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Two Problems on Bipartite Graphs

Albert Bush

TWO PROBLEMS ON BIPARTITE GRAPHS

by

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Under the Direction of Dr. Yi Zhao

ABSTRACT

Erdös proved that every graph G has a bipartite, spanning subgraph B such that $d_B(v) \geq \frac{d_G(v)}{2}$ for any $v \in V(G)$. Bollobás and Scott conjectured that every graph G has a balanced, bipartite, spanning subgraph B such that $d_B(v) \geq \frac{d_G(v)-1}{2}$. We prove this for graphs with maximum degree 3.

However, the majority of this paper is focused on bipartite graph tiling. We prove a conjecture of Zhao that implies an asymptotic version Kühn and Osthus' tiling result when restricted to a bipartite graph H. Specifically, we prove for any bipartite graph H on h vertices, if G is a bipartite graph on $2n = mh$ vertices and $\delta(G) \geq (1 - \frac{1}{\chi^*(H)})n + \gamma n$, then G contains an H-tiling where $\chi^*(H)$ is either the chromatic number or the critical chromatic number of H.

INDEX WORDS: Graph tiling, Balanced bipartite subgraph, Graph packing

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DEDICATION

This thesis is dedicated to the mathematicians in the past who inspired, and to the mathematicians of the future who will hopefully build mountains out of our molehills.

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I am deeply grateful to Dr. Zhao for his patience and guidance. Thank you to Dr. Chen and Dr. Hattingh for their involvement on my committee. I am also thankful to my parents for always encouraging me to pursue my interests regardless of any foreseeable success. Finally, I would like to thank my Katja, for understanding.

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INTRODUCTION

1.1 Notation

We will use the following notation throughout the paper. $G = (V, E)$ is a graph, and $G[X, Y]$ is a bipartite graph with vertex set $V = X \cup Y$. We assume all graphs are simple. When referring to the vertex set or the edge set of some graph, we write $V(G)$, $E(G)$, or simply V or E. Moreover, the size of the vertex set is denoted $v(G)$ and is often called the order of G. The size of the edge set is denoted $e(G)$ and is often called the size of G. If we are working with two sets of vertices A and B, we will denote $E(A, B)$ as the set of edges with one end in A and the other in B. Similarly, $e(A, B)$ is the number of such edges. We will refer to the complete graph on r vertices as an r-clique and denote it K_r. The complete bipartite graph with one side of order s and the other of order t will be denoted $K_{s,t}$ and is sometimes called a bipartite clique. The degree of a vertex v is denoted by $\deg(v)$ or $d(v)$. Since we will often refer to subgraphs of G, we will often specify the degree of v in a specific subgraph G' as $d_{G'}(v)$. The maximum degree in of a vertex in G is denoted by $\Delta(G)$. The minimum degree of a vertex is $\delta(G)$. The chromatic number of a graph is $\chi(G)$. We will refer to an H-tiling of a graph G as a decomposition of G into vertex disjoint copies of H. This is also sometimes referred to as a perfect H-tiling, an H-packing, or an H-factor.

1.2 Two Problems

Bipartite graphs form a rich subject of study in graph theory. They are often starting points for broader theorems as they can contain the complexity of more general graphs but are often easier to study. Every graph has many spanning, bipartite subgraphs. Erdös proved that, in fact, every graph G has a bipartite, spanning subgraph F such that $d_F(v) \geq \frac{d_G(v)}{2}$ for any $v \in V(G)$ [9]. Our first theorem is a small step towards proving Bollobás and Scott's conjecture [4] that every G has a bipartite, spanning, balanced subgraph B such that $d_B(v) \geq \frac{d_G(v)-1}{2}$.

However, the majority of this paper is focused on bipartite graph tiling. We prove a conjecture of Zhao [22] that can be characterized as a bipartite version of Kühn and Osthus' result [17] for arbitrary graph tiling. Kühn and Osthus prove that given a graph G on n and a graph H on h vertices, if $\delta(G) \geq (1-\frac{1}{\chi^*(H)})n+O(1)$, then G contains an H-tiling. One will notice the similarity of our result in which we prove for any bipartite graph H on h vertices, if G is a bipartite graph on $2n = mh$ vertices and $\delta(G) \geq (1-\frac{1}{\chi^*(H)})n+\gamma n$, then G contains an H-tiling where $\chi^*(H)$ is either the chromatic number or the critical chromatic number of H. For emphasis, we point out that important difference: in our graph G, n is not the number of vertices in G but the number of vertices in one of the two partitions of G. Additionally, this result implies an asymptotic version of Kühn and Osthus' result for bipartite graphs. In general, tiling in multi-partite graphs tends to provide looser minimum degree conditions than the general case. In fact, combined with the aforementioned Bollobás-Scott conjecture, it essentially implies Kühn and Osthus' result for bipartite graphs up to a constant.

BOLLOBÁS-SCOTT CONJECTURE

2.1 Introduction

Judicious partitioning problems in graph theory generally involve maximizing or minimizing a certain quantity while partitioning the graph in a certain way. For example, one of the first and most widely known results of this kind is due to Erdös [9] and is stated as follows: For any graph G, G has a spanning, bipartite subgraph B such that $d_B(v) \geq \frac{d_G(v)}{2}$. Thus, in this problem, we partition the graph into two parts and try to maximize the degree of each vertex. More recent results of similar flavor tend to ask about the size (the number of edges) of a bipartite subgraph (see [1] [21] for more). For example, a well-known result of Edwards [8] gives an upper bound on the number of edges in a bipartite subgraph of G. The contrasting difference between the Erdös' result and Edwards' result is that the degree of each vertex is a local value whereas the number of edges is a global value. There have been many recent advances in the latter area, but in this paper, we focus on the former. Mainly, we ask, given a graph G, what is the largest minimum degree possible in a spanning, *balanced*, bipartite subgraph? As stated in [4], $K_{2\ell+1,m}$ shows us that we cannot achieve the same result that we had for any bipartite subgraph – that is, we cannot achieve half the minimum degree of the original. However, Bollobás and Scott conjectured that perhaps we can achieve *almost* half.

Conjecture 1. *For any graph G, there exists a spanning, balanced, bipartute subgraph B such that*

$$d_B(v) \geq \frac{d_G(v) - 1}{2}.$$

The general form of this problem is quite difficult. Through a relatively simple probabilistic argument, one can get the following similar result (see eg. [18], [3] for a proof).

Theorem 2. *There exists an integer n_0 such that for all $n \geq n_0$ the following holds: let G be a graph on $2n$ vertices. G contains a spanning, bipartite subgraph $B[X_1, X_2]$ where*

$|X_1| = |X_2| = n$ and

$$d_B(v) \geq \frac{d_G(v)}{2} - 4\sqrt{n \log n}$$

We present the proof of this by using a similar technique to [18]. We use the following lemma from [18] which is an application of the large deviation bound for hypergeometric distributions (see [12] for more details).

Lemma 3. *Given $n \in \mathbb{N}$, and sets $N \subseteq V$ with $|V| \geq n$, let Y be a subset of V which is obtained by successively selecting n elements of V at random without repitition. Let $X = |N \cap Y|$. The following inequalities hold: (i) if $0 < \alpha \leq \frac{3}{2}$ then we have*

$$Prob(|X - E[X]| \geq \alpha E[X]) \leq 2e^{-\frac{\alpha^2}{3}E[X]}.$$

(ii) If $\alpha > \frac{3}{2}$, we have

$$Prob(X > \alpha E[X]) \leq e^{-c\alpha E[X]}$$

where c is an absolute constant.

We now present the proof of Theorem 2.

Proof. Let G be a graph on $2n$ vertices. Consider a random, spanning, balanced, bipartition $B[X_1, X_2]$ of G. Without loss of generality, we show that the probability that a randomly chosen vertex $v \in X_1$ has degree in B at least $\frac{d_G(v)}{2} - 4\sqrt{n \log n}$ is near 1. This will show that we can add up the probability for all vertices and obtain a positive probability. Hence, for a large enough n, there exists an outcome where such a bipartition exists.

The expected value of the degree of a vertex v in B is $\frac{d(v)}{2} = E[d_B(v)]$. Let $\gamma = 4\sqrt{\frac{\log n}{n}}$. We now consider two cases:

Case 1: $d_G(v) \geq \frac{4\gamma n}{3}$. In this case, we use part (i) of the previous lemma as follows: let $X = d_B(v)$, $\alpha = \frac{2\gamma n}{d_G(v)}$, and $E[X] = \frac{d_G(v)}{2}$. By the assumption on the size of $d_G(v)$, we get that $\alpha \leq \frac{3}{2}$. Thus, we get that

$$Prob(|d_B(v) - \frac{d_G(v)}{2}| \geq \gamma n) \leq 2e^{-\frac{\alpha^2}{3}E[d_B(v)]} = 2e^{\frac{-4\gamma^2 n^2}{3d_G(v)^2}\cdot\frac{d_G(v)}{2}}$$

$$= 2e^{\frac{-2}{3}\frac{\gamma^2 n^2}{d_G(v)}} = 2e^{\frac{-32n\log n}{3d_G(v)}} \leq 2e^{\frac{-32\log n}{3}} = \frac{2}{n^{\frac{32}{3}}}$$

So, with probability at least $1 - \frac{2}{n^10}$, vertex v has the appropriate degree.

Case 2: $d_G(v) < \frac{4\gamma n}{3}$. In this case, we use part (ii) of the lemma as follows:

$$Prob(|d_B(v) - \frac{d_G(v)}{2}| \geq \gamma n) = Prob(d_B(v) \geq \frac{d_G(v)}{2} + \gamma n)$$

$$\leq Prob(d_B(v) \geq \gamma n) \leq e^{-c\gamma n} = e^{-4c\sqrt{n\log n}}$$

In Case 1, we had a probability less than $e^{-\frac{-32}{3}\log n}$. In Case 2, we have an even lower probability since we have a factor of $\sqrt{n}$ in the exponent making Case 2 even more favorable. Thus, in both cases, we have a probability of at least $1 - \frac{2}{n^10}$ that v has degree in B of at least $\frac{d_G(v)}{2} - 4\sqrt{n\log n}$. Thus, it occurs with positive probability that all vertices have the appropriate degree. So the outcome that there exists such a partition exists. □

This establishes strong motivation for the Bollobás-Scott conjecture. It has already been shown that the conjecture is true for cubic graphs [5]. In this paper, we provide some small progress in this area by proving the following: 1) the conjecture is true for graphs with maximum degree less than or equal to three; 2) the conjecture is true if and only if it is true for even graphs; 3) the conjecture holds by a trivial proof for trees.

2.2 Our Result

As stated in the previous section, we first tackle the problem for graphs with maximum degree less than or equal to 3. We do this by proving a slightly stronger statement. When G

has minimum degree 2, we show it has a spanning, balanced, bipartition of minimum degree at least 1. This implies that the conjecture is true for graphs of maximum degree less than or equal to 3. Our technique is to use induction on the number of edges and vertices. We consider several cases. In each case, we find two adjacent vertices who have a certain degree. We remove them from the graph, add some edges if necessary, apply induction, and then consider the resulting balanced, bipartite graph. Through adding the two removed vertices back into this bipartition, we are able to obtain the desired graph.

Theorem 4. *Every simple graph G with $\delta(G) \geq 2$ contains a spanning, balanced bipartition B with $\delta(B) \geq 1$.*

Proof. First, we assume G is connected. If not, consider each connected component separately. We perform induction on $e(G)$ and $v(G)$ while considering the following cases.

Base Case: The smallest connected graph G such that $\delta(G) \geq 2$ is C_3. Remove any edge, and we have the desired bipartition.

Case 1: Suppose there exists an edge $e = \{a, b\} \in E(G)$ with $d(a) \geq 3$ and $d(b) \geq 3$. We can delete e, and apply induction. The graph $G - e$ is a graph with minimum degree at least 2, and thus, by the induction hypothesis, it has a spanning, balanced bipartite graph B with $\delta(B) \geq 1$. This graph is also a spanning balanced bipartition of G, so we are done.

We now suppose there is no such edge e with both ends of degree at least 3.

Case 2: There exists an edge $e = \{a, b\} \in E(G)$ with $d(a) = d(b) = 2$. First, assume a and b have a common neighbor c. If $d(c) = 2$, then $G = C_3$ since G is connected. If $d(c) \geq 3$, then we consider $G' = G[V - \{a, b\}]$. If the minimum degree of G' is at least two – this happens when $d_G(c) \geq 4$ – then we simply apply the induction hypothesis, add a and b to each side of the resulting bipartition, and we are done.

Note: we commonly use the term **critical** to describe *an edge added to G* such that, when we consider the resulting bipartite subgraph $B'[X, Y]$, the deletion of this edge will make the degree of an incident vertex less than 1.

Otherwise, suppose $d_G(c) = 3$ which implies $d_{G'}(c) = 1$. In this case, consider $c' = N_{G'}(c)$. Add an edge from c to a vertex $c'' \in N(c') - c$. Because $d(a) = d(b) = 2$, $c'' \notin N(a) \cup N(b)$. So, G' now has minimum degree 2.

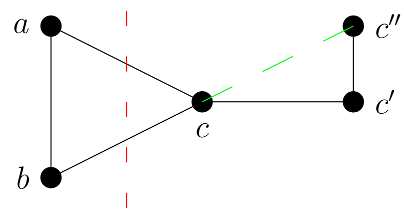

Figure 2.1: Vertex Cut $\{a, b\}$, and Added Edge $\{c, c''\}$

Inductively, we get a spanning, balanced bipartition of G'. If the edge $\{c, c''\}$ is not critical to c'' in the bipartition, then we add a to one side of the partition and b to the other side – one of which will give an edge to c – remove the edge $\{c, c''\}$, and we are done. So suppose $\{c, c''\}$ is critical to c''. In this case, we swap c and c''. We argue that swapping these vertices does not affect the degree of any other vertices. Since the edge was critical to c'', its only neighbor in the bipartition must be c. So swapping c'' affects no other vertex. The only other neighbor c can have is c'. However, since both c and c'' were neighbors of c' in the original graph, swapping them does not affect the degree of c'. The degree of c'' is now above 1. Moreover, c' is a neighbor of both c and c'', so his degree remains the same after swapping. Now, remove $\{c, c''\}$ in the bipartition. Add b to the same side as c and a to the opposite side of the bipartition which ensures that $d(c) \geq 1$.

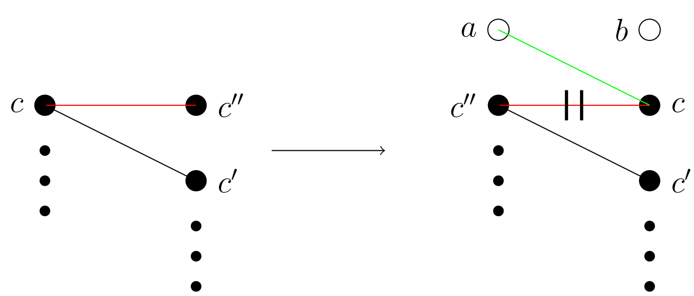

Figure 2.2: Bipartite Graph Before and After

If $d(a) = d(b) = 2$ and they have no neighbor c in common, then there are several cases to consider. Let $w = N(a) - b$ and $u = N(b) - a$. We first consider the case where $\{w, u\} \notin E(G)$. Let G' be the induced subgraph on $V - \{a, b\}$ as before. Add an edge between w and u to ensure that the minimum degree of G' is at least 2. Apply the induction hypothesis to obtain $B'[X, Y]$. If $\{w, u\}$ is critical to w or u – thus w and u are on opposite sides – we add a to the same side as u (thus making it adjacent to w) and b to same side as w (making it adjacent to u).

Suppose $\{w, u\} \in E(G)$. If $d(w) = 2$ and $d(u) = 2$, then the graph is C_4 which is already a balanced bipartition. If $d(w) \geq 3$ and $d(u) \geq 3$, then we are in Case 1. Thus, without loss of generality, suppose $d(w) = 2$, and $d(u) \geq 3$. Let $v \in N(u) - \{b, w\}$. Let G' be the induced subgraph on $V - \{a, b\}$ with the added edge $\{w, v\}$.

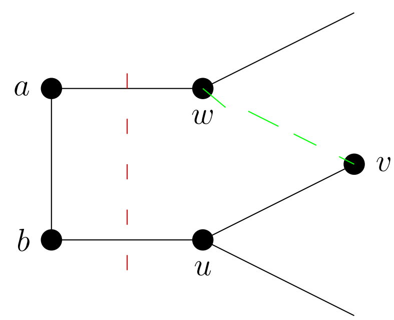

Figure 2.3: Vertex Cut $\{a, b\}$, and Added Edge $\{w, v\}$

We apply the induction hypothesis to G' to get a spanning, balanced, bipartite subgraph B'. If the edge $\{w, v\}$ is critical to w (this happens when w and u are on the same side) then we form the desired graph B by adding a to the opposite side of w and b to the same side as w. If the edge $\{w, v\}$ is critical to v, this means v is on the same side as u. Thus, we swap w and v. Swapping does not affect the degrees of any other vertices because the only neighbor of v is w. The only possible neighbors of w are v and u, but swapping w and v will not affect the degree of u. So, we add a and b to the respective sides to obtain the desired bipartition.

Case 3: Assume every edge in G has one end with degree 2, and one end with degree at least 3. Let $\{a, b\}$ be an edge with $d(a) \geq 3$, and $d(b) = 2$. Let $w = N(b) - a$. Thus, $d(w) \geq 3$. Let $U = N(a) - b$. Every $u \in U$ must have degree 2, which clearly implies $w \notin U$. We partition U into two sets: U_1 and U_2 where $U_1 = U \cap N(w)$ and $U_2 = U - U_1$.

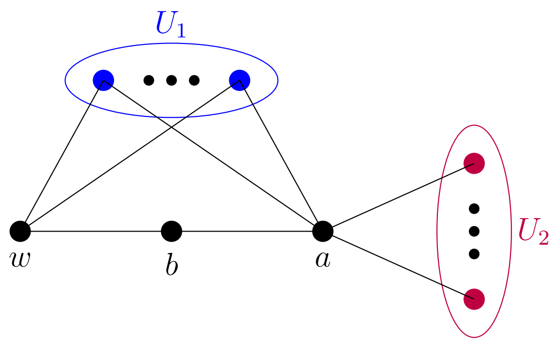

Figure 2.4: Case 3

Again, we let G' be the induced subgraph on $V - \{a, b\}$.

If $U_1 = \emptyset$, we add edges from every vertex in U_2 to w, apply induction and consider the resulting bipartite graph $B'[X, Y]$. Removing the added edges, the only vertices with degree 0 will be either w or vertices in U_2 (neighbors of a) whose only neighbor in B' was w. Thus, they must all be on the same side of the bipartition. Add a to the other side. Now, they all have degree at least 1. Add b to the opposite side of a, and we're done.

Now, suppose $U_1 \neq \emptyset$. Denote some vertex in U_1 as x. We add edges from every vertex in $U - \{x\}$ to x. We know $|U - \{x\}| \neq \emptyset$ because $d_G(a) \geq 3$. Now, apply induction and consider the resulting bipartite graph $B'[X, Y]$.

We consider the two possibilities: (1) x and w are on opposite sides in the bipartition, or (2) x and w are on the same side of the bipartition.

Suppose x and w are on opposite sides in the bipartition. All critical edges must be incident with x and some vertex in U on the same side as w. In this case, we simply add a to the same side as x and b to the opposite side, and remove all added edges. We know x was already incident with w, so it has degree at least 1. All the vertices in U on the opposite side of x are now incident with a. So we now have a spanning balanced bipartition with minimum degree at least 1.

Suppose x and w are on the same side of the bipartition and there is a critical edge in the bipartition – if there is no critical edge, simply remove the added edges and we are done. All critical edges must be incident with x and some other vertex in U_2 – the vertices in U_1 cannot be incident with a critical edge because they are connected to w. So, we mention that any vertices other than x that are incident on a critical edge are on the same side. Thus, let $\{x, u\} \in E(B')$ denote a critical edge. Swap x and u. Additionally, add b to the same side as x and a to the opposite side. Delete all the added edges. Since $x \in U_1$, x is now incident with w so they both have degree at least one. Because $\{x, u\}$ was a critical edge for u and $d_G(u) \geq 2$, the other neighbors of u in G must be on the same side of the bipartition before the swap. After swapping, u is now incident with its other neighbors in G. All other vertices – including x – that were incident on a critical edge are now adjacent to a. Thus, we now have the desired bipartition.

□

Corollary 5. *Let G be a graph with maximum degree $\Delta(G) \leq 3$. G has a spanning, balanced, bipartite subgraph B such that $d_B(v) \geq \frac{d_G(v)-1}{2}$.*

Proof. Induction on $v(G)$. Let G be a graph on n vertices. If G has minimum degree 2, we apply the previous result and are done. If G has minimum degree 1, we do the following. Let v be a vertex of degree 1. Let u be the neighbor of v in G. If $d(u) = 3$, cut v, apply induction, and add v to the smaller side of the resulting bipartition (if both sides are already of equal size, add v to either side). If $d(u) = 2$, we do the following. Let P be the shortest path starting at vertex v and ending at vertex x where every vertex in $P - \{v, x\}$ has degree 2 and x has degree 3. Let $P' = P - \{x\}$. Thus, when we cut P' from G, it affects no vertices other than x. However, since x has degree 3, the minimum degree of $G - P'$ is still 2. We apply induction. Let $B[X, Y]$ be the resulting bipartition, and without loss of generality, suppose $|X| \geq |Y|$. Add P' to G by starting by adding v to Y, then u to X, and so on, alternating the sides that we add the vertices to. The resulting bipartition is balanced, and all vertices have the appropriate degree.

□

2.3 Remarks on the General Conjecture

The reader may immediately ask if we can generalize this technique to the degree 4 case. That is, let G be a graph with minimum degree at least 4. Can we show that it contains a spanning, balanced, bipartite subgraph of with minimum degree at least 2? The authors considered this, but it seems that the above technique heavily relied on only needing to ensure each vertex had a single edge incident to it in the resulting bipartition. Additionally, it used the triangular structure guaranteed by a a vertex of degree 2 and its 2 neighbors. Thus, the degree 4 case would definitely be much more complex. A perfunctory glance reveals that the number of cases would be quite overwhelming with no clear generalization in sight. It seems that one can get a flavor of the difficulty of the conjecture just by looking at the 4-regular case. The following reduction of the problem justifies this claim.

The observant reader may notice that if a vertex in G has odd degree, it is allowed to have less than half of its degree in the bipartition. However, if it has even degree, it must have at least half of its degree in the bipartition. This allows us to perform the following reduction of the conjecture.

Observation 6. *Conjecture 1 is true for any graph G if and only if it is true for all even graphs.*

Proof. If Conjecture 1 is true, then obviously every even graph G has a spanning, balanced, bipartition B such that $d_B(v) \geq \frac{d_G(v)-1}{2}$. Thus, we look at the converse. Suppose the conjecture is true for even graphs. Let G be an arbitrary graph. If G is even, then we are done. So, suppose G has some odd degree vertices. There must be an even number of these vertices. Arbitrarily pair up these vertices and add an edge between each pair to obtain the graph G'. G' is even. Using the hypothesis, G' has a balanced bipartition B' such that

$d'_B(v) \geq \frac{d'_G(v)-1}{2}$. Now, remove the added edges to obtain the bipartite subgraph B. Let v be a vertex with degree $2k+1$ in G. In B, it has degree greater than or equal to

$$\frac{d'_G(v)-1}{2} - 1 = \frac{d_G(v)+1-1}{2} - 1 = \frac{2k+1}{2} - 1 = k - \frac{1}{2}$$

But $k - \frac{1}{2}$ is not an integer. Thus, v must have degree at least $k = \frac{d_G(v)-1}{2}$. Thus, G has the desired bipartition.

□

Lastly, there is a rather trivial special case that the conjecture is true. If the graph is a tree, a simple induction argument similar to Corrolary 3 gives us our desired result.

Observation 7. *For any tree T, T has a balanced bipartite subgraph B such that $d_B(v) \geq \frac{d_T(v)-1}{2}$.*

Proof. Let T be a tree. We proceed by induction on $e(T)$ and $v(T)$.

Case 1: Suppose T has 2 adjacent vertices of odd degree. Remove the edge between them, apply induction. It is easy to check that the resulting bipartition is sufficient. Thus, every parent of a leaf must be of even degree.

Case 2: T has a vertex v that is only a parent to leaves. Such a vertex must exist because if we root the tree at an arbitrary vertex, then look at the lowest level of T, the lowest level must consist of all leaves. Take the parent of any of those leaves, and we must have an even degree vertex whose only neighbors can be the one neighbor above him in the branching (his parent), and the neighbors below him. Since we are on the second to lowest level of the tree, everyone below him must be leaves.

Now, if v has degree 2, cut him and his adjacent leaf u. Apply induction, add v to the side of the bipartition opposite his parent. Add u to the other side. One can check that we have achieved the minimum degree.

If v has degree greater than 2, cut 2 neighboring leaves. Apply induction and add each cut leaf to opposite sides of the bipartition.

□

Induction seems to be the most natural approach to this problem – both Edwards and Erdös used it in their initial problems. However, many judicious partitioning results take advantage of more advanced techniques – probabilistic and otherwise. Thus, perhaps a different approach will yield more light to this problem. We also mention that a similar conjecture exists for balanced partitions into k parts with $k > 2$.

BIPARTITE GRAPH TILING

3.1 History

We will denote G as a graph on n vertices and H as a graph on h vertices. The first graph tiling (also commonly referred to as graph packing) result is due to Dirac who solved a seemingly unrelated problem on Hamilton paths [7]. It states that a graph G has a Hamilton path if each vertex has degree $\frac{n}{2}$ or greater. Taking such a Hamilton path also provides a matching (or a 1-factor; or a K_2-tiling) in the obvious manner. A decade later, Corrádi and Hajnal deduced minimum degree conditions to guarantee and K_3-tiling (or a triangle factor) [6]. The next obvious step was to find minimum degree conditions for a K_4 factor, but Hajnal and Szemerédi put a quick end to the tradition by giving minimum degree conditions that ensure a K_r-tiling for all integers r [10]. Two decades later, Alon and Yuster impressively applied Szemerédi's Regularity Lemma in order to prove minimum degree conditions that guarantee an H-factor for an arbitrary H [2],[3]. Now that things have settled, important tools such as the Blow-up Lemma of Komlós, Sárközy, and Szemerédi [14] have standardized the approach to graph tiling problems.

Recent work has been focused on graph tiling with a slightly different flavor. Rather than working with an arbitrary graph G and tiling it with some graph H, we will take an r-partite graph G and tile it with an r-partite graph H. One may wonder why this may be different, and a short answer is that multipartite results are stronger than the corresponding general results when tiling r-partite graphs.

3.2 Definitions

We will need the following definitions in the subsequent sections, and we gather them here for convenience. We denote $\sigma(H) = a$ as the size of the smallest color class over all proper colorings of H; *hcf* denotes the highest common factor and is sometimes known as the *gcd*. The critical chromatic number of a graph H, denoted $\chi_{cr}(H)$ is defined as follows:

$$\chi_{cr}(H) = \frac{(\chi(H) - 1)h}{h - \sigma(H)}$$

Note that $\chi(H) - 1 < \chi_{cr}(H) \leq \chi(H)$ with the last inequality being equal if and only if H has perfectly balanced color classes. The density between two sets of vertices A and B is denoted $d(A, B)$, and it is defined as

$$d(A, B) = \frac{e(A, B)}{|A||B|}$$

Occasionally, similar notation will be used for degree, but it will be clear from context whether density or degree applies.

We follow Kühn and Osthus in defining $hcf(H)$. Let H be an ℓ-chromatic graph with connected components $C_1, \ldots, C_k$. We define $hcf_c(H)$ as the highest common factor of the set of integers $\{|C_1|, \ldots, |C_k|\}$. We define $hcf_\chi(H)$ as follows. Given a proper coloring C of H, denote the sizes of the color classes of C as $x_1 \leq x_2 \leq \ldots \leq x_\ell$. Let $D(C) = \{x_{i+1} - x_i | i = 1, \ldots, \ell - 1\}$. So $D(C)$ is a set of integers. Let $D(H) = \cup D(C)$ where the union ranges over all proper colorings of H. Now, $hcf_\chi(H)$ is the highest common factor of $D(H)$. Note that if $D(H) = \{0\}$, we set $hcf_\chi(H) = \infty$. Lastly, we define $hcf(H)$ as follows. We saw $hcf(H) = 1$ if $\chi(H) \neq 2$ and $hcf_\chi(H) = 1$. If $\chi(H) = 2$, we say $hcf(H) = 1$ if both $hcf_c(H) = 1$ and $hcf_\chi(H) \leq 2$. Otherwise, we saw $hcf(H) \neq 1$.

We are now able to define $\chi^*(H)$ which determines the minimum degree needed for a perfect H-tiling. If $hcf(H) = 1$, then we say $\chi^*(H) = \chi_{cr}(H)$. If $hcf(H) \neq 1$, then we say $\chi^*(H) = \chi(H)$. In this paper, we will be only concerened with the case that $\chi^*(H) = \chi_{cr}(H)$. Thus, we work with the assumption that $hcf(H) = 1$. Additionally, since our H is always bipartite, this implies $hcf_c(H) = 1$ and $hcf_\chi(H) \leq 2$.

3.3 Recent Activity

Kühn and Osthus recently published a result that, similar to Alon and Yuster's original result, provides sufficient degree conditions to tile an arbitrary graph G with some smaller graph H [17]. The difference between their result and Alon and Yuster's is that Kühn and Osthus give a *tight* condition by using Komlós' insight into the importance of the critical chromatic number, and also, they prove the result up to a constant depending on H rather than a constant depending on $v(G)$. It was well-known that the chromatic number played a pivotal role in the minimum degree condition, but its role was not consistent until Komlös discovered that the critical chromatic number is also important [13]. Part of the significance of Kühn and Osthus' result is determining exactly when either the chromatic number or the critical chromatic number is the relevant parameter.

Now, on a slightly different note, Zhao answered the following question: given a balanced, bipartite graph G on $2n$ vertices, what minimum degree will guarantee a $K_{s,s}$-packing? The answer is $\sim \frac{n}{2} + c$ [22]. Thus, bipartite tiling of a clique seems to require only half of the vertices on one side of the partition whereas tiling a clique in general requires half of $v(G)$. This idea holds for arbitrary graphs H as well. The main focus of this paper will be to combine the work of Zhao by using his techniques on bipartite tiling and the work of Kühn and Osthus in the use of the critical chromatic number to solve the bipartite tiling problem for an arbitrary bipartite H.

Theorem 8. *Let H be a bipartite graph on h vertices such that $hcf(H) = 1$. Let G be a balanced bipartite graph on $2n = mh$ vertices for some $m \in \mathbb{Z}^+$. For any $\gamma' > 0$, there exists a positive integer m_0 such that if $m \geq m_0$ and*

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)}\right) n + \gamma' n$$

then G contains a perfect H-packing.

One may wonder what if $hcf(H) \neq 1$. In that case, we use Zhao's theorem to tile G with copies of $K_{h,h}$. Thus, if $hcf(H) \neq 1$, we always need a factor of $\frac{n}{2}$ where n is the size of each partition of G. The following constructions also show that this is roughly best possible.

3.4 Lower Bound

We first look at the lower bound of our main theorem – Theorem 8. The following constructions prove that we need not consider the case when $hcf(H) \neq 1$ as mentioned above, and that when $hcf(H) = 1$, our minimum degree condition is tight up to a factor of γn.

Theorem 9. *Let H be any bipartite graph on h vertices. We assume G to be a balanced bipartite graph on $2n = mh$ vertices where $m \in \mathbb{Z}$. If $hcf(H) \neq 1$, then there exists a G such that $\delta(G) = \lceil \frac{n}{2} \rceil - 1$ and G does not contain an H-factor. If $hcf(H) = 1$, then there exists a G such that $\delta(G) = \left(1 - \frac{1}{\chi_{cr}(H)}\right) n - 1$ and G does not contain an H-factor.*

Proof. To prove the tightness of the lower bound, we give the following four constructions.

Construction 1. *Let $hcf_c(H) \geq 3$. Let $G = K_{\lceil \frac{n}{2} \rceil - 1, \lceil \frac{n}{2} \rceil - 1} \cup K_{\lceil \frac{n}{2} \rceil + 1, \lceil \frac{n}{2} \rceil + 1}$. G does not contain a perfect H-packing.*

Proof. Since $hcf_c(H) \geq 3$, and any component of H must fit entirely into one of the two connected components of G, we can deduce the following. The size of the components of G differ by either 1 or 2 depending on whether n is even or odd. However, the size of the components of H differ by multiples of $hcf_c(H)$ which is larger than 3. Thus, there is no way to arrange the components nor the copies of H to even out the sizes of the components of G. So G contains no perfect H-packing.

□

Construction 2. *Let $hcf_c(H) = 2$. If n is odd, let $G = K_{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil} \cup K_{\lfloor \frac{n}{2} \rfloor, \lfloor \frac{n}{2} \rfloor}$. If n is even, let $G = K_{\frac{n}{2}+1, \frac{n}{2}+1} \cup K_{\frac{n}{2}-1, \frac{n}{2}-1}$. G does not contain a perfect H-packing.*

Proof. **Case 1:** n is odd. Because $hcf_c(H) = 2$, each component of H is even-sized. However, n is odd, so one of the components of G is odd. So, we have even sized components going into odd sized components. Thus, there can be no perfect packing.

Case 2: n is even. Similarly, each component of H is even, but each component of G is odd. So there can be no perfect packing.

□

Construction 3. *Let $hcf_c(H) = 1$, $hcf_\chi(H) \geq 3$. Let $G = K_{\lfloor \frac{n}{2} \rfloor+1, \lceil \frac{n}{2} \rceil -1} \cup K_{\lceil \frac{n}{2} \rceil -1, \lfloor \frac{n}{2} \rfloor +1}$. G does not contain a perfect H-packing.*

Proof. First, we defined $hcf_{\chi,c}(H)$. Let H have connected components

$$C_1[X_1, Y_1], C_2[X_2, Y_2], \ldots, C_k[X_k, Y_k]$$

Let $S = \{||X_i| - |Y_i|| : i = 1, \ldots, k\}$. Let $hcf_{\chi,c}(H)$ be the highest common factor of the set of integers S. We now prove if $hcf_\chi(H) \geq 3$ and $hcf_c(H) = 1$, then $hcf_{\chi,c}(H) \geq 3$. Note: this does not imply $hcf_{\chi,c}(H) \geq hcf_\chi(H)$. The reason we prove this is that if the differences in the sizes of the partitions in the components was relatively prime, then we could easily adjust the sizes of the components of G by carefully arranging the components of H to attain an H-packing. However, we prove that this cannot happen if $hcf_\chi(H) \geq 3$ and $hcf_c(H) = 1$.

Thus, suppose $hcf_\chi(H) \geq 3$ and $hcf_c(H) = 1$. First, $hcf_{\chi,c}(H) \neq 2$. If it did, then this would mean for each component $C_i[X_i, Y_i]$, $|X_i| + |Y_i|$ is even. This means $|X_i|$ and $|Y_i|$ have the same parity. Thus, $|X_i| - |Y_i|$ is even for all $i = 1, \ldots, k$. But this implies $hcf_c(H) = 2$ contradicting our assumption.

Now, we show $hcf_{\chi,c}(H) \neq 1$. To do this, we need the following lemma:

Lemma 10. *Let $\{a_1, a_2, \ldots, a_k\}$ be a set of positive integers. Let $A = \{a_1 \pm a_2 \pm \ldots \pm a_k\}$. That is, A is the set of all combinations of adding and subtracting the elements $a_1, \ldots, a_k$. Then,*

$$hcf(a_1, a_2, \ldots, a_k) \leq hcf(A) \leq 2 \cdots hcf(a_1, \ldots, a_k)$$

Proof of Lemma. The first inequality, $hcf(a_1, a_2, \ldots, a_k) \leq hcf(A)$ follows immediately by considering the following. Let $hcf(a_1, \ldots, a_k) = p$. Then p factors out of $a_1 \pm a_2 \pm \ldots \pm a_k$. So, $p \leq hcf(A)$.

For the second inequality, we suppose $hcf(A) > 2 \cdots hcf(a_1, \ldots, a_k) = 2p$. Thus, $hcf(A) = pq$ for some integer $q > 2$. Let a_i be the first term in any sum in A without q as a factor. We know a_i exists, otherwise pq would be equal to the $hcf(a_1, \ldots, a_k)$. Consider the sum $a_1 + a_2 + \ldots - a_i + \ldots + a_k = a_1 + a_2 + \ldots + a_i + \ldots + a_k - 2a_i$. Since pq is a common factor, pq factors out of $a_1 + a_2 + \ldots + a_k$. However, pq does not factor out of $2a_i$ unless $q = 2$. This is a contradiction since we assumed $q > 2$.

$\square$

Now, let $B = \{a_i = ||X_i| - |Y_i|| : i = 1, \ldots, k\}$. The highest common factor of B is equal to $hcf_{\chi,c}(H)$. Let $A = \{a_1 \pm a_2 \pm \ldots \pm a_k\}$. The highest common factor of A is equal to $hcf_{\chi}(H)$. By the previously proved lemma, if $hcf_{\chi,c}(H) = 1$, then $hcf_{\chi}(H) \leq 2$. This contradicts our assumption that $hcf_{\chi}(H) \geq 3$.

So, we have established $hcf_{\chi,c}(H) \geq 3$. Also, the sizes of the components of G differ by at most 2. Now, the claim becomes clear because if $hcf_{\chi,c} \geq 3$ and we can only adjust the relative sizes of the components of G by $hcf_{\chi,c}(H) \geq 3$, then we can never get a perfect H-packing.

$\square$

Construction 4. *Let $hcf(H) = 1$. Let $G = K_{\frac{m\sigma(H)}{2}-1, \frac{mh-m\sigma(H)}{2}+1} \cup K_{\frac{mh-m\sigma(H)}{2}+1, \frac{m\sigma(H)}{2}-1}$. Then $\delta(G) = \left(1 - \frac{1}{\chi_{cr}(H)}\right)n - 1$, and G has no perfect H-packing.*

Proof. Let H be a graph with components $C_1, C_2, \ldots, C_k$. By contradiction, suppose G has a perfect H-packing. Then, one can see that

$$\sigma(G) \geq m \sum_{i=1}^{k} \sigma(C_i) = m\sigma(H)$$

This comes from the fact that one can simply arrange the mk packed components of G in the same way that one arranges the color classes of G to attain $\sigma(G)$. However, it is easy to see that $\sigma(G) = m\sigma(H) - 2$ by simply placing the 2 components of size $\frac{m\sigma(H)}{2} - 1$ in the same color class. This is a contradiction.

$\square$

3.5 Regularity Lemma and Other Tools

The Regularity Lemma [20] and the Blow-up Lemma [14] are the backbone of our proof. They allow us to gain convenient structural properties from an arbitrary graph G. Before stating the lemmas, we define ϵ-regularity, and (ϵ, δ)-super-regularity.

Definition 11. *Let $\epsilon > 0$. Suppose graph G has disjoint vertex sets X and Y. We say the pair (X, Y) is ϵ-regular if for every $A \subseteq X$ and $B \subseteq Y$ satisfying $|A| > \epsilon|X|$, $|B| > \epsilon|Y|$ we have $|d(A, B) - d(X, Y)| < \epsilon$. The pair (X, Y) is (ϵ, δ)-super-regular if (X, Y) is ϵ-regular and $d(x, Y) > \delta$ for every $x \in X$ and $d(y, X) > \delta$ for every $y \in Y$ where $d(x, Y)$ and $d(y, X)$ are the density.*

Now we are ready to state the bipartite form of Szemer edi's Regularity Lemma (see the survey in [16]).

Regularity Lemma. *For every $\epsilon > 0$, there exists an $M \in \mathbb{R}^+$ such that if $G = (X, Y; E)$ is any bipartite graph with $|X| = |Y| = n$, and $d \in [0, 1]$ is any real number, then there is a partition of X into cluster $X_0, X_1, \ldots, X_k$ and a partition of Y into $Y_0, Y_1, \ldots, Y_k$ and a spanning subgraph $G' = (X, Y; E')$ with the following properties:*

- $k \leq M$
- $|X_0|, |Y_0| \leq \epsilon n$
- $|X_i| = |Y_i| = N \leq \epsilon n$ *for all* $i \geq 1$
- $deg_{G'}(v) > deg_G(v) - (d + \epsilon)n$ *for all* $v \notin X_0 \cup Y_0$
- *All pairs (X_i, Y_j), $1 \leq i, j \leq k$, are ϵ-regular under G', each with density either 0 or greater than d.*

The Blow-up Lemma is an incredibly useful tool for graph tiling, especially when combined with the Regularity Lemma as it essentially says that if a complete bipartite version of a graph can be tiled by some H, then an (ϵ, δ)-regular version of a graph can be tiled as well.

Blow-up Lemma. *Given a graph G or order n and parameters $\delta, \Delta > 0$, there exists an $\epsilon > 0$ such that the following holds: Let N be an arbitrary positive integer, and let us replace the vertices of G with pairwise disjoint N-sets $V_1, V_2, \dots, V_n$. We construct two graphs on the same vertex set $V = \cup V_i$. The graph $K(G)$ is obtained by replacing all edges of G with copies of the complete bipartite graph $K_{N,N}$ and a less dense graph G' is constructed by replacing the edges of G with some (ϵ, δ)-super-regular pairs. If a graph H with maximum degree $\Delta(H) \leq \Delta$ can be embedded into $K(G)$, then it can be embedded into G'.*

Lastly, we need a result of Kühn and Osthus [17] that will essentially allows us to tile a subgraph of G with some graph H as long as that subgraph satisfies certain properties. We state the bipartite version of their lemma.

Kühn and Osthus Lemma. *Let H be a bipartite graph on h vertices such that $hcf(H) = 1$, $\sigma(H) = a$, and $h - \sigma(H) = b$ for some integers a and b. Let $0 < d \ll \beta \ll \frac{a}{b}, 1 - \frac{a}{b}, \frac{1}{a+b}$ be positive constants. Suppose that $F[U_1, U_2]$ is a complete bipartite graph with such that the following hold:*

(1) $|F| \gg h$

(2) $|F|$ *is divisible by* h

(3) $(1 - \beta^{\frac{1}{10}})|U_2| \leq \frac{a}{b}|U_1| \leq (1 - \beta)|U_2|$

Then F contains a perfect H-packing.

3.6 Outline of Proof

Before we get into the details of the proof of Theorem 8, we provide the reader with a broad outline that will guide us through the steps we will use to obtain our desired H-packing. Let H be a graph on h vertices. Let $\sigma(H)$ be the size of the smallest color class of H over all the color classes. Also, we use $hcf_c(H)$ and $hcf_\chi(H)$ which were defined earlier.

- **Step 1:** We apply the Regularity Lemma to G to obtain a spanning subgraph, G'. The resulting subgraph has the many useful properties ensured by the Regularity Lemma such as ϵ-regularity between clusters, appropriate densities, exact sizes, and so on.

- **Step 2:** From the graph G', we obtain the reduced graph R. From the minimum degree of G, we calculate the minimum degree of R.

- **Step 3:** We take a maximum matching M in R. From the minimum degree of R, we obtain a bound on the minimum size of the matching.

- **Step 4: (Decomposition Algorithm Part 1)** From the extra term in the minimum degree, we use integers p and q such that the ratio $\frac{p}{q}$ is slightly more than $\frac{\sigma(H)}{h-\sigma(H)}$, and these integers will satisfy the requirements of our decomposition lemma. The reason for these integers is that we want cluster pairs with ratio slightly better than the most extreme coloring of H. This will allow us some room to add or delete vertices in certain clusters as well as remove some copies of H. Now, to start the decomposition, we find two bipartite subgraphs between the matched and unmatched clusters in R (one from the left unmatched vertices to their neighbors, and one from the right unmatched vertices to their neighbors). These sugraphs will be such that the unmatched vertices will have degree p, and the matched vertices will have degree less than or equal to $q-p$.

- **Step 5: (Decomposition Algorithm Part 2)** Using the above bipartite subgraphs of R, we break up adjacent clusters into pairs of clusters with ratio $\frac{p}{q}$ or significantly more balanced cluster-pairs. We will then also break up these balanced cluster-pairs into several cluster pairs with ratio $\frac{p}{q}$.

- **Step 6:** We now want super-regularity between each cluster. Thus, we move vertices that have low degree into the adjoining cluster pair to the exceptional sets.

- **Step 7:** After ensuring super-regularity, we need to get rid of the exceptional sets. We do this by finding appropriate clusters to move each vertex too. Also, in order not to change any of the original properties of any of the clusters, we are careful not to choose any cluster too many times.

- **Step 8:** In preparation for invoking Kühn and Osthus' Lemma, we insure that that each cluster pair is divisibly by h.

- **Step 9:** Now the only cluster-pairs that are left are the $p : q$-ratio pairs. We show they satisfy the conditions given in the Kühn and Osthus paper, then use their following lemma to prove they can be perfectly tiled by H.

3.7 Proof of Upper Bound

Proof. We assume n is large. Let γ' be some positive real number. Without loss of generality, we greatly simplify our calculations and prove a slightly stronger statement by using $\gamma := \frac{1}{z}$ rather than the true value of γ' where z is the smallest integer greater than $\frac{1}{\gamma'}$. We also use the following parameters:

$$\epsilon \ll d \ll \gamma < \gamma' \ll 1$$

where $\frac{64h^2q^4\epsilon}{a} < d < \frac{\gamma}{4q^2}$ where we state the value of x is given at the beginning of Step 4.

Recalling the following notation, $\sigma(H) = a$ and $h - \sigma(H) = b$, routine calculation shows the minimum degree of G in terms of a and b:

$$\delta(G) \geq \left(1 - \frac{1}{\chi_{cr}(H)}\right) n + \gamma n = \left(\frac{a}{a+b} + \gamma\right) n.$$

Step 1: We apply the Regularity Lemma to G with parameters ϵ and d. We obtain a spanning subgraph, G', with clusters of vertices $X_1, Y_1, \ldots X_k, Y_k$, each of size $N \leq \epsilon n$ and exceptional sets X_0 and Y_0 each of size less than or equal to ϵn. Also, every pair of clusters (X_i, Y_j) is ϵ-regular, and the density between X_i and Y_j is either 0 or greater than d. Lastly, the degrees of the vertices in G' are very close to their degrees in G:

$$d_{G'}(v) > d_G(v) - (d+\epsilon)n = \left(\frac{a}{a+b} + \gamma - d - \epsilon\right) n.$$

Step 2: Let $R = R[X,Y]$ be the reduced graph of G' which we define as the graph where each vertex corresponds to a cluster in G', and we say there is an edge between X_i and Y_j if the density between X_i and Y_j is at least d. We calculate $\delta(R)$ as follows: let X_i be an arbitrary vertex in R. Let x be an arbitrary vertex in the cluster X_i. In G', x is adjacent to $(\frac{a}{a+b} + \gamma - d - \epsilon)n$ vertices. Disregarding the exceptional sets, x is adjacent to at least $(\frac{a}{a+b} + \gamma - d - 2\epsilon)n$ vertices. Now, we divide by N to get the minimum number of clusters that x is adjacent to. Using the following set of inequalities

$$n(1-\epsilon) \leq n - |X_0| \leq \sum_{i=1}^{k} |X_i| = kN \leq n$$

to obtain that $k \leq \frac{n}{N}$, we get that x is adjacent to at least

$$\left(\frac{a}{a+b} + \gamma - d - \epsilon\right)\frac{n}{N} \geq \left(\frac{a}{a+b} + \gamma - d - \epsilon\right) k \geq \left(\frac{a}{a+b} + \frac{\gamma}{2}\right) k$$

vertices. Now, if x has an edge to a cluster, then the density between X_i and that cluster is greater than 0. The Regularity Lemma then guarantees that it must be at least d. Thus, X_i is adjacent to that cluster in R. So, we deduce the minimum degree for R:

$$\delta(R) \geq \left(\frac{a}{a+b} + \frac{\gamma}{2}\right) k.$$

Step 3: Let M be a maximum matching in R. Denote M_X and M_Y as the matched vertices from X and Y. Denote U_X and U_Y as the unmatched vertices from X and Y. Now, we must prove a claim that will guarantee us that the size of the maximum matching $|E(M)| \geq 2 \cdot \delta(R) = 2(\frac{a}{a+b} + \frac{\gamma}{2})k$ and consequently, $|U_X|, |U_Y| \leq (\frac{b-a}{a+b} - \gamma)k$.

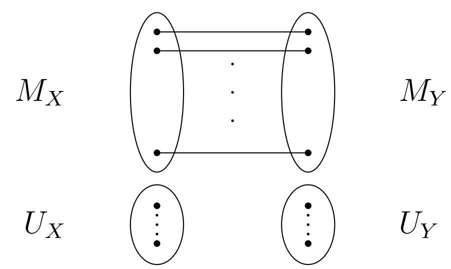

Figure 3.1: A Maximum Matching M in R

Claim. *Let $G[X,Y]$ be a bipartite graph on n vertices with minimum degree δ such that $|X| \le |Y|$. Then, G has a matching of size at least $min\{2\delta, |X|\}$.*

Proof. **Case 1:** $\delta < \frac{|X|}{2}$. Let M be a maximum matching in G. Let $X = X_1, \ldots, X_n$ and $Y = Y_1, \ldots, Y_n$. We say $X_i \sim Y_j$ if there is an edge between them in the matching. Let U_X and U_Y be the unmatched vertices. By reindexing, we obtain for some integer ℓ, $M = \{X_1 \sim Y_1, X_2 \sim Y_2, \ldots, X_\ell \sim Y_\ell\}$, $U_X = \{X_{\ell+1}, \ldots, X_k\}$, $U_Y = \{Y_{\ell+1}, \ldots, Y_k\}$.

Obviously there are no edges between U_X and U_Y, otherwise we can extend the matching. Now, in the following, we simply formalize the following idea: there can be no edge between a neighbor of U_X and a neighbor of U_Y or we can extend the matching as well. More formally, let $X' \in U_X$ and $Y' \in U_Y$. Let $I_{X'} = \{i : X' \sim Y_i\}$ and $I_{Y'} = \{j : Y' \sim Y_j\}$. Let t be in the intersection of $I_{X'}$ and $I_{Y'}$. We thus expand the matching with edges $X' \sim Y_t$ and $Y' \sim X_t$ contradicting the maximality of M. So, $I_{X'}$ and $I_{Y'}$ are disjoint subsets of $1, \ldots, \ell = [\ell]$.

However, it is clear that,

$$|I_{X'}|, |I_{Y'}| \ge \delta$$

Thus, $|E(M)| \ge |I_{X'} \cup I_{Y'}| \ge 2 \cdot \delta$.

Case 2: $\delta \ge \frac{|X|}{2}$. We assume there is a vertex $X_i \in X$ that is not included in M. Otherwise, we are done. Since $|Y| \ge |X|$, there must also be a vertex $Y_j \in Y$ that is not included in M. Following the reasoning above, their neighborhoods must be disjoint. However, their neighborhoods are at least of size $\delta \ge \frac{X}{2}$. Thus, every vertex in X is covered in the matching. □

Now, using the above claim, on R, we achieve the lower bound of a maximum matching in R.

Step 4: (Decomposition Algorithm Part 1) In preparation for our short-term goal which is to decompose adjacent cluster pairs such that they have the appropriate ratio. Let $p = 2az + b$ and $q = 2bz$ be integers. We see p and q satisfy the following inequality:

$$\frac{a}{b} < \frac{\gamma}{4} + \frac{a}{b} \leq \frac{p}{q} = \frac{a}{b} + \frac{\gamma}{2}$$

where the reader may recall that we set $\gamma = \frac{1}{z}$ in the beginning. In Step 5, we will prove that that such integers will satisfy our decomposition lemma.

We first prove that we can find bipartite subgraphs P_1 and P_2 of R that satisfy certain properties. P_1 will be a bipartite subgraph with vertex sets U_X and $N(U_X) \subset M_Y$. Moreover, for any vertex $x \in U_X$, $d(x) = p$, and for any vertex $y \in N(U_X)$, $d(y) \leq q - p$. We define P_2 similarly: P_2 will have vertex sets U_Y and $N(U_Y) \subset M_X$; for any $x' \in U_Y$, $d(x') = p$, and for any $y' \in N(U_Y)$, $d(y') \leq q - p$. We will only prove that we can find P_1 because the proof for P_2 will be the same.

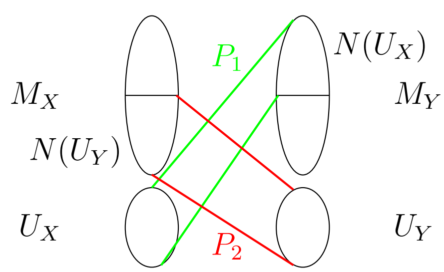

Figure 3.2: Finding P_1 and P_2

We will find such a subgraph by the greedy algorithm. Arbitrarily order the vertices in U_X. For each vertex in U_X, we find p neighbors in $N(U_X)$ with the restriction that we cannot choose any vertex in $N(U_X)$ more than $q - p$ times. Thus, for the ith vertex in U_X, we must have p vertices in $N(U_X)$ that have not been chosen $q - p$ times. Let m be the number of vertices in $N(U_X)$ that have been chosen $q - p$ times. It suffices to show that

$\delta(R) \geq m+p$. Since $m \leq \frac{(i-1)p}{q-p}$, it suffices to show that $|N(U_X)| \geq \frac{(i-1)p}{q-p}+p$ or equivalently $|N(U_X)| - \frac{(i-1)p}{q-p} \geq p$. From the bounds we got in step 3, we get that

$$|N(U_X)| - \frac{(i-1)p}{q-p} \geq \left(\frac{a}{a+b}+\frac{\gamma}{2}\right)k - \frac{p}{q-p}\left(\frac{b-a}{a+b}-\gamma\right)k. \tag{3.1}$$

However, since we can take k to be arbitrarily large, if we can show that

$$\left(\frac{a}{a+b}+\frac{\gamma}{2}\right) - \frac{p}{q-p}\left(\frac{b-a}{a+b}-\gamma\right) > 0$$

then surely (3.1) will be greater than p. Algebraic manipulation shows the above equation is equal to

$$\frac{qa-pb}{a+b}+\frac{(q+p)\gamma}{2}$$

Thus, if we can show

$$qa-pb > \frac{-(q+p)(a+b)\gamma}{2}$$

we will be done. From the inequality given at the beginning of Step 4, $\frac{p}{q} \leq \frac{a}{b}+\frac{\gamma}{2}$ which is equivalent to $qa-pb \geq -\frac{bq\gamma}{2}$. So, we show that

$$-\frac{bq\gamma}{2} > \frac{-(q+p)(a+b)\gamma}{2}$$

This is easily verified algebraically:

$$-\frac{bq\gamma}{2} = -\frac{(b)(2bz)}{2z} = -b^2 > -\left(a+b+\frac{a+b}{2z}\right)(a+b) = \frac{-(q+p)(a+b)\gamma}{2}.$$

Thus, the greedy algorithm is sufficient to find such subgraphs P_1 and P_2. This concludes the first part of the Decomposition Algorithm.

Step 5: (Decomposition Algorithm Part 2) Now that we have the subgraphs P_1 and P_2, we detail how we want to break up the clusters. Again, we only give the details on P_1 because P_2 follows the exact same procedure. We decompose every $x \in U_X$ into p

subclusters and adjoin each subcluster to a unique neighbor of x. Since $d(x) = p$ for each x, we know we can always find p unique neighbors. However, we do not adjoin each subcluster of x to the entire cluster. Instead, we adjoin it to a subcluster of size $\frac{N}{q}$. Thus, the ratio between the sizes of these two subclusters is $\frac{p}{q}$.

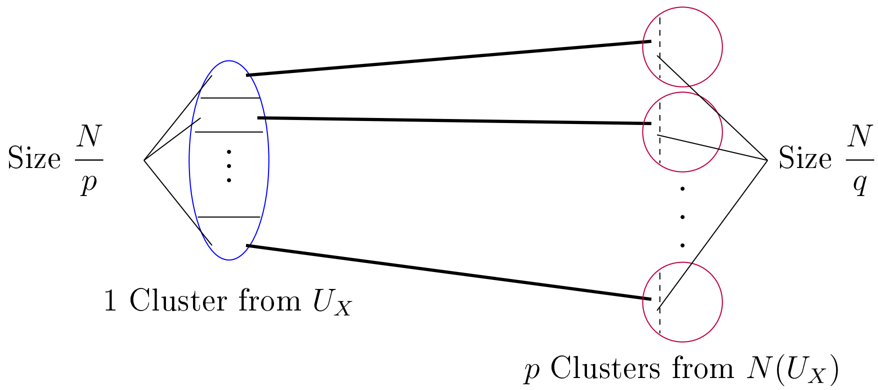

Figure 3.3: Decomposing One Cluster in U_X

Now, because $d(y) \leq q - p$ for every $y \in N(U_X)$, no cluster gets chosen more than $q - p$ times. Thus, at the end of the algorithm, the left over clusters have at least $N-(q-p)\frac{N}{q} = \frac{pN}{q}$ vertices. We adjoin these clusters to their neighbors in the matching which again makes a cluster-pair with ratio at least $\frac{p}{q}$.

However, some of these cluster pairs may have ratios significantly more than $\frac{p}{q}$ (ie. when a cluster in $N(U_X)$ gets picked less than $q - p$ times). Thus, we detail how to break up each of these cluster pairs into smaller ones that each have ratio $\frac{p}{q}$.

Let Y_j be a cluster in $N(U_X)$ that has degree $i < q-p$ in P_1. If $i = q-p$ we simply match up the remaining $\frac{pN}{q}$ vertices in Y_j with cluster X_j so that we have a cluster-pair of ratio $\frac{p}{q}$. Thus, suppose $0 \leq i < q - p$. We decompose Y_j as follows. First, $\frac{iN}{q}$ vertices are used in cluster pairs with one side of size $\frac{N}{p}$ from U_X and the other side from Y_j of size $\frac{N}{q}$ as detailed above. Now, we match up the remaining $N - \frac{iN}{q}$ vertices in Y_j with the cluster X_j (it's matched neighbor) of size N. From these 2 clusters, we will obtain at most 3 cluster-pairs of ratio $\frac{p}{q}$.

Take $\frac{iN}{q-p}$ vertices from X_j and match them with a subcluster of size $\frac{ipN}{q(q-p)}$ from Y_j. This makes a cluster pair with ratio $\frac{p}{q}$. Now, the number of remaining vertices in X_j is

$$N - \frac{iN}{q-p}$$

and the remaining vertices in Y_j is

$$N - \left(\frac{iN}{q} + \frac{ipN}{q(q-p)}\right) = N - \frac{iN}{q-p}.$$

So, there are equal amount of vertices remaining from X_j and Y_j. Now, we make two more cluster pairs with ratio $\frac{p}{q}$ by pairing together $(N - \frac{iN}{q-p})(\frac{p}{q+p})$ vertices from one cluster with $(N - \frac{iN}{q-p})(\frac{q}{q+p})$ from the other. Then, pairing the rest of the vertices left over results in another $\frac{p}{q}$-ratio cluster-pair.

Before breaking up these adjacent clusters, we had ϵ-regular pairs with density at least d. However, after breaking them up, we may no longer have such properties. However, since we broke the clusters up into relatively large sizes (larger than ϵN), the density is still within ϵ of d. Additionally, the cluster-pairs are still ϵ'-regular for some value of ϵ' that is fairly close to ϵ. The smallest a cluster can be after this step is

$$min\left\{\frac{N}{q}, \frac{p}{q^2-p^2}N\right\}.$$

Thus, the Slicing Lemma [16] guarantees that the cluster-pairs are still ϵ'-regular with $\epsilon' = max\{q\epsilon, \frac{q^2-p^2}{p}\epsilon\}$.

To summarize, what we have done is completely decomposed G' into a new graph R' such that R' is a perfect matching of clusters where the relative size of each matched pair is exactly our desired ratio: $p : q$.

Step 6: We now proceed towards super-regularity between each cluster-pair. Thus, we remove vertices that have low density into their adjoining cluster. More specifically, for

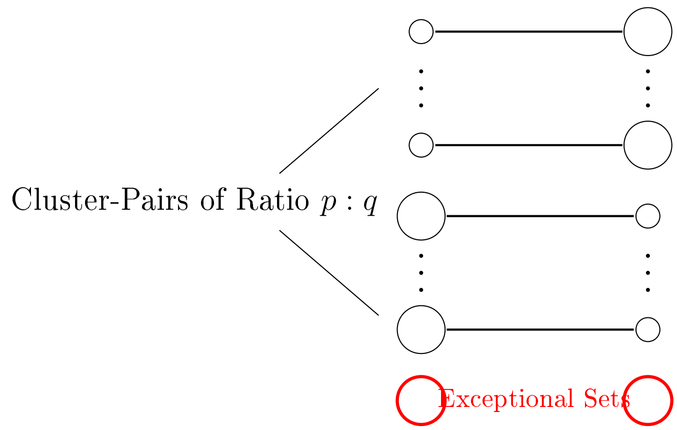

Figure 3.4: Graph R' After Decomposition

any cluster pair C_x and C_y we move any vertex $x \in C_x$ (or $y \in C_y$) such that $deg(x, C_y) < (d - 2\epsilon')|C_y|$. For any cluster C_x, ϵ'-regularity guarauntees we have at most $\epsilon'|C_x|$ vertices in such a cluster with density less than $d - 2\epsilon'$ into its adjoining cluster pair. However, when we remove these vertices, we may end up adjusting the ratio between the cluster-pairs. We want the ratio to be exactly $\frac{p}{q}$ even after removing vertices.

Let C_x and C_y be an arbitrary cluster pair in R' where C_x is the smaller of the two. After removing i vertices from C_x and j vertices from C_y, we have two potential cases:

Case 1: $j > \frac{q}{p}i$. In this case, remove $\frac{p}{q}j - i$ arbitrary vertices from C_x so that C_x loses a total of $\frac{p}{q}j$ vertices.

Case 2: $j \leq \frac{q}{p}i$. In this case, remove $\frac{q}{p}i - j$ arbitrary vertices from C_y so that C_y loses a total of $\frac{q}{p}i$ vertices. Thus, in both cases, we maintain the ratio $\frac{p}{q}$.

After doing this step, every cluster-pair now has (ϵ'', d')-super-regularity where we determine ϵ'' and d' as follows. The smallest a cluster can be in Step 5 is either $\frac{N}{q}$ or $(\frac{p}{q+p})(N - \frac{iN}{q-p})$ depending on the values of p and q. Both of these are bigger than $\frac{N}{q^2}$. So, a cluster is always bigger than

$$\frac{N}{q^2} - \frac{q}{p}\epsilon' N \geq \frac{N}{2q^2}.$$

So, again using the Slicing Lemma, $\epsilon'' = 2q^2\epsilon$. Also, each vertex in a cluster still has a density into its adjoining cluster of at least $(d - 2\epsilon' - \frac{q}{p}\epsilon')N > \frac{d}{2} = d'$. Thus, each cluster pair is still $(2q^2\epsilon, \frac{d}{2})$-super-regular.

In total, we moved at most $\epsilon'(\sum |C_x|)$ vertices to X_0 where the sum ranges over all clusters on one side of the bipartition. Similarly, we moved at most $\epsilon'(\sum |C_y|)$ vertices to Y_0 where the sum ranges over all clusters on the other side of the bipartition. Thus, $|X_0|, |Y_0| \leq \epsilon n + \epsilon' n \leq 2\epsilon' n$.

Step 7: We now need to get rid of the exceptional sets by moving each vertex in the exceptional set to an appropriate cluster. First, we define adjacency between individual vertices and clusters. We say the vertex v is adjacent to a cluster C (written $x \sim C$) if $d(x, C) \geq d'|C|$. To get rid of X_0 and Y_0, for each vertex $x \in X_0$ (or Y_0), we find a cluster that x is adjacent to. We move x to that cluster and remove a copy of H containing x. However, to ensure that each cluster-pair remains super-regular, we do not choose any cluster more than $\frac{dN}{8q^2h}$ times.

First, we prove that we have enough clusters to do this. We have at most $2\epsilon' n$ vertices in X_0 and $2\epsilon' n$ vertices in Y_0. There are more than k cluster pairs. If we have all $4\epsilon' n$ vertices adjacent to the same $(\frac{a}{a+b} + \frac{\gamma}{2})k$ cluster pairs, we must show there are enough cluster pairs to satisfy all the vertices in the exceptional set.

Thus, we must prove

$$\left(\frac{a}{a+b} + \frac{\gamma}{2}\right) k \frac{dN}{8q^2h} > 4\epsilon' n.$$

The calculation goes as follows:

$$\left(\frac{a}{a+b} + \frac{\gamma}{2}\right) k \frac{dN}{8q^2h} > d\left(\frac{a}{8h^2q^2} + \frac{\gamma}{16q^2h}\right) kN$$

$$> d\left(\frac{a}{8h^2q^2}\right) kN > d\left(\frac{a}{8h^2q^2}\right)(n - \epsilon n) = (dn - d\epsilon n)\left(\frac{a}{8h^2q^2}\right)$$

$$> \left(\frac{dn}{2}\right)\left(\frac{a}{8h^2q^2}\right) > d\left(\frac{na}{16q^2h^2}\right) > 4q^2\epsilon n \geq 4\epsilon' n.$$

We remind the reader that in the beginning, we required $d > \frac{64h^2q^4\epsilon}{a}$. This and the fact that $\epsilon' \leq q^2\epsilon$ was used in the last step in the calculation.

Now, we show that after removing vertices from a cluster pair in the above manner, each vertex still has many neighbors in the adjoining cluster. Let C_x and C_y be some cluster pair. C_x may lose up to $\frac{dN}{8q^2h} \cdot b < \frac{dN}{8q^2}$ vertices. Since originally, the clusters were ϵ'-regular, each vertex had a minimum density into the adjoining cluster of at least d'. Now, after removing the exceptional sets, any vertex in C_x (or C_y) still has density at least

$$\frac{d}{2} - \frac{dN}{8q^2|C_y|} > \frac{d}{2} - \frac{dN}{8q^2 \cdot \frac{N}{2q^2}} = \frac{d}{2} - \frac{d}{4} = \frac{d}{4}$$

since $|C_y| > \frac{N}{2q^2}$. Hence, we still have super-regularity between clusters.

Lastly, the ratio may be slightly altered after removing these copies of H from each cluster. So, in the following calculation we determine upper and lower bounds for the ratio which will be useful in Step 9. First, we assume $|C_x|$ is the smaller cluster in the cluster pair. Let us assume C_x loses $ub + wa$ vertices which implies C_y lose $ua + wb$ vertices where $u + w \leq \frac{dN}{8q^2h}$. Before this step, the ratio between the clusters was

$$\frac{|C_x|}{|C_y|} = \frac{p}{q} = \frac{a}{b} + \frac{\gamma}{2}.$$

After removing copies of H as above, ratio between $|C_x|$ and $|C_y|$

$$\frac{|C_x| - ub - wa}{|C_y| - ua - wb} \geq \frac{|C_x| - ub - wa}{|C_y|} = \frac{|C_x|}{|C_y|} - \frac{ub + wa}{|C_y|}$$

$$\geq \frac{a}{b} + \frac{\gamma}{4} - \frac{(u+w)h}{|C_y|} \geq \frac{a}{b} + \frac{\gamma}{4} - \frac{dNh}{8q^2h|C_y|} \geq \frac{a}{b} + \frac{\gamma}{4} - \frac{dN}{8q^2|C_y|}.$$

Now, we do not know the value of $|C_y|$, but we know it is larger than $\frac{N}{2q^2}$. Substituting this in along with the fact that $d \ll \gamma$, we get the lower bound for the ratio between a cluster pair.

$$\frac{a}{b}+\frac{\gamma}{4}-\frac{dN}{8q^2|C_y|}\geq\frac{a}{b}+\frac{\gamma}{4}-\frac{dN}{\frac{8N}{2q^2}}=\frac{a}{b}+\frac{\gamma}{4}-\frac{d}{4}$$

$$>\frac{a}{b}+\frac{\gamma}{4}-\frac{\gamma}{8}=\frac{a}{b}+\frac{\gamma}{8}$$

The inequalities proving the upper bound are as follows:

$$\frac{|C_x|-ua-wb}{|C_y|-wa-ub}\leq\frac{|C_x|-(u+w)a}{|C_y|-wa-ub}\leq\frac{|C_x|-(u+w)a}{|C_y|-(u+w)b}$$

$$=\frac{|C_x|-(u+w)a}{|C_y|-((u+w)a)\frac{b}{a}}\leq\frac{|C_x|-(u+w)a}{|C_y|-((u+w)a)\frac{q}{p}}=\frac{p}{q}$$

Hence, we can eliminate the exceptional sets by removing a small number of copies of H each of which contains a vertex from an exceptional set. Lastly, although we do not prove it until Step 9, each cluster also remains relatively large enough to keep the super-regular property.

Step 8: We are now left with all super-regular cluster pairs with ratio near $\frac{p}{q}$. However, we want the total number of vertices in each cluster pair to be divisible by h. This will allow us to tile these clusters using Kühn and Osthus' Lemma which relies on the Blow-up Lemma. We use the fact that $hcf_c(H)=1$. We recall what this means.

Denote the components of H as follows: $H=C_1[U_1,W_1],\ldots,C_t[U_t,W_t]$. We say that $hcf_c(H)=1$ if $hcf(|C_1|,\ldots,|C_t|)=1$ (where hcf – the highest common factor – is also commonly known as the greatest common divisor). Reindexing if necessary, this implies that there exists nonnegative integers $a_1,\ldots,a_t$ such that the following holds:

$$a_1|C_1|+\ldots+a_j|C_j|=1+a_{j+1}|C_{j+1}|+\ldots+a_t|C_t| \tag{3.2}$$

In order to ensure that the size of each cluster pair in R' is divisible by h, it is enough to show how to increase or decrease a cluster pair's size by 1 modulo h. Let B_1 and B_2 denote two cluster pairs in R'. We will decrease the size of B_1 by 1 modulo h and increase the size of B_2 by 1 modulo h. To do this, we remove a total of $2s$ copies of H where $s = max\{a_i : i = 1, \ldots t\}$. However, we selectively choose where the components of H come from. Since the cluster pairs are super-regular, we know that we can find copies of these components.

From B_1 we remove $s - a_1$ copies of C_1, $s - a_2$ copies of C_2, $\ldots$, $s - a_j$ copies of C_j, $s + a_{j+1}$ copies of C_{j+1}, $\ldots$, $s + a_t$ copies of C_t. Thus, B_1 loses

$$(s - a_1)|C_1| + \ldots + (s - a_j)|C_j| + (s + a_{j+1})|C_{j+1}| \ldots + (s + a_t)|C_t|$$

$$= s(|C_1| + \ldots + |C_t|) - (a_1|C_1| + \ldots + a_j|C_j|) + (a_{j+1}|C_{j+1}| + \ldots + a_t|C_t|)$$

vertices. However, by using (3.2) and substituting in we see that B_1 loses

$$s(|C_1| + \ldots + |C_t|) - (1 + a_{j+1}|C_{j+1}| + \ldots + a_t|C_t|) + a_{j+1}|C_{j+1}| + \ldots + a_t|C_t|$$

$$= s(|C_1| + \ldots + |C_t|) - 1$$

$$= s \cdot h - 1$$

vertices. So, B_1 loses $s \cdot h - 1 \equiv -1 \pmod{h}$. Similarly, from B_2 we remove $s + a_1$ copies of C_1, $\ldots$, $s + a_j$ copies of C_j, $\ldots s - a_t$ copies of C_t which, through a similar calculation, shows us that we lose $s \cdot h + 1 \equiv 1 \pmod{h}$. Continuing in this fashion, we can ensure that every cluster pair is divisible by h.

Step 9: The last thing to do is to prove that we can invoke Kühn and Osthus' lemma. First, we claim each cluster is still (ϵ'', d'')-super-regular, where $\epsilon'' = 2q^2\epsilon'$ and $d'' = \frac{d}{4}$. The smallest a cluster can be in Step 5 is either $\frac{N}{q}$ or $(\frac{p}{q+p})(N - \frac{iN}{q-p})$ depending on the values of p and q. Both of these are bigger than $\frac{N}{q^2}$. So, after the rest of the steps, a cluster is always bigger than

$$\frac{N}{q^2} - \frac{q}{p}\epsilon' N - \frac{dNb}{8q^2h} - (h-1)(sh+1) \tag{3.3}$$

Now, $(h-1)(sh+1)$ is essentially a constant, so it is much less than ϵN. Also, recall that $\epsilon' \leq q^2\epsilon$ and $\gamma \gg d \gg \epsilon$. So, we see that (3.3) is greater than

$$N\left(\frac{1}{q^2} - \frac{q^3}{p}\epsilon - \frac{dN}{8q^2} - \epsilon N\right) > N\left(\frac{1}{q^2} - \frac{\gamma}{q^2}\right) = \frac{N}{2q^2}.$$

So, again using the Slicing Lemma, $\epsilon'' = 2q^2\epsilon$ just as we had earlier.

The density is as follows: as we saw in Step 7, each vertex still had density at least $\frac{d}{4}$ into its adjoining cluster-pair. Using similar ideas as above, after Step 8, a vertex in $|C_x|$ may have lost $(h-1)(sh+1) < \epsilon N$ more neighbors. However, this lets us conclude that it still has more than

$$\frac{d|C_y|}{4} - \epsilon N > \frac{dN}{8q^2} - \epsilon N > \frac{dN}{8q^2} - \frac{dN}{16q^2} = \frac{dN}{16q^2}$$

neighbors in the adjoining cluster. Thus, it still has density at least $\frac{dN}{16q^2|C_y|} > \dfrac{dN}{8} = d''$.

So, since we have (ϵ'', d'') super-regular pairs, we can use the Blow-up Lemma to treat all of our clusters as complete bipartite graphs (per the requirement in the lemma). To prove $|F| \gg h$, we prove that even the smallest cluster still has many more than h vertices. This was done above since each cluster has at least $\frac{N}{2q^2} \gg h$ vertices. Also, the Kühn and Osthus Lemma requires divisibility by h. We already ensured $|F|$ is divisible by h in the previous step.

Now, for any cluster pair C_x and C_y, we know that

$$\frac{\gamma}{8} + \frac{a}{b} \leq \frac{|C_x|}{|C_y|} \leq \frac{a}{b} + \frac{\gamma}{2}.$$

Using this and doing some algebra results in

$$(1 - \frac{\gamma b}{\frac{\gamma b}{2} + 2a})|C_x| \leq \frac{a}{b}|C_y| \leq |C_x|(1 - \frac{\frac{\gamma b}{2}}{2(a + \frac{b\gamma}{2})}$$

In Kühn and Osthus' Lemma, let $\beta = \frac{\gamma b}{4a+2b\gamma}$, and we get that

$$(1 - \beta^{\frac{1}{10}})|C_x| \leq \frac{a}{b}|C_y| \leq (1 - \beta)|C_x|.$$

We can now use their lemma, and that concludes the proof.

$\square$

BIBLIOGRAPHY

[1] N. Alon, B. Bollobás, M. Krivelevich, and B. Sudakov. Maximum cuts and judicious partitions in graphs without short cycles. *J. Combin. Theory Series B* **88** (2003), 329-346.

[2] N. Alon and R. Yuster. Almost H-factors in dense graphs. *Graphs and Combinatorics* **8** (1992), 95-107.

[3] N. Alon and R. Yuster. H-factors in dense graphs. *J. Combin. Theory Ser. B* **66** (1996), 269-282.

[4] B. Bollobás and A. D. Scott. Problems and results on judicious partitions. *Rand. Struc. Alg.* **21** (2002), 414-430.

[5] H.C. Sun, J. Ni, G.D. Chen. Spanning balanced bipartite subgraphs of regular graphs: on R. Häggkvist's conjecture. (Chinese) *Nanjing Daxue Xuebao* **4** (1980), 16-20.

[6] K. Corradi and A. Hajnal. On the maximal number of independent circuits in a graph. *Acta Math. Hung.* **14** (1963), 423-439.

[7] G. Dirac. Some theorem on abstract graphs. *Proc. London Math. Soc.* **2** (1952), 69-81.

[8] C.S. Edwards. An improved lower bound on the number of edges in a largest bipartite subgraph. *Proc. 2nd Czech. Symp. on Graph Theory.* Prague (1975), 167-181

[9] P. Erdös. On some extremal problems in graph theory. *Israel Journal Math.* **3** (1965), 113-116.

[10] A. Hajnal and E. Szemerédi. Proof of a conjecture of Erd́os. *Combinatorial theory and its applications, II (Proc. Colloq., Balatonf:ured, 1969*, 601-623. *North-Holland, Amsterdam,* 1970.

[11] P. Hall. On representatives of subsets. *J. London Math. Soc.* **10** (1935), 26-30.

[12] S. Janson, T. Luczak, and A. Ruciński. *Random Graphs*, Wiley-Interscience 2000.

[13] J. Komlós. Tiling Turan theorems. *Combinatorica,* **20** (2000), 203-218.

[14] J. Komlós, G. N. Sárk:ozy, and E. Szemerédi. Blow-up Lemma. *Combinatorica* **17** (1997), 109-123.

[15] J. Komlós, G. N. Sárközy, and E. Szemerédi. Proof of the Alon-Yuster conjecture. Combinatorics (Praque, 1998). *Discrete Math.* **235** (2001), 255-269.

[16] J. Komós and M. Simonovits. Szemerédi's Regularity Lemma and its applications in graph theory. *DIMACS Technical Report* (1996).

[17] D. Kühn and D. Osthus. The minimum degree threshold for perfect graph packings. *Combinatorica* **29** (2009), 65-107.

[18] D. Kühn and D. Osthus. Matchings in hypergraphs of large minimum degree. *Journal of Graph Theory* **51** iss. 4, (2006), 269-280.

[19] A. Shokoufandeh and Y. Zhao. Proof of a conjecture of Komlós. *Random Structures & Algorithms* **23** (2003), no.2, 180-205.

[20] E. Szemerédi. Regular partitions of graphs. *Problèmes Combinatoires et Théorie des Graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*, 399-401, Colloq. Internat. CNRS, 260, Pairs, 1978.

[21] B. Xu and X. Yu. Judicious k-partitions of graphs. *J. Combin. Theory Series B* **99** (2009), 324-337.

[22] Y. Zhao. Bipartite graph tiling. *SIAM J. Disc. Math.* **23** vol.2, (2009), 888-900.