Georgia State University
ScholarWorks @ Georgia State University

Mathematics Dissertations | Department of Mathematics and Statistics

12-16-2019

Maximally Even Tilings: Theory and Algorithms

Jeremiah D. Kastine
Georgia State University

Follow this and additional works at: https://scholarworks.gsu.edu/math_diss

Recommended Citation

Kastine, Jeremiah D., "Maximally Even Tilings: Theory and Algorithms." Dissertation, Georgia State University, 2019.
https://scholarworks.gsu.edu/math_diss/63

This Dissertation is brought to you for free and open access by the Department of Mathematics and Statistics at ScholarWorks @ Georgia State University. It has been accepted for inclusion in Mathematics Dissertations by an authorized administrator of ScholarWorks @ Georgia State University. For more information, please contact scholarworks@gsu.edu.

MAXIMALLY EVEN TILINGS: THEORY AND ALGORITHMS

by

JEREMIAH KASTINE

Under the direction of Mariana Montiel, PhD

ABSTRACT

This dissertation combines two previously separate topics from the field of mathematical music theory: rhythmic tiling canons and maximally even set/rhythms. In particular, it will investigate the existence, classification, and construction of rhythmic tiling canons in which the composite rhythm is maximally even.

INDEX WORDS: Mathematical musical theory, rhythmic canons, tiling, maximally even sets

MAXIMALLY EVEN TILINGS: THEORY AND ALGORITHMS

by

JEREMIAH KASTINE

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

in the College of Arts and Sciences

Georgia State University

2019

Copyright by
Jeremiah Kastine
2019

MAXIMALLY EVEN TILINGS: THEORY AND ALGORITHMS

by

JEREMIAH KASTINE

Committee Chair: Mariana Montiel

Committee: Francisco (Paco) Gómez

Michael Stewart

Hein van der Holst

Electronic Version Approved:

Office of Graduate Studies

College of Arts and Sciences

Georgia State University

December 2019

ACKNOWLEDGEMENTS

I would like to thank...

- my wife, Val, for her love, support, and encouragement. For several years, I had abandoned the idea of earning a PhD, but she never stopped telling me, "You should/will do it one day."

- my kids, Sebbie and Lucy, who, as infants, inspired me to study the mathematical properties of rhythmic canons. It was whilst rocking and singing them back to sleep for fifth time of the night that I had most creative thoughts.

- Mariana for taking such an interest in my ideas and letting me run with them, for encouraging me to publish and present at conferences, and for helping me talk through and clarify my thoughts.

- Paco for introducing me to the concept of maximal evenness, and for agreeing to serve on my committee from afar.

- Dr. Stewart and Dr. van der Holst for agreeing to serve on my committee even though my topic is quite far removed from their respective fields of research.

- the instructors/professors at Mt. Hood Community College, Portland State University, and Georgia State University that helped me through my academic journey and helped me become a better mathematician: Marshall Tuttle, Philip Demarois, the graduate teaching assistant who taught my first "proofs" class at PSU, Gerardo Lafferriere, John Erdman, Sean Larson, Robert Clewley, Imre Patyi, and (of course) Mariana Montiel.

- my supervisors at Isothermal Community College and Georgia Highlands College for giving me the flexibility and financial support to pursue my research and this degree.

TABLE OF CONTENTS

LIST OF FIGURES

Chapter 1

INTRODUCTION

1.1 Notation

This dissertation concerns rhythmic canons, which are musical structures consisting of two or more translations of a single rhythmic theme. Figure 1.1 gives an example of a rhythmic canon in conventional music notation. In this example, we see that the rhythmic

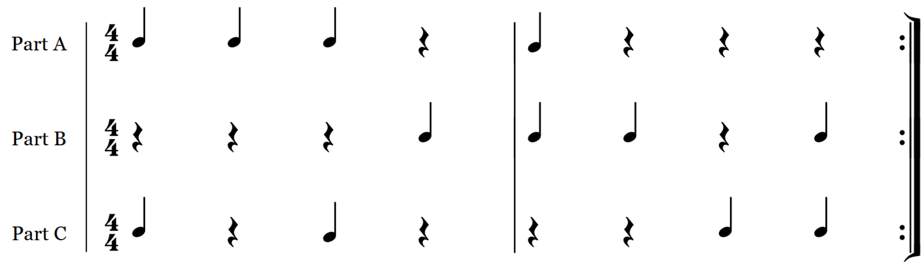


Figure (1.1) A rhythmic canon.

pattern of Part A (note-note-note-rest-note-rest-rest-rest) is translated 3 beats to the right in Part B and 6 beats to the right in Part C.

For the purposes of this dissertation, we will adopt the following more useful and compact form for displaying such musical scores.

	0	1	2	3	4	5	6	7	
Part A	1	2	3	·	4	·	·	·	:||
Part B	·	·	·	1	2	3	·	4	:||
Part C	3	·	4	·	·	·	1	2	:||

We can also describe the rhythmic canon above as an ordered pair of sets with a subscript, $(\{0,1,2,4\},\{0,3,6\})_8$, in which the first component is the rhythmic theme, the second component is the set of translations, and the subscript indicates the number of beats.

In addition to this musical notation, the following mathematical notation will be used throughout this dissertation.

- $\equiv_n$ will denote equivalence modulo n.
- $\text{mod}_n(a)$ will denote the least non-negative number equivalent to a modulo n.
- Interval notation will be used to indicate sets of consecutive integers. For example, $[a, b)$ will denote the set of all integers x such that $a \leq x < b$.
- We will use $\oplus_n : [0, n)^2 \to [0, n)$ to denote the function defined by $\oplus_n(x, y) = \text{mod}_n(x+y)$.
- It will sometimes be more convenient to represent ordered pairs (x, y) as $\binom{x}{y}$.
- $\lfloor x \rfloor$ will denote the floor of x (the greatest integer less than or equal to x) and $\lceil x \rceil$ will denote the ceiling of x (the least integer greater than or equal to x).
- $|X|$ will denote the number of elements in the set X.

1.2 Informal description of the central problem

Researchers in the field of mathematical musical theory have shown much interest in rhythmic canons that "tile" the space of beats in which they are situated, as in the following example.

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	
Part A	1	2	3	·	4	·	·	·	5	·	·	·	·	·	·	:‖
Part B	·	·	·	·	·	1	2	3	·	4	·	·	·	5	·	:‖
Part C	·	·	·	5	·	·	·	·	·	·	1	2	3	·	4	:‖

In this rhythmic canon, which can be expressed as $(\{0, 1, 2, 4, 8\}, \{0, 5, 10\})_{15}$, each beat is occupied by exactly one note.

Interest in rhythmic tiling canons was spurred by Vuza's seminal papers on the subject [19–22], and many researchers have continued to investigate some challenging open questions regarding their construction and properties. Special issues on the topic of tiling by the Journal of Mathematics and Music [1] and Perspectives of New Music [2] would serve as good resources to those wanting to learn more about the history, theory, and musical applications of rhythmic tiling canons.

One of the most important results about rhythmic tiling canons is that most of them are periodic, meaning that for most rhythmic tiling canons $(A, B)_n$ we have either $\text{mod}_n(A+p) = A$ or $\text{mod}_n(B+p) = B$ for some $p \in [1, n)$ (e.g., $\{0, 5, 10\}$ in the example above). In particular, the only lengths n that permit aperiodic tiling are 72, 108, 120, 144,... [3,6,13,17].

From a musical standpoint, periodic rhythmic tiling canons are often quite monotonous and lacking in character. The composer of rhythmic tiling canons has the following three options available:

1. settle for the monotonous rhythms of periodic tilings,

2. employ aperiodic tilings, the length of which may surpass the audience's capacity for processing, or

3. relax the rules of rhythmic tiling.

In this dissertation, I will relax the rules by allowing "holes" (unoccupied beats) in the composite rhythmic pattern while maintaining the non-overlapping requirement, as in the rhythmic canon of $(\{0, 2, 6, 7, 11\}, \{0, 3\})_{13}$ given below.

	0	1	2	3	·	5	6	7	·	9	10	11	·	
Part A	1	·	2	·	·	·	3	4	·	·	·	5	·	:||
Part B	·	5	·	1	·	2	·	·	·	3	4	·	·	:||

Notice that in this small example we have already achieved aperiodicity.

Another desirable property of the preceding example is that the notes and rests of the composite rhythm are distributed in a relatively even manner: 4 notes, 1 rest, 3 notes, 1 rest, 3 notes, 1 rest. Thus, the musical effect of strict tiling, in which the notes are perfectly evenly distributed, is to some extent maintained.

Evenness is also interesting from a mathematical perceptive. In particular, the mathematical music theory community has produced a fair amount of research on the properties and construction of so-called "maximally even sets/rhythms". The precise definition of such rhythms and a development of their theory are given in Chapter 3. The goal of this dissertation is to study rhythmic tiling canons in which the composite rhythm is maximally even.

Chapter 2

RHYTHMIC TILING

2.1 Definitions and properties

Definition 1. Let $n \geq 1$ and $A, B \subseteq [0, n)$ such that the restriction of $\oplus_n$ to $A \times B$ is bijective. Then we will say that $(A, B)_n$ is an n-note full tiling. We will also say that A is a modulo n tiling complement of B and vice-versa.

Example 2. The restriction of $\oplus_{15}$ to $\{0, 1, 2, 4, 8\} \times \{0, 5, 10\}$ is represented in the following addition table.

$\oplus_{15}$	0	1	2	4	8
0	0	1	2	4	8
5	5	6	7	9	13
10	10	11	12	14	3

Each value in $[0, 15)$ appears exactly once in the output entries of this table. So $(\{0, 1, 2, 4, 8\}, \{0, 5, 10\})_{15}$ is a 15-note full tiling. (Indeed, this is the same example used in Figure 1.1 at the beginning of Section 1.2.)

Definition 3. Let $n \geq 1$ and $A, B \subseteq [0, n)$.

- We will call $(B, A)_n$ the dual of $(A, B)_n$.
- Let $t_1, t_2 \in \mathbb{Z}$. We will call $(\mathrm{mod}_n(A - t_1), \mathrm{mod}_n(B - t_2))_n$ a translation of $(A, B)_n$.
- Let u be a modulo n unit. We will call $(\mathrm{mod}_n(uA), \mathrm{mod}_n(uB))_n$ a unit multiple of $(A, B)_n$.

Proposition 4. $(A, B)_n$ is a full tiling if and only if its dual, translations, and unit multiples are full tilings.

2.2 Conditions T1 and T2

Most of the research about rhythmic tiling has made use of algebraic methods for studying and constructing full tilings [9, 13, 15]. This approach has proved very fruitful and has even allowed researchers to make progress toward proving (or disproving) Fuglede's conjecture in one dimension [5, 13]. In this section, I will give only a brief overview of this approach, as I have not found any way to generalize or make use of it in the construction of maximally even tilings.

Definition 5. For a finite set of non-negative integers A, we define a corresponding polynomial $A(x) = \sum_{k\in A} x^k$.

Definition 6. For $n \geq 1$, the n-th cyclotomic polynomial $\Phi_n(x)$ is a polynomial with integer coefficients defined by

$$\Phi_n(x) = \prod_{\substack{k\in[1,n]\\ \gcd(k,n)=1}} (x - e^{2i\pi\frac{k}{n}}).$$

Proposition 7. [9] $(A, B)_n$ is a full tiling if and only if $A(1)B(1) = n$ and $\Phi_t(x)$ divides $A(x)$ or $B(x)$ for each factor $t > 1$ of n.

Proposition 8. [9] Let $n \geq 1$ and $A \subseteq [0, n)$. Let S_A^n to be the set of prime powers s dividing n such that $\Phi_s(x)$ divides $A(X)$. Define the following properties.

- T1: $A(1) = \prod_{s\in S_A^n} \Phi_s(1)$
- T2: If $s_1, s_2, ..., s_m \in S_A^n$ are powers of distinct primes, then $\Phi_{s_1 s_2 \ldots s_m}(x)$ divides $A(x)$.

The following hold.

1. If T1 and T2 hold, then A has a modulo n tiling complement.
2. If A has a modulo n tiling complement, then T1 holds.
3. If A has a modulo n tiling complement and $|A|$ has no more than two distinct prime factors, then T2 holds.

Conjecture 9. [9] A has a modulo n tiling complement if and only if T1 and T2 hold.

Example 10. Consider the case of $n = 24$ and $A = \{0, 1, 2, 4, 5, 6\}$. It should be clear that A has no modulo 24 tiling complement. Note that $A(x) = \Phi_8(x)\Phi_3(x)$, so $S_A^n = \{8, 3\}$. Condition T1 is satisfied since $A(1) = 6 = 2 \cdot 3 = \Phi_8(1)\Phi_3(1)$. However, condition T2 is not satisfied, since $\Phi_{24}(x)$ does not divide $A(x)$. This example shows that T1 (by itself) is not sufficient for the existence of a tiling complement.

Example 11. Consider the case of $n = 24$ and $A = \{0, 1, 3, 4, 5, 6\}$. It should be clear that A has no modulo 24 tiling complement. Note that $A(x) = \Phi_2(x)(1 + x^3 + x^5)$, where $1 + x^3 + x^5$ has no cyclotomic factors, so $S_A^n = \{2\}$. Condition T2 is trivially satisfied. However, condition T1 is not satisfied, since $A(1) = 6$ but $\Phi_2(1) = 2$. This example shows that T2 (by itself) is not sufficient for the existence of a tiling complement.

Example 12. Consider the case of $n = 24$ and $A = \{0, 1, 2, 6, 7, 8\}$. Note that $A(x) = \Phi_4(x)\Phi_3(x)\Phi_{12}(x)$, so $S_A^n = \{4, 3\}$. T1 is satisfied since $A(1) = 6 = 2 \cdot 3 = \Phi_4(x)\Phi_3(x)$, and T2 is satisfied since $\Phi_{12}(x)$ divides $A(x)$. Therefore, a modulo 24 tiling complement should exist. In this case, it is easy to see that $\{0, 3, 12, 15\}$ works.

Conditions T1 and T2 can be used to design algorithms for constructing full tilings [4, 5, 9, 15]. However, since these algorithms are difficult to describe and their approaches do not extent to the maximally even tilings that we are presently interested in, we will not elaborate on them here.

2.3 De Bruijn-type tilings

In this section we look at a construction due to De Bruijn [16] that does not produce all full tilings for a given value of n but is extremely simple and efficient to carry out.

Proposition and Definition 13. Let $\{c_i\}_{i=1}^{j} \subseteq [2, \infty)$, and choose a partition of $[1, j]$ into two sets E and F. If we let $d_0 = 1$ and $d_i = \Pi_{i=1}^{j} c_i$ for $i \in [1, j]$ and

$$A = \sum_{i \in E} d_{i-1}[0, c_i)$$
$$B = \sum_{i \in F} d_{i-1}[0, c_i)$$
$$n = d_j,$$

then $(A, B)_n$ is a full tiling. (We define $\sum_{i \in \emptyset} d_{i-1}[0, c_i) = \{0\}$.) We will call a full tiling of this form a De Bruijn-type tiling.

Example 14. Define $\{c_i\}_{i=1}^{4}$ as follows

i	1	2	3	4
c_i	2	4	3	2

and partition $[1, 4]$ into $E = \{1, 3\}$ and $F = \{2, 4\}$. Carrying out the calculations indicated in Proposition and Definition 13, we have

i	0	1	2	3	4
d_i	1	2	8	24	48

and

$$A = d_0[0, c_1) + d_2[0, c_3) = 1[0, 2) + 8[0, 3) = \{0, 1, 8, 9, 16, 17\}$$
$$B = d_1[0, c_2) + d_3[0, c_4) = 2[0, 4) + 24[0, 2) = \{0, 2, 4, 6, 24, 26, 28, 30\}$$
$$n = d_4 = 48.$$

One can verify that $(A, B)_n$ is a full tiling.

Chapter 3

MAXIMALLY EVEN RHYTHMS

The concept of "maximally even sets" was introduced by John Clough and Jack Douthett in 1991 [8]. Interest in the closely related topic of well-formed scales [7] also picked up around the same time. The goal of this chapter is to develop the theory of maximally even set and rhythms that will be used in the remainder of the dissertation.

3.1 Definition and existence by construction

There exists a surprising variety of equivalent characterizations of maximally even sets and rhythms [10]. For the purposes of this dissertation, the following will suffice.

Proposition and Definition 15. Let $k \leq n$, and let r be a strictly increasing function from $[0, k)$ onto $X \subseteq [0, n)$. We will say that r is a k-note n-beat rhythm. For $i \in \mathbb{Z} \setminus [0, k)$, we will take $r(i)$ to mean $r(\mathrm{mod}_k(i))$. The following are equivalent.

1. $r(j) - r(i) \in \left\{ \left\lfloor \frac{(j-i)n}{k} \right\rfloor, \left\lceil \frac{(j-i)n}{k} \right\rceil \right\}$ for all $0 \leq i \leq j < k$.
2. $r(j) - r(i) \in \left\{ \left\lfloor \frac{(j-i)n}{k} \right\rfloor, \left\lceil \frac{(j-i)n}{k} \right\rceil \right\}$ for all $i, j \in [0, k)$.
3. $\mathrm{mod}_n(r(j) - r(i)) \in \left\{ \left\lfloor \frac{\mathrm{mod}_k(j-i)n}{k} \right\rfloor, \left\lceil \frac{\mathrm{mod}_k(j-i)n}{k} \right\rceil \right\}$ for all $i, j \in \mathbb{Z}$.

If these conditions hold, then we will say that r is a k-note n-beat maximally even rhythm and that its range X is an order k modulo n maximally even set.

Proof. We will show that $1 \Rightarrow 2 \Rightarrow 3 \Rightarrow 1$.

- Suppose that condition 1 holds, and let $i, j \in [0, k)$. If $i \leq j$, then condition 2 follows directly from condition 1. On the other hand, if $i > j$, then

$$\begin{aligned} r(j) - r(i) &= -(r(i) - r(j)) \\ &\in -\left\{ \left\lfloor \frac{(i-j)n}{k} \right\rfloor, \left\lceil \frac{(i-j)n}{k} \right\rceil \right\} \\ &= \left\{ \left\lceil \frac{(j-i)n}{k} \right\rceil, \left\lfloor \frac{(j-i)n}{k} \right\rfloor \right\}. \end{aligned}$$

since $-\lfloor x \rfloor = \lceil -x \rceil$ and $-\lceil x \rceil = \lfloor -x \rfloor$. So condition 2 holds.

- Suppose that condition 2 holds, and let $i, j \in \mathbb{Z}$. According to the definition above $r(j) = r(j')$ where $j' = \mathrm{mod}_k(j)$ and $r(i) = r(i')$ where $i' = \mathrm{mod}_k(i)$. If $i' \leq j'$, then

$$\begin{aligned} \mathrm{mod}_n(r(j) - r(i)) &= r(j') - r(i') \\ &\in \left\{ \left\lfloor \frac{(j'-i')n}{k} \right\rfloor, \left\lceil \frac{(j'-i')n}{k} \right\rceil \right\} \\ &= \left\{ \left\lfloor \frac{\mathrm{mod}_k(j-i)n}{k} \right\rfloor, \left\lceil \frac{\mathrm{mod}_k(j-i)n}{k} \right\rceil \right\}. \end{aligned}$$

If $i' > j'$, then

$$\begin{aligned} \mathrm{mod}_n(r(j) - r(i)) &= r(j') - r(i') + n \\ &\in \left\{ \left\lfloor \frac{(j'-i')n}{k} \right\rfloor, \left\lceil \frac{(j'-i')n}{k} \right\rceil \right\} + n \\ &= \left\{ \left\lceil \frac{(j'-i'+k)n}{k} \right\rceil, \left\lfloor \frac{(j'-i'+k)n}{k} \right\rfloor \right\} \\ &= \left\{ \left\lceil \frac{\mathrm{mod}_k(j-i))n}{k} \right\rceil, \left\lfloor \frac{\mathrm{mod}_k(j-i)n}{k} \right\rfloor \right\}. \end{aligned}$$

So condition 3 holds.

- Suppose that condition 3 holds, and let $0 \leq i \leq j < k$. Then

$$\begin{aligned} r(j) - r(i) &= \operatorname{mod}_n(r(j) - r(i)) \\ &\in \left\{ \left\lfloor \frac{\operatorname{mod}_k(j-i)n}{k} \right\rfloor, \left\lceil \frac{\operatorname{mod}_k(j-i)n}{k} \right\rceil \right\} \\ &= \left\{ \left\lfloor \frac{(j-i)n}{k} \right\rfloor, \left\lceil \frac{(j-i)n}{k} \right\rceil \right\}. \end{aligned}$$

  So condition 1 holds.

□

Next we will show that there exists a k-note n-beat maximally even rhythm for any $k \leq n$. We will do so by giving an explicit construction noted in [8].

Proposition 16. For $n \geq k$, the following defines a k-note n-beat maximally even rhythm.

$$r^{\text{flr}}_{k,n}(i) = \left\lfloor \frac{in}{k} \right\rfloor$$

Proof. First recall that

$$\lfloor x \rfloor + \lfloor y \rfloor \leq \lfloor x + y \rfloor \leq \lfloor x \rfloor + \lfloor y \rfloor + 1$$

for all $x, y \in \mathbb{R}$.

Let $i, j \in [0, k)$. Then

$$\begin{aligned} r^{\text{flr}}_{k,n}(j) - r^{\text{flr}}_{k,n}(i) &= \left\lfloor \frac{jn}{k} \right\rfloor - \left\lfloor \frac{in}{k} \right\rfloor \\ &= \left\lfloor \frac{(j-i)n}{k} + \frac{in}{k} \right\rfloor - \left\lfloor \frac{in}{k} \right\rfloor \\ &\geq \left\lfloor \frac{(j-i)n}{k} \right\rfloor + \left\lfloor \frac{in}{k} \right\rfloor - \left\lfloor \frac{in}{k} \right\rfloor \\ &= \left\lfloor \frac{(j-i)n}{k} \right\rfloor. \end{aligned}$$

If $\frac{(j-i)n}{k}$ is an integer, then

$$\begin{aligned} r^{\text{flr}}_{k,n}(j) - r^{\text{flr}}_{k,n}(i) &= \left\lfloor \frac{(j-i)n}{k} + \frac{in}{k} \right\rfloor - \left\lfloor \frac{in}{k} \right\rfloor \\ &= \frac{(j-i)n}{k} + \left\lfloor \frac{in}{k} \right\rfloor - \left\lfloor \frac{in}{k} \right\rfloor \\ &= \frac{(j-i)n}{k} \\ &= \left\lceil \frac{(j-i)n}{k} \right\rceil . \end{aligned}$$

If $\frac{(j-i)n}{k}$ is not an integer, then

$$\begin{aligned} r^{\text{flr}}_{k,n}(j) - r^{\text{flr}}_{k,n}(i) &= \left\lfloor \frac{(j-i)n}{k} + \frac{in}{k} \right\rfloor - \left\lfloor \frac{in}{k} \right\rfloor \\ &\leq \left\lfloor \frac{(j-i)n}{k} \right\rfloor + \left\lfloor \frac{in}{k} \right\rfloor + 1 - \left\lfloor \frac{in}{k} \right\rfloor \\ &= \left\lfloor \frac{(j-i)n}{k} \right\rfloor + 1 \\ &= \left\lceil \frac{(j-i)n}{k} \right\rceil . \end{aligned}$$

So $r^{\text{flr}}_{k,n}$ is a k-note n-beat maximally even rhythm. $\square$

Proposition 17. For $n \geq k$, the following defines an order k modulo n maximally even set.

$$X^{\text{flr}}_{k,n}(i) = \left\{ \left\lfloor \frac{in}{k} \right\rfloor : i \in [0, k) \right\}$$

3.2 Uniqueness up to translation and another construction

In this section we will show that there is a unique k-note n-beat maximally even rhythm up to translation. This was proved in [10], but we will prove it here in a different way, essentially using the fact that when k and n are relatively prime, any k-note n-beat maximally even rhythm is "well-formed" [7].

Proposition and Definition 18. Let r be a k-note n-beat rhythm and $c \in \mathbb{Z}$. It is easy to see that there exists $d \in [0, k)$ such that $r'(i) = \mathrm{mod}_n(r(i+d) - c)$ defines the k-note n-beat rhythm. We will say that r' a modulo n translation of r.

In particular, if $d \in \mathbb{Z}$, then the function defined by $r'(i) = \mathrm{mod}_n(r(i+d) - r(d))$ is a modulo n translation of r.

Proposition 19. If r is a k-note n-beat maximally even rhythm, then so are its modulo n translations.

Proof. Let $c \in [0, n)$, and choose $d \in [0, k)$ such that $r'(i) = \mathrm{mod}_n(r(i+d) - c)$ is a k-note n-beat maximally even rhythm. For $i, j \in \mathbb{Z}$, we have the following.

$$\begin{aligned}\mathrm{mod}_n(r'(j) - r'(i)) &= \mathrm{mod}_n(r(\mathrm{mod}_k(j+d)) - r(\mathrm{mod}_k(i+d))) \\ &\in \left\{ \left\lfloor \frac{\mathrm{mod}_k(j-i)n}{k} \right\rfloor, \left\lceil \frac{\mathrm{mod}_k(j-i)n}{k} \right\rceil \right\}\end{aligned}$$

So r' is a k-note n-beat maximally even rhythm. □

Proposition and Definition 20. Let X be an order k subset of $[0, n)$. For $c \in \mathbb{Z}$, we will call $X' = \mathrm{mod}_n(X - c)$ a modulo n translation of X. If X is an order k modulo n maximally even set, then so is X'.

Proposition 21. If r is a k-note n-beat maximally even rhythm and $h \in [0, k)$, then there are exactly $k - \mathrm{mod}_k(hn)$ values of $i \in [0, k)$ such that

$$\mathrm{mod}_n(r(i+h) - r(i)) = \left\lfloor \frac{hn}{k} \right\rfloor$$

Proof. If $hn \equiv_k 0$, then $\left\lfloor \frac{hn}{k} \right\rfloor = \left\lceil \frac{hn}{k} \right\rceil = \frac{hn}{k}$. So $\mathrm{mod}_n(r(i+h) - r(i)) = \left\lfloor \frac{hn}{k} \right\rfloor$ for all $k = k - \mathrm{mod}_k(hn)$ values of $i \in [0, k)$.

Now suppose that $hn \not\equiv_k 0$. First note that $\sum_{i=0}^{k-1} \mathrm{mod}_n(r(i+h) - r(i)) \equiv_n 0$. Since r is a k-note n-beat maximally even rhythm, each term in the series above is either $\left\lfloor \frac{hn}{k} \right\rfloor$ or

$\left\lceil \frac{hn}{k} \right\rceil$. Let a be the number of terms that are equal to $\left\lfloor \frac{hn}{k} \right\rfloor$. Then we have

$$a\left\lfloor \frac{hn}{k} \right\rfloor + (k-a)\left\lceil \frac{hn}{k} \right\rceil \equiv_n 0$$

$$a\left(\left\lceil \frac{hn}{k} \right\rceil - 1\right) + (k-a)\left(\left\lceil \frac{hn}{k} \right\rceil\right) \equiv_n 0$$

$$a \equiv_n k\left\lceil \frac{hn}{k} \right\rceil$$

$$a \equiv_n k\left(\frac{hn + \mathrm{mod}_k(-hn)}{k}\right)$$

$$a \equiv_n \mathrm{mod}_k(-hn)$$

$$a = k - \mathrm{mod}_k(hn)$$

□

Proposition 22. Let $k < n$ with $\gcd(k,n) = 1$, and let $\alpha \in [0,n)$ be a modulo n multiplicative inverse of k. Then $r^{\mathrm{inv}}_{k,n}(i) = \mathrm{mod}_n(\alpha \mathrm{mod}_k(-ni))$ defines a k-note n-beat maximally even rhythm, and every k-note n-beat maximally even rhythm is a modulo n translation of $r^{\mathrm{inv}}_{k,n}$.

Proof. Let r be a k-note n-beat maximally even rhythm, and choose $\beta \in [0,k)$ such that $\alpha k - \beta n = 1$. By the previous proposition, there is $k - \mathrm{mod}_k(\beta n) = k - (k-1) = 1$ value of $i \in [0,k)$ such that $\mathrm{mod}_n(r(i+\beta) - r(i)) = \left\lfloor \frac{\beta n}{k} \right\rfloor$. Call that value i_0. Let $d = \beta + i_0$ and $r'(i) = \mathrm{mod}_n(r(i+d) - r(d))$. Then for $j = 0, 1, ..., k-2$, we have $\mathrm{mod}_k((j+1)\beta + i_0) \neq i_0$

and

$$\begin{aligned} r'((j+1)\beta) - r'(j\beta) &\equiv_n r((j+1)\beta + d) - r(j\beta + d) \\ &\equiv_n r((j+1)\beta + i_0 + \beta) - r(j\beta + i_0 + \beta) \\ &\equiv_n r((j+1)\beta + i_0 + \beta) - r((j+1)\beta + i_0) \\ &\equiv_n \left\lceil \frac{\beta n}{k} \right\rceil \\ &\equiv_n \left\lceil \alpha - \frac{1}{k} \right\rceil \\ &\equiv_n \alpha \end{aligned}$$

So for $i \in [0, k)$ we have

$$\begin{aligned} r'(\beta i) - r'(0) &\equiv_n \sum_{j=0}^{i-1} \left[r'(\beta(j+1)) - r'(\beta j) \right] \\ &\equiv_n \sum_{j=0}^{i-1} \alpha \\ &\equiv_n \alpha i \\ &\equiv_n \alpha \mathrm{mod}_k(-n\beta i) \end{aligned}$$

So $r'(i) - r'(0) = \mathrm{mod}_n(\alpha \mathrm{mod}_k(-ni)) = r^{\mathrm{inv}}_{k,n}(i)$ for all $i \in [0, k)$. Since r is a modulo n translation of r', which is a modulo n translation of $r' - r'(0)$, which is equal to $r^{\mathrm{inv}}_{k,n}(i)$, we find that r is a modulo n translation of $r^{\mathrm{inv}}_{k,n}(i)$. Since $r^{\mathrm{inv}}_{k,n}(i)$ is a modulo n translation of r, we find that $r^{\mathrm{inv}}_{k,n}(i)$ is a k-note n-beat maximally even rhythm. □

Proposition 23. Let $k < n$ with $\gcd(k, n) = 1$, and let $\alpha \in [0, n)$ be a modulo n multiplicative inverse of k. Then $X^{\mathrm{inv}}_{k,n} = \mathrm{mod}_n(\alpha[0, k))$ is an order k modulo n maximally even set, and every order k modulo n maximally even set is a modulo n translation of $X^{\mathrm{inv}}_{k,n}$.

The next few propositions follow from the more detailed investigation of the structural properties of maximally even rhythms found in [11].

Proposition 24. Let r be a k-note n-beat maximally even rhythm. Let $g \geq 1$ be a common divisor of k and n, and let $k' = \frac{k}{g}$ and $n' = \frac{n}{g}$. Let r' be the restriction of r to $[0, k')$. Then r' is a k'-note n'-beat maximally even rhythm, and

$$r(i) = n' \left\lfloor \frac{i}{k'} \right\rfloor + r'(i)$$

for all $i \in [0, k)$. (Recall that $r'(i)$ means $r'(\mathrm{mod}_{k'}(i))$)

Proof. For $i, j \in [0, k')$, we have

$$r'(j) - r'(i) = r(j) - r(i) \in \left\{ \left\lfloor \frac{(j-i)n}{k} \right\rfloor, \left\lceil \frac{(j-i)n}{k} \right\rceil \right\} = \left\{ \left\lfloor \frac{(j-i)n'}{k'} \right\rfloor, \left\lceil \frac{(j-i)n'}{k'} \right\rceil \right\},$$

so r' is a k'-note n'-beat maximally even rhythm. For $i \in [0, k)$ we have

$$r(i) - r(\mathrm{mod}_{k'}(i)) \in \left\{ \left\lfloor \frac{(i - \mathrm{mod}_{k'}(i))n}{k} \right\rfloor, \left\lceil \frac{(i - \mathrm{mod}_{k'}(i))n}{k} \right\rceil \right\},$$

but

$$\frac{(i - \mathrm{mod}_{k'}(i))n}{k} = \frac{\left(\left\lfloor \frac{i}{k'} \right\rfloor k' \right) n'}{k'} = n' \left\lfloor \frac{i}{k'} \right\rfloor,$$

so

$$r(i) - r(\mathrm{mod}_{k'}(i)) = n' \left\lfloor \frac{i}{k'} \right\rfloor,$$

and

$$r(i) = n' \left\lfloor \frac{i}{k'} \right\rfloor + r'(i).$$

□

Proposition 25. Let X be an order k modulo n maximally even set. Let $g \geq 1$ be a common divisor of k and n, and let $k' = \frac{k}{g}$ and $n' = \frac{n}{g}$. Let $X' = X \cap [0, n')$. Then

$X = \bigcup_{j\in[0,g)}(n'j + X')$ and X' is an order k' modulo n' maximally even set.

Proposition 26. Let $k' \leq n'$, $g \geq 1$, $k = k'g$, and $n = n'g$. Let r' be a k'-note n'-beat maximally even rhythm, and define

$$r(i) = n' \left\lfloor \frac{i}{k'} \right\rfloor + r'(i)$$

for $i \in [0, k)$. Then r is a k-note n-beat maximally even rhythm.

Proof. For $i, j \in [0, n)$, let $i' = \mathrm{mod}_{n'}(i)$ and $j' = \mathrm{mod}_{n'}(j)$, and note that

$$\begin{aligned}
r(j) - r(i) &= \left(n' \left\lfloor \frac{j}{k'} \right\rfloor + r'(j)\right) - \left(n' \left\lfloor \frac{i}{k'} \right\rfloor + r'(i)\right) \\
&= \left(n' \left(\frac{j-j'}{k'}\right) + r'(j')\right) - \left(n' \left(\frac{i-i'}{k'}\right) + r'(i')\right) \\
&= \frac{(j-j'-i+i')\,n'}{k'} + r'(j') - r'(i') \\
&\in \frac{(j-j'-i+i')\,n'}{k'} + \left\{\left\lfloor \frac{(j'-i')n'}{k'} \right\rfloor, \left\lceil \frac{(j'-i')n'}{k'} \right\rceil\right\} \\
&\in \left\{\left\lfloor \frac{(j-i)n'}{k'} \right\rfloor, \left\lceil \frac{(j-i)n'}{k'} \right\rceil\right\} \\
&\in \left\{\left\lfloor \frac{(j-i)n}{k} \right\rfloor, \left\lceil \frac{(j-i)n}{k} \right\rceil\right\}.
\end{aligned}$$

So r is a k-note n-beat maximally even rhythm. $\square$

Proposition 27. Let $k' \leq n'$, $g \geq 1$, $k = k'g$, and $n = n'g$. Let X' be an order k' modulo n'-beat maximally even set, and define

$$X = \bigcup_{j\in[0,g)} (n'j + X').$$

Then X is an order k modulo n-beat maximally even set.

Proposition 28. Let g be a divisor of n, and let $n' = \frac{n}{g}$. Let X' be a subset of $[0, n')$ and $X = \cup_{j\in[0,g)}(n'j + X')$. Let $c \in \mathbb{Z}$, and let $u \in \mathbb{Z}$ be a modulo n unit. Then

$$\mathrm{mod}_n(uX - c) = \bigcup_{j\in[0,g)} (n'j + \mathrm{mod}_{n'}(uX' - c)).$$

Proof. Let $x_0 \in \mathrm{mod}_n(uX - c)$. Then we can write $x_0 = ux - c - mn$ for some $x \in X$ and $m \in \mathbb{Z}$. Then we can write $x = n'j_0 + x'$ for some $j_0 \in [0, g)$ and $x' \in X'$. So

$$\begin{aligned} x_0 &= u(n'j_0 + x') - c + mn \\ &= un'j_0 + ux' - c + mgn' \\ &\equiv_{n'} ux' - c, \end{aligned}$$

and since $x_0 \in [0, n)$, we can write $x_0 = n'j_1 + \mathrm{mod}_{n'}(ux' - c)$ where $j_1 \in [0, g)$. So $x_0 \in \bigcup_{j\in[0,g)}(n'j + \mathrm{mod}_{n'}(uX' - c))$. Therefore, $\mathrm{mod}_n(uX - c) \subseteq \bigcup_{j\in[0,g)}(n'j + \mathrm{mod}_{n'}(uX' - c))$. Since both of these sets are of the same finite cardinality, $|X|$, equality holds. □

Proposition 29. Let $n > k$ with $g = \gcd(k, n) > 1$, and let $k' = \frac{k}{g}$ and $n' = \frac{n}{g}$. Then

$$X^{\mathrm{inv}}_{k,n} = \bigcup_{j\in[0,g)} \left(n'j + X^{\mathrm{inv}}_{n',k'}\right)$$

defines a order k modulo n maximally even set, and every order k modulo n maximally even set is a modulo n translation of $X^{\mathrm{inv}}_{k,n}$.

Proof. Let X be an order k modulo n maximally even set. Let $X' = X \cap [0, n')$. By Proposition 25, X' is an order k' modulo n' maximally even set, and

$$X = \bigcup_{j\in[0,g)} (n'j + X')\,.$$

By Proposition 23, X' is a modulo n' translation of $X^{\mathrm{inv}}_{n',k'}$, so we can choose $c \in [0, n')$ such that

$$X' = \mathrm{mod}_{n'}\left(X^{\mathrm{inv}}_{n',k'} - c\right).$$

Using the previous proposition, we find that

$$\begin{aligned}
X &= \bigcup_{j\in[0,g)} \left(n'j + \mathrm{mod}_{n'}\left(X^{\mathrm{inv}}_{n',k'} - c\right)\right) \\
&= \mathrm{mod}_n\left(\bigcup_{j\in[0,g)} \left(n'j + X^{\mathrm{inv}}_{n',k'}\right) - c\right) \\
&= \mathrm{mod}_n\left(X^{\mathrm{inv}}_{n,k} - c\right)
\end{aligned}$$

Therefore, X is a modulo n translation of $X^{\mathrm{inv}}_{n,k}$, which means that $X^{\mathrm{inv}}_{n,k}$ is an order k modulo n maximally even set. □

3.3 More evenness preserving operations

In this section we demonstrate a few more operations that preserve evenness. The following four propositions were proved in [12].

Proposition 30. If X is an order k modulo n maximally even set, then $\mathrm{mod}_n(-X)$ is an order k modulo n maximally even set.

Proof. Let r be the k-note n-beat rhythm with range X.

- If $0 \in X$, then the k-note n-beat rhythm with range $\mathrm{mod}_n(-X)$ is defined by $r'(i) = \mathrm{mod}_n(-r(\mathrm{mod}_k(-i)))$, and r' is maximally even, since for $i, j \in \mathbb{Z}$, we have the following.

$$\begin{aligned}
\mathrm{mod}_n(r'(j) - r'(i)) &= \mathrm{mod}_n(r(\mathrm{mod}_k(-i)) - r(\mathrm{mod}_k(-j))) \\
&\in \left\{\left\lfloor \frac{\mathrm{mod}_k(j-i)n}{k} \right\rfloor, \left\lceil \frac{\mathrm{mod}_k(j-i)n}{k} \right\rceil\right\}
\end{aligned}$$

- If $0 \notin X$, then the k-note n-beat rhythm with range $\mathrm{mod}_n(-X)$ is defined by $r'(i) = n - r(k-1-i)$, and r' is maximally even, since for $i, j \in [0, k)$, we have the following.

$$\begin{aligned} r'(j) - r'(i) &= r(k-1-i) - r(k-1-j) \\ &\in \left\{ \left\lfloor \frac{(j-i)n}{k} \right\rfloor, \left\lceil \frac{(j-i)n}{k} \right\rceil \right\} \end{aligned}$$

□

Proposition 31. If X is an order k modulo n maximally even set, then $[0, n) \setminus X$ is an order $n - k$ modulo n maximally even set.

Proof. Since every order k modulo n maximally even set is a modulo n translation of $X^{\text{inv}}_{k,n}$, it will suffice to show that $[0, n) \setminus X^{\text{inv}}_{k,n}$ is an order $n - k$ modulo n maximally even set.

Let $g = \gcd(k, n)$, $k' = \frac{k}{g}$, and $n' = \frac{n}{g}$. Let α be a modulo n' multiplicative inverse of k', and recall that

$$X^{\text{inv}}_{k',n'} = \mathrm{mod}_{n'}(\alpha[0, k'-1]).$$

Next note that $-\alpha$ is a modulo n' multiplicative inverse of $n' - k'$, so

$$X^{\text{inv}}_{n'-k',n'} = \mathrm{mod}_{n'}\left(-\alpha[0, n'-k'-1]\right).$$

By Propositions 20 and 30, the following is an order $n' - k'$ modulo n' maximally even set.

$$\begin{aligned} \mathrm{mod}_{n'}\left(\alpha k' - X^{\text{inv}}_{n'-k',n'}\right) &= \mathrm{mod}_{n'}\left(\alpha k' - \mathrm{mod}_{n'}\left(-\alpha[0, n'-k'-1]\right)\right) \\ &= \mathrm{mod}_{n'}\left(\alpha k + \alpha[0, n'-k'-1]\right) \\ &= \mathrm{mod}_{n'}\left(\alpha[k', n'-1]\right) \\ &= [0, n'-1] \setminus X^{\text{inv}}_{k',n'} \end{aligned}$$

Finally note that

$$\begin{aligned}[0,n-1]\setminus X^{\mathrm{inv}}_{k,n} &= [0,n-1]\setminus\left(\bigcup_{j\in[0,g)}\left(n'j+X^{\mathrm{inv}}_{n',k'}\right)\right)\\ &= \bigcup_{j\in[0,g)}\left(n'j+[0,n'-1]\setminus X^{\mathrm{inv}}_{n',k'}\right)\end{aligned}$$

is an order $n-k$ modulo n maximally even set. □

Proposition 32. Let r be a k-note n-beat maximally even rhythm, where $k=k_1k_2$. Then $r_m(i)=r(k_2i+m)$ defines a k_1-note n-beat maximally even rhythm for each $m\in[0,k_2)$.

Proof. Let $i,j\in[0,k_1)$.

$$\begin{aligned}r'(j)-r'(i) &= r(k_2j+m)-r(k_2i+m)\\ &\in \left\{\left\lfloor\frac{(k_2j-k_2i)n}{k}\right\rfloor, \left\lceil\frac{(k_2j-k_2i)n}{k}\right\rceil\right\}\\ &= \left\{\left\lfloor\frac{(j-i)n}{k_1}\right\rfloor, \left\lceil\frac{(j-i)n}{k_1}\right\rceil\right\}\end{aligned}$$

So r' is a k_1-note n-beat maximally even rhythm. □

Proposition 33. Let X be an order k modulo n maximally even set, and let $k=k_1k_2$. Then

$$X=\bigcup_{i=1}^{k_2}X_i,$$

where X_i are pairwise disjoint k_1-note n-beat maximally even sets.

Next we will introduce a pair of an evenness preserving operations that have not previously been described in the literature on maximally even rhythms. These operation will be very important in the context of maximally even tilings.

Example 34. Consider the $[k=10]$-note $[n=13]$-beat maximally even rhythm r defined below.

i	0	1	2	3	4	5	6	7	8	9
$r(i)$	0	1	2	4	5	6	7	9	10	11

The table below give the values of $\mathrm{id}_{[0,k)} + r$.

i	0	1	2	3	4	5	6	7	8	9
$i + r(i)$	0	2	4	7	9	11	13	16	18	20

One can check that this is a [k=10]-note [k+n=23]-beat maximally even rhythm. From a musical perspective, this operation inserts a beat of rest between each pair of consecutive notes. We will refer to this operation as *step expansion*.

Now consider the table below that gives the values of $(1 + \lceil n/k \rceil)\mathrm{id}_{[0,k)} - r$.

i	0	1	2	3	4	5	6	7	8	9
$3i - r(i)$	0	2	4	5	7	9	11	12	14	16

One can check that this is a [k=10]-note [$(1+\lceil n/k \rceil)k$–n=17]-beat maximally even rhythm. From a musical perspective, this operation swaps the steps between consecutive notes. (What was originally short-short-long-short-... becomes long-long-short-long...) We will refer to this operation as *step swapping*.

Proposition 35. If r is a k-note n-beat maximally even rhythm, then

- $h\mathrm{id}_{[0,k)} + r$ is a k-note $(hk + n)$-beat maximally even rhythm for $h \geq 1 - \left\lfloor \frac{n}{k} \right\rfloor$, and
- $h\mathrm{id}_{[0,k)} - r$ is a k-note $(hk - n)$-beat maximally even rhythm for $h \geq 1 + \left\lceil \frac{n}{k} \right\rceil$.

Proof. We will prove the first part only. The second can be proved in a similar way. First notice that

$$hk + n \geq \left(1 - \left\lfloor \frac{n}{k} \right\rfloor\right) k + n = k + \left(n - \left\lfloor \frac{n}{k} \right\rfloor k\right) = k + \mathrm{mod}_k(n) \geq k.$$

Also, $h\mathrm{id}_{[0,k)} + r$ is increasing, since it is the sum of increasing functions. Finally, for $0 \leq$

$i < j < k$ we have

$$\begin{aligned}
(h\mathrm{id}_{[0,k)} + r)(j) - (h\mathrm{id}_{[0,k)} + r)(i) &= (hj + r(j)) - (hi + r(i)) \\
&= h(j-i) + (r(j) - r(i)) \\
&\in h(j-i) + \left\{ \left\lfloor \frac{(j-i)n}{k} \right\rfloor, \left\lceil \frac{(j-i)n}{k} \right\rceil \right\} \\
&= \left\{ \left\lfloor \frac{(j-i)(hk+n)}{k} \right\rfloor, \left\lceil \frac{(j-i)(hk+n)}{k} \right\rceil \right\}.
\end{aligned}$$

□

3.4 Unit multiplication of maximally even sets

In this section we will examine the affect of unit multiplication on maximally even set and rhythms. To my knowledge, this operation has not previously been considered by researchers.

Example 36. Consider the following 9-note 15-beat maximally even set.

$$X^{\mathrm{flr}}_{9,15} = \{0, 1, 3, 5, 6, 8, 10, 11, 13\}$$

If we multiply by the modulo 15 unit 4, we get the following, which is yet another 9-note 15-beat maximally even set.

$$\mathrm{mod}_{15}(4X^{\mathrm{flr}}_{9,15}) = \{0, 2, 4, 5, 7, 9, 10, 12, 14\}$$

But if we multiply by the modulo 15 unit 8, we get the following, which is no longer a 9-note 15-beat maximally even set.

$$\mathrm{mod}_{15}(8X^{\mathrm{flr}}_{9,15}) = \{0, 3, 4, 5, 8, 9, 10, 13, 14\}$$

The next few propositions describe under what circumstances unit multiplication preserves maximal evenness.

Proposition 37. Let X be an order k modulo n maximally even set, and let u be a modulo n unit. Let $g = \gcd(k, n)$, $k' = \frac{k}{g}$, and $n' = \frac{n}{g}$. If $k' = 1$, $k' = n' - 1$, or $u \equiv_{n'} \pm 1$, then $\mathrm{mod}_n(uX)$ is an order k modulo n maximally even set.

Proof. It will suffice to prove the result for the case of $X = X^{\mathrm{inv}}_{n,k}$. By Propositions 29 and 28,

$$\begin{aligned} \mathrm{mod}_n\left(uX^{\mathrm{inv}}_{n,k}\right) &= \mathrm{mod}_n\left(u \bigcup_{j\in[0,g)} \left(n'j + X^{\mathrm{inv}}_{n',k'}\right)\right) \\ &= \bigcup_{j\in[0,k)} \left(n'j + \mathrm{mod}_n\left(uX^{\mathrm{inv}}_{n',k'}\right)\right) \\ &= \bigcup_{j\in[0,k)} \left(n'j + \mathrm{mod}_{n'}(u\alpha[0, k'-1])\right), \end{aligned}$$

where α is a modulo n' multiplicative inverse of k'. By Proposition 27, we just need to show that $\mathrm{mod}_{n'}(u\alpha[0, k'-1])$ is an order k' modulo n' maximally even set.

- If $k' = 1$, then $\mathrm{mod}_{n'}(u\alpha[0, k'-1]) = \{0\}$ is an order k' modulo n' maximally even set.
- If $k' = n' - 1$, then

  $$\mathrm{mod}_{n'}(u\alpha[0, k'-1]) = \mathrm{mod}_{n'}(u\alpha[0, n'-2]) = [0, n') \setminus \{\mathrm{mod}_n(u\alpha(n'-1))\}$$

  is an order k' modulo n' maximally even set.
- If $u \equiv_{n'} 1$, then $\mathrm{mod}_{n'}(u\alpha[0, k'-1]) = X^{\mathrm{inv}}_{n',k'}$ is an order k' modulo n' maximally even set.

- If $u \equiv_{n'} -1$, then

$$\begin{aligned}
\mathrm{mod}_{n'}(u\alpha[0,k'-1]) &= \mathrm{mod}_{n'}(-\alpha[0,k'-1]) \\
&= \mathrm{mod}_{n'}(\alpha[-k'+1,0]) \\
&= \mathrm{mod}_{n'}(\alpha[0,k-1]+\alpha(k'-1)) \\
&= \mathrm{mod}_{n'}(X^{\mathrm{inv}}_{n',k'}+\alpha(k'-1))
\end{aligned}$$

  is an order k' modulo n' maximally even set.

□

Proposition 38. Let $1 < k < n-1$ with $\gcd(k,n) = 1$. Let X be an order k modulo n maximally even set. Then

$$|X \cap \mathrm{mod}_n(X+y)| = k-1$$

if and only if $ky \equiv_n \pm 1$.

Proof. It will suffice to prove the result in the case of $X = X^{\mathrm{inv}}_{k,n}$. Let α be a modulo n multiplicative inverse of k, and note that

$$\begin{aligned}
|X \cap \mathrm{mod}_n(X+y)| &= |\mathrm{mod}_n(\alpha[0,k-1]) \cap \mathrm{mod}_n(\alpha[0,k-1]+y)| \\
&= |\mathrm{mod}_n([0,k-1]) \cap \mathrm{mod}_n([0,k-1]+ky)| \\
&= \begin{cases}
k-\mathrm{mod}_n(ky) & \mathrm{mod}_n(ky) \in [0,\min(k,n-k)] \\
k-(n-k) & \mathrm{mod}_n(ky) \in (n-k,k) \\
0 & \mathrm{mod}_n(ky) \in (k,n-k) \\
k-\mathrm{mod}_n(-ky) & \mathrm{mod}_n(ky) \in [\max(k,n-k),n)
\end{cases}
\end{aligned}$$

Suppose that $|X \cap \mathrm{mod}_n(X+y)| = k-1$. Since $1 < k < n-1$, $k-(n-k) \neq k-1$ and $0 \neq k-1$, so we must have either $k-\mathrm{mod}_n(ky) = k-1$ or $k-\mathrm{mod}_n(-ky) = k-1$, which imply that $ky \equiv_n \pm 1$.

Now suppose that $ky \equiv_n \pm 1$. Since $1 < k < n-1$, $\mathrm{mod}_n(ky) \notin (n-k, k)$ and $\mathrm{mod}_n(ky) \notin (k, n-k)$. If $\mathrm{mod}_n(ky) \in [0, \min(k, n-k)]$, then $\mathrm{mod}_n(ky) = 1$ and $|X \cap \mathrm{mod}_n(X+y)| = k-1$. If $\mathrm{mod}_n(ky) \in [\max(k, n-k), n)$, then $\mathrm{mod}_n(ky) = k-1$. So $\mathrm{mod}_n(-ky) = 1$ and $|X \cap \mathrm{mod}_n(X+y)| = k-1$. □

Proposition 39. Let X be an order k modulo n maximally even set. Let $g = \gcd(k,n)$, $k' = \frac{k}{g}$, and $n' = \frac{n}{g}$. If u is a modulo n unit such that $\mathrm{mod}_n(uX)$ is an order k modulo n maximally even set, then $k' = 1$, $k' = n'-1$, or $u \equiv_{n'} \pm 1$.

Proof. By Proposition 25,

$$X = \bigcup_{j \in [0,g)} (n'j + X').$$

where X' is an order k' modulo n' maximally even set. By Proposition 28,

$$\mathrm{mod}_n(uX) = \bigcup_{j \in [0,g)} (n'j + \mathrm{mod}_n(uX')).$$

Since $\mathrm{mod}_n(uX)$ is an order k modulo n maximally even set, Proposition 25 tells us that $\mathrm{mod}_n(uX')$ is an order k' modulo n' maximally even set.

Suppose that $k' \neq 1$ and $k' \neq n'-1$. Let y be a modulo n' multiplicative inverse of k', and let v be a modulo n' multiplicative inverse of u. By the previous proposition,

$$|X \cap \mathrm{mod}_n(X + vy)| = |\mathrm{mod}_n(uX) \cap \mathrm{mod}_n(uX + y)| = k' - 1,$$

and $vyk' \equiv_{n'} \pm 1$. Then $v \equiv_{n'} \pm 1$, and $u \equiv_{n'} \pm 1$. □

Next we will look at the affect of unit multiplication on maximally even rhythms.

Definition 40. Let $n \geq 1$, and let u be a modulo n unit. Let $\pi_{n,u}$ be the permutation on $[0,n)$ defined by $\pi_{n,u}(i) = \mathrm{mod}_n(ui)$. We will call any permutation of this form, a unit multiplication permutation.

Proposition 41. Let r be a k-note n-beat rhythm, and let u be a modulo n unit. Then there exists a unique permutation π on $[0, k)$ such that $\pi_{n,u} \circ r \circ \pi$ is a k-note n-beat rhythm.

Example 42. Consider the 9-note 15-beat maximally even rhythm r defined below.

i	0	1	2	3	4	5	6	7	8
$r(i)$	0	1	3	5	6	8	10	11	13

The following gives the values of $\pi_{15,2} \circ r$.

i	0	1	2	3	4	5	6	7	8
$(\pi_{15,2} \circ r)(i)$	0	2	6	10	12	1	5	7	11

This destroys the strictly increasing order, so we need a permutation π on $[0, 9)$ to fix it.

i	0	1	2	3	4	5	6	7	8
$\pi(i)$	0	5	1	6	2	7	3	8	4
$(\pi_{15,2} \circ r \circ \pi)(i)$	0	1	2	5	6	7	10	11	12

In this case, π is equal to the unit multiplication permutation $\pi_{9,5}$.

Now consider $\pi_{15,4} \circ r$.

i	0	1	2	3	4	5	6	7	8
$(\pi_{15,4} \circ r)(i)$	0	4	12	5	9	2	10	14	7

Again, the strictly increasing order is destroyed, so we need a (different) permutation π on $[0, 9)$ to fix it.

i	0	1	2	3	4	5	6	7	8
$\pi(i)$	0	5	1	3	8	4	6	2	7
$(\pi_{15,4} \circ r \circ \pi)(i)$	0	2	4	5	7	9	10	12	14

In this case, π is not a unit multiplication permutation.

We will see later that whether or not π is a unit multiplication permutation, determines whether an important property of maximally even tilings (regularity) is preserved under unit multiplication.

In the remainder of this section, we will characterize the cases in which π is a unit multiplication permutation.

Proposition 43. Let r be k-note n-beat maximally even rhythm with $r(0) = 0$. Let $u \in \mathbb{Z}$ be a modulo n unit that divides $n - jk$ for some $j \in \left[1, \frac{n}{k}\right]$. Then there exists $v \in \mathbb{Z}$ such that $\pi_{n,u} \circ r \circ \pi_{k,v}$ is a k-note n-beat rhythm.

Proof. Since u divides $n - jk$, we can choose $c \in \mathbb{Z}$ such that $cu = n - jk$. Since u is an modulo n unit, u and n are relatively prime, and we can choose $c', c'' \in \mathbb{Z}$ such that $c'u + c''n = 1$. Then

$$c'u + c''(cu + jk) = 1$$
$$(c' + c''c)u + c''jk = 1$$

So u and k are relatively prime, u is a modulo k unit, and we can let v be a modulo k multiplicative inverse of u.

We will first show that $\pi_{n,u} \circ r^{\text{flr}}_{k,n} \circ \pi_{k,v}$ is a k-note n-beat rhythm. Using the fact that $\left\lfloor \frac{a}{b} \right\rfloor b = a - \text{mod}_b(a)$, can calculate the following.

$$\begin{aligned}
&(\pi_{n,u} \circ r^{\text{flr}}_{k,n} \circ \pi_{k,v})(i) \\
=&\text{mod}_n\left(u\left\lfloor \frac{\text{mod}_k(vi)n}{k} \right\rfloor\right) \\
=&\text{mod}_n\left(u\left(\frac{\text{mod}_k(vi)n - \text{mod}_k(vin)}{k}\right)\right) \\
=&\text{mod}_n\left(\frac{u\text{mod}_k(vi)n - u\text{mod}_k(vin)}{k}\right) \\
=&\text{mod}_n\left(\frac{u\left(vi - \left\lfloor \frac{vi}{k} \right\rfloor k\right)n - u\text{mod}_k(vi(cu + jk))}{k}\right)
\end{aligned}$$

$$
\begin{aligned}
&=\mathrm{mod}_n\left(\frac{uvin - u\left\lfloor\frac{vi}{k}\right\rfloor kn - u\mathrm{mod}_k(ic)}{k}\right)\\
&=\mathrm{mod}_n\left(\frac{uvin - u\mathrm{mod}_k(ic)}{k} - u\left\lfloor\frac{vi}{k}\right\rfloor n\right)\\
&=\mathrm{mod}_n\left(\frac{uvin - u\mathrm{mod}_k(ic)}{k}\right)\\
&=\mathrm{mod}_n\left(\frac{\left(\left\lfloor\frac{uv}{k}\right\rfloor k + \mathrm{mod}_k(uv)\right) in - u\mathrm{mod}_k(ic)}{k}\right)\\
&=\mathrm{mod}_n\left(\frac{\left\lfloor\frac{uv}{k}\right\rfloor kin + in - u\mathrm{mod}_k(ic)}{k}\right)\\
&=\mathrm{mod}_n\left(\frac{in - u\mathrm{mod}_k(ic)}{k} + \left\lfloor\frac{uv}{k}\right\rfloor in\right)\\
&=\mathrm{mod}_n\left(\frac{in - u\mathrm{mod}_k(ic)}{k}\right)\\
&=\mathrm{mod}_n\left(\frac{in - u\left(ic - \left\lfloor\frac{ic}{k}\right\rfloor k\right)}{k}\right)\\
&=\mathrm{mod}_n\left(\frac{i(n-uc) + u\left\lfloor\frac{ic}{k}\right\rfloor k}{k}\right)\\
&=\mathrm{mod}_n\left(\frac{i(jk) + u\left\lfloor\frac{ic}{k}\right\rfloor k}{k}\right)\\
&=\mathrm{mod}_n\left(ij + u\left\lfloor\frac{ic}{k}\right\rfloor\right)
\end{aligned}
$$

Keeping in mind that $u > 0 \Rightarrow c \geq 0$ and $u < 0 \Rightarrow c \leq 0$, we have

$$0 \leq ij + u\left\lfloor\frac{ic}{k}\right\rfloor < kj + u\left\lfloor\frac{kc}{k}\right\rfloor = kj + uc = n,$$

so that

$$(\pi_{n,u} \circ r^{\mathrm{flr}}_{k,n} \circ \pi_{k,v})(i) = ij + u\left\lfloor\frac{ic}{k}\right\rfloor$$

which is strictly increasing since for $i \in [0, k-2]$,

$$\begin{aligned}
&(\pi_{n,u} \circ r^{\text{flr}}_{k,n} \circ \pi_{k,v})(i+1) - (\pi_{n,u} \circ r^{\text{flr}}_{k,n} \circ \pi_{k,v})(i) \\
&= \left((i+1)j + u\left\lfloor \frac{(i+1)c}{k} \right\rfloor\right) - \left(ij + u\left\lfloor \frac{ic}{k} \right\rfloor\right) \\
&= j + u\left(\left\lfloor \frac{(i+1)c}{k} \right\rfloor - \left\lfloor \frac{ic}{k} \right\rfloor\right) \\
&\geq j
\end{aligned}$$

is positive. So $\pi_{n,u} \circ r^{\text{flr}}_{k,n} \circ \pi_{k,v}$ is a k-note n-beat rhythm.

Now choose $d \in [0, k)$ such that $r(i) = r^{\text{flr}}_{k,n}(i+d) - r^{\text{flr}}_{k,n}(d)$. Then

$$\begin{aligned}
&(\pi_{n,u} \circ r \circ \pi_{k,v})(i) \\
&= \operatorname{mod}_n\left(ur^{\text{flr}}_{k,n}(vi+d) - ur^{\text{flr}}_{k,n}(d)\right) \\
&= \operatorname{mod}_n\left((\pi_{n,u} \circ r^{\text{flr}}_{k,n} \circ \pi_{k,v})(i+ud) - (\pi_{n,u} \circ r^{\text{flr}}_{k,n} \circ \pi_{k,v})(ud)\right)
\end{aligned}$$

is a k-note n-beat rhythm. □

Proposition 44. Let r be a k-note n-beat maximally even rhythm with $r(0) = 0$. Let $u \in \mathbb{Z}$ be an modulo n unit and $v \in \mathbb{Z}$ be a modulo k unit such that $r' = \pi_{n,u} \circ r \circ \pi_{k,v}$ is a k-note n-beat rhythm. Then u is equivalent modulo n to a divisor of $n - jk$ where $j \in \left[1, \frac{n}{k}\right]$.

Proof. Using the fact that r' is increasing, v is a modulo k unit, and Proposition 21, we obtain the following.

$$\begin{aligned} n &= \sum_{i\in[0,k)} \mathrm{mod}_n\left(r'(i+1) - r'(i)\right) \\ &= \sum_{i\in[0,k)} \mathrm{mod}_n\left(ur(vi+v)) - ur(vi)\right) \\ &= \sum_{i\in[0,k)} \mathrm{mod}_n\left(ur(i+v) - ur(i)\right) \\ &= \sum_{i\in[0,k)} \mathrm{mod}_n\left(u[r(i+v) - r(i)]\right) \\ &= (k - \mathrm{mod}_k(vn))\mathrm{mod}_n\left(u\left\lfloor\frac{vn}{k}\right\rfloor\right) + \mathrm{mod}_k(vn)\mathrm{mod}_n\left(u\left\lceil\frac{vn}{k}\right\rceil\right) \end{aligned}$$

Consider the following cases.

- Suppose that $\mathrm{mod}_n\left(u\left\lceil\frac{vn}{k}\right\rceil\right) = \mathrm{mod}_n\left(u\left\lfloor\frac{vn}{k}\right\rfloor\right)$. Then $\left\lceil\frac{vn}{k}\right\rceil \equiv_n \left\lfloor\frac{vn}{k}\right\rfloor$. If $\left\lceil\frac{vn}{k}\right\rceil = \left\lfloor\frac{vn}{k}\right\rfloor + 1$, then we have the trivial case where $n = k = u = v = 1$ and result holds for $j = 1$. If $\left\lceil\frac{vn}{k}\right\rceil = \left\lfloor\frac{vn}{k}\right\rfloor$, then $k|vn$, which means that $vn \equiv_k 0$. Since v is a modulo k unit, $n \equiv_k 0$, and the result holds for $j = \frac{n}{k}$.

- Now suppose that $\mathrm{mod}_n\left(u\left\lceil\frac{vn}{k}\right\rceil\right) > \mathrm{mod}_n\left(u\left\lfloor\frac{vn}{k}\right\rfloor\right)$. Then

$$\begin{aligned} n &= (k - \mathrm{mod}_k(vn))\mathrm{mod}_n\left(u\left\lfloor\frac{vn}{k}\right\rfloor\right) + \mathrm{mod}_k(vn)\mathrm{mod}_n\left(u\left\lceil\frac{vn}{k}\right\rceil\right) \\ &= k\mathrm{mod}_n\left(u\left\lfloor\frac{vn}{k}\right\rfloor\right) + \mathrm{mod}_k(vn)\left(\mathrm{mod}_n\left(u\left\lceil\frac{vn}{k}\right\rceil\right) - \mathrm{mod}_n\left(u\left\lfloor\frac{vn}{k}\right\rfloor\right)\right) \\ &= jk + \mathrm{mod}_k(vn)\mathrm{mod}_n(u) \end{aligned}$$

  for $j = \mathrm{mod}_n\left(u\left\lfloor\frac{vn}{k}\right\rfloor\right)$. So u is equivalent modulo n to $\mathrm{mod}_n(u)$ which divides $n - jk$, where

$$1 \leq j = \frac{n - \mathrm{mod}_k(vn)\mathrm{mod}_n(u)}{k} \leq \frac{n}{k}.$$

- Now suppose that $\mathrm{mod}_n\left(u\left\lceil\frac{vn}{k}\right\rceil\right) < \mathrm{mod}_n\left(u\left\lfloor\frac{vn}{k}\right\rfloor\right)$. Then

$$\begin{aligned}
n &= (k - \mathrm{mod}_k(vn))\mathrm{mod}_n\left(u\left\lfloor\frac{vn}{k}\right\rfloor\right) + \mathrm{mod}_k(vn)\mathrm{mod}_n\left(u\left\lceil\frac{vn}{k}\right\rceil\right) \\
&= \mathrm{mod}_k(-vn)\mathrm{mod}_n\left(u\left\lfloor\frac{vn}{k}\right\rfloor\right) + (k - \mathrm{mod}_k(-vn))\mathrm{mod}_n\left(u\left\lceil\frac{vn}{k}\right\rceil\right) \\
&= k\mathrm{mod}_n\left(u\left\lceil\frac{vn}{k}\right\rceil\right) + \mathrm{mod}_k(-vn)\left(\mathrm{mod}_n\left(u\left\lfloor\frac{vn}{k}\right\rfloor\right) - \mathrm{mod}_n\left(u\left\lceil\frac{vn}{k}\right\rceil\right)\right) \\
&= k\mathrm{mod}_n\left(u\left\lceil\frac{vn}{k}\right\rceil\right) + \mathrm{mod}_k(-vn)(n - \mathrm{mod}_n(u)) \\
&= jk - \mathrm{mod}_k(-vn)(\mathrm{mod}_n(u) - n)
\end{aligned}$$

for $j = \mathrm{mod}_n\left(u\left\lceil\frac{vn}{k}\right\rceil\right)$. So u is equivalent modulo n to $\mathrm{mod}_n(u) - n$ which divides $n - jk$, where

$$1 \leq j = \frac{n + \mathrm{mod}_k(-vn)(\mathrm{mod}_n(u) - n)}{k} \leq \frac{n}{k}.$$

□

Chapter 4

MAXIMALLY EVEN TILINGS

4.1 Definition and existence

Definition 45. Let $n \geq 1$ and $A, B \subseteq [0, n)$ such that the restriction of $\oplus_n$ to $A \times B$ is injective. Let $k = |A \times B| = |\text{mod}_n(A + B)| = |A||B|$. Then we will say that $(A, B)_n$ is a k-note n-beat partial tiling with range $\text{mod}_n(A + B)$. Furthermore,

- we will say that $(A, B)_n$ is a maximally even tiling if its range is an order k modulo n maximally even set,
- $(A, B)_n$ is a full tiling if its range is all of $[0, n)$ (i.e., when $n = k$), and
- we will say that $(A, B)_n$ is trivial if $|A| = 1$ or $|B| = 1$.

Definition 46. We will define the dual, translations, and unit multiples of partial tilings in the same way that we did for full tilings in Definition 3. In addition, for $t \geq 1$, we will call $(tA, tB)_{tn}$ an augmentation of $(A, B)_n$.

Proposition 47. $(A, B)_n$ is a partial tiling if and only if is dual, translations, unit multiples, and augmentations are partial tilings. Moreover,

- duality and translation always preserve maximal evenness,
- Proposition 39 gives the cases in which unit multiplication preserves maximal evenness, and
- augmentation preserves maximal evenness if and only if $k = |A||B|$ divides n.

Proposition 48. If $n \geq k$ and $k = k_1k_2$, then there exists a k-note n-beat maximally even tiling $(A, B)_n$ with $|A| = k_1$ and $|B| = k_2$.

Proof. Let X be an order k modulo n maximally even set. By Proposition 33, $X = \bigcup_{i=1}^{k_2} X_i$, where X_i are pairwise disjoint k_1-note n-beat maximally even sets. By Proposition 29, for each $i \in [1, k_2]$ we can choose $c_i \in [0, n)$ such that $X_i = \mathrm{mod}_n(X_{k_1,n}^{\mathrm{inv}} + c_i)$. If we let $A = X_{k_1,n}^{\mathrm{inv}}$ and $B = \{c_i : i \in [1, k_2]\}$, then $(A, B)_n$ is a k-note n-beat maximally even tiling with $|A| = k_1$ and $|B| = k_2$. □

Example 49. Let us show how we can construct a 15-note 17-beat maximally even tiling $(A, B)_{17}$ with $|A| = 5$ and $|B| = 3$. First we need to choose an order 15 modulo 17 maximally even set. Let us choose the following.

$$X_{15,17}^{\mathrm{flr}} = \{0, 1, 2, 3, 4, 5, 6, 7, 9, 10, 11, 12, 13, 14, 15\}$$

The proof of Proposition 33 shows us how to write $X_{15,17}^{\mathrm{flr}}$ as $\bigcup_{i=1}^{k_2} X_i$ where

$$X_1 = \{0, 3, 6, 10, 13\}$$

$$X_2 = \{1, 4, 7, 11, 14\}$$

$$X_3 = \{2, 5, 9, 12, 15\}$$

are order 5 modulo 17 maximally even sets. By Proposition 29, each of these sets is a modulo 17 translation of $X_{5,17}^{\mathrm{inv}} = \{0, 4, 7, 11, 14\}$. In particular,

$$X_1 = \mathrm{mod}_n(X_{5,17}^{\mathrm{inv}} + 6)$$

$$X_2 = \mathrm{mod}_n(X_{5,17}^{\mathrm{inv}} + 7)$$

$$X_3 = \mathrm{mod}_n(X_{5,17}^{\mathrm{inv}} + 15).$$

So $(\{0, 4, 7, 11, 14\}, \{6, 7, 15\})_{17}$ is a 15-note 17-beat maximally even tiling.

4.2 Regular vs irregular maximally even tilings

In the remainder of this chapter we investigate an important classification of partial tilings (and maximally even tiling in particular). Some partial tiling canons can be formed by simply inserting beats of rest into a full tiling canon. We will call such partial tilings *regular*, while partial tilings that cannot be formed in such a manner will be called *irregular*.

Consider the rhythmic canon of the maximally even tiling $(\{0,1,4,5,9\},\{0,6\})_{13}$.

	0	1	2	·	4	5	6	7	·	9	10	11	·	
Part A	1	2	·	·	3	4	·	·	·	5	·	·	·	:‖
Part B	·	·	5	·	·	·	1	2	·	·	3	4	·	:‖

If we remove the beats of rest, we get the following.

	0	1	2	3	4	5	6	7	8	9	
Part A	1	2	·	3	4	·	·	5	·	·	:‖
Part B	·	·	5	·	·	1	2	·	3	4	:‖

In this case the rhythmic imitation is maintained, and we get the rhythmic canon of the full tiling $(\{0,1,3,4,7\},\{0,5\})_{10}$. Thus, $(\{0,1,4,5,9\},\{0,6\})_{13}$ is regular, since it can be formed by inserting rests into the rhythmic canon of a full tiling.

Now consider the rhythmic canon of another maximally even tiling $(\{0,1,5,7,9\},\{0,3\})_{13}$.

	0	1	·	3	4	5	·	7	8	9	10	·	12	
Part A	1	2	·	·	·	3	·	4	·	5	·	·	·	:‖
Part B	·	·	·	1	2	·	·	·	3	·	4	·	5	:‖

If we remove the beats of rest we get the following.

	0	1	2	3	4	5	6	7	8	9	
Part A	1	2	·	·	3	4	·	5	·	·	:‖
Part B	·	·	1	2	·	·	3	·	4	5	:‖

In this case the rhythmic imitation is not maintained. For example, notes 3 and 4 are one beat apart in Part A but two beats apart in Part B. Thus, $(\{0,1,5,7,9\},\{0,3\})_{13}$ is irregular, since it cannot be formed by inserting rests into a full tiling.

Given a regular maximally even tiling, we can apply the step expansion and swapping operations that were discussed in Example 34 and Proposition 35 to derive/describe an infinite family regular maximally even tilings. For example, below are the rhythmic canons of the regular maximally even tiling $(\{0,1,4,5,9\},\{0,6\})_{13}$, its single step expansion $(\{0,2,7,9,16\},\{0,11\})_{23}$, and its step swapping $(\{0,2,5,7,12\},\{0,9\})_{17}$.

	0	1	2	·	4	5	6	7	·	9	10	11	·	
Part A	1	2	·	·	3	4	·	·	·	5	·	·	·	:‖
Part B	·	·	5	·	·	·	1	2	·	·	3	4	·	:‖

	0	·	2	·	4	·	·	7	·	9	·	11	·	13	·	·	16	·	18	·	20	·	·	
Part A	1	·	2	·	·	·	·	3	·	4	·	·	·	·	·	·	5	·	·	·	·	·	·	:‖
Part B	·	·	·	·	5	·	·	·	·	·	·	1	·	2	·	·	·	·	3	·	4	·	·	:‖

	0	·	2	·	4	5	·	7	·	9	·	11	12	·	14	·	16	
Part A	1	·	2	·	·	3	·	4	·	·	·	·	5	·	·	·	·	:‖
Part B	·	·	·	·	5	·	·	·	·	1	·	2	·	·	3	·	4	:‖

In Section 4.4 we will show that all regular maximally even tilings are step expansions and/or swappings of those in which the number of beats n is no more than one and a half times the number of notes k. Thus, a search for regular maximally even tilings can be narrowed down to $n \in [k, 1.5k]$.

In Section 4.5 we will show that irregular k-note n-beat maximally even tilings must satisfy $n \le 2(n - \gcd(k,n))$, which again narrows a search to a relatively small finite number of possibilities for n.

4.3 Function representation of partial tilings

Example 50. Notice that our informal description of regularity required us to refer to the rhythmic canons of partial tilings. So in order to proceed with a formal investigation of regularity we will need a mathematically precise way of describing the score of a rhythmic canon. In [14] I used weighted cycles to do this, but I have since realized a couple of drawbacks to this approach. So I take a different approach here.

Consider $(\{0,1,3,7\},\{0,5,10\})_{15}$, the rhythmic canon of which is given below.

	0	1	2	3	·	5	6	7	8	·	10	11	12	13	·	
Part A	1	2	·	3	·	·	·	4	·	·	·	·	·	·	·	:‖
Part B	·	·	·	·	·	1	2	·	3	·	·	·	4	·	·	:‖
Part C	·	·	4	·	·	·	·	·	·	·	1	2	·	3	·	:‖

We can describe this structure with the pair of functions s and r given below.

$\binom{x}{y}$	$\binom{1}{A}$	$\binom{2}{A}$	$\binom{4}{C}$	$\binom{3}{A}$	$\binom{1}{B}$	$\binom{2}{B}$	$\binom{4}{A}$	$\binom{3}{B}$	$\binom{1}{C}$	$\binom{2}{C}$	$\binom{4}{B}$	$\binom{3}{C}$
$s\binom{x}{y} = z$	0	1	2	3	4	5	6	7	8	9	10	11
$r(z)$	0	1	2	3	5	6	7	8	10	11	12	13

The function s maps each note of each part to a unique value in $[0,12)$, determining the order in which the notes occur. The function r is a 12-note 15-beat rhythm. The following proposition and definition gives several equivalent ways to confirm that such a pair of functions does indeed represent a rhythmic canon.

Proposition and Definition 51. Let X and Y be finite sets, $k = |X \times Y|$, and $s : X \times Y \to [0,k)$ be a one to one correspondence. Let r be a k-note n-beat rhythm. Let $q = r \circ s$. Then the following are equivalent.

1. $q\binom{x_1}{y_1} - q\binom{x_1}{y_2} \equiv_n q\binom{x_2}{y_1} - q\binom{x_2}{y_2}$ for all $\binom{x_1}{y_1}, \binom{x_2}{y_2} \in X \times Y$.

2. $q\binom{x_1}{y_1} - q\binom{x_2}{y_1} \equiv_n q\binom{x_1}{y_2} - q\binom{x_2}{y_2}$ for all $\binom{x_1}{y_1}, \binom{x_2}{y_2} \in X \times Y$.

3. $q\binom{x}{y} \equiv_n q\binom{x}{y_0} + q\binom{x_0}{y} - q\binom{x_0}{y_0}$ for a fixed $\binom{x_0}{y_0} \in X \times Y$ and all $\binom{x}{y} \in X \times Y$.

If the conditions above hold, then we will say that ${}_n(r, s)$ is a k-note n-beat partial tiling.

- When we refer to the domain of a partial tiling ${}_n(r, s)$, we will mean the domain of s, and when we refer to the range of a partial tiling ${}_n(r, s)$, we will mean the range of r.

- In the case that $n = k$, then $r = \mathrm{id}_{[0,k)}$, and we will say that ${}_n(r, s)$ is a full tiling.

- We will say that ${}_n(r, s)$ is regular if ${}_k(\mathrm{id}_{[0,k)}, s)$ is a full tiling. Otherwise, we will say that ${}_n(r, s)$ is irregular.

- If r is a k-note n-beat maximally even rhythm, then we will say that ${}_n(r, s)$ is a maximally even tiling.

Proof. Suppose that the first condition is satisfied. Then for all $\binom{x_1}{y_1}, \binom{x_2}{y_2} \in X \times Y$ we have the following.

$$\begin{aligned}
&q\begin{pmatrix}x_1\\y_1\end{pmatrix} - q\begin{pmatrix}x_2\\y_1\end{pmatrix}\\
\equiv_n &q\begin{pmatrix}x_1\\y_1\end{pmatrix} - q\begin{pmatrix}x_1\\y_2\end{pmatrix} + q\begin{pmatrix}x_1\\y_2\end{pmatrix} - q\begin{pmatrix}x_2\\y_2\end{pmatrix} + q\begin{pmatrix}x_2\\y_2\end{pmatrix} - q\begin{pmatrix}x_2\\y_1\end{pmatrix}\\
\equiv_n &q\begin{pmatrix}x_2\\y_1\end{pmatrix} - q\begin{pmatrix}x_2\\y_2\end{pmatrix} + q\begin{pmatrix}x_1\\y_2\end{pmatrix} - q\begin{pmatrix}x_2\\y_2\end{pmatrix} + q\begin{pmatrix}x_2\\y_2\end{pmatrix} - q\begin{pmatrix}x_2\\y_1\end{pmatrix}\\
\equiv_n &q\begin{pmatrix}x_1\\y_2\end{pmatrix} - q\begin{pmatrix}x_2\\y_2\end{pmatrix}
\end{aligned}$$

So the second condition is satisfied.

Now suppose that the second condition is satisfied. Then for fixed $\binom{x_0}{y_0} \in X \times Y$ and all $\binom{x}{y} \in X \times Y$ we have the following.

$$\begin{aligned} q\binom{x}{y} &\equiv_n q\binom{x}{y} - q\binom{x_0}{y} + q\binom{x_0}{y} \\ &\equiv_n q\binom{x}{y_0} - q\binom{x_0}{y_0} + q\binom{x_0}{y} \\ &\equiv_n q\binom{x}{y_0} + q\binom{x_0}{y} - q\binom{x_0}{y_0} \end{aligned}$$

So the third condition is satisfied.

Now suppose that the third condition is satisfied for a fixed $\binom{x_0}{y_0} \in X \times Y$. Then for all $\binom{x_1}{y_1}, \binom{x_2}{y_2} \in X \times Y$ we have the following.

$$\begin{aligned} & q\binom{x_1}{y_1} - q\binom{x_1}{y_2} \\ \equiv_n & \left[q\binom{x_1}{y_0} + q\binom{x_0}{y_1} - q\binom{x_0}{y_0}\right] - \left[q\binom{x_1}{y_0} + q\binom{x_0}{y_2} - q\binom{x_0}{y_0}\right] \\ \equiv_n & q\binom{x_0}{y_1} - q\binom{x_0}{y_2} \\ \equiv_n & \left[q\binom{x_2}{y_0} + q\binom{x_0}{y_1} - q\binom{x_0}{y_0}\right] - \left[q\binom{x_2}{y_0} + q\binom{x_0}{y_2} - q\binom{x_0}{y_0}\right] \\ \equiv_n & q\binom{x_2}{y_1} - q\binom{x_2}{y_2} \end{aligned}$$

So the first condition is satisfied. $\square$

Definition 52. Let ${}_n(r, s)$ be a partial tiling with domain $X \times Y$. Let $\sigma_1 : X' \to X$ and $\sigma_2 : Y' \to Y$ be bijections, and define $\sigma\binom{x}{y} = (\sigma_1(x), \sigma_2(y))$. We will call ${}_n(r, s \circ \sigma)$ a relabeling of ${}_n(r, s)$. (From a musical point of view, this just renames the parts and notes.)

Definition 53. Let $(A, B)_n$ be a k-note n-beat partial tiling with range C. Let r be the k-note n-beat rhythm with range C and $s = r^{-1} \circ \oplus_n|_{A\times B}$. We will call ${}_n(r, s)$ the function representation of $(A,B)_n$.

Definition 54. Let ${}_n(r,s)$ be a k-note n-beat partial tiling with domain $X\times Y$. Let $q=r\circ s$. Choose $(x_0,y_0)\in X\times Y$ and $a_0,b_0\in[0,n)$ such that $a_0+b_0\equiv_n q(x_0,y_0)$. Define the following.

$$A=\left\{\mathrm{mod}_n\left(q\binom{x}{y_0}-q\binom{x_0}{y_0}+a_0\right):x\in X\right\}$$

$$B=\left\{\mathrm{mod}_n\left(q\binom{x_0}{y}-q\binom{x_0}{y_0}+b_0\right):y\in Y\right\}$$

We will call $(A,B)_n$ a subset representation of ${}_n(r,s)$.

Proposition 55. Let $(A,B)_n$ be a partial tiling with range C.

- Let ${}_n(r,s)$ be the function representation of $(A,B)_n$. Then ${}_n(r,s)$ is a partial tiling with range C. (In particular, if $(A,B)_n$ is maximally even, then so is ${}_n(r,s)$.)
- Let $(A',B')_n$ be a subset representation of ${}_n(r,s)$. Then $(A',B')_n$ is a range-invariant translation of $(A,B)_n$.

Proof. By definition r is the k-note n-beat rhythm with range $C=\mathrm{mod}_n(A+B)$ and $s=r^{-1}\circ\oplus_n|_{A\times B}$. Let $q=r\circ s$. Then for $\binom{a_1}{b_1},\binom{a_2}{b_2}\in A\times B$ we have the following.

$$\begin{aligned}
q\binom{a_1}{b_1}-q\binom{a_1}{b_2} &\equiv_n \mathrm{mod}_n(a_1+b_1)-\mathrm{mod}_n(a_1+b_2)\\
&\equiv_n b_1-b_2\\
&\equiv_n \mathrm{mod}_n(a_2+b_1)-\mathrm{mod}_n(a_2+b_2)\\
&\equiv q\binom{a_2}{b_1}-q\binom{a_2}{b_2}
\end{aligned}$$

So ${}_n(r,s)$ is a partial tiling.

By definition, there exist $(x_0,y_0)\in A\times B$ and $a_0,b_0\in[0,n)$ such that $a_0+b_0\equiv_n q(x_0,y_0)$ with

$$A'=\left\{\mathrm{mod}_n\left(q\binom{x}{y_0}-q\binom{x_0}{y_0}+a_0\right):x\in X\right\}$$

and

$$B' = \left\{ \mathrm{mod}_n \left(q\binom{x_0}{y} - q\binom{x_0}{y_0} + b_0 \right) : y \in Y \right\}$$

Note that

$$\begin{aligned} A' &= \{\mathrm{mod}_n(\mathrm{mod}_n(x + y_0) - \mathrm{mod}_n(x_0 + y_0) + a_0) : x \in A\} \\ &= \{\mathrm{mod}_n(x - x_0 + a_0) : x \in A\} \\ &= \mathrm{mod}_n(A - x_0 + a_0) \end{aligned}$$

is a modulo n translation of A. In the same way, $B' = \mathrm{mod}_n(B - y_0 + b_0)$ is a modulo n translation of B. Finally, note that the range of $(A', B')_n$ is

$$\begin{aligned} \mathrm{mod}_n(A' + B') &= \mathrm{mod}_n(A - x_0 + a_0 + B - y_0 + b_0) \\ &= \mathrm{mod}_n \left(A + B + q\binom{x_0}{y_0} - (a_0 + b_0) \right) \\ &= \mathrm{mod}_n(A + B) \end{aligned}$$

Therefore, $(A', B')_n$ is a range-invariant translation of $(A, B)_n$. □

Example 56. Consider the partial tiling $(A, B)_n = (\{0, 2, 4, 8, 9\}\{0, 3\})_{13}$. Then its function representation ${}_n(r, s)$ is as follows.

(x, y)	$\binom{0}{0}$	$\binom{2}{0}$	$\binom{0}{3}$	$\binom{4}{0}$	$\binom{2}{3}$	$\binom{4}{3}$	$\binom{8}{0}$	$\binom{9}{0}$	$\binom{8}{3}$	$\binom{9}{3}$
$s(x, y)$	0	1	2	3	4	5	6	7	8	9
$r(s(x, y))$	0	2	3	4	5	7	8	9	11	12

One can check that is indeed a partial tiling. Let us choose $(x_0, y_0) = (4, 3)$, $a_0 = 1$, and $b_0 = 6$, so that $a_0 + b_0 \equiv 7 \equiv_n q(x_0, y_0)$. Then letting

$$A' = \left\{ \mathrm{mod}_n \left(q\binom{x}{3} - q\binom{4}{3} + 1 \right) : x \in A \right\} = \{1, 5, 6, 10, 12\}$$

and

$$B' = \left\{ \operatorname{mod}_n \left(q\binom{4}{y} - q\binom{4}{3} + 6 \right) : y \in B \right\} = \{3, 6\},$$

we can see that $(A', B')_n$ is indeed a range-invariant translation of $(A, B)_n$.

Proposition 57. Let ${}_n(r, s)$ be a partial tiling with range Z.

- Let $(A, B)_n$ be a subset representation of ${}_n(r, s)$. Then $(A, B)_n$ is a partial tiling with range Z. (In particular, if ${}_n(r, s)$ is maximally even, then $(A, B)_n$ is maximally even.)
- Let ${}_n(r', s')$ be the function representation of $(A, B)_n$. Then ${}_n(r', s')$ is a relabeling of ${}_n(r, s)$.

Proof. Let $X \times Y$ be the domain of ${}_n(r, s)$, and let $q = r \circ s$. By definition, there exist $(x_0, y_0) \in X \times Y$ and $a_0, b_0 \in [0, n)$ such that $a_0 + b_0 \equiv_n q\binom{x_0}{y_0}$ with

$$A = \left\{ \operatorname{mod}_n \left(q\binom{x}{y_0} - q\binom{x_0}{y_0} + a_0 \right) : x \in X \right\}$$

and

$$B = \left\{ \operatorname{mod}_n \left(q\binom{x_0}{y} - q\binom{x_0}{y_0} + b_0 \right) : y \in Y \right\}.$$

Suppose $a_1 + b_1 \equiv_n a_2 + b_2$ for some $\binom{a_1}{b_1}, \binom{a_2}{b_2} \in A \times B$. By definition,

$$a_i = \operatorname{mod}_n \left(q\binom{x_i}{y_0} - q\binom{x_0}{y_0} + a_0 \right)$$

for $x_i \in X$, and

$$b_i = \operatorname{mod}_n \left(q\binom{x_0}{y_i} - q\binom{x_0}{y_0} + b_0 \right)$$

$y_i \in Y$. Substituting these values into $a_1 + b_1 \equiv_n a_2 + b_2$ yields the following.

$$q\binom{x_1}{y_0} + q\binom{x_0}{y_1} \equiv_n q\binom{x_2}{y_0} + q\binom{x_0}{y_2}$$

$$q\binom{x_1}{y_0} - q\binom{x_2}{y_0} \equiv_n q\binom{x_0}{y_2} - q\binom{x_0}{y_1}$$

$$q\binom{x_1}{y_2} - q\binom{x_2}{y_2} \equiv_n q\binom{x_1}{y_2} - q\binom{x_1}{y_1}$$

$$q\binom{x_2}{y_2} \equiv_n q\binom{x_1}{y_1}$$

$$q\binom{x_2}{y_2} = q\binom{x_1}{y_1}$$

Since q is one-to-one, $x_1 = x_2$ and $y_1 = y_2$. So

$$a_1 = \mathrm{mod}_n\left(q\binom{x_1}{y_0} - q\binom{x_0}{y_0} + a_0\right) = \mathrm{mod}_n\left(q\binom{x_2}{y_0} - q\binom{x_0}{y_0} + a_0\right) = a_2,$$

and similarly, $b_1 = b_2$. Therefore, $(A, B)_n$ is a partial tiling. Furthermore, the range of $(A, B)_n$ is

$$\begin{aligned}
&\mathrm{mod}_n(A + B)\\
&= \mathrm{mod}_n\left(\left\{\mathrm{mod}_n\left(q\binom{x}{y_0} - q\binom{x_0}{y_0} + a_0\right) : x \in X\right\}\right.\\
&\qquad\qquad \left.+\left\{\mathrm{mod}_n\left(q\binom{x_0}{y} - q\binom{x_0}{y_0} + b_0\right) : y \in Y\right\}\right)\\
&= \mathrm{mod}_n\left(\left\{q\binom{x}{y_0} - q\binom{x_0}{y_0} + a_0 + q\binom{x_0}{y} - q\binom{x_0}{y_0} + b_0 : (x, y) \in X \times Y\right\}\right)\\
&= \mathrm{mod}_n\left(\left\{q\binom{x}{y} - q\binom{x_0}{y} + a_0 + q\binom{x_0}{y} - q\binom{x_0}{y_0} + b_0 : (x, y) \in X \times Y\right\}\right)\\
&= \mathrm{mod}_n\left(\left\{q\binom{x}{y} + a_0 + b_0 - q\binom{x_0}{y_0} : (x, y) \in X \times Y\right\}\right)\\
&= \mathrm{mod}_n\left(\left\{q\binom{x}{y} : (x, y) \in X \times Y\right\}\right)\\
&= Z.
\end{aligned}$$

By the previous proposition, ${}_n(r', s')$ is a partial tiling. Let $q' = r' \circ s'$. By definition, r' is the n-beat rhythm with range Z. But so is r, so $r' = r$. Also by definition,

$$s' = (r')^{-1} \circ \oplus_n|_{A \times B} = r^{-1} \circ \oplus_n|_{A \times B}.$$

Define

$$\sigma_1(x) = \mathrm{mod}_n \left(q\binom{x}{y_0} - q\binom{x_0}{y_0} + a_0 \right)$$

for $x \in X$,

$$\sigma_2(y) = \mathrm{mod}_n \left(q\binom{x_0}{y} - q\binom{x_0}{y_0} + b_0 \right)$$

for $y \in Y$, and

$$\sigma\binom{x}{y} = (\sigma_1(x), \sigma_2(y))$$

for $\binom{x}{y} \in X \times Y$. Then for $(a, b) \in A \times B$, we have the following.

$$\begin{aligned}
q'\binom{a}{b} &\equiv_n q'\binom{a}{b_0} + q'\binom{a_0}{b} - q'\binom{a_0}{b_0} \\
&\equiv_n (a + b_0) + (a_0 + b) - (a_0 + b_0) \\
&\equiv_n a + b \\
&\equiv_n \left[q\binom{\sigma_1^{-1}(a)}{y_0} - q\binom{x_0}{y_0} + a_0 \right] + \left[q\binom{x_0}{\sigma_2^{-1}(b)} - q\binom{x_0}{y_0} + b_0 \right] \\
&\equiv_n \left[q\binom{\sigma_1^{-1}(a)}{y_0} + q\binom{x_0}{\sigma_2^{-1}(b)} - q\binom{x_0}{y_0} \right] + \left[a_0 + b_0 - q\binom{x_0}{y_0} \right] \\
&\equiv_n q\binom{\sigma_1^{-1}(a)}{\sigma_2^{-1}(b)} + 0 \\
&\equiv_n (q \circ \sigma^{-1})\binom{a}{b}
\end{aligned}$$

Applying $r^{-1} = (r')^{-1}$ to both sides, we get $s' = r \circ \sigma^{-1}$. So ${}_n(r', s')$ is a relabeling of ${}_n(r, s)$. □

Example 58. Consider the partial tiling ${}_{10}(r, s)$ defined as follows.

(x, y)	$\binom{1}{A}$	$\binom{3}{B}$	$\binom{4}{B}$	$\binom{2}{A}$	$\binom{1}{B}$	$\binom{3}{A}$	$\binom{4}{A}$	$\binom{2}{B}$
$s(x, y)$	0	1	2	3	4	5	6	7
$r(s(x, y))$	1	2	3	4	6	7	8	9

Choose $(x_0, y_0) = (4, B)$, $a_0 = 1$, and $b_0 = 2$, so that $a_0 + b_0 \equiv_{10} = 3 \equiv_{10} q(4, B)$. Then letting

$$A = \left\{ \mathrm{mod}_{10} \left(q\binom{x}{B} - q\binom{4}{B} + 1 \right) : x \in X \right\} = \{0, 1, 4, 7\}$$

and

$$B = \left\{ \mathrm{mod}_{10} \left(q\binom{4}{y} - q\binom{4}{B} + 2 \right) : y \in X \right\} = \{2, 7\},$$

we can see that $(A, B)_{10}$ is indeed a partial tiling.

Then we can calculate the function representation ${}_n(r', s')$ of $(A, B)_n$ as follows.

(x,y)	$\binom{4}{7}$	$\binom{0}{2}$	$\binom{1}{2}$	$\binom{7}{7}$	$\binom{4}{2}$	$\binom{0}{7}$	$\binom{1}{7}$	$\binom{7}{2}$
$s'(x,y)$	0	1	2	3	4	5	6	7
$r'(s'(x,y))$	1	2	3	4	6	7	8	9

This is a relabeling of ${}_n(r, s)$ by the following mappings.

x	0	1	4	7
$\sigma_1(x)$	3	4	1	2

y	2	7
$\sigma_2(y)$	B	A

Proposition 59. Let ${}_{n_1}(r_1, s)$ and ${}_{n_2}(r_2, s)$ be partial tilings. Let $j_1, j_2 \in \mathbb{Z}$ such that $r = j_1 r_1 + j_2 r_2$ is strictly increasing, and let $n = j_1 n_1 + j_2 n_2$. Then ${}_n(r, s)$ is a partial tiling.

Proof. Let $q_1 = r_1 \circ s$, $q_2 = r_2 \circ s$, and $q = r \circ s$. For $\binom{x_1}{y_1}, \binom{x_2}{y_2} \in X \times Y$, we have the following.

$$\begin{aligned}
&q\binom{x_1}{y_1} - q\binom{x_1}{y_2}\\
&\equiv_n \left[j_1q_1\binom{x_1}{y_1} + j_2q_2\binom{x_1}{y_1}\right] - \left[j_1q_1\binom{x_1}{y_2} + j_2q_2\binom{x_1}{y_2}\right]\\
&\equiv_n j_1\left[q_1\binom{x_1}{y_1} - q_1\binom{x_1}{y_2}\right] + j_2\left[q_2\binom{x_1}{y_1} - q_2\binom{x_1}{y_2}\right]\\
&\equiv_n j_1\left[q_1\binom{x_2}{y_1} - q_1\binom{x_2}{y_2}\right] + j_2\left[q_2\binom{x_2}{y_1} - q_2\binom{x_2}{y_2}\right]\\
&\equiv_n \left[j_1q_1\binom{x_2}{y_1} + j_2q_2\binom{x_2}{y_1}\right] - \left[j_1q_1\binom{x_2}{y_2} + j_2q_2\binom{x_2}{y_2}\right]\\
&\equiv_n q\binom{x_2}{y_1} - q\binom{x_2}{y_2}
\end{aligned}$$

So ${}_n(r, s)$ is a partial tiling. □

Example 60. Consider the partial tilings ${}_7(r_1, s)$ and ${}_9(r_2, s)$ defined below.

(x, y)	$\binom{1}{A}$	$\binom{1}{B}$	$\binom{2}{A}$	$\binom{2}{B}$	$\binom{3}{A}$	$\binom{3}{B}$
$s(x, y)$	0	1	2	3	4	5
$r_1(s(x, y))$	0	1	3	4	5	6
$r_2(s(x, y))$	0	2	3	5	6	8

Now let us calculate ${}_{4(7)-1(9)}(4r_1 - 1r_2, s)$.

(x, y)	$\binom{1}{A}$	$\binom{1}{B}$	$\binom{2}{A}$	$\binom{2}{B}$	$\binom{3}{A}$	$\binom{3}{B}$
$s(x, y)$	0	1	2	3	4	5
$(4r_1 - 1r_2)(s(x, y))$	0	2	9	11	14	16

One can check that this is indeed a partial tiling.

4.4 Regular maximally even tilings

Proposition 61. Let ${}_n(r, s)$ be a k-note regular maximally even tiling. Then

- ${}_{hk+n}(h\,\mathrm{id}_{[0,k)} + r, s)$ is a regular maximally even tiling for all $h \geq 1 - \left\lfloor \frac{n}{k} \right\rfloor$, and

- ${}_{hk-n}(h\,\mathrm{id}_{[0,k)} - r, s)$ is a regular maximally even tiling for all $h \geq 1 + \left\lceil \frac{n}{k} \right\rceil$.

Proof. This result follows directly from Propositions 59 and 35. □

Proposition 62. For every k-note n-beat regular maximally even tiling ${}_{n}(r, s)$ there exists a regular maximally even tiling ${}_{n'}(r', s)$ with $n' \leq 1.5k$ such that ${}_{n}(r, s)$ can be expressed as

- ${}_{hk+n'}(h\,\mathrm{id}_{[0,k)} + r', s)$ for some $h \geq 0$, or

- ${}_{hk-n'}(h\,\mathrm{id}_{[0,k)} - r', s)$ for some $h \geq 3$.

Proof. Consider the following cases.

- Suppose that $\mathrm{mod}_k(n) \leq .5k$. Let $h = \left\lfloor \frac{n}{k} \right\rfloor - 1$, $n' = -hk + n$, and $r' = -h\,\mathrm{id}_{[0,k)} + r$. By Proposition 61, ${}_{n'}(r', s)$ is a regular maximally even tiling with

$$n' = \left(1 - \left\lfloor \frac{n}{k} \right\rfloor\right) k + n = k + \left(n - \left\lfloor \frac{n}{k} \right\rfloor k\right) = k + \mathrm{mod}_k(n) \leq k + .5k = 1.5k.$$

  Then note that $hk + n' = n$, $h\,\mathrm{id}_{[0,k)} + r' = r$, and $h = \left\lfloor \frac{n}{k} \right\rfloor - 1 \geq 0$.

- Suppose that $\mathrm{mod}_k(n) > .5k$. Let $h = 1 + \left\lceil \frac{n}{k} \right\rceil$, $n' = hk - n$, and $r' = h\,\mathrm{id}_{[0,k)} - r$. By Proposition 61, ${}_{n'}(r', s)$ is a regular maximally even tiling with

$$\begin{aligned} n' &= \left(1 + \left\lceil \frac{n}{k} \right\rceil\right) k - n \\ &= \left(2 + \left\lfloor \frac{n}{k} \right\rfloor\right) k - n \\ &= 2k + \left(\left\lfloor \frac{n}{k} \right\rfloor k - n\right) \\ &= 2k - \mathrm{mod}_k(n) \\ &< 2k - .5k \\ &= 1.5k. \end{aligned}$$

  Then note that $hk - n' = n$, $h\,\mathrm{id}_{[0,k)} - r' = r$, and $h = 1 + \left\lceil \frac{n}{k} \right\rceil \geq 3$.

$\square$

Example 63. Consider the regular maximally even tiling ${}_{142}(r, s)$ defined below.

(x,y)	$\binom{1}{A}$	$\binom{2}{A}$	$\binom{3}{C}$	$\binom{1}{B}$	$\binom{2}{B}$	$\binom{3}{A}$	$\binom{1}{C}$	$\binom{2}{C}$	$\binom{3}{B}$
$s(x,y)$	0	1	2	3	4	5	6	7	8
$r(s(x,y))$	0	16	32	48	64	79	95	111	127

Then ${}_{142}(r, s)$ can be expressed as ${}_{17(9)-11}(17\mathrm{id}_{[0,9)} - r', s)$, where ${}_{11}(r', s)$ is a the regular maximally even tiling given below.

(x,y)	$\binom{1}{A}$	$\binom{2}{A}$	$\binom{3}{C}$	$\binom{1}{B}$	$\binom{2}{B}$	$\binom{3}{A}$	$\binom{1}{C}$	$\binom{2}{C}$	$\binom{3}{B}$
$s(x,y)$	0	1	2	3	4	5	6	7	8
$r'(s(x,y))$	0	1	2	3	4	6	7	8	9

4.5 Irregular maximally even tilings

In this section, we will give an important necessary condition for the existence of an irregular k-note n-beat tiling.

Proposition 64. Let $n \geq k$, and let $\gcd(n, k) = g$. Then there exists $i \in [0, k)$ such that

$$\left\lceil \frac{in}{k} \right\rceil = \left\lfloor \frac{(i+1)n}{k} \right\rfloor$$

if and only if $n \leq 2(k - g)$.

Proof. Suppose that $\left\lceil \frac{in}{k} \right\rceil = \left\lfloor \frac{(i+1)n}{k} \right\rfloor = j$ for some $i \in [0, k)$. Let $n' = n/g$ and $k' = k/g$. Then we have the following.

$$j - 1 < \frac{in}{k} = \frac{in'}{k'} \leq j \leq \frac{(i+1)n'}{k'} = \frac{(i+1)n}{k} < j + 1$$

So we can write

$$\frac{in'}{k'} = j - 1 + \frac{r_1}{k'} \quad \text{and} \quad \frac{(i+1)n'}{k'} = j + \frac{r_2}{k'}$$

where $0 < r_1 \leq k'$ and $0 \leq r_2 < k'$. Now note that

$$\begin{aligned}
\frac{n}{k} &= \frac{n'}{k'} \\
&= \frac{(i+1)n'}{k'} - \frac{in'}{k'} \\
&= \left(j + \frac{r_2}{k'}\right) - \left(j - 1 + \frac{r_1}{k'}\right) \\
&= 1 + \frac{r_2}{k'} - \frac{r_1}{k'} \\
&\leq 1 + \frac{k'-1}{k'} - \frac{1}{k'} \\
&= 2 - \frac{2}{k'}
\end{aligned}$$

which yields $n \leq 2(k-g)$.

Now suppose that $n \leq 2(k-g)$. Choose $h, j \in \mathbb{Z}$ such that $hn + jk = g$. Use the division algorithm to write $h = kq + i$ where $0 \leq i < k$. Note that

$$\frac{in}{k} = \frac{(h-kq)n}{k} = \frac{hn - kqn}{k} = \frac{g - jk - kqn}{k} = \frac{g}{k} - j - qn.$$

Since $1 \leq g \leq k$, we have $\left\lceil \frac{in}{k} \right\rceil = 1 - j - qn$. Also,

$$\begin{aligned}
1 - j - qn &< \frac{g+n}{k} - j - qn \\
&= \frac{(i+1)n}{k}
\end{aligned}$$

and

$$\begin{aligned}\frac{(i+1)n}{k} &= \frac{g+n}{k} - j - qn \\ &\leq \frac{g+2(k-g)}{k} - j - qn \\ &= 2 - \frac{g}{k} - j - qn \\ &< 2 - j - qn\end{aligned}$$

So we have $\left\lfloor \frac{(i+1)n}{k} \right\rfloor = 1 - j - qn$. Thus, $\left\lceil \frac{in}{k} \right\rceil = \left\lfloor \frac{(i+1)n}{k} \right\rfloor$.

$\square$

Proposition 65. If a k-note n-beat maximally even tiling is irregular, then $n \leq 2(k - \gcd(n,k))$.

Proof. Let ${}_n(r,s)$ be a k-note n-beat irregular maximally even tiling, and let $q = r \circ s$. Since ${}_n(r,s)$ is irregular, ${}_k(\mathrm{id}_{[0,k)}, s)$ is not a tiling. So we can choose $x_1, x_2 \in X$, and $y_1, y_2 \in Y$ such that

$$s\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - s\begin{pmatrix} x_1 \\ y_2 \end{pmatrix} \not\equiv_k s\begin{pmatrix} x_2 \\ y_1 \end{pmatrix} - s\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

Let $i = \mathrm{mod}_k \left(s\binom{x_1}{y_1} - s\binom{x_1}{y_2} \right)$ and $j = \mathrm{mod}_k \left(s\binom{x_2}{y_1} - s\binom{x_2}{y_2} \right)$. Without loss of generality we can assume that $i < j$. Since r is a k-note n-beat maximally even rhythm we have the following.

$$
\begin{aligned}
&\mathrm{mod}_n\left(q\binom{x_1}{y_1}-q\binom{x_1}{y_2}\right)\\
=&\mathrm{mod}_n\left(r\left(s\binom{x_1}{y_1}\right)-r\left(s\binom{x_1}{y_2}\right)\right)\\
\in&\left\{\left\lfloor\frac{\mathrm{mod}_k\left(s\binom{x_1}{y_1}-s\binom{x_1}{y_2}\right)n}{k}\right\rfloor,\left\lceil\frac{\mathrm{mod}_k\left(s\binom{x_1}{y_1}-s\binom{x_1}{y_2}\right)n}{k}\right\rceil\right\}\\
=&\left\{\left\lfloor\frac{in}{k}\right\rfloor,\left\lceil\frac{in}{k}\right\rceil\right\}
\end{aligned}
$$

Similarly we have the following.

$$
\mathrm{mod}_n\left(q\binom{x_2}{y_1}-q\binom{x_2}{y_2}\right)\in\left\{\left\lfloor\frac{jn}{k}\right\rfloor,\left\lceil\frac{jn}{k}\right\rceil\right\}
$$

Now note that

$$
\begin{aligned}
\mathrm{mod}_n\left(q\binom{x_1}{y_1}-q\binom{x_1}{y_2}\right)&\le\left\lceil\frac{in}{k}\right\rceil\\
&\le\left\lfloor\frac{(i+1)n}{k}\right\rfloor\\
&\le\left\lfloor\frac{jn}{k}\right\rfloor\\
&\le\mathrm{mod}_n\left(q\binom{x_2}{y_1}-q\binom{x_2}{y_2}\right).
\end{aligned}
$$

Since ${}_n(r,s)$ is a partial tiling, $q\binom{x_1}{y_1}-q\binom{x_1}{y_2}\equiv_n q\binom{x_2}{y_1}-q\binom{x_2}{y_2}$. So equality holds throughout the previous series of inequalities. Therefore,

$$
\left\lceil\frac{in}{k}\right\rceil=\left\lfloor\frac{(i+1)n}{k}\right\rfloor.
$$

So by the previous proposition, $n \leq 2(k - \gcd(n,k))$. $\square$

Example 66. Suppose that we wanted to find all irregular 12-note maximally even tilings. According to Proposition 65, we would only need to look for those of length $n = 13, 14, 15, 16, 17, 19$. This is a very useful narrowing of our search.

4.6 Unit multiplication

Definition 67. Let ${}_n(r,s)$ be a k-beat n-note partial tiling and u be a modulo n unit. By Proposition 41, there is a unique permutation π on $[0,k)$ such that $\pi_{n,u} \circ r \circ \pi$ is strictly increasing. We will call $\mathrm{MULT}_u[{}_n(r,s)] =_n (\pi_{n,u} \circ r \circ \pi, \pi^{-1} \circ s)$ a unit multiple of ${}_n(r,s)$.

Proposition 68. Let ${}_n(r,s)$ be a k-beat n-note partial tiling and u be a modulo n unit. Then $\mathrm{MULT}_u[{}_n(r,s)]$ is a k-beat n-note partial tiling.

Proposition 69. Let ${}_n(r,s)$ be a k-note n-beat maximally even tiling. Let u be a modulo n unit that is equivalent modulo n to a divisor of $n - jk$ for some $j \in \left[1, \frac{n}{k}\right]$. Then $\mathrm{MULT}_u[{}_n(r,s)]$ is regular if and only if ${}_n(r,s)$ is regular.

Proof. By Proposition 43, there exists a modulo k unit v such that

$$\mathrm{MULT}_u[{}_n(r,s)] =_n (\pi_{n,u} \circ \pi_{k,v}, \pi_{k,v}^{-1} \circ s).$$

Suppose that ${}_n(r,s)$ is regular. Then ${}_k(\mathrm{id}_k, s)$ is a full tiling. Let v' be a modulo k multiplicative inverse of v. For $\binom{x_1}{y_1}, \binom{x_2}{y_2} \in X \times Y$, we have the following.

$$\begin{aligned}
(\pi_{k,v}^{-1} \circ s)\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - (\pi_{k,v}^{-1} \circ s)\begin{pmatrix} x_1 \\ y_2 \end{pmatrix} &\equiv_k v' s\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - v' s\begin{pmatrix} x_1 \\ y_2 \end{pmatrix} \\
&\equiv_k v' \left(s\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} - s\begin{pmatrix} x_1 \\ y_2 \end{pmatrix} \right) \\
&\equiv_k v' \left(s\begin{pmatrix} x_2 \\ y_1 \end{pmatrix} - s\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right) \\
&\equiv_k v' s\begin{pmatrix} x_2 \\ y_1 \end{pmatrix} - v' s\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \\
&\equiv_k (\pi_{k,v}^{-1} \circ s)\begin{pmatrix} x_2 \\ y_1 \end{pmatrix} - (\pi_{k,v}^{-1} \circ s)\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}
\end{aligned}$$

So ${}_k(\mathrm{id}_k, \pi_{k,v}^{-1} \circ s)$ is a full tiling. Therefore, $\mathrm{MULT}_u[_n(r, s)]$ is regular. The converse can be proved in a similar way. $\square$

Chapter 5

ALGORITHMS

In this chapter we will describe a brute-force algorithm for finding maximally even tilings, and then we will develop a more efficient algorithm that seems to produce the same results.

5.1 A brute-force search: `METv1`

In this section we will describe a general method for finding all partial tilings with a given range, and then apply that method to the task of finding maximally even tilings.

Definition 70. Let A_1 and A_2 be sets of non-negative integers. We will say that $A_1 \geq A_2$ if

$$\sum_{i \in A_1} 2^{-i} \geq \sum_{i \in A_2} 2^{-i}.$$

This defines a total order on sets of non-negative integers. (This is essentially a lexicographical order on sets of non-negative integers.) We will say that a partial tiling $(A, B)_n$ is in standard form if

- A and B are greatest among their respective modulo n translations and
- $A \geq B$.

Example 71. Every partial tiling can be put in standard form by translation and duality. Consider the partial tiling $(\{0, 2, 11\}, \{3, 4, 10, 11\})_{13}$. Note that

- $\{0, 2, 4\}$ is the greatest modulo 13 translation of $\{0, 2, 11\}$,
- $\{0, 1, 6, 7\}$ is the greatest modulo 13 translation of $\{3, 4, 10, 11\}$, and
- $\{0, 1, 6, 7\} \geq \{0, 2, 4\}$.

Thus, $(\{0,1,6,7\},\{0,2,4\})_{13}$ is the standard form representation of $(\{0,2,11\},\{3,4,10,11\})_{13}$.

Proposition 72. Let $n \geq 1$, $A \subseteq [0,n)$, and $j,t \in [0,n)$. If $A \geq \mathrm{mod}_n(A-t)$, then $A \cap [0,j] \geq \mathrm{mod}_n(A \cap [0,j] - t)$. (Thus, if A is greatest among its modulo n translations, then so is $A \cap [0,j]$.)

Proof. Suppose that $A \geq \mathrm{mod}_n(A-t)$. Then

$$\sum_{i \in A} 2^{-i} \geq \sum_{i \in \mathrm{mod}_n(A-t)} 2^{-i}.$$

Consider the following cases.

- Suppose that $j \geq t$. Then

$$\begin{aligned}
\sum_{i \in A \cap [0,j]} 2^{-i} &= \left(\sum_{i \in A} 2^{-i}\right) - \left(\sum_{i \in A \cap (j,n)} 2^{-i}\right) \\
&\geq \left(\sum_{i \in \mathrm{mod}_n(A-t)} 2^{-i}\right) - \left(\sum_{i \in A \cap (j,n)} 2^{-(i-t)}\right) \\
&= \left(\sum_{i \in \mathrm{mod}_n(A-t)} 2^{-i}\right) - \left(\sum_{i \in \mathrm{mod}_n(A \cap (j,n) - t)} 2^{-i}\right) \\
&= \sum_{i \in \mathrm{mod}_n(A \cap [0,j] - t)} 2^{-i}
\end{aligned}$$

- Suppose that $j < t$. Then

$$\begin{aligned}
\sum_{i \in A \cap [0,j]} 2^{-i} &\geq \sum_{i \in A \cap [0,j]} 2^{-(i-t+n)} \\
&= \sum_{i \in \mathrm{mod}_n(A \cap [0,j] - t)} 2^{-i}
\end{aligned}$$

Therefore, $A \cap [0,j] \geq \mathrm{mod}_n(A \cap [0,j] - t)$. $\square$

Algorithm 73. Let C be an order k subset of $[0, n)$. Define the following main procedure and recursive subroutine.

`PT`

1. Output the standard form of the trivial partial tiling $(C, \{0\})_n$.
2. For each $0 < i < j < n$, call `PTrec`$(\{0, i\}, \{0, j\}, 1)$.

`PTrec`(A, B, STAGE)

1. If the restriction of $\oplus_n$ to $A \times B$ fails to be injective, then exit this subroutine.
2. If the range of $(A, B)_n$ fails to lie within a modulo n translation of C, then exit this subroutine.
3. If A or B fail to be greatest among their respective modulo n translations, then exit this subroutine.
4. If the least divisor of k greater than or equal to $|A|$ times the least divisor of k greater than or equal to $|B|$ is greater than k, then exit this subroutine.
5. If $|A||B| = k$, then output $(A, B)_n$ and exit this subroutine.
6. If $\mathrm{STAGE} = 1$, then do the following.
   (a) If $|A| = |B|$, then do the following.
      i. Call `PT`$(A \cup \{i\}, B, 1)$ for each $i \in (\max A, n)$.
      ii. If $|A|$ divides k, then call `PT`$(A, B \cup \{j\}, 3)$ for each $j \in (\max B, n)$.
   (b) Otherwise, if $|A| \neq |B|$, then do the following.
      i. Call `PT`$(A, B \cup \{j\}, 1)$ for each $j \in (\max B, n)$.
      ii. If $|B|$ divides k, then call $(A \cup \{i\}, B, 2)$ for each $i \in (\max A, n)$.
7. If $\mathrm{STAGE} = 2$, then call `PT`$(A \cup \{i\}, B, 2)$ for each $i \in (\max A, n)$.
8. If $\mathrm{STAGE} = 3$, then call `PT`$(A, B \cup \{j\}, 3)$ for each $j \in (\max B, n)$.

Proposition 74. The `PT` algorithm produces all standard form partial tilings whose ranges are modulo n translations of C.

Justification. Each component set of any nontrivial solution in standard form must contain 0 and at least one other value, where the least nonzero element of the first component is less than the least nonzero element of the second component. Thus, we can use $\{0, i\}$ and $\{0, j\}$ with $i < j$ as a starting point (since we will continue appending values to each component in increasing order).

At each recursive step, we check that the two sets are on track to being a partial tiling whose range is a modulo n translation of C. Proposition 72 allows us to check that each component is greatest among its modulo n translations. We also check that the cardinalities of the two components fit within a factorization of k. (For example, a partial tiling in which $|A| = 3$ and $|B| = 3$ cannot be expanded to a 22-note partial tiling.) If these criteria are met, then we can either output the result (if an entire modulo n translation of C has been covered) or continue appending.

The appending process comes in three stages. During stage 1, we alternate between the two components, appending to A first, then B, then A, and so on. When we reach a point where $|A|$ is a divisor of k, we can branch off to stage 2, in which we only append to B. Likewise, when we reach a point where $|B|$ is a divisor of k, we can branch off to stage 3, in which we only append to A.

Algorithm 75. Given $k_{\min} \leq k_{\max}$. Define the following procedure.

`METv1`

1. For each $k \in [k_{\min}, k_{\max}]$ and each n such that $k \leq n \leq 1.5k$ or $k \leq n \leq 2(k - \gcd(k, n))$, use `PT` to find partial tilings whose ranges are modulo n translations of $C = X^{\mathrm{flr}}_{k,n}$.

Proposition 76. The `METv1` algorithm produces all k-note maximally even tilings for $k \in [k_{\min}, k_{\max}]$ up to translation, duality, and step expansion/swapping of the regular ones.

Justification. Recall that all regular maximally even tilings are step expansions/swappings of those with $k \leq n \leq 1.5k$ and that we have $k \leq n \leq 2(k - \gcd(k, n))$ for all irregular maximally even tilings.

Discussion. The appendix shows how I have implemented `METv1` in Excel/VBA. It takes my computer approximately 100 minutes to complete this procedure for $k \in [4, 18]$, and it produces a total of 4093 standard form maximally even tilings. Going much further beyond $k = 18$ quickly becomes intractable.

5.2 Unit multiplication and `METv2`

Over the course of the next few sections, we will develop a procedure for finding maximally even tilings that is significantly more efficient than `METv1`. Our hope will be to derive all maximally even tilings from De Bruijn-type full tilings, since these can be calculated relatively quickly. In this section we will see how to construct partial tilings that are unit multiples of regular partial tilings based on De Bruijn-type full tilings, and then we will apply that method to the task of finding maximally even tilings.

Algorithm 77. Let $n \geq 1$. Define the following procedure.

`DeB`

1. Let $p_1 p_2 ... p_j$ be the prime factorization of n.
2. For each nontrivial partition of $[1, j]$ into two sets E and F with $1 \in E$, and each sequence $\{c_i\}_{i=1}^{j}$ formed by permuting $\{p_i\}_{i=1}^{j}$ and satisfying $c_i \leq c_{i+1}$ when i and $i+1$ are in the same member of the partition, do the following.
   (a) Let $d_0 = 1$ and $d_i = \Pi_{i=1}^{j} c_i$ for $i \in [1, j]$.
   (b) Let $A = \{0\} + \sum_{i \in E} d_{i-1}[0, c_i)$.
   (c) Let $B = \{0\} + \sum_{i \in F} d_{i-1}[0, c_i)$.
   (d) Output $(A, B)_n$.

Proposition 78. The `DeB` algorithm produces all nontrivial n-note De Bruijn-type full tilings up to duality.

Justification. This follows directly from Proposition and Definition 13, with the requirement that $c_i \leq c_{i+1}$ when i and $i+1$ are in the same member of the partition simply eliminating some redundant results.

Example 79. Suppose that we want to construct a [k=10]-note [n=13]-beat maximally even tiling with range

$$C = X^{\text{flr}}_{10,13} = \{0, 1, 2, 3, 5, 6, 7, 9, 10, 11\}.$$

- Let us choose the following $t \in C$, modulo n unit u, and k-note De Bruijn-type full tiling.

  $$t = 3, u = 4, (A, B)_k = (\{0, 2, 4, 6, 8\}, \{0, 1\})_{10}$$

  Consider the following transformation of C.

  $$C' = \text{mod}_n(u(C - t)) = \{0, 1, 2, 3, 5, 6, 8, 9, 11, 12\}$$

  Let r be the k-note n-beat rhythm with range C'. Also, let ${}_k(\text{id}_{[0,k)}, s)$ be the function representation of $(A, B)_k$. Then ${}_n(r, s)$ is as follows.

(x, y)	$\binom{0}{0}$	$\binom{0}{1}$	$\binom{2}{0}$	$\binom{2}{1}$	$\binom{4}{0}$	$\binom{4}{1}$	$\binom{6}{0}$	$\binom{6}{1}$	$\binom{8}{0}$	$\binom{8}{1}$
$s(x, y)$	0	1	2	3	4	5	6	7	8	9
$r(s(x, y))$	0	1	2	3	5	6	8	9	11	12

  We can check that ${}_n(r, s)$ is a partial tiling, and its subset representation is $(A', B')_n = (\{0, 2, 5, 8, 11\}, \{0, 1\})_{13}$. Let $v = 10$ be the multiplicative inverse of $u = 4$ modulo $n = 13$. Then the following is a 10-note 13-beat a maximally even tiling with range C.

  $$(\text{mod}_n(vA'), \text{mod}_n(vB' + t))_n = (\{0, 2, 6, 7, 11\}, \{0, 3\})_{13}$$

- Now let us choose the following $t \in C$, modulo n unit u, and k-note De Bruijn-type full tiling. (The difference from the first part of this example is the value of t.)

$$t = 2, u = 4, (A, B)_k = (\{0, 2, 4, 6, 8\}, \{0, 1\})_{10}$$

Consider the following transformation of C.

$$C' = \mathrm{mod}_n(u(C - t)) = \{0, 2, 3, 4, 5, 6, 7, 9, 10, 12\}$$

Let r be the k-note n-beat rhythm with range C'. Also, let ${}_k(\mathrm{id}_{[0,k)}, s)$ be the function representation of $(A, B)_k$. Then ${}_n(r, s)$ is as follows.

(x, y)	$\binom{0}{0}$	$\binom{0}{1}$	$\binom{2}{0}$	$\binom{2}{1}$	$\binom{4}{0}$	$\binom{4}{1}$	$\binom{6}{0}$	$\binom{6}{1}$	$\binom{8}{0}$	$\binom{8}{1}$
$s(x, y)$	0	1	2	3	4	5	6	7	8	9
$r(s(x, y))$	0	2	3	4	5	6	7	9	10	11

Unfortunately, this is not a partial tiling (e.g., $q(0, 1) - q(0, 0) \not\equiv_{13} q(2, 1) - q(2, 0)$), so we cannot continue further as we did in the previous case.

While not all combinations of $t \in C$, modulo n unit u, and De Bruijn-type full tiling yield results, we will find that this is still a very powerful means of constructing partial tilings with a given range.

Algorithm 80. Let $C \subseteq [0, n)$ with $|C| = k$. Define the following procedure.

`UnitMult`

1. Output the standard form of the trivial partial tiling $(C, \{0\})_n$.
2. Generate a list of k-note De Bruijn-type full tilings using `DeB`.
3. For each modulo n unit $u \in [0, n)$, each $t \in C$, and each k-note De Bruijn-type full tiling $(A, B)_n$ generated in the previous step, do the following.
   (a) Let r be the k-note n-beat rhythm with range $\mathrm{mod}_n(u(C - t))$.
   (b) Let ${}_k(\mathrm{id}_{[0,k)}, s)$ be the function representation of $(A, B)_k$.

(c) If ${}_n(r,s)$ is a partial tiling, then do the following.

i. Let $(A', B')_n$ be the subset representation of ${}_n(r,s)$, and let v be a modulo n inverse of u.

ii. Output the standard form of $(\mathrm{mod}_n(vA'), \mathrm{mod}_n(vB'+t))_n$.

Algorithm 81. Given $k_{\min} \leq k_{\max}$. Define the following procedure.

`METv2`

1. For each $k \in [k_{\min}, k_{\max}]$ and each n such that $k \leq n \leq 1.5k$ or $k \leq n \leq 2(k - \gcd(k,n))$, use `UnitMult` to find partial tilings whose ranges are modulo n translations of $C = X^{\mathrm{flr}}_{k,n}$.

Discussion. The appendix shows how I have implemented `METv2` in Excel/VBA. It takes my computer approximately 4 minutes to complete this procedure for $k \in [4, 18]$ (versus approximately 100 minutes for `METv1`), but it only produces 1501 standard form maximally even tilings (versus 4093 found by `METv1`). The following is one of the maximally even tilings that `METv2` fails to find.

$$(\{0,1,4,7,8\},\{0,2\})_{12}$$

Indeed, one can check that this not a unit multiple of any regular partial tiling (De Bruijn-based or otherwise). So additional techniques will be required.

5.3 Multiplexation and `METv3`

In the context of full tilings, the operation of multiplexation has proved useful [5]. In this section, we describe how multiplexation can be applied to the construction of partial tilings.

Proposition and Definition 82. Let $C \subseteq [0,n)$, and let $d|n$. Let $C_i = \frac{(C-i)\cap(d\mathbb{Z})}{d}$ for each $i \in [0,d)$, and let $\mathcal{I}$ be the set of $i \in [0,d)$ such that $C_i \neq \emptyset$. Suppose that $(A, B_i)_{n/d}$ is a partial tiling with range C_i for each $i \in \mathcal{I}$. Then

$$(A', B')_n = (dA, \cup_{i\in\mathcal{I}}(dB_i + i))_n$$

is a partial tiling with range C. We will call $(A', B')_n$ the multiplexation of $\{(A, B_i)_{n/d}\}_{i\in\mathcal{I}}$. (Of course, we can also switch the roles of the first and second components.)

Proof. Suppose that $\mathrm{mod}_n(a'_1 + b'_1) = \mathrm{mod}_n(a'_2 + b'_2) = c$ for $(a'_1, b'_1), (a'_2, b'_2) \in A' \times B'$. Since a'_1 and a'_2 are multiples of d, we have $b'_1 \equiv_d b'_2 \equiv_d c$. If we let $i_0 = \mathrm{mod}_d(c)$, then we have $b'_1, b'_2 \in dB_{i_0} + i_0$ and $c \in C_{i_0}$.

Choose $a_1, a_2 \in A$ such that $a'_1 = da_1$ and $a'_2 = da_2$. Choose $b_1, b_2 \in B_{i_0}$ such that $b'_1 = db_1 + i_0$ and $b'_2 = db_2 + i_0$. Then we have the following.

$$a'_1 + b'_1 \equiv_n a'_2 + b'_2$$

$$da_1 + db_1 + i_0 \equiv_n da_2 + db_2 + i_0$$

$$d(a_1 + b_1) \equiv_n d(a_2 + b_2)$$

$$a_1 + b_1 \equiv_{n/d} a_2 + b_2$$

Since $(A, B_{i_0})_{n/d}$ is a partial tiling, $(a_1, b_1) = (a_2, b_2)$, which in turn implies that $(a'_1, b'_1) =$

(a_2', b_2'). Thus $(A', B')_n$ is a partial tiling, and its range is

$$\begin{aligned}
\mathrm{mod}_n(dA + \cup_{i\in\mathcal{I}}(dB_i + i)) &= \cup_{i\in\mathcal{I}}\mathrm{mod}_n(dA + dB_i + i) \\
&= \cup_{i\in\mathcal{I}}\mathrm{mod}_n(d(A + B_i) + i) \\
&= \cup_{i\in\mathcal{I}}\mathrm{mod}_n(dC_i + i) \\
&= \cup_{i\in\mathcal{I}}\mathrm{mod}_n\left(d\left(\frac{(C-i)\cap(d\mathbb{Z})}{d}\right) + i\right) \\
&= \cup_{i\in\mathcal{I}}\mathrm{mod}_n\left(C \cap (d\mathbb{Z} + i)\right) \\
&= C.
\end{aligned}$$

$\square$

Example 83. Let $n = 12$, $C = \{0, 1, 2, 3, 4, 6, 7, 8, 9, 10\}$, and $d = 2$. Calculate the following.

$$C_0 = \frac{(C-0)\cap(2\mathbb{Z})}{d} = \{0, 1, 2, 3, 4, 5\}$$

$$C_1 = \frac{(C-1)\cap(2\mathbb{Z})}{d} = \{0, 1, 3, 4\}$$

Note that $(\{0, 1\}, \{0, 2, 4\})_6$ is a partial tiling with range C_0 and $(\{0, 1\}, \{0, 3\})_6$ is a partial tiling with range C_1. The multiplexation of these partial tilings is $(\{0, 2\}, \{0, 1, 4, 7, 8\})_{12}$. Putting this in standard form, we get $(\{0, 1, 4, 7, 8\}, \{0, 2\})_{12}$, which is precisely the maximally even tiling that was previously pointed out to be unreachable by the `METv2` procedure!

Algorithm 84. Let $C \subset [0, n)$ with $|C| = k$. Define the following procedure.

`Multiplex`

1. For each divisor d of n, do the following.

   (a) Let $C_i = \frac{(C-i)\cap(d\mathbb{Z})}{d}$ for each $i \in [0, d)$, and let $\mathcal{I}$ be the set of $i \in [0, d)$ such that $C_i \neq \emptyset$.

(b) If the greatest common multiple of $\{|C_i|\}_{i\in\mathcal{I}}$ is greater than one, then do the following.

i. For each $i \in \mathcal{I}$, use `UnitMult` to generate a set S_i of standard form partial tilings $(A, B)_{n/d}$ whose ranges are modulo n translations of C_i.

ii. For each $((A, B_i)_n)_{i\in\mathcal{I}} \in \bigtimes_{i\in\mathcal{I}} S_i$ and every $(t_i)_{i\in\mathcal{I}}$ such that the range of each $(A, \mathrm{mod}_{n/d}(B_i - t_i))_n$ is C_i, output the standard form of the following.

$$(dA, \cup_{i\in\mathcal{I}}(d\mathrm{mod}_{n/d}(B_i - t_i) + i))_n$$

iii. For each $((A_i, B)_n)_{i\in\mathcal{I}} \in \bigtimes_{i\in\mathcal{I}} S_i$ and every $(t_i)_{i\in\mathcal{I}}$ such that the range of each $(\mathrm{mod}_{n/d}(A_i - t_i), B)_n$ is C_i, output the standard form of the following.

$$(\cup_{i\in\mathcal{I}}(d\mathrm{mod}_{n/d}((A_i - t_i) + i), dB)_n$$

Algorithm 85. Given $k_{\min} \le k_{\max}$. Define the following procedure.

`METv3`

1. For each $k \in [k_{\min}, k_{\max}]$ and each n such that $k \le n \le 1.5k$ or $k \le n \le 2(k - \gcd(k, n))$, use `Multiplex` to find partial tilings whose ranges are modulo n translations of $C = X^{\mathrm{flr}}_{k,n}$.

Discussion. The appendix shows how I have implemented `METv3` in Excel/VBA. It takes my computer approximately 9 minutes to complete this procedure for $k \in [4, 18]$ (versus approximately 100 minutes for `METv1`), but it only produces 4063 standard form maximally even tilings (versus 4093 found by `METv1`). The following is one of the maximally even tilings that `METv3` fails to find.

$$\{0, 1, 16\}, \{0, 3, 7, 10, 14, 17\}_{21}$$

Indeed, one can check that this not a multiplexation of unit multiples of any regular partial tiling based on De Bruijn-type full tilings. So additional techniques will be required.

5.4 Periodic expansion and `METv4`

In this section, we look at a technique that helps us construct periodic partial tilings. Despite the fact that our musical motivation for studying partial tilings was to escape periodicity, we will find that the addition of this technique gives us the mathematical satisfaction of producing maximally even tilings that were unreachable by `METv3`.

Proposition and Definition 86. Let $C \subseteq [0,n)$, and let $d|n$ with $\mathrm{mod}_n(C-d) = C$. Let $(A', B')_d$ be a partial tiling with range $C' = C \cap [0,d)$. Choose $h_{a'} \in [0, n/d)$ for each $a' \in A'$. Then

$$(A, B)_n = (\{dh_{a'} + a' : a' \in A'\}, \cup_{i=0}^{n/d-1}(di + B'))_n$$

is a partial tiling with range C. We will call it a periodic expansion of $(A', B')_d$. (Of course, we can also switch the roles of the first and second components.)

Proof. Suppose that $a_1 + b_1 \equiv_n a_2 + b_2$ for $(a_1, b_1), (a_1, b_1) \in A \times B$. Then

$$\begin{aligned} a_1 &= dh_{a'_1} + a'_1 \\ b_1 &= di_1 + b'_1 \\ a_2 &= dh_{a'_2} + a'_2 \\ b_2 &= di_2 + b'_2 \end{aligned}$$

for some $a'_1, a'_2 \in A'$, $h_{a'_1}, h_{a'_2}, i_1, i_2 \in [0, n/d)$, and $b'_1, b'_2 \in B'$. Then

$$\begin{aligned} a_1 + b_1 &\equiv_n a_2 + b_2 \\ dh_{a'_1} + a'_1 + di_1 + b'_1 &\equiv_n dh_{a'_2} + a'_2 + di_2 + b'_2 \\ a'_1 + b'_1 &\equiv_d a'_2 + b'_2 \end{aligned}$$

which implies that $(a_1', b_1') = (a_2', b_2')$, since $(A', B')_d$ is a partial tiling, which in turn implies that $(a_1, b_1) = (a_2, b_2)$. Therefore, $(A, B)_n$ is a partial tiling, and its range is

$$\begin{aligned}
\mathrm{mod}_n(A+B) &= \mathrm{mod}_n(\{dh_{a'} + a' : a' \in A'\} + \cup_{i=0}^{n/d-1}(di + B')) \\
&= \cup_{i=0}^{n/d-1}\mathrm{mod}_n(\{dh_{a'} + a' + di + b' : a' \in A', b \in B'\}) \\
&= \cup_{i=0}^{n/d-1}\{\mathrm{mod}_d(a' + b') + di : a' \in A', b \in B'\} \\
&= \cup_{i=0}^{n/d-1}(C' + di) \\
&= C.
\end{aligned}$$

□

Example 87. Let $n = 21$, $C = X^{\text{flr}}_{18,21}$, and $d = 7$. Note that $\mathrm{mod}_{21}(C - 7) = C$ and let $C' = C \cap [0, 7) = X^{\text{flr}}_{6,7}$. Note that $(A', B')_d = (\{0, 1, 2\}, \{0, 3\})_7$ is a partial tiling with range C'. Choose the following.

$$h_0 = 0, h_1 = 0, h_2 = 2$$

Then the following is a periodic expansion of $(\{0, 1, 2\}, \{0, 3\})_7$.

$$(\{dh_a + a : a \in A\}, \cup_{i=0}^{j-1}(di + B))_n = (\{0, 1, 14\}, \{0, 3, 7, 10, 14, 17\})_{21}$$

Notice that this is precisely the maximally even tiling that was pointed out to be unreachable by the `METv3` procedure!

Algorithm 88. Let $C \subset [0, n)$. Define the following procedure.

`Periodic`

1. For each divisor d of n such that $\mathrm{mod}_n(C - d) = C$, do the following.
   (a) Use `Multiplex` to find partial tilings with range $C' = C \cap [0, n/d)$.
   (b) For each partial tiling $(A', B')_n$ found in the previous step and each

$\{h_{a'}\}_{a' \in A'} \subseteq [0, n/d)$, output the standard form of following

$$(A, B)_n = (\{dh_{a'} + a' : a' \in A'\}, \cup_{i=0}^{n/d-1}(di + B'))_n$$

Algorithm 89. Given $k_{\min} \le k_{\max}$. Define the following procedure.

`METv4`

1. For each $k \in [k_{\min}, k_{\max}]$ and each n such that $k \le n \le 1.5k$ or $k \le n \le 2(k - \gcd(k, n))$, use `Periodic` to find partial tilings whose ranges are modulo n translations of $C = X_{k,n}^{\text{flr}}$.

Discussion. The appendix shows how I have implemented `METv4` in Excel/VBA. It takes my computer approximately 10 minutes to complete this procedure for $k \in [4, 18]$ (versus approximately 100 minutes for `METv1`), and it produces all 4093 found by `METv1`! However, at some point beyond $k = 18$, `METv4` will fail. Indeed, a construction by Szabo can be used to produce an aperiodic 900-note full tiling has been constructed that cannot be formed by multiplexation of smaller tilings [5, 13, 18]. Such a tiling cannot be constructed through the techniques used in `METv4` (periodic expansion, multiplexation, and unit multiplication of De Bruijn-type tilings).

Chapter 6

CONTINUING WORK

6.1 A stronger necessary condition for irregularity

Recall that Proposition 65 gives us a necessary condition for the existence of an irregular k-note n-beat maximally even tiling: $n \leq 2(k - \gcd(k, n))$. However, this condition is not sufficient. For example, in the case of $k = 18$, the inequality is satisfied for

$$n = 19, 20, 21, 22, 23, 24, 25, 26, 28, 29, 31, 32,$$

but upon examining the solutions found by `METv1`, we find that irregular 18-note n-beat maximally even tilings exist only for

$$n = 19, 20, 21, 23, 25.$$

Thus, it may be possible to discover a stronger necessary condition for the existence of irregular maximally even tilings. In this section, we will look at one possible approach to this goal.

Example 90. Let us attempt to construct an irregular 18-note 29-beat maximally even tiling and see how we fail. Without loss of generality, we can choose the range to be $X^{\text{flr}}_{18,29}$ given below.

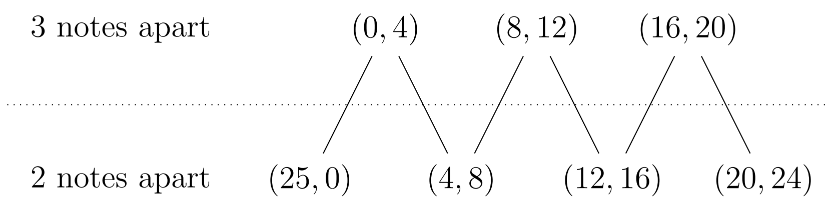
$$0 \;\; 1 \;\cdot\; 3 \;\; 4 \;\cdot\; 6 \;\cdot\; 8 \;\; 9 \;\cdot\; 11 \;\; 12 \;\cdot\; 14 \;\cdot\; 16 \;\; 17 \;\cdot\; 19 \;\; 20 \;\cdot\; 22 \;\cdot\; 24 \;\; 25 \;\cdot\; 27 \;\cdot\; :\|$$

The necessary condition $n \leq 2(k - \gcd(k, n))$ ensures there exists a value (or values) of $i \in [0, 18)$ such that $\lceil \frac{29i}{18} \rceil = \left\lfloor \frac{29(i+1)}{18} \right\rfloor$. In particular, this happens for $i = 2, 5, 7, 10, 12, 15$.

When $i = 2$ we have $\left\lceil \frac{29(2)}{18} \right\rceil = \left\lfloor \frac{29(3)}{18} \right\rfloor = 4$, which means that, of the pairs of notes in this rhythm that are 4 beats apart, some may be 2 notes apart and some may be 3 notes apart. By examination, we find that there are seven pairs of notes that are 4 beats apart: $(4, 8)$, $(12, 16)$, $(20, 24)$, and $(25, 0)$ are 2 notes apart, while $(0, 4)$, $(8, 12)$, and $(16, 20)$ are 3 notes apart. To construct an irregular maximally even tiling that takes advantage of this irregularity, we must choose a subset of these pairs that meets the following criteria.

- We need to use at least one pair from each of the two categories (2 note apart and 3 notes apart).
- We may not use two pairs with a common value (due to the non-overlapping requirement of partial tiling).

The following graph, with edges between pairs sharing a common value, helps us identify sets of pairs that meet these criteria.



By inspection, one quickly finds that the maximum number of pairs that can be chosen satisfying the given criteria is 3. Thus, any irregular 18-note 29-beat maximally even tiling taking advantage of the 4-beat irregularity can have no more than 3 parts. For example, one possible set of pairs is $\{(0, 4), (12, 16), (20, 24)\}$, which would yield the following "skeleton" of an irregular 18-note 29-beat maximally even tiling with range $X^{\text{flr}}_{18,29}$.

	0	1	·	3	4	·	6	·	8	9	·	11	12	·	14	·	16	17	·	19	20	·	22	·	24	25	·	27	·	
Part A	0	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	:‖
Part B	·	·	·	·	·	·	·	·	·	·	·	·	0	·	·	·	1	·	·	·	·	·	·	·	·	·	·	·	·	:‖
Part C	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	0	·	·	·	1	·	·	·	·	:‖

Let us carry out a similar analysis for the irregularity corresponding to $i = 5$, in which some of the notes that are 9 beats apart are 5 notes apart while others are 6 notes apart.

The graph showing the relationship between these pairs is given below.

5 notes apart $(20, 0)$

6 notes apart $(11, 20)$ $(0, 9)$ $(3, 12)$ $(8, 17)$ $(16, 25)$ $(24, 4)$

This time we find that a set of no more than 5 pairs can be chosen satisfying the criteria. Thus, any irregular 18-note 29-beat maximally even tiling taking advantage of the 9-beat irregularity can have no more than 5 parts.

Continuing along these lines we get the following.

notes apart	$2, 3$	$5, 6$	$7, 8$	$10, 11$	$12, 13$	$15, 16$
beats apart	4	9	12	17	20	25
max parts	3	5	4	4	5	3

Thus, any irregular 18-note 29-beat maximally even tiling can have no more than 5 parts. But, noting that a partial tiling is irregular if and only if its dual is irregular, and noting the possible factorizations of 18, we can conclude that no irregular 18-note 29-beat maximally even tiling can exist.

The argument outlined in the previous example can be used to disqualify several values of n satisfying the necessary condition given in Proposition 65 (and therefore implies a significantly stronger necessary condition) but still fails to identify some values of n for which irregular 18-note n-beat maximally even tilings do not exist (thus falling short of sufficiency).

For example, the following "skeleton" of an irregular 18-note 22-beat maximally even tiling can be constructed.

	0	1	2	3	4	·	6	7	8	9	·	11	12	13	14	15	·	17	18	19	20	
Part A	0	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	:‖
Part B	·	·	·	·	0	·	1	·	·	·	·	·	·	·	·	·	·	·	·	·	·	:‖
Part C	·	·	·	·	·	·	·	0	·	1	·	·	·	·	·	·	·	·	·	·	·	:‖
Part D	·	·	·	·	·	·	·	·	·	·	·	0	·	1	·	·	·	·	·	·	·	:‖
Part E	·	·	·	·	·	·	·	·	·	·	·	·	0	·	1	·	·	·	·	·	·	:‖
Part F	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	0	·	1	·	·	·	:‖

Since this skeleton features 6 parts, the argument used in the previous example cannot be applied to discount the possibility of expanding the skeleton to a complete tiling with range $X^{\text{flr}}_{18,22}$.

In the future, I plan to investigate the following questions.

- In what cases can the argument in the previous example be employed?
- Is there an extension of that argument that might be used to describe an even stronger necessary condition?

6.2 New algorithms

As noted at the end of the previous chapter, `METv4` (which finds all maximally even tilins that are periodic expansions of multiplexations of unit multiples of regular partial tilings based on De Bruijn type full tilings) does manage to find all maximally even tilings for $k \in [4, 18]$ but does fail at some point beyond $k = 18$ (at least by $k = 900$). In the future, I plan to investigate the following questions.

- What is the smallest value of k for which `METv4` fails?
- What properties do those maximally even tilings missed by `METv4` possess?
- Can we leverage any of those properties to define a `METv5` that gets us even further?

While the `METv?` series of algorithms seeks to find all maximally even tilings, it might be interesting to develop algorithms that focus specifically on finding irregular maximally even tilings, which are are much more rare than regular ones. On one hand, the fact that irregular maximally even tilings are rare means that they are harder to find, but on the other hand it means there are fewer of them to be found.

- Can we modify `METv1` or `METv4` to focus on finding irregular maximally even tilings in such a way that their performance can be significantly improved?
- Could the approach used in the previous section ("look for a skeleton first") be useful?

6.3 Potential to engage varied audiences

In this last section, I would like to make the case that this topic of "maximally even tilings" has the potential to appeal to a wide variety of audiences.

- The concepts of maximal evenness and rhythmic tiling (without the formal definitions and theory) can be explained to and understood by the layperson in a relatively short amount of time; and with some guidance and a simple Excel spreadsheet to interact with, anyone (even kids!) can play around with composing their own maximally even tilings.
- As a quick teaser, I have plans to develop a (rudimentary) video game, in which the player must essentially perform a maximally even rhythmic canon. This will be a powerful tool in drawing interest to this topic.
- Musicians and composers have already taken interest in the topics of maximal evenness and tilings independently, and I hope that the combination of these two topics will inspire some great music in my lifetime!
- The mathematics used in this dissertation is relatively elementary and is well within the grasp of undergraduate mathematics students; and due to the novelty of the topic,

I believe that there is still plenty of useful work that could be carried out as a part of student research projects.

- While this dissertation does not employ any "high powered" mathematics, the answers to the questions presented in the previous two sections very well might. I believe that a professional mathematician could make a career out of exploring the topic of maximally even tilings. After all, that is what I am hoping to do!

REFERENCES

[1] Journal of mathematics and music. 3(2), 2009.

[2] Perspectives of new music. 49(2), 2011.

[3] A102562. `http://oeis.org/A102562`, 2014. Accessed: 9/24/2018.

[4] Carlos Agon and Moreno Andreatta. Modeling and tiling rhythmic canons in the openmusic visual programming language. *Perspectives of New Music*, 49(2):66–92, 2011.

[5] Emmanuel Amiot. New perspectives on rhythmic canons and the spectral conjecture. *Journal of Mathematics and Music*, 3(2):71–84, 2009.

[6] Moreno Andreatta. Constructing and formalizing tiling rhythmic canons:a historical survey of a “mathemusical” problem. *Perspectives of New Music*, 49(2):33–65, 2011.

[7] Norman Carey and David Clampitt. Aspects of well-formed scales. *Music Theory Spectrum*, 11(2):187–206, 1989.

[8] John Clough and Jack Douthett. Maximally even sets. *Journal of Music Theory*, 35(1/2):93–173, 1991.

[9] Ethan M Coven and Aaron Meyerowitz. Tiling the integers with translates of one finite set. *Journal of Algebra*, 212(1):161–174, 1999.

[10] Erik D. Demaine, Francisco Gomez-Martin, Henk Meijer, David Rappaport, Perouz Taslakian, Godfried T. Toussaint, Terry Winograd, and David R. Wood. The distance geometry of music. *Computational Geometry*, 42(5):429 – 454, 2009. Special Issue on the Canadian Conference on Computational Geometry (CCCG 2005 and CCCG 2006).

[11] Francisco Gomez-Martin, Perouz Taslakian, and Godfried Toussaint. Structural properties of euclidean rhythms. *Journal of Mathematics and Music*, 3(1):1–14, 2009.

[12] Francisco Gomez-Martín, Perouz Taslakian, and Godfried Toussaint. Interlocking and euclidean rhythms. *Journal of Mathematics and Music*, 3(1):15–30, 2009.

[13] Franck Jedrzejewski. Tiling the integers with aperiodic tiles. *Journal of Mathematics and Music*, 3(2):99–115, 2009.

[14] Jeremy Kastine. Maximally even tilings. In *Mathematics and Computation in Music 7th International Conference Proceedings*, 2019.

[15] Mihail N. Kolountzakis and Mátè Matolcsi. Algorithms for translational tiling. *Journal of Mathematics and Music*, 3(2):85–97, 2009.

[16] Melvyn Nathanson. Additive systems and a theorem of de bruijn. *The American Mathematical Monthly*, 121, 01 2013.

[17] A. Sands. On the factorization of abelian groups ii. *Acta Mathematica Academiae Scientiarum Hungarica*, 13:153–159, 1962.

[18] Sándor Szabó. A type of factorization of finite abelian groups. *Discrete mathematics*, 54(1):121–124, 1985.

[19] Dan Tudor Vuza. Sets and regular complementary unending canons (part one). *Perspectives of New Music*, 28(2):22–49, 1991.

[20] Dan Tudor Vuza. Sets and regular complementary unending canons (part three). *Perspectives of New Music*, 30(2):102–124, 1992.

[21] Dan Tudor Vuza. Sets and regular complementary unending canons (part two). *Perspectives of New Music*, 30(1):184–207, 1992.

[22] Dan Tudor Vuza. Sets and regular complementary unending canons (part four). *Perspectives of New Music*, 31(1):270–305, 1993.

APPENDIX

This appendix describes how to implement the algorithms given in Chapter 5 using Visual Basic for Application (VBA) within Excel.

`PT` implementation

The steps below describe how to implement `PT` in Excel/VBA for $|C| \leq 36$.

1. Open an new Excel workbook, and save it as an Excel Macro-Enabled Workbook (*.xlsm) named "Maximally Even Tilings".

2. Rename the default worksheet ("Sheet1") as "PT".

3. Enter the following formula in cell G2.

   ```
   =IF(AND(G$1<>"",$F2<>""),MOD(G$1+$F2,$B$1),"")
   ```

4. Copy G2 and paste to G2:AO36.

5. Enter the following formula in cell AR1.

   ```
   =IF(A1="","",MOD(A1-AQ$1,$B$1))
   ```

6. Enter the following formula in cell AS1.

   ```
   =IF(AR1="","",COUNTIF($F$1:$AO$36,AR1))
   ```

7. Copy AR1:AS1 and paste to AR1:AS36.

8. Enter the following code for the worksheet "PT".

   ```
   Option Explicit
   Private WF As WorksheetFunction
   Private k As Long
   Private n As Long
   Private A As Range
   Private B As Range
   ```

```
Private p As Long
Public Sub PT()
    Set WF = Application.WorksheetFunction
    k = WF.Count(Range("A:A"))
    n = Range("B1")
    Set A = Range("F1:AO1")
    Set B = Range("F1:F36")
    p = Period(Range("A1:A36"), n)
    Range("D:D").ClearContents
    A.ClearContents
    B.ClearContents
    A.Cells(1) = 0
    Range("D1") = Trivial(Range("A1:A36"), n)
    Dim i As Long, j As Long
    For i = 1 To n - 1
    For j = i + 1 To n - 1
        A.Cells(2) = i
        B.Cells(2) = j
        PTrec 1
    Next j
    Next i
    Range("D:D").RemoveDuplicates Columns:=1, Header:=xlNo
    Range("D:D").Sort Key1:=Range("D1"), Order1:=xlAscending, Header:=xlNo
    Columns(4).AutoFit
    Columns(4).HorizontalAlignment = xlCenter
End Sub
Private Sub PTrec(ByVal Stage As Long)
    DoEvents
    Dim i As Long
    For i = 0 To p - 1
        Range("AQ1") = i
        If WF.Max(Range("AS:AS")) > 1 Then Exit Sub
        If WF.Sum(Range("AS:AS")) = WF.Count(A) * WF.Count(B) Then Exit For
    Next i
    If i = p Then Exit Sub
    If GreatestTrans(A, n) <> 0 Then Exit Sub
    If GreatestTrans(B, n) <> 0 Then Exit Sub
    If NextDiv(WF.Count(A), k) * NextDiv(WF.Count(B), k) > k Then Exit Sub
    If WF.Count(A) * WF.Count(B) = k Then
        AppendToRange Range("D:D"), StandardForm(A, B, n, Range("AQ1"), p)
        Exit Sub
    End If
    If Stage = 1 Then
        If WF.Count(A) = WF.Count(B) Then
            NextStep A, 1
            If k Mod WF.Count(A) = 0 Then NextStep B, 3
        Else
```

```
            NextStep B, 1
            If k Mod WF.Count(B) = 0 Then NextStep A, 2
        End If
    ElseIf Stage = 2 Then
        NextStep A, 2
    ElseIf Stage = 3 Then
        NextStep B, 3
    End If
End Sub
Private Sub NextStep(R As Range, ByVal Stage As Long)
    Dim i As Long
    For i = WF.Max(R) + 1 To n - 1
        AppendToRange R, i
        PTrec Stage
        RemoveFromRange R
    Next i
End Sub
Private Function NextDiv(A As Variant, B As Variant) As Long
    For NextDiv = A To B
        If MyMod(B, NextDiv) = 0 Then Exit Function
    Next NextDiv
End Function
```

9. Insert a new module ("Module1") and enter the following code. (This module includes subroutines that are not necessarily called by `PT` but will be used by later procedures.)

```
Option Explicit
Private WF As WorksheetFunction
Public Function Trivial(R As Range, n As Long) As String
    Dim t As Long
    t = GreatestTrans(R, n)
    Trivial = StringOf(R, t, n) & "|" & "0" & "|" & t
End Function
Public Function StringOf(R As Range, t As Long, n As Long) As String
    Dim i As Long
    Set WF = Application.WorksheetFunction
    Dim A(1 To 36) As Variant
    For i = 1 To 36
        If R.Cells(i) <> "" Then A(i) = MyMod(R.Cells(i) - t, n)
    Next i
    SortArray A
    For i = 1 To 36
        If A(i) <> "" Then
            If StringOf <> "" Then StringOf = StringOf & ","
            StringOf = StringOf & A(i)
        End If
```

```
    Next i
End Function
Public Function GreatestTrans(R As Range, n As Long) As Long
    Dim t As Long, MaxVal As Double, fVal As Double
    MaxVal = 0
    For t = 0 To n - 1
        fVal = fRange(R, n, t)
        If fVal > MaxVal Then
            GreatestTrans = t
            MaxVal = fVal
        End If
    Next t
End Function
Public Function fRange(R As Range, n As Long, t As Long) As Double
    Dim L As Long, i As Long
    L = Application.WorksheetFunction.Count(R)
    For i = 1 To L
        fRange = fRange + 2 ^ -MyMod(R.Cells(i) - t, n)
    Next i
End Function
Public Function MyMod(x As Variant, n As Variant)
    MyMod = x - n * Int(x / n)
End Function
Public Sub SortArray(A As Variant)
    Dim done As Boolean, i As Long, x As Long
    Do
        done = True
        For i = LBound(A) To UBound(A) - 1
            If A(i + 1) <> "" And A(i + 1) < A(i) Then
                x = A(i)
                A(i) = A(i + 1)
                A(i + 1) = x
                done = False
            End If
        Next i
    Loop Until done
End Sub
Public Sub AppendToRange(R As Range, x As Variant)
    R.Cells(Application.WorksheetFunction.CountA(R) + 1) = x
End Sub
Public Sub RemoveFromRange(R As Range)
    R.Cells(Application.WorksheetFunction.CountA(R)) = ""
End Sub
Public Function Factor(ByVal n As Long) As Variant
    Dim A() As Variant
    ReDim A(4, 0)
    If n = 1 Then
```

```
        A(0, 0) = 1
        Exit Function
    End If
    Dim j As Long, i As Long
    j = 0
    For i = 2 To n
        If n Mod i = 0 Then
            A(j, 0) = i
            j = j + 1
            n = n / i
            i = 1
        End If
    Next i
    Do Until j = 5
        A(j, 0) = ""
        j = j + 1
    Loop
    Factor = A
End Function
Public Function Component(s As String, i As Long) As String
    Dim A() As String
    A = Split(s, "|")
    Component = A(i - 1)
End Function
Public Sub StringToRange(s As String, R As Range)
    R.ClearContents
    Dim A() As String, i As Long
    A = Split(s, ",")
    For i = LBound(A) To UBound(A)
        R.Cells(i - LBound(A) + 1) = CLng(A(i))
    Next i
End Sub
Public Function MultInvMod(x As Variant, n As Variant) As Variant
    For MultInvMod = 1 To n - 1
        If x * MultInvMod Mod n = 1 Then Exit Function
    Next MultInvMod
End Function
Public Function StandardForm(R1 As Range, R2 As Range, n As Long, _
t As Long, p As Long) As String
    Dim s1 As String, s2 As String, t1 As Long, t2 As Long
    t1 = GreatestTrans(R1, n)
    t2 = GreatestTrans(R2, n)
    s1 = StringOf(R1, t1, n)
    s2 = StringOf(R2, t2, n)
    If fString(s1) > fString(s2) Then
        StandardForm = s1 & "|" & s2 & "|" & MyMod(t1 + t2 + t, p)
    Else
```

```
        StandardForm = s2 & "|" & s1 & "|" & MyMod(t1 + t2 + t, p)
    End If
End Function
Public Function Period(R As Range, n As Long) As Long
    Dim x As Double
    Dim y As Double
    For Period = 1 To n
        x = fRange(R, n, Period)
        y = fRange(R, n, 0)
        If x = y Then
            Exit Function
        End If
    Next Period
End Function
Public Function fString(s As Variant) As Double
    Dim A() As String, i As Long
    A = Split(s, ",")
    For i = LBound(A) To UBound(A)
        fString = fString + 2 ^ -CLng(A(i))
    Next i
End Function
```

10. Enter the desired range C in column A and n in cell B1.

11. Run `PT` and partial tilings will be output to column D in the form “A|B|t” where $(A, B)_n$ is a partial tiling with range $\mathrm{mod}_n(C - t)$.

`METv1` implementation

The steps below describe how to implement `METv1` in Excel/VBA for $k_{\min} = 4$ and $k_{\max} = 18$.

1. Create a new worksheet and rename it “METv1”.

2. Enter the following code for the worksheet “METv1”.

```
Option Explicit
Public WF As WorksheetFunction
Public Sub METv1()
    Set WF = Application.WorksheetFunction
    Dim c As Long, i As Long, k As Long, n As Long, t As Double, _
    j As Long, L As Long
    Cells.ClearContents
```

```
    c = 0
    For k = 4 To 18
        For n = k To 2 * k
            If n <= 1.5 * k Or n <= 2 * (k - WF.Gcd(k, n)) Then
                t = Now
                c = c + 1
                Cells(1, c) = k
                Cells(2, c) = n
                Cells(3, c) = "Calculating..."
                Sheets("PT").Range("A:B").ClearContents
                For j = 1 To k
                    Sheets("PT").Cells(j, "A") = _
                    WF.Floor((j - 1) * n / k, 1)
                Next j
                Sheets("PT").Range("B1") = n
                Sheets("PT").PT
                L = WF.CountA(Sheets("PT").Range("D:D"))
                Cells(4, c) = L
                For i = 1 To L
                    AppendToRange Columns(c), Sheets("PT").Cells(i, "D")
                Next i
                Cells(3, c) = (Now - t) * 24 * 60
                Columns(c).AutoFit
                Columns(c).HorizontalAlignment = xlCenter
            End If
        Next n
    Next k
    Cells(3, c + 1) = WF.Sum(Range(Cells(3, 1), Cells(3, c)))
    Cells(4, c + 1) = WF.Sum(Range(Cells(4, 1), Cells(4, c)))
End Sub
```

3. Run `METv1`. For each pair of k and n, a column will be produced, in which the first cell is k, the second cell is n, the third cell in the time in minutes to complete the search, the fourth cell is the number of distinct results found, and below that is a list of those results in the form "A|B|t" where $(A, B)_n$ is a partial tiling with range $\mathrm{mod}_n(C - t)$.

`DeB` implementation

The steps below describe how to implement `DeB` in Excel/VBA for $n \leq 36$.

1. Create a new worksheet and rename it "DeB".

2. Enter the following code for the worksheet "DeB".

```
Option Explicit
Private WF As WorksheetFunction
Public Sub DeB()
    Set WF = Application.WorksheetFunction
    Range("D:F").ClearContents
    Range("F1") = 0
    DeBrec 1
    Range("D:D").RemoveDuplicates Columns:=1, Header:=xlNo
    Range("D:D").Sort Key1:=Range("D1"), Order1:=xlAscending, Header:=xlNo
    Columns(4).AutoFit
    Columns(4).HorizontalAlignment = xlCenter
End Sub
Private Sub DeBrec(j As Long)
    DoEvents
    If j = WF.Count(Range("B1:B5")) Then
        Range("G:G").ClearContents
        If WF.Sum(Range("F1:F5")) <> 0 Then DeBrec2 0
        Exit Sub
    End If
    Cells(j + 1, "F") = 0
    DeBrec j + 1
    Cells(j + 1, "F") = 1
    DeBrec j + 1
    Cells(j + 1, "F") = ""
End Sub
Private Sub DeBrec2(j As Long)
    DoEvents
    Dim i As Long
    If j = WF.Count(Range("B:B")) Then
        DeBoutput
        Exit Sub
    End If
    For i = 1 To 5
        If Cells(i, "B") <> Cells(i + 1, "B") Then
            If WF.CountIf(Range("G:G"), Cells(i, "B")) < _
            WF.CountIf(Range("B:B"), Cells(i, "B")) Then
                If Cells(WF.Max(j, 1), "F") <> Cells(j + 1, "F") Or _
                Cells(i, "B") >= Cells(WF.Max(j, 1), "G") Then
                    Cells(j + 1, "G") = Cells(i, "B")
                    DeBrec2 j + 1
                    Cells(j + 1, "G") = ""
                End If
            End If
        End If
    Next i
End Sub
Private Sub DeBoutput()
```

```
    Dim i As Long
    Range("J:K").ClearContents
    Cells(1, "J") = 0
    Cells(1, "K") = 0
    For i = 1 To WF.Count(Range("G:G"))
        If Cells(i, "F") = 0 Then
            AddToSet Range("J1:J36"), Cells(i, "H"), Cells(i, "G")
        Else
            AddToSet Range("K1:K36"), Cells(i, "H"), Cells(i, "G")
        End If
    Next i
    Dim n As Long
    n = Range("A1")
    AppendToRange Range("D:D"), StandardForm(Range("J1:J36"), _
    Range("K1:K36"), n, 0, 1)
End Sub
Private Sub AddToSet(R As Range, x As Variant, y As Variant)
    Dim CardR As Long, m As Long, j As Long
    CardR = WF.Count(R)
    For m = 1 To y - 1
    For j = 1 To CardR
        AppendToRange R, R.Cells(j) + m * x
    Next j
    Next m
End Sub
```

3. Select B1:B5, type the following formula, and press ctrl-shift-enter.

```
=Factor(A1)
```

4. Enter the value `1` in H1 and the following formula in H2.

```
=IF(G2="","",PRODUCT(G$1:G1))
```

5. Copy H2 and paste to H2:H5.

6. Enter the desired value of n in cell A1, and run `DeB`. De Bruijn-type full tilings will be output to column D in the form “A—B—0” where $(A, B)_n$ is a full tiling.

`UnitMult` implementation

The steps below describe how to implement `UnitMult` in Excel/VBA for $|C| \leq 36$.

1. Create a new worksheet and rename it “UnitMult”.

2. Enter the following code for the worksheet “UnitMult”.

```
Option Explicit
Private WF As WorksheetFunction
Private n As Long
Private k As Long
Public Sub UnitMult()
    Dim L As Long, u As Long, i As Long, j As Long
    Set WF = Application.WorksheetFunction
    n = Range("B1")
    k = Range("F1")
    Range("D:D").ClearContents
    Range("D1") = Trivial(Range("A1:A36"), n)
    Sheets("DeB").Range("A1") = k
    Sheets("DeB").DeB
    L = WF.CountA(Sheets("DeB").Range("D:D"))
    For i = 1 To L
        StringToRange Component(Sheets("DeB").Cells(i, "D"), 1), _
        Range("L1:UT1")
        StringToRange Component(Sheets("DeB").Cells(i, "D"), 2), _
        Range("K2:K37")
        For u = 1 To n - 1
            If WF.Gcd(n, u) = 1 Then
                Range("G1") = u
                For j = 1 To k
                    If Cells(j, "A") < Period(Range("A1:A36"), n) Then
                        DoEvents
                        Range("H1") = Cells(j, "A")
                        If WF.CountIf(Range("M3:AU37"), False) = 0 _
                        Then UnitMultOutput
                    End If
                Next j
            End If
        Next u
        Range("D:D").RemoveDuplicates Columns:=1, Header:=xlNo
    Next i
    Range("D:D").Sort Key1:=Range("D1"), Order1:=xlAscending, Header:=xlNo
    Columns(4).AutoFit
    Columns(4).HorizontalAlignment = xlCenter
End Sub
Private Sub UnitMultOutput()
    Dim v As Long, t As Long, i As Long, x As Long
    v = MultInvMod(Range("G1"), n)
    t = Range("H1")
    Range("AW:AX").ClearContents
```

```
    For i = 1 To 36
        If Range("L2:AU2").Cells(i) <> "" Then AppendToRange _
        Range("AW:AW"), MyMod(Range("L2:AU2").Cells(i) * v, n)
        If Range("L2:L37").Cells(i) <> "" Then AppendToRange _
        Range("AX:AX"), MyMod(Range("L2:L37").Cells(i) * v + t, n)
    Next i
    AppendToRange Range("D:D"), _
    StandardForm(Range("AW1:AW36"), Range("AX1:AX36"), n, 0, _
    Period(Range("A1:A36"), n))
End Sub
```

3. Enter the following formula in F1:

```
=COUNT(A:A)
```

4. Enter the following formula in I1. Copy I1 and paste to I1:I36.

```
=IF(A1="","",MOD(G$1*(A1-H$1),B$1))
```

5. Enter the value `0` in cell L2.

6. Enter the following formula in cell L3. Copy L3 and paste to L3:L37.

```
=IF(K3="","",SMALL($I:$I,K3+1))
```

7. Enter the following formula in cell M2. Copy M2 and paste to M2:AU2.

```
=IF(M1="","",SMALL($I:$I,M1+1))
```

8. Enter the following formula in cell M3. Copy M3 and paste to M3:AU37.

```
=IF(AND(M$1<>"",$K3<>""),MOD(M$2+$L3-SMALL($I:$I,MOD(M$1+$K3,$F$1)+1),$B$1)
                         =0,"")
```

9. Enter the set C in column A and n in cells A1, and run `UnitMult`. Partial tilings will be output to column D in the form "A|B|t" where $(A, B)_n$ is a partial tiling with range $\text{mod}_n(C - t)$.

`METv2` implementation

The steps below describe how to implement `METv2` in Excel/VBA for $k_{\min} = 4$ and $k_{\max} = 18$.

1. Copy the worksheet "METv1", and rename it "METv2".
2. In the code for "METv2", replace all occurrences of "PT" with "UnitMult".
3. Run `METv2`. For each pair of k and n, a column will be produced, in which the first cell is k, the second cell is n, the third cell in the time in minutes to complete the search, the fourth cell is the number of distinct results found, and below that is a list of those results in the form "A|B|t" where $(A, B)_n$ is a partial tiling with range $\mathrm{mod}_n(C - t)$.

`Multiplex` implementation

The steps below describe how to implement `Multiplex` in Excel/VBA for $|C| \leq 36$.

1. Create a new worksheet and rename it "Multiplex".
2. Enter the following code for the worksheet "Multiplex".

```
Option Explicit
Private WF As WorksheetFunction
Private n As Long
Public Sub Multiplex()
    n = Range("A1")
    Dim d As Long, j As Long, i As Long, h As Long, m As Long, L As Long, _
    p As Long
    Set WF = Application.WorksheetFunction
    Range("D:D").ClearContents
    n = Range("B1")
    For d = 1 To n
        If n Mod d = 0 And GCDcard(d) > 1 Then
            Range("F:L").ClearContents
            For j = 0 To d - 1
                DoEvents
                Sheets("UnitMult").Range("A:A").ClearContents
                For m = 1 To 36
                    If Cells(m, "A") <> "" And Cells(m, "A") Mod d = j Then
```

```
                AppendToRange Sheets("UnitMult").Range("A:A"), _
                (Cells(m, "A") - j) / d
            End If
        Next m
        If WF.Count(Sheets("UnitMult").Range("A:A")) = 0 Then
            AppendToRange Range("L:L"), j
        Else
            p = Period(Sheets("UnitMult").Range("A1:A36"), n / d)
            Sheets("UnitMult").Range("B1") = Range("B1") / d
            Sheets("UnitMult").UnitMult
            L = WF.CountA(Sheets("UnitMult").Range("D:D"))
            For i = 1 To L
            For h = 0 To n / d / p - 1
                AppendToRange Range("G:G"), j
                AppendToRange Range("H:H"), Component( _
                Sheets("UnitMult").Cells(i, "D"), 1)
                AppendToRange Range("I:I"), Component( _
                Sheets("UnitMult").Cells(i, "D"), 2)
                AppendToRange Range("J:J"), Component( _
                Sheets("UnitMult").Cells(i, "D"), 3) + h * p
                AppendToRange Range("G:G"), j
                AppendToRange Range("H:H"), Component( _
                Sheets("UnitMult").Cells(i, "D"), 2)
                AppendToRange Range("I:I"), Component( _
                Sheets("UnitMult").Cells(i, "D"), 1)
                AppendToRange Range("J:J"), Component( _
                Sheets("UnitMult").Cells(i, "D"), 3) + h * p
            Next h
            Next i
        End If
    Next j
    L = WF.CountA(Range("H:H"))
    For i = 1 To L
        Cells(i, "F") = fString(Cells(i, "H"))
    Next i
    Range("F:J").Sort Key1:=Range("F1"), Order1:=xlAscending, _
    Key2:=Range("G1"), Order2:=xlAscending, Header:=xlNo
    Range("N:N").ClearContents
    For i = 1 To L
        If Cells(i, "G") = 0 Then
            Cells(1, "N") = i
            MultiplexRec i, Cells(i, "F"), 1, d
        End If
    Next i
End If
Range("D:D").RemoveDuplicates Columns:=1, Header:=xlNo
Next d
```

```
    Range("D:D").Sort Key1:=Range("D1"), Order1:=xlAscending, Header:=xlNo
    Columns(4).AutoFit
    Columns(4).HorizontalAlignment = xlCenter
End Sub
Private Function GCDcard(d) As Long
    Dim i As Long, j As Long, c As Long
    For i = 0 To d - 1
        c = 0
        For j = 1 To 36
            If Cells(j, "A") <> "" And MyMod(Cells(j, "A"), d) = i Then _
            c = c + 1
        Next j
        If GCDcard = 0 Then
            GCDcard = c
        Else
            GCDcard = WF.Gcd(GCDcard, c)
        End If
    Next i
End Function
Private Sub MultiplexRec(ByVal R As Long, fVal As Double, _
j As Long, d As Long)
    DoEvents
    If j = d Then
        MultiplexOutput d
        Exit Sub
    End If
    If WF.CountIf(Range("L:L"), j) > 0 Then
        MultiplexRec R, fVal, j + 1, d
        Exit Sub
    End If
    Do Until Cells(R, "F") <> fVal
        If Cells(R, "G") = j Then
            AppendToRange Range("N:N"), R
            MultiplexRec R, fVal, j + 1, d
            RemoveFromRange Range("N:N")
        End If
        R = R + 1
    Loop
End Sub
Private Sub MultiplexOutput(d As Variant)
    Range("P:Q").ClearContents
    Dim A() As String, j As Long, L As Long, i As Long
    ReDim A(1 To 36)
    A = Split(Cells(Cells(1, "N"), "H"), ",")
    For j = LBound(A) To UBound(A)
        AppendToRange Range("P:P"), CLng(A(j)) * d
    Next j
```

```
    L = WF.Count(Range("N:N"))
    For i = 1 To L
        ReDim A(1 To 36)
        A = Split(Cells(Cells(i, "N"), "I"), ",")
        For j = LBound(A) To UBound(A)
            AppendToRange Range("Q:Q"), _
            MyMod((CLng(A(j)) + Cells(Cells(i, "N"), "J")) * d + _
            Cells(Cells(i, "N"), "G"), n)
        Next j
    Next i
    AppendToRange Range("D:D"), StandardForm(Range("P:P"), Range("Q:Q"), _
    n, 0, Period(Range("A1:A36"), n))
End Sub
```

3. Enter the set C in column A and n in cells A1, and run `Multiplex`. Partial tilings will be output to column D in the form "A|B|t" where $(A, B)_n$ is a partial tiling with range $\text{mod}_n(C - t)$.

`METv3` implementation

The steps below describe how to implement `METv3` in Excel/VBA for $k_{\min} = 4$ and $k_{\max} = 18$.

1. Copy the worksheet "METv1", and rename it "METv3".

2. In the code for "METv3", replace all occurrences of "PT" with "Multiplex".

3. Run `METv3`. For each pair of k and n, a column will be produced, in which the first cell is k, the second cell is n, the third cell in the time in minutes to complete the search, the fourth cell is the number of distinct results found, and below that is a list of those results in the form "A|B|t" where $(A, B)_n$ is a partial tiling with range $\text{mod}_n(C - t)$.

`Periodic` implementation

The steps below describe how to implement `Periodic` in Excel/VBA for $|C| \leq 36$.

1. Create a new worksheet and rename it "Periodic".

2. Enter the following code for the worksheet "Periodic".

```
Option Explicit
Private WF As WorksheetFunction
Private n As Long, d As Long
Public Sub Periodic()
    Set WF = Application.WorksheetFunction
    Dim h As Long, L As Long, i As Long, s As String
    Range("D:D").ClearContents
    n = Range("B1")
    For d = 1 To n
        If n Mod d = 0 And fRange(Range("A1:A36"), n, 0) = _
        fRange(Range("A1:A36"), n, d) Then
            Sheets("Multiplex").Range("A:B").ClearContents
            For i = 1 To 36
                If Cells(i, "A") < d Then _
                Sheets("Multiplex").Cells(i, "A") = Cells(i, "A")
            Next i
            Sheets("Multiplex").Range("B1") = d
            Sheets("Multiplex").Multiplex
            L = WF.CountA(Sheets("Multiplex").Range("D:D"))
            For i = 1 To L
                s = Sheets("Multiplex").Cells(i, "D")
                Range("F:I").ClearContents
                StringToRange Component(s, 1), Range("F1:F36")
                StringToRange Component(s, 2), Range("G1:G36")
                Range("H1") = Component(s, 3)
                Range("I1") = 0
                PeriodicRec
                Range("F:I").ClearContents
                StringToRange Component(s, 2), Range("F1:F36")
                StringToRange Component(s, 1), Range("G1:G36")
                Range("H1") = Component(s, 3)
                Range("I1") = 0
                PeriodicRec
            Next i
            Range("D:D").RemoveDuplicates Columns:=1, Header:=xlNo
        End If
    Next d
    Range("D:D").Sort Key1:=Range("D1"), Order1:=xlAscending, Header:=xlNo
    Columns(4).AutoFit
    Columns(4).HorizontalAlignment = xlCenter
End Sub

Private Sub PeriodicRec()
    DoEvents
    Dim i As Long
    If WF.Count(Range("I:I")) = WF.Count(Range("F:F")) Then
```

```
        PeriodicRecOutput
        Exit Sub
    End If
    For i = 0 To n / d - 1
        AppendToRange Range("I:I"), i
        PeriodicRec
        RemoveFromRange Range("I:I")
    Next i
End Sub
Private Sub PeriodicRecOutput()
    Dim L As Long, i As Long, j As Long
    Range("K:L").ClearContents
    L = WF.Count(Range("F:F"))
    For i = 1 To L
         AppendToRange Range("K:K"), Cells(i, "F") + d * Cells(i, "I")
    Next i
    L = WF.Count(Range("G:G"))
    For j = 0 To n / d - 1
    For i = 1 To L
        AppendToRange Range("L:L"), Cells(i, "G") + d * j
    Next i
    Next j
    AppendToRange Range("D:D"), StandardForm(Range("K1:K36"), _
    Range("L1:L36"), n, Range("H1"), Period(Range("A1:A36"), n))
End Sub
```

3. Enter the set C in column A and n in cells A1, and run `Periodic`. Partial tilings will be output to column D in the form "A|B|t" where $(A, B)_n$ is a partial tiling with range $\text{mod}_n(C - t)$.

`METv4` implementation

The steps below describe how to implement `METv3` in Excel/VBA for $k_{\text{min}} = 4$ and $k_{\text{max}} = 18$.

1. Copy the worksheet "METv1", and rename it "METv4".
2. In the code for "METv4", replace all occurrences of "PT" with "Periodic".
3. Run `METv4`. For each pair of k and n, a column will be produced, in which the first cell is k, the second cell is n, the third cell in the time in minutes to complete the search,

the fourth cell is the number of distinct results found, and below that is a list of those results in the form “A|B|t” where $(A, B)_n$ is a partial tiling with range $\mathrm{mod}_n(C - t)$.