.21, $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring $\varphi^* = \varphi/P_{y_p}(\eta_m, \gamma_{m2}, \varphi)$. Thus $\varphi^*(f_n) = \eta'_n$. Moreover, T in φ^* satisfies MP and R2 (1) since $\gamma_{m2} \notin \varphi(T - T_{n,q})$, $\eta_m \notin \varphi(T_{y_{p-1}} - T_{n,q} - f_n)$ and $\varphi^*(f_n) = \eta'_n$. Moreover, $\gamma_{m2} \notin \varphi^*(T - T_{n,q})$ because $\gamma_{m2} \notin \varphi(T - T_{n,q})$ and no η_m edge in T is recolored in φ^* . By Claim 2.3.19, $|\bar{\varphi}^*(T_{y_{p-2}}) - \varphi^*(T_{y_{p-2}} - T_{n,q})| \geq 11 + 2n$. Hence there exists $\beta \in \bar{\varphi}^*(T_{y_{p-2}}) - \Gamma^q$ such that $\beta \notin \varphi^*(T - T_{n,q})$. By Claim 2.3.20, $P_{v(\gamma_{m2})}(\beta, \gamma_{m2}, \varphi^*) = P_{v(\beta)}(\beta, \gamma_{m2}, \varphi^*)$, and $P_{y_p}(\beta, \gamma_{m2}, \varphi^*)$ is a different path than above. Let $\varphi^{**} = \varphi^*/P_{y_p}(\beta, \gamma_{m2}, \varphi^*)$. Applying Claim 2.3.21, we see that the coloring φ^{**} is $T_{n,q}$ -stable, and $T_{n,q}$ satisfies MP and R2 under φ^{**} . Moreover, since $\gamma_{m2}, \beta \notin \varphi^*(T - T_{n,q})$, $\gamma_{m2}, \beta \notin \varphi^{**}(T - T_{n,q})$ and therefore T satisfies MP and R2 (1) under φ^{**} . Now $\beta \in \bar{\varphi}^{**}(y_p)$. By Claim 2.3.20, $P_{v(\gamma_{k1})}(\beta, \gamma_{k1}, \varphi^{**}) = P_{v(\beta)}(\beta, \gamma_{k1}, \varphi^{**})$, and $P_{y_p}(\beta, \gamma_{k1}, \varphi^{**})$ is a different path than above. Finally, we let $\varphi^{***} = \varphi^{**}/P_{y_p}(\beta, \gamma_{k1}, \varphi^{**})$. Since $\gamma_{k1} \notin \varphi^{**}(T_{y_{p-1}} - T_{n,q})$, by Claim 2.3.21, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{***} . Moreover, since $\beta \notin \varphi^{**}(T - T_{n,q})$ and $\gamma_{k1} \notin \varphi^{**}(T_{y_{p-1}} - T_{n,q})$, T is still an ETT satisfying MP and R2 (1) under φ^{***} . However, under φ^{***} we have $\varphi^{***}(e_p) = \beta$, $\gamma_{k1} \in \bar{\varphi}^{***}(y_p)$ and $v(\beta) \prec_\ell y_{p-1}$. Note that the case $q = 0$ and $p = 1$ will not happen here because we had the assumption $\theta \in \Gamma^q$. Hence

$(T_{n,q}, y_0, e_1, \dots, e_{p-2}, y_{p-2}, e_p, y_p)$ is a counterexample smaller than T , giving a contradiction.

Case 2.1.2. $\theta \in \overline{\varphi}(y_{p-1})$. In this case $\theta \neq \eta_m$ and $\theta \notin \overline{\varphi}(T_{n,q})$.

Note that η_m is only possibly used by a connecting edge in $T - T_n$. Moreover, $\gamma_{m1} \notin \varphi(T - T_{n,q})$ by R2 (1) and the fact $\theta \in \overline{\varphi}(y_{p-1})$. By Claim 2.3.20, $P_{v(\gamma_{m1})}(\eta_m, \gamma_{m1}, \varphi) = P_{y_{p-1}}(\eta_m, \gamma_{m1}, \varphi)$ and $P_{y_p}(\eta_m, \gamma_{m1}, \varphi)$ is a different path. Let $\varphi^* = \varphi / P_{y_p}(\eta_m, \gamma_{m1}, \varphi)$. By Claim 2.3.21, $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Moreover, $\gamma_{m1} \notin \varphi(T - T_{n,q})$ and η_m is only used possibly by a connecting edge in $T - T_n$ ensures T satisfies MP and R2 (1) under φ^* . In addition, $\gamma_{m1} \notin \varphi^*(T - T_{n,q})$. Since $\gamma_{m1} \in \overline{\varphi^*}(T_{n,q})$, by applying Claim 2.3.20 again, we have $P_{v(\gamma_{m1})}(\theta, \gamma_{m1}, \varphi^*) = P_{y_{p-1}}(\theta, \gamma_{m1}, \varphi^*)$ and $P_{y_p}(\theta, \gamma_{m1}, \varphi^*)$ is a different path. Let $\varphi^{**} = \varphi^* / P_{y_p}(\theta, \gamma_{m1}, \varphi^*)$. By applying Claim 2.3.21, we see that $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{**} . Moreover, T satisfies MP and R2 (1) under φ^{**} since $\gamma_{m1} \notin \varphi^*(T - T_{n,q})$ and θ may only be used by f_n in $T - T_{n,q}$ if $\theta = \eta'_n$ and $q = 0$. Note that under φ^{**} , $\theta \in \overline{\varphi^{**}}(y_p) \cap \overline{\varphi^{**}}(y_{p-1})$ and $\varphi^{**}(e_p) = \gamma_{m1}$. If $\theta \in D_{n,q}$, then under φ^{**} we have Case 2.1.1. So we may assume $\theta \notin D_{n,q}$, which will be handled in Case 2.2.1 below.

Case 2.2. $\alpha \in \overline{\varphi}(y_p) \cap \overline{\varphi}(y_{p-1})$ and $\alpha \notin D_{n,q}$.

Case 2.2.1. $\theta \notin \overline{\varphi}(y_{p-1})$.

In this case, $T_{y_p, y_{p-1}} := (T_{n,q}, y_0, e_1, \dots, y_{p-2}, e_p, y_p, e_{p-1}, y_{p-1})$ is also an ETT in φ satisfies MP and R2 except for the case where $\theta \in \Gamma_m^q$ with $\eta_m \in D_{n,q}$ and $\eta_m \in \overline{\varphi}(y_{p-1})$. We first assume there does not exist m such that $\theta \in \Gamma_m^q$ with $\eta_m \in D_{n,q}$ and $\eta_m \in \overline{\varphi}(y_{p-1})$. Note that by R2, we either have $\theta \notin \Gamma^q$ or $\theta \in \Gamma_m^q$ with $\eta_m \in D_{n,q}$ and $\eta_m \in \overline{\varphi}(T_{y_{p-2}})$. By Claim 2.3.19, we have $|\overline{\varphi}(T_{y_{p-2}}) - \varphi(E(T_{y_{p-2}} - T_{n,q}))| \geq 2n + 11$. So there exists a color $\beta \in \overline{\varphi}(T_{y_{p-2}}) - D_{n,q}$ such that $\beta \notin \varphi(T - T_{n,q})$. We claim that $\beta \notin \overline{\varphi}(y_p)$. Otherwise, because we either have $\theta \notin \Gamma^q$ or $\theta \in \Gamma_m^q$ with $\eta_m \in D_{n,q}$ and $\eta_m \in \overline{\varphi}(T_{y_{p-2}})$, $(T_{n,q}, y_0, e_1, \dots, e_{p-2}, y_{p-2}, e_p, y_p)$ is a counterexample smaller than T , giving a contradiction. Since $\alpha, \beta \notin D_{n,q}$ and $\beta \notin \varphi(T - T_{n,q})$, by Claim 2.3.20 $P_{v(\beta)}(\alpha, \beta, \varphi) = P_{y_{p-1}}(\alpha, \beta, \varphi)$. Applying Claim 2.3.20 to $T_{y_p, y_{p-1}}$, we see that $P_{v(\beta)}(\alpha, \beta, \varphi) = P_{y_p}(\alpha, \beta, \varphi)$. So, $P_{v(\beta)}(\alpha, \beta, \varphi)$ has three endvertices

$v(\beta)$, y_{p-1} and y_p , a contradiction. Hence, we may assume $\theta = \gamma_{m1}$ with $\eta_m \in D_{n,q}$ and $\eta_m \in \overline{\varphi}(y_{p-1})$, which in turn gives $\gamma_{m2}, \alpha \notin \varphi(T - T_{n,q})$ by R2 (1) and η_m is only used possibly by a connecting edge in $T - T_n$.

By Claim 2.3.20, $P_{v(\gamma_{m2})}(\alpha, \gamma_{m2}, \varphi) = P_{y_{p-1}}(\alpha, \gamma_{m2}, \varphi)$ and $P_{y_p}(\alpha, \gamma_{m2}, \varphi)$ is a different path from above. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \gamma_{m2}, \varphi)$. Since $\gamma_{m2} \in \overline{\varphi}(T_{n,q})$, by applying Claim 2.3.21, we see that $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Moreover T satisfies MP and R2 (1) under φ^* because $\gamma_{m2}, \alpha \notin \varphi(T - T_{n,q})$. Moreover, $\gamma_{m2}, \alpha \notin \varphi^*(T - T_{n,q})$. If $\eta_m \in \overline{\varphi^*}(y_p)$, then with $\eta_m \in \overline{\varphi^*}(y_p) \cap \overline{\varphi^*}(y_{p-1})$ and $\varphi^*(e_p) = \gamma_{m1} \notin \overline{\varphi^*}(y_{p-1})$, where we have Case 2.1.1. Hence $\eta_m \notin \overline{\varphi^*}(y_p)$. Now by Claim 2.3.20, $P_{v(\gamma_{m2})}(\eta_m, \gamma_{m2}, \varphi^*) = P_{y_{p-1}}(\eta_m, \gamma_{m2}, \varphi^*)$, and $P_{y_p}(\eta_m, \gamma_{m2}, \varphi^*)$ is different from the path above. Let $\varphi^{**} = \varphi^*/P_{y_p}(\eta_m, \gamma_{m2}, \varphi^*)$. Since $\gamma_{m2} \in \overline{\varphi^*}(T_{n,q})$, by applying Claim 2.3.21, we see that $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{**} . Additionally, T satisfies MP and R2 (1) because $\gamma_{m2} \notin \varphi^*(T - T_{n,q})$ and η_m is only possibly used by f_n in $T - T_n$. Under φ^{**} , we have $\eta_m \in \overline{\varphi^{**}}(y_p) \cap \overline{\varphi^{**}}(y_{p-1})$, $\gamma_{m1} = \varphi^{**}(e_p) \notin \overline{\varphi^{**}}(y_{p-1})$, which also leads us back to Case 2.1.1.

Case 2.2.2. $\theta \in \overline{\varphi}(y_{p-1})$. In this case, $\alpha \notin \varphi(T - T_{n,q})$ because $\alpha \notin D_{n,q}$.

We first assume $\theta = \eta_m$ with $\eta_m \in D_{n,q}$. By R2 (1), $\gamma_{m1} \notin \varphi(T - T_{n,q})$. By Claim 2.3.20, $P_{v(\gamma_{m1})}(\alpha, \gamma_{m1}, \varphi) = P_{y_{p-1}}(\alpha, \gamma_{m1}, \varphi)$, and $P_{y_p}(\alpha, \gamma_{m1}, \varphi)$ is a different path. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \gamma_{m1}, \varphi)$. Since $\gamma_{m1} \in \overline{\varphi}(T_{n,q})$ and $\gamma_{m1}, \alpha \notin \varphi(T - T_{n,q})$, we see that $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* by Claim 2.3.21. Moreover T satisfies MP and R2 (1) since $\alpha, \gamma_{m1} \notin \varphi(T - T_{n,q})$. Note that $\gamma_{m1} \notin \varphi^*(T - T_{n,q})$. By Claim 2.3.20 again, $P_{v(\gamma_{m1})}(\eta_{m1}, \gamma_{m1}, \varphi^*) = P_{y_{p-1}}(\eta_m, \gamma_{m1}, \varphi^*)$, and $P_{y_p}(\eta_{m1}, \gamma_{m1}, \varphi^*)$ is a different path. Let $\varphi^{**} = \varphi^*/P_{y_p}(\eta_{m1}, \gamma_{m1}, \varphi^*)$. Since $\gamma_{m1} \in \overline{\varphi}(T_{n,q})$, by applying Claim 2.3.21, $T_{n,q}$ an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{**} . Because $\eta_m \neq \alpha$, $\eta_m \notin \varphi(T_{y_{p-1}} - T_{n,q} - f_n)$ implies $\eta_m \notin \varphi^*(T_{y_{p-1}} - T_{n,q} - f_n)$. The fact that T satisfies MP and R2 (1) under φ^{**} follows from the fact that $\gamma_{m1} \notin \varphi^*(T - T_{n,q})$, $\eta_m \notin \varphi^*(T_{y_{p-1}} - T_{n,q} - f_n)$ and $\eta_m \in \overline{\varphi^{**}}(y_{p-1})$. Note that in φ^{**} , we have $\eta_m \in \overline{\varphi^{**}}(y_p) \cap \overline{\varphi^{**}}(y_{p-1})$, $\gamma_{m1} = \varphi^{**}(e_p) \notin \overline{\varphi^{**}}(y_{p-1})$, which is Case 2.1.1. So, we assume $\theta \notin D_{n,q}$.

By Claim 2.3.19, we either have $|\overline{\varphi}(T_{y_{p-2}}) - \Gamma^q \cup D_{n,q} \cup \varphi(T_{y_{p-2}} - T_{n,q})| \leq 4$, or there exist 7 distinct η_i with $\eta_i \in D_{n,q}$ and $\eta_i \in \overline{\varphi}(T_{y_{p-2}})$ such that all colors $\eta_i, \gamma_{i1}, \gamma_{i2} \notin \varphi(T_{y_{p-2}} - T_{n,q})$. Thus there exists a color $\beta \in \overline{\varphi}(T_{y_{p-2}}) - D_{n,q}$ with $\beta \notin \varphi(T - T_{n,q})$ such that either $\beta \notin \Gamma^q$ or $\beta = \gamma_{r1} \in \Gamma^q$ and $\eta_r \in \overline{\varphi}(T_{y_{p-2}})$. By Claim 2.3.20, $P_{v(\beta)}(\alpha, \beta, \varphi) = P_{y_{p-1}}(\alpha, \beta, \varphi)$ and $P_{y_p}(\alpha, \beta, \varphi)$ is a different path from above. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \beta, \varphi)$. Applying Claim 2.3.21, we see that $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Moreover T satisfies MP and R2 (2) under φ^* since $\alpha, \beta \notin \varphi(T - T_{n,q})$. Under φ^* , we have $\beta \in \overline{\varphi}^*(y_p) \cap \overline{\varphi}^*(v(\beta))$ and $v(\beta) \neq y_{p-1}$. Note that $\theta \notin \varphi^*(T_{y_{p-1}} - T_{n,q})$ and $\beta \notin \varphi^*(T - T_{n,q})$. Since $\beta, \theta \notin D_{n,q}$, by Claim 2.3.20, $P_{v(\beta)}(\theta, \beta, \varphi^*) = P_{y_{p-1}}(\theta, \beta, \varphi^*)$ and $P_{y_p}(\theta, \beta, \varphi^*)$ is different path other than above. Let $\varphi^{**} = \varphi/P_{y_{p-1}}(\theta, \beta, \varphi^*)$. Applying Claim 2.3.21 again, we see that $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{**} . Since $\theta \notin \varphi^*(T_{y_{p-1}} - T_{n,q})$ and $\beta \notin \varphi^*(T - T_{n,q})$, we have R2 (1) satisfied if $\beta \notin \Gamma^q$. For the case when $\beta = \gamma_{r1} \in \Gamma^q$, we have $\eta_r \in \overline{\varphi}(T_{y_{p-2}})$ which in turn gives R2 (1). Under φ^{**} , $\theta \in \overline{\varphi}^{**}(y_p) \cap \overline{\varphi}^{**}(y_{p-1})$, $\varphi^{**}(e_p) = \beta \notin \varphi^{**}(y_{p-1})$, which goes back to Case 2.2.1.

Case 2.3. $\alpha \in \overline{\varphi}(y_p) \cap \overline{\varphi}(v)$ for a vertex $v \prec_\ell y_{p-1}$.

Claim 2.3.25. We may assume $\alpha \notin \varphi(T - T_{n,q})$ such that either $\alpha \notin D_{n,q} \cup \Gamma^q$, or $\alpha = \eta_k \in \overline{\varphi}(T)$ with $\eta_k \in D_{n,q}$ and $\gamma_{k1}, \gamma_{k2} \notin \varphi(T - T_{n,q})$.

Proof. By Claim 2.3.19, we have $|\overline{\varphi}(T_{y_{p-2}} - D_{n,q} \cup \Gamma^q \cup \varphi(T_{y_{p-2}} - T_{n,q}))| \geq 4$ or there exists index k such that η_k, γ_{k1} and $\gamma_{k2} \in \overline{\varphi}(T_{y_{p-2}}) - \varphi(T - T_{n,q})$. The first inequality implies that there exists a color $\beta \in \overline{\varphi}(T_{y_{p-2}}) - D_{n,q} \cup \Gamma^q \cup \varphi(T - T_{n,q})$. If the second case happens, we take $\beta = \eta_k$. If $\beta \in \overline{\varphi}(y_p)$, we are done. Hence we assume $\beta \notin \overline{\varphi}(y_p)$. Let $P := P_{y_p}(\alpha, \beta, \varphi)$. We will show one of the following two statement holds.

- a: Under the $T_{n,q}$ -stable coloring $\varphi^* = \varphi/P$, T is an ETT satisfying MP and R2 with $\beta \notin \varphi(T - T_{n,q})$.
- b: In φ , there exist a non-elementary ETT T' satisfying MP and R2 with the same ladder, splitters and Γ sets as T where $T_{n,q} \subseteq T'$, but $p(T') < p(T)$.

Note that Statement a gives Claim 2.3.25 while Statement b gives a contradiction. Note that $\beta \notin \Gamma^q$ by the choice of β in Claim 2.3.19. We proceed with the proof by considering three cases: $\alpha \notin \Gamma^q$, $\alpha \in \Gamma^q - \varphi(T - T_{n,q})$ and $\alpha \in \Gamma^q \cap \varphi(T - T_{n,q})$.

If $V(P) \cap V(T_{y_{p-1}}) = \emptyset$, by Lemma 2.3.6, $T_{y_{p-1}}$ is an ETT satisfying MP and R2 with R2 (2) satisfied for $T_{n,q}$ under $\varphi^* = \varphi/P$. Moreover since $\alpha, \beta \neq \varphi(e_p)$ and $\beta \notin \varphi(T - T_{n,q})$, T satisfies R2 (1) and $\beta \notin \varphi^*(T - T_{n,q})$, so statement a holds. Hence we assume $V(P) \cap V(T_{y_{p-1}}) \neq \emptyset$. Along the order of P from y_p , let u be the first vertex in $V(T_{y_{p-1}})$ and P' be the subpath joining u and y_p . Let

$$\begin{aligned} T' &= T_{y_{p-2}} \cup P' && \text{if } u \neq y_{p-1}, \text{ and} \\ T' &= T_{y_{p-1}} \cup P' && \text{if } u = y_{p-1}. \end{aligned}$$

Note that $e_0 \notin T'$ may happen when $q > 0$, but it is easy to see that T' is still an ETT with the same ladder as T and q splitters where $T_{n,q} \subseteq T'$. Thus T' satisfies MP. Moreover $T_{n,q}$ still satisfies R2 up to itself under φ because we did not change the coloring.

Case I: $\alpha \notin \Gamma^q$. Since $\alpha, \beta \in \overline{\varphi}(T_{y_{p-2}})$ and $\alpha, \beta \notin \Gamma^q$, T' is an ETT satisfying R2 (1). Hence statement b holds and gives a contradiction to the minimality of $p(T)$.

Case II: $\alpha \in \Gamma^q \cap \varphi(T - T_{n,q})$. Assume $\alpha = \gamma_{m1} \in \Gamma^q$ with $\eta_m \in D_{n,q}$. Since $\varphi(e_p) \neq \alpha$, $\alpha \in \varphi(T_{y_{p-1}} - T_{n,q})$. Therefore we must have $\eta_m \in \overline{\varphi}(T_{y_{p-2}})$ by R2 (1). Furthermore, $\beta \in \overline{\varphi}(T_{y_{p-2}})$. Therefore T' satisfies R2 (1). Hence statement b holds.

Case III: $\alpha \in \Gamma^q - \varphi(T - T_{n,q})$. Then $\alpha, \beta \notin \varphi(T - T_{n,q})$. Let $\varphi^* = \varphi/P$. By Claim 2.3.20, P is a path different from $P_{v(\alpha)}(\alpha, \beta, \varphi) = P_{v(\beta)}(\alpha, \beta, \varphi)$. Hence $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* by Claim 2.3.21. Note that in this case $\varphi(f) = \varphi^*(f)$ for every edge f in $E(T - T_{n,q})$ and $\overline{\varphi}(v') = \overline{\varphi}^*(v')$ for each $v' \in T_{y_{p-1}}$. Therefore T is an ETT satisfying MP and R2 (1) under φ^* . Moreover, $\beta \notin \varphi^*(T - T_{n,q})$. Hence statement a holds. \square

Now let φ and α be as the claim above. We then consider two cases.

Case 2.3.1. $\theta = \varphi(e_p) \notin \overline{\varphi}(y_{p-1})$.

Recall that $T^* = (T_{n,q}, e_0, y_0, e_1, y_1, \dots, e_{p-2}, y_{p-2}, e_p, y_p)$. In this case, T^* is an ETT satisfies MP. Note that T' also satisfies condition R2 with the exception $\theta = \gamma_{m1}$ and $\eta_m \in \overline{\varphi}(y_{p-1})$ with $\eta_m \in D_{n,q}$, which gives a contradiction to the minimality of $p(T)$. Hence we may assume $\theta = \gamma_{m1}$ and $\eta_m \in \overline{\varphi}(y_{p-1})$ with $\eta_m \in D_{n,q}$. By R2 (1), we have $\gamma_{m1} \notin \varphi(T_{y_{p-1}} - T_{n,q})$. By Claim 2.3.20, $P_v(\alpha, \gamma_{m1}, \varphi) = P_{v(\gamma_{m1})}(\alpha, \gamma_{m1}, \varphi)$ and $P_{y_p}(\alpha, \gamma_{m1}, \varphi)$ is different from the path above. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \gamma_{m1}, \varphi)$. By Claim 2.3.21, $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Since $\alpha \notin \varphi(T - T_{n,q})$, we have $\varphi^*(f) = \varphi(f)$ for every edge $f \in T_{y_{p-1}} - T_{n,q}$. We have $\varphi^*(f) = \alpha$. By our choice on α we have $\alpha \notin \Gamma^q$. Therefore T is an ETT satisfying MP and R2 (1) under φ^* . Moreover, $\gamma_{m1} \notin \varphi^*(T - T_{n,q})$. Note $\gamma_{m1} \in \overline{\varphi^*}(y_p) \cap \overline{\varphi^*}(v(\gamma_{m1}))$. Since $\eta_m \in \overline{\varphi}(y_{p-1})$, we have $\eta_m \notin \varphi(T - T_{n,q})$. Let $\varphi^{**} = \varphi^*/P_{y_p}(\eta_m, \gamma_{m1}, \varphi^*)$. Applying Claim 2.3.20 and Claim 2.3.21, we can show as before that $T_{n,q}$ is an ETT and satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{**} . Since $\eta_m, \gamma_{m1} \notin \varphi(T - T_{n,q})$, T satisfies MP and R2 (1) under φ^{**} . Note that $\eta_m \in \overline{\varphi^{**}}(y_p) \cap \overline{\varphi^{**}}(y_{p-1})$. So, under φ^{**} we go back to Case 2.1.

Case 2.3.2. $\theta = \varphi(e_p) \in \overline{\varphi}(y_{p-1})$.

We first assume $\theta = \varphi(e_p) = \eta_m$ with $\eta_m \in D_{n,q}$. By R2 (1), $\gamma_{m1} \notin \varphi(T - T_{n,q})$. By Claim 2.3.20, $P_{v(\gamma_{m1})}(\alpha, \gamma_{m1}, \varphi) = P_v(\alpha, \gamma_{m1}, \varphi)$ and therefore $P_{y_p}(\alpha, \gamma_{m1}, \varphi)$ is a different path. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \gamma_{m1}, \varphi)$. By Claim 2.3.21, $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Since $\alpha, \gamma_{m1} \notin \varphi(T - T_{n,q})$, under coloring φ^* , $T_{n,q}$ can be extended to T as an ETT with MP and R2 (1) being are satisfied. Now $\gamma_{m1} \in \overline{\varphi^*}(y_p)$ and $\eta_m, \gamma_{m1} \notin \varphi^*(T_{y_{p-1}} - T_{n,q})$. Similarly, by applying Claim 2.3.20 and Claim 2.3.21, we can show that under the $T_{n,q}$ -stable coloring $\varphi^{**} = \varphi^*/P_{y_p}(\eta_m, \gamma_{m1}, \varphi^*)$, $T_{n,q}$ is also an ETT and satisfying MP and R2 up to itself. Moreover T satisfies MP and R2 (1) since $\eta_m, \gamma_{m1} \notin \varphi^*(T_{y_{p-1}} - T_{n,q})$ and $\eta_m \in \overline{\varphi^{**}}(y_{p-1})$. Now $\eta_m \in \overline{\varphi^{**}}(y_p) \cap \overline{\varphi^{**}}(y_{p-1})$ and $\varphi^{**}(e_p) = \gamma_{m1} \notin \overline{\varphi^{**}}(y_{p-1})$, which is dealt in Case 2.1.1.

We now consider the case $\theta = \varphi(e_p) \notin D_{n,q}$. Since $\theta \in \overline{\varphi}(y_{p-1})$ and $T_{y_{p-1}}$ is elementary,

we have $\theta \notin \Gamma^q$, so $\theta \notin D_{n,q} \cup \Gamma^q$. Suppose $\alpha \neq D_{n,q}$. Then, $\alpha \neq D_{n,q} \cup \Gamma^q$ by Claim 2.3.25. By Claim 2.3.20, $P_{v(\alpha)}(\alpha, \theta, \varphi) = P_{y_{p-1}}(\alpha, \theta, \varphi)$ and $P_{y_p}(\alpha, \theta, \varphi)$ is a different path than the one above. Let $\varphi^* = \varphi / P_{y_p}(\alpha, \theta, \varphi)$. By Claim 2.3.21, T is an ETT satisfying MP and R2 under the $T_{n,q}$ -stable coloring φ^* . Now $\theta \in \overline{\varphi^*}(y_p) \cap \overline{\varphi^*}(y_{p-1})$, which is dealt in Case 2.1. Hence we may assume $\alpha = \eta_m$ with $\eta_m \in D_{n,q}$. By Claim 2.3.25, we have $\gamma_{m1}, \gamma_{m2}, \eta_m \notin \varphi(T - T_{n,q})$. By Claim 2.3.20, $P_{v(\gamma_{m1})}(\alpha, \gamma_{m1}, \varphi) = P_v(\alpha, \gamma_{m1}, \varphi)$ and $P_{y_p}(\alpha, \gamma_{m1}, \varphi)$ is different from the path above. Let $\varphi^* = \varphi / P_{y_p}(\alpha, \gamma_{m1}, \varphi)$. By Claim 2.3.21, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Since $\gamma_{m1}, \eta_m \notin \varphi(T - T_{n,q})$, it is easy to check that $T_{n,q}$ can be extended to T as an ETT under φ^* with MP and R2 (1) being satisfied. Note $\gamma_{m1} \notin \varphi^*(T - T_{n,q})$. Hence by Claim 2.3.20, $P_{v(\gamma_{m1})}(\theta, \gamma_{m1}, \varphi^*) = P_{y_{p-1}}(\theta, \gamma_{m1}, \varphi^*)$ and $P_{y_p}(\theta, \gamma_{m1}, \varphi^*)$ is a different path. Let $\varphi^{**} = \varphi^* / P_{y_p}(\theta, \gamma_{m1}, \varphi^*)$. Again by Claim 2.3.21, $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{**} . Note that from coloring φ^* to coloring φ^{**} , in $T - T_{n,q}$, e_p is the only edge changed color and its colors has been changed from θ to γ_{m1} . Since $\eta_m = \alpha \in \varphi^{**}(v)$, T also an ETT satisfying MP and R2 (1) under φ^{**} , Now $\theta \in \overline{\varphi^{**}}(y_p) \cap \overline{\varphi^{**}}(y_{p-1})$, which is dealt in Case 2.2. This completes Case 2. \square

In the remainder of the proof, let $I_\varphi = \{i \geq 0 : \overline{\varphi}(y_p) \cap \overline{\varphi}(y_i) \neq \emptyset\}$ and let $j = p(T)$. Clearly $I_\varphi = \emptyset$ when $\{v : \overline{\varphi}(y_p) \cap \overline{\varphi}(v) \neq \emptyset\} \subset V(T_{n,q})$. For convention, we denote $\max I_\varphi = -1$ when $I_\varphi = \emptyset$. By the assumption of $p(T)$, we have $j \geq 1$ and y_{j-1} is not incident to e_j . We let y_{j-2} to be the last vertex in $T_{n,q}$ if $j = 1$, and in this case $T_{y_{j-2}} = T_{n,q}$.

Case 3. $p(T) \leq p - 1$ and $\max(I_\varphi) \geq p(T)$.

This case is similar to Case 1 and can be handled in the same fashion: We first show that $\max(I_\varphi) = p - 1$ and replace color $\varphi(e_p)$ by α to get a smaller counterexample. We omit the details.

Case 4. $p(T) \leq p - 1$ and $\max(I_\varphi) < p(T)$.

Let $j = p(T)$. Then $j \geq 1$ and $e_j \notin E_G(y_{j-1}, y_j)$. Let $\min(I_\varphi) = i$ if $I_\varphi \neq \emptyset$.

Claim 2.3.26. *We may assume there exist $\alpha \in \overline{\varphi}(y_p) \cap \overline{\varphi}(T_{y_{j-2}})$ such that either $\alpha \notin \Gamma^q$, or $\alpha = \gamma_{m1} \in \Gamma^q$ with $\eta_m \in D_{n,q}$ and $v(\eta_m) \preceq_\ell y_{j-2}$.*

Proof. We first consider the case when $I_\varphi \neq \emptyset$. Since we assume $\max(I_\varphi) < j$, $i \leq j - 1$. If $i < j - 1$, then $j - 2 \geq 0$ and we have $\alpha \in \overline{\varphi}(y_p) \cap \overline{\varphi}(T_{y_{j-2}})$ with $\alpha \notin \Gamma^q$ because we have a color in $\overline{\varphi}(y_p) \cap \overline{\varphi}(T_{y_{j-2}} - T_{n,q})$. Assume $i = j - 1$. Then we have a color $\alpha \in \overline{\varphi}(y_i) \cap \overline{\varphi}(y_p)$, and therefore $\alpha \notin \overline{\varphi}(T_{n,q})$. Thus $\alpha \notin \Gamma^q$. By Claim 2.3.19, there exists a color $\beta \in \overline{\varphi}(T_{y_{j-2}})$ such that $\beta \notin \varphi(T_{y_{j-1}} - T_{n,q})$ and either $\beta \notin D_{n,q} \cup \Gamma^q$ or $\beta = \eta_k \in D_{n,q}$ and additionally $\gamma_{k1}, \gamma_{k2}, \eta_k \notin \varphi(T_{y_{j-1}} - T_{n,q})$.

We first consider the case $\alpha \in D_{n,q}$, say $\alpha = \eta_m$ with $\eta_m \in D_{n,q}$. By R2 (1), $\gamma_{m1} \notin \varphi(T_{y_i} - T_{n,q})$. By Claim 2.3.20, $P_{v(\gamma_{m1})}(\eta_m, \gamma_{m1}, \varphi) = P_{y_i}(\eta_m, \gamma_{m1}, \varphi)$ and $P_{y_p}(\eta_m, \gamma_{m1}, \varphi)$ is different from the path above. Let $\varphi^* = \varphi / P_{y_p}(\eta_m, \gamma_{m1}, \varphi)$. By Claim 2.3.21, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Since $\gamma_{m1} \notin \varphi(T_{y_i} - T_{n,q})$, $\eta_m \in \overline{\varphi^*}(y_i)$ and η_m is used only possibly by a connecting edge in $T_{y_i} - T_n$, $T_{n,q}$ can be extended to T with MP and R2 (1) being satisfied under φ^* . Note that $\gamma_{m1} \notin \varphi^*(T_{y_{j-1}} - T_{n,q})$. We have $\gamma_{m1} \notin \varphi^*(T_{v(\beta)} - T_{n,q})$. Since $\beta \notin \Gamma^q$, $\beta \neq \gamma_{m1}$. Moreover $\beta \neq \eta_m$ because $\eta_m \in \overline{\varphi}(y_{j-1})$ and $\beta \in \overline{\varphi}(T_{y_{j-2}})$. Thus $\beta \notin \varphi^*(T_{y_{j-1}} - T_{n,q})$ because $\beta \notin \varphi(T_{y_{j-1}} - T_{n,q})$. By Claim 2.3.20, $P_{v(\gamma_{m1})}(\beta, \gamma_{m1}, \varphi^*) = P_{v(\beta)}(\beta, \gamma_{m1}, \varphi^*)$ and $P_{y_p}(\beta, \gamma_{m1}, \varphi^*)$ is a different path. Let $\varphi^{**} = \varphi^* / P_{y_p}(\beta, \gamma_{m1}, \varphi^*)$. By Claim 2.3.21, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{**} . From $\beta, \gamma_{m1} \notin \varphi^*(T_{y_{j-1}} - T_{n,q})$ we get $\beta, \gamma_{m1} \notin \varphi^{**}(T_{y_{j-1}} - T_{n,q})$. Recall that we have $\eta_m \in \overline{\varphi}(y_i) = \overline{\varphi^*}(y_i) = \overline{\varphi^{**}}(y_i)$. Thus T is an ETT satisfying MP and R2 (1) under φ^{**} . Now $\beta \in \overline{\varphi^{**}}(y_p) \cap \varphi^{**}(v(\beta))$, so Claim 2.3.26 holds.

We now consider the case $\alpha \notin D_{n,q}$. Then $\alpha \notin D_{n,q} \cup \Gamma^q$. We first assume that $\beta \notin D_{n,q}$. Then $\beta \notin D_{n,q} \cup \Gamma^q$ by our choice of β . Since $\beta \notin \varphi(T_{y_{j-1}} - T_{n,q})$. By Claim 2.3.20, $P_{v(\beta)}(\alpha, \beta, \varphi) = P_{y_i}(\alpha, \beta, \varphi)$ and $P_{y_p}(\alpha, \beta, \varphi)$ is different from the path above. By Claim 2.3.21, T satisfies MP and R2 under the $T_{n,q}$ -stable coloring φ^* because we have $\alpha, \beta \notin D_{n,q} \cup \Gamma^q$. It is seen that, under φ^* , Claim 2.3.26 holds by β .

We now assume $\beta = \eta_m$ with $\eta_m \in D_{n,q}$. By our choice of β , we have $\eta_m, \gamma_{m1}, \gamma_{m2} \notin \varphi(T_{y_i} - T_{n,q})$. By Claim 2.3.20, $P_{v(\gamma_{m1})}(\alpha, \gamma_{m1}, \varphi) = P_{y_i}(\alpha, \gamma_{m1}, \varphi)$ and $P_{y_p}(\alpha, \gamma_{m1}, \varphi)$ a

different path. Let $\varphi^* = \varphi/P_{y_p}(\alpha, \gamma_{m1}, \varphi)$. Because $\gamma_{m1} \notin T_{y_i} - T_{n,q}$, by Claim 2.3.21, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* . Since $\alpha, \gamma_{m1} \notin \varphi(T_{y_i} - T_{n,q})$, we have $\alpha, \gamma_{m1} \notin \varphi^*(T_{y_i} - T_{n,q})$. Since $\eta_m \in \overline{\varphi}(T_{y_{j-2}})$, $T_{n,q}$ can be extended to T as an ETT satisfying MP and R2 (1). Since $\eta_m, \gamma_{m1} \notin \varphi(T_{y_i} - T_{n,q})$ and $\gamma_{m1}, \alpha \notin \varphi(T_{y_i} - T_{n,q})$, $\eta_m, \gamma_{m1} \notin \varphi^*(T_{y_i} - T_{n,q})$. Thus by Claim 2.3.20 and Claim 2.3.21, $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{**} . Moreover, T is an ETT and satisfies MP and R2 (1) because we have $\eta_m, \gamma_{m1} \notin \varphi^*(T_{y_i} - T_{n,q})$ and $\eta_m \in \overline{\varphi}(T_{y_{j-2}}) = \overline{\varphi}^*(T_{y_{j-2}}) = \overline{\varphi}^{**}(T_{y_{j-2}})$. Now we have as claimed under φ^{**} .

We then consider the case $I_\varphi = \emptyset$. If $\alpha \notin \Gamma^q$, we are done. Hence we assume $\alpha = \gamma_{m1} \in \Gamma^q$ with $\eta_m \in D_{n,q}$. We first assume that $\eta_m \notin \overline{\varphi}(T_{y_{p-1}})$. Then $\gamma_{m1} \notin \varphi(T - T_{n,q})$ by R2 (1). By Claim 2.3.19, there exists a color $\beta \in \overline{\varphi}(T_{y_{p-2}})$ with $\beta \notin \varphi(T - T_{n,q})$ such that either $\beta \notin D_{n,q} \cup \Gamma^q$ or $\beta = \eta_k \in D_{n,q}$ and additionally $\gamma_{k1}, \gamma_{k2}, \eta_k \notin \varphi(T - T_{n,q})$. By Claim 2.3.20, $P_{v(\gamma_{m1})}(\beta, \gamma_{m1}, \varphi) = P_{v(\beta)}(\beta, \gamma_{m1}, \varphi)$ and $P_{y_p}(\beta, \gamma_{m1}, \varphi)$ a different path. By Claim 2.3.21, $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring $\varphi^* := \varphi/P_{y_p}(\beta, \gamma_{m1}, \varphi)$. Since $\beta, \gamma_{m1} \notin \varphi(T - T_{n,q})$, we have $\beta, \gamma_{m1} \notin \varphi^*(T - T_{n,q})$ and $T_{n,q}$ can be extended to T under φ^* with MP and R2 (1) being satisfied. Note that under φ^* we have Claim 2.3.26 if $v(\beta) \preceq_\ell y_{j-2}$ or Case 3 if $y_j \preceq_\ell v(\beta)$, or the case $I_\varphi \neq \emptyset$ if $v(\beta) = y_{j-1}$, where we can proceed as before.

Now we assume that $\eta_m \in \overline{\varphi}(T_{y_{p-1}})$. Note that $\eta_m \notin \overline{\varphi}(T_{n,q})$ since $\eta_m \in D_{n,q}$. Without loss of generality, we can assume that $\eta_m \in \overline{\varphi}(y_k)$ for some $0 \leq k \leq p-1$. If $k < j-1$, we have Claim 2.3.26, hence we assume $k \geq j-1$. By Claim 2.3.20, $P_{v(\gamma_{m1})}(\eta_m, \gamma_{m1}, \varphi) = P_{y_k}(\eta_m, \gamma_{m1}, \varphi)$ and $P_{y_p}(\eta_m, \gamma_{m1}, \varphi)$ a different path than above. By Claim 2.3.21, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring $\varphi^* := \varphi/P_{y_p}(\eta_m, \gamma_{m1}, \varphi)$. Since $\gamma_{m1}, \eta_m \notin \varphi(T_{y_k} - T_{n,q})$ and $\eta_m \in \overline{\varphi}(y_k)$, T satisfies MP and R2 (1) in φ^* because the edges of T colored different in φ^* and φ is in $T_{y_p} - T_{y_k}$, and they are colored by γ_{m1} or η_m in both colorings φ and φ^* . If $k > j-1$, we have Case 3. If $k = j-1$, we have the case $I_\varphi \neq \emptyset$, where we can proceed as before. \square

We then consider the color φ satisfies Claim 2.3.26. By Claim 2.3.19, there exists a

color $\beta \in \overline{\varphi}(T_{y_{j-2}})$ with $\beta \notin \varphi(T_{y_j} - T_{n,q})$ such that either $\beta \notin D_{n,q} \cup \Gamma^q$ or $\beta = \eta_k \in \overline{\varphi}(T_{y_{j-2}})$ with $\eta_k, \gamma_{k1}, \gamma_{k2} \notin \varphi(E(T_{y_j} - T_{n,q}))$ with $\eta_k \in D_{n,q}$. Note that $\beta \notin \Gamma^q$ by our choice. Now we consider the path $P := P_{y_p}(\alpha, \beta, \varphi)$. First we consider the case $V(P) \cap V(T_{y_{j-1}}) \neq \emptyset$. Along the order of P from y_p , let u be the first vertex in $V(T_{y_{j-1}})$ and P' be the subpath joining u and y_p . Let

$$\begin{aligned} T' &= T_{y_{j-2}} \cup P' && \text{if } u \neq y_{j-1}, \text{ and} \\ T' &= T_{y_{j-1}} \cup P' && \text{if } u = y_{j-1}. \end{aligned}$$

Again note that $e_0 \notin T'$ may happen when $q > 0$, but it is easy to see that T' is still an ETT satisfying MP with the same ladder as T and where $T_{n,q} \subseteq T'$ under φ because $\beta, \alpha \in \overline{\varphi}(T_{y_{j-2}})$. Moreover $T_{n,q}$ satisfies R2 up to itself under φ because T satisfies R2 under φ .

Case I: $\alpha \notin \Gamma^q$. Since $\alpha, \beta \notin \Gamma^q$, T' is an ETT satisfying R2 (1), giving a contradiction to the minimality of $p(T)$.

Case II: $\alpha \in \Gamma^q$. Then by Claim 2.3.26, $\alpha = \gamma_{m1} \in \Gamma^q$ with $\eta_m \in D_{n,q}$ and $v(\eta_m) \prec_\ell y_{j-2}$. Then $\eta_m \in \overline{\varphi}(T_{y_{j-2}})$. Therefore T' is an ETT satisfying R2 (1), giving a contradiction to the minimality of $p(T)$.

□

Therefore we have $V(P) \cap V(T_{y_{j-1}}) = \emptyset$. Let $\varphi^* = \varphi/P$. Then φ^* is $T_{y_{j-1}}$ -stable and $T_{n,q}$ satisfies MP and R2 up to itself under φ^* by Lemma 2.3.6. Moreover, since $V(P) \cap V(T_{y_{j-1}}) = \emptyset$ and e_j is adjacent to $V(T_{y_{j-1}})$, we have $\varphi(f) = \varphi^*(f)$ for every $f \in T_{y_j}$ and $\overline{\varphi}(v') = \overline{\varphi^*}(v')$ for every $v' \in T_{y_{j-1}}$. Since $\beta \notin \varphi(T_{y_j} - T_{n,q})$, $\beta \notin \varphi^*(T_{y_j} - T_{n,q})$. Moreover, if $\beta = \eta_m$, then $\eta_m, \gamma_{m1}, \gamma_{m2} \notin \varphi^*(T_{y_j} - T_{n,q})$ because in this case $\eta_m, \gamma_{m1}, \gamma_{m2} \notin \varphi(T_{y_j} - T_{n,q})$. Recall that $\beta \notin \Gamma^q$ by our choice of β . If $\alpha \notin \Gamma^q$, T satisfies MP and R2 (1) under φ^* since $\beta, \alpha \in \overline{\varphi}(T_{y_{j-2}})$. If $\alpha \in \Gamma^q$, by Claim 2.3.26, $\alpha = \gamma_{m1} \in \Gamma^q$ with $\eta_m \in D_{n,q}$ and $v(\eta_m) \prec_\ell y_{j-2}$. Therefore T satisfies MP and R2 (1) since $\beta, \eta_m \in \overline{\varphi}(T_{y_{j-2}})$. Note that $\beta \notin \varphi^*(T_{y_j} - T_{n,q})$ and $\beta \in \overline{\varphi^*}(y_p) \cap \overline{\varphi^*}(v(\beta))$, where $v(\beta) \prec_l y_{j-2}$. Denote $v = v(\beta)$ for convenience. Let $\gamma \in \overline{\varphi}(y_j)$.

Then $\gamma \notin \Gamma^q$. We then denote $\varphi^* = \varphi$ and consider the following two cases.

Case 4.1. $\gamma \notin D_{n,q}$.

Case 4.1.1. $\beta \notin D_{n,q}$.

Then $\beta, \gamma \notin \Gamma^q \cup D_{n,q}$. By Claim 2.3.20, $P_{v(\beta)}(\beta, \gamma, \varphi) = P_{y_j}(\beta, \gamma, \varphi)$ and $P_{y_p}(\beta, \gamma, \varphi)$ a different path than above. let $\varphi^* = \varphi/P_{y_p}(\beta, \gamma, \varphi)$. Then by Claim 2.3.21 T is an ETT satisfying MP and R2 under the $T_{n,q}$ -stable coloring φ^* . Now $\gamma \in \overline{\varphi^*}(y_p) \cap \overline{\varphi^*}(y_j)$, where we reach Case 3.

Case 4.1.2. $\beta = \eta_m \in D_{n,q}$.

Recall that we proved $\gamma_{m1}, \gamma_{m2} \notin \varphi((Ty_j - T_{n,q}))$ before starting Case 4.1. By Claim 2.3.20, $P_{v(\beta)}(\beta, \gamma_{m1}, \varphi) = P_{v(\gamma_{m1})}(\beta, \gamma_{m1}, \varphi)$ and therefore $P_{y_p}(\beta, \gamma_{m1}, \varphi)$ a different path. Let $\varphi^* = \varphi/P_{y_p}(\gamma_{m1}, \beta, \varphi)$. Then $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* by Claim 2.3.21. Moreover, T satisfies MP and R2 (1) under φ^* since $\eta_m = \beta, \gamma_{m1} \in \overline{\varphi}(Ty_{j-2})$ and $\beta, \gamma_{m1} \notin \varphi(Ty_j - T_{n,q})$. Moreover, $\gamma_{m1} \notin \varphi^*((Ty_j - T_{n,q}))$. Since $\gamma \in \overline{\varphi}(y_j)$, $\gamma \neq \eta_m$ and $\gamma \neq \gamma_{m1}$. Since $\gamma \notin D_{n,q}$, $\gamma \notin \varphi(Ty_j - T_{n,q})$. Since $\gamma \neq \eta_m$ and $\gamma \neq \gamma_{m1}$, $\gamma \notin \varphi^*(Ty_j - T_{n,q})$. Since $\gamma_{m1}, \gamma \notin \varphi^*((Ty_j - T_{n,q}))$, by Claim 2.3.20 again, $P_{v(\beta)}(\gamma, \gamma_{m1}, \varphi^*) = P_{v(\gamma_{m1})}(\gamma, \gamma_{m1}, \varphi^*)$ and therefore $P_{y_p}(\gamma, \gamma_{m1}, \varphi^*)$ a different path. Let $\varphi^{**} = \varphi^*/P_{y_p}(\gamma_{m1}, \gamma, \varphi^*)$. By Claim 2.3.21, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{**} . Since $\eta_m \in \overline{\varphi^*}(Ty_{j-2})$, $\gamma \notin \Gamma^q$ and $\gamma, \gamma_{m1} \notin \varphi(Ty_j - T_{n,q})$, T satisfies MP and R2 (1) under φ^{**} . However, we have $\gamma \in \overline{\varphi^{**}}(y_p) \cap \overline{\varphi^{**}}(y_j)$, where we reach Case 3.

Case 4.2. $\gamma = \eta_m \in D_{n,q}$.

Then $\gamma_{m1}, \gamma_{m2} \notin \varphi(Ty_j - T_{n,q})$ by R2 (1). By Claim 2.3.20, $P_{v(\beta)}(\beta, \gamma_{m1}, \varphi) = P_{v(\gamma_{m1})}(\beta, \gamma_{m1}, \varphi)$ and $P_{y_p}(\beta, \gamma_{m1}, \varphi)$ is a different path than above. Let $\varphi^* = \varphi/P_{y_p}(\gamma_{m1}, \beta, \varphi)$. Then $T_{n,q}$ is an ETT satisfying MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^* by Claim 2.3.21. Moreover, T satisfies MP and R2 (1) under φ^* since $\eta_m = \gamma \in \overline{\varphi}(Ty_j)$, $\beta \notin \Gamma^q$ and $\beta, \gamma_{m1} \notin \varphi(Ty_j - T_{n,q})$. Moreover, we have $\gamma_{m1} \notin \varphi^*(Ty_j - T_{n,q})$. Since $\gamma_{m1}, \beta \in \overline{\varphi}(Ty_{j-2})$ and $\gamma \in \overline{\varphi}(y_j)$, $\gamma_{m1} \neq \gamma$ and $\beta \neq \gamma$. Thus $\gamma \notin \varphi^*(Ty_j - T_{n,q} - f_n)$. Since $\gamma_{m1} \notin \varphi^*(Ty_j - T_{n,q})$,

by Claim 2.3.20, $P_{v(\beta)}(\gamma, \gamma_{m1}, \varphi^*) = P_{v(\gamma_{m1})}(\gamma, \gamma_{m1}, \varphi^*)$ and therefore $P_{y_p}(\gamma, \gamma_{m1}, \varphi^*)$ is a different path. Let $\varphi^{**} = \varphi^*/P_{y_p}(\gamma_{m1}, \gamma, \varphi^*)$. By Claim 2.3.21, $T_{n,q}$ satisfies MP and R2 up to itself under the $T_{n,q}$ -stable coloring φ^{**} . Since $\eta_m \in \overline{\varphi}^*(T_{y_j})$, $\gamma, \gamma_{m1} \notin \varphi(T_{y_j} - T_{n,q} - f_n)$ and φ^{**} is $T_{n,q}$ -stable (thus it is (T_n, D_n, φ) -stable), T satisfies MP and R2 (1) under φ^{**} . Now we have $\eta_m \in \overline{\varphi}^{**}(y_p) \cap \overline{\varphi}^{**}(y_j)$, where we reach Case 3.

This completes the proof of Case 4. Now for all cases we arrive at a contradiction, which proved statement A. \square

Finally we prove the last Proposition which is an inductive proof of (A1) (2).

Proposition 7. *Suppose (A1) (1), (A2), (A3), (A4) and (A5) hold for ETTs satisfying MP with at most n -rungs and (A1) (2) holds for ETTs satisfying MP with at most $n - 1$ -rungs. Let T be a closed ETT satisfying MP with n -rungs with last coloring φ_n . Then every $\alpha \in \overline{\varphi}(T)$ is interchangeable with any color β in T under every (T, D_n, φ_n) -stable coloring φ .*

Proof. We assume $\alpha \in \overline{\varphi}(T)$ is not interchangeable with β in T . In this case $|\partial_\beta(T)| = \text{odd}$. Hence T has odd number of (α, β) exit paths because T is elementary by (A1) (1) and closed for α . Let u, v, w be exits from three (α, β) exit paths for T with $u \prec_l v \prec_l w$. Without loss of generality, we may assume $T_n \vee T_n^d \subset T$ if $\Theta_n = \text{PE}$. We claim that $w \in T - T_n$ if $\Theta_n = \text{SE}$ or RE , and $w \in T - T_n \vee T_n^d$ if $\Theta_n = \text{PE}$. Suppose on the contrary our claim does not hold. First we consider the case $\Theta_n = \text{SE}$ or RE . Let $\alpha^* \in \overline{\varphi}(T_n)$. Since T is closed under φ_n , T is closed for both α and α^* under φ . Hence $\varphi^* = \varphi/(G - T, \alpha, \alpha^*)$ is (T, D_n, φ_n) -stable and therefore it is (T_n, D_n, φ_n) -stable. Because $\Theta_n = \text{SE}$ or RE , φ_n is $(T_n, D_{n-1}, \varphi_{n-1})$ -stable. Thus φ^* is $(T_n, D_{n-1}, \varphi_{n-1})$ -stable. Note that we have three (α^*, β) exit paths for T_n under φ^* . Moreover, T_n is elementary under φ^* because φ^* is $(T_n, D_{n-1}, \varphi_{n-1})$ -stable and T_n is elementary under φ_{n-1} by (A1) (1). Therefore, there are at least two (α^*, β) paths intersecting T_n , a contradiction to (A1) (2) for T_n which is an ETT with $n - 1$ rungs under φ_{n-1} . Now we assume $\Theta_n = \text{PE}$. Similarly, we have a color $\alpha^* \in \overline{\varphi}(T_n \cap T_n^d)$. Note that T is closed for both α and α^* under φ . Therefore $\varphi^* = \varphi/(G - T, \alpha, \alpha^*)$ is $(T_n \vee T_n^d, D_n, \varphi_n)$ -stable and therefore is $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable. Moreover $T_n \vee T_n^d$ is elementary under φ^* . Note

that we have three (α^*, β) exit paths for $T_n \vee T_n^d$ under φ^* , $T_n \vee T_n^d$ being elementary under φ^* implies that there are at least two (α^*, β) paths intersecting $T_n \vee T_n^d$. Thus we have a contradiction with Lemma 2.3.2 (3), because $\alpha^* \in \overline{\varphi}(T_n \cap T_n^d)$ and φ^* is $(T_n \cup T_n^d, D_n, \varphi_n)$ -stable. Now we have as claimed.

Let $\gamma \in \overline{\varphi}(w)$. Note that $\varphi^* = \varphi/(\alpha, \gamma, G - T)$ is (T, D_n, φ_n) -stable and therefore it is (T_n, D_n, φ_n) -stable. By (A5), T is still an ETT satisfying MP under φ^* . Moreover, under φ^* , we have $P_w^{ex}(\gamma, \beta, \varphi^*) = P_w(\gamma, \beta, \varphi^*) = P_w^{ex}(\alpha, \beta, \varphi)$, $P_u^{ex}(\gamma, \beta, \varphi^*) = P_u^{ex}(\alpha, \beta, \varphi)$ and $P_v^{ex}(\gamma, \beta, \varphi^*) = P_v^{ex}(\alpha, \beta, \varphi)$ are three (γ, β) exit paths for T . Let the three other end vertices of $P_w^{ex}(\gamma, \beta, \varphi^*)$, $P_u^{ex}(\gamma, \beta, \varphi^*)$ and $P_v^{ex}(\gamma, \beta, \varphi^*)$ not in $T_{n,j}$ be w_2 , u_2 and v_2 respectively. Let u' be the vertex in $P_u^{ex}(\gamma, \beta, \varphi^*)$ next to u , and the edge connecting u and u' be f_u ; and v' be the vertex in $P_v^{ex}(\gamma, \beta, \varphi^*)$ next to v , and the edge connecting v and v' be f_v . Note that f_v and f_u are colored β in φ^* . Let $\varphi^2 = \varphi^*/P_w^{ex}(\gamma, \beta, \varphi^*)$. Since $w \in T - T_n$ if $\Theta_n = \text{SE}$ or RE , $w \in T - T_n \vee T_n^d$ if $\Theta_n = \text{PE}$, and $P_w(\gamma, \beta, \varphi^*) \cap T = w$, we have that φ^2 is (T_n, D_n, φ) -stable if $\Theta_n = \text{SE}$ or RE and φ^2 is $(T_n \cup T_n^d, D_n, \varphi)$ -stable if $\Theta_n = \text{PE}$. Moreover, T_w is an ETT satisfying MP by (A5). Note that under φ^2 , $\beta \in \overline{\varphi^2}(w)$. Therefore $\{T_w, f_u, u', f_v, v'\}$ is an ETT satisfying MP. By (A5), we can keep condition MP by keeping extending $\{T_w, f_u, u', f_v, v'\}$ by TAA until it is closed. Let the resulting ETT be T^2 . Clearly T^2 satisfies MP. By (A1) (1), T^2 is elementary because it has n rungs. If one of w_2, u_2, v_2 is in T^2 , then γ must be missing at that vertex since $\beta \in \overline{\varphi^2}(T^2)$. Therefore we must have all three vertices w_2, u_2, v_2 are in T^2 . However, all of them miss either γ or β in φ^2 , which gives a contradiction to the elementary property by (A1) (1). Thus none of the vertices above are in T^2 . Hence each of $P_u^{ex}(\gamma, \beta, \varphi^*)$, $P_v^{ex}(\gamma, \beta, \varphi^*)$ and $P_w^{ex}(\gamma, \beta, \varphi^*)$ contains a (γ, β) exit path of T^2 . Let u_1, v_1, w_1 be the exits for the (γ, β) exit paths contained in the three paths above respectively. We without loss of generality assume $u_1 \prec_f v_1 \prec_f w_1$. Note that $w_1 \neq w$ since we already have $w \prec_f u' \prec_f v'$ in T^2 . Note that $P_{u_1}^{ex}(\gamma, \beta, \varphi^2)$ and $P_{v_1}^{ex}(\gamma, \beta, \varphi^2)$ are sub-paths of $P_u^{ex}(\alpha, \beta, \varphi)$ and $P_v^{ex}(\alpha, \beta, \varphi)$ and are shorter than those two. Moreover, since $w_1 \in T^2 - T_n$ if $\Theta_n = \text{SE}$ or RE and $w_1 \in T^2 - T_n \vee T_n^d$ with $T_n \cup T_n^d$ still being a half closure of T_n by Lemma 2.3.4 if $\Theta_n = \text{PE}$, we can continue the proof process again for T^2 inductively as

we did for T . Continue in this fashion, we will reach a contradiction because we will obtain shorter and shorter exit paths until finally all the ends are contained in.

□

Now all (A1) to (A5) are proved inductively, we finished the main Theorem.

PART 3

MINIMUM RANKS OF NON-NEGATIVE SIGN PATTERNS

In this chapter, we will establish a connection between sign patterns and point-line configurations and use it to prove some results on the minimum ranks of non-negative sign patterns.

An important part of combinatorial matrix theory is the study of sign pattern matrices. A *sign pattern (matrix)* (respectively, *nonnegative sign pattern (matrix)*) is a matrix whose entries are from the set $\{+, -, 0\}$ (respectively, $\{+, 0\}$). For a real matrix B , $\text{sgn}(B)$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by $+$ (respectively, $-$, 0). For a sign pattern matrix \mathcal{A} , the *qualitative class* of \mathcal{A} , denoted $Q(\mathcal{A})$, is defined as

$$Q(\mathcal{A}) = \{A \mid A \text{ is a real matrix with } \text{sgn}(A) = \mathcal{A}\}.$$

A *signature sign pattern* is a diagonal sign pattern matrix whose diagonal entries are from the set $\{+, -\}$. Two $m \times n$ sign pattern \mathcal{A}_1 and \mathcal{A}_2 are said to be *signature equivalent* or *diagonally equivalent* if there exist signature sign patterns \mathcal{D}_1 and \mathcal{D}_2 such that $\mathcal{A}_2 = \mathcal{D}_1 \mathcal{A}_1 \mathcal{D}_2$.

A square $n \times n$ sign pattern is called a *permutation sign pattern* if each row and column contains exactly one $+$ entry and $n - 1$ zero entries. Two $m \times n$ sign pattern matrices \mathcal{A}_1 and \mathcal{A}_2 are said to be *permutationally equivalent* if there exist permutation sign patterns \mathcal{P}_1 and \mathcal{P}_2 such that $\mathcal{A}_2 = \mathcal{P}_1 \mathcal{A}_1 \mathcal{P}_2$.

The product $\mathcal{P}\mathcal{D}$ of a permutation sign pattern \mathcal{P} and a signature sign pattern \mathcal{D} is called a *signed permutation (sign pattern)*. Two $m \times n$ sign pattern matrices \mathcal{A}_1 and \mathcal{A}_2 are said to be *signed permutationally equivalent* if there exist signed permutation sign patterns \mathcal{P}_1 and \mathcal{P}_2 such that $\mathcal{A}_2 = \mathcal{P}_1 \mathcal{A}_1 \mathcal{P}_2$.

The minimum rank of a sign pattern matrix \mathcal{A} , denoted $mr(\mathcal{A})$, is the minimum of the ranks of the real matrices in $Q(\mathcal{A})$. Similarly, the rational minimum rank of a sign pattern \mathcal{A} , denoted $mr_{\mathbb{Q}}(\mathcal{A})$, is defined to be the minimum of the ranks of the rational matrices in $Q(\mathcal{A})$. The minimum ranks of sign pattern matrices have been the focus of a large number of papers (see e.g. [1, 2, 3, 4, 5, 6, 12, 14, 15, 25, 27, 28, 38]), and they have important applications in areas such as communication complexity [1, 31, 32], machine learning [16], neural networks [10], combinatorics [12, 20, 43], and discrete geometry [29].

It is clear that $mr(\mathcal{A}) \leq mr_{\mathbb{Q}}(\mathcal{A})$ for every sign pattern \mathcal{A} . When $mr(\mathcal{A}) = mr_{\mathbb{Q}}(\mathcal{A})$, we say that the minimum rank of \mathcal{A} can be *realized rationally*. It is known (see [2, 5, 28, 38]) that for every $m \times n$ sign pattern \mathcal{A} with $mr(\mathcal{A}) \leq 2$ or $mr(\mathcal{A}) \geq n - 2$, its minimum rank can be realized rationally. However, it is shown in [27] and [12] that there exist sign patterns with minimum rank 3 whose rational minimum rank is greater than 3. In contrast, using a correspondence established in [12] between sign patterns with minimum rank $r \geq 2$ and point-hyperplane configurations in \mathbb{R}^{r-1} and Steinitz's theorem (see [43]) on the rational realizability of 3-polytopes, we show in Section 2 that for every nonnegative sign pattern of minimum rank at most 4, the minimum rank and the rational minimum rank are equal, but there are nonnegative sign patterns with minimum rank 5 whose rational minimum rank is greater than 5. We also establish several other interesting properties of nonnegative sign patterns with minimum rank 3. In Section 3, we find upper bounds on the entries of some integer matrices achieving the minimum ranks of nonnegative sign patterns with minimum rank 3 or 4.

Consider a nonnegative sign pattern \mathcal{A} . Observe that if \mathcal{A} contains a zero row or zero column, then deletion of the zero row or zero column preserves the minimum rank. Similarly, if two nonzero rows (or columns) of \mathcal{A} are identical, then deleting such a row (or column) also preserves the minimum rank. Clearly, deletion of a zero or duplicate row or column does not affect rational realizability of the minimum rank. Indeed, the deletion process can be reversed easily to create a matrix in the nonnegative sign pattern class of the original nonnegative sign pattern that achieves the minimum rank. Following [28], we say that a

nonnegative sign pattern is *condensed* if it does not contain a zero row or a zero column and no two rows or two columns of it are identical. Clearly, given any nonzero nonnegative sign pattern \mathcal{A} , we can delete the zero rows and columns, and delete all except the first row or column from each maximal collection of identical nonzero rows or columns of \mathcal{A} to get the condensed nonnegative sign pattern matrix \mathcal{A}_c of \mathcal{A} with the same minimum rank.

$$\text{For example, for the sign pattern } \mathcal{A} = \begin{bmatrix} 0 & + & + & + & + \\ + & 0 & 0 & + & + \\ + & 0 & 0 & + & + \\ + & + & + & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \text{ we have } \mathcal{A}_c = \begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}.$$

Since every nonnegative sign pattern and its condensed sign pattern have the same minimum rank and the same rational minimum rank, without loss of generality, in most of the subsequent discussions, we may assume that the nonnegative sign patterns involved are condensed.

3.1 Rational realizability of the minimum ranks of nonnegative sign patterns

As shown in [12], every sign pattern with minimum rank $r \geq 2$ corresponds to a point-hyperplane configuration in \mathbb{R}^{r-1} . The following lemma shows a very special feature of the point-hyperplane configurations corresponding to nonnegative sign patterns: all the points in the configuration are in the same closed half space bounded by any of the hyperplanes in the configuration.

Clearly, for an $m \times n$ nonnegative sign pattern \mathcal{A} and a signature sign pattern \mathcal{D} with order n , $\text{mr}(\mathcal{A}) = \text{mr}(\mathcal{A}\mathcal{D})$.

Lemma 3.1.1. *Let \mathcal{A} be an $m \times n$ condensed nonnegative sign pattern with $\text{mr}(\mathcal{A}) = r \geq 2$. Then there exist a suitable signature sign pattern \mathcal{D} of order n and a real matrix $B \in Q(\mathcal{A}\mathcal{D})$,*

such that B has a factorization $B = UV$ of the following form

$$U = \begin{bmatrix} 1 & u_{11} & \cdots & u_{1r} \\ 1 & u_{21} & \cdots & u_{2r} \\ \vdots & \vdots & \vdots & \vdots \\ 1 & u_{m1} & \cdots & u_{mr} \end{bmatrix}, \text{ and } V = [v_1, \dots, v_n] = \begin{bmatrix} v_{r1} & v_{r2} & \cdots & v_{rn} \\ \vdots & \vdots & \vdots & \vdots \\ v_{11} & v_{12} & \cdots & v_{1n} \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Proof. Let $B_1 \in Q(\mathcal{A})$ be a real matrix with $\text{rank}(B_1) = r$ and let U_1 be a matrix whose columns form a maximal linearly independent list of columns of B_1 . Then U_1 has size $m \times r$ and there exists an $r \times n$ matrix V_1 such that $B_1 = U_1V_1$. Since \mathcal{A} is condensed, B_1 does not have any zero row or column. Note that each row of U_1 is nonzero and nonnegative. Since the total number of row vectors of U_1 and the column vectors of V_1 is the finite number $m + n$, there exist suitable Givens rotation matrices (through some small positive angles) $R(\theta_2, 1, 2), R(\theta_3, 1, 3), \dots, R(\theta_r, 1, r)$ of order r such that the real orthogonal matrix $G = R(\theta_2, 1, 2)R(\theta_3, 1, 3) \dots R(\theta_r, 1, r)$ satisfies the following two conditions.

- (i) The first column of U_1G^T has only positive entries.
- (ii) The last row of GV_1 has no zero entries.

Hence, there is a diagonal matrix D_1 with all diagonal entries positive and a diagonal matrix D_2 with all diagonal entries nonzero such that all the entries in the first column of $D_1U_1G^T$ are equal to 1, and all the entries in the last row of GV_1D_2 are equal to 1. Now, let $U = D_1U_1G^T$, $V = GV_1D_2$ and $B = UV$. Since multiplication on the left by D_1 preserves the sign pattern, we see that $B \in Q(\mathcal{AD})$, where $\mathcal{D} = \text{sgn}(D_2)$. This completes the proof. \square

It is shown in [12] that every sign pattern with minimum rank $r \geq 2$ corresponds to a point-hyperplane configuration in \mathbb{R}^{r-1} . In particular, for the sign pattern \mathcal{AD} in the preceding lemma, through the special factorization $B = UV$, by identifying the i th row of U with the point $p_i = (u_{i2}, \dots, u_{ir}) \in \mathbb{R}^{r-1}$ ($i = 1, \dots, m$) and identifying the j th column v_j of V with the hyperplane h_j in \mathbb{R}^{r-1} satisfying the equation $[1 \ x_1 \ \dots \ x_{r-1}]v_j = 0$ ($j = 1, \dots, n$), we get an m point- n hyperplane configuration in \mathbb{R}^{r-1} , denoted $C_{\mathcal{A}}$. Furthermore, p_i is above

(respectively, below, in) h_j if and only if the (i, j) entry of \mathcal{AD} is $+$ (respectively, $-$, 0).

Conversely, given any point-hyperplane configuration C in \mathbb{R}^{r-1} consisting of m points p_1, \dots, p_m and n nonvertical (i.e., the normal vector is not perpendicular to the x_{r-1} -axis) hyperplanes h_1, \dots, h_n , we may write $p_i = (u_{i2}, \dots, u_{ir})$, and suppose that h_j is given by the equation $[1 \ x_1 \ \dots \ x_{r-1}]v_j = 0$, where the last component of v_j is 1. Let U, V be defined as in Lemma 3.1.1. Then $B = UV$ is a matrix of rank at most r . Furthermore, $\mathcal{A} = \text{sgn}(B) = [a_{ij}]$ is an $m \times n$ sign pattern with $\text{mr}(\mathcal{A}) \leq r$ such that $a_{ij} = +$ (respectively, $-$, 0) if and only if p_i is above (respectively, below, in) h_j . Alternatively, each hyperplane in the configuration can be oriented by prescribing one of its two sides as the positive side. Then the configuration gives rise to a sign pattern $\mathcal{A} = [a_{ij}]$ such that $a_{ij} = +$ (respectively, $-$, 0) if and only if p_i is on the positive side of (respectively, on the negative side of, in) h_j . As observed in [12], re-orienting the hyperplanes amounts to negating certain columns of the corresponding sign pattern. Two point-hyper plane configurations are said to be *equivalent* if their corresponding sign patterns are signed permutationally equivalent.

Lemma 3.1.2. *Let \mathcal{A} be a condensed nonnegative sign pattern with minimum rank 3, and let its corresponding point-hyperplane configuration C consist of m points p_1, \dots, p_m and n lines l_1, \dots, l_n in \mathbb{R}^2 . Let $P = \{p_1, p_2, \dots, p_m\}$ and let $L = \{l_1, l_2, \dots, l_n\}$. Then*

- (1) *All the points of P are in the same closed halfplane bounded by l_j , for each $j = 1, \dots, n$ (namely, no two points of P are on opposite sides of any line l_j in L);*
- (2) *If three distinct points $p_{i_1}, p_{i_2}, p_{i_3} \in P$ are collinear with p_{i_2} between p_{i_1} and p_{i_3} , then the only possible line in L passing through p_{i_2} is the line containing p_{i_1}, p_{i_2} , and p_{i_3} ;*
- (3) *There are at most three points of P on the same line in L and there are at most 3 lines in L passing through the same point in P .*

Proof. (1). Note that as in the preceding discussion, every column in \mathcal{AD} is either nonnegative or nonpositive. Assume that there exist two points p_i and p_k on opposite sides of l_j . Then the j th column of \mathcal{AD} must have at least one $+$ entry and one $-$ entry, which is a contradiction.

(2). Assume that three distinct points $p_{i_1}, p_{i_2}, p_{i_3} \in P$ are collinear with p_{i_2} between p_{i_1} and p_{i_3} , and there is a line $l_j \in L$ passing through p_{i_2} but not containing p_{i_1} . Then p_{i_1} and p_{i_3} are on opposite sides of l_j , contradicting (1).

(3). Assume that there are four distinct points, p_1, p_2, p_3, p_4 of P appearing in this order on the same straight line $l_j \in L$. By (2), there is no other line in L that passes through p_2 or p_3 . As a result, p_2 and p_3 are on the same side of all other lines in L . It follows that the rows of \mathcal{A} corresponding to p_2 and p_3 are identical, contradicting the fact that \mathcal{A} is row condensed. To show that there are at most 3 lines in L passing through the same point in P , we consider the point-hyperplane configuration C' corresponding to A^T . Note that C' may be obtained from C by transforming the n lines in L into n points p'_1, \dots, p'_n by the dual transform (which transforms the line not passing through the origin given by $\{x \in \mathbb{R}^2 \mid \langle a, x \rangle = 0\}$ to the point a and vice versa, see [29]) and also by transforming the m points in P into the m lines by the dual transform. Since concurrent lines are transformed to collinear points under the dual transform, the last part of (3) follows. \square

As an immediate consequence, we obtain the following result on the number of zero entries in a condensed nonnegative sign pattern with minimum rank 3.

Theorem 3.1.3. *Let \mathcal{A} be an $m \times n$ condensed nonnegative sign pattern with $mr(\mathcal{A}) = 3$. Then each row and column of \mathcal{A} has at most 3 zero entries. Hence there are at most $\min\{3m, 3n\}$ zero entries in \mathcal{A} .*

Proof. By Lemma 3.1.2, each point in $C_{\mathcal{A}}$ is on at most three lines and each line passes through at most 3 points. Thus each row and column of \mathcal{A} has at most 3 zero entries since the rows and columns correspond to points and hyperplanes in $C_{\mathcal{A}}$, respectively. \square

We remark that the upper bound on the number of zero entries given in the preceding theorem is the best possible. For example, let C be the point-hyperplane configuration in \mathbb{R}^2 whose hyperplanes are (the extensions of) the edges of a fixed n -gon G (with the side including the interior of G being the positive side) and whose points are the vertices of G and the midpoints of the edges of G . Then the resulting condensed nonnegative sign pattern

\mathcal{A} corresponding to C has size $2n \times n$ and minimum rank 3 (see Theorem 3.1.9 below) and has exactly $3n$ zero entries (with exactly 3 zero entries in each column).

It is also worth noting that every condensed nonnegative sign pattern \mathcal{A} with minimum rank 3 must have at least 3 zero entries. Indeed, if a condensed nonnegative sign pattern \mathcal{A} with minimum rank 3 has at most 2 zero entries, then \mathcal{A} must be 3×3 (otherwise, \mathcal{A} would have two positive rows or columns) and there must be two zeros entries on distinct rows and columns. It follows that up to permutational equivalence, $\mathcal{A} = \begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & + \end{bmatrix}$, so that $\text{mr}(\mathcal{A}) = 2$, a contradiction. The minimum number of zero entries, 3, is achieved by nonnegative sign patterns such as $\begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$ and $\begin{bmatrix} + & + & + \\ 0 & + & + \\ 0 & 0 & + \end{bmatrix}$.

We now arrive at one of the key facts of this paper.

Theorem 3.1.4. *Let \mathcal{A} be an $m \times n$ nonnegative sign pattern with $\text{mr}(\mathcal{A}) = 3$. Then $\text{mr}_{\mathbb{Q}}(\mathcal{A}) = 3$.*

Proof. Without loss of generality, we may assume that \mathcal{A} is a condensed nonnegative sign pattern.

By Lemma 3.1.1, there exist a suitable signature pattern \mathcal{D} and a matrix $B \in Q(\mathcal{AD})$ such that $B = UV$, where

$$U = \begin{bmatrix} 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \\ \vdots & \vdots & \vdots \\ 1 & a_m & b_m \end{bmatrix}, \text{ and } V = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ d_1 & d_2 & \cdots & d_n \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

We associate \mathcal{A} with its point-line configuration $C_{\mathcal{A}} = P \cup L$, where $P = \{p_1, \dots, p_m\}$ and $L = \{l_1, \dots, l_n\}$, as in Lemma 3.1.2. We complete the proof by finding a rational point-line configuration (namely, a configuration whose points are all rational points and each of whose lines contains two distinct rational points of \mathbb{R}^2) that is equivalent to $C_{\mathcal{A}}$.

Consider the convex polytope $K = \text{conv}(P)$. The fact that $\text{mr}(\mathcal{A}) = 3$ ensures that not all the points in P are collinear (see [12]). Hence, K is a polygonal region, whose boundary

is a polygon with $k \geq 3$ vertices p_{i_1}, \dots, p_{i_k} (in counterclockwise order). For convenience, we may assume that each line of L is oriented so that its positive side contains the center of mass of K . Observe that each point of P is either a vertex of $\text{conv}(P)$, or a point in the relative interior of an edge of K , or an interior point of K . Let

$$\begin{aligned} P_0 &= \{p \in P \mid p \text{ is a vertex of } K \}, \\ P_1 &= \{p \in P \mid p \text{ is in the relative interior of an edge of } K \}, \text{ and} \\ P_2 &= \{p \in P \mid p \text{ is an interior point of } K \}. \end{aligned}$$

Clearly, $P = P_0 \cup P_1 \cup P_2$, and $P_0 = \{p_{i_1}, \dots, p_{i_k}\}$. Note that if a point $p_{i_0} \in P_2$, then it corresponds to a row of \mathcal{A} that is positive (namely, all the entries in that row are positive). Thus there is at most one point in P_2 , and replacing such a point (if any) with the center of mass of K results in an equivalent configuration. In the case a point $p_i \in P_1$, by Lemma 3.1.2 (1), the only possible line in L containing p_i is the line extending the edge containing p_i ; such a point p_i can be replaced with the midpoint of the edge of K containing this point and the resulting configuration is equivalent to the original one. We now replace the 2-polytope K with consecutive vertices p_{i_1}, \dots, p_{i_k} by a 2-polytope K' with k consecutive rational vertices $p'_{i_1}, \dots, p'_{i_k}$ (also in counterclockwise order) such that no edge is vertical. If $P_2 \neq \emptyset$ with $P_2 = \{p_{i_0}\}$, we let p'_{i_0} be the center of mass of K' . If a point $p_j \in P_1$, then there are vertices p_{i_t} and $p_{i_{t+1}}$ (with the understanding that $i_{k+1} = i_1$) such that p_j is an interior point of the edge $p_{i_t}p_{i_{t+1}}$, and we let p'_j be the midpoint of $p'_{i_t}p'_{i_{t+1}}$. Let $P' = \{p' \mid p \in P\}$, and for each $k \in \{0, 1, 2\}$, let $P'_k = \{p' \mid p \in P_k\}$.

Observe that every line in L contains exactly 0, or 1, or 2 vertices of $K = \text{conv}(P)$. If a line $l_{j_0} \in L$ does not intersect K , then its corresponding column of \mathcal{A} is positive (and hence, there is at most one such line in L), and this line can be replaced with any horizontal line below K ; in this case, we define l'_{j_0} to be a rational horizontal line below K' . If a line l_j in L passes through two consecutive vertices p_{i_t} and $p_{i_{t+1}}$ of K , then we let l'_j be the line through p'_{i_t} and $p'_{i_{t+1}}$. If a line $l_i \in L$ passes through exactly one vertex of p_{i_t} of K , then we define l'_i to be any nonvertical rational line passing through p'_{i_t} that has only one intersection point

with $K' = \text{conv}(P')$. Let $L' = \{l' \mid l \in L\}$. We orient each line l'_j in L' such that the interior of K' is on the positive side l'_j .

Thus we obtain a rational point-line configuration $C' = P' \cup L'$, such that for all i, j , p_i is on the positive side of (respectively, on the negative side of, in) l_j if and only if p'_i is on the positive side of (respectively, on the negative side of, in) l'_j . Hence, the rational point-line configuration C' is equivalent to C . Using C' , we can define rational matrices U' and V' of sizes $m \times 3$ and $3 \times n$ respectively as in the discussion preceding Lemma 3.1.2, and the sign pattern of the resulting rational matrix $B' = U'V'$ is $\mathcal{A}\mathcal{D}'$, where \mathcal{D}' is a signature sign pattern with negative diagonal entries corresponding to the lines in L' that are above the interior of $\text{conv}(P')$. It follows that $B'\mathcal{D}'$ is a rational matrix in $Q(\mathcal{A})$ achieving the minimum rank 3.

□

There is an interesting relationship between the numbers of rows and columns of a condensed nonnegative sign pattern with minimum rank 3.

Theorem 3.1.5. *Let \mathcal{A} be an $m \times n$ condensed nonnegative sign pattern with $mr(\mathcal{A}) = 3$. Then $n \leq 2m + 1$ and $m \leq 2n + 1$, where each upper bound is tight.*

Proof. By considering \mathcal{A}^T instead, it suffices to show that $n \leq 2m + 1$. By Lemma 3.1.2 and the proof of Theorem 3.1.4, it is easy to see that for a point-line configuration $C = P \cup L$ corresponding to the sign pattern \mathcal{A} , since there are m points in P , there is at most one line in L disjoint with $\text{conv}(P)$, at most m lines in L passing through exactly one vertex of $\text{conv}(P)$, and at most m lines in L that are extensions of edges of $\text{conv}(P)$. Moreover, every line in L must be one of the above types. It follows that $|L| \leq 2m + 1$, namely, $n \leq 2m + 1$.

Note that for the point-line configuration obtained by taking the points to be the vertices of a fixed m -gon and the lines to be the edges of the m -gon along with a horizontal line below the m -gon and m additional lines each of which is a line intersecting the m -gon at only one point, then the corresponding condensed nonnegative sign pattern is $m \times (2m + 1)$ and has minimum rank 3 (see Theorem 3.1.9). This completes the proof. □

We now arrive at another key result of this paper, based on Steinitz's theorem ([43]) on the rational realizability of 3-polytopes. Basic terms and facts about convex polytopes may be found in [43].

Theorem 3.1.6. *Let \mathcal{A} be an $m \times n$ nonnegative sign pattern with $mr(\mathcal{A}) = 4$. Then $mr_{\mathbb{Q}}(\mathcal{A}) = 4$.*

Proof. Without loss of generality, we assume that \mathcal{A} is condensed. Similarly as in the proof of Theorem 3.1.4, it suffices to find a rational point-hyperplane configuration in \mathbb{R}^3 equivalent to $C_{\mathcal{A}} = P \cup H$, where $P = \{p_1, \dots, p_m\} \subset \mathbb{R}^3$ and $H = \{h_1, \dots, h_n\}$ consists of hyperplanes. Let $K = \text{conv}(P)$, which must be a 3-polytope [12]. Then each point of P is either a vertex of K , or an interior point of K , or in the relative interior of either an edge or a facet of K . In other words, each point of P is in the relative interior of a face of K of dimension 0, 1, 2, or 3. Of course, a 0-dimensional face is a vertex, a 1-dimensional face is an edge, a 2-dimensional face is a facet, and the 3-dimensional face is K itself. Let the vertices of K be p_{i_1}, \dots, p_{i_k} . By Steinitz's Theorem (see [43]), there is a rational 3-polytope K' whose vertices are rational perturbations $p'_{i_1}, \dots, p'_{i_k}$ of the corresponding vertices of K and whose face-lattice is isomorphic to that of K under the mapping $p_{i_t} \mapsto p'_{i_t}$. By using suitable rational Givens rotations if necessary, we may assume that no facet of K' is vertical (namely, the normal vector of any facet is not perpendicular to the z -axis).

The fact that \mathcal{A} is condensed ensures that there is at most 1 point in P in the relative interior of each face of K . For each point $p \in P$ in the relative interior of a face F of K , we define p' to be the center of mass of the corresponding face F' of K' . Clearly, each p' is a rational point since all the vertices of K' are rational points and every face of K' is the convex hull of a subset of the vertices of K' .

Since \mathcal{A} is nonnegative, by Lemma 3.1.1, in the configuration $C_{\mathcal{A}} = P \cup H$ no two vertices of K are on opposite sides of any hyperplane $h \in H$, and hence the intersection of each hyperplane $h \in H$ with K is either empty or is a face F of K , with $\dim(F) = 0, 1$, or 2.

Consider a hyperplane $h \in H$. If $h \cap K = \emptyset$, let h' be any rational horizontal hyperplane

below K . If $\dim(h \cap K) = 0$ with $h \cap K = \{p_{i_t}\}$, let h' be a rational hyperplane (namely, a hyperplane passing through a rational point and having a rational normal vector) such that $h' \cap K' = p'_{i_t}$. More generally, if $h \cap K = F$ for some face F of K , let h' be a rational hyperplane such that $h' \cap K' = F'$. This is clearly possible since F' is rational and when $\dim(F') = \dim(F) \leq 1$, there is a cone of possible choices of normal vectors of h' ([43]).

It is evident that the rational configuration $C' = P' \cup H'$ is equivalent to $C_{\mathcal{A}}$. We can use C' to reverse the process of Lemma 3.1.1 to create two rational factors U and V of sizes $m \times 4$ and $4 \times n$ respectively, to obtain a rational matrix $B = UV$ of rank at most 4 whose sign pattern is signature equivalent to \mathcal{A} . It follows that $\text{mr}_{\mathbb{Q}}(\mathcal{A}) \leq 4$. Therefore, $\text{mr}_{\mathbb{Q}}(\mathcal{A}) = 4$. \square

The argument in the preceding proof can be adapted to show the following result, whose formal proof is omitted as the main ideas are the same.

Theorem 3.1.7. *Let \mathcal{A} be a condensed nonnegative sign pattern with minimum rank $r \geq 5$. Then $\text{mr}_{\mathbb{Q}}(\mathcal{A}) = r$ if and only if for a point-hyperplane representation $C_{\mathcal{A}} = P \cup H$, where P is the set of all the points of $C_{\mathcal{A}}$, the face lattice of the convex polytope $K = \text{conv}(P)$ is isomorphic to that of a rational convex polytope K' .*

The construction of the possible hyperplanes in H' (according to their intersections with K') in the proof of Theorem 3.1.6 reveals an upper bound on n for any $m \times n$ condensed nonnegative sign pattern with minimum rank 4. By considering the transpose, we get the dual upper bound.

Theorem 3.1.8. *Let \mathcal{A} be an $m \times n$ condensed nonnegative sign pattern with $\text{mr}(\mathcal{A}) = 4$. Then*

$$n \leq \sum_{k=0}^3 \binom{m}{k} = \frac{1}{6}m^3 + \frac{5}{6}m + 1 \quad \text{and} \quad m \leq \sum_{k=0}^3 \binom{n}{k} = \frac{1}{6}n^3 + \frac{5}{6}n + 1.$$

As indicated in the preceding theorems, the study of the minimum ranks of nonnegative sign pattern leads to investigation of convex polytopes. We now show that due to the two-way correspondence between sign pattern matrices and point-hyperplane configurations, a

convex polytope also determines a nonnegative sign pattern naturally, and this approach has been exploited previously, see [17].

Theorem 3.1.9. *Let K be a convex polytope of dimension $d \geq 1$ that has m vertices and n facets, embedded in \mathbb{R}^d . Let P be the set of vertices of K and let H be the set of hyperplanes each of which contains precisely one facet of K , with each hyperplane oriented so that the interior of K is on its positive side. Then the $m \times n$ nonnegative sign pattern \mathcal{A} corresponding to the point-hyperplane configuration $P \cup H$ has minimum rank $d+1$ and up to permutational equivalence, \mathcal{A} contains an upper triangular submatrix of order $d+1$ all of whose diagonal entries are positive.*

Proof. As indicated in the discussion preceding Lemma 3.1.2, using the configuration $P \cup H$, we get an $m \times (d+1)$ matrix U using the coordinates of the points in P and a $(d+1) \times n$ matrix V using equations of the hyperplanes in H . Hence, the matrix $B = UV$ has rank at most $d+1$. Since the nonnegative sign pattern \mathcal{A} corresponding to the configuration $P \cup H$ is signed permutationally equivalent to $\text{sgn}(B)$, we have $\text{mr}(\mathcal{A}) \leq d+1$.

We prove the opposite inequality $\text{mr}(\mathcal{A}) \geq d+1$ and the result about triangular submatrices by induction on d . For $d=1$, a 1-polytope K is just a line segment in \mathbb{R}^1 , and K has two vertices $p_1 < p_2$, which are also the facets of K . Let $h_1 = p_2$ and $h_2 = p_1$, with the positive side containing the midpoint of K . Then the nonnegative sign pattern corresponding to the configuration $\{p_1, p_2\} \cup \{h_1, h_2\}$ is $\begin{bmatrix} + & 0 \\ 0 & + \end{bmatrix}$, which has minimum rank 2 and is upper triangular with positive diagonal entries. Thus the result holds for $d=1$.

Suppose that $d \geq 2$ and the result holds for every $(d-1)$ -polytope. Let K be any d -polytope, with m vertices and n facets. By rotating K suitably if necessary, we may assume that K has a unique highest vertex (namely, the vertex with the largest x_d coordinate), denoted p_0 . Let K_0 be a vertex figure (see [43]) of K at p_0 , namely, K_0 is the intersection of a hyperplane h_0 slightly below the point p_0 such that h_0 strictly separates p_0 from the remaining vertices of K . Clearly, h_0 intersects every edge of K with p_0 as an endpoint at an interior point of the edge. It is well known that the vertex figure K_0 of K has dimension

$\dim(K) - 1 = d - 1$ (see [43]). Note that every vertex of K_0 is on an edge of K that has p_0 as an endpoint. By the induction hypothesis, K_0 has d vertices v_1, \dots, v_d and d facets F_1, \dots, F_d (viewed as hyperplanes) which form a configuration that gives rise to an upper triangular nonnegative sign pattern of order d with all diagonal entries positive. For each $i = 1, \dots, d$, let p_i be the vertex of K below h_0 on the extension of p_0v_i , and let $h_i = \text{conv}(\{p_0\} \cup F_i)$. Then h_i is contained in a facet of K , $i = 1, \dots, d$. Let h_{d+1} be any facet of K not containing p_0 . We identify each set h_i with the unique hyperplane containing it. Then the subconfiguration consisting of the points p_1, \dots, p_d, p_0 and the hyperplanes h_1, \dots, h_d, h_{d+1} yields an upper triangular nonnegative sign pattern of order $d + 1$ with all diagonal entries positive. Since such an upper triangular sign pattern is a submatrix of \mathcal{A} up to permutation equivalence, it follows that $\text{mr}(\mathcal{A}) \geq d + 1$. Since we also have $\text{mr}(\mathcal{A}) \leq d + 1$, we conclude that $\text{mr}(\mathcal{A}) = d + 1$. This complete the proof. \square

We point out that the preceding result resolves an open problem posed in [17], as this result implies the existence of a fooling-set submatrix (which includes a nonsingular triangular matrix as a special case, see [17]) of order $d + 1$ of the nonnegative sign pattern (or the (0,1)-matrix) determined by the convex polytope K of dimension d , while the previously known lower bound for the order of a largest fooling-set submatrix is \sqrt{d} .

Note that in general, up to permutation equivalence, a nonnegative sign pattern \mathcal{A} with minimum rank r may not have a triangular submatrix of order r with all diagonal entries positive. For instance, the sign pattern $\begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$ has minimum rank 3, but it is not permutationally equivalent to an upper triangular matrix.

In view of Theorems 3.1.7 and 3.1.9 and the fact that for each $d \geq 4$ there are d -polytopes that are not rationally realizable (see [43]), the following result is immediate.

Theorem 3.1.10. *For each integer $r \geq 5$, there exists a nonnegative sign pattern \mathcal{A} with $\text{mr}(\mathcal{A}) = r$ and $\text{mr}_{\mathbb{Q}}(\mathcal{A}) > r$.*

3.2 Integer realization of the minimum rank

As shown in Section 2, for each nonnegative sign pattern \mathcal{A} with minimum rank at most 4, its minimum rank can be realized rationally, and hence, there exist integer matrices in $Q(\mathcal{A})$ realizing the minimum rank. We now give upper bounds on the entries of some integer matrices in $Q(\mathcal{A})$ realizing the minimum rank.

General (not necessarily nonnegative) sign patterns with minimum rank 2 are characterized in [28]. The condensed nonnegative sign patterns \mathcal{A} with $\text{mr}(\mathcal{A}) = 2$ are quite simple. By considering the corresponding point-hyperplane configuration in \mathbb{R}^1 (in which a hyperplane is also a point), we see that each row and each column contains at most one zero entry. Suppose that \mathcal{A} has three or more zero entries, then up to permutation equivalence, \mathcal{A} contains the sign pattern $\begin{bmatrix} 0 & + & + \\ + & 0 & + \\ + & + & 0 \end{bmatrix}$ as a submatrix, which has minimum rank 3, a contradiction. Thus \mathcal{A} contains one or two zero entries and no two zero entries can occur in the same row or the same column. It follows that up to permutation equivalence, \mathcal{A} contains $\begin{bmatrix} 0 & + \\ + & + \end{bmatrix}$ or $\begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix}$ as a submatrix, and all other entries are $+$. Consequently, \mathcal{A} has two or three rows and two or three columns, and its minimum rank is achieved by an integer matrix in $Q(\mathcal{A})$ with entries from the set $\{0, 1, 2\}$. Thus we arrive at the following result.

Theorem 3.2.1. *Let \mathcal{A} be any nonnegative sign pattern such that $\text{mr}(\mathcal{A}) = 2$. Then its condensed sign pattern \mathcal{A}_c has at most three rows (columns) and contains at least one and at most two zero entries, with no two zero entries on the same row or column, and there is an integer matrix in $Q(\mathcal{A})$ with entries from the set $\{0, 1, 2\}$ that achieves the minimum rank of \mathcal{A} .*

In order to achieve the minimum rank of a condensed nonnegative sign pattern of size $m \times n$ with minimum rank 3 by an integer matrix, by following the steps of the proof of Theorem 3.1.4, it suffices to construct a 2-polytope (a convex polygonal region) in \mathbb{R}^2 with m integral vertices whose coordinates are even integers and the slopes of whose non-horizontal edges are odd integers, so that the midpoint of each edge is also an integral point and there is an integral interior point of the 2-polytope.

Lemma 3.2.2. *Let m be a positive integer with $m \geq 3$ and let $t = \lceil \frac{m}{4} \rceil$. Then there is a 2-polytope K in \mathbb{R}^2 with $4t \geq m$ integral vertices with even coordinates such that the absolute values of the x -coordinates of the vertices of K are at most $2t$, the absolute values of the y -coordinates of the vertices of K are at most $2t^2$, and the slopes of the edges of K are odd integers with absolute value at most $2t - 1$. Further, the y -intercepts of the extensions of the edges of K are at most $4t^2$ in absolute value, and at each vertex of K , there is a line with even slope that intersects K at exactly one point and has y -intercept with absolute value at most $4t^2$.*

Proof. Consider the graphs of $y = f_1(x) = \frac{1}{2}x^2 - 2t^2$ and $y = f_2(x) = 2t^2 - \frac{1}{2}x^2$ over the interval $[-2t, 2t]$. These curves meet at the points $(\pm 2t, 0)$ and form the boundary of a convex set in \mathbb{R}^2 symmetric about the x -axis and the y -axis. Let K be the 2-polytope whose vertices are the following $4t$ points on the two curves above: $(2k, \pm(2t^2 - 2k^2))$, $k = -t, -t + 1, \dots, t - 1, t$. Obviously, the coordinates of all the vertices of K are even integers; the absolute values of the x -coordinates of the vertices of K are at most $2t$; and the absolute values of the y -coordinates of the vertices of K are at most $2t^2$. Note that the slopes of the edges of K are odd integers with absolute values at most $2t - 1$. Observe that the absolute values of the y -intercepts of the extensions of the edges of K are integers that increase as the edges are further away from the y -axis. Thus the largest y -intercept absolute value of the edge extensions of K is $2t(2t - 1)$. Further, at the vertices with x -coordinate $\pm 2t$, the line with slope $2t$ is a supporting line of K with y -intercept $\pm 4t^2$ whose intersection with K is a vertex. For every vertex whose x -coordinate has absolute value less than $2t$, its incident edges of K have odd integer slopes that differ by 2, so there is a supporting line of K through such a vertex such that its intersection with K is just a point, its slope is an even integer, and its y -intercept has absolute value less than $4t^2$; in fact, this supporting line is the tangent line to one of the two curves given above. \square

Theorem 3.2.3. *Let \mathcal{A} be a condensed nonnegative sign pattern of size $m \times n$ with $mr(\mathcal{A}) = 3$. Let $t = \lceil \frac{m}{4} \rceil$. Then there is an integer matrix $B = [b_{ij}] \in Q(\mathcal{A})$ with entries bounded above*

by $10t^2$ that achieves the minimum rank of \mathcal{A} .

Proof. Let $C = \{p_1, \dots, p_m\} \cup \{l_1, \dots, l_n\}$ be a point-hyperplane configuration corresponding to \mathcal{A} . Suppose that the 2-polytope $\hat{K} = \text{conv}(\{p_1, \dots, p_m\})$ has s vertices. If necessary, we may delete an odd number of vertices from the top (deleting the vertices with the largest y -coordinates first) of K and delete the bottom vertex of K , to obtain a 2-polytope K' that has the remaining s vertices of K as its vertices. Clearly, either the top (bottom) vertex of K is retained in K' or K' has two top (bottom) vertices with the same y -coordinate. Note that K' is symmetric about the y -axis. As in the proof of Theorem 3.1.4, we may replace \hat{K} with the integral 2-polytope K' and apply the preceding Lemma to construct an equivalent integral point-line configuration $C' = \{p'_1, \dots, p'_m\} \cup \{l'_1, \dots, l'_n\}$ (all of whose points have integral coordinates and all of its lines have integer slopes and integer y -intercepts). As far as the points of C' are concerned, this construction is possible since each vertex of K' as well as the midpoint of each edge of K' is an integral point (as the coordinates of the vertices of K' are even integers) and the midpoint of the intersection of the y -axis with K' is an integral point in the interior of K' . The line $y = -2t^2 - 1$ is an integral line below K' , and Lemma 3.2.2 ensures that the needed integral lines that support K' are available, and all such lines have integer slopes with absolute values at most $2t$ and have y -intercepts with absolute values most $4t^2$. As in the proof of Theorem 3.1.4, the integral point-line configuration C' gives rise to integer matrices

$$U = \begin{bmatrix} 1 & a_1 & b_1 \\ 1 & a_2 & b_2 \\ \vdots & \vdots & \vdots \\ 1 & a_m & b_m \end{bmatrix}, \text{ and } V = \begin{bmatrix} c_1 & c_2 & \cdots & c_n \\ d_1 & d_2 & \cdots & d_n \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

where (a_i, b_i) are the coordinates of p'_i , and c_j and d_j are the negatives of the y -intercept and the slope of l'_j , respectively. By Lemma 3.2.2, we have $|a_i| \leq 2t$, $|b_i| \leq 2t^2$, $|c_j| \leq 4t^2$, and $|d_j| \leq 2t$, for all i and j . It follows that the (i, j) -entry of the matrix $B' = [b'_{ij}] = UV$ satisfies

$|b'_{ij}| = |c_j + a_i d_j + b_i| \leq |c_j| + |a_i| |d_j| + |b_i| \leq 4t^2 + 2t^2 + 2t^2 = 10t^2$. Since multiplying certain columns of B_1 by -1 yields a nonnegative integer matrix $B \in Q(\mathcal{A})$, the desired conclusion follows. \square

It is known [33] that the combinatorial type of every 3-polytope with m vertices can be realized in integer grid of width $O(2^{7.55m})$. We use this result to derive an upper bound for the entries of some integer matrix that achieves the minimum rank of a nonnegative sign pattern with minimum rank 4.

Theorem 3.2.4. *Let \mathcal{A} be a condensed nonnegative sign pattern of size $m \times n$ with $mr(\mathcal{A}) = 4$. Then there is an integer matrix $B \in Q(\mathcal{A})$ with entries at most $O(m2^{22.65m})$ that achieves the minimum rank of \mathcal{A} .*

Proof. Let $C = P \cup H = \{p_1, \dots, p_m\} \cup \{h_1, \dots, h_n\}$ be a point-hyperplane configuration corresponding to \mathcal{A} in \mathbb{R}^3 . Then the 3-polytope $K = \text{conv}(\{p_1, \dots, p_m\})$ has at most m vertices. By [33], the combinatorial type of K can be achieved by an integral 3-polytope K' (with all the vertices being integral points) in an integer grid of width $O(2^{7.55m})$ in the first orthant. Expanding K' 12 times if necessary, we may assume that the coordinates of each vertex of K' are multiples of 12. We follow the procedure in the proof of Theorem 3.1.6 to construct an integral point-hyperplane configuration $C' = P' \cup H'$ equivalent to C (where all points in P' have integer coordinates and all the hyperplanes in H' are given by linear equations with integer coefficients). Each point in P' is either a vertex of K' or a point in the relative interior of a face of K' . Since the coordinates of the vertices are multiples of 12, the midpoint of each edge of K' is an integer point, the center of mass of any triangle whose vertices are some vertices of a facet of K' is an integral point, and the interior of K' contains the integral center of mass of a tetrahedron whose vertices are four noncoplanar vertices of K' . Thus all the points in P' can be constructed in the same integer grid.

We now construct integral hyperplanes in H' . Of course, a hyperplane is an integral hyperplane provided it passes through an integral point and it has an integral normal vector. The intersection of each hyperplane in H' with K' is either empty or is a face of K' of

dimension at most 2. An integral hyperplane not intersecting K' is given by $z = 0$. A hyperplane containing a facet of K' contains two consecutive edges of the facet and a normal vector of the hyperplane is given by the cross product of the two integer vectors obtained by treating the two consecutive edges as vectors (we may orient the edges of the facet in counterclockwise fashion when viewed from the outside so that the resulting normal vector is pointing outward). It follows that each coordinate of an integral normal vector of each facet is at most $O(2^{2 \cdot 7.55m}) = O(2^{15.10m})$. Since an equation of a hyperplane in \mathbb{R}^3 passing through the point (x_0, y_0, z_0) and having a normal vector (b, c, d) is given by $-(bx_0 + cy_0 + dz_0) + bx + cy + dz = 0$, the hyperplane has an equation of the form $a + bx + cy + dz = 0$, where the coefficients are integers and $|a| \leq O(2^{3 \cdot 7.55m}) = O(2^{22.65m})$, and $|b|, |c|, |d| \leq O(2^{15.10m})$.

A hyperplane whose intersection with K' is an edge of K' may be constructed with its normal vector being the sum of the two outward integral normal vectors of the two facets containing this edge, and hence, an equation with integer coefficients of such a hyperplane satisfies the same conditions as above.

Finally, a hyperplane whose intersection with K' is a vertex of K' may be constructed with its normal vector being the sum of the outward integral normal vectors of the at most $m - 1$ facets containing this vertex. Hence, the hyperplane has an equation of the form $a + bx + cy + dz = 0$, where the coefficients are integers and $|a| \leq O(m2^{3 \cdot 7.55m}) = O(m2^{22.65m})$, and $|b|, |c|, |d| \leq O(m2^{15.10m})$.

The configuration C' gives rise to two integer matrices

$$U = \begin{bmatrix} 1 & x_1 & y_1 & z_1 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & x_m & y_m & z_m \end{bmatrix}, \text{ and } V = \begin{bmatrix} a_1 & \cdots & a_n \\ b_1 & \cdots & b_n \\ c_1 & \cdots & c_n \\ d_1 & \cdots & d_n \end{bmatrix},$$

where (x_i, y_i, z_i) are the integer coordinates of the point p'_i in P' , and $a_j + b_jx + c_jy + d_jz = 0$ is an equation with integer coefficients of the hyperplane h'_j in H' . As shown above, $|x_i|, |y_i|, |z_i| \leq O(2^{7.55m})$, $|a_j| \leq O(m2^{3 \cdot 7.55m}) = O(m2^{22.65m})$, and $|b_j|, |c_j|, |d_j| \leq$

$O(m2^{15.10m})$. It follows that the entries of the integer matrix $B_1 = [b'_{ij}] = UV$ of rank 4 satisfy

$$|b'_{ij}| = |a_j + b_j x_i + c_j y_i + d_j z_i| \leq 4O(m2^{22.65m}) = O(m2^{22.65m}).$$

Since multiplying certain columns of B_1 by -1 yields a nonnegative integer matrix $B \in Q(\mathcal{A})$, the desired conclusion follows. \square

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