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An In-Depth Investigation Of How An Undergraduate Mathematics Major Student Learns The Concept Of Proof

Shanah K. Grant

AN IN-DEPTH INVESTIGATION OF HOW AN UNDERGRADUATE MATHEMATICS MAJOR STUDENT LEARNS THE CONCEPT OF PROOF

by

SHANAH GRANT

Under the Direction of Draga Vidakovic, PhD

ABSTRACT

Mathematical proof is of high importance in the advanced proof-based courses which mathematics majors must take in order to graduate. Investigating how a competent student learns the concept of proof may be very beneficial in the pedagogical approaches of proof courses. In this study, the Self-Regulated Learning (SRL) theory and the Action-Process-Object-Schema (APOS) theoretical framework were employed. A competent mathematics major student was observed during two semesters – Bridge to Higher Math and Analysis. The observational data was triangulated through follow up discussions after class observations and a final interview at the end of the semester. The results of data analysis indicated that the participating student was successful in writing valid proofs in the Bridge to Higher Math course but only memorized the proofs in the Analysis course. Results showed that a mismatch in the student's learning style and the instructor's teaching style in the Analysis course negatively affected the student's level of self-regulation and thus attributed to him not moving past the Action conception stage of understanding for the content covered in the course. A lack of conceptual understanding was also a difficulty that arose for the student when learning proof concepts. There was a positive correlation between the student's level of self-regulation and course grade. The student's responses to the SRL questionnaire were used to develop a generalized linear regression model to estimate the student's success based on his/her level of self-regulation. Self-efficacy proved to be the only significant component for the model. From the view of APOS theory, his conception of a proof was at mostly at the Process conception stage of understanding in the Bridge to Higher Math course and was predominately at the Action conception stage of understanding in the Analysis course. Suggestions on how to incorporate self-regulated learning in the classroom and APOS theory in the pedagogical approaches for proof courses were made.

INDEX WORDS: Self-regulated learning theory, SRL, APOS theory, Proof understanding, Proof difficulties, Grade success estimator

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SHANAH GRANT

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MAJOR STUDENT LEARNS THE CONCEPT OF PROOF

by

SHANAH GRANT

Committee Chair: Draga Vidakovic

Committee: Valerie Miller

Michael Stewart

Vladimir Bondarenko

Electronic Version Approved:

Office of Graduate Services

College of Arts and Sciences

Georgia State University

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DEDICATION

This is dedicated to my family, parents, and husband. I do not take being the first holder of a doctoral degree in the family for granted. I did it!

Mommy, *"mi tek mi book and get ah good education!"*

Daddy, your many sacrifices were not in vain.

To my husband, thank you for your patience and support.

To my son, remember,

"I can do all things through Christ who strengthens me." -Philippians 4:13.

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1 INTRODUCTION

Students enroll in college courses in the hopes of gaining information to help them in their day-to-day life. One may argue that mathematics serves as the foundation that links all courses together. In general, mathematics can be described as the search for structures and patterns that simplifies the world we live in (Griffiths, 2000). With that said, mathematics is an essential tool in our everyday life. Collegiate mathematics educators would agree that one of the most important aspects of advanced mathematics is the concept of proof. Proofs are used to confirm truths and are essential in mathematical reasoning (Smith, Eggen, & St. Andre, 2011). Nonetheless, whether one considers themselves a mathematician or not, he or she uses proof in everyday life. Be it when he or she tries to determine the better deal between buying two medium cans of baked beans versus one large can or when he or she tries to decide between yielding at the yellow light or racing through the light before it turns red. Take, for example, when someone tries to decide if they should yield at the yellow light. An informal logical proof is formulated to make a decision. The line of reasoning used may be something like, “It takes around 3 seconds for the light to change from yellow to red. The state law says drivers should slow down when the light turns yellow, but I am running late for work. If I go at a faster speed, I will be able to get through the light before it turns red. I will accelerate as opposed to slow down and stop.” There are many other instances and situations in our daily lives in which we use logical reasoning to help us arrive at a suitable decision.

While a proof in general is seen as a logical line of reasoning, formally a mathematical proof can be defined in many ways. In this study, we adopt the definition from Griffiths, (2000), that a mathematical proof is a formal and logical line of reasoning that begins with a set of axioms and moves through logical steps to a conclusion. Mathematicians and mathematics educators use formal definitions and axioms woven to form a mathematical proof. In fact, advanced mathematics courses are universally taught in an axiom-definition-theorem-proof format, in which the instructor presents the definitions of concepts, introduces axioms, states theorems related to the concepts and axioms presented, and then formulate the proofs of these theorems. According to Richeson (2008), a definition is precise. It is an unambiguous description of the meaning of a mathematical term. It characterizes the meaning of a word by giving all the properties and only those properties that must be true. An axiom, on the other hand, is a statement that is assumed to be true without proof. These are the basic building blocks from which all theorems are proven. Lastly, a theorem is a mathematical statement that is proved using rigorous mathematical reasoning.

1.1 Statement of the problem

How does one comprehend mathematical proof and thus develop his or her understanding of proof and the ability to effectively write a proof? There has been a great deal of research revolving around this question. Mathematical proof is of high importance at the university level, specifically, in the advanced proof-based courses which mathematics majors must take in order to graduate (Mejía-Ramos, Lew, de la Torre, & Weber, 2017). After all, proofs are used to confirm truth (Griffiths, 2000; Samkoff & Weber, 2015; Harel & Sowder, 2007; Csíkos, 1999; Smith, Eggen, & St. Andre, 2011) and as a form of instructional communication in the classroom (Mejía-Ramos, Lew, de la Torre, & Weber, 2017) at the advanced level. As undergraduate mathematics majors advance through the mathematics curriculum, they are expected to learn the art of proving and develop the ability to confirm mathematical truth and justifications themselves. It is expected that students will comprehend proof and learn how to effectively prove theorems from advanced mathematics courses. However, after completing higher level mathematics courses, a significant number of mathematics majors still find it difficult to write a proof on their own (Zazkis, Weber, & Mejía-Ramos, 2015). While a substantial number of researchers have focused on the notion of mathematical proof such as proof assessment (Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012), the strategies students use while proving (Zazkis, Weber, & Mejía-Ramos, 2015) , and how students read proof, very little work has been done on how students comprehend proof (Samkoff & Weber, 2015).

It is assumed that, after reading a formal proof from the textbook or after seeing an instructor prove a theorem, students will understand and thus learn the techniques required to prove a theorem. Research has revealed this is not so (Weber & Mejia-Ramos, 2014; Samkoff & Weber, 2015). Though an extensive amount of research has been conducted around proof, the question of what is the most effective way to teach students how to read and comprehend proof still remains unanswered. We believe that investigating how a competent mathematics major student learns and comprehends proof will illuminate how to develop teaching methods to improve students' comprehension of proof and thus equip them with the knowledge and skills necessary to efficiently write a proof. Moreover, understanding how successful students learn has the potential to inform teaching and learning and possibly give insight into the types of instructional practices needed for educators to better assist less successful students as they learn the concept of proof (McMillian, 2010). In this study, we aim to do an in-depth analysis of how a competent student in mathematics learns and comprehends proof.

1.2 Purpose of the study

Introduction to Proof (ITP) courses are used to bridge the gap between computational mathematical courses and advanced proof courses. They serve as the building blocks for proof on which students build as they advance in their mathematics program. As a result, it is important that students are taught the proper techniques for proving in their ITP course. Investigating how a competent student learns in an ITP course and later transfers the techniques developed to a higher-level proof course may be very beneficial in the pedagogical approaches when teaching the proof-based courses. We define competency as the ability to *"deal with different situations by drawing on concepts, knowledge, information, procedures, and methods. It incorporates many elements, [mobilizes] knowledge, and strategically marshals capabilities in accordance with the specific nature of the situation*" (Goudreau, et al., 2009, p. 3). Hence a competent student in mathematics has the ability to strategically recall the knowledge and methods required to solve mathematical problems. In addition to improving instructional practices, studying how a competent student learns the concept of proof may also shed some light on how to better assist students who struggle with learning the concept of proof (Greene & Azevedo, 2007) . Through this research we hope to answer the following questions:

1. What learning strategies does a competent student in mathematics use when learning about proof and proof techniques in proof-based courses?
    a. What is the work ethic and study habit of a competent mathematics major student as he or she learns the concept of proof?
    b. How does a competent student in mathematics develop his/her understanding of proof concepts?
2. How can we use the knowledge obtained about how a competent student learns and understands proof to help design pedagogical approaches?
    a. What approaches to learning new concepts in proof courses, used by a competent mathematics major student, could be used in teaching these concepts?

b. What challenges in learning new concepts in proof courses, encountered by a competent mathematics major student, could be used in teaching these concepts?

A single-case research design is appropriate as we will be doing an intensive study of how a competent student learns the concept of proof in the hope of answering the above questions. We will discuss this choice of methodology more in depth in the methodology section below. This study will be based on two theoretical frameworks. First, we will be employing the Self-Regulated Learning (SRL) theory to gain clarity in answering our research questions. SRL is recognized as a tool that can be used as an important predictor of student academic motivation and achievement (Zumbrunn, Tadlock, & Roberts, 2011; Pintrich & De Groot, 1990). SRL is a central conceptual framework used to understand students' cognition, motivation and emotions as they try to learn (Panadero, 2017). Self-regulated learning is also identified as the process students use to activate and sustain their cognition personally; it affects heavily, their behavior towards learning (Schunk & Zimmerman, 2008). As such, the SRL theory will be used to help identify the work ethic and study habits of a competent mathematics major student as he learns proof concepts and to help us delve into the mind of the student.

In addition to the SRL theory, the Action Process Object Schema (APOS) theory, refers to a framework of learning mathematics in which students go through four stages, Action – Process – Object – Schema (Arnon et al., 2014). If a student is able to successfully go through each of these stages, particularly arriving at the Process or Object stage of understanding a mathematical proof, then he or she is considered to have a good understanding of the concept they are being taught. Both the APOS and SRL theoretical frameworks will be described in further details in the subsequent section.

We hope this study will reveal detailed information of the strategies a mathematics major student used in learning and comprehending proof concepts, a knowledge that will be a significant addition to the body of current research on improving the instruction of advanced mathematics courses.

1.3 Theoretical perspective

While Foundationalism, which is a view that all knowledge or justified belief rest ultimately on a foundation of non-inferential knowledge or justified belief (Hasan & Fumerton, 2016), and Interpretivism, in which the goal is to understand and interpret the meanings in human behavior rather than to generalize and predict causes and effects (Edirisingha, 2016), are both suitable epistemological stances for our study, this research will be grounded in the epistemology of constructivism. Constructivism "is the view that all knowledge, and therefore all meaningful reality as such, is contingent upon human practices, being constructed in and out of interaction between human beings and their world, and developed and transmitted within an essentially social construct" (Crotty, 1998, p. 42).

This epistemology is fitting as we believe that, in addition to our analysis of the student's study habits, and performance on tests, quizzes and homework problems, we will gain insight on how a student learns and understands the concept of proof by observing how the student interacts with his or her professor and classmates in the classroom. More explicitly, we will investigate how the student constructs knowledge while interacting with his or her professor and/or peers, how he or she completes homework problems, and how the student studies and reads his or her textbook and or other resources.

1.3.1 SRL conceptual framework

Now we will take a closer look at the SRL theory, a grounded theory, that will be employed to inform this body of research. Grounded theory may be defined as an approach for developing theory that is ‘grounded in data systematically gathered and analyzed’ (Crabtree, 2006). Grounded theory is also fitting for this research as it allows for the data to reveal evidence of the type of student that is successful in effectively writing a proof. The SRL theory will shed light on the cognition, metacognition and motivation of the student as he or she learns the concept of proof. Furthermore, grounded theory allows us to let the evidence from the data emerge for conclusions to be drawn during analysis (McMillian, 2010).

SRL refers to the self-directive processes and self-beliefs that enables learners to transform their mental abilities, such as verbal aptitude, into an academic performance skill (Zimmerman, 2008; Davoodi, Khaefi, & Sadighi, 2017). The SRL theory has been used in examining the different characteristics that has an influence on successful learning (McMillian, 2010). While there are many definitions of self-regulated learning, there are three main components that each definition includes (Pintrich & De Groot, 1990; McMillian, 2010; Greene & Azevedo, 2007; Zimmerman, 2008,). These are:

1. Cognitive strategies that students use to learn, remember, and understand the material.
2. Metacognitive strategies for planning, monitoring and modifying their cognition.
3. Student management and control of their effort on classroom academic tasks.

The cognitive strategies include what students use to learn, remember, and understand course materials (rehearsal, elaboration, and organization – this may be seen as note taking techniques, highlighting key words/methods, identifying main ideas related to a proof technique, or para phrasing a proof for one's understanding). The metacognitive strategies refer to how students plan and organize their time, monitor their learning and regulate their cognition as they learn (information seeking, time management and critical thinking – this may be seen as students going to YouTube videos for assistance, making a study schedule, self-testing, and test strategies). The third component breaks down the students' motivation to learn. The motivation for a student to learn draws on his/her emotional reaction to assignments, how interested he/she is in the subject and the level of importance the assignment and or class is to him/her.

One can use the SRL theory to understand the learning process in order to better assist those who struggle with it. In the case of this research, it will be used to assist in answering the question of "What is the work ethic and study habit(s) of a competent mathematics student as he or she learns the concept of proof?" More specifically, we hope that the SRL theory will reveal the kind of habits that will enable the student to succeed in proof courses. If we are able to determine the cognitive and metacognitive aspects, along with the motivation, of a competent student as he or she learns the concept of proof, suggestions can be made about pedagogical approaches in teaching proof-based courses that can potentially help students in developing these SRL behaviors and thus stand a chance to be successful in proof courses.

While researching why academic achievement was so low across the United States, the U.S. Department of Education began to investigate what drove students' personal responsibilities (Schunk & Zimmerman, 1994). This research resulted in the emergence of the self-regulated learning theory (Zimmerman & Schunk, 2001). Mentioned previously, students are said to be self-regulated according to their level of metacognition, motivation and how they actively behave towards their learning (Zimmerman & Schunk, 2001; Zumbrunn, Tadlock, & Roberts, 2011; Zimmerman, 1989; Greene & Azevedo, 2007). In essence, SRL models are comprised of three phases, illustrated in Figure 1.1: the forethought and planning phase, the performance control phase, and the reflection on performance phase (Zimmerman & Schunk, 2001; Zumbrunn, Tadlock, & Roberts, 2011).

In the first phase, the forethought and planning phase, students decipher the tasks they are given and set personal goals to complete the task. This is particularly important as it can enhance the students' commitment to achieving the desired goal (Zimmerman & Schunk, 2001). For example, entering a proof-based class, a student may set a goal of mastering each proving technique outlined in the syllabus. Next, he or she may set mini goals such as do all odd problems in the homework section associated with each technique. At this stage however, since the student is just being introduced to proof style courses or is still learning how to assimilate, he or she may not know the best ways to approach the task of learning each technique. In this case, the instructor or more experienced peers will have to step in and instruct the student on how to do this (Zumbrunn, Tadlock, & Roberts, 2011).

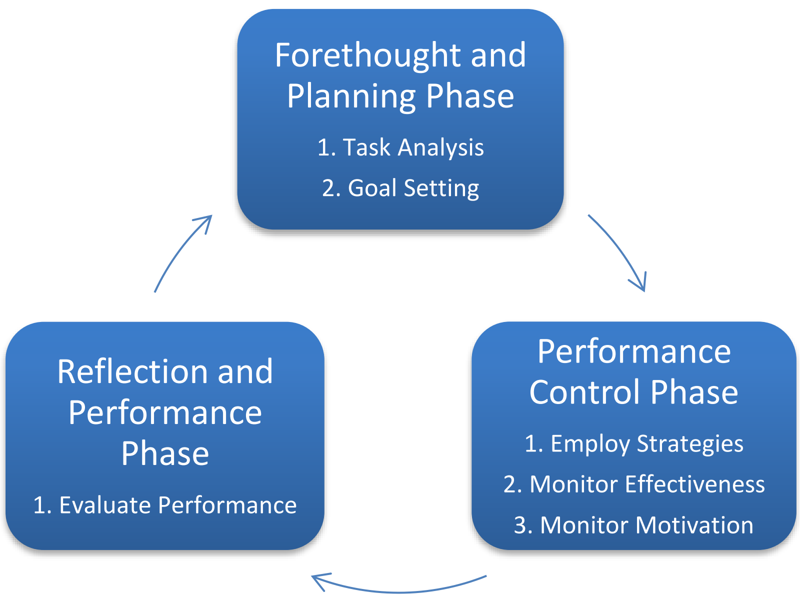

Figure 1.1 Illustration of a cycle of the SRL model.

Performance control is the second phase of SRL. In this phase, students determine strategies to advance their learning task and assess how effective these strategies are. They also evaluate their motivation to complete the desired task at this phase. For instance, a student who aims to learn the technique of direct proof for, he/she may choose to complete all odd problems associated with direct proofs in the textbook. Furthermore, the student may divide the number of odd problems evenly across a certain number of days before the first exam; and can monitor how much progress he or she is making based of the number of problems he or she has completed as the days go by. However, if the student needs to come up with new strategies, but do not know how to do so, the student may stick to old, familiar strategies that may not work for the new tasks. Specifically, in textbooks for proof-based courses, there may not be solutions to the odd problems in the back of the textbook as there are for odd problem in lower level mathematics courses. Hence, while the strategy of doing all odd problems in the chapter relating to a particular topic worked for lower level computational courses, that may not work for a proof-based course. In this case, teacher monitoring and instructor feedback may help the student develop new strategies (Zumbrunn, Tadlock, & Roberts, 2011).

In the final phase, reflection and performance, students evaluate their performance on the learning task with respect to the effectiveness of the strategies they chose (Zumbrunn, Tadlock, & Roberts, 2011). As learners pursue their goals, it is important that they feel as though they are making progress on their goal (Zimmerman & Schunk, 2001). This phase heavily influences the student's future planning and goals and thus initiates the start of the cycle again (Zumbrunn, Tadlock, & Roberts, 2011).With the previous example of a student learning the technique of direct proof, the student may reflect on his/her performance on the first exam. If the student mastered all questions that required a direct proof, he or she may continue to use the prior strategies for studying proof concepts as the course progresses and set similar goals for other techniques such as proof by contradiction, proof by induction etc. If the student did not master all of the questions related to direct proof, he/she will try to develop a new technique for studying for future tests.

1.3.1.1 The role of self-efficacy and motivation in SRL

Self-efficacy and motivation plays an essential role in self-regulated learning. Self-efficacy may be defined as a student's belief in their capabilities to learn and skills to perform a given task (Zimmerman & Schunk, 2001; Zumbrunn, Tadlock, & Roberts, 2011; Li et al., 2018; Schunk, 1985). Social cognitive theorists assume that self-efficacy is a key variable affecting self-regulated learning (Zimmerman, 1989). If a student does not think he or she is capable of completing a task, then he or she will refrain from attempting the task. For instance, suppose on the first day of class a student sees the material introduced and thinks it is too much for them to comprehend. The student may resort to dropping the class. If he or she decides to stay enrolled, the student may not put forth much effort in studying or doing assignments as he or she will see it as a waste of time since they would fail the task even if they tried it. On the other hand, if the student believes that he or she possesses the skills and capability to do well in the class, he or she will put forth the effort to complete assignments and stands a better chance to pass the class.

Motivation is another key component of self-regulated learning. Motivation comes from the Latin word, *moveré*, which means "to move". As a result, there are multiple twists to the definition of motivation. We particularly prefer the definition that motivation is "an internal state or condition (sometimes described as a need, desire, or want) that serves to activate or energize behavior and give it direction" (Ganah, 2012, p. 250). Self-motivation essentially refers to students completing tasks in the absence of external rewards. There are a number of factors that affects and influence motivation for students to learn and engage in the learning process. These include: interest in the subject, perception of its usefulness, general desire to achieve, self-confidence and self-esteem and persistence and patience (Ganah, 2012). Self-motivation takes place when a student independently uses one or more strategies to keep themselves on task to achieve their learning goal (Zumbrunn, Tadlock, & Roberts, 2011). When students are motivated, they develop interest in the subject and are thus able to see its usefulness. Motivated students generally have high self-esteem and a strong desire to achieve. As a result of this desire to achieve, the student sets his or her own learning objectives without the instructor and works diligently to achieve these goals.

If a student is able to self-regulate his or her learning through the forethought and planning phase, the performance control phase and the reflection and performance phase, they stand a high chance of being successful in their courses. More specifically, highly motivated students who possess high self-esteem and positive self-efficacy are more likely to be successful in their courses. Research suggests that there is as essence for motivation, self-efficacy, and goal orientation in the learning process (Greene & Azevedo, 2007). Self-regulated learning theory is a useful framework for examining the variety of student characteristics that influence successful learning (McMillian, 2010). In particular, it was used to identify the characteristics of the student observed as he learned the concept of proof.

1.3.1.2 SRL model prediction

The ability to model students' success may help to improve their academic performance. Specifically, being able to estimate how well or poorly a student may perform in a course, could aid instructors in their preparation of course materials. In addition, if instructors are able to assess the factors that contribute to students' academic success, then they may be able to better assist an "at risk" student, by closely monitoring the student and taking proactive measures to keep the student on track (Huang & Fang, 2010; Alyahyan & Düştegör, 2020).

These proactive activities may include scheduled office hour visits, additional recitation sessions, extra homework assignments, incorporating more active learning pedagogical approaches, etc. (Huang & Fang, 2010).

Moreover, using the SRL model of a student's academic success may be very helpful to instructors and students alike. One way to utilize the SRL model of students' academic success in a proof course is to determine which component(s) of the SRL model – motivation, self-efficacy, cognitive strategy use and self-regulation, contribute(s) a student's outcome in the course. While there are many methods such as data mining, artificial intelligence, and machine learning that can be used to predict students' grades (Alyahyan & Düştegör, 2020; Huang & Fang, 2010), we used the statistical method of regression analysis. Multivariate linear regression is usually the technique used as it is the easiest of all the methods for researchers to understand and translate (Huang & Fang, 2010). We used the stepwise variable selection method in the regression model to derive the significant predictors in our study.

There are two types of stepwise algorithm: the forward step regression and the backward step regression. In the forward stepwise regression model, one begins with no candidate variable for the model and start to introduce the variable that makes the most significant improvement in the fit. One way to quantify the model's improvement of the fit of the model, is to consider the deviance for the model fit. The deviance is adjusted for the number of predictors and refers to the amount of variability in the response variable that has not been explained (or not accounted for) by the model in question. The deviance increases only when the new variable added improves the model fit and decreases otherwise. Deviance is a preferred metric for multiple linear regression models because it is adjusted for the number of predictors.

Furthermore, one stops adding variables when the deviance is not being improved. The forward stepwise regression is usually used when there are a lot of variables to be considered.

The backward stepwise regression, however, does the opposite of the forward stepwise regression. That is, instead of starting with no candidate variables, one starts with all candidate variables and based on a test of significance (usually at point nine five level of significance), one chooses the variable that is the least statistically significant (or the one with the highest p-value) as the variable to be removed from the model. This process continues until a stopping rule is reached. A stopping rule is reached when all the predictors have statistical significance in the model - significant $p - value$ of below$\propto = 0.05$. More explicitly, if the significant value of the predictor is less than five percent, then the predictor is relevant for the model and vice versa (Faraway, 2014; Zoubir, 1993). After the criteria of a significant value of $\propto = 0.05$ or less is met, one selects the variable that increases the deviance the most.

A regression analysis is usually used to explain relationships between a single variable Y, usually referred to as the response variable, and dependent variables $X_1, \dots ,X_n$. When $n = 1$, we have a simple regression and when $n > 1$, we have a multiple regression (Faraway, 2014). To model a response variable with p predictors, a general form of the model would be:

$$y = f\left(X_{1,}X_{2,} \dots ,X_n\right) + \varepsilon$$

$$y = logit(n) = log\left(\frac{n}{1 - n}\right)$$

where, n is the probability of success, f is an unknown function that may be linear or non-linear and ε is the error of the model (Faraway, 2014). With the above form, Y may have many possibilities. It is usually assumed to take the linear form

$$Y = \beta_0 + \beta_1 X_1 + \cdots + \beta_n X_n + \varepsilon$$

where β_i are unknown parameters to be determined and β_0 can be thought of as the baseline average of the response when no other predictors are present. For this study, the predictors were motivation, self-efficacy, cognitive strategy use and self-regulation. If we can determine which, if not all, of these predictors is/are important in assessing a student's potential performance in a proof course, we may assist in improving such factors to aid students in improving their conceptual understanding in proof-based courses. That is, we may be able to assist instructors how to plan to support students even before the class begins.

1.3.2 APOS theoretical framework

While SRL is a grounded theory, the APOS theoretical framework is a constructivism epistemology that will be used to further inform this study. As a result of the constructivism epistemology, Ed Dubinsky's Action-Process-Object-Schema (APOS) constructivist theoretical framework was used as the other lens to inform this research. The focus of the APOS theoretical framework is on what might be taking place in the minds of students as they learn mathematical concepts. It describes how mathematical concepts can be learned, as well as how one constructs his or her understanding of mathematical concepts mentally. While the APOS theory may be used by instructors to design lesson plans and to evaluate how students learn as it relates to mathematical problems, in this study, it will be used for the latter.

Arnon et al. (2014) state in their research that when a student is introduced to a concept, it is first conceived as an Action. At the Action stage, an individual performs steps externally with cues and instructions that order the steps of what to do next. At this stage, the student cannot skip any step nor can he or she imagine the steps. For instance, take a student who is learning the technique of direct proof. In the example of the steps for employing the technique of direct proof below, P and Q are referred to as propositions. In mathematics, propositions are defined as statements that have exactly one truth value: either true or false (Smith, Eggen, & St. Andre, 2011). The symbol " $\Rightarrow$ " means implies. Together, $P \Rightarrow Q$ forms the conditional statement, "If P, then Q", meaning if the proposition P is true, then that implies that the proposition Q is also true. The steps outlining the process of direct proof follows:

Direct Proof of $P \Rightarrow Q$

Proof.

1. Assume P.
2. Use P with other known statements (axioms and definitions) to prove Q.
3. Therefore, Q.
4. Thus $P \Rightarrow Q$.

For instance, let x be an integer. Suppose a student is asked to prove that if x is odd, then $x+1$ is even. At the Action stage for the method of direct proof, the student will follow the steps accordingly. Here P is the proposition is "x is odd" while Q is the proposition "$x+1$ is even."

Proof.

1. Assume x is odd.
2. Let x be odd. Then by definition, $x = 2n + 1$ for some integer n. Then, $x + 1 = (2n + 1) + 1$ for some integer n. Since $(2n + 1) + 1 = 2n + 2 = 2(n + 1)$, then $x + 1$ is a product of 2 and an integer.
3. Therefore, $x + 1$ is even.
4. Thus if x is odd, then $x + 1$ is even.

At the Action stage the student will not be able to do a direct proof without following these four steps in sequential order and would rely on the cues given in each step to proceed to the next step. Actions are fundamental to the APOS theory and are vital for the conception of other structures (Arnon et al., 2014). It is important to point out that a student who does not exhibit even the Action level of understanding based on APOS theory is said to be at the pre-Action conception of understanding.

After performing Actions repetitively, the student should interiorize these Actions to form Processes in order to arrive at the Process stage. The student will gradually move from merely repeating the Actions with dependence on the external cues to interiorize them. At this stage, he or she is able to visualize the steps in his or her mind without actually doing them. At the Process stage, steps can be skipped and or reversed. A Process is essentially "a mental structure that performs the same operation as the Action being interiorized, but wholly in the mind of the individual, thus enabling her or him to imagine performing the transformation without having to execute each step explicitly" (Arnon et al., 2014, p. 21). For instance, looking at the method of direct proof above, after completing a series of direct proofs step by step, line by line, the student who progresses to the Process stage no longer needs examples nor cues of what to do next in a direct proof. He or she would have interiorized the Action into a Process and can now describe the method of direct proof without necessarily doing every step on the paper. For instance, the student at the Process stage may give the following answer: "To prove that if x is an odd number, then the expression $x + 1$ is even, we will start with the assumption of an odd number, write it in algebraic form as $2n + 1$, then using algebra we will show that when we add 1 to this number we will get the number of the form $2m$, i.e. an even number. Therefore, this would prove that if x is an odd number, then $x + 1$ is an even number." Notice that the student is not relying on cues and has developed the proof mentally, without performing every algebraic step and writing the solution on paper.

The third stage, Object, may be seen as the most difficult stage to reach of the APOS theory as it relates to encapsulation of an existing mental Process. According to dictionary.com, encapsulation is the action of enclosing something in or as if in a capsule. When a student is able to apply an Action to a Process, we say the student has encapsulated the Process into an Object. Unfortunately, “as reported in various APOS-based studies, the mechanism of encapsulation is the most difficult” (Arnon et al., 2014, p. 22). A student who moves to the Object stage has encapsulated their mental Process as a static structure to which Actions can be applied. Referring back to the example of the student learning the technique of direct proof. At the Object stage, the student is able to view direct proof as one of many possible methods to prove a statement and compare it, for example to some other methods of proof. Furthermore, the student at the Object stage may give an answer along the following lines: “If I take the direct proof of the statement ‘if $x\ is\ odd,$ then $x + 1$ is even’ described above and compare it to proof by contrapositive of the same statement, I can see that in direct proof I would start by assuming that $x\ +\ 1$ is odd and arrive at x is even. While in the case by proof by contraposition, I would start by assuming $x + 1$ is even and arrive at $x\ is\ odd$.” One should note that a student can de-encapsulate an Object back into a Process as needed. Additionally, during the de-encapsulation Process, two Objects can be de-encapsulated, their Processes coordinated into one or more Processes, and the coordinated Process(es) encapsulated to form a new Object (Arnon et al., 2014).

Schema, the final stage of the APOS theory, is the stage where the student coordinates and organizes all concepts and knowledge related to the concept being learned. The coherence of a Schema is determined by how well the student learned the concept and is able to successfully apply what is being learned to a particular situation. Arnon et al., (2014) describe Schema as structures that contain the descriptions, organizations, and exemplifications of the mental structures that an individual has constructed regarding a mathematical concept. In the example above concerning direct proof, all description, organization, and examples that the student constructed when going through the A-P-O stages while learning direct proof makes up his or her schema of direct proof.

I will now discuss the mental mechanisms that facilitate the transitions within the APOS theory. These mechanisms are interiorization, encapsulation, coordination, reversal, de-encapsulation, and thematization (Arnon et al., 2014). How these mental mechanisms intertwine in the APOS theory is described in the cyclic figure below:

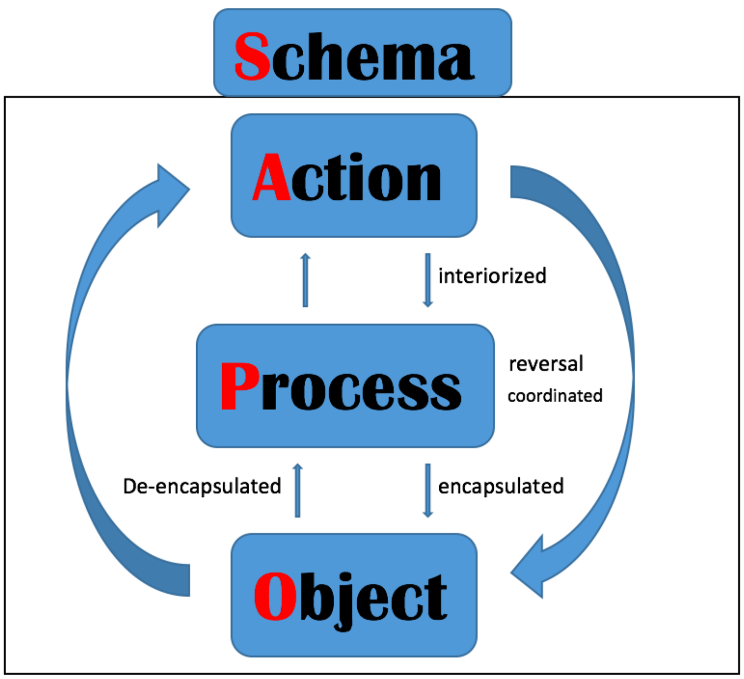

Figure 1.2 Illustration of the cycle of the APOS theory.

The cycle of learning mathematics starts when a student reflects on previous knowledge as objects. This is referred to as reflective abstraction. According to Piaget (Arnon, et al., 2014), reflective abstraction involves awareness and contemplative thought about mathematical content and operation on that content from a low cognitive stage. The content may be reconstructed and reorganized to a higher-level stage that results in operations that becomes new content to which new operations can be applied (Arnon et al., 2014). Piaget's reflective abstraction is what prompted Ed Dubinsky's interest in how individuals learn mathematics concepts and later lead to the APOS theoretical framework. APOS theory is a model that is primarily used for describing how students learn and mentally construct their understanding of mathematical concepts (Arnon et al., 2014).

Arnon et al. (2014) further state that the APOS Theory is based on the premise that an individual can learn any mathematical concept provided the structures necessary to understand those concepts have been built. In short, Actions are interiorized to form mental Processes. These Processes are then encapsulated to form cognitive Objects. Furthermore, since learning mathematics is not linear "a Process can be reversed to construct another Process, two Processes may be coordinated to form a new Process, and a Schema can be organized into a cognitive Object" (Arnon et al., 2014, p. 25).

1.4 Summary of chapter

In the beginning of this chapter, an introduction to the concept of proof and the need to study how a successful student learns the concept of proof were discussed. Furthermore, the research questions to be answered were highlighted. In addition, an elaboration on the theoretical perspective on the frameworks – SRL conceptual framework and the APOS theory, that will be used as lenses to inform this research was given. How self-regulated a student is coupled with how a student goes through the stages of APOS may shed light on the type of student who succeeds in proof courses. Moreover, with the assistance of grade prediction, this research may also assist in the development of curriculum to facilitate learning for students who are not successful in proof-based courses.

2 LITERATURE REVIEW

In this section, I will be discussing what has been done in the past to address the lack of conceptual understanding among students in mathematics programs. I will first start by discussing the difficulties students face when learning the concept of proof, then I will discuss what has been done in the classroom to address these difficulties. Thereafter, I will report on what research has been conducted relating to the SRL conceptual framework and the APOS theoretical framework.

2.1 Literature on students' difficulties with the concept of proof

A great deal of research reveals that undergraduate students face difficulties when understanding the concept of proof (Samkoff & Weber, 2015; Dreyfus, 1999; Selden & Selden, 2011; Weber & Mejia-Ramos, 2014). A major assumption behind pedagogical practices in advanced mathematics classrooms is that mathematics majors can learn the concept of proof by reading and studying the proofs that their professors present. Unfortunately, research suggests that mathematics majors may learn little from studying these proofs (Weber & Mejia-Ramos, 2014). Weber and Mejia-Ramos (2014, p. 1) further quote,

> "*If you need evidence that we have a problem, let one of your B students . . . explain the statement and proof of a theorem from a section in the book that you have skipped. My students, at least, do not have the innate ability to read and understand what they have read. When I ask them to read a problem and explain it to me, the majority just recite the same words back again.*"

In essence, instead of comprehending, mathematics majors actually get confused by the proofs they read. One of the major reasons for this is because students lack the necessary strategies for reading and comprehending proofs (Weber & Mejia-Ramos, 2014). Research also shows that there is a disconnect between what mathematics majors believe they should do when studying a proof and what mathematicians and mathematics educators expect of them. For instance, Weber and Mejia-Ramos (2014) audiotaped twenty-eight mathematics majors who completed a transition-to-proof course while validating ten mathematical arguments. The students ranged between sophomore and juniors. They were asked: Did they feel they understood the argument? Did they find the argument convincing? Did they think the argument constituted a mathematical proof? The results found by Weber and Mejia-Ramos (2014) will now be discussed.

As it relates to students and proof, the results revealed that only 16 of the 28 participants indicated that a good mathematical argument should explicitly list all the logical details and justifications within a proof. Mathematics majors do not believe that they are to justify statements within a proof. They think reading the proof is enough and believe that understanding a proof consist of them being able to verify how each statement in the proof follows after the other. Additionally, students spend a very short time browsing and reading a proof. Weber and Mejia-Ramos (2014) revealed that students usually spend under two minutes reading a proof for comprehension. Lastly, the authors noted that mathematics majors do not see it as their responsibility to draw diagrams to aid in the process of comprehending a proof. Instead, if a diagram is needed to facilitate the comprehension of a proof, they think it must be provided in the proof.

In contrast to the students' belief and expectations, mathematicians and mathematics educators expect students to fill in the necessary gaps missing in a proof and that this should not be provided in the proof. Mathematicians and mathematics educators mentioned that they present proofs to students to explain why theorems are true and to highlight different techniques for proving. Hence, explicit justification of each line is not usually presented. While students spend under two minutes reading a proof for comprehension, mathematicians and mathematics educators expect them to spend between two to fifteen minutes studying a proof. Furthermore, mathematicians and mathematics educators emphasized that students would need to draw diagrams to help them understand some of the proofs they encounter, which is contrary to what the students believe as they think if a diagram is needed, it should be presented in the proof for them.

In the study described above, we see that one of the major barriers that hinders students' proof comprehension is the disconnect between what the students think is expected of them while they learn the concept of proof and what mathematicians and mathematics educators expect of them. In recent times, students are now being asked to explain their reasoning within a proof in order to show that they know and understand the logic behind what they are proving (Dreyfus, 1999). In his paper, Dreyfus (1999) discussed why students cannot prove. He explained, similarly to Weber and Mejia-Remos (2014), that there is a misunderstanding in what is expected from students by their instructor and what the student thinks is expected of them. Research across the board reveals that most students (high school and collegiate) do not know what a proof is and the purpose of it. In fact, research shows that less than fifteen percent of students comprehend the meaning of a mathematical proof. Some of these reasons include students not being able to use the appropriate language to express what they are asked to prove, and or they may lack the understanding or the clarity of what to prove. In his study for example (Dreyfus, 1999), a student was asked:

Determine whether the following statement is true or false, and explain:

If $\{v_1, v_2, v_3, v_4\}$ *is linearly independent, then* $\{v_1, v_2, v_3\}$ *is also linearly independent.*

The student responded with: “True because taking down a vector does not help linear independence.” In the example from Dreyfus (1999, p. 88) we see that the student used “taking down” as opposed to “omitting” and “help” as opposed to “produce” in their explanation suggesting that the student may not have the linguistic ability to explain what it is he or she is thinking. Additionally, the student’s answer is vague and not completely correct which may suggest a lack of knowledge to understand this problem. The question that arises in this sense is, what do instructors look for when they say “explain,” “justify” or “prove?” Mathematicians and mathematics educators sometimes fail to take into consideration that college students do not read mathematics research papers, nor do they normally interact with mathematicians. Instead, they are only exposed to their textbook and what their instructor lectures, as well as homework problems, test questions and feedback from instructors and participation in class. As a result, this is what shapes their notion of proof. Whatever their teachers accept as a suitable answer, along with the feedback they get on graded assignments and what they see in textbooks greatly shape their proving techniques.

While one of the purposes of proofs is to provide justification, verification and or explanation, there is not a clear understanding or distinction between these words. More explicitly, there is not a clear difference in the terms explanation, proof and argument. The table below shows the difference and similarities among these terms as taken from (Dreyfus, 1999). Furthermore, nothing clearly separates the definition and use of an explanation, from an argument nor from a proof. According to dictionary.com, an explanation is a statement or account that makes something clear, while an argument is a reason or set of reasons given with the aim of persuading others that an idea is right or wrong. In the classroom however, neither an explanation nor an argument is usually accepted as a formal proof. Dreyfus (1999) goes further

Table 2.1 Similarities and differences between the definition of explanation, proof and argument (Dreyfus, 1999).

Explanation	Proof	Argument
A statement or account that makes something clear.	A formal and logical line of reasoning that begins with a set of axioms and moves through logical steps to a conclusion.	A reason or set of reasons given with the aim of persuading others that an idea is right or wrong.
Answers the question why.	Answers the question why.	Does not answer the question why.
Does not serve as a basis for understanding.	Affirms basis for understanding.	Does not serve as a basis for understanding.
Uses examples, models, visuals etc.; attempts to illustrate why a concept in mathematics is true.	May call for explanation used to highlight the central idea of a proof.	May call for examples in persuasion.
May be interwoven with proof; shares a close relationship with proof.	May be interwoven with explanation; shares a close relationship with explanation.	May not be interwoven with explanation.
Descriptive – is used to give reason.	Assesses the strength of the reason.	Assesses the strength of the reason.

to state that what is accepted as an effective proof (formal proof) is based on the culture and era in which the proof is analyzed. Additionally, how a proof is accepted may be looked at as a social act. For example, looking at the development of the Cauchy proof, that the limit of a converging sequence of continuous functions is continuous. Since the concept of limit and continuity was not fully developed, what was accepted as true during Cauchy's era, raises the eyebrows of today's mathematicians. Furthermore, there needs to be a clear universal distinction between an explanation, argument and proof. If this is not clearly stated, students will give arguments and explanations as proofs and think this is correct. Additionally, there should be a thorough evaluation of what is expected from students from both the students' perspective and the instructor's perspective. Moreover, the goal that the students set for themselves during the proving process is often inconsistent with what their course instructor expects of them (Samkoff & Weber, 2015). The two should be the same.

Similar difficulties students face when learning the concept of proof, as discussed above, have been summarized by Selden and Selden (2011) in their study. These authors included students not knowing what a proof is, the lack of ability to unpack, understand, and interpret definitions and theorems correctly, the lack of ability to check one's logic, not possessing some relevant content knowledge related to the problem in question, not having a rich concept image of relevant ideas, and not being able to "feel" for the content and what kinds of properties and theorems are important when proving a statement, as the main difficulties students face with the concept of proof. To elaborate on each difficulty, I will start with students not knowing what a proof is. Selden and Selden (2011) explained that in their research of seventeen secondary high school mathematics teachers with three to twenty years of teaching experience, the teachers did not see the importance of teaching proof for promoting understanding or giving insight. These authors went further to say that the teachers saw two-column geometry proofs as ideal formal proofs and informal proofs were explanations or empirically-based arguments. Recall that explanations are not accepted as formal proofs by mathematicians and most mathematics educators (Dreyfus, 1999). All the teachers that participated in the study, accepted informal proofs as valid proofs from students in lower-level mathematics classes. Selden and Selden (2011) went further to state that this is an unfortunate consequence where students develop the belief that checking several examples constitutes a proof. However, this is not the case when they enter an ITP course or beyond.

Next, Selden and Selden (2011) spoke on students' lack of ability to unpack, understand and interpret definitions and theorems correctly. According to these authors, students get confused about the role of definition in mathematics. The authors went further to say that students often miss important features of prospective examples. For instance, when a number of preservice elementary teachers were asked whether $F = 151 \times 157$ is prime, they noted that both 151 and 157 are prime numbers but then concluded that their product was prime. A prime number is any number that is divisible by one and itself only. Thus, the product of two factor, other than one and the number in question, cannot be prime. In this instance, the students did not apply the definition of prime correctly. In addition to being able to interpret definitions correctly, some students may not be able to make a distinction between everyday definitions and mathematical definitions as it relates to proofs. When proving a theorem, all parts of a definition should be considered, and the necessary parts applied to the proof in question. Similarly, undergraduate students often do not use relevant theorems and also fail to interpret the context of the theorem correctly.

Subsequently, these authors highlighted the difficulty of undergraduate students often ignoring relevant hypotheses or applying the converse of a statement when the statement does not hold (Selden & Selden, 2011). The authors go further to note that students often call on well-known theorems that they know and try to apply them to problems that do not necessarily require that specific theorem. For instance, the Pinching or Squeeze Theorem, sometimes referred to as the Sandwich Theorem, is a famous theorem that may be used in calculus or analysis to confirm the limit of a function. Furthermore, the Pinching Theorem may be used to prove $\lim_{j\to\infty}(2+\frac{cos j}{j}) = 2$. To do so, one would compare the function in question to a function that is smaller than $\lim_{j\to\infty}(2+\frac{cos j}{j})$, for example $\lim_{j\to\infty}(2-\frac{1}{j})$, and a function that is larger than $\lim_{j\to\infty}(2+\frac{cos j}{j})$, for example $\lim_{j\to\infty}(2+\frac{1}{j})$, whose limits are easier to calculate and or known. According to the Pinching Theorem, if a function lies between two functions on an interval and the limit of the two end functions are the same, then the limit of the function in between (forming a sandwich) must be the same as the limit of the end functions. Looking closer at proving $\lim_{j\to\infty}(2+\frac{cos j}{j}) = 2$, since $\lim_{j\to\infty}(2-\frac{1}{j}) = 2$ and $\lim_{j\to\infty}(2+\frac{1}{j}) = 2$, then by the pinching theorem, $\lim_{j\to\infty}(2+\frac{cos j}{j}) = 2$. Since the Pinching Theorem is a famous theorem, students may resort to using it to *Prove every convergent sequence is bounded*. In this instance, since the question is not asking to confirm a limit, the Pinching Theorem would not necessarily apply to this problem however, students may incorrectly use it because the Pinching theorem may have been emphasized in their lessons.

Another difficulty Selden and Selden (2011) discussed in their study is the lack of ability for students to check their own logic. The authors asked eight undergraduate students in a transition-to-proof course to explain the statement of a theorem in their own words, give examples and then to try and prove it. Initially, only two students were successful in doing this. The students were further shown four “proofs” and were asked to think aloud as they read each proof and then decide if the proof was valid or not. It was not until the fourth read, in accompany with probing from the interviewer that the students correctly answered 81 percent of the questions collectively. In the study, students revealed that when reading a proof, they attempt to read the proof line by line to follow the validity of each line and check to ensure that no mathematical computations were left out. The authors additionally mentioned that students admitted that while reading a proof, some go through the proof using examples while others only look for a sense of understanding to determine if the proof made sense or not. Consequently, the students were not able to check the logic in their proofs. If students are not able to check the logic of proofs, then they cannot be expected to produce a logically correct proof.

The next difficulty discussed by Selden and Selden (2011) is students not possessing some relevant content knowledge related to the problem in question and thus not having a rich concept image of relevant ideas (Samkoff & Weber, 2015). When constructing a proof, it is imperative that students have the knowledge required to do the proof. This knowledge is associated with their concept image. Concept image describes the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes (Tall & Vinner, 1981). This may include multiple examples, non-examples, facts, properties, relationships, diagrams and visualization that a student may associate with a concept (Selden & Selden, 2011). Typically, after students are presented with a topic accompanied by examples in an upper-level undergraduate classroom, they are expected to be able to use the definitions appropriately. However, if students are not able to develop a sound and correct concept image about the concept being taught, they will not be able to apply the techniques properly in order to produce an “acceptable” proof.

On the other hand, students may in fact possess the knowledge needed to prove a statement, but do not know what information in the definition is important, how to use the information they have and what information is useful in coming up with the proof of a statement. This is the last difficulty discussed by Shelden and Shelden (2011). For instance, they explained in their study that mathematicians often do not draw on the most useful information, in an axiom or definition, needed to prove a statement at the right time. The authors further stated that students are not able to "feel" for the content and determine what kinds of properties and theorems are important when proving a statement. According to the authors, it is not easy for students to see the relevance and usefulness of the knowledge in order to bring it to a problem or proof. The authors additionally noted that this may be due to the fact that undergraduates mainly study proofs that are complete and focus on the details as opposed to spotting the importance of the result of a theorem and how it fits in the mathematics world. Moreover, a student may know the necessary content required to prove a statement but fail to apply it correctly.

Weber (2003) shares similar sentiments as Selden and Selden (2011). In his article, Weber highlighted students' difficulties with proof as inadequate cognitive development/poor conceptual understanding, “notational difficulties”, ineffective proof strategies, and “sociomathematical norms”. The difficulty of inadequate cognitive development and poor conceptual understanding described by Weber (2003) is parallel to Selden and Selden's (2011) theory of a lack of knowledge and the correct concept image needed to prove a statement. Weber (2003) notes that students may be able to state a concept's definition and still have little to no understanding of the concept. For instance, he went further to explain that students from a study conducted in 1994 could not describe certain concepts in their own words nor generate a single example of these concepts. As a result, when these same students were asked to write proofs about a concept, they did not know how to begin.

Next, I will be discussing the difficulties students have with quantifiers as reported by Weber (2003). Weber uses "notational difficulties" to refer to students' difficulties with the use of multiple quantifiers. Quantifiers may be defined as words, expressions, or phrases that indicate the number of elements that a statement pertains to. In mathematics, there are two quantifiers: 'there exists' ($\exists$) and 'for all' ($\forall$) (Pennington, 2003-2018). The quantifier 'there exists' is usually used to refer to at least one element that exists that has certain properties. Whereas, the 'for all' quantifier usually refers to a universal group of elements that has some particular set of properties. Students have difficulties knowing when to use these quantifiers correctly. For instance, Weber (2003) asked 61 students from an ITP course to read and explain a formal proof that contained quantifiers, students were successful at this less than 10 percent of the time. This author goes further to state that research has illustrated how extracting meaning from a quantified logical statement is a very difficult and complex process for students and thus can hinder them during the proving process.

Conclusively, with regards to ineffective proof strategies, Weber (2003) states that in order to construct non-trivial proofs, undergraduates need strategies and heuristics to help them to decide how they should attack problems. In essence, students would have to read a plethora of proofs in order to become masters of which technique to use and when. In short, students lack the understanding of what forms of logical methods are permissible and struggle with which to choose, when to abandon a strategy and when to pursue a strategy. (Samkoff & Weber, 2015).

2.2 Literature on classroom curriculum intervention in proof-based courses

With all the difficulties listed above, there seems to be an urgent need for research on ways in which students can be successfully taught to improve their reading comprehension (Samkoff & Weber, 2015). Zazkis, Weber and Mejia-Ramos (2015) aimed to identify approaches that undergraduates successfully use to prove theorems in undergraduate mathematics proof courses. In their paper, Zazkis, Weber and Mejia-Ramos (2015) highlighted two proof reading strategies they observed from students as they learned the concept of proof. These two strategies were the targeted strategy and the shotgun strategy. In the targeted strategy, students develop a strong understanding of what is to be proven, determine an appropriate plan based on their understanding, decipher reasons why the plan will be successful and then execute the proof. In the shotgun strategy on the other hand, students attempt the proof without fully understanding (immediately after reading the proof) what is to be proven. They try multiple plans and move on from plan to plan if they meet a stumbling block. In other words, the student does not spend sufficient time understanding the statement of the proof before he or she begins proving (Selden & Selden, 2011). Zazkis, Weber and Mejia-Ramos (2015) noted that mathematicians' proving strategies may rely on experiences and understandings that most undergraduates may lack which leads to naïve applications of these strategies by undergraduates. Furthermore, the authors illustrated how these two particular proving approaches have the potential to help mathematics majors overcome or avoid impasses on the proving process. Unfortunately, they found preference for shotgun or targeted strategies may be task or situation dependent and not a static personal preference. So, no conclusion was drawn on how or when to use either strategy.

I will now discuss the findings of Samkoff and Weber (2015) in their study on an instructional intervention on proof comprehension. In their research, they noted that previous research on proof reading has covered measuring mathematics majors' success at determining if a proof argument is valid. Furthermore, the authors reported on past research that covered asking a student whether a proof argument was convincing and used the response of the student to give insight on the student's standards of conviction. Samkoff and Weber (2015) further discussed the proof-reading strategies discovered by Zazkis, Weber and Mejia-Ramos (2015) but highlighted that sometimes students do not benefit from these strategies because they are not implemented properly. In their study, Samkoff and Weber (2015) incorporated the model from (Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012) for characterizing and assessing students' understanding of proof in advanced mathematics. Since the expectation of the instructor differs from that of the student, Mejia-Ramos and his co-authors saw it fit to come up with an assessment instrument on proof comprehension. In most cases, students' comprehension of proof is usually judged based on their ability to regenerate a proof or to modify a proof in some way to prove an analogous theorem. In this regards, students' level of comprehension of proof is solely subjected to the instructor or grader's take on how well the student can memorize, alter and regenerate a proof.

In their model, Mejia-Ramos et al. (2012) characterized and assessed students' understanding of proof in advanced mathematics. There were seven dimensions to understanding a proof. These dimensions were split into 2 categories - local dimensions (three categories) and holistic dimensions (four categories). The local dimension focused on understanding that can be gathered from carefully reading a small number of statements in the proof. These three categories were:

1. Meaning of terms and statements: stating the definitions of terms used in the theorem statement and proof by identifying trivial implications of a given statement.
2. Justification of claims: understanding why each claim is made in the proof follows from previous ones and being able to identify claims that follow from a given statement.
3. Logical status of statements and proof framework: understanding the logical relation

between the assumptions and conclusions in a proof, identifying the proof technique being used, and conceptualizing the proof in terms of its proof framework.

Research suggests that students do not achieve this local understanding when checking for correctness. This is partly because they do not understand the theorem before reading the proof nor validity of the justification from line to line (Weber, 2015). The second set of dimensions, holistic dimensions, focused on ways to understand a proof and how to synthesize the entire proof or entire parts of the proof as a coherent whole. These four categories include:

1. Identifying the modular structure: understanding how a proof can be broken into mathematically independent parts or sub-proofs, and how these parts logically relate to one another.
2. Illustrating with examples: understanding how a sequence of inferences can be applied to verify that a general theorem is true for a specific example.
3. Summarizing via high-level ideas: understanding the overarching logical structure of the proof and being able to summarize a proof in terms of these ideas.
4. Transferring the general ideas or methods to another context: being able to use the ideas or methods in the proof to establish a different theorem.

As stated before, when mathematics majors validate proofs, they focus on line-by-line checks rather than studying the overarching methods used in the proof. This suggests that mathematics majors usually focus on developing a local understanding as opposed to a holistic understanding (Weber, 2015). These seven dimensions were analyzed and summarized into 6 strategies for students to comprehend the concept of proof. Students were presented with various proofs and instructed to use the strategies listed. The strategies investigated were

1. Understand the Theorem Statement
2. Try to prove the theorem statement before reading its proof
3. Considering the proof framework used in the proof
4. Partitioning the proof into parts or sub-proofs
5. Using examples to make sense of statements within the proof
6. Comparing the method in the proof to one's own methods

These strategies, with the exception of strategy four, appeared to be beneficial to the students as they learned the concept of proof. The downfall of these strategies lies in the fact that they may not be used to answer proof comprehension tests. To further test the proof-reading strategies above, Weber (2015) videotaped four advanced mathematics students reading 6 proofs who were:

(i) Trying to prove a theorem before reading its proof,

(ii) Identifying the proof framework being used in the proof,

(iii) Breaking the proof into parts or sub-proofs,

(iv) Illustrating difficult assertions in the proof with an example, and

(v) Comparing the method used in the proof with one's own approach.

Weber also had 83 professors comment on whether they wanted their students to use the strategies listed above. The professors in the study said they usually did not discuss with students how they should be reading proofs and that they only assessed students' understanding of a proof in a superficial manner. For instance, in Weber's study, the researchers presented students with superficially deductive proofs and asked them to determine whether or not these proofs were correct. Next, students (or teachers) were asked to evaluate different types of arguments against some criterion, such as whether they find the argument to be personally convincing or whether the argument would qualify as a proof. The goal of Weber's study was for researchers to determine if students can distinguish between valid and invalid proof arguments and to understand the processes students use to make these judgments. His study revealed that students struggle with comprehending proofs because they focus heavily on the calculations of the proof as opposed to the big picture.

After asking the 83 mathematics professors from universities across the US how they felt about each strategy, over 80 percent favored strategies two- trying to prove the theorem statement before reading its proof, four- partitioning the proof into parts or sub-proofs, and 6 - comparing the method in the proof to one's own methods. Overall, Weber (2015) concluded that the three strategies favored can be improved if students can be taught to apply them correctly. Unfortunately, the link between identifying these strategies and teaching them to students is not straightforward.

2.3 SRL and the classroom

In this section, I will discuss SRL as it has been implemented in the classroom. I will start by discussing the work of Sun, Xieb, and Andermanb (2018). In their research, the authors studied the relationships between academic achievement and three key self-regulatory constructs (prior domain knowledge, self-efficacy, and the use of learning strategies) in two Calculus flipped classrooms. A flipped classroom is one in which students learn content material online prior to going to class. The content is usually learned through online instructional videos and assigned reading in texts. This gives the instructor more time in class to engage with the students to assist them in applying what they learned prior to attending class.

Sun, Xieb, and Andermanb (2018) used the SRL theoretical framework to inform their research. The authors pointed out that self-regulated learning is seen as an integrated learning process guided by a set of motivational beliefs, behaviors, and meta-cognitive activities that are planned and adopted to support the pursuit of personal goals. The authors chose to adopt the SRL framework desigened by Winne and Hadwin in their 1998 and 2008 reports. Winne and Hadwin's self-regulated learning model consists of four stages: task definition, goal setting and planning, enactment, and adaption, with each stage occurring within a micro-cognitive system that includes fives processes: conditions, operations, products, evaluation, and standards (Sun, Xieb, & Andermanb, 2018). Sun, Xieb, and Andermanb (2018) reported that researchers identified prior domain knowledge, self-efficacy and the use of learning strategies as significant components of students' self-regulated learning.

Prior domain knowledge refers to prior knowledge that students bring to the material they are going to be taught. According to Sun, Xieb, and Andermanb (2018), there is a significant relationship between prior knowledge and self-efficacy, the use of learning strategies and academic achievement. The authors further stated that self-efficacy was positively related to the use of cognitive and meta-cognitive learning strategies. In their study, the authors hypothesized that prior domain knowledge influences self-efficacy, learning strategies and achievement directly. They aimed to assess the relationship between the three constructs: prior domain knowledge, self-efficacy, and the use of learning strategies and academic achievement in the pre-class and in-class learning environments of the flipped classroom. There were 151 undergraduate students in the study. There were two courses observed, Calculus I and Calculus II, which were designed by the same instructor, with similar activities. To consider students' prior domain knowledge, students were asked to complete a self-report of the highest-level mathematics class they completed in high school. The authors mesured three domains of self-efficacy – Mathematics Self-efficacy (MSE), Collaborative Learning Self-efficacy (CLSE), and Internet Self-efficacy (ISE).

The results of the study revealed that MSE and the use of help seeking strategies significantly impacted the students' mathematical learning achievements. The results further showed that students who were more self-efficacious in learning mathematics were more likely to achieve at higher levels in both pre-class and in-class environments. This is in support of previous research that revealed that students' strong belief in their ability to learn mathematics can assist them in generating appropriate goals for completing online lectures and engaging in group-based mathematics learning activities, guide their behaviors, and eventually lead them to higher achievemnet on assignments (Sun, Xieb, & Andermanb, 2018). Furthermore, in their results, they saw that students who passed higher upper level mathematics courses in high school, reported higher self-efficacy for learning mathematics. For students who do not possess high self-efficacy, Sun, Xieb, and Andermanb (2018) suggest that teachers should pay keen attention to these students and identify their learning needs based on pre-assessments. Additionally, instructors can design learning tasks in the group-based format for students to observe their peers. Lastly, instructors can provide positive feedback on students' progress in solving in-class mathematics problems. In short, this study illuminated the importance of students' self-regulated learning process in classes.

In line with Sun et al., in his paper, Schunk (1985) reported on the effects of social comparison, such as working in a group setting on students' performance in the classroom. Schunk focused how self-efficacy relates to classroom success. Furthermore, Schunk touched on how self-efficacy can affect motivation. Specifically, in his paper, he revealed that repeated success contributes to a raise of self-efficacy. Schunk (1985) described the components he believed were important to motivate students as they learned. These included student characteristics, expectancies, task engagement variables and efficacy cues. As it relates to students' characteristics, Schunk highlighted that the way students approach learning a task varies based on the student's abilities, interests, personal characteristics and prior experience. Specifically, the latter characteristics can influence students' self-efficacy for learning new materials. Moreover, he noted that outcome expectancies and self-efficacy are usually related as students who views themselves with the ability to perform well, expect positive reactions from their teachers after a successful performance which promotes self-efficacy. In relation to task engagement variables, Schunk (1985) noted that self-efficacy is usually endorsed when a student develops a new skill and in contrast, students who encounter difficulties in processing new information eventually doubt their capabilties and thus leads to low self efficacy. In the last component, efficacy cues, Schunk (1985) discussed how success raises self-efficacy, while failure lowers it. He continued in his report noting that the difficulty of a task has a moderate impact on the outcome of a student's self-efficacy.

In addition, he noted that efforts aimed at skill improvement begin with the asssistance of instructor's corrective feedback or assistance from other students. These feedback include comments such as, "That is correct," or "You have improved." However, students who master tasks with little to no help generally develop a higher self-efficacy than those who get assistance. Essentially, motivated learning is categorized by an interactive relationship between self-efficacy and how students learn. To conclude his report, Schunk (1985) discussed how goal setting and rewards affect a student's self-efficacy. For instance, goals that are attainable and in closer proximity of completion, have a greater contribution to self-efficacy than goals that are nontangible and not proximal. This is because the ability to measure success is easier to gauge and achieve. In relation to rewards, these should be tangible for students and not be vague such as a participation grade.

Next, in their paper, Xiao, Yao, and Wang (2019) discussed the relationship between SRL and college students. The authors noted that SRL is one of the most investigated topics among educators and psychologists. Xiao, Yao, and Wang (2019) also noted that not a lot of research was conducted on how SRL affects university students. The authors made it a point to report on some of the studies that have been done on SRL and students in the past. For instance, Karabenick and Knapp, (1991) reported that students who have high self-efficacy are more likely to use cognitive and metacognitive strategies while learning. Lindner et al. (1992) investigated how significant self-regulation was among university students. The report of Lindner et al. (1992) revealed that SRL was important for academic success and that there was a correlation between SRL and student’s grade point average (Xiao, Yao, & Wang, 2019). There may also be a correlation between one’s gender and success due to self-regulation. For example, Virtanen & Nevgie (2010) found that since male and female students have personality traits, incentives, instruction preferences, and enjoy different incentives these factors may contribute to females having higher SRL strategies in the forethought and planning phase (Xiao, Yao, & Wang, 2019). Essentially, Xiao et al. found that self-regulated learning plays an important role for university students as they learn. This is so because at the university level, students are able to set their own study schedule. Additionally, highly self-regulated students are proven to be more successful than student who do not self-regulate.

For an investigation of how self-regultaion impact students in the Eastern countries such as China, I will now discuss the work done by Li, Ye, Tang, and Zhou, (2018). Most reserch focused on students in Western conturies thus Li et al. decided to bridge that gap and conduct a student on students in an Eastern country, China. In their study, Li, Ye, Tang, and Zhou, (2018) analyzed 59 empirical studies in the China National Knowledge Infrastructure, the Wanfang Database, and the Vip Paper Check System. The criteria for inclusion in the study were in-person teaching in the classroom, students were enrolled in elementary, junior high, or senior high students in China, and a report of the correlation coefficients, sample sizes, means, and standard deviations given. When coding, the papers were assessed for the following:

1. The basic information (title of the study, publication year and journal),
2. The type of SRL strategies,
3. The type of academic achievement,
4. The educational stage (elementary, junior high school and senior high school),
5. Proportion of female,
6. The phase of SRL.

Li, and Ye coded the articles independently. In the study, the authors concluded that for Chinese students in elementary and high school, the SRL strategies that contributed to learning the most were self-efficacy, task strategies (breaking up task into parts, etc.) and self-reflections. They found that SRL worked better for students in relation to science disciplines such as mathematics and that the performance phase and the reflection and performance phase had a larger effect on academic achievement. They also reported that junior high school level may be critical for SRL development in students and that during the time period of 1998 to 2016, the effect size of SRL was decreasing. Suggestions on how to incorporate SRL in the classroom included encouragement from teachers for students to evaluate their performance and to reflect on their learning actions. It is essential that instructors receive training on SRL theory in order to be able to help students maximize their full learning potential (Li, Ye, Tang, & Zhou, 2018). In essence, the study revealed that while self-efficacy, grade goal setting, and effort regulation are particularly important for learners in Western countries, for Chinese students, self-efficacy, task strategies, and self-evaluation were the most important SRL strategies.

2.4 SRL and model prediction

In this section, I will discuss what has been done in reference to grade predictions and the SRL model. I will start by reporting on the findings of the study reported by Harding, et al. (2019). The purpose of their work was to answer the research questions 1) Do SRL behaviors predict academic performance? And 2) Are there differences in the use of SRL behaviors across grade levels 5, 6, 7, and 8? To answer these question, 4232 students from public schools in Victoria, Australia completed an SRL questionnaire online that measured SRL behavior in terms of level of quality for each student. The questionnaire was coded based on a hierarchial scale determined by responses described by Krathwohl's taxonomy, Bloom's taxonomy, the Dreyfus model and the work done by Ryan and Deci (2000). For an investigation of the relationship between SRL and content ability across the different grades, an assessment system was provided by the Assessment Research Center at the University of Melbourne. Mathematics and English were the content of the assessments. Linear regression with a significance level of five percent was used to analyze the relationship between SRL and content ability. This was done seperately for each subject.

The results showed that grade (particularly age) did have a significant effect on SRL use. That is, students in higher grade levels had lower use of SRL practices. The lowest of each grade level studied was 8th grade. The authors also found that grade had a significant impact on both content ability for mathematics and English. After using simple regression to determine whether

SRL intentions and motivation predicted academic performance for mathematics or reading comprehension, it was determined that effect size was small for both subjects. However SRL was reported to have a higher predictive power on mathematics than English. In essence the authors found that SRL did impact academic performance in all grade levels tested. Additionally, in relation to the second research question, the authors found that SRL did vary among grade levels. Their study also revealed that self-efficacy was strongly linked to performance.

Next, I will report on the work done by Los and Scheweinle (2019). In their paper, the authors used theories of motivation and environment to predict academic success of students. Particularly, the paper focused on the way in which instrumental environment interacts with SRL factors, such as motivation and self-efficacy, to predict academic success. Particularly, how these factors predict academic outcome. Additionally, the authors investigated how the instrumental environment predicts the way and to what extent the factors relate to academic success. There were 315 college student participants and 32 instructor participants. The students answered questions pertaining to the current course they were taking at the time. The student instruments evaluated academic outcome (measured at the beginning of the course), self-efficacy, personal achievement goal orientation, learning approach, student's perceived responsibility for learning and self-regulation. While instructors' instrument evaluated instructor self-efficacy, achievement goal orientation for teaching, approaches to instruction and instructors' perceived responsibility for learning. Emails with the survey were sent to instructors asking them to fill out the survey, as well as ask their students to complete the student survey.

A hierarchical linear model with students nested within classes was used. There was an 8.795 variance in course grade that may be attributed to the difference in instructors. Multiple models were compared using the deviances. The authors found that self-efficacy, goal orientation, self-efficacy for SRL, learning approach, perceived responsibility for learning, and self-regulation, predicted student's self-reported academic outcome. Among all, self-efficacy was found to be the greatest predictor. Additionally, the authors found that instructor mastery goal orientation was the greatest instructional environment predictor of students' academic outcome. Moreover, instructors who used a performance approach goal orientation increased students' self-efficacy for SRL and thus academic outcome. The authors reported that performance feedback from instructors is necessary to increase students' self-efficacy.

2.5 APOS theory and proof difficulties

In this section I will discuss research done on analyzing how students comprehend proof using the APOS theoretical framework. In particular, I will be discussing work done by Syamsuri, Purwanto, Subanji, and Irawati (2017). In their paper, the authors discussed the thinking process of students who are unable to effectively construct a mathematical proof. The authors reported a quadrant model, to describe students' classification of proof production. The quadrant model categorizes proof construction based on thinking processes and how developed a student's concept image and concept definition is as it relates to formal proofs. Recall, concept image refers to all the mental pictures and associated properties and processes a student has related to a concept (Tall & Vinner, 1981). Concept definition, on the other hand, is the students' definition of a concept based on his/her concept image (Syamsuri, Purwanto, Subanji, & Irawati, 2016).

Students who can effectively construct a proof, that is, students with a rich concept image and who have a good connection between concept image and concept definition are categorized in Quadrant I. Furthermore, students in Quadrant I can correctly construct a formal proof (a proof accepted by the mathematical community). In the next quadrant, Quadrant II, are students who have a rich concept image but a poor connection between concept image and concept definition. As a result, these students may have errors in their proof that cause them to leave out important information in the proof. In Quadrant III are students who have just close to the required concept image and thus a poor connection between concept image and concept definition. Students in this category may write a proof with extraneous information that may not be related to the proof. Similar, to the students in Quadrant III, the students in Quadrant IV have minimal concept image and a poor connection between concept image and concept definition. However, the students in Quadrant IV are said to have incorrect proof structure in formal proof construction. That is, these students may not know the correct approach to start a proof. These students may write a complete proof, but the method is not accurate and statements in the proof are not correct (Syamsuri, Purwanto, Subanji, & Irawati, 2016).

To further differentiate between the quadrants, I have adopted a proof validation excerpt from Syamsuri, Purwanto, Subanji, and Irawati (2016) and placed a possible response a student may give who fits in each quadrant in Figure 2.1 below. Students were asked to validate, with reasoning, the following proof:

For any positive integer n, if n^2 is a multiple of 3, then n is a multiple of 3.

Proof.

Because n is a multiple of 3, it can be written $n = 3m$, m an integer. Consequently, $n^2 = (3m)^2 = 9m^2 = 3 * (3m^2) = 3k$, k an integer, $k = 3m^2$. Thus $n^2 = 3k$, which implies n^2 is a multiple of 3 and further n is a multiple of 3.

In Quadrant I, the student has a good connection between the concept images and concept definitions required for this proof. In Quadrant II, the student does not completely understand the concepts and did not recognize that what is to be proven was assumed. As a result, this led to wrong calculations.

Quadrant II This proof is true. Since $6^2 = 3k$ and 36 is divisible by 3. Since we want to prove that n is a multiple of 3, all steps of the proof are correct. This is direct proof.	Quadrant I This proof is false. The converse of the proof was proven. That is, if *n is a multiple of* 3 then n^2 *is a multiple of* 3 instead of the desired if n^2 *is a multiple of* 3*, then n is a multiple of* 3.
Quadrant III This proof is true. Since n^2 is a multiple of 3, then n is a multiple of 3. For example, take $m = 4$, then the statement is correct. The logic of the proof is clear using direct proof.	Quadrant IV This proof is true. Since n^2can be expressed as $n^2 = 3k$ and for every number that is a multiple of 3 will generate numbers that are also multiples of 3. Further, the value $(3m)^2$ may be simplified as done in the proof.

Figure 2.1 Illustration of a student's response from each quadrant in Quadrant Model.

The response in Quadrant III shows a student who does not comprehend fully the concepts needed for the proof. Thus arriving at errors in computational procedures and extraneous statements such as, *"For example, take $m = 4$, then the statement is correct."* For the last quadrant, Quadrant IV, starting with the wrong approach, and making false claims, this student has errors in his / her logical deduction.

In their article, Syamsuri, Purwanto, Subanji, and Irawati (2017) analyzed students in Quadrant III. That is, students who were not able to construct a formal proof, had insufficient concept and incorrect proof structure in formal proof construction. The research was conducted on 26 students majoring in mathematics education in a public university in Indonesia. The authors collected their data using a think-aloud protocol for students as they attempted to solve the proving task given them. The students were later interviewed after their responses were analyzed. Each student was asked to:

Prove: For any positive integers m&n, if m^2 and n^2 are divisible by 3, then

$m + n$ *is divisible by 3.*

The authors reported that they choose this particular problem because multiple proof methods such as direct proof, contradiction, proof by contrapositive etc. could be used to prove this task. Two students' results were discussed in depth. Both students arrived at the "Action stage" but because they had little conceptual understanding of the concepts required for this proof, neither of them were able to perform interiorization of "Action" into "Process" (Syamsuri, Purwanto, Subanji, & Irawati, 2017). In fact, the Actions performed by each student to solve the proof task, were not Actions as defined in the APOS theoretical construct. For example, the students merely picked multiple pairs of numbers that were not divisible by three and showed that the square of their sum was also not divisible by three. Student one was convinced that her proof was solid and accurate. The authors reported that the students could not construct a formal proof of the statement because they could not pass the Action stage to encapsulate their Actions into a Process. Furthermore, the authors suggested that students should be assisted in refining the encapsulation Process. They also suggested that instructors assist students in refining their proof structure and conceptual understanding.

Next, I will report on the work done by Arnawra et al., (2007). In their study the authors investigated the effects of the APOS theory teaching method on students in an Elementary Algebra course. The study reported on prior results that revealed that students had difficulies with concept in Elementary Algebra. These concepts included coset, and quotient group, to name a few. They also noted difficulties that students had relating to proof involving the latter concepts. For instance, the authors noted that students often confused normality with commutativity and that while they could perform calculations involving these concepts, it was difficult for them to think as cosets as objects they can manipulate.

Furthermore, in their study, Arnawra et al., (2007) administered a quasi-experimental nonrandomized pretest-posttest control group design. There were two groups, the control group and the experimental group. Both groups received the pre-test. Then the experimental group was taught using the APOS theory instruction. Particularly, the ACE teaching cycle. The control group, however, were taught using the traditional teacher approach. After which, both groups were given the post test. In total, there were 180 students from the department of mathematics at a university in Indonesia. The instructors who taught using the APOS theory instruction taught in a way that:

1. Topics were designed to revolved around the APOS active learning
2. The instructor acted as a facilitator who supported and guided the students as they learned with hints.
3. There existed a multi-direction interaction between students and instructor and students learning from their peers.

Whereas the instructors who taught using the traditional approach, taught in such a way that:

1. The topics were designed from the textbook, or lecture notes,
2. The instructor directly explained all mathematical ideas.
3. All interactions included the instructor.

The results revealed that the students who received instruction using the APOS theory instruction performed better that the students who did not.

2.6 Summary of chapter

In this chapter, I discussed several difficulties students face when trying to comprehend mathematics and proof. As a result of these difficulties, I have highlighted what has been done previously to address these difficulties. I also discussed the importance of SRL in the classroom.

Lastly, I examined research conducted on students' understanding of proof using the APOS theoretical framework. Along with the difficulties mentioned in this chapter, I plan to highlight how a successful student learns the content of proof in the hopes of adding to this body of literature and to possibly help students overcome the difficulties discussed.

3 METHODOLOGY

Along with the SRL conceptual framework and the APOS theoretical framework, a single-case study design was used to answer the explanatory question of how an undergraduate mathematics major student learns the concept of proof. A case study was appropriate as I observed the student during an extended period of time in his Introduction to Proof course and Analysis course. I will also report on mini conversations we had as he developed his proof conception (Nock, Michel, & Valerie, 2007).

Yin (2003) states that for case studies, there are five components that are of extreme importance. These components are: 1. a study's questions; 2. its propositions, if any; 3. its unit(s) of analysis; 4. the logic linking the data to the propositions; and 5. the criteria for interpreting the findings. As it relates to the study questions, one must decide the types of questions the research is seeking to answer. To this end, I sought to answer explanatory questions such as "what," and "how." For example, I sought to answer the question of what I can gather from how a competent student learns and comprehends proof concepts. As for the research proposition, the researcher must examine what the scope of the study is. In this study, I examined the work ethics and understanding of a student enrolled in two proof courses. The third component, unit of analysis, refers to what or who will be investigated. For this study, the unit of analysis was one successful mathematics major student. Furthermore, Yin (2003) mentions linking the data to the propositions and the criteria for interpreting the findings. I will explore the latter in the data collection methods section.

Yin (2003) goes on further to discuss five reasons why a single-case study design may be appropriate. These reasons are:

1. When there is a critical case in testing a well-formulated theory.

2. When the case represents an extreme case of a unique case.
3. When a single-case is the representative or typical case.
4. When the investigator has the opportunity to observe and analyze a phenomenon previously inaccessible to scientific investigation
5. When one is studying the same single-case at two or more different points in time.

In this study, I used the single-case study design for numbers three and five on the list. The case of how the subject of this study, a successful student in mathematics, learns and understand proof may be a single representative of how a typical successful student learns and understands proof. Additionally, this research aimed to use one student, eighteen years or older, enrolled in ITP course and a higher-level proof course for which the ITP is the foundation.

3.1 Data collection methods

Reverting back to linking the data to the propositions and the criteria for interpreting the findings, one needs to decide what data is necessary in order to support the propositions, and to reflect on the criteria for interpreting the findings (Rowley, 2002). Recall that the propositions of this study were the work ethic and the development of the student's understanding as he learned the concept of proof. With this in mind, I conducted participant observations. Kawulich (2005) describes participant observation as "the process enabling researchers to learn about the activities of the people under study in the natural setting through observing and participating in those activities." For this study, the competent student was observed during his learning in his natural setting for learning – the classroom. I strictly observed and did not participate. It is important to note that I engaged the student in casual conversations to probe the student's thoughts and or concerns as he learned different concepts relating to proof. Additionally, I gathered copies of the student's notes, quizzes, tests, homework etc. that may help to explain my observations.

Though I made observations of what was taking place in the classroom while the student was learning, I recorded what was being observed. This data is referred to as field notes. Once I returned from each class session, I wrote down a description of the conversations I had with the participating student, the activities that took place in class, as well as any ideas, strategies, reflections, hunches or patterns I may have noticed during my observations (Bogdan & Biklen, 2007). Bogdan and Biklen (2007) state that field notes provide the context for development of sampling guidelines and interview guides which leads us to our third type of data, an audio recorded semi-structured interview. At the end of each of the two semesters, the student was interviewed to expound on my classroom observations throughout the semester. I chose a semi-structured interview so as to help guide the interview while giving the student the freedom to answer the questions in his own way. Though I could have taken notes during the interview, it would have been difficult to jot down notes while asking questions resulting in poor note taking and distraction during the interview. Thus, the interviews were audio recorded and later transcribed to allow for a smooth interview (Cohen & Crabtree, 2016). I would also like to point out here that while each of these data sources may stand on its own, to get a robust description of the data, I used all three. Furthermore, more recently, researchers use interviews or participant observation as supplementary data to see how documents get interpreted by real people instead of by an imaginary audience (Bogdan & Biklen, 2007).

The last thing I will address is the criteria for interpreting the findings. As mentioned in depth in the theoretical perspective section earlier, I used the APOS theory and SRL conceptual framework as the criteria to determine the level of knowledge that was developed by the student throughout the study. Additionally, being that there is not a lot of research on how a successful student learns proof, from this study, using the lenses of the SRL conceptual framework and the APOS theoretical framework, I have gained insight on the methods and strategies that were effective for a competent mathematics major student when he learned the concept of proof. Combining the misconceptions and struggles students have when learning the concept of proof from previous studies, with the insight I gained about how a successful student learns the concept of proof, using the APOS theoretical framework and the SRL conceptual framework, I suggested teaching implications that can possibly assist students who struggle when learning and comprehending the concept of proof.

3.2 Participant

This research took place on the campus of a local university in the metro Atlanta area. There are multiple instructors who taught proof-based courses at the university of interest; however, I selected a student from instructors who I believed were more comfortable with an observer in the classroom. Furthermore, this site was chosen due to ease of access to the students in question. The instructor of the first course (ITP course) was asked to rank the students enrolled in the course based on prior grades in undergraduate mathematics courses. For this study, I did purposeful sampling. I chose the particular student I believed was able to facilitate the expansion of developing a curriculum that is insightful for students learning proof (Bogdan & Biklen, 2007). The student with the strongest background reflected from the most grades of A in prior math courses was listed as number one on the list. The student with the second highest number of grade A in prior math courses was listed second on the list and so on. The first student who was asked to participate in the study, agreed. It is significant to point out that the first student on the list was not in country to start the study. As a result, the second student on the list was asked to be a part of the study. Only one student was recruited. This student was also the subject for the Analysis I course. It was stressed to the student that this was a strictly voluntary consent and that he could have removed himself from the study at any point in time if he felt uncomfortable. The student was informed that there were no risks involved in the study and that his confidentiality is of extreme importance and as a result, I did everything in my power to ensure it was guaranteed. The student was also informed of the interview at the end of the semester.

The student who volunteered to be a part of our study was given the pseudonym name "Henry". As recently mentioned, Henry was the second student on the list. He received A's for almost all his math courses (except one) and fit the criteria we were looking for as a "successful student". He started out as an Economics major and then switched to mathematics because of the "challenge" it brings. He was not always good at mathematics and thought excelling in mathematics was an inherent thing until his high school mathematics teacher showed him differently. Henry expressed that it was after he started practicing and developed good logic, that he became good at mathematics.

3.3 Quality

To ensure that I have a well-rounded set of data, a diverse data set was used (participant observation, field notes, and an audio recorded semi-formal interview). I triangulated the data using these sources. Triangulation is the use of multiple data sources in a study to produce understanding (Cohen and Crabtree, 2006). Cohen and Crabtree (2006) further states in their article that rather than seeing triangulation as a method for validation or verification, qualitative researchers generally use this technique to ensure that an account is rich, robust, comprehensive and well-developed. I used triangulation in the latter sense to ensure quality in this study.

3.4 Summary of chapter

This chapter covered the methodology that was used in this research. I discussed the choice of this methodology and why it was best suited for this study, as well as the data collection methods, participant, and how I ensured quality in this study. The report that was done on the student in this study is not necessarily a reflection of all successful students in mathematics and how they learn the concept of proof.

4 ANALYSIS AND RESULTS

Previously, I mentioned what data was collected and how it was used. In this section I will discuss my analysis of the data I collected. Recall that I designed the study based on two frameworks, the self-regulated learning conceptual framework and the APOS theoretical framework. These two frameworks were used in the analysis and data interpretation. To start my analysis, I first transcribed the mini conversations I had with Henry before and/or after class. I also transcribed the recorded interviews that were conducted at the end of each semester. Afterwards, I coded the transcribed conversations. Following coding, I analyzed Henry's tests, quizzes and homework to confirm or deny what was revealed in our conversations. I will explain in more details each of these steps in the sections below.

4.1 SRL conceptual framework

In this section I will analyze and interpret the data that will help in answering the research questions:

> What is the work ethic and study habit of a competent mathematics major student as he or she learns the concept of proof?

Henry was observed for two semesters - summer in Bridge to Higher Mathematics (the ITP course) and Fall in Analysis.[1] For the two semesters, his self-regulation and outcome in the classes were compared. The first set of data that will be examined is the transcribed conversations with Henry before and/or after class, paired with the transcribed interviews. After transcription, the conversations were coded. There were two stages of coding. In the first coding stage, I highlighted parts of the transcribed data a specific color to represent self-efficacy,

[1] The word Analysis beginning with an upper-case A will refer to the course Analysis taken by the student in Fall 2018 as a distinction from the word analysis as in data analysis.

motivation, forethought and planning, performance control or, reflection and performance of the SRL framework. Table 4.1 below shows an example of a color-coded portion of the transcribed conversations. This illustration was chosen because it shows four out of the five components, being analyzed, highlighted together. All SRL phrases in the transcribed data were highlighted pink. However, the text colors varied for each component. For instance, for self-efficacy, the text was highlighted purple, for self-efficacy the text remained black, for

Table 4.1 Illustration of an example of a coded portion of the transcribed discussions about the student's Analysis course for the SRL framework.

Took me a while…I messed it up on the test…but like going over it made me understand it better…so yeah…that's what I hit that first…and as far as the right ones…I don't have to focus on them a lot…	SRL – Reflection and Performance
Henry: yeah I did that for math 3000?…no…I don't think I had…I don't think I had nearly the amount of time to do that…just 'cause everything was blasted in my face back to back…	SRL – Reflection and Performance
Henry: I'm studying tonight…	SRL – Forethought and Planning / Two Monday before the test
I was pretty confident but there are some topics I feel I definitely should have looked at more…and…ah but I guess it's the first one so I really can't say much…	SRL - Confidence
I have a better understanding of how things should go…are now moving forward and	SRL – Reflection and Performance
Henry: yeah I looked at the homework problems and I noticed that a lot of the test that we were doing…the problems on the test were homework problems…	SRL – Performance Control SRL - Reflection and Performance
so but I did everything else…	SRL – Performance Control

forethought and planning, the text was highlighted yellow, for performance control, the text was highlighted blue and for reflection and performance, the text was highlighted red. In this phase of coding, when coding for self-efficacy, I looked for parts of the conversations where the student discussed his confidence in his ability to do well in the courses, as well as his enjoyment in each class (McMillian, 2010; Zimmerman & Schunk, 2001; Zumbrunn, Tadlock, & Roberts, 2011;

Sun, Xieb, & Andermanb, 2018). For motivation, I looked for instances where the student took initiative to do things on his own or expressed intrinsic values relating to the course (Ganah, 2012; McMillian, 2010). Since the forethought and planning phase is where the student sets goals and tasks that will help him achieve his goals (Zumbrunn, Tadlock, & Roberts, 2011; Li, Ye, Tang, & Zhou, 2018), when coding for this phase, I looked for parts of the conversations where the student set tasks for himself to complete. Recall that in the performance control phase, the student employs strategies to complete his tasks (Zumbrunn, Tadlock, & Roberts, 2011; Li, Ye, Tang, & Zhou, 2018). As a result, when coding for performance control, I looked for instances where the student did what he planned to do. Lastly, in the reflection and performance phase, I looked for parts of the conversation where the student reflected on his performance on a quiz, test or homework problem and evaluated his performance (Zumbrunn, Tadlock, & Roberts, 2011; Li, Ye, Tang, & Zhou, 2018).

To further explain how I coded the transcribed conversations, as an example I will give a brief description of the coded data in Table 4.1. In the first row, coded as reflection and performance, Henry was explaining how he performed on a previous test and what he typically does when he gets back a graded assignment from his professor. He explained that he first looks at what he “messed up on” to help him understand the concepts better. This part of the data was coded as reflection and performance because he was reviewing how he performed on the exam, making note of the topics he could improve on. Similarly, in the second row, Henry was reflecting on his performance in Bridge to Higher Math course the previous semester, while in the fifth and sixth rows, he was making note of how he performed on the first test in Analysis and how to move forward when studying for test two. In the third row, I coded “I’m studying tonight” as forethought and planning since he is setting a plan to study for the second test. This was two weeks prior the test date. In the fourth row, Henry was expressing his confidence about the test he had about test one. I classified this statement as self-efficacy, as self-confidence is a form of self-efficacy (Schunk, 1985; Schunk & Zimmerman, 2008; Schunk & Zimmerman, 1994; Li, Ye, Tang, & Zhou, 2018; Los & Schweinle, 2019). Lastly, in rows 6 and 7, when Henry expressed that he completed a task in preparation for a test or exam, I recorded those phrases as performance control.

From this snippet of the data, I observed that Henry was doing fairly well at regulating himself as the Analysis course began. Furthermore, he reflected on his performance in Bridge to Higher Math to better prepare for the Analysis course. He additionally reflected on how he performed on his first test in Analysis and used that to adjusted his preparation for test two. From the table, one can observe that he completed tasks he set for himself in preparation for studying for test two. I will further elaborate on what the outcome of Henry's self-regulation was for each course later in this chapter.

After the first phase of coding, in the second phase, I looked through the first coded document to see if there was a pattern for each component. For example, I found that in instance where Henry discussed his self-efficacy, he used words such as "*Doing well*," "*Did pretty well*," and "*Pretty confident*." When screening the transcribed conversations for motivation, I found phrases similar to "*I feel like working hard*," "*Wanted job as a grader*," and "*have to teach myself*." When analyzing the data for the forethought and planning phase, phrases such as "*I will do it over the weekend*", "*I'm going to do that tomorrow*," and "*Gonna do that next week*," etc. surfaced. Moreover, when looking at the data for performance control, I found statements along the lines of "*I went to*" "*I went over*," and "*I looked at.*" Lastly, when looking through the conversations related to the reflection and performance phase, phrases such as "*I didn't work* *enough on*," "*I know where I messed up*" and "*just doing the homework…wasn't enough*." In the document where I condensed the statements Henry made in reference to each component of the SRL conceptual framework, I grouped the statements according to the phrases above. In the sections below, I will explain further my findings for each component of the SRL conceptual framework.

4.1.1 Self-regulation

Recall, as mentioned in chapter one, there are three main components of the SRL conceptual framework (Pintrich & De Groot, 1990; McMillian, 2010; Greene & Azevedo, 2007; Zimmerman, 2008). These are:

1. Cognitive strategies that students use to learn, remember, and understand the material.
2. Metacognitive strategies for planning, monitoring and modifying their cognition.
3. Student management and control of their effort on classroom academic tasks.

The cognitive strategies include what students use to learn, remember, and understand course materials; the metacognitive strategies refer to how students plan and organize their time, monitor their learning and regulate their cognition as they learn, and the third component breaks down the students' motivation to learn. These components may be categorized into three phases of the SRL conceptual framework - the forethought and planning phase, the performance control phase and the reflection and performance phase (Zumbrunn, Tadlock, & Roberts, 2011; Xiao, Yao, & Wang, 2019; Li, Ye, Tang, & Zhou, 2018). As stated before, students decipher the tasks they are given and set personal goals to complete the task in the forethought and planning phase. In the second phase, performance control, students determine strategies to advance their learning task and assess how effective these strategies are. While in the final phase, reflection and performance, students evaluate their performance on learning tasks (Zimmerman & Schunk, 2001; Zumbrunn, Tadlock, & Roberts, 2011; Xiao, Yao, & Wang, 2019, Li, Ye, Tang, & Zhou, 2018). In conjunction with the three phases of the SRL conceptual framework, self-efficacy and motivation plays a vital role in self-regulated learning (McMillian, 2010; Zimmerman & Schunk, 2001; Zumbrunn, Tadlock, & Roberts, 2011; Sun, Xieb, & Andermanb, 2018). Self-efficacy refers to the student's belief in their capabilities to learn and skills to perform a given task (Zimmerman & Schunk, 2001; Zumbrunn, Tadlock, & Roberts, 2011). Whereas motivation refers to the intrinsic values the student places on learning (Ganah, 2012; McMillian, 2010).

To analyze Henry's self-regulation in relation to the three phases of the SRL conceptual framework, I looked through the coded conversations and interviews for instances where Henry addressed the three phases of the SRL conceptual framework – forethought and planning, performance control, and reflection and performance (Zumbrunn, Tadlock, & Roberts, 2011; Zimmerman & Schunk, 2001). After which, I looked through the coded conversations and interviews for instances where he addressed his motivation and self-efficacy for the courses (McMillian, 2010; Zimmerman & Schunk, 2001; Zumbrunn, Tadlock, & Roberts, 2011; Sun, Xieb, & Andermanb, 2018). Using the transcribed data from the conversations and interviews I had with Henry, I compared and contrasted how he exhibited each phase of the SRL framework while enrolled in the Bridge to Higher Math course and the Analysis course.

Overall, between both the Bridge to Higher Math course and the Analysis course, Henry and I had eighteen mini conversations and two end of semester interviews. For the Bridge to Higher Math course, we had ten mini conversations, while we had eight mini conversations in the Analysis course. Since there was not a consistent number of conversations for each class, I used conversations that lasted for more than one minute. That narrowed down the number of conversations to seventeen, ten of which were from Bridge to Higher Math and seven from the

Analysis course. It is important to point out that the number of conversations between classes for each test were not consistent for the courses. For this reason, I considered the average number of statements Henry made in relation to the three SRL phases. That is, for each test, I counted the number of statements Henry made relating to forethought and planning, performance control, and reflection and performance, and took the average for each phase per the number of conversations counted for that test.

4.1.1.1 Self-regulation in Bridge to Higher Math

To delve into my analysis, I will start by discussing the statements Henry made that related to the forethought and planning phase for test one in the Bridge to Higher Math course. In the Bridge to Higher Math course, there were a total of three conversations counted before test one. In these conversations, I counted the number of times Henry mentioned anything associated with his plans to prepare for test one. Since these included conversations at the beginning of the course, I also included statements Henry made that focused on how he planned to approach the course as a whole.

For example, Henry shared that in his previous mathematics course, Discrete Mathematics, the instructor wanted him to write proofs in a particular way. For the Bridge to Higher Math course however, Henry expressed that he was happy that the instructor did not mind how students constructed the proofs. He further stated that the Bridge to Higher Math instructor was *"not specific about proof. She doesn't want me to specify the obvious.*" This was coded as forethought and planning because Henry considered what was required for him to succeed in the course and planned to follow what the instructor required of him (Zumbrunn, Tadlock, & Roberts, 2011; Li, Ye, Tang, & Zhou, 2018). When asked if he tried any of the homework problems early in the semester, prior to the first test, Henry's response of "*I will do it now and* *over the weekend*," was categorized as forethought and planning since he was planning to do this in preparation for an upcoming quiz (Zumbrunn, Tadlock, & Roberts, 2011; Li, Ye, Tang, & Zhou, 2018).

On average, Henry had a total of two phrases per conversation in which he mentioned what he planned to do for the course or for test one. Furthermore, as it relates to his preparation for test one, he mentioned that he did what he sought out to do on average once per conversation. For the first test, Henry made a grade of 75 percent and after getting back his test, he reflected on how he performed on test one, on average once in the conversations after the test. Henry expressed his disappointment in his test score and so moving forward to study for test two, he made plans of studying and doing homework problems ten times. Specifically, he stated:

> "*I didn't take the effort to go out and…you know understand that, which is why I guess on the test, that's where I lost all of my points,*"

when reflecting on why he did not get a higher grade.

As the course progressed, looking at test two, Henry's forethought and planning increased on average by 50 percent. Precisely, he mentioned plans to prepare for test two on an average three times per conversation (we had three conversations before test two). Furthermore, Henry set new goals to accomplish for test two. For instance, Henry stated,

> *"I'd go to her. I can't visit her during office hours because her office hour…I* have *another class. I'll have to set an appointment with her…ahm go over all the problems that I find ahhh…I made a mistake on. Find out what I essentially, you know, was not clear in my mind and yeah, I'm gonna work on those moving forward. Yeah."*

He did not mention going to the instructor's office in preparation for test one. Here we see Henry's goals shifting as the semester progressed. He saw that studying on his own was not enough and made plans to visit the instructor to help him better prepare for the upcoming exam. Henry also realized that he needed to study more for test two, this is evident when he said,

> *"I'd have to go over more of those cause that's something I feel I need to…you know…really drill into my head apart from that…I think I need to go over more induction. Induction is the tricky part…It's more something I need to put more focus on."*

In these instance, Henry realized that he needed to focus more on the new topics introduced in order to do well on the upcoming test. He expressed that he was too confident in his preparation for test one and here we see that he adjusted his study style.

On average, Henry's performance control for the second exam increased 100 percent for test two. Henry reported on following through with his plan to look into topics he thought he needed to focus more on with statements like "*I've started going over the material.*" In fact, he mentioned that he started reviewing the material roughly a week before the date of the exam. For test one, he did not do much studying. However, we see in preparation for test two, he went to the instructor's office and started going over the material days in advanced before the test. Henry's grade for test two increased by 17 points from 75 to 92. In this instance, we note that an increase in self-regulation, resulted in an increase in Henry's test grade (Duckworth & Carlson, 2013; Pintrich & De Groot, 1990).

As the class progressed, Henry continued to self-regulate himself. However, his level of self-regulation for test three was roughly the same as his level of self-regulation for test two. To the contrary, for test three, Henry's forethought and planning went down. On average, Henry mentioned plans to prepare for test three roughly once per conversation (there were three conversation). His study habit seemed to be similar to that of test two as he made statements such as, *"Ah I think I'm gonna focus…ahm…I'm gonna practice section three of course,"* during our conversations pertaining to his preparation for test three. Furthermore, he mentioned that he followed through with his study plans on average, two times per conversation. For instance, he stated he, *"went over the homework and the concepts"* and that he *"prepped pretty well…went over all the problems from the homework."* He studied and did homework problems on average twice as much as he planned to. For exam three however, his grade dropped 24 points from 92 percent to 68 percent.

Next, in preparation for test four, the final exam, Henry reflected on his performance on test three on average two times per conversation. Being disappointed with his test three score, Henry put planning in place to prepare for the final exam. On average, he mentioned plans such as, *"so I can go in today…I can go in tomorrow… and…not on Wednesday because I think she probably have a final,"* four times per conversation (there was one conversation before the final). His performance control was on average, roughly the same as tests two and three. That is, he mentioned executing his plans to study on average two times per conversation. Though his self-regulation seemed to be the same as that of test three, Henry achieved a higher test score on the final than he did on test three. Specifically, his final exam grade went up by seventeen points. He made an eighty five percent on the final exam. In short, it was when Henry did most self-regulating that he scored the highest on a test (Duckworth & Carlson, 2013; Li, Ye, Tang, & Zhou, 2018; Los & Schweinle, 2019). Table 4.2 gives a summary of the average number of phrases Henry stated per test reflecting each of the three phrases of the SRL conceptual framework.

Based on the data we note that there is a correlation in the level of self-regulation and academic success (Duckworth & Carlson, 2013; Li, Ye, Tang, & Zhou, 2018; Los & Schweinle, 2019; Sahranavard, Miri, & Salehiniya, 2018; Xiao, Yao, & Wang, 2019; Lindner & Harris, 1992).

Table 4.2 Summary of average number of statements Henry made per test contributing to each of the three phrases of the SRL conceptual framework for the Bridge to Higher Math course.

Bridge to Higher Math		
Tests	**SRL Phase**	**Averages Number of Phrases Per Test**
Test 1 Grade: 75%	Forethought and Planning	2
	Performance Control	1
	Reflection and Performance	1
Test 2 Grade: 92%	Forethought and Planning	3
	Performance Control	2
	Reflection and Performance	2
Test 3 Grade: 68%	Forethought and Planning	1
	Performance Control	2
	Reflection and Performance	2
Final Exam Grade: 85%	Forethought and Planning	1
	Performance Control	2
	Reflection and Performance	2
End of Semester Interview	Forethought and Planning	3
	Performance Control	2
	Reflection and Performance	5

Henry’s overall grade in the Bridge to Higher Math class was a B. Based on the level of self-regulation, this may be expected since he did not have a consistently high level of self-regulation throughout the course (Duckworth & Carlson, 2013).

During the end of semester interview for the Bridge to Higher Math course, Henry made comments on his performance in the course and what he could do to improve his grade. To that end, in the interview, there were five instances where he reflected on his final exam and or the class as a whole. In his reflections, Henry mentioned the concepts he thought students needed to know in order to be successful in the Bridge to Higher Math course. He also commented on topics that were challenging for him, and how he approached learning new concepts in the class. When I asked Henry to list the topics, he considered essential for students to know in order to do well in the course, he listed concepts such as arithmetic, the commutative and associative property of addition and multiplication, set theory and sequences. In relation to topics that were challenging, Henry listed topics such as converse, the well ordering principle, modular arithmetic, combinations, even and odd parities, induction, partition and equivalence classes.

Moreover, I asked Henry about his study habits in the course. As a part of his performance control, Henry responded,

> *“So I would go over my notes…I think say about two times a week because I’d also you know…you know it’s also about going over that thing inside your head outside of class…going over the homework…doing the homework…not necessarily all of them but at least…ahm…so a couple of problems you know you can definitely do them in your head… you don’t need to write the proof down for it.”*

I further, asked Henry for advice he would give to students who may be looking to enroll in Bridge to Higher Math. When asked how he would advise students who were preparing to take the Bridge to Higher Math course how to study new concepts he mentioned that,

> "*First of all, it is extremely important to get the framework of the proof and the idea and the theory in your mind 'cause…you're always gonna get confused.*"

In essence, Henry recommended that upcoming students focus on getting the theory of the proofs presented to them and, study their notes at least four to five hours outside of class. Specifically, Henry stated,

> *"So, if you're taking 15 credits…you need to have 15 times 3 which is 45 credits…45 hours outside of class. Students are smart enough to assess where they stand…or in some cases…they are kind of deluded about how good they are at something."*

So overall, Henry suggested that students study three times the number of credit hours the course is worth. Looking ahead at the Analysis course for the upcoming fall, Henry talked about plans for the course. He stated that students should source out materials required for upcoming courses and try to get ahead of the class. When asked if he planned to do that himself, Henry replied, "*Oh I need to follow through with that…I don't have a choice*." This was recorded as forethought and planning for the upcoming Analysis course.

4.1.1.2 Self-regulation in Analysis

Now I will look at Henry's self-regulation in the Analysis course. Making a grade of B in the Bridge to Higher Math course, Henry was determined earn an A in the Analysis course. As a result, he talked about his plans for studying and doing homework problems on average 6 times (three conversations were counted) before the first test in the Analysis course. When asked how he was doing in the class, Henry mentioned that he was reading the textbook. This is a change from how he started in the Bridge to Higher Math course. Henry stated,

"*I'll go through the text book…I'll probably go through it again…after now…because I really…yeah I need to go over it."*

Here one can see that Henry shifted his study habit from merely planning to do homework problems in the Bridge to Higher Math course, to planning to read the textbook multiple times in the Analysis course. In another instance, Henry mentioned that he was planning to become a grader in order to learn the material. Specifically, he noted,

"What I am trying to do for that is…what I've read a lot from… stock exchange and stuff is become a grader for first quarter of analysis…I think I'm gonna be more proactive this time…I've already started…I just have to keep going."

Unlike the Bridge to Higher Math course, he planned to be proactive in the sense of reading ahead of the class and doing practice problems in the textbook.

While Henry was planning to be proactive in the class by reading ahead of class, doing all homework problems and to become a grader, on average Henry mentioned completing these tasks once per conversation. The option to be a grader was not a possibility for Henry because the department lacked funding. To that end, the goal of becoming a grader was unattainable. Prior to learning that he could not be a grader, Henry mentioned that he planned to do all the homework problems. When asked if he was still planning to do all the homework despite of the fact that he can no longer be a grader he noted,

"Yeah yeah…I am still doing the homework…I've done everything up until this point."

This indicates that he was still regulating his learning even though he could not be a grader. In reference to reading and staying ahead of the class, Henry told me he was two months ahead of the class stating, "*I'm actually two…a month or two away…ahead of my whole class*." This was completely different from what he did in the Bridge to Higher Math course. Recall in the Bridge to Higher Math course, he never mentioned studying ahead of the class. Specifically, he studied when an exam or quiz was upcoming. Furthermore, Henry made plans to visit the instructor's office hours and to seek out help from his peers. For instance, he noted that, he was "*gonna just pop into his office every now and then*," before test one. He did not mention visiting the instructor's office hours in Bridge to Higher Math until test two.

In the first conversations before test one, Henry reflected quite a bit on his performance in the Bridge to Higher Math course. Specifically, he reflected on average four times per conversation. When asked if he thought he could make a good grade in the Analysis course, he confidently said yes, and noted, *"I know…where I messed up in Bridge."* In another instance, he mentioned that,

> *"[Bridge to Higher Math] really like showed me…ah…well [Dr. G's] class wasn't harder itself but it was more like well if you're gonna go out and do more work…you're gonna get rewarded for it…you perform better on the exams…ahm…it kind of gave me an example to actually teach myself. Her exams were like if you do the homework or do more problems… you're gonna get rewarded for it…so yeah that's one of the reasons why… you'll see it again from doing it…maybe not explicitly but at least….you know actually, it's gonna help you."*

Here, one can see that Henry was using his experience in the Bridge to Higher Math course to make plans for studying in the Analysis course. With the adjusted self-regulation Henry made an 80 percent on test one. This is five points higher than his test one grade in the Bridge to Higher

Math course. This contributes to the positive correlation between self-regulation and exam grade (Duckworth & Carlson, 2013; Los & Schweinle, 2019; Alotaibi, Tohmaz, & Jabak, 2017).

Similarly, his preparation for test two in Bridge to Higher Math, Henry's level of self-regulation increased. When Henry got back his results for test one, he reflected on his performance on average two times per conversation. He mentioned that there were,

> "*some topics I feel I definitely should have looked at more…and…ah but I guess it's the first one so I really can't say much*."

He also noted that with test one, he had an understanding of what was expected of him moving forward.

Looking closely at the other two phases of the SRL framework, Henry's forethought and planning decreased from an average of 6 statements per conversations in preparation for test one to an average of three statements per conversations in preparation for test two. Moreover, Henry's level of performance control increased from an average of once per conversation, to an average of three times mentioned per conversation. That was the exact average for his forethought and planning. This implies that Henry, on average, mentioned that he completed the same number of tasks he set out to do in preparation for test two.

For instance, when conversing about his homework problems, Henry stated that he was "*gonna do them this week because I know he's gonna collect them next week…I spoke to him about that."* It is important to point out that there is a possibility that Henry no longer completed his homework problems since he no longer had the opportunity to become a grader. As seen above, he was behind on the homework problems as opposed to being ahead when he had the intent to become a grader. An example of a plan that he made and executed during our conversations for test two was to visit the instructor's office hours. He mentioned multiple times that he wanted to talk about concepts he was having trouble understanding. For example, he stated, that he was "*probably gonna email him so that's gonna be one thing…or maybe I'll go to…I'm not free Wednesday…maybe I'll go to him on Monday next week…and ask him about it."* He later reported that he did visit with the instructor during office hours. On multiple occasions, Henry expressed that he did homework problems in preparation for the upcoming test. He also noted that, he "*did the homework…but just like doing them I noticed wasn't enough…probably doing them more times*." Recall that he never spoke on doing the homework problems multiple times in the Bridge to Higher Math course. Again, we can see that Henry adjusted his study habits as the class progressed. Compared to his test one grade of 80 percent in the Analysis course, Henry's grade on test two went up to 87 percent. This may be attributed to his adjustment to studying and his level of self-regulation (Duckworth & Carlson, 2013; Xiao, Yao, & Wang, 2019).

Reflecting on test two, in preparation for test three, Henry mentioned that he "*prepared… quite a while…at least a week back*," for the test by doing homework problems. He also mentioned that he "*practiced…quite a lot…for like two hours or something,"* in preparation for the test. This seems like an increase in the performance phase led to an increase in his test score (Li, Ye, Tang, & Zhou, 2018). As it relates to test three, in the Analysis course, test three was a take home exam. During the preparation time for this exam, Henry had a lot of other tasks to complete outside of the Analysis course. For example, he told me that he had a project to do for his research assignment and that his GRE test was close to the due date of the take home test. More specifically, Henry responded, "*I don't have time for it…I'm taking the GRE on Monday,*" when I asked about his progress on the take-home exam. It is important to point out that Henry had three weeks to complete the take home exam. It appears as though the GRE and his research obligations heavily affected his level of self-regulation and preparation for the take home exam. For instance, he mentioned plans to do the exam on average, once per conversation (we only had one conversation). Compared to his previous test preparations, this was lower than usual for Henry. Furthermore, we were only able to have one conversation before the test was due because of Henry's busy schedule and the Thanksgiving break. With what appeared to be lack of preparation and plans to "study" for the take-home exam, Henry made a 63 on the exam. This may be attributed to his lack of self-regulation for this test (Duckworth & Carlson, 2013; Los & Schweinle, 2019; Alotaibi, Tohmaz, & Jabak, 2017).

As in the Bridge to Higher Math course, test four in Analysis was the final exam. Henry and I were unable to have any conversations in regard to his preparation for the final exam. As result, I cannot report on how self-regulated he was for the final exam. After the Thanksgiving break, the semester ended quickly, and consequently the final exam quickly approached. Henry's grade for the final exam was substantially low. Specifically, he scored 52 percent on his final exam. It is worth pointing out that 52 was the lowest grade Henry made on a test in either of the two classes I observed. Looking at the trend from the previous tests where Henry's test score went down, one may attribute this low test grade to a low level of self-regulation (Duckworth & Carlson, 2013; Li, Ye, Tang, & Zhou, 2018; Los & Schweinle, 2019; Sahranavard, Miri, & Salehiniya, 2018; Xiao, Yao, & Wang, 2019; Lindner & Harris, 1992). A summary of the average number of statements Henry made per test contributing to each of the three phrases of the SRL conceptual framework for the Analysis course is shown in Table 4.3.

During the end of semester interview for the Analysis course, Henry reflected on his overall performance in the course, as well as what he could have done to improve his grade. He also reflected on the concepts he thought students needed to know in order to be successful, topics that were challenging for him, and how he approached learning new concepts in the class. To that end, the concepts that he mentioned students needed to know in order to be successful in the Analysis course were: sequences, series, tests for divergence and *"the end of calc two to be more specific."* When asked about his work ethics, study habits and his approach to new

Table 4.3 Summary of average number of statements Henry made per test contributing to each of the three phrases of the SRL conceptual framework for the Analysis course.

Analysis		
Tests	**SRL Phase**	**Averages Number of Phrases Per Test**
Test 1 **Grade: 80%**	Forethought and Planning	6
	Performance Control	1
	Reflection and Performance	4
Test 2 **Grade: 87%**	Forethought and Planning	3
	Performance Control	3
	Reflection and Performance	2
Test 3 **Grade: 63%**	Forethought and Planning	1
	Performance Control	3
	Reflection and Performance	2
Final Exam **Grade: 52%**	Forethought and Planning	0
	Performance Control	0
	Reflection and Performance	0
End of Semester Interview	Forethought and Planning	1
	Performance Control	5
	Reflection and Performance	1

concepts, Henry mentioned that he rehearsed "*stuff*" in his head and suggested that students "*study every day for two hours…three hours for this class."* He also noted that he took notes on what was important. More specifically, he noted,

> "*Yeah I don't usually take a lot of…I take notes on what I think is important…because then otherwise I'm able to focus on what the professor is teaching.*"

Suggestions for students preparing to take the Analysis course included for them to,

> "*Study every single day for two hours…three hours…as much as [they] can…it might feel like…[they] already know this though…no [they] don't know it…if [they] think [they]* know i*t…[they] don't know it…make sure [they] understand the main ideas…like really understand them…and not just think [they] understand them…ahh…what else…the proofs…do all the proof…do all the proofs…like if there's fifty proofs…make sure [they]* do all *fifty proofs…at least two to three times over… [it] doesn't matter if [their] hand hurts…just do them…and ahm…make the sheet like I said…make a sheet of what you think you need to focus on…what's important…what's not…where you're comfortable with…and ah…and then also…if even though you've already practiced somethings two to three times over…before…two days before the test…a day before the test or something just do it two more times over…because I actually made the mistake of not practicing…doing that."*

We can see here that the advice Henry suggested for students preparing to take the Analysis course, was a little different from what he suggested for students who are preparing to take the Bridge to Higher Math course. In the interview at the end of the Analysis course, Henry mentioned, similarly as for Bridge to Higher Math, that it is important for students to get the main idea of the proof. On the contrary, he emphasized that students should practice the proofs multiply times, regardless of the number of proofs, noting a difference in preparation was needed. Needless to say, he earned a C in the Analysis course and was very disappointed about this grade. Recall that prior to the Bridge to Higher Math course, Henry made all A's in his mathematics courses. Based on our conversations, it appears as though Henry's level of self-regulation for both classes was roughly the same but the outcome for the classes were different. When Henry realized he did not make an A in the Bridge to Higher Math course, he was very disappointed. Even more so, when he came to the realization that he made a C in the Analysis course, he was devastated and ashamed. One contributing factor that may have led to such a low grade in the Analysis course may be due to his level of understanding and expectation in the course. I will attempt to explain how his level of understanding in the course may have affected his outcome in this class later section 4.1.2.2.1

Comparing Henry's performance in both courses, when Henry was most self-regulated, he made his highest grade in each class (Duckworth & Carlson, 2013; Li, Ye, Tang, & Zhou, 2018; Los & Schweinle, 2019; Sahranavard, Miri, & Salehiniya, 2018; Xiao, Yao, & Wang, 2019; Lindner & Harris, 1992). To the contrary, when he did the least self-regulating, he made his lowest grade in the courses. Thus, further confirming that there is a correlation between his performance on tests and his level of self-regulation as seen in Duckworth and Carlson (2013), Li et al., (2018), Los and Schweinle, (2019), Sahranavard et al., (2018), Lindner and Harris (1992), and Xiao et at., (2019). Henry's forethought and planning phase in the two classes were considerably different. In the Bridge to Higher Math course, he mentioned putting plans in place to prepare to pass the class roughly twice per conversation. While in contrast in the Analysis course, he mentioned he made on average 6 plans (total) for preparing to do well in the Analysis course. The high level of planning in the Analysis course was due to the fact that he reflected on his overall performance in the Bridge to Higher Math course the semester before and, determined that he needed to study more in the Analysis course in order to earn an A out of the class. He mentioned executing his plans on average once per conversation for both tests during the semester (excluding the final interview). Furthermore, after getting back test one in Bridge to Higher Math, he reflected on his performance three times during our conversation. This was the same number of times he reflected on test one in the Analysis course.

As a means to triangulate what Henry said he did as he studied in each course, he was given an SRL questionnaire as used in Pintrich & De Groot (1990). During one of our conversations throughout the semester, Henry was asked to fill out this questionnaire. In filling out this questionnaire, Henry rated himself as a self-regulated student. The questionnaire had a series of questions reflecting self-regulated learning. Each question had a rating from one to seven where one represented not at all true for me and seven represented very true of me. Based on the questionnaire, Henry's responses gave him an average score of four point three out of seven for the Bridge to Higher Math course. In comparison, his responses gave him an average score of four point seven out of seven for Analysis. Though very close, Henry scored himself less self-regulated in the Bridge to Higher Math course than in the Analysis course. This result is confirmed by what was analyzed from the transcribed conversations and interview I had with Henry. Based on the transcribed data, we saw that Henry had a fair sense of his own self-regulation for these courses as reported in the questionnaire. Overall, based on Henry's grades from both courses, his level of self-regulation needed to be improved in each class. This adds to the body of work done in previous research that revealed a correlation between SRL and academic success (Duckworth & Carlson, 2013; Li, Ye, Tang, & Zhou, 2018; Los & Schweinle, 2019; Sahranavard, Miri, & Salehiniya, 2018; Xiao, Yao, & Wang, 2019; Lindner & Harris, 1992). In the next sections, I will look at Henry's motivation and self-efficacy in each class.

4.1.2 Motivation

Recall, motivation plays a vital role in the self-regulated process. Self-motivation is affected by a student's interest in the subject, perception on the importance of the subject, the student's desire to achieve, how the student feels about himself/herself, his/her persistence and, his/her level of patience. In this section, I will report about how Henry's motivation in the courses impacted his self-regulation and as a result, the outcome of his grade in each course. As previously mentioned, when coding for motivation, I looked through the transcribed conversations and interviews for instances where Henry took initiative to do things on his own or expressed intrinsic values relating to the course (Ganah, 2012; McMillian, 2010; Pintrich & De Groot, 1990).

4.1.2.1 Motivation in Bridge to Higher Math

Starting with the Bridge to Higher Math course, Henry was intrinsically motivated to get a high grade in the course. He was a straight A student (except for one class) prior to the Bridge to Higher Math course and he was determined to continue on his straight A streak with the Bridge to Higher Math course. At the beginning of the class, Henry identified what the instructor required of him on exams and quizzes. In one of our conversations, he pointed out that the instructor was *"not… specific about proof,"* and that *"she [did not] want [him] to specify the obvious."* His motivation to do well in the course pushed him to identify what the instructor looked for during exams in order for him to do well in the course. Henry later mentioned that he did not do well on his first exam in the course before Bridge to Higher Math (Discrete Math) because the instructor *"wanted [him] to write answers in a particular way."* As a result, owing to his intrinsic motivation, he made it a point to understand what the Bridge to Higher Math instructor expected of him (Alotaibi, Tohmaz, & Jabak, 2017; Zumbrunn, Tadlock, & Roberts, 2011).

As the course progressed, Henry adjusted his study habit in order to do well in the course. As reported previously in section 4.1.1, when Henry made a low grade on the first exam, he reviewed his test, and reached out to the instructor for assistance. He additionally changed his study habits. For the first test, Henry neglected to do all homework problems - he skipped around the assignments. As a result, he made a 75 percent on the first test. However, because he was motivated to get a higher grade on test two, he did all the homework problems assigned when he studied for test two. This resulted in a higher grade on exam two. Opting to do all the homework problems confirms that intrinsic motivation had a positive effect on Henry's self-regulation and thus exam grade (Alotaibi, Tohmaz, & Jabak, 2017; Pintrich & De Groot, 1990; Zumbrunn, Tadlock, & Roberts, 2011).

To the contrary, his grade on test three was lower than that of test two. It is important to point out that Henry stated that he had to study for another class during the time he prepared for exam three. This may or may not have affected his performance on test three. After seeing his grade on test three, Henry was again motivated to improve his grade for the final exam, and so he visited the instructor's office hours and asked for help (Pintrich & De Groot, 1990). As a result, he made an 85 percent on the final exam. It is worth pointing out that Henry revealed to me that he wanted to take the Graduate Record Examinations (GRE) and that he knew the material from the Bridge to Higher Math course would be on the GRE exam. This too was an intrinsic motivation for him to learn the content from the course thus leading to him working hard to get a good grade in the class. The intrinsic motivation of doing well in the course and learning the material in the course were contributing factors to Henry's level of SRL and thus lead to good grade in the course (Alotaibi, Tohmaz, & Jabak, 2017; Pintrich & De Groot, 1990; Zumbrunn, Tadlock, & Roberts, 2011).

4.1.2.2 Motivation in Analysis

As the Analysis course began, Henry's level of motivation was eminent. He was determined to do better in the Analysis course than he did in the Bridge to Higher Math course (recall as a prior A student, he made a B in the Bridge to Higher Math course). During our first conversation at the beginning of the Analysis course, Henry stated that he wanted to become a grader for the current Analysis course. He went further to say he read online that becoming a grader would help him learn and retain the material. He specifically said he wanted to become a grader,

> *"because then [he] would have all those proofs in [his] head to write down and [he wouldn't] have to worry so much about that."*

Here we see where Henry placed enough value of the course to want to become a grader in order to learn the material 'better.' Becoming a grader was another form of intrinsic motivation to do well in the course (Pintrich & De Groot, 1990; Alotaibi, Tohmaz, & Jabak, 2017; Zumbrunn, Tadlock, & Roberts, 2011). As mentioned earlier in the previous section, due to a lack of funding, the option of becoming a grader for the Analysis course was not a possibility for Henry.

In addition to wanting to earn a good grade in the class, Henry mentioned his plans to take the GRE subject test again. He went further to say, *"[Analysis] is…about a fourth of the portion"* of the test. Moreover, he stated that, *"it's important to know this stuff,"* when referring to the content on the GRE subject test. This confirmed that Henry had an additional motive to learn the course content. Midway through the semester however, Henry noted that he was no longer going to pursue mathematics in graduate school. Instead, he was planning to pursue a computer science degree. More specifically, he told me *"[he's] not taking… [the GRE subject test] …there's just too much."* The opportunity of becoming a grader along with the choice of not taking the GRE subject exam appeared to have affected Henry's motivation to learn the material and thus affected his ability to do well in the course (Alotaibi, Tohmaz, & Jabak, 2017; Pintrich & De Groot, 1990; Zumbrunn, Tadlock, & Roberts, 2011).

After reviewing Henry's motivation in the course based on our conversations and interviews, I analyzed the results from the SRL questionnaire. Henry scored himself an average of 6.1 for intrinsic value for the Bridge to Higher Math course. This confirms that he was motivated to do well in the course. This 6.1 out of 7 may be considered as a high level of motivation which may explain why he did fairly well in the course (Alotaibi, Tohmaz, & Jabak, 2017; Pintrich & De Groot, 1990; Zumbrunn, Tadlock, & Roberts, 2011). As it relates to the Analysis course, Henry scored himself a 5.8 out of 7 for his intrinsic value for the course. This was lower than that of the Bridge to Higher Math course and confirms a correlation between the motivation and grade earned in courses (Ganah, 2012; Alotaibi, Tohmaz, & Jabak, 2017; Pintrich & De Groot, 1990; Zumbrunn, Tadlock, & Roberts, 2011).

4.1.2.2.1 Effects of teaching styles and learning styles on motivation

Next, I will attempt to explain how Henry's grade in the Analysis course may have been affected by his level of understanding. As students enroll in a class, they unconsciously seek varying type of understanding for the course material. For instance, some students may seek to learn just the method or mechanics of how to do a problem. More specifically, some students are only interested in learning the rules to a problem without knowing the reasoning behind the rules.

This type of understanding is in the literature referred to as instrumental understanding (Skemp, 1978). The other type of understanding that students may seek, is relational understanding. As in the definition of the word relation, for this type of understanding, students are interested in learning how to relate to the problem in such a way that they know the method required to solve a problem and, why that method works (Skemp, 1978).

For instance, in his paper, Skemp (1978) gave an example of two sets of students learning in a music class. Suppose there were students being taught the elements of music. One set of students are taught the five-line stave, the musical notes by drawing them on paper and how to manipulate music, also on paper. For these students, this is all they are introduced to when learning music once per day, five times per week and told the importance of music. Skemp (1978) explained in his paper that the depth of knowledge for these students would be shallow and as a result, the students would find the music class to be boring and would be forced to memorize what they are taught. With learning like this, students tend to give up on the content being taught. In contrast, take a set of students who are taught to associate sounds to the musical notes. First, they make the sounds themselves thus enabling them to imagine the sounds whenever they see or write it on paper. Without having to rely on memorization and more on how they relate to the material, these students would be able to differentiate between melodies and harmonies in music. According to Skemp (1978), these students would find their learning to be pleasurable and would continue to pursue learning it.

Comparing the above example from Skemp and Henry's learning in the Bridge to Higher Math course and the Analysis course, Henry was the type of student who sought to learn relationally. For example, in the Bridge to Higher Math Course, Henry presented himself as a student who was seeking relational understanding. Throughout the semester, he focused on learning the theory and understanding how and why each proof method worked. This is also evident with the tips that he gave students who were planning to take the Bridge to Higher Math course, that were mentioned in section 4.1.1. He suggested that students should learn *"the theory"* behind the proof. The theory Henry was referring to is specifically how to solve the problem and why the method works for that particular problem. The instructor of the Bridge to Higher Math course taught the course in such a way that facilitated the learning objectives Henry set for himself. That is, Henry studied for relational understanding in the course and the instructor's pedagogical approaches encouraged that type of learning. This encouraged and motivated him to continue to persist at learning the material. That is considered as a match between Henry's learning and his instructor's teaching style (Skemp, 1978).

To the contrary, the Analysis instructor taught in a different style. While Henry still aimed to learn in such a way that he could relate to the problems, he quickly realized that his study method had to change in order for him to get the grade he aimed for out of the course (a grade of A). For example, Henry expressed his disappointment in the challenge, or lack thereof, that he was getting in the class. Recall that he expressed in one of the mini converstaions that he felt like the class was *"so easy, it's humiliating."* When asked to expound on what he meant, Henry stated,

> *"most of the problems follow a similar problem…they follow somewhat with the logic that [the instructor] already [gave]…I really don't know how I am learning anything new in [this class]…How am I seeing anything new? How is this gonna help me?"*

With this realization, Henry did not *"feel happy with the class,"* and as a result, he had to seek outside resource for additional practice problems that were more challenging to him. Here we see a mismatch in the teaching style of the instructor and the understanding that Henry sought

(Skemp, 1978). The professor taught the course instrumentally, while Henry was seeking relational understanding. Skemp (1978) stated that this mismatch between student's learning style and instructor's teaching style was 'dangerous.'

The mismatch in teaching and learning styles influenced Henry's motivation in a negative way as the course progressed. While Henry was seeking outside resources to gain relational understanding, this was more challenging than he anticipated and as a result, he resorted to doing the homework problems over and over to learn the mechanics of the problems. This was evident when he mentioned that students should write the proofs multiple times. More specifically, he stated that students should

> *"do all the proofs…like if there's fifty proofs…make sure you do all fifty proofs…at least two to three times over."*

This was not in line with the advice he gave for students who were preparing to take the Bridge to Higher Math course. Needless to say, Henry was not able to master the Analysis course in this way and as a result, he made a C in the course.

In this section, I reported on how Henry's motivation to learn in the courses may have affected the outcomes in his grades for the classes. Based on his scores on the SRL questionnaire, Henry seemed to be more motivated to learn in the Bridge to Higher Math class. Moreover, the match between his desire for relational understanding and the supported teaching style of the instructor may have also contributed to his motivation to learn in the Bridge to Higher Math course. In the Analysis course on the other hand, since there was a mismatch between the way Henry was seeking to learn compared to how the instructor taught the course, Henry's motivation, and thus grade in the course, was affected. I also reported on other factors that affected Henry's motivation to learn in each course. These factors included his intention to earn a mathematics degree, as well as his intention to switch from earning a graduate mathematics degree (he decided this during the Analysis course), his plan to take the GRE subject test, and his plan to become a grader. In the next section, I will discuss Henry's self-efficacy and how it affected his outcome in each class.

4.1.3 Self-efficacy

In addition to motivation, self-efficacy plays an important role in the self-regulated learning process. As stated before, self-efficacy refers to a student's belief in his or her abilities to complete a task (Zimmerman, 1989; Zimmerman & Schunk, 2001; Zumbrunn, Tadlock, & Roberts, 2011; Los & Schweinle, 2019; Li, Ye, Tang, & Zhou, 2018, Schunk, 1985). In this section, I will report on Henry's self-efficacy in each class. I will be using the same text analyzed in the sections 4.1.1 and 4.1.2, but from a different view point of the SRL framework. While coding the conversations and interviews with Henry, I looked for instances where Henry mentioned or commented on his ability to complete assignments in each course. For example, I looked for instances where he commented on his confidence to perform well on assignments, as well as when he expressed his interest in the course or lack thereof. I will first report on the Bridge to Higher Math course and then follow with the Analysis course.

4.1.3.1 Self-efficacy in Bridge to Higher Math

As the Bridge to Higher Math course began, Henry was very confident in his abilities to do well in the course. For instance, during the first conversation, he expressed that he enjoyed Discrete Mathematics (the prerequisite course for Bridge to Higher Math) and that he did well in it. He went further to explain that he was originally an economics major, pursuing a mathematics minor but switch to be a mathematics major because he developed an intuition for the subject. More specifically, he stated,

> *"I was not good in math until like high school. But then I had a teacher who was truly supportive of me. He really helped me do well…get better in high school…through junior year…through senior year. He really helped me a lot…was very supportive and until then I thought being bad in math was inherently a thing but then he showed me it's basically just practicing and developing mathematical intuition and since then I've really enjoyed doing math. Ahm…I was a math minor as an econ major…ahm…but then I realized I didn't want to pursue economics…I want to pursue…or finance…and I wanna pursue math…I was doing really well in my math classes up until then…you know…I took a lot of classes in mathematics but I still scored an A+ in all of them…and I thought let's try this out…maybe it will open up new avenues…I enjoy doing this…let's do this…this seems much more challenging now…and I enjoy that challenge."*

In this excerpt, one can see that because Henry had an instructor who *supported* him through his difficulties with mathematics throughout his junior and senior years in high school, he grew a liking to it. It appears as though this is where his self-efficacy in relation to the subject began. Instructors have a strong influence on students' self-efficacy (Los & Schweinle, 2019). Furthermore, the latter response revealed that because of his previous prior knowledge and existing cognitive structure, Henry had a high level of self-efficacy prior to entering the Bridge to Higher Math course (Nurjanah & Dahlan, 2018; Sun, Xieb, & Andermanb, 2018).

Now I will look closely at how Henry's confidence transferred to his performance in the Bridge to Higher Math course. In our conversations throughout the semester, Henry expressed numerous times that he found the material of the Bridge to Higher Math course interesting. For instance, he made statements such as, "*Proofs are really interesting…that's the best part"* and *"interesting…it's not really a challenge…it's interesting."*

Expressing that he thought the concepts in the course were interesting, is a positive contributing factor to his level of self-efficacy in the course (Nuutila, et al., 2020). Moreover, Henry was so confident in his abilities that, he underestimated how much he needed to study for the first exam administered in the course. In fact, he expressed that he was too confident about his abilities to do well on test one. Specifically, after getting back exam one, Henry was asked about his preparation for the test. He explained stating,

"*I got cocky about it. I was pretty overconfident, and I think that came back to bite me.*"

It appears he was too confident in his abilities which led to him not preparing adequately for the exam (Seifert & Sutton, 2009). As a result, he made a 75 percent. He was not content and believed he could have made a better grade if he prepared better.

As the class progressed, Henry was asked how he felt about the material in the course. He responded saying, "*I feel confident-ish*." In this instance, we see that Henry's level self-efficacy decreased from what it was when he started the course. He expressed that he needed to study more. In fact, he went further to say,

"*I'd have to go over more of those cause that's something I feel I need to…you know…really drill into my head*."

From this excerpt we can see that because Henry was not familiar with the material, he was not feeling confident at this point in the course. However, because he had a high level of self-efficacy, he planned to put effort into learning the concepts (Schunk, 1985; Los & Schweinle, 2019; Alotaibi, Tohmaz, & Jabak, 2017).

Later in the course, Henry was again asked how he felt about the material in the course. Specifically, he was asked this question as he was preparing to take the second exam. His response to this question was,

> *"I feel like I am enjoying the class and I feel like working hard just because of the amount of effort [Instructor G] puts in and the care with which…I kind of enjoy the entire class, not just any… yeah I kind of like all of the topics that she does.*"

In the above excerpt, we see that Henry was enjoying the topics taught in the course thus far. Thus, contributing to his level of self-efficacy. Furthermore, it appears as though he was inspired to put effort in the course because of the instructor. We see here again that Henry's instructor affected his level of self-efficacy for the subject. Particularly, the instructor's enthusiasm and possibly her self-efficacy affected Henry's self-efficacy (Los & Schweinle, 2019). Continuing with the mini conversations as the semester went on, Henry was asked to express which topics he liked the least. For this question, his response was,

> "*There is nothing I really don't like. I think I learned over time that…you know…if you don't like something, you kind of create a mental barrier with it and that's always gonna stick in your head….I always think you should approach it mutually first and if you still don't develop a liking to it, doesn't necessarily mean you dislike it, it's just whatever, it's just there…yeah, I've learned to come with an open mind."*

The latter excerpt reveals that Henry's high level of self-efficacy enabled him to have a positive outlook (*open mind)* on challenging problems (Nuutila, et al., 2020). Owing to Henry's high level of self-efficacy, he made it a point to put more effort into studying for exam two than he did for exam one (Alotaibi, Tohmaz, & Jabak, 2017; Nuutila, et al., 2020; Pintrich & De Groot, 1990; Ahmad, Hussain, & Azeem, 2012). As a result, he earned a 92 percent on test two.

By the time test three came around, Henry seemed to have gotten overwhelmed with the magnitude of material that was covered for exam three. This is evident when he said,

> *"I was just like…this is just way too much…it's a lot of material and I think…ah…that's one of the reasons probably…why I couldn't do well."*

Here we see that he started to doubt his abilities and was trying to figure out why he did not do well on exam three. He went further to say,

> "*I think that's one of the reasons why…I was surprised…I thought I did poorly when she posted the grade but, I looked at the class average…it was like a 52…okay looks like I'm not the only guy…the class average was like a 52 or 55.*"

In retrospect, Henry made a 68 percent on test three. In conversation, while looking over his test three exam, Henry noted that he was confident going into the exam. However, even though he said he felt confident about "everything", his grade did not reflect this confidence. Overconfidence, as seen in exam one, could have been a contributing factor as to why Henry scored so low on exam three even though he felt confident going into the exam (Seifert & Sutton, 2009). An additional explanation for the decrease in his test score might be as a result of a decreased level of self-efficacy due to the content covered for test three (Schunk, 1985).

Finishing off the course, Henry made 85 percent on the final exam. During the end of semester interview, Henry commented on how he thought the course went for him. He expressed,

> "*I'm not confident about getting a good grade in this class to be completely honest with you and I can't let my grades fall down again. I can take at most…I've never gotten more than one…I've never gotten a B in math…yeah…except for math stat one…which was in general was just hard because the professor made it hard."*

In this instance, one can see that there was a decrease in Henry's level of self-efficacy from what it was at the beginning of the course. Recall that when the course began, Henry was confident in his abilities to do well in the course. One may go further to say; he was confident that he could earn an A in the class. However, towards the end of the course, we see his confidence decreased perhaps because of the magnitude of the content covered. Nonetheless, we can see that Henry kept persisting as he made a 85 percent on the final exam (Alotaibi, Tohmaz, & Jabak, 2017; Nuutila, et al., 2020; Pintrich & De Groot, 1990).

It is important to note that even though Henry's level of self-efficacy varied throughout the course, it was still high enough for him to continue to work hard towards achieving a good grade (Alotaibi, Tohmaz, & Jabak, 2017; Nuutila, et al., 2020; Schunk, 1985; Los & Schweinle, 2019). During the end of semester interview, he was asked to rate himself in the course. He rated himself in the top five compared to other students. In actuality, when comparing the grades of all the students in the course, Henry was indeed number five in the course. This illustrates that he was self-aware and confident in his abilities in the class (Pintrich & De Groot, 1990).

4.1.3.2 Self-efficacy in Analysis

I will now report on Henry's level of self-efficacy in the Analysis course. At the beginning of the semester, Henry was asked to express how he felt about the upcoming course. Specifically, he was asked if he thought the course was important. He responded saying, "*Oh yeah*." Recall, if students think the course they are taking is important, then they are more likely to try the assignments (Pintrich & De Groot, 1990). With this in mind, one may say that Henry's level of self-efficacy was high as the course began. Particularly, Henry was asked if he thought it was possible for him to make a good grade in the course. His response was, "*Oh definitely…*

definitely…definitely because I know what I…where I messed up in bridge." This response further confirms that he had a high level of self-efficacy as the course began.

As the class progressed, similarly to the Bridge to Higher Math course, Henry's level of self-efficacy changed. Specifically, when Henry was asked if he found any of the topics he was learning to be difficult, he responded,

> *"No… I mean I think his homework is…I don't know why he's playing them so low ball…I don't know…all the homework are like so low ball…like it's so…I'm sorry…I feel like it's so easy, it's humiliating."*

With this response, it appears as though Henry was not enjoying the course as much as he would like. Recall that Henry expressed that the challenge of mathematics is what made him switch to major in it. Since the homework problems were not of much challenge to him, one may say he was not enjoying the course up to this point. It is important to note that Henry mentioned that he thought the course was easy as a result of the homework lacking challenge. Specifically, when asked why he thought the homework was easy, Henry responded,

> "*Because he's already discussed some… to put it in more blatant terms…he's giving us homework he's already discussing."*

This confirms that the lack of challenge has led to a decrease in Henry's level of self-efficacy regarding his interest in the course (Nuutila, et al., 2020) . Since the instructor gave hints and or did the homework problems in class, Henry was not challenged to think of the solutions himself. Henry went further to say,

> *"I don't feel happy with the class…because how am I seeing anything new? How is this gonna help me?"*

In this excerpt, Henry was expressing his discontent with the course. This may have further contributed to a decreased level of interest in the course. Furthermore, because the assignments were easy for Henry, he was losing interest in the course (Nuutila, et al., 2020).

Roughly one month into the course Henry admitted that he was not feeling confident in the course. With this, one can see that it appears his interest and confidence in the course was going down. He expressed that he was not understanding the material, as well as he hoped. Explicitly, he stated,

> *"Like I understand the beta structure you know…like epsilon, big 'n' and converging… diverging…all of that stuff…but ah…and Cauchy…but ah like I don't know…somehow…I don't feel confident with it…I need to…I need to practice more…I just picked up…I started Rudin…textbook*."

His lack of confidence, and thus decreased level of self-efficacy led to a decrease in how much effort he put to learn the material in the course (Alotaibi, Tohmaz, & Jabak, 2017; Nuutila, et al., 2020; Pintrich & De Groot, 1990). It is important to note that even though Henry's confidence in his abilities to complete the assignments in the course went down, he continued to express that he saw the value of the course. For instance, when asked if there were any topics he found that were challenging, he stated,

> *"This is like the building block…of mathematics…this is the time to put your big boy pants on…it's no longer like vanilla calculus…it's important to know this stuff."*

While he was not enjoying the course, because he saw the importance and the value of the course, he still put forth effort in excelling in it (Pintrich & De Groot, 1990).

The conversations reported thus far were before exam one in the course. After the first exam, Henry commented that he was "*pretty confident*" about the exam but felt he could have studied the material more. Nonetheless, despite what seemed to be his decreased level of interest, Henry made 80 percent on the first exam. This indicates his intrinsic value for the class positively affected his self-efficacy (Pintrich & De Groot, 1990). Reflecting back on the exam and how he performed, Henry expressed that he "*got cocky about it*" and "*mess[ed] up*." Here again, we see that overconfidence affected how he prepared for the exam and thus affected his grade on the exam (Seifert & Sutton, 2009). At this point in the semester, Henry was asked specifically, what grade he thought he could make in the course. He responded by saying, *"I think I can get an A."* One may interpret this as an increased level of self-efficacy. Recall earlier in the semester, Henry expressed that he was not confident in the class and that he was not happy. Based on his latter comment however, it appears that because he got an 80 percent on the exam, he gained back the confidence he had in the beginning of the course to do well.

Progressing through the course in preparation for test two, it seemed as though Henry's level or self-regulation in regard to his confidence went back down. For example, when asked how he felt about the material for test two, he responded with,

> *"I kind of do somewhat understand them…ahm…there are still some things here and there that I feel like…I take for granted."*

In this instance, "*somewhat understand*" may be interpreted as a lack of confidence. Recall that when he was previously asked about the material in the course, he expressed that he felt that the assignments were easy. In his latter response however, he was expressing that there were topics that he did not completely understand. Particularly, the material was not so "easy" anymore. In addition, when he was asked if he felt confident about the upcoming test, his response was, "*no*." This decrease in self-efficacy was due to a lack of understanding in the material of the course up to this point (Schunk, 1985).

Nonetheless, he scored 87 percent on test two. This may be attributed to the fact that his overall self-efficacy was strong enough to propelled him to study for the exam even though he was not feeling confident (Alotaibi, Tohmaz, & Jabak, 2017; Nuutila, et al., 2020; Pintrich & De Groot, 1990). This is evident in his response when asked how he felt about his preparation for test two. Particularly, he responded saying he felt "*very good*" about his preparation and that he felt prepared because he *"was preparing for quite a while…at least a week back."* Moving along in the semester, test three was a take home exam. Henry made a 63 percent on the take-home exam. It is important to note, as mentioned in section 4.1.1.1, that he expressed that he did not have time to do the exam, as he was preparing to take the GRE exam in addition to other obligations to commit to. This may have been the contributing factor to such a low grade on the exam.

In the end of semester interview, Henry expressed that he did not know his position in the course compared to the other students. Recollect that for the Bridge to Higher Math course, he commented that he thought he was in the top five compared to other students and the grades of the course reflected such. In this case, he expressed that he did not know where he stood compared to the other students. More specifically, he replied, *"I really don't know to be honest."* In this response, it appears as though he was not as self-aware as he was in the Bridge to Higher Math course. During the end of semester interview, Henry seemed to have lost confidence in himself as the course ended. Specifically, he glimpsed the title of my dissertation, "*An in-depth investigation of how an undergraduate mathematics major student learns the concept of proof*," and commented,

> *"I'm not as successful as you…I'm successful…what a joke…I'm not a successful student…kind of think I'm successful."*

Remember that in the beginning of the semester, Henry was convinced that he could earn an A in the course. In this instance, he seemed to lack the confidence that this was still possible. A lack of understanding, particularly relational understanding, may have led to a decrease in self-efficacy and thus a low grade in the course.

Next, I will briefly discuss how Henry scored himself on self-efficacy in the SRL questionnaire from Pintrich & De Groot (1990) given in each course. Recall that the questionnaire had a series of questions reflecting intrinsic values (motivation), self-efficacy and the three components of self-regulated learning. Each question had a rating from one to seven where, one represented not all true for me and seven represented very true for me. The questions were mixed in on the questionnaire and then separated and averaged. Based on the questionnaire, Henry rated himself a 5.4 out of 7 for the Bridge to Higher Math course and a 3.4 out of 7 for the Analysis course. Based on what we saw in each course Henry had a good sense of his own self-efficacy for each course (Pintrich & De Groot, 1990). Furthermore, based on the data, Henry appeared to be more confident and self-aware in the Bridge to Higher Math course than he was in the Analysis course.

In this section, I discussed Henry's level of self-efficacy throughout the courses. Before the Bridge to Higher Math course, Henry was a confident student who enjoyed the subject of mathematics. From the data, we saw that he started both courses with a high level of self-efficacy, however, his self-efficacy fluctuated as the courses progressed. Furthermore, it appeared as though he had a higher level of self-efficacy in the Bridge to Higher Math course than that of the Analysis course. Based on the data, this may have led to him putting in more effort to learn in the Bridge to Higher Math course and thus receiving a higher grade in that course over the Analysis course. In the next section, I will discuss how the SRL questionnaire was used to model Henry's chance of success in each of the courses.

4.1.4 SRL success predictor

In this section, I will report on how a model selection was done for a generalized logistic regression model based on Henry's response on the SRL - questionnaire (Pintrich & De Groot, 1990). Henry used the SRL - questionnaire to rate his level of self-regulation. As adopted from Pintrich & Groot (1990), the questionnaire had 44 questions used to rate Henry on his self-efficacy, cognitive strategy use, intrinsic value (motivation), self-regulation, and test anxiety for each course. Each question had a rating from one to seven where one represented not at all true for me and seven represented very true of me. There were 9 questions related to self-efficacy, self-regulation and intrinsic values each, 13 questions related to cognitive strategies and four questions related to test anxiety. While Henry reported on his self-efficacy, cognitive strategy use, intrinsic value (motivation), self-regulation, and test anxiety for each course, for this research, only the self-efficacy, cognitive strategy use, intrinsic value (motivation), and self-regulation components were used as factors for the regression model. The questions pertaining to test anxiety were not used since test anxiety was not a focus in this study. After eliminating the questions relating to test anxiety, only 40 questions remained. However, since there were only 9 questions relating to self-efficacy, self-regulation and intrinsic values, to match the numbers, only the first 9 questions for cognitive strategy were used in the analysis.

The model selection was done using RStudio software. In this study, I am interested in what factors of the SRL model will affect Henry's success in proof-based courses. That is to say, I am interested in identifying if he will be successful (i.e., good student) or if he will not be successful in the courses. Since there are two possible outcomes, the generalized logistic regression model is the best model to use for this assessment. Specifically, the generalized logistic regression model has binary outputs of zero (not successful) and one (successful). The backward stepwise regression as discussed in section 1.3.2.2 was used to identify the relevant predictor(s) from the SRL – questionnaire (Faraway, 2014). Four candidate variables - motivation, self-efficacy, cognitive strategy use, and self-regulation, were used. The forward stepwise regression is usually used when a large number of variables are being analyzed. Since there were only four variables to be considered, we used the backward stepwise regression method. To determine which predictor was relevant for the best fit model, the test of significance was conducted at a point nine five level, and the predictors with the $p-value$ that failed the $\propto= 0.05$ cut off, were removed from the model. That is, if the significant value of a predictor was less than five percent, then the predictor was not relevant for the model.

Beginning with the backward elimination method, in the first step, all predictors were considered. The results follow in Table 4.4.

Table 4.4 Illustration of backward elimination method with predictors intrinsic value, self-efficacy, self-regulation and cognitive strategy use.

Predictors	Coefficients	P-value
(Intercept)	-12.0528	0.0563
Intrinsic Value	1.4899	0.1706
Self-Efficacy	0.7167	0.2097
Self-Regulation	-0.2101	0.6566
Cognitive Strategies	0.2863	0.4769
	Deviance	15.131

Looking at the $p - values$, we see that the predictor with the highest $p - value$ greater than five percent is self-regulation, with a $p - value$ of 0.06566. As a result, the predictor self-regulation was removed from the model and the model was refitted. After which, the results were reported in the table below.

Table 4.5 Illustration of backward elimination method with predictors intrinsic value, self-efficacy, and cognitive strategies.

Predictors	Coefficients	P-value
(Intercept)	-11.8956	0.0526
Intrinsic Value	1.2966	0.1816
Self-Efficacy	0.8231	0.1387
Cognitive Strategies	0.1644	0.5680
	Deviance	15.341

As done previously, the predictor with the highest $p - value$ greater than five percent was removed from the model. In this iteration, the predictor with the highest $p - value$ was cognitive strategy use. For that reason, the predictor cognitive strategy was removed from the model and the model was refitted. The results are displayed in Table 4.6.

Table 4.6 Illustration of backward elimination method with predictors intrinsic value, and self-efficacy.

Predictors	Coefficients	P-value
(Intercept)	-10.5628	0.0549
Intrinsic Value	1.1641	0.2164
Self-Efficacy	0.8735	0.1046
	Deviance	15.673

Since all remaining predictors did not have a $p - value$ less than $\propto = 0.05$, another iteration was done, removing the predictor with the higher $p - value$ greater than five percent. In this instance, that predictor was intrinsic value (motivation). The result of the final iteration is reported in the table that follows.

Table 4.7 Illustration of backward elimination method with the predictor self-efficacy.

Predictors	Coefficients	P-value
(Intercept)	-5.1490	0.0452
Self-Efficacy	1.1187	0.0381
	Deviance	17.699

In the last iteration, we see that the final remaining predictor was self-efficacy with a $p - value$ of $\propto = 0.0381$. This means self-efficacy was the only significant factor using the five percent significance level (Lent, Brown, & Larkin, 1986; Huang & Fang, 2010; Li, Ye, Tang, & Zhou, 2018). It is worth noting that one would move to comparing the second criteria for the model selection - deviance. However, since there was only one predictor with a significant value of $\propto = 0.05$ or less, then we stop at this model.

Nonetheless, one should observe that the deviance did increase as the irrelevant predictors were removed from the model. Specifically, in the first iteration, the deviance was 15.131, in the second iteration, the deviance was 15.341, in the third iteration, the deviance was 15.673, and in the last iteration, the deviance was 17.699. The logistic model takes the form:

$$y = 1.1187 * (self\_efficacy) - 5.1490. \qquad 4.1$$

Recall,

$$y = logit(n) = log\left(\frac{n}{1-n}\right)$$

where, n is the probability of success. In equation 4.1, y is the log odds function, $n = \frac{e^y}{1+e^y}$ is the odds and e^{β_i} is the odds ratio. Specifically, the term -5.1490 represents the baseline of Henry's success in reference to SRL. Hence the probability of Henry succeeding in either course without self-efficacy was $n = \frac{e^{-5.1490}}{1+e^{-5.1490}} = 0.0058$. This represents the odds (0.58 percent) of Henry's success in either proof course if he lacked self-efficacy. On the other hand, $n = \frac{e^{1.1187}}{1+e^{1.1187}} = 0.75$ represents the odds of Henry's success increasing for every unit level increase in his self-efficacy. More specifically, Henry's level of self-efficacy, based on the SRL questionnaire for the Bridge to Higher Math course, was 5.44. This means, $y = 1.1187 * (5.44) - 5.1490 = 0.94$ and $n = \frac{e^{0.94}}{1+e^{0.94}} = 0.72$. Thus, his chance of being successful in the Bridge to Higher Math course was 72%. In contrast, his level of self-efficacy for the Analysis course based on the questionnaire was 3.44. More explicitly, $y = 1.1187 * (3.44) - 5.1490 = -1.3$ and $n = \frac{e^{-1.3}}{1+e^{-1.3}} = .21$, equating to a 21% chance of success in the Analysis course. Comparing the two, we see that Henry was more likely to do better in the Bridge to Higher Math course than he would in the Analysis course. This was based on his level of confidence in his abilities to do the course work in each course.

In this section, we saw that based on the stepwise backward elimination method and the $p - value$ with five percent significance level, the predictor that was most important in determining if Henry would be successful in the Bridge to Higher Math and the Analysis course, was self-efficacy. This is in line with what was found by Pintrich and De Groot (1990), Lent et al. (1986), Huang and Fang (2010), Harding et al., (2019), Los and Schweinle (2019), Ahmad et al. (2012) and Li et (2018). Furthermore, this model may be used by instructors of proof courses to identify students who have a low percentage of success in the course so as to take the necessary measures to ensure that they are successful in learning the content in the course.

4.1.5 A closer look at cognitive and metacognitive strategies

As previously mentioned in sections 1.3.1 and 4.1.1, there are three components common to the many versions of definitions of self-regulated learning. These include cognitive strategies (what students use to learn, remember, and understand the material), metacognitive strategies (how students plan, monitor and modify their cognition) and management and control of efforts on classroom academic tasks (Pintrich & De Groot, 1990; McMillian, 2010; Greene & Azevedo, 2007; Zimmerman, 2008). While all of these components are intertwined in the three phases of the SRL model discussed in section 4.1.1, in this section I will report more explicitly on the cognitive and metacognitive components of Henry's self-regulated learning strategies. The management and control of efforts on classroom academic tasks component was thoroughly reported in subsection 4.1.1 since this component is the main focus of the SRL model.

4.1.5.1 Cognitive strategies

To that end, I will first report on Henry's cognitive strategies in both courses. Unlike the previous sections, in this chapter, I am reporting on Henry's cognitive and metacognitive strategies together since they were similar for both courses. As a refresher, Henry's cognitive strategies includes strategies that he used to learn, remember and understand the course materials. This included his note taking techniques, him highlighting key words and or methods, him identifying the main ideas related to a proof technique or him paraphrasing a proof for his own understanding (Pintrich & De Groot, 1990; McMillian, 2010; Greene & Azevedo, 2007; Zimmerman, 2008). As it relates to note taking technique, Henry did not write a lot of notes in either course. Traditionally, students take notes and try to learn simultaneously. Henry on the other hand, took minimal notes. During the final interview after the Analysis course, Henry was asked to explain why he did not write notes and attempt to learn concurrently. He responded by saying,

> *"yeah I don't usually take a lot of…I take notes on what I think is important…because then otherwise I'm not able to focus on what the professor is teaching… yeah…I need to focus…'cause I can't hear and write at the same time…I can either hear or I can either write. So, I'd rather hear…see what he is doing…then write down what I don't understand out of it…because ah…relatively it's easier for me to remember things when I'm solving them on my own…or studying on my own."*

In this excerpt, Henry was explaining that he recognized that he could not focus on what the instructor was teaching while he wrote the information the instructor is covering. He instead, listened to what the instructors were teaching and jotted down what he did not understand. He went further to explain that he did not take a lot of notes in general but instead wrote what he did not understand, what he considered to be important or what he thought he would forget. In addition to what he did not comprehend, Henry revealed that he also made note of what he thought was important. For example, he wrote hints for the homework or hints of what he thought would be on an upcoming test. In addition to what he thought was important and hints the instructor gave, Henry wrote notes on things he thought he would forget for recollection when he studied. In essence, Henry stated that he did not write everything the instructor wrote on the board during class.

Henry was asked similar questions during the Bridge to Higher Math course, and his responses were similar to that of the responses reported in the Analysis course above. Furthermore, at the beginning of the Analysis course, Henry was specifically asked about what he did to help him learn in the courses. For instance, he was asked if he highlighted information while he studied or read through his notes or the textbook. His response to that question was as follows:

> *"[I] rehearse stuff in my head…I am not very fancy…like I keep it nice and simple…like I'll go over it in my head when I'm doing it actually…I don't use highlighters."*

In this portion of the transcribed data, it appears as though the highlighting technique did not work for Henry. Instead, he rehearsed the proofs in his head to retain the concepts being taught. More specifically, Henry mentioned the following in a conversation we had during the semester he took the Analysis course,

> *"I keep repeating some of the facts in my head….and it's like if I can recall the logic without looking at a piece of paper…even when I'm like…I don't know…doing my dishes or walking…that's when I know…okay…that I've understood the topic…but as far as everything tying in together…ah if I can just look at any problem that's given to me…and if I can trace back…I can solve it….but more importantly I can trace my own logic for that."*

This response confirms that Henry tries to focus on understanding the main idea and logic of the concept in his head before writing it down.

4.1.5.2 Metacognitive strategies

Now I will report on Henry's metacognitive strategies for both courses. Similarly, to his cognitive strategies, his metacognitive strategies were more or less the same for both classes. For instance, his metacognitive strategies included doing the homework problems in preparation for the quizzes and exams. While he did not specifically make a study schedule nor study time table, he used the homework problems given in each class to pace himself for the upcoming exams.

Occasionally, he would prioritize his homework problems and study time for exams based on what his work load was like for the semester. More specifically, during a conversation in the Bridge to Higher Math course, he mentioned,

> *"So yeah, that's how I manage my time. I felt like this, at this point, things were not as challenging in this class…which is why I felt like you know…if I can pull that time out and put it over here (Math Stat II)…it would be much more beneficial."*

In this particular case, Henry was explaining that he prioritized his assignments based on what he thought needed the most attention. In this particular excerpt, he was explaining that he opted to study Math Stat II and not Bridge to Higher Math because he was familiar with the material being taught in the Bridge to Higher Math course at the time and was struggling in the Math Stat II course. There was a similar instance in the Analysis course where he placed more emphasis on studying for the GRE and conducting his research assignment instead of preparing for his take home exam.

As for monitoring his learning, as mentioned in section 4.1.1, he occasionally adjusted his study habit in each course according to his performance on exams. More specifically, when he made a low grade on exam one in both classes, he changed the length of time he studied and or the way he studied to ensure that he got a higher grade on the upcoming exam. He did not mention much about information seeking but, expressed that he wished advanced mathematics courses had assistance on Khan Academy. Furthermore, he expressed,

> *"I sometimes go to Khan Academy…but ahmm…I don't know if they have videos for this class…I don't know…this is like upper level undergraduate math."*

With this excerpt we see that Henry was expressing his desire to use external sources to assist in his learning. Though it appears as though none was available for advanced math courses, one may say, based on the latter response, Henry would have taken advantage of external sources if he had access to them.

In this section, I briefly discussed Henry's cognitive and metacognitive strategies in both courses. His cognitive strategies included being attentive in class during the lecture and noting information he found of interest, challenging and or helpful hints for exams and quizzes. In addition, as opposed to highlighting material in the text book or his notes, he focused on learning the logic of the proofs in his head. As for his metacognitive strategies, while he did not make a time table nor study schedule, he paced himself using the homework problems given. Moreover, he adjusted his study habits after exams to improve his grade on the upcoming exam. Based on our conversations throughout both semesters, it appears as though Henry often placed other things above the courses.

4.2 APOS theoretical framework

In this section I will analyze and interpret the data that will help in answering the research questions:

- How does a competent student in mathematics develop his/her understanding of proof concept?
- What challenges in learning new concepts in proof courses, encountered by a competent mathematics student, could be used in teaching these concepts?
- What approaches to learning new concepts in proof courses, used by a competent mathematics student, could be used in teaching these concepts?

The data that will be analyzed consists of conversations and interviews I had with Henry. In addition, I will be analyzing the work Henry did on homework problems, quizzes and exams. Recall, in the previous sections, sections 4.1.1 – 4.1.6, I analyzed Henry's performance in the

Bridge to Higher Math course and the Analysis course using the self-regulated conceptual framework. In this section, I will be analyzing Henry's performance in the Bridge to Higher Math and the Analysis course using the APOS theoretical framework. Using the lens of APOS, I will be looking at how Henry constructed his understanding of proof concept.

When a student is first introduced to a proof concept, he or she usually begins at the Action stage of APOS theory. Recall, at the Action stage, students are merely repeating, step by step, how they have been taught to prove a particular theorem or, how they have memorized a particular procedure or proof technique (Asiala, et al., 1996; Arnon, et al., 2014; Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007). For example, in a proof course, a student may be introduced to proof by contradiction. For example, at the Action stage, a student will only be able to do a proof by contradiction by following the steps outlined by the instructor or in a textbook (Chamberlain & Vidakovic, 2020). For instance, a student at the Action stage may memorize the following steps as outlined by Chamberlain and Vidakovic (2020, p. 6) to prove a statement of $P \to Q$ by contradiction:

1. "Convert the statement into symbolic notation;
2. Identify the assumption (P), the conclusion (Q), and write the statement in the `If P, then Q' form;
3. Write the negation of the conclusion;
4. Assume the negation of the conclusion and the assumption are both true;
5. Symbolically manipulate the conclusion to get a statement that contradicts the assumption; and
6. State that a contradiction has been made and thus the proof is done."

At this stage, the student would not be able to do a contradictory proof requiring any slight deviation from the steps above. When a student is able to interiorize the Actions of using the outline of the proof given (memorized or written on paper) to complete a particular proof then, he or she will arrive at the Process stage. More specifically, when the student does not need to think of the memorized steps or use the notes with the outlined proof, we say that he/she can think of it in his/her mind. Furthermore, he/she has interiorized the above described Action into a Process of proving by contradiction. (Chamberlain & Vidakovic, 2020).

At the Process stage of APOS, a student is able to mentally conceive how to do a proof by contradiction and can verbally explain how to do it without needing to physically perform the Actions (Chamberlain & Vidakovic, 2020). That is, when the student can explain the procedure of proof by contradiction without having to write down the solution, then he/she is at the Process stage of APOS. More specifically, as stated in Chamberlain and Vidakovic (2020, p. 7), the student at the Process stage would be able to explain how to do a proof by contradiction in the following way:

1. "Assume the premise and the negation of the conclusion are true;
2. Show that step [one] leads to a mathematical absurdity, i.e. a contradiction; and
3. Conclude the statement to be proved is true."

In this instance, the student is able to bypass the steps outlined at the Action stage and can summarize in his/her own words how to do a proof by contradiction.

When the student has encapsulated the process of proof by contradiction, then that student has arrived at the Object stage. That is, the student is able to view proof by contradiction as an object to which Processes can be applied. For example, the student who arrives at the Object stage of proof by contradiction, sees proof by contradiction as a method of proof that can be applied to different problems. In addition, a student who is able to compare proof by contradiction to other methods of proof - [2]direct proof, proof by induction or proof by contraposition for instance, that student has also arrived at the Object stage of APOS for proof by contradiction. Furthermore, the student is able to view the method itself as a static structure to be acted on (Chamberlain & Vidakovic, 2020; Asiala, et al., 1996; Arnon, et al., 2014). It is important to note that since learning is not linear, sometimes students may need to de-encapsulate an Object back to a Process to gain understanding of another concept he or she is learning (Arnon, et al., 2014; Chamberlain & Vidakovic, 2020; Asiala, et al., 1996). For example, a student at the Object stage of proof by contradiction may need to de-encapsulate the Object back to a Process when learning proof by contraposition. Additionally, he/she may develop different Processes relating to a proof – negation, implication, existence etc., that needs to be coordinated together into the Process of proof by contradiction (Chamberlain & Vidakovic, 2020). The coordination of all concepts and knowledge relating to proof by contradiction, makes up the student's Schema of proof by contradiction.

Similarly, to how Henry's self-regulation was reported through the SRL conceptual framework, I will be reporting on Henry's understanding of the proof concepts he learned over the two semesters through the lens of the APOS theoretical framework. As previously mentioned, the conversations between Henry and I were transcribed and coded. In a similar way to how the conversations were coded for the SRL framework, they were also coded using the APOS framework. That is, I highlighted parts of the transcribed data a specific color to represent phrases that appeared to reflect the Action, Process or Object stages of APOS (Arnon, et al., 2014; Asiala, et al., 1996; Chamberlain & Vidakovic, 2020). Table 4.8 below shows an example

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[2] Direct proof, proof by induction and proof by contraposition are different methods of proof that mathematics students are often introduced to in their Discrete math course and or their Bridge to Higher Math course.

of a color-coded portion of the transcribed data to reflect the APOS theoretical framework. This

Table 4.8 Illustration of an example of a coded portion of the transcribed discussions about the student's Bridge to Higher Math course for the APOS framework.

me it's basically just practicing	APOS - ACTION
Grad student: So you practiced a lot? Henry: Yeah...up until then...ahm...after that, once you understand...because i think discrete and proof is more so developed by you practicing algebra more than anything else.	APOS - ACTION
: I went over some of the examples	APOS - ACTION
if I felt like I was able to go through the examples **mentally** for the first or second step and after I saw that leading to a clear pathway resolution I skipped that problem and I moved on to the ones that I found actually a little challenging	APOS – PROCESS (mentally)
on the problem I was working on, then I would grab paper and pencil and work it all the way out	APOS - ACTION
Contrapositive and Contradiction...well contradiction is the best actually.	APOS – Object?

illustration was chosen because it shows all three components highlighted together. All phrases related to APOS were highlighted yellow, while the text colors varied for each stage. More specifically, for the phrases that reflected the Action stage, the text was highlighted green, for the phrases that reflect the Process stage, the text was highlighted pink and for phrases that reflected the Object stage, the text was highlighted orange. In this phase of coding, when coding for the Action stage, I looked for instances where Henry mentioned that he was doing activities in solving problems that indicated the Action stage of APOS. For example, when he mentioned that he was practicing problems or doing problem over and over, that portion of the text was highlighted green and coded as Action (Arnon, et al., 2014; Asiala, et al., 1996; Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007). When he mentioned that he was doing problems in his head or mentally, that portion of the text was highlighted pink and coded as Process (Arnon, et al., 2014; Asiala, et al., 1996). When he [3]appeared to be describing proofs as an Object, that portion of the text was highlighted orange and coded as Object (Arnon, et al., 2014; Asiala, et al., 1996). More specifically, looking at Table 4.8, in the first two rows, Henry was explaining how he studied to become a competent student in mathematics. He explained that he practiced a lot of problems, specifically algebra, in order to get better at mathematics. Since it appeared as though he performed the Action of practicing physically, this portion of the conversation was coded as the Action stage (Arnon, et al., 2014; Asiala, et al., 1996).

In the third row, Henry mentioned that he went over problems while studying for an upcoming exam. This was coded as Action, since "going over problems" we interpret as him physically solving problems he had been introduced to in class to commit to memory. In the fourth row, Henry was explaining how he studied for exam one in the Bridge to Higher Math course. Since he made reference to doing the problems mentally, this was coded as the Process stage (Arnon, et al., 2014; Asiala, et al., 1996; Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007). That is, it appeared as though he interiorized the Actions of practicing the problems into a mental Process. In the fifth row, since Henry mentioned he "grabbed paper and pencil" to work on problems physically, this was coded as the Action conception stage (Arnon, et al., 2014; Asiala, et al., 1996; Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007). Recall that if a student has to physically write down the solution of a problem, this indicates that he or she is at the Action stage. In the last row, we interpret that Henry was comparing proof by contradiction to proof by contraposition as methods of proving. Thus, this was coded as Object where comparison is the Action performed on two Objects - proof by contradiction and proof by contraposition (Arnon, et al., 2014; Asiala, et al., 1996; Arnawra, Sumarno, Kartasasmita, &

[3] We use the term "appears" as we can only report on what we perceive Henry to be thinking..

Baskoro, 2007). It is important to point out that some questions may only require the Action conception stage of APOS. An example of these questions is a question that prompts a student to give regurgitate a definition. Equally important to note is that a student who does not exhibit at least the Action stage, is said to be at the pre-Action conception stage.

4.2.1 Development of understanding in Bridge to Higher Math through the lens of APOS

To report on Henry's development of understanding in the Bridge to Higher Math course I will go through a detailed timeline of our conversations throughout the semester, his quizzes, and his exams. The timeline is illustrated in Figure 4.1. In the timeline, Con. -represents conversations that I had with Henry, Q – represents the quizzes he took throughout the semester, T – represents the tests he took throughout the semester, and F – represents the final exam. More specifically: on June fourth, when the class began, we had our first conversation, coded Con. 1; on June 8th, we had conversation two, coded Con. 2; on June 11th, Henry took quiz one; on June 15th, we had conversation three; on June 18th, Henry took test one; on June 20th, we had conversation four; on June 22 we had conversation five; on June 25th, Henry took quiz two; on June 27th we had conversation 6; on July second Henry took test two, on July 6th Henry and I had

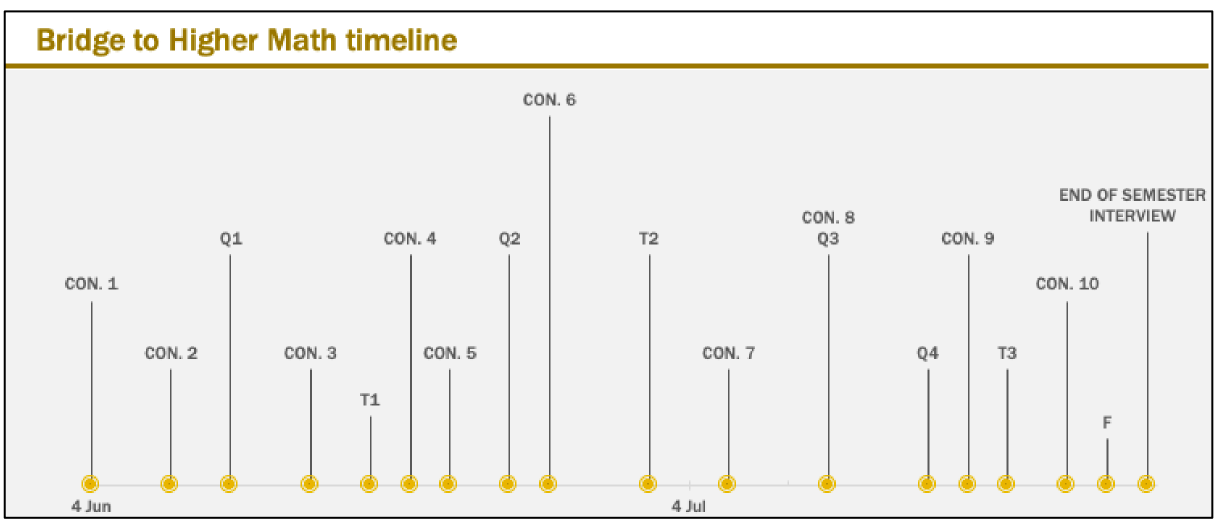


Figure 4.1 Illustration of timeline of Henry and I's conversations throughout the semester, his quizzes, exams and the end of semester interview for the Bridge to Higher Math course.

conversation 7; on July 11th, Henry took quiz three and we had conversation 8; on July 18th, Henry took quiz four and we had conversation 9; on July 20th, Henry took test three; on July 23rd we had the 10th and final mini conversation for the semester; on July 25th, Henry took the final exam while on July 27, I conducted the end of semester interview with Henry. I would like to point out that the height of the lines in the timeline in Figure 4.1 does not have any significance. Each specific height was used for placement convenience.

I will now report on the stages Henry appeared to exhibit at each instance listed in the timeline in Figure 4.1. The figure shows that Henry and I had 10 mini conversations, in addition to the four quizzes and four exams (including the final exam) that were administered in the course. I will analyze the data in three sections to correspond with the three data set - quizzes, exams and transcribed conversations to include the end of semester interview. Particularly, I will give illustrations of each stage of APOS theory exhibited by Henry for each category, followed by a summary of my analysis of all quiz and exam problems, as well as all transcribed data. All questions from the quizzes and exams will be in the appendix.

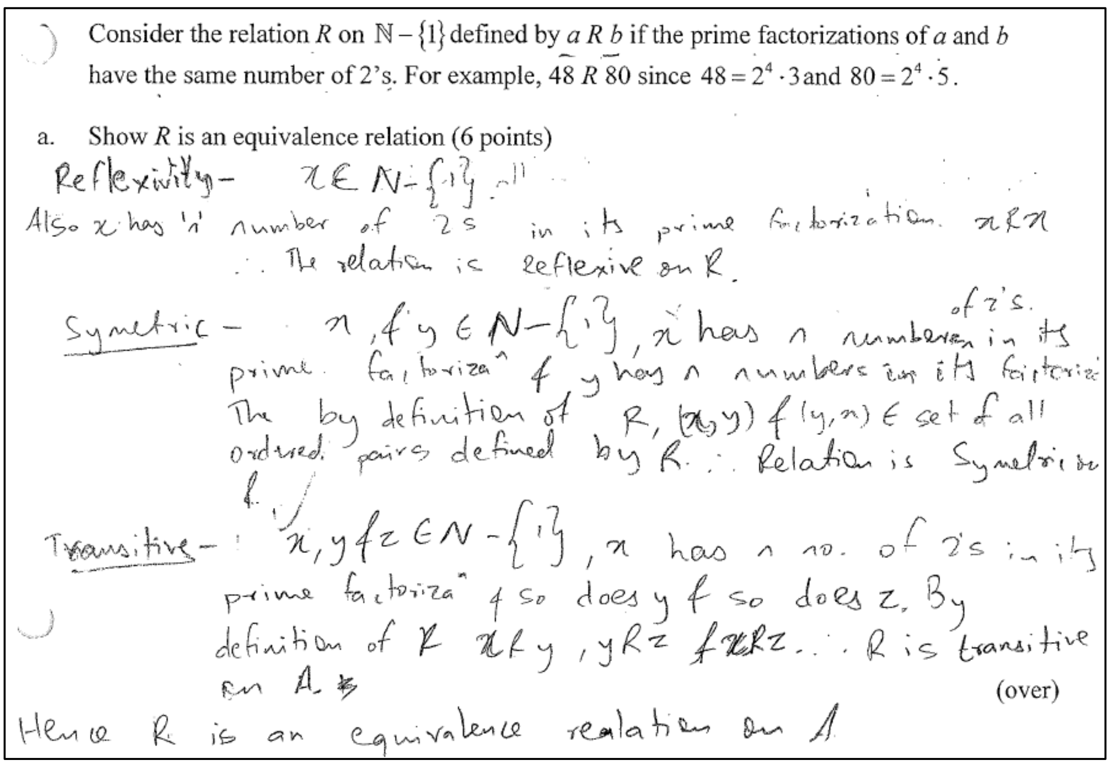
The fundamental concepts covered throughout the semester were the truth table, even and odd parities, quantifiers, arithmetic, Fibonacci numbers, set theory, family of sets, integer proofs, equivalence relation, partition, modulo arithmetic, functions, summation, converse, proof by contradiction, proof by contraposition, proof by induction, and the well ordering principles (WOP). In the illustrations of my analysis, it is my aim to cover as many of these concepts as possible, so as to give the reader an idea of how Henry's understanding developed in relation to the different concepts taught. What follows are the illustrations from each data set starting with the quizzes from the course.

4.2.1.1 Quizzes

Quizzes serve as one of the check points to examine if students are learning the concepts taught in class. As it relates to the quizzes, Henry exhibited the Action and Process stages of conception of understanding of APOS. For a representation of Henry's work at the Action conception stage of understanding on the quizzes, I am presenting the work he did on quiz four question two, part *a*. For this problem, Henry was prompted to:

> *Consider the relation R on $\mathbb{N} - \{1\}$ defined by $a\ R\ b$ if the prime factorization of a and b have the same number of $2's$. For example, $48\ R\ 80$ since $48 = 2^4 * 3$ and $80 = 2^4 * 5$. Show R is an equivalence relation.*

Below, is a picture of Henry's work.



Consider the relation R on $\mathbb{N} - \{1\}$ defined by $a\ R\ b$ if the prime factorizations of a and b have the same number of 2's. For example, $48\ R\ 80$ since $48 = 2^4 \cdot 3$ and $80 = 2^4 \cdot 5$.

a. Show R is an equivalence relation (6 points)

Reflexivity – $x \in N - \{1\}$

Also x has 'n' number of 2s in its prime factorization. xRx

∴ The relation is reflexive on R.

Symetric – $x, \& y \in N - \{1\}$, x has n numbers of 2's in its prime factoriza^n & y has n numbers in its factoriza The by definition of R, (x, y) & $(y, x) \in$ set of all ordered pairs defined by R. ∴ Relation is Symetric on R.

Transitive – $x, y \& z \in N - \{1\}$, x has n no. of 2's in its prime factoriza^n & so does y & so does z. By definition of R xRy, yRz & xRz. ∴ R is transitive on A.

(over)

Hence R is an equivalence realation on A

Figure 4.2 Illustration of evidence showing what appears to be Henry exhibiting the Process conception stage of an equivalence relation.

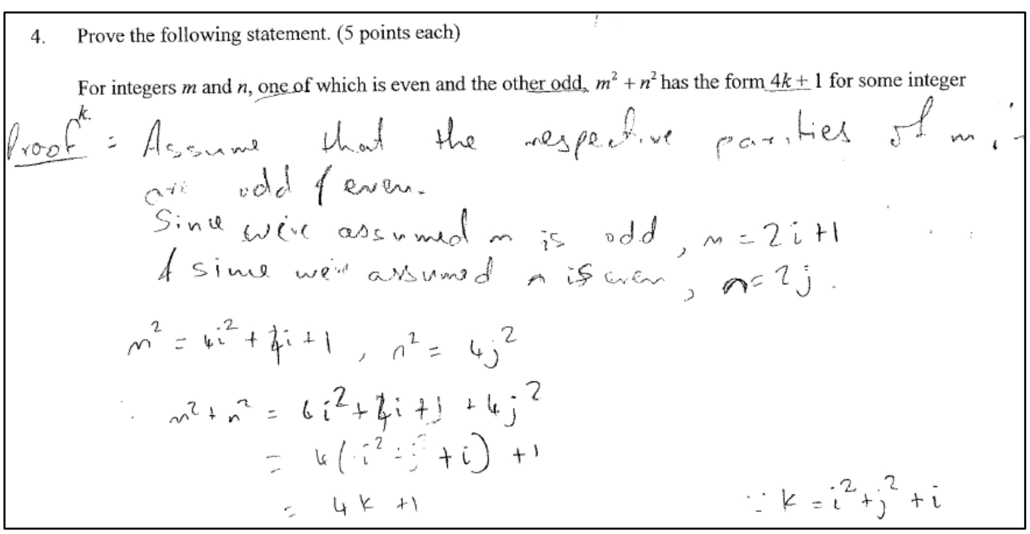
Before I analyze Henry's response, we note that according to Smith, Eggen, & St. Andre, (2011) a relation R on a set A is an equivalence relation on A iff R is:

(i) reflexive on A iff for all $x \in A$, $x\ R\ x$.

(ii) symmetric on A iff for all $x, y \in A$, if $x\ R\ y$ then $y\ R\ x$.

(iii) transitive on A iff for all x, y and $z \in A$, if $x\ R\ y$, $y\ R\ z$ then $x\ R\ z$.

Observe that for the proof, Henry wrote each property, reflexivity, symmetry, and transitivity and then proceeded with the proof of each property. Writing the properties to illustrate what needs to be proved, is an indication that Henry was referencing the definition for a relation to be reflexive. As a result, this proof was coded as Henry exhibiting the Action conception stage of understanding of equivalence relation using the method of direct proof (Asiala, et al., 1996; Arnon, et al., 2014; Chamberlain & Vidakovic, 2020; Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007). To illustrate an example Henry's exhibition of the Process conception stage of understanding of APOS theory, I will report on Henry's work for quiz one, problem number four. Figure 4.3 displays Henry's work. In this particular problem, Henry was



4. Prove the following statement. (5 points each)

For integers m and n, one of which is even and the other odd, $m^2 + n^2$ has the form $4k + 1$ for some integer k.

Proof: Assume that the respective parities of m, n are odd & even.
Since we've assumed m is odd, $m = 2i+1$
& since we've assumed n is even, $n = 2j$.

$m^2 = 4i^2 + 4i + 1$, $n^2 = 4j^2$

$\therefore m^2 + n^2 = 4i^2 + 4i + 1 + 4j^2$

$= 4(i^2 + j^2 + i) + 1$

$= 4k + 1$ $\therefore k = i^2 + j^2 + i$

Figure 4.3 Illustration of evidence showing what appears to be Henry exhibiting the Process conception stage of the concept of direct proof on quiz one.

asked to:

Prove the following statement.

For integers m and n, one of which is even and the other odd, $m^2 + n^2$ has the form $4k + 1$ for some integer.

Henry did a direct proof of this statement. Observe that he was not doing the method of direct proof step by step. Recall the steps outlining the process of direct proof are (Smith, Eggen, & St. Andre, 2011):

Direct Proof of $P \Rightarrow Q$

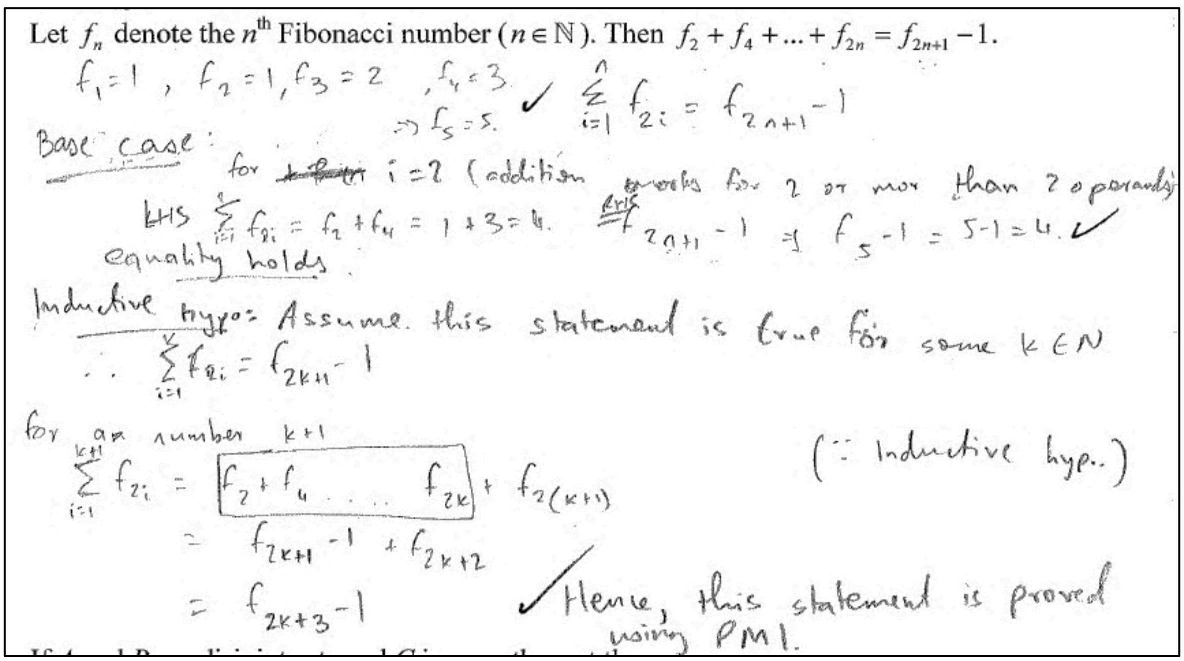
Proof.

1. Assume P.
2. Use P with other known statements (axioms and definitions) to prove Q.
3. Therefore, Q.
4. Thus $P \Rightarrow Q$.

Henry did not use any cues of the above steps of the method of direct proof to complete his proof. More specifically, he started by assuming that m was odd and n was even. Observe also that he rewrote this statement in his own words and not as it appeared in the question. That is, instead of using, "*for integers m and n, one of which is even and the other odd,"* for P, he rewrote "*for integers m and n, one of which is even and the other odd*." He continued by stating his supportive arguments and then his conclusion. Since he completed these steps without having to explicitly state them nor using cues, this work done by Henry was coded as the Process conception stage of understanding of APOS theory using the method of direct proof (Arnon, et al., 2014; Asiala, et al., 1996; Chamberlain & Vidakovic, 2020; Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007). My analysis of Henry's work on these quiz problems is representative of my analysis of work done by Henry on all quiz problem for the semester.

4.2.1.2 Tests

When exams are given in class, students are expected to apply what they have learned from practicing homework problems, taking quizzes and what the instructor taught in class. As previously stated, there were four exams given in the course. Similarly, to the analysis of the quizzes, Henry only exhibited the Actions and Process conception stages on the exams. To illustrate an example of Henry's work representing the Action conception stage of understanding exhibited by Henry, I am presenting the work he did for test two, problem four, part *c*.



Let f_n denote the n^{th} Fibonacci number ($n \in \mathbb{N}$). Then $f_2 + f_4 + \ldots + f_{2n} = f_{2n+1} - 1$.

$f_1 = 1$, $f_2 = 1$, $f_3 = 2$, $f_4 = 3$ $\Rightarrow f_5 = 5$. ✓ $\sum_{i=1}^{n} f_{2i} = f_{2n+1} - 1$

Base case: for $i = 2$ (addition works for 2 or more than 2 operands)

LHS $\sum_{i=1}^{2} f_{2i} = f_2 + f_4 = 1 + 3 = 4$. RHS $f_{2n+1} - 1 \Rightarrow f_5 - 1 = 5 - 1 = 4$. ✓

equality holds.

Inductive hypo: Assume this statement is true for some $k \in \mathbb{N}$

$\therefore \sum_{i=1}^{k} f_{2i} = f_{2k+1} - 1$

for a number $k+1$

$\sum_{i=1}^{k+1} f_{2i} = \boxed{f_2 + f_4 \ldots f_{2k}} + f_{2(k+1)}$ ($\because$ Inductive hyp..)

$= f_{2k+1} - 1 + f_{2k+2}$

$= f_{2k+3} - 1$ ✓ Hence, this statement is proved using PMI.

Figure 4.4 Illustration of evidence showing what appears to be Henry exhibiting the Action conception stage on test two problem four.

In this problem, the prompt was:

Let f_n *denote the* n^{th} *Fibonacci number* ($n \in \mathbb{N}$)*. Then* $f_2 + f_4 + \cdots + f_{2n} = f_{2n+1} - 1$.

Recall the steps for the method of proof by induction are (Smith, Eggen, & St. Andre, 2011)

Proof:

(i) (Basis step) Show that $P(1)$ is true.

(ii) (Inductive step) Suppose $P(n)$ for some $n \in \mathbb{N}$.

…

Therefore $P(n+1)$.

(iii) Therefore, by the principle of mathematical induction, for all $n \in \mathbb{N}$, $P(n)\ is\ true.$

Observe that Henry wrote out the terms "base case" and "inductive step" as cues while he proved the statement using proof by induction. He proceeded by proving the base where $i = 2$. Since this was a summation statement, the base case would be the first sum. That is, the sum of the first two terms. Next, he moved on to the inductive step by assuming the summation holds for k terms and proceeded to prove the summation holds for $k + 1$ terms. He concluded the proof by stating "hence proved by [4]PMI." Because Henry explicitly wrote the terms "base case" and "inductive step" as cues for his proof, this proof was coded as the Action conception stage of understanding of APOS theory for Fibonacci number using the method of proof by induction (Arnon, et al., 2014; Asiala, et al., 1996; Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007).

It is worth pointing out that, mathematicians usually explicitly write "basis step" and "inductive step" when doing a proof by induction. For instance, in the instructor's solution for the exam, these two steps were explicitly written for most of the problems done using proof by induction. One may argue that this proof should be coded as the Process stage however, I coded this proof as the Action stage because he closed the proof by saying "hence proved by PMI," which is the last explicit step in the method of proof by induction. Clear evidence of him exhibiting the Process conception stage of understanding of the method of proof by induction would be him completing the proof without cues and/or skipping one or more of the steps

[4] PMI is short for principle of mathematical induction.

(Arnon, et al., 2014; Asiala, et al., 1996; Chamberlain & Vidakovic, 2020; Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007).

For an illustration Henry's work at the Process conception stage, I am presenting his work for problem 16, part *b* of the final exam.

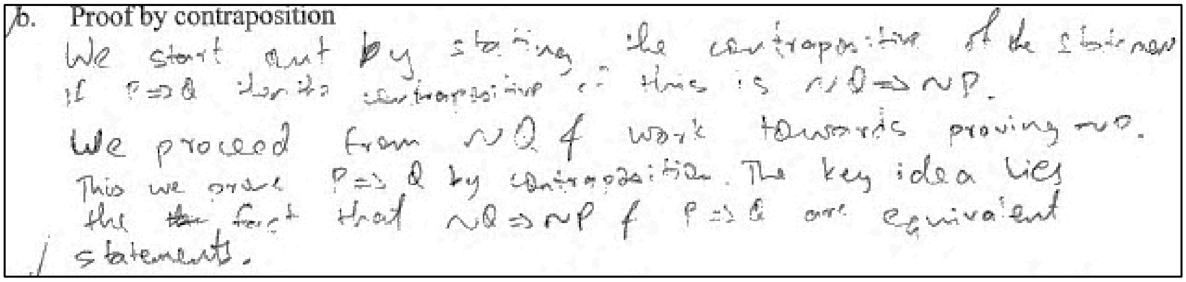

b. Proof by contraposition
We start out by stating the contrapositive of the statement
if $P \Rightarrow Q$ then its contrapositive i.e. this is $\sim Q \Rightarrow \sim P$.
We proceed from $\sim Q$ & work towards proving $\sim P$.
This we prove $P \Rightarrow Q$ by contraposition. The key idea lies
the ~~the~~ fact that $\sim Q \Rightarrow \sim P$ & $P \Rightarrow Q$ are equivalent
statements.

Figure 4.5 Illustration of evidence showing what appears to be Henry exhibiting the Process conception stage of proof by contraposition.

For this problem, Henry was prompted to:

Give a description for each of the following types of proof for a statement of the form $P \Longrightarrow Q$ or $P(n) \Longrightarrow Q(n)$, as appropriate.

b. Proof by contraposition

In the above question, the instructor asked a non-traditional question for a proof course. In his response, Henry was able to describe the process of proof by contraposition correctly. Recall that in a proof by contraposition of the statement $P \Rightarrow Q$ one proves [5]$\sim Q \Rightarrow \sim P$ since the two statements are equivalent. Henry highlighted this fact in his response stating, "the key idea is the fact that $\sim Q \Rightarrow \sim P$ and $P \Rightarrow Q$ are equivalent." Since he was able to successfully write an explanation of the process of proof by contraposition, this response was coded as the Process conception stage of APOS for proof by contraposition.

[5] The symbol $\sim$ represents the negation of the proposition. That is $\sim Q \Rightarrow \sim P$ means not Q implies not P.

4.2.1.3 Conversations and end of semester interview

During the conversations with Henry, I sought additional insight on how he learned proof concept in the course. In contrast to the quizzes and exams, Henry exhibited the Action, Process and Object conception stage of understanding. To illustrate each of these stages, I am presenting an excerpt from the conversation Henry and I had on June 15th. A quiz was administered on this day. Henry was asked about his performance on the quiz. During this conversation, there was one instance where he appeared to be exhibiting the Action conception stage of APOS, two instances where he appeared to be exhibiting the Process conception stage, and one instance where he appeared to be exhibiting the Object conception stage. The excerpt follows below. In the excerpt, I am "Grad Student."

Grad Student: *So, the quiz that we had, how did you prepare for that?*

Henry: *I went over some of the examples…ahm…I didn't go through all of them….ahm…I glanced over the examples that I had that were in the text book…if I felt like I was able to go through the examples* ***mentally*** *for the first or second step and after I saw that leading to a clear pathway resolution, I skipped that problem and I moved on to the ones that I found actually a little challenging. Of course, I didn't do all the challenging ones but that's how I kind of approached it.*

Grad Student: *Okay so like maybe mentally just see if you had the idea of what the proof is looking for?*

Henry: *Yeah. And if I felt like my logic was breaking down, ahm…on the problem I was working on, then I would grab paper and pencil and work it all the way out.*

Grad Student: *Okay, were there any difficult questions when you were going through that you were like…uuumm….and you skipped it? Or did you figure out everything?*

Henry: *Yeah, I did, because you know it was on a weekend. There were a couple of difficult ones that I did skip over umm…yeah so I didn't do all the difficult ones.*

Grad Student: *Okay. So, do you think skipping over it did anything for you in your preparation? Or…*

Henry: *I felt, well my performance. You saw my score, right?*

Grad Student: *Yeah.*

Henry: *I guess it didn't hurt me. But I felt like I should have done it.*

Grad Student: *And then, so far in the class you would say you haven't found anything challenging?*

Henry: *Not really, no.*

Grad Student: *Okay, have you found anything that you like more than the other?*

Henry: *Proofs.*

Grad Student: *You like the proofs?*

Henry: *Proofs are really interesting. That's the best part.*

Grad Student: *When you actually have to do a proof? Which proof do you prefer and why?*

Henry: *Contrapositive and Contradiction…well contradiction is the best actually.*

Furthermore, when asked how he studied for the quiz, Henry's responded,

> *"I went over some of the examples…ahm…I didn't go through all of them….ahm…I glanced over the examples that I had that were in the textbook…if I felt like I was able to go through the examples mentally for the first or second step and after I saw that leading*

to a clear pathway resolution I skipped that problem and I moved on to the ones that I found actually a little challenging."

In the above portion of the excerpt, Henry mentioned that while studying, if he could go through the examples *mentally*, then he would move to the next problem. This is a sign of him interiorizing the Actions of performing proofs into Processes while he studied. That is, he did not have to physically write the proofs over and over nor try to commit them to memory. Thus this part of the conversation was coded as Henry exhibiting the Process conception stage of understanding of APOS theory for the concepts he studied (Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007; Arnon, et al., 2014; Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007; Chamberlain & Vidakovic, 2020). As the conversation progressed however, Henry expressed that for some questions, he had to physically write the solutions. Particularly he said,

"Yeah. And if I felt like my logic was breaking down, ahm…on the problem I was working on, then I would grab paper and pencil and work it all the way out."

Unlike what he stated previously, that he if he was able to do the proofs mentally in his head he would skip over that proof, in this excerpt he stated that he had instances where he had to explicitly write the proofs. I interpreted that as him performing at the Action conception stage of understanding for some of the proof concept he was studying. Moreover, since he had to physically write down the proofs, it appears as though he did not interiorize the Actions associated with those proofs into Processes.

Later on in the conversation, Henry was asked which proof technique he preferred and why. He responded saying, *"contrapositive and contradiction…well contradiction is the best actually."* Here, it [6]appears as though Henry was comparing the methods of proof by

[6] We use the word appear because the comparison is not explicit.

contradiction and proof by contraposition. That is, it appeared as though Henry encapsulated each of the Processes of proof by contradiction and proof by contraposition into Objects. With this comparison, Henry seemed to acknowledge the methods of contradiction and contraposition as Objects that problems can be applied to. For those reasons, this statement was coded as Henry exhibiting the Object conception stage of understanding of the APOS theory for proof by contradiction and proof by contraposition (Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007; Arnon, et al., 2014; Asiala, et al., 1996; Chamberlain & Vidakovic, 2020). My analysis of this excerpt reflects my analysis of excerpts from the conversations that were transcribed. Tables 4.9 and 4.10 summarize all of my analysis of all quiz, and exam problems, as well as conversations

Table 4.9 Part 1 of 2 of a detailed summary of the timeline explaining Henry's development of understanding for particular concepts through the lens of the APOS theory in the Bridge to Higher Math course.

	Truth Table	Discrete Math	Even and Odd parities	Quantifiers	Arithmetic Manipulation	Fibonacci numbers	Set Theory	Family of sets	Integers
4-Jun	*Process* *Direct Proof*								
8-Jun		*Action* *N/A*							
11-Jun			Action|Process Converse | Direct						
15-Jun									
18-Jun			ACTION | PROCESS CONVERSE | DIRECT	PROCESS CONTRAPOSITION					
20-Jun									
22-Jun									
25-Jun									
27-Jun									
2-Jul					ACTION INDUCTION	ACTION INDUCTION	PROCESS DIRECT PROOF	ACTION INDUCTION	PROCESS CONTRADICTION
6-Jul									
11-Jul					*Process* *Direct Proof*		Process Direct Proof		
16-Jul									
18-Jul									
20-Jul									
23-Jul									
25-Jul			PROCESS DIRECT PROOF		ACTION INDUCTION	ACTION INDUCTION			
27-Jul			*Process* *Contradiction (WOP)*						

and the end of semester interview. Table 4.9 shows a timeline of how Henry's understanding varied through the semester, through the lens of APOS theory, for the concepts of truth table, even and odd parities, quantifiers, arithmetic manipulation, Fibonacci numbers, set theory, family of sets, integers and how he studied in Discrete Math. Table 4.10 on the other hand, displays a timeline tracking Henry's development of understanding changed as the semester progressed for the concepts such as equivalence relation, partition, modulo arithmetic, functions, summation, converse, contradiction, contraposition, induction, the well ordering principle and general topics Henry may have not specified during our conversations.

Table 4.10 Part 2 of 2 of a detailed summary of the timeline explaining Henry's development of understanding for particular concepts through the lens of the APOS theory in the Bridge to Higher Math course.

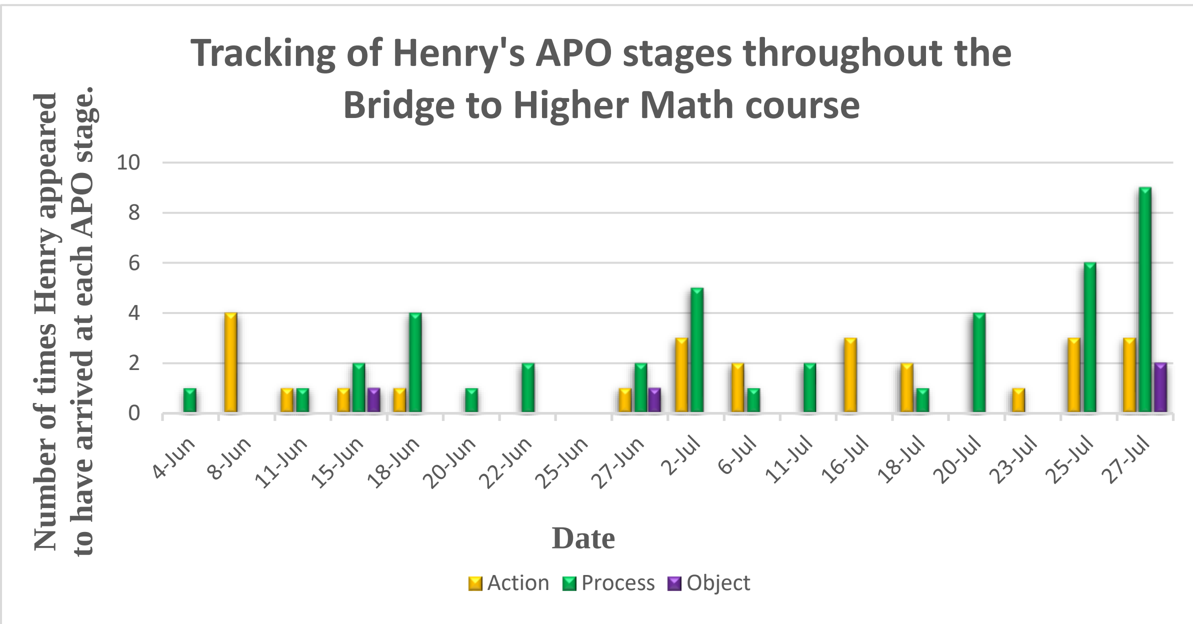
	Eqivalence Relation	Partition	Modulo Arithmatic	Function	Summation	Converse	Contradiction	Contrapositon	Induction	General	WOP
4-Jun											
8-Jun											
11-Jun											
15-Jun							Object N/A	Object N/A		Action/Process N/A	
18-Jun											
20-Jun						Process N/A		Process N/A			
22-Jun									Process N/A		
25-Jun											
27-Jun							Process/Object N/A		Process N/A	Action N/A	
2-Jul											
6-Jul										Action N/A	Process N/A
11-Jul	Action N/A										
16-Jul	Action Dierct	Action N/A									
18-Jul										Action N/A	
20-Jul	PROCESS N/A		PROCESS N/A	PROCESS DIRECT PROOF							
23-Jul										Action N/A	
25-Jul	ACTION DIRECT PROOF		PROCESS CONTRADICTION	PROCESS DIRECT PROOF				PROCESS N/A	PROCESS N/A		
27-Jul	Process Contradiction (WOP)		Object Contradiction | Induction		Process Induction			Process N/A		Action/Process N/A	

To the left of the table, the date each event (quiz, exam, or conversation) occurred is listed, while at the top are the concepts covered. To distinguish between what data was analyzed for each report in the table, different font styles and text colors were used. For instance, for conversations, the regular font style but italic was used. For exams, bold – caps font was used, while for quizzes, regular bold font was used. In reference to the conception stages of APOS theory, text colors were used to distinguish each. Specifically, yellow was used to represent the Action conception stage, green was used to illustrate the Process conception stage, and purple represents the Object conception stage of understanding. There were four instances where Henry exhibited the Action and Process conception stage for a concept simultaneously. This text was highlighted blue.

In addition to recoding the stage Henry exhibited for a particular concept, I also noted the method of proof Henry used when applicable. For instances, such as in our conversations, where a method of proof was not specified, N/A was used. Particularly, tracking Henry's development for even and odd parities, on June 11th from the analysis of a quiz, Henry exhibited the Action and Process conception stages for the converse of a statement and direct proofs. As a result, this text was regular font but bold and highlighted blue. On June 18th on exam one, Henry also exhibited the Action and Process conception stages. However, since the analysis was of an exam, the text was bold, all caps, still highlighted blue. By the end of the semester on July 25th, Henry exhibited only the Process conception on the final exam. Thus, the text was bold, all caps but highlighted green. Lastly, in the end of semester interview, Henry exhibited the Process conception stage of understanding for even and odd parities resulting in the text being regular font but italic. This coding was used throughout the table.

In this section, I analyzed Henry's understanding of the proof concept he learned in the Bridge to Higher Math course. Throughout the semester Henry had 23 instances where he appeared to exhibit the Action conception stage, 41 instances where he appeared to exhibit the

Process conception stage and four instances where he appeared to exhibit the Object conception stage. A quantified summary of my findings is broken down in the histogram in Figure 4.6. More specifically, on June fourth, Henry appeared to exhibit the Action conception stage once; on June 8th, Henry appeared to exhibit the Process conception stage four times; on June 11th, he appeared to exhibit the both the Action and Process stage once; on June 15th he appeared to exhibit the Action stage and Object stage once and the Process stage twice; on June 18th, he appeared to exhibit the Action stage once and the Process stage four time; on June 20th, he



appeared to exhibit the Process stage once; on June 22, he appeared to exhibit the Process stage

Figure 4.6 Frequency of each of the APO stages Henry exhibited at a particular date.

twice; on June 27th, he appeared to exhibit the Action and Object stage once and the Process stage twice; on July second, he appeared to exhibit the Action stage three times and the Process stage five times; on July 6th, he appeared to exhibit the Action stage twice and the Object stage once; on July 11th, he appeared to exhibit the Process stage twice; on July 16th, he appeared to exhibit the Action stage three times; on July 18th, he appeared to exhibit the Action stage twice and the process stage once; on July 20th, he appeared to exhibit the Process conception stage four times; on July 23rd, he appeared to exhibit the Action stage once; on July 25th, he appeared to exhibit the Action stage three times and the Process stage six times; and on July 27th, he appeared to exhibit the Action stage three times, the Process stage 9 times and the Object stage three times. In this research, only A-P-O was used.

4.2.2 Development of understanding in Analysis through the lens of APOS

In the previous section, I reported on my analysis of Henry's understanding of proof concept in the Bridge to Higher Math course. I will now report on my analysis of his understanding of proof concept in the Analysis course. As done for the Bridge to Higher Math course, I analyzed Henry's solutions to homework problems, exam questions and excerpts from our conversations throughout the semester, as well as the final interview. It is important to point out that during the end of semester interview, Henry answered four proof questions aloud. This was also a part of the analysis. A timeline outlining when Henry received each homework problem, took an exam or had a conversation with me is illustrated in Figure 4.7.

In the timeline, Con. -represents the conversations that I had with Henry, HW – represents the homework problems Henry was assigned throughout the semester, T – represents the tests he took throughout the semester, T.H – represents the take home exam and F – represents the final exam. More specifically: on August 20th, I had the first conversation of the semester with Henry and the instructor assigned homework number one; on August 22nd, the instructor assigned homework problem numbers two, three, four and five; on August 27th, the

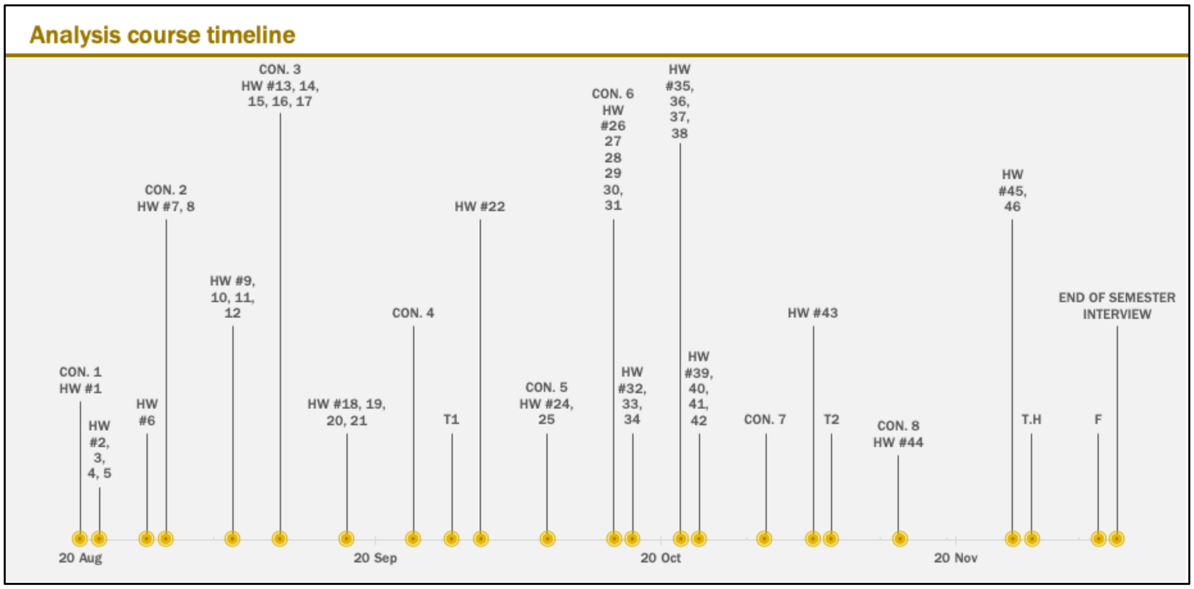


Figure 4.7 Illustration of timeline of Henry and I's conversations throughout the semester, his quizzes, exams and the end of semester interview for the Analysis course.

instructor assigned homework problem number 6; on August 29, we had conversation two and the instructor assigned homework number 7 and 8; on September fifth, the instructor assigned homework numbers 9, 10 and 11; on September 10th, we had conversation three and the instructor assigned homework numbers 13 and 14; on September 17th the instructor assigned homework numbers 18, 19, 20, and 21; on September 24th, we had conversation number four; on September 28, Test one was administered; on October first, homework problem number 22 was given; on October 8th we had conversation number five and homework number 24 and 25 were assigned; on October 15th' we had conversation 6 and homework problem numbers 26 - 31 were assigned, on October 17th, homework numbers 32 - 34 were assignment; on October 22nd, homework problems 35, - 38 were given; on October 24th, homework numbers 39 - 42 were assigned; on October 31, Henry and I had conversation 7; on November fifth, homework number 43 was assigned; on November 7th, test two was administered; on November 14th, we had conversation 8 and homework number 44 was given; on November 26th, homework number 45 and 46 were given; on November 28th the take home exam was collected, on December fifth, the final exam was administered and on December 7th, Henry took part in the end of semester interview. I would like to point out that the height of the lines in the timeline in Figure 36 does not have any significance. Each specific height was used for placement convenience.

As outlined in the Figure 4.7, there were 46 homework problems, four tests - including the final exam, 8 mini conversations and the end of semester interview. As a result, I will be doing my analysis in three categories: homework problems, tests, and conversations. In the category homework, I will analyze Henry's homework solutions. In the category tests, I will analyze Henry's solutions on the course exams and in the category conversations, I will analyze the transcribed conversations that I had with Henry throughout the semester. All analysis will be done through the lens of APOS theory.

Throughout the semester, the underlying concepts covered were the $\varepsilon - N$ definition of the limit of a sequence, boundedness of a sequence (supremum/infimum), convergence of a sequence, Cauchy Sequence, tests for convergence of a series (Ratio, Comparison, Alternating, P-Series, and Root), calculating/proving the limit of a sequence, $\varepsilon - \delta$ definition of the limit of a sequence, continuity at a point and briefly, metric spaces. In my analysis, I will attempt to illustrate all of the stages of conception of understanding exhibited by Henry in different context in terms of the problem concept. That is, I will aim to cover the broad range of concepts, mentioned earlier, that were covered in the Analysis course. In addition, since I am tracking how Henry's understanding of proof concept developed throughout the semester, I will also note the proof method Henry used when he solved problems on the homework problems and exams. These methods include direct proof, proof by induction, proof by contraposition, indirect proof, and the converse of a statement.

Recall, there are four stages of APOS theory – Action, Process, Object and Schema. For my analysis, I will go through the APO stages of APOS theory exhibited by Henry based on the three set of data (homework, exams and conversations). However, there was evidence of Henry at the [7]pre-Action conception of understanding of APOS theory and thus that stage was introduced in my analysis. I will provide representative and illustrative samples of Henry's pre-Action, Action, Process and Object if it appears. Furthermore, illustrations of the various stages of APOS theory will be given from the three different data sources. First, I will give illustrations of each stage, if exhibited, on homework problems, then tests – including the final exam, then conversations – including the end of semester interview.

4.2.2.1 Homework problems

Homework problems are meant for students to learn and practice the concepts taught in class. As mentioned earlier, there were 46 problems assigned throughout the semester. Henry exhibited only the Action and Process conception stages of understanding in the work he did on the homework problems. That is, he did not exhibit the Object conception stage of understanding on work done on the homework problems. Starting with the Action conception stage of understanding, I will analyze Henry's work for homework problem one. In problem number one, Henry was prompted to:

Prove [8]*(use* $\varepsilon - N$*) that* $\lim_{j\to\infty} a_j = 3$, *where* $a_j = \frac{1}{j} + 3, j = 1, 2, 3, \ldots$

The phrase *"*$\varepsilon - N$*"* refers to the $\varepsilon - N$ definition of the limit of a sequence that states, if $\forall \varepsilon > 0,\ \exists\, N \in \mathbb{N}$ such that if $j > N \Longrightarrow |a_j - \alpha| < \varepsilon$ then $\lim_{j\to\infty} a_j = \alpha$. In other words, the

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[7] There was evidence illustrating Henry attempting to memorize steps or procedures of methods presented in the Analysis course but was unsuccessful. These instances were coded as the Pre-Action conception since he was not completely at the Action conception stage.

[8] In the Analysis course, students were presents with definitions, corollary, propositions, and theorems and prompted to prove statements using these definitions, corollary, prepositions, and theorems.

sequence $\{a_j\}$ converges to α. In Figure 4.8, I am presenting a comparison of Henry's solution for homework problem one and the instructor's solution to a similar problem done in class.

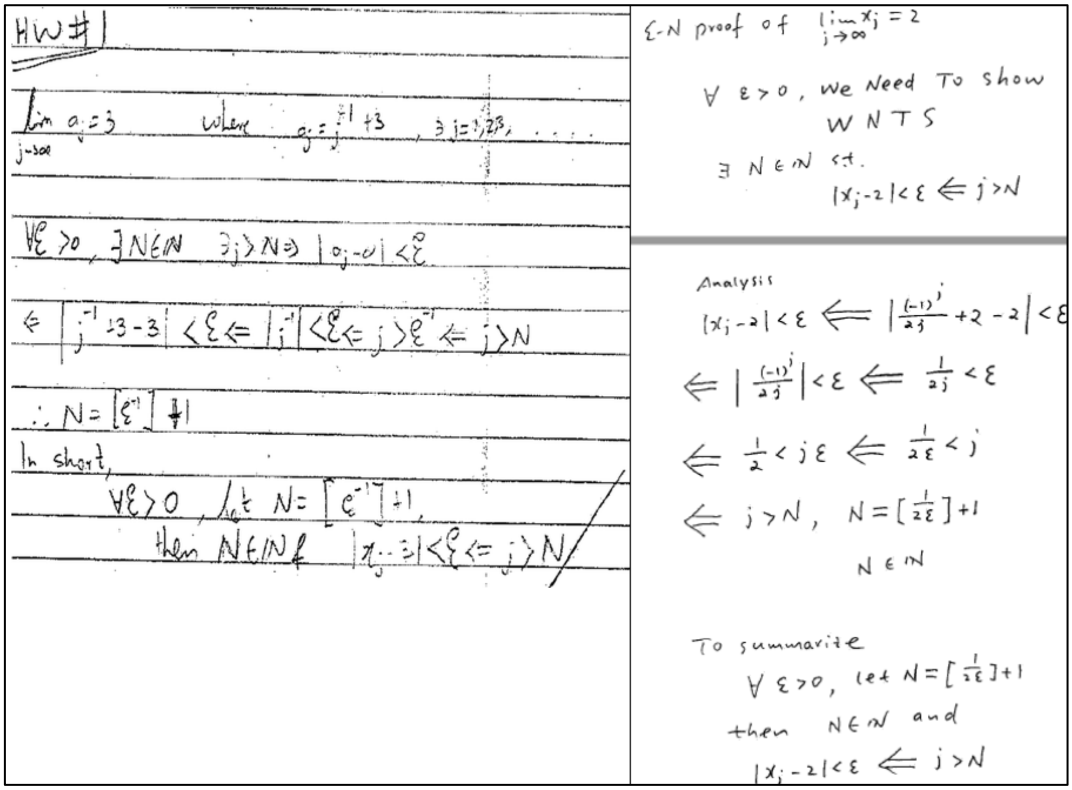


Figure 4.8 Comparison of Henry's solution for homework problem one and the instructor's solution to a similar problem done in class.

To the left of Figure 4.8 is Henry's solution of the proof for homework problem one, while to the right, is the instructor's proof of *given* $x_j = \frac{(-1)^j}{2j} + 2$, $\lim_{j\to\infty} x_j = 2$. Observe that Henry used a similar method, backward implication[9], $\Longleftarrow$, as done by the instructor. In this instance, Henry's proof of $\lim_{j\to\infty} \frac{1}{j} + 3 = 3$, and the instructor's proof of $\lim_{j\to\infty} \frac{(-1)^j}{2j} + 2 = 2$ are very similar. In fact, it appears as though Henry followed exactly the method/format of the

[9] This is a method of indirect proof, in which the instructor worked backward to find big N. Some instructors consider this as scratch to the proof. However, the instructor for the Analysis course considered it acceptable. This may be referred to an indirect method of proof.

instructor. Though he used his own words, he used cues from the format used by the instructor. For instance, in the beginning of the proof, Henry started with the phrase "$\forall \varepsilon > 0, \exists\, N \in \mathbb{N}$" as done by the instructor. Continuing with the proof, Henry used the backward implication method, inserting the arrows, $\Longleftarrow$, in similar points of the instructor's proof. Additionally, Henry's selection of N to be [10]$[\varepsilon^{-1}] + 1$ is replicative of the instructor's choice of N in his proof. Lastly, the final line in Henry's proof resembles the last line of the instructor's proof. For instance, Henry wrote "in short" instead of "to summarize," which was used by the instructor. In addition, Henry used the exact same wording the instructor used after the phrase "to summarize," tailored to his problem. As a result, this solution was coded as Henry exhibiting the Action conception stage of understanding of the APOS theory for the $\varepsilon - N$ definition of limit of a sequence using the method of indirect proof (Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007; Arnon, et al., 2014; Asiala, et al., 1996; Chamberlain & Vidakovic, 2020).

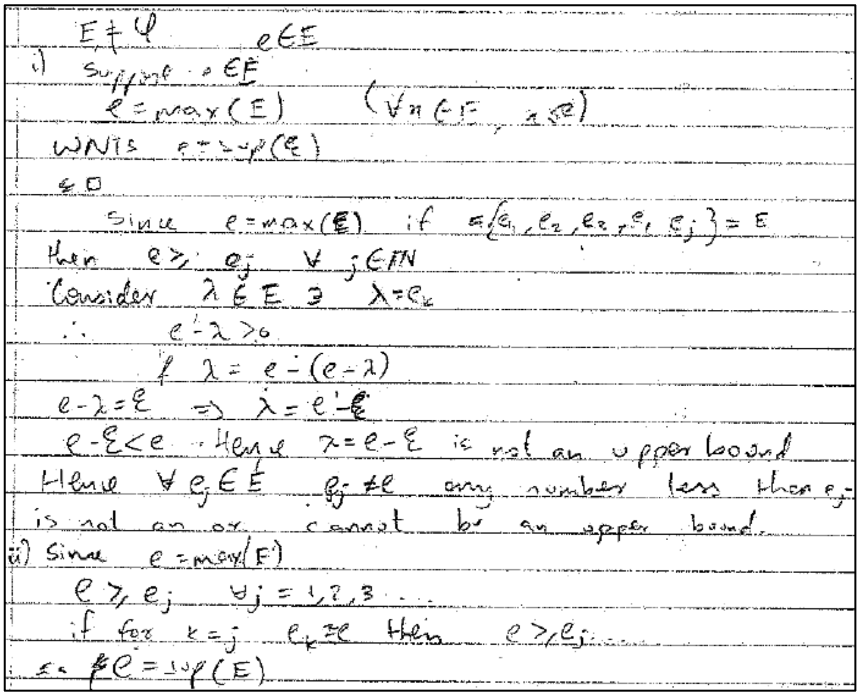
The following example comes from the work Henry did on homework problem 9 and illustrates a situation in which Henry exhibited the Process conception stage of understanding. An in depth analysis of Henry's solution is shown. In this problem, Henry was prompted to prove:

> *Given* $E \neq \emptyset, E \subseteq \mathbb{R}$, *suppose* $e \in E$, $e = max\, E$ $(\forall\, x\ \in E, x\ \leq e)$. *Show that* $e = sup\, E$.

This problem was assigned on September 5th. During the class, the instructor gave the definition of the supremum of a set E (denoted $sup\ E\ or\ Lub\ E$). Specifically, he noted that if [11]$E \neq \emptyset, E \subseteq \mathbb{R}$ and E is bounded above ($\exists\, M \in \mathbb{R}$ such that $x\ \leq M\ \forall\, x\ \in E$), then there exists some

[10] The greatest integer function, $[x]$ denotes the largest integer greater than or equal to x.
[11] The symbols in this statement are read, if the set E is a non-empty set and a proper subset of the set of real numbers…

β that is the least upper bound of E. Though the instructor gave hints for this problem, he did not demonstrate


$E \neq \varphi$ $e \in E$

i) Suppose $e \in E$

$e = max(E)$ $(\forall x \in E, x \leq e)$

WNTS $e = sup(E)$

$\leq \square$

Since $e = max(E)$ if $\{e_1, e_2, e_3, \ldots e_j\} = E$

then $e \geq e_j$ $\forall j \in \mathbb{N}$

Consider $\lambda \in E \ni \lambda = e_k$

$\therefore$ $e - \lambda > 0$

$\lambda = e - (e - \lambda)$

$e - \lambda = \varepsilon$ $\Rightarrow$ $\lambda = e - \varepsilon$

$e - \varepsilon < e$. Hence $\lambda = e - \varepsilon$ is not an upper bound

Hence $\forall e_j \in E$ $e_j \neq e$ any number less than e_j is not an or cannot be an upper bound.

ii) Since $e = max(E)$

$e \geq e_j$ $\forall j = 1, 2, 3 \ldots$

if for $k = j$ $e_k \neq e$ then $e \geq e_j \ldots$

so $e = sup(E)$

Figure 4.9 . Illustration of evidence showing what appears to be Henry exhibiting the Process conception stage of supremum of a set.

the solution to any similar problem. Nonetheless, Henry was able to come up with a complete and correct solution to the problem. Henry started the proof by stating the hypothesis and applying the definition of the maximum of a set. He then picked an arbitrary element, λ, in the set E, and used algebraic manipulation to show that λ, and thus all other elements of the set E, must be less than or equal to e. Hence $e = sup\, E$. This indicates that he interiorized the Actions of solving problems associated with boundedness of a set, done in class or on prior homework problems, into a Process. Furthermore, he was able to apply that Process to the solution of this problem. For this reason, this solution was coded as Henry exhibiting the Process conception stage of understanding for the supremum of a set for the APOS theory using the method of direct proof (Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007; Arnon, et al., 2014; Asiala, et al., 1996; Chamberlain & Vidakovic, 2020). My analysis of Henry's work on these two homework problems represents my analysis of all 46 homework problems from the semester.

4.2.2.2 Tests

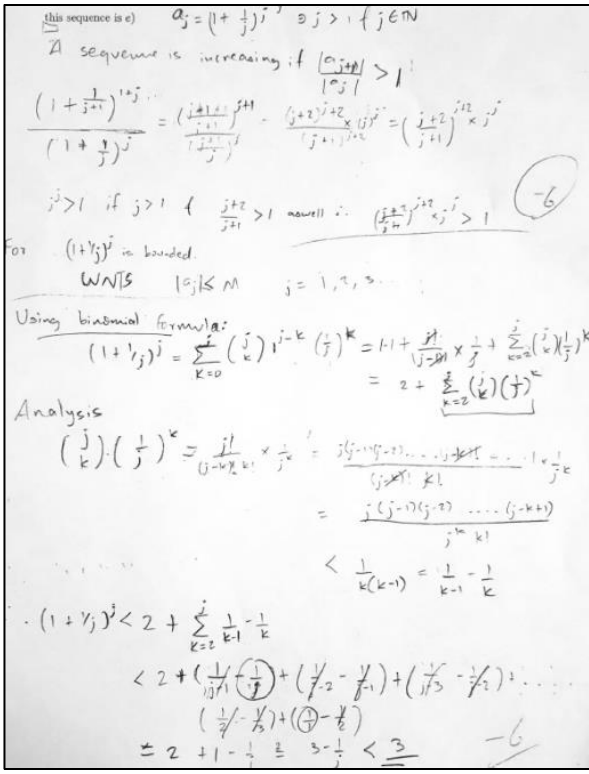
While homework problems are given for students to practice the concepts they learn in a given course, tests are administered to give students the opportunity to apply what they have learned in class or outside of class. On the exams given in the Analysis course, Henry exhibited the pre-Action, Action, and Process conception stages of understanding. As such, I will now analyze three of Henry's proofs from the exams he took in the Analysis course that illustrate each of these stages respectively. Exam three was a take-home exam. In this problem, Henry was prompted to:

Prove that the sequence $\{(1+\frac{1}{j})^j\}$ *is increasing and bounded.*

It is important to note that it is common for students to use outside sources such as the internet to assist them in solving difficult math problems. In this case, students at the Action conception stage would copy the solution of the problem they searched exactly the same or use cues from the solution they find to solve the problem. While a student at the Process conception stage understanding[12] would be able to formulate a proof based on his or her understanding of what he or she found online. With that said, I went to google and searched question two of the take-home exam. Upon searching question two, I found several solutions, one of which looked similar to Henry's solution. For comparison, Henry's solution and one of the solutions found on the internet are presented in Figure 4.10.

[12] It is important to note that Henry could have googled the solutions to the homework problems. There is no way to verify this. Hence why the phrase "appears to exhibit" is used in my analysis.

In the first part of the proof, it appears as though Henry was attempting to prove that *the sequence* $\{(1+\frac{1}{j})^j\}$ *is increasing.* Though I found a solution for this proof on the internet, Henry's proof did not reflect the proof I found online[13]. Observe that Henry was unsuccessful when he tried to show $(1+\frac{1}{j+1})^{j+1} \geq (1+\frac{1}{j})^j \Leftrightarrow \frac{|(1+\frac{1}{j+1})^{j+1}|}{|(1+\frac{1}{j})^j|} \geq 1$.



$$\left(1+\frac{1}{n}\right)^n = 1+1+\sum_{k=2}^{n}\binom{n}{k}\cdot\left(\frac{1}{n}\right)^k.$$

$$\binom{n}{k}\cdot\left(\frac{1}{n}\right)^k = \frac{n(n-1)\cdots(n-k+1)}{n^k}\cdot\frac{1}{k!} < \frac{1}{k(k-1)} = \frac{1}{k-1}-\frac{1}{k}.$$

$$\left(1+\frac{1}{n}\right)^n < 2+\sum_{k=2}^{n}\left(\frac{1}{k-1}-\frac{1}{k}\right) = 3-\frac{1}{n} < 3.$$

Figure 4.10 Henry's proof of problem two on the take home exam (left) and a similar proof found on the internet (right).

In this instance, it appears as though Henry was not even at the Action conception stage of understanding for showing a sequence is increasing using the method of direct proof. For that reason, this was coded as the pre-Action conception of understanding of APOS for increasing sequence. Moving to part two of the question where Henry proved that the sequence is bounded, in my search, I found a similar solution to Henry's proof on the internet, shown on the right of

[13] There is a possibility that this version of Henry's proof was online, but it did not appear in my search.

Figure 4.10. The two proofs have very similar reasoning. Specifically, the use of the binomial expansion of $(1+\frac{1}{j})^j$. This illustrates Henry mirroring the method he possibly found online. Thus, this proof was coded as Henry appearing to exhibit the Action conception stage of understanding of a sequence being bounded using the method of direct proof.

Next, I will provide detailed analysis of Henry's work on problem six on the exam. Question six from the final exam that prompted to:

Prove (use $\varepsilon-\delta$) that $f(x)$ is continuous at $x=1$ for the function

$$f(x)=\begin{cases}3x \ \ if \ \ x \ \ is \ \ rational\\ \frac{3}{x} \ \ if \ \ x \ \ is \ irrational.\end{cases}$$

It is significant to point out that this problem was not on any of the previous homework problems nor tests. However, the instructor did a similar problem in class on November 12th

proving that the function,

$$f(x)=\begin{cases}x^2 \ \ if \ \ x \ \ is \ \ rational\\ 3x \ \ if \ \ x \ \ is \ irrational\end{cases}$$

is continuous at $x=0, and\ x=3$ *and not continuous at every* $x\neq 3\ and\ x\neq 0$.

I am presenting both the instructor's proof and Henry's proof in Figure 4.11. For the purpose of comparing the methods used by Henry and the instructor, for the instructor's proof, I am only showing a portion of the instructor's proof that related to proving that the function *is continuous at* $x=0, and\ x=3$. Looking at the two proofs we see that the method used by Henry and the instructor were very different. Furthermore, in the instructor's proof, he did not use the $\varepsilon-\delta$ definition. Henry on the other hands, did use the $\varepsilon-\delta$ definition in his solution (the question required him to).

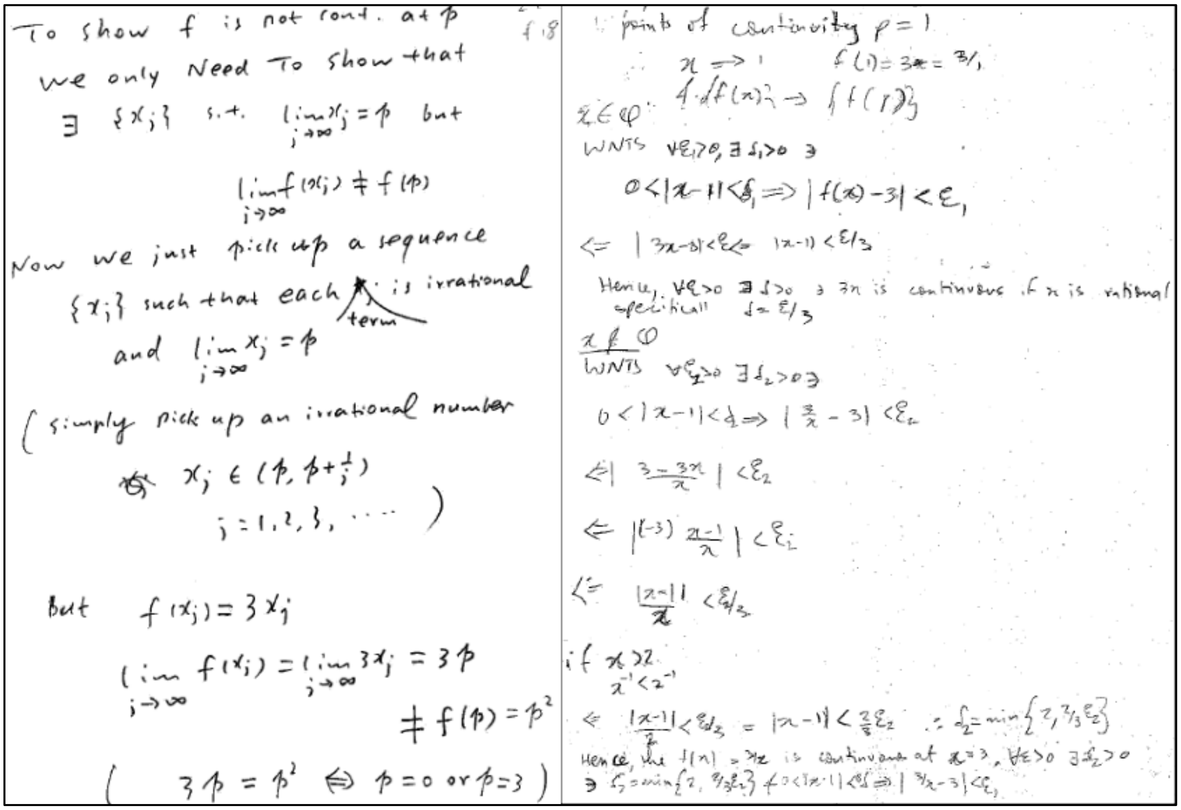


Figure 4.11 Illustration of a comparison of the instructor's proof of a problem similar to question six on the final exam (left) and Henry's proof of problem six of the final exam (right).

Additionally, it is worth noting that the only problem presented in the course that was similar to problem six on the final exam in the one presented in Figure 4.11. That is to say, Henry would have to exhibit at least the Process conception stage of APOS to be able prove this statement correctly using the $\varepsilon - \delta$ definition. Thus, this solution was coded as Henry's exhibiting the Process conception stage of APOS for using the $\varepsilon - \delta$ definition to prove continuity at a point using the method of indirect proof (Arnon, et al., 2014; Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007; Asiala, et al., 1996; Chamberlain & Vidakovic, 2020). What has been presented in this section is illustrative of my analysis of all other exam questions.

4.2.2.3 Conversations and end of semester interview

The conversations I had with Henry throughout the semester served as a medium to get an idea of how the course was going for Henry. The end of the semester interview was used to triangulate the data from Henry's homework problem and exams, as well as the conversations we had. Similarly, to what was done for the homework problems and tests, I will present illustrations of Henry's exhibition of the stages that appeared in the excerpts of the transcribed conversations and end of semester interview. In this section, I will also include think aloud solutions to four problems Henry answered during the end of semester interview. During the interview Henry explained his solutions for the final exam. In particular, for problem one part one of the final, he exhibited the pre-Action conception of APOS theory. In this question, Henry was prompted to:

Use $\varepsilon - N$ to prove that $\lim\limits_{j\to\infty} \frac{j^2+2}{2j^2-j+5} = \frac{1}{2}$.

His solution is presented in Figure 4.12. In addition, an excerpt from our conversation is below.

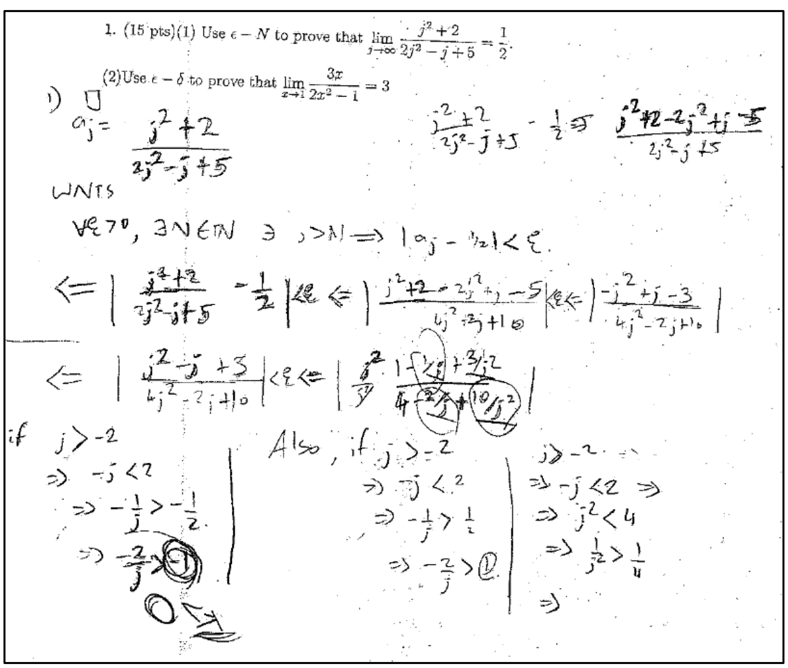


Figure 4.12 Illustration of Henry's proof of Use $\varepsilon - N$ to prove that $\lim\limits_{j\to\infty} \frac{j^2+2}{2j^2-j+5} = \frac{1}{2}$ *on the final exam.*

I choose to include my questions and responses to give the read a clear overview of Henry's explanation of his solution. In the excerpt, I am the "Grad Student":

Henry: *So this is…I got stuck here…so what happened…I tried finding…ahm…so I got this expression…I ration…ahmm…ahmm…I brought it under…you know…one fraction…ahm…factored the* j^2 *out…tried finding a bound for these…. all of these terms four j terms…this this this and this… I would have to find…so if…as…as…as…if I found a bound for this…an upper bound…I don't get a reasonable lower bound for this and for this…cause these are squared terms… which is why I was running around in circle trying to figure out how to do this…but didn't work out…I wasn't able to do it…this problem."*

Grad Student: Wait a minute…before you ah…this method did you see it before?

Henry: *Yeah yeah…*

Grad Student: He [the instructor] did it in class before…did he do this problem or something similar in class or…

Henry: *Not this problem…like some…something similar to this…*

Grad Student: Similar…

Henry: *Maybe we didn't have as many j terms in the denominator…*

Grad Student: *mhm…*

Henry: *Or power j terms…*

Grad Student: mhm…

Henry: *Ah especially ones with negative round…cause you see how I have to flip the inequality by multiplying by a negative…*

Grad Student: *mhm*

Henry: *And that's what was making all the problems…you see this is greater…this is a lower bound…*

Grad Student: *mhm…*

Henry: *I need an upper bound…I need something that's k…that's greater than whatever…this messy stuff here…I just wasn't getting it…*

Grad Student: *mhm…*

Henry: *So yeah…*

Grad Student: *So, he kinda did something like this in class…so you were trying to remember the technique…*

Henry: *Yeah, I did homework as well…right I practiced…this a lot so…*

Grad Student: *mhm…*

Henry: *Yeah…*

From this excerpt, we see that Henry was not able to do this problem on the final exam because the problem varied from the ones he "*practiced…a lot.*" It appears as though Henry attempted to memorize his solutions from the homework problems by writing the solutions repeatedly but was unable to recall the solution to this particular problem (or problems similar). This is a clear indication of the pre-Action conception of understanding of APOS theory. Because he was unable to recall the solution to complete and correctly solve the given problem, this portion of the excerpt was coded as Henry exhibiting the pre-Action conception of understanding of APOS theory for using the $\varepsilon - N$ definition to prove the limit of a sequence using the method of indirect proof.

Next, I will present an illustration of Henry exhibiting the Process conception stage of understanding. This illustration is also from the end of semester interview and a solution Henry explained from the final exam. It is important to note that Henry was not able to complete the solution to this problem on the final exam. This question was also a question Henry was asked to solve out loud. In this question, Henry was prompted to:

a. State the definition that a sequence $\{a_j\}$ in $\mathbb{R}$ is a Cauchy Sequence.

b. Prove that the sequence $\{x_j\}$ defined by $x_j = \int_1^j \frac{sinx}{x} dx$, $j = 1,2,3, \ldots$, is a Cauchy sequence.

First, I am presenting the excerpt from Henry's proof of this problem aloud. His rendition of the proof is as follows:

There exists j, k…there exists j and k contained in the naturals…such that…j is…if j is greater than k…is greater than some…no….for…there exists an n contained in the naturals such that j is… j is greater than k is greater than n…tells me a_j minus a_k is less than epsilon…such as…that's a Cauchy Sequence…prove that the sequence is a Cauchy Sequence…ahm…x_j equals integral one to j…sine of x over xdx….where j is contained in the naturals…is…we need to show…for all epsilon positive…there exists an n in the natural such that if j is greater than k is greater than n…I know…x…j minus x…k is less than some number epsilon…so move forward…x_j so that's one to j, sine of x over x dx is infimum? of…sine of x dx…k…then you split this…from one to j…then you go from one to k…sine of x dx over x…plus k to j…sine of x dx over x minus sine…one to k…sine of x…d…over x dx…so out with this bad boy here…k_j…now…if that is the case…I can integrate this function by parts…I will bound this…the integral…sum …what…hmm…u equals one over x…dv equals sine x…therefore you have…u integral of sine is negative cosine of x…v is 1 over x…goes from k to j…minus the integral of…hmm…v which is again…negative cosine of x dx…v…du…derivative of that is ahm…

negative one over x^2 …making negative positive x^2 …so you can easily calculate that…minus cosine…so I can flip this inequality and I can go from sine of…sine of…cosine k over k…hence cosine of j over j…minus…the integral of cosine xdx over x and this thing goes from k to j…can't forget that…now…we can use…we can…triangular inequality…equal it to thing…absolute value of cosine of x…and then this was supposed to be…plus minus here…yep…plus and then minus over here….yep so I can add everything…I don't wanna see negatives there….cosine of k…over k…plus …cos…absolute value of negative cosine j over j…plus…I can bound that integral…so essential…can I see an eraser? so it's less than or equal to…I can do that in the next step...for now let's just bound this…the absolute value of the integral of cosine x over x from j to k…continuing forward….we bound all of these…cosine is contained…the absolute value of cosine is less than or equal to one…so I can bound all of these cosines from one to j…one…plus one over k…and then I can…bound…this integral inside…which would still have to be less than or equal to that value…so I'm not messing up my inequality…and then I keep going…one over k…and so if I …I can bound this between…again…sine of x is again bounded by 1 at the top so I can bound that…sin absolute value of x dx…one to j plus one over k…integral of one over x…negative one over x^2 ...goes from j to k…you have one over j…hmmm…is that right…

*Flips page to look at entire problem**

*Long pause as he reviews his previous work**

Hmm…a mistake here…so that's that…yep…this was supposed to be an x^2 …no wonder…it's supposed to go down to x…not…yeah…so x^2 …x^2 …cause ln of x…was not…make sense…yeah…so that's where I am…this will be x^2 …put an x^2 where there *needs to be one…okay…the integral of that is negative one over x…goes from j to k…now…which is plus one over j…minus one over k…this cancels out…so I'm left with two over j…is less than epsilon…which means j equals two over epsilon…j is greater than two epsilon cause I flip the inequality…j half is less than one over epsilon and I multiply by two…then yeah j is greater than epsilon…which is correct…therefore n equals …the function of two over epsilon plus one hence…for all epsilon greater than zero…there exists an n contained in the naturals such that j is greater than k…is greater than n…tells me…a_j minus a_k less than epsilon…more specifically…n equals the greatest integer function of two over epsilon plus one…that's one hundred percent right…okay…you can look at that if you want.*

Henry's written work is shown in Figure 4.13. Analyzing Henry's "think aloud," there are

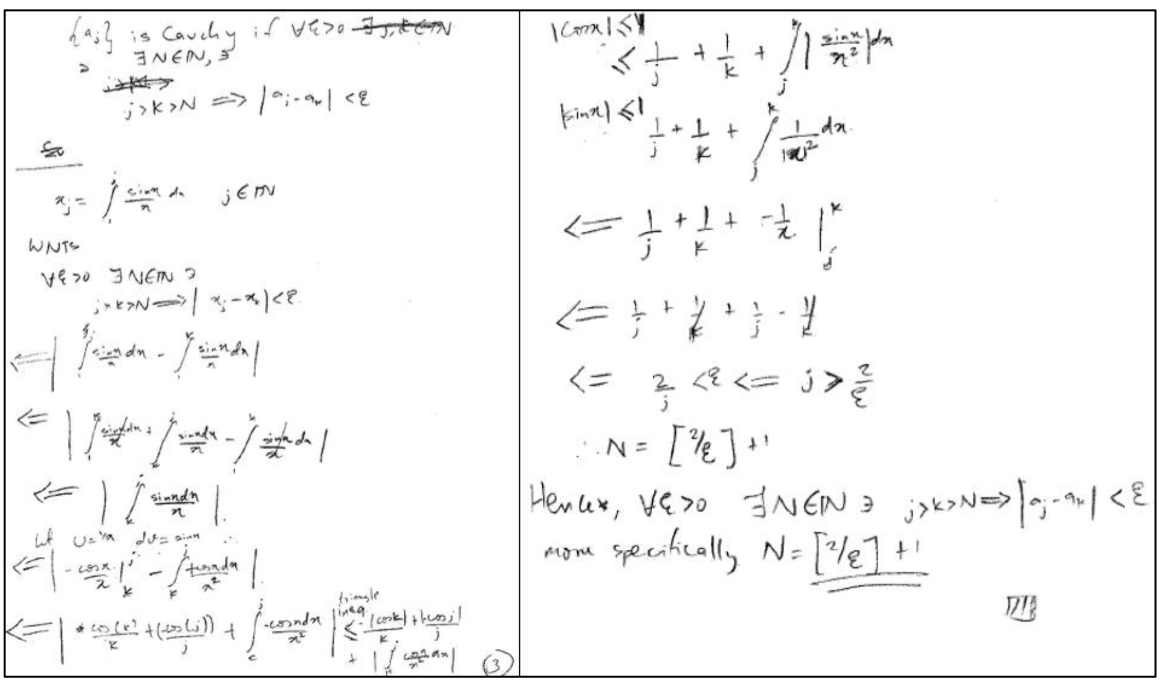
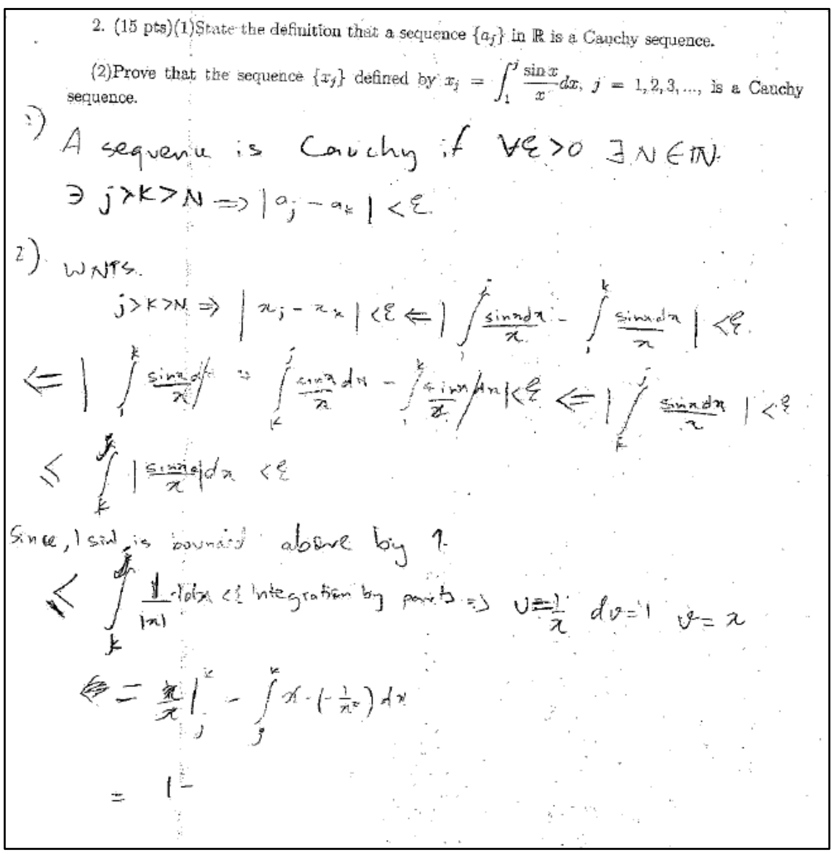


Figure 4.13 Illustration of Henry's proof of $x_j = \int_1^j \frac{\sin x}{x}\,dx$*, j=1,2,3,…, is a Cauchy during the end of semester interview.*

indications where it appears as though Henry was recalling the proof from memory. Particularly, when he said, *"yep so I can add everything…I don't wanna see negatives there."* Henry noting that he did not "want to see negatives" was an indication that he knew what the solution should be. Specifically, that there should not be negative values in the proof. This is interpreted as Henry recalling the proof from memory. Furthermore, as he continued the proof, he noticed that he made an error. For instance, he noted *"a mistake here…so that's that…yep… this was supposed to be an* x^2 *…no wonder…it's supposed to go down to x."* Noting what the proof was "supposed to be" further indicates that Henry memorized the solution to the proof. To confirm that this was an illustration of the Action conception stage of understanding, I am also presenting an excerpt from the portion of the interview where Henry explained why he was unable to solve this problem on the final exam but was able to solve it during the end of semester interview. I am also presenting his proof of this problem done on the final exam in Figure 4.14.



2. (15 pts)(1)State the definition that a sequence $\{a_j\}$ in $\mathbb{R}$ is a Cauchy sequence.

(2)Prove that the sequence $\{x_j\}$ defined by $x_j = \int_1^j \frac{\sin x}{x}dx$, $j = 1, 2, 3, \ldots$, is a Cauchy sequence.

1) A sequence is Cauchy if $\forall \varepsilon > 0\ \exists N \in \mathbb{N}$ $\ni j > k > N \Rightarrow |a_j - a_k| < \varepsilon$.

2) WNTS.

$j > k > N \Rightarrow |x_j - x_k| < \varepsilon \Leftarrow \left| \int_1^j \frac{\sin x}{x} dx - \int_1^k \frac{\sin x}{x} dx \right| < \varepsilon$

$\Leftarrow \left| \int_1^k \frac{\sin x}{x} + \int_k^j \frac{\sin x}{x} dx - \int_1^k \frac{\sin x}{x} dx \right| < \varepsilon \Leftarrow \left| \int_k^j \frac{\sin x}{x} dx \right| < \varepsilon$

$\leq \int_k^j \left|\frac{\sin x}{x}\right| dx < \varepsilon$

Since, $|\sin|$ is bounded above by 1.

$\leq \int_k^j \frac{1}{|x|} dx < \varepsilon$ Integration by parts $\Rightarrow u = \frac{1}{x}$ $dv = 1$ $v = x$

$= \frac{x}{x}\Big|_j^k - \int_j^k x \cdot \left(-\frac{1}{x^2}\right) dx$

$= 1$

Figure 4.14 Illustration of Henry's proof of $x_j = \int_1^j \frac{\sin x}{x} dx$*,*
j=1,2,3,…, on the final exam.

The excerpt of Henry explaining his proof aloud is below.

Henry: *Cauchy was pretty much definition…*

Grad student: *mhm…*

Henry: *Ahm…a_j is Cauchy if there exists for all epsilon positive there is an n in the natural such that j is greater than k is greater than n implies a_j minus a_k is less than epsilon…*

Grad student: *mhm…*

Henry: *Then this thing [pointing on the solution in Figure 43]*…like I said…I was doing it this morning because it hurt me so much that I wasn't able to do on the final…so yeah…I pretty much memorized it…which is why when I knew…I came here…no wait…that's ln of x so I need to go back and check…*

Grad student: *mhm…*

Henry: *I had a mistake here…that should have been x^2…so I fixed all of it…I came to the right thing…*

Grad student: *So, what did you do…you came home after the final and memorized everything…*

Henry: *No, I didn't have time to do that… 'cause I had a final right after…. this morning…right…*

Grad Student: *mhm…*

In this particular excerpt, Henry admitted that he memorized the solution to the problem the morning before he came to the end of semester interview[14]. This further confirmed that the excerpt from the think aloud proof was an exhibition of the Action conception stage of understanding of Cauchy Sequence using the method of direct proof (Arnon, et al., 2014;

[14] Henry was not asked to do proofs allowed in end of semester interview for the Bridge to Higher course nor did he know I was going to ask him to solve problems during the interview. He memorized the solution on his own.

Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007; Asiala, et al., 1996; Chamberlain & Vidakovic, 2020).

Continuing with my analysis, I am presenting an excerpt from conversation four that Henry and I had on September 24th. During the conversation, Henry was asked about his study habits. Specifically, how he knew he understood the material he was studying. His response was,

> *"So I think a good pointer for me to know if everything is cemented in my head is…one…this is something I do all the time…I keep repeating some of the facts in my head….and it's like if I can recall the logic without looking at a piece of paper…even when I'm like…I don't know…doing my dishes or walking…that's when I know…okay…that I've understood the topic."*

In this quote, we see that Henry was noting that he studied by trying to mentally recall the information he was studying without writing down the information on paper. Particularly, Henry's reference of having the material "*cemented in [his] head*" correlates to the interiorization of an Action into a Process. Furthermore, being able to Process the material without needing to explicitly write it down on paper is an indication of Henry exhibiting the Process conception stage of understanding in APOS theory[15]. Thus, this excerpt was coded as Henry exhibiting the Process conception stage of APOS for the concepts he was referring to (unclear at this point). Henry continued to say,

> *"But as far as everything tying…together…ah if I can just look at any problem that's given to me…and if I can trace back…I can solve it….but more importantly I can trace my own logic for that…correctly… aah…that's when I know things kind of tie together."*

[15] This is in relation to whatever concept Henry was attempting to study at this point. In this instance, he did not specify.

Here Henry was continuing to describe how he studied. Tracing back "*his own logic*" of a problem in his head may be interpreted as Henry reversing the Process of interiorization. For instance, going from the Process (or Object) conception stage of a concept back to the Actions the Processes (or Objects) came from. In this case, he would have to be at the Process stage to be able to revert back to the Actions. Hence, this too was coded as the Process conception stage of understanding for the concepts Henry studied up to this point of the course.

In the last illustration for this type of data, I am presenting an excerpt from one of the proofs Henry did out loud during the end of semester interview. For this particular problem, Henry solved:

Prove that if x is odd, then $x + 1$ is even.

Henry's solution to this problem is shown in Figure 4.15. In this particular instance, Henry was asked to explain the method he used in this proof and why he chose to use that particular method.

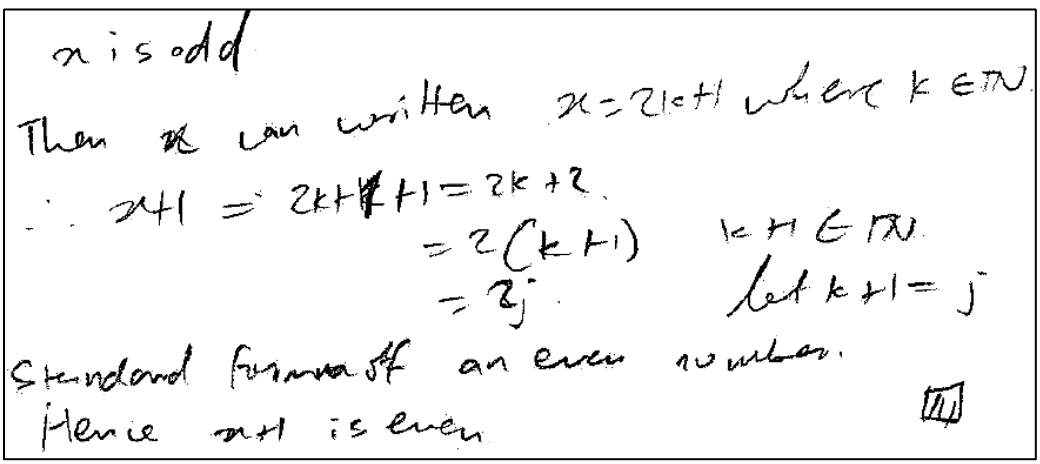


Figure 4.15 Illustration of Henry's proof of x is odd, then x+1 is even.

The portion of the transcribed data with Henry's explanation is below:

Grad student: *mhm…first you could tell me what is this method and why you choose this way…*

Henry: *Ahm…I think this is pretty much standard parity proofs…where you…if x is odd you pick 2k +1…if x is even you pick 2k…and then you go where you wanna with* *that…but that's pretty much standard parity proofs…ah…proofs based on standard parity laws…ahm…with the [direct proof]…*

Grad student: *What kind of proof is this?*

Henry: *Ah…what kind of proof this is…*

Grad student: *mhm…*

Henry: *Ahm…direct proof…*

Grad student: *Could you do it any other way?*

Henry: *Ahm…let's see…if x is odd and x plus one is even…hmm…yeah you could…you could do like contrapositive…but that would kinda be unnecessary because this is pretty straight forward…*

Grad student: *mhm…*

Henry: *You could say like…x plus one is not odd…so like x is odd…*

Grad student: *You can try this one… [Henry appeared to need a pencil so I was offering one]**

Henry: *No that's fine…*

Grad student: *Okay…*

Henry: *x is odd…implies x plus one even right…*

Grad student: *mhm…*

Henry: *So, contrapositive would be x plus one not even…x plus one is odd…therefore x must be even…*

Grad student: *mhm…*

Henry: *If x must be even…and then you go from there…*

Grad student: *mhm…*

Henry: *If x plus one is even…you…x plus k plus one…ahm…ah…k plus one…again x…you know since this is odd you can write this as 2k+1…you do some quick algebra…you know…if x equals 2k…standard for of an even integer…therefore x must be even…*

Grad student: *mhm…*

Henry: *But…it's kinda over kill…*

Grad student: *mhm*

Henry: *You can do a direct proof…*

In the first part of Henry's response from the excerpt, Henry appeared to be thinking of the method of proof by induction as a method to which some Actions could be applied. Specifically, in this illustration, he appeared to de-encapsulated the Object (direct proof) and used the Process of applying the induction method for which certain conditions are satisfied. Furthermore, he later acknowledged that he solved the problem directly. When I asked Henry if he could have done the proof another way, his response of*, "yeah…you could do like contrapositive,"* indicated that he also viewed proof by contraposition as an Object to which this problem could be applied. Further in the excerpt, Henry de-encapsulated the Object, proof by contraposition, back into a Process. Moreover, this may be translated as a comparison of proof by contraposition and direct proof. A comparison of methods indicates that Henry encapsulated the Process of proof by contraposition into an Object that may be applied to solving different problems. For aforementioned reasons, this excerpt was coded as Henry exhibiting the Object conception stage of understanding of even and odd parities for the methods of proof by induction, direct proof and proof by contraposition (Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007; Arnon, et al., 2014; Asiala, et al., 1996; Chamberlain & Vidakovic, 2020).

Furthermore, this was the only instance where Henry clearly exhibited the Object conception stage of understanding for the semester and it was in reference to a concept covered in the Bridge to Higher Math course. Similar to my analysis of the work Henry did on the homework problems and on the exam questions, my analysis of the above excerpts represents my analysis of all of the conversations I had with Henry including the interview.

My analysis of Henry's work on each homework problem, test question, conversation/interview excerpts and solution to the end of interview problems is summarized in Tables 4.11 and 4.12. The coding for the summary tables for the Analysis course, is reflective of the coding done in Tables 4.9 and 4.10. That is, to the right of the table in a timeline tracking when each piece of data was analyzed and at the top of the tables is a list of the concepts that surfaced during my analysis. A list that includes all homework problems, exams and think aloud proofs for the Analysis course is presented in the appendix. To distinguish each of the three data sets in the table, I used three different font styles. As with Tables 4.9 and 4.10, for the data analyzed from the homework problems, the text in the table has a regular font but is bold. For the data analyzed from the exams, the text is in all caps and bold. For the conversation and end of semester interview on the other hand, the text was normal font but italic. Each stage of APOS was color coded. For instance, orange was used to illustrate when Henry appeared to exhibit the pre-Action stage; yellow was used to denote for when Henry appeared to exhibit the Action conception stage of understanding and green was used to denote when Henry showed evidence of the Process conception stage of understanding. There were specific days that Henry appeared to exhibit more than one stage of APOS for a particular concept. These instances were coded accordingly. The color gray was used to signify when Henry appeared to exhibit both the pre-Action and Action conception stages and the color pink correlated to when Henry appeared to exhibit both the pre-Action and Process conception stages. Lastly, the color teal was used to highlight when Henry exhibited both the Process and Object stages of conception of understanding and the color blue was used to represent the Action and Process stages together,

Table 4.11 Part 1 of 2 of a detailed summary of the timeline explaining Henry's development of understanding for particular concepts through the lens of the APOS theory in the Analysis course.

	Epsilon big N to find limit of sequence	Arithmetic Inequality	Boundedness of a sequence (sup/inf)	Convergence of a sequence	Increasing sequence	Limit of a sequence	Cauchy Sequence	Ratio Test for convergence	Comparison Test for convergence
20-Aug	Action Indirect Proof								
22-Aug	Action Indirect Proof	Process Direct Proof							
27-Aug	Process Direct Proof								
29-Aug			*Process* *Contradiction*						
5-Sep			Process Direct Proof	Process Direct Proof					
10-Sep	*Action* *N/A*		*Action* *N/A*	*Action* *N/A*	Process Induction	Action Direct Proof			
17-Sep	Action Indirect Proof					Process Indirect Proof	Action Indirect Proof		
24-Sep	*Action | Process* *N/A*		*Action | Process* *N/A*	*Action | Process* *N/A*			*Action* *N/A*		
28-Sep	PRE-ACTION | PROCESS Direct Proof		PRE-ACTION DIRECT PROOF	PRE-ACTION DIRECT PROOF			PRE-ACTION DIRECT PROOF		
8-Oct	*Pre-Action | Action | Process* *N/A*		Action Indirect Proof						
15-Oct				Process Direct Proof					Process Direct Proof
17-Oct								Action Direct Proof	Action Direct Proof
22-Oct				Action Direct Proof					
24-Oct	Action Indirect Proof								
5-Nov	Action Indirect Proof								
7-Nov	PRE-ACTION DIRECT PROOF		PRE-ACTION DIRECT PROOF						PRE-ACTION DIRECT PROOF
26-Nov									
28-Nov	PROCESS DIRECT PROOF	ACTION INDIRECT PROOF	ACTION DIRECT PROOF	PRE-ACTION DIRECT PROOF	PRE-ACTION DIRECT PROOF				
5-Dec	ACTION DIRECT PROOF			ACTION DIRECT PROOF			ACTION N/A		
7-Dec	*Pre-Action | Action* *Direct Proof*		*Pre-Action | Action* *Direct Proof*	*Pre-Action | Action* *Direct Proof*		*Pre-Action | Action* *Direct Proof*	*Action* *Direct Proof*		

while the color blue was used to represent the Action and Process stages together. I made it a point to note the proof techniques Henry used, where applicable, in each analysis. The phrase N/A was used mainly for instances where Henry and I had conversations and I was not able to associate a method of proof to his response.

Table 4.12 Part 2 of 2 of a detailed summary of the timeline explaining Henry's development of understanding for particular concepts through the lens of the APOS theory in the Analysis course.

	Alternative test for convergence	Epsilon delta definition of limit of a sequence	P-Series Test for converge	Root Test for convergence	Continuity at a point	Metric space	Even and odd parities	Modular arithmetic	General Concepts
20-Aug									*Action* *N/A*
22-Aug									
27-Aug									
29-Aug									*Action* *N/A*
5-Sep									
10-Sep									
17-Sep									
24-Sep									
28-Sep									
8-Oct									
15-Oct									
17-Oct									
22-Oct									
24-Oct	Action Direct Proof								
5-Nov									
7-Nov		ACTION DIRECT PROOF	ACTION DIRECT PROOF	PRE-ACTION DIRECT PROOF	ACTION DIRECT PROOF				
26-Nov					Action Direct Proof	Action Direct Proof			
28-Nov		PROCESS DIRECT PROOF			PROCESS DIRECT PROOF				
5-Dec		PROCESS INDIRECT PROOF			PROCESS INDIRECT PROOF				
7-Dec		*Action* *Direct Proof*			*Pre-Action* *Direct Proof*		*Process|Object* *Direct|Contraposition|Converse*	*Process* *Direct Proof*	*Action* *N/A*

Table 4.11 reports on Henry' development of understanding for $\varepsilon - N$ definition, arithmetic inequality, boundedness of a sequence (supremum/infimum), convergence of a sequence, showing a sequence is increasing, finding the limit of a sequence, Cauchy Sequence and using the Ratio Test, and Comparison Test for convergence. On the other hand, Table 4.12 illustrates Henry's development of the concepts Alternating test for convergence, $\varepsilon - \delta$ definition of a limit, P-Series test, Root test, continuity at a point, metric space, even and odd parities, modular arithmetic and general concepts.

Looking closely at Henry's development of understanding for $\varepsilon - N$ definition, based on my analysis, it appears that Henry exhibited the Action stage of understanding, using the method of indirect proof, in the beginning of the semester on August 20th. As such, this text was highlighted yellow in regular font and bold, indicating that this analysis was done of the work Henry did on a homework problem. Henry appeared to advance to the Process conception stage of understanding using the method of direct proof thereafter on August 27th. To illustrate this in Table 4.11, this text was highlighted green in regular font and bold (analysis of a homework problem). However, owing to the fact that learning is not linear, Henry appeared to revert back to the Action conception stage on September 10th. To illustrate this in the table, this text was highlighted yellow but italic indicating that this analysis was done of the excerpt from a conversation Henry and I had. Thereafter, it appears as though Henry was going back and forth between the pre-Action, Action and Process conception stages of understanding. For example, on September 24, from the analysis of a conversation, Henry appeared to exhibit both the Action and Process conception stages of understanding using the method of direct proof. Thus, that text in the table was highlighted blue and italic. Since on September 28 Henry when Henry took exam one, he exhibited the pre-Action and Process stage of understanding for the $\varepsilon - N$ definition, the text is in bold capital letters. From my analysis of one of our conversations, Henry exhibited the pre-Action, Action, and Process conception stage of understanding. As a result, this text was highlighted red and italic.

By the end of the semester, it appears as though Henry did not pass the Action conception stage of understanding for $\varepsilon - N$ definition. This is indicated in the Table 4.11 with the orange highlight and bold caps text for the analysis of the take home exam indicating that he was exhibiting the pre-Action conception using direct proof. As the semester ended, during the end of semester interview, Henry exhibited the pre-Action and Action conception stage of understanding and so the texted was highlighted gray in italic. These colors and font styles were used throughout the table to organize the analysis of Henry's work on the homework problems, exam and conversations/end of semester interview. A quantified summary tracking Henry's developmental understanding throughout the semester is broken down in the histogram in Figure 4.16.

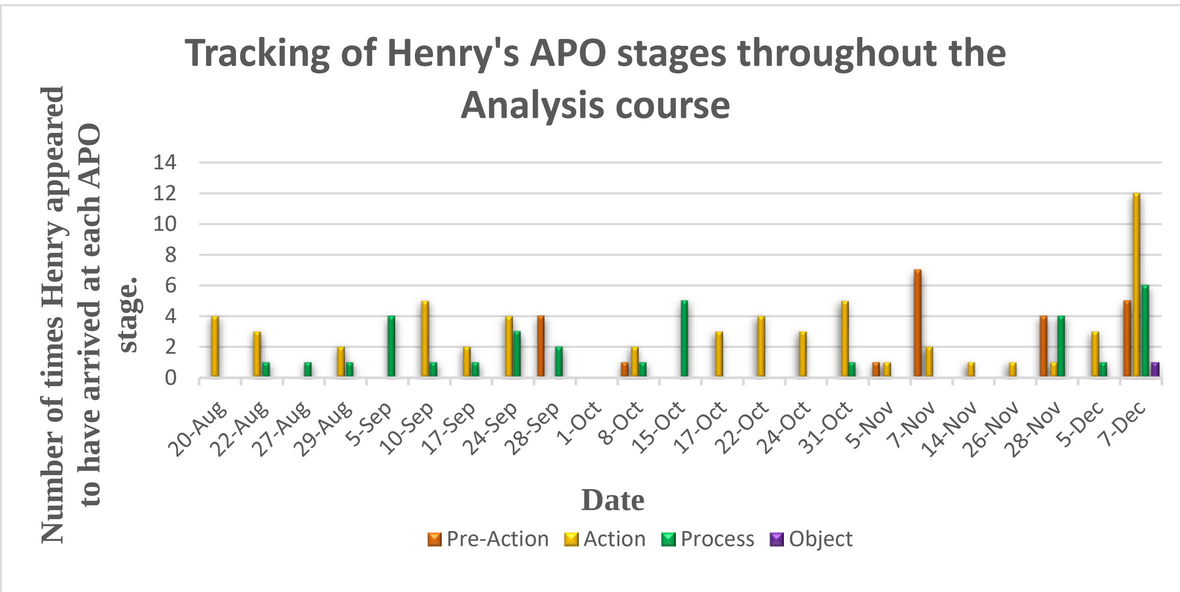


Figure 4.16 Frequency of each of the APO stages Henry exhibited at a particular date in the Analysis course.

Specifically, on August 20th, Henry appeared to exhibit the Action stage four times; on August 22nd, Henry appeared to exhibit the Action stage three times and the Process stage once; on August 27th, Henry appeared to exhibit the Process stage once; on August 29th, Henry appeared to exhibit the Action stage twice and the Process stage once; on September fifth, Henry appeared to exhibit the Process stage four times; on September tenth, Henry appeared to exhibit the Action stage five times and the Process stage once; on September 17th, Henry appeared to exhibit the Action stage twice and the Process stage once; on September 24th, Henry appeared to exhibit the Action stage four times and the Process stage three times; on September 28th, Henry appeared to exhibit the pre-Action stage four times and the Process stage two times; on October 1st, Henry did not appear to exhibit any of the stages of APOS; on October 8th, Henry appeared to exhibit the pre-Action stage once, the Action stage twice and the Process stage once; on October 15th, Henry appeared to exhibit the Process stage five times; on October 17th, Henry appeared to exhibit the Action stage three; on October 22nd, Henry appeared to exhibited the Action stage four times; on October 24th, Henry appeared to exhibit the Action stage three times; on October 31st, Henry appeared to exhibit the Action stage five times and the Process stage once; on November 5th, Henry appeared to exhibit the Action stage once; on November 7th, Henry appeared to exhibit the pre-Action seven times, and the Action stage two times; on both November 14th and November 26th, Henry appeared to exhibit the Action stage once; on November 28th, Henry appeared to exhibit the pre-Action stage five times, the Action stage once and the Process stage four times; on December 5th, Henry appeared to exhibit Action stage three times and the Process stage once and lastly on December 7th, Henry exhibit the pre-Action stage five times, the Action stage twelve times, the Process stage six times and the Object stage once.

4.2.3 Summary of chapter

In this chapter I analyzed Henry's homework, quizzes, exams and the transcripts of my conversations with him, as well as the end of semester interviews from both the Bridge to Higher Math course and the Analysis course using the lens of the SRL conceptual framework and the APOS theoretical framework. I also analyzed Henry's responses of the SRL questionnaire. It appeared as though Henry's self-regulation fluctuated throughout both courses. However, he was more self-regulated in the Bridge to Higher Math course potentially leading to a better grade in the course than the Analysis course. In addition, I used Henry's level of self-regulation throughout each course to design an SRL model to predict Henry's grade based on his self-regulation. Of all the components of self-regulation, it was determined that self-efficacy was the only factor that contributed to the outcome of Henry's grade in each course. He had a relatively high level of self-efficacy in the Bridge to Higher Math course, and thus got a B in the course, while he had a low level of self-efficacy in the Analysis course, resulting in a C in the course. Moreover, it appeared as though Henry exhibited the APO stage of the APOS framework in the Bridge to Higher Math course. However, for the concepts taught in the Analysis course, Henry did not appear to pass the Action stage which may have attributed to his getting a C in the course. In the next chapter, I will discuss my findings, as well as my conclusions and recommendations.

5 DISCUSSION

Thus far, I have introduced the purpose of this research study and what has been done in relation to the research questions. I also reported my analysis of the data - Henry’s work on homework problems, quizzes, exams and excerpts from transcribed conversation and the end of semester interviews. SRL conceptual framework and APOS theoretical framework were used to analyze the data. In this chapter, I will discuss the answers of the two research questions of this study per my data analysis. In the first section of the chapter, I will discuss the results of data analysis to answer research question one, followed by research question two in the section to follow. I will also be discussing my findings in relation to what research has been done thus far and the contributions of this project to the literature. After which, I will conclude with my recommendations for future research based on the findings from the data and the literature of what has been done.

5.1 Research question one

In this section I will discuss the results of my findings based on the data and answer the research question:

What learning strategies does a competent student in mathematics use when learning about proof and proof techniques in proof-based courses?

5.1.1 Part a) What is the work ethic and study habit of a competent mathematics major student as he or she learns the concept of proof?

I will be using the results of my data analysis from the lens of the SRL framework to answer this question. Particularly, I will discuss Henry’s level of self-regulation as it relates to the phases of SRL, his motivation, his self-efficacy, as well as his cognitive and metacognitive strategies in the courses.

5.1.1.1 Self-regulation

Henry's level of self-regulation was dependent on how well he performed on the exams in each course. Particularly, he was not pleased with his grade on the first exam in either class. After reflecting on his performance on each exam, he saw that he had to adjust how he studied or study more. While doing the homework problems in the Bridge to Higher Math course helped in his preparation for the quizzes and exam, for the Analysis course he identified that just doing the homework problems was not enough. In this instance, his style of studying in order to succeed in the Analysis course was changed from doing the homework for understanding, to doing the homework for memory. We saw that this was a result of the mismatch teaching and learning styles reported by Skemp (1978). This was a confirmed difficulties faced by students in proof-based courses as reported by Weber & Majia-Ramos, (2014) and Dreyfus, (1999). A summary of how Henry self-regulated his learning is the Bridge to Higher Math course and the Analysis course is shown in Tables 5.1 and 5.2 respectively.

Table 5.1 Summary of Henry's self-regulation in the Bridge to Higher Math course

	Forethought and Planning	**Performance Control**	**Reflection and Performance**	**Test Score**
Test 2	Up ⬆	Up ⬆	UP ⬆	Up ⬆
Test 3	Down ⬇	Same ▬	Same ▬	Down ⬇
Test 4	Same ▬	Same ▬	Same ▬	Up ⬆

Table 5.2 Summary of Henry's self-regulation in the Analysis course

	Forethought and Planning	Performance Control	Reflection and Performance	Test Score
Test 2	Down	Up	Down	Up
Test 3	Down	Same	Same	Down
Test 4	Down	Down	Down	Down

Comparing his preparation for test two and test one in both classes, Henry increased his self-regulation. As a result, his grade on test two, for both classes, was higher than that of test one. Naturally, with an increase in Henry's grade, one may expect all three phases of Henry's self-regulation to increase. To the contrary, this was not the case for both classes. In the Bridge to Higher Math course, all three phases did increase. This is in opposition to what was found by Li et al., (2018) where their study revealed that an increase in only performance control improved student's success. In the Analysis course on the other hand, only the performance control phase increased. This implies that an increase in performance control does in fact leads to an increase in students' overall performance, as is in line with what was reported by Li et al., (2018). Importantly, forethought and planning went down for test two in the Analysis course, but Henry's test two grade increased, nonetheless. From this observation, one may conclude that the forethought and planning phase was not of high importance for Henry self-regulation process as found for Chinese students reported by Alotaibi, Tohmaz, and Jabak (2017).

Looking closely at the third test for both classes, both of Henry's test scores went down. He did not do much forethought and planning for test three in either course. In both courses, he made plans to study for test three on average once per conversation. This was a low number compared to his preparation for test two. Additionally, he mentioned that he followed through with his plans on average twice in the Bridge to Higher Math course, and on average three times in the Analysis course. His forethought and planning went down for test three in both classes, while performance control remained the same. To that end, Henry’s grades for test three in both classes were lower than the grades he received for test two. Since his forethought and planning was what decreased on average (leading to a low test grade), does this mean that the forethought and planning phase is of higher important in the self-regulation process? As mentioned previously, the latter was reported by Alotaibi et al., (2017). In their study, Alotaibi et al., (2017) found that when students set goals and develop a plan to execute those goals, the other phases of SRL will fall into place and thus increase academic success. To the contrary, Li et al. (2018) found that performance control was the most important factor among Chinese students with the reflection and performance phase the second most important and the forethought and planning phase the least important. Our study revealed that when all three phases of SRL increased, Henry’s exam grade increased as well. Additionally, an increase in his performance control phase led to an increase in his grade.

Lastly, in preparation for the final exam for the Bridge to Higher Math course, Henry mentioned that he made plans to study on average once per conversation. On the other hand, since we did not get the chance to have a conversation before the final exam for the Analysis course, I could not determine Henry’s level of self-regulation for it. Comparing his grades on the final exam, in Bridge to Higher Math, Henry scored 85 percent on the final exam, while in the Analysis course he made a 52 percent. It is a possibility that Henry was not able to do much self-regulating for the final exam in the Analysis course, hence the low grade (Duckworth & Carlson, 2013; Li, Ye, Tang, & Zhou, 2018; Los & Schweinle, 2019; Sahranavard, Miri, & Salehiniya, 2018; Xiao, Yao, & Wang, 2019; Lindner & Harris, 1992).

The results from the questionnaire confirmed my analysis of the transcribed conversations and interviews I had with Henry. Based on the transcribed data, we saw that Henry was aware of his own level of self-regulation for each course as reported on the questionnaire. Overall, based on Henry's grades from both courses, his level of self-regulation correlated with his academic success.. This result adds to the body of work done in previous research that revealed a positive correlation between self-regulation and academic success (Duckworth & Carlson, 2013; Li, Ye, Tang, & Zhou, 2018; Los & Schweinle, 2019; Sahranavard, Miri, & Salehiniya, 2018; Rowley, 2002; Lindner & Harris, 1992).

Based on Henry's response to the SRL questionnaire, a regression model was developed. This model showed that self-efficacy was the only significant factor of the SRL components (i.e., forethought and planning phase, performance phase, reflections and performance phase, motivation) to determine academic success. This is what was reported by Pintrich and De Groot (1990), Lent et al (1986), Huang and Fang (2010), Harding et al., (2019), Los and Schweinle (2019), Ahmad et al (2012) and Li et (2018). Furthermore, based on the model, he had more confidence in his ability to do well in the Bridge to Higher Math course than in the Analysis course. Moreover, his confidence spilled over in his course work. Also based on the regression model, Henry had a 72% chance of being successful in the Bridge to Higher Math course and a 21% chance of being successful in the Analysis course. Consequently, he earned a higher grade in the Bridge to Higher Math course than that of the Analysis course.

In this section, the results showed that Henry was a self-regulated learner. Depending on how well he performed on his exams and or quizzes, he adjusted his level of self-regulation accordingly. We saw that the more he self-regulated, the better he performed on his exams. This revealed a correlation between self-regulation and academic achievement. Overall, the results showed that the forethought and planning phase, as well as the performance control phase were important in achieving academic success. Lastly, the regression model showed that the only component of SRL that significantly contributed to Henry's success in the proof-based courses was self-efficacy.

5.1.1.2 Motivation

The intrinsic motivation of doing well in the Bridge to Higher Math course and learning the material for the GRE exam were contributing factors to Henry's level of SRL. We saw that Henry was highly motivated throughout the course and this led him to persist even when the course seemed challenging. This motivation thus resulted in him earning a good grade in the course (Alotaibi, Tohmaz, & Jabak, 2017; Pintrich & De Groot, 1990; Zumbrunn, Tadlock, & Roberts, 2011). We also saw how the instructors' pedagogical approaches also affected Henry's motivation to learn (Los & Schweinle, 2019). That is, since the instructor of the Bridge to Higher Math course taught in a way that facilitated Henry's learning style, he was motivated to learn. In the Analysis course as mentioned before, the mismatch in teaching and learning styles affected Henry's motivation to learn in a negative way. Specifically, he was less motivated to do homework problems. Nonetheless, due to his intrinsic motivation, he persisted to learn by seeking help from outside resources. This result of intrinsic motivation leading to a tenacity to learn, adds to the results of the work reported by (Alotaibi, Tohmaz, & Jabak, 2017; Pintrich & De Groot, 1990; Zumbrunn, Tadlock, & Roberts, 2011).

Lastly, Henry scored his level of motivation as 6.1 one out of 7 for the Bridge to Higher Math course and 5.8 out of 7 in the Analysis course, confirming that he was more motivated in the Bridge to Higher Math course than the Analysis course. Along with the mismatch teaching and learning styles, other factors that may have affected Henry's motivation negatively in the Analysis course were not being able to become a grader, along with the choice of not taking the GRE subject exam. These factors affected his motivation and thus his grade in the Analysis course (Alotaibi, Tohmaz, & Jabak, 2017; Pintrich & De Groot, 1990; Zumbrunn, Tadlock, & Roberts, 2011).

In relation to motivation, we saw that due to intrinsic motivation, Henry persisted to learn throughout both courses, even when the content covered was challenging. This was evident also when he continued to study in the Analysis course, even though he was not learning in the way he desired. My analysis of the results also revealed that outside factors such as not being able to become a grader and the choice of not taking the GRE subject exam, affected his motivation.

5.1.1.3 Self-efficacy

As the Bridge to Higher Math course began, we saw that prior knowledge played a role in Henry's self-efficacy. This was reported by Nurjanah and Dahlan (2018). We also saw where because Henry's instructors, from high school and Bridge to Higher Math, appeared to exude self-efficacy in their ability to help him learn, this affected his level of self-efficacy in a positive way. This result was confirmed by work done by Los and Schweinle, (2019). Additionally, because he had interest in the concepts being taught in the Bridge to Higher Math course, this contributed to his efficacy in the course. The latter result supports what was reported Nuutila, et al., (2020). In both courses, for exam one, Henry's overconfidence led him not to study adequately and consequently scoring lower than he anticipated on both exams. This result adds to the body of literature that overconfidence can have a negative impact on students' performance found by Seifert and Sutton, (2009). However, after reflecting on his grade for exam one, due to his high level of self-efficacy, Henry put a considerably substantial effort into studying for exam two in both courses. This speaks to the work done by Alotaibi et al., (2017), Nuutila, et al., (2020), Pintrich and De Groot, (1990), Ahmad, and Hussain et al., (2012). When the course content was overwhelming or difficult, owing to his high level of self-efficacy, Henry still persisted to learn the concepts covered in both courses. This result confirmed what was found by Schunk, (1985), Los and Schweinle, (2019), and Alotaibi et al., (2017).

Even though Henry was an efficacious student, his level of self-efficacy decreased in both classes when he was faced with challenging concepts. Particularly, in the Analysis course, his confidence and thus interest in the course decreased. This led to a decrease in the amount of effort he put into studying for the exams. The latter result confirms the report of Alotaibi et al., (2017), Nuutila, et al., (2020), as well as Pintrich and De Groot, (1990). Additionally, in the Analysis course, we saw that when Henry was not enjoying the course, this affected his interest and thus self-efficacy confirming that intrinsic value positively affects self-efficacy as reported by Pintrich and De Groot, (1990). Additional factors leading to a decrease in self-efficacy was a lack of understanding due to the mismatch between teaching and learning styles as reported by Schunk, (1985) and Pintrich and De Groot, (1990).

Looking closely at the role of self-efficay in Henry's level of self-regulation, we saw that the instructors' level of self-efficacy afftected Henry's level of self-efficacy. Henry's interest in each course, contrubuted to his self-efficacy. On the other hand, when Henry was overconfident in his abilities, he negleted to study appropriately. This affected his grade on the exams. Lastly, the results showed that when the material proved to be challenging, Henry's self-efficacy went down.

5.1.1.4 Cognitive and metacognitive strategies

As for cognitive and metacognitive strategies, Henry expressed that his cognitive strategies were thinking about the concepts covered in class over and over until it made sense in his head. When he approached a mental block, he would get paper and pen to solve the problems. Additionally, he would seek help from the instructors when he deemed it necessary. We also saw how Henry used the method of partitioning proofs when studying based on his solutions on exams in the Analysis course. This adds to the body of work by Weber (2015). In addition, Henry stated that he did not take notes simultaneously as the instructor taught. Instead, he made notes of hints, important facts he needed to know and what he thought he would forget. For Henry, it was more important (and useful) if he focused on what the instructor was presenting instead of writing complete notes.

Henry's metacognitive strategies included his prioritizing his assignments based on what he thought needed the most attention. For instance, he focused on Math Stat II when he thought the material in the Bridge to Higher Math course was not challenging. Additionally, he adjusted his study habits when he made a low grade on exams. That is, his plans to study changed as per his grades on the exams.

In answering question one, part a, we saw that the work ethics and study habits of a competent mathematics major student involved a high level of self-regulation. As a competent student, Henry adjusted his study habits as needed per exam in each course. Additionally, he focused on learning the theory of the concepts taught by recollecting the content in his head over and over until he understood the concepts. In relation to note taking, Henry focused more on following what the instructors were teaching instead of taking notes and studying simultaneously.

5.1.2 Part b) How does a competent student in mathematics develop his/her understanding of proof concepts?

To answer this question, I will be using the lens extended by APOS theoretical framework. First, I will discuss how Henry developed his understanding of particular mathematical concepts as well about proof techniques covered in the Bridge to Higher Math course followed by the Analysis course.

5.1.2.1 Bridge to Higher Math course

Throughout the Bridge to Higher Math course, Henry was grappling between the Action and Process conception stage of understanding. Specifically, on June fourth, he exhibited the Process conception stage of understanding using the method of direct proof for the truth table, while on June 8th, he exhibited the Action conception stage of understanding for topics covered in Discrete Math. Pertaining to even and odd parities, Henry exhibited at both the Action and Process conception stages of understanding the converse of a statement and using the method of direct proof respectively. Similarly, on June 18th, on exam one, he exhibited the Action and Process conception stage of understanding. As the semester ended, on the final exam, Henry exhibited the Process conception stage of understanding for even and odd parities using the method of direct proof. Also, on June 18th, Henry exhibited the Process conception stage of understanding for quantifiers using the method of proof by contradiction. This indicates that inability to use quantifiers, as reported by Weber (2003), was not a difficulty for Henry in understanding proof concept. He confirmed this analysis in the end of semester interview using the method of contradiction (WOP).

Henry first exhibited the Action conception stage of understanding for problems involving arithmetic manipulations on July second and progressed to the Process conception stage using the method of direct proof on July 11th. Soon after however, on the final exam on July 25th, he appeared that he reversed back to the Action conception stage using the method of proof by induction. In relation to Fibonacci numbers, Henry only exhibited the Action conception stage using the method of proof by induction on exam two the final exam. For problems involving set theory, Henry only exhibited the Process conception stage of understanding. He did so on exam two on July second and quiz three on July 11th. For family of sets, Henry only exhibited the Action conception stage of understanding using the method of proof by induction on exam three that was administered on July second. Once throughout the semester, Henry exhibited the Process conception stage of understanding for questions concerning integers. This was on exam one taken on July second.

In the latter part of the semester, equivalence relation was introduced. Henry was struggling between the Action and Process conception stages of understanding for the latter concept. Specifically, on July 11th, and July 16th, Henry exhibited the Action conception stage of understanding. On July 20th however, he exhibited the Process conception stage of understanding. He reverted back and exhibited the Action conception stage of understanding of concepts included on the final exam but exhibited the Process conception stage of understanding during the end of semester interview. For the concept partition, Henry only exhibited the Action conception stage on July 16th. Modulo arithmetic was the only concept for which Henry exhibited the Object conception stage. For the first time in the semester, on exam three, he exhibited the Process conception stage of understanding and shortly after, on the final exam administered on July 25th. Subsequently, in the end of semester interview, he exhibited the Object conception stage of modulo arithmetic. For the concepts function and summation, Henry only exhibited the Process concept stage. Moreover, on exam three on July 20th and the final exam on July 25th, he exhibited the process conception stage of understanding for functions. On July 27th during the end of semester interview was when Henry exhibited the Process conception stage of understanding for summation.

While Henry used different methods to prove the various concepts covered throughout the course, he was also tested on his knowledge for these concepts. He exhibited the Action conception stage of understanding for all method of proofs considered in my analysis. For instance, on June 20th, Henry exhibited the Process conception stage of understanding for the converse of a statement. On June 15th, he exhibited the Object conception stage of understanding for proof by contradiction but exhibited both the Process and Object conception stages during our conversation on June 27th. In relation to proof by contraposition, similarly to proof by contradiction, he exhibited the Object conception stage on June 15th. However, he de-encapsulated the Object of proof by contradiction back into a Process on June 20th and stayed at the Process conception stage of understanding throughout the semester. He also exhibited the Process conception stage of understanding on July 25th and July 27th. For the method of proof by induction, Henry only exhibited the Process conception stage of understanding. He did so on June 22nd, June 27th, and July 25th. Finally, on the method of proof by contradiction using the well ordering principle, Henry exhibited the Process conception stage of understanding on July 6th. A summary of Henry's level of conception at the end of the semester is shown in Figure 5.1. In Figure 5.1 we see that Henry was able to go through only the A-P-O stage of the APOS theoretical framework over the course of the Bridge to Higher Math course. Particularly, as the semester ended, Henry exhibited the Action conception stage of understanding for topics relating to Discrete Math, arithmetic manipulation, Fibonacci numbers, family of sets, and partition. On the other hand, as the semester came to an end, it appeared as though he interiorized and exhibited the Process conception stage of understanding for the truth table, even odd parities, quantifiers, integers, equivalence relation, functions, summation, contraposition, proof by induction, equivalence class, and the well ordering principle. Though he continually encapsulated and de-encapsulated the Processes of modular arithmetic, proof by contradiction and proof by contraposition, the only concept that he exhibited the Object stage of conception for at the ended the semester was modulo arithmetic. We saw throughout the semester that he was performing at mostly the Action conception stage for the method of proof by induction. For the methods of direct proof, proof by contradiction, proof by contraposition, and proof by

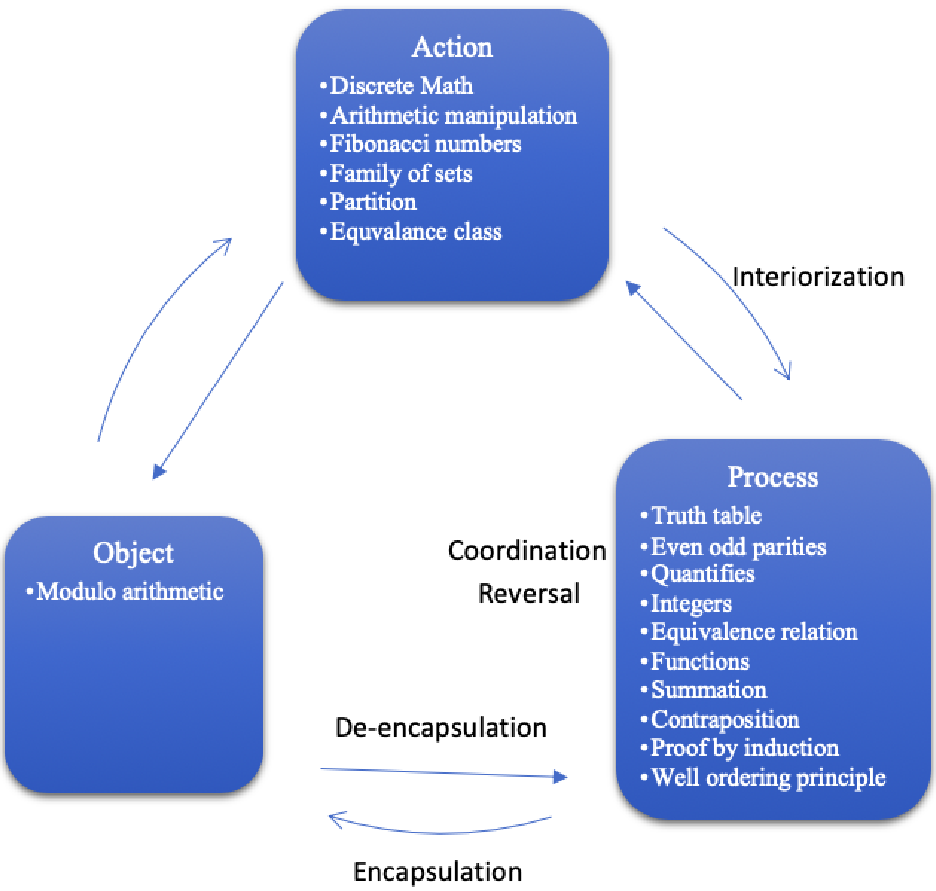


Figure 5.1 Summary of Henry's level of understanding in the Bridge to Higher Math course through the lens of APOS.

contradiction using the WOP, Henry was performing at the Process stage for majority of the semester. For long durations of the semester, Henry exhibited the Process stage. This may explain why he obtained a B in the course.

It is important to emphasize that Henry exhibited at least the Process conception stage of understanding for the method of proofs covered in the course. However, for concepts such as arithmetic manipulation, Fibonacci numbers, and family of sets, he exhibited the Action conception stage using the method of proof by induction. This suggests that even though students may have an understanding of proof techniques, because they do not have a good concept image and or concept definition the concepts they are proving, this hinders them from producing a body of proof. This adds to the results reported by Selden and Selden (2011), Samkoff and Weber (2015) and Weber (2003). Of the 23 instances in the semester where Henry exhibited the Process conception stage of understanding, he used the method of direct proof 7 times, the method of proof by contradiction four times. Considering only when a method of proof was applied to a question, Henry used the method of direct proof 52 percent of the time, proof by induction 22 percent of the time, proof by contradiction 17 percent of the time, and proved the converse of a statement 9 percent of the time. Importantly, the stage a student exhibits depends also on the task since some tasks may require only an action conception of understanding or at most process conception of understanding.

In the Bridge to Higher Math course, Henry predominately exhibited the Process conception stage of understanding for most of the content covered in the course. He was able to go through the APO stages of APOS. He appeared to interiorize Actions related to the truth table, even and odd parities, quantifiers, integers, equivalence relation, functions, summation, proof by contraposition, proof by induction and WOP. Unfortunately, by the end of the semester, the only

Process he appeared to encapsulate was modulo arithmetic.

5.1.2.2 Analysis course

For the analysis of how Henry developed his understanding of proof concepts in the Analysis course, I introduced the pre-Action conception[16] stage of understanding. Looking closely at how Henry's understanding of each concept covered in the Analysis course, I will start by discussing the $\varepsilon - N$ definition of the limit of a sequence. For the concept $\varepsilon - N$ definition of a limit, Henry exhibited pre-Action, Action and Process conception of understanding. He started the semester on August 20th exhibiting the Action conception stage using the method of direct proof. Subsequently after, on July 20th, he exhibited the Action conception stage of understanding and on August 27th, it appeared as though he interiorized these Actions into a Process. However, during our conversation on September 10th, he exhibited the Action conception stage and did so again on September 17th. On September 24th, he exhibited both the Action and Process conception stages of understanding. Moreover, on exam one, he exhibited the pre-Action and Process conception stage of understanding while on October 8th, he exhibited the pre-Action, Action and Process conception stages of understanding during our conversation. Subsequently after, on October 24th and November 7th, he exhibited the Action conception stage of understanding of $\varepsilon - N$ definition of a limit. On test two he exhibited the pre-Action conception of understanding and on the three, he appeared to exhibit the Process conception stage of understanding and on the final exam he exhibited the Action conception stage of understanding. During the final interview however, Henry did exhibit more than the Action conception stage of understanding for the $\varepsilon - N$ definition of the limit of a sequence. In fact, he

[16] For concepts that Henry did not exhibit at least the Action conception stage of understanding for, this was labeled as Pre-Action conception of understanding.

exhibited both the pre-Action and Action conception stage of understanding.

As it relates to arithmetic inequality, Henry appeared to exhibit the Process conception stage of understanding on August 22nd. Later in the semester in November 28th, he exhibited the Action conception stage of understanding. Moving on to the concept of boundedness, Henry appeared to have exhibited the Process conception stage of understanding on August 29th and September fifth on his work done on homework problems. However, on September 10th during our conversation, he exhibited the Action conception stage of understanding. Thereafter, on September 24th, he exhibited both the Action and Process conception stages on understanding. By the time the test one came around, it was revealed that Henry exhibited the pre-Action conception of understanding. On October 8th, he appeared to have progressed to the Action conception stage but exam two revealed that he still exhibited the pre-Action conception of understanding. On exam two, he exhibited the Action conception stage and during the end of semester interview, he exhibited both the pre-Action and Action conception stage of understanding. When determining the convergence of a sequence, Henry appeared to exhibit the Process conception stage of understanding on September fifth on a homework assignment. On September 10th however, he exhibited the Action conception stage and on September 24th, he exhibited both the Action and Process conceptions stages of understanding. However, exam one on September 24th soon revealed that he exhibited the pre-Action conception of understanding.

Later on, October 15th, Henry appeared to exhibit the Process conception stage of understanding on a homework problem but the Action conception stage shortly after on October 22nd on a different homework problem. On the take home exam, Henry exhibited the pre-Action conception of understanding and on the final Exam administered on December fifth, Henry exhibited the Action conception stage of understanding. Furthermore, the final exam revealed that Henry exhibited the Action conception stage of understanding (he exhibited both the Pre-Action and Action conception stages of understanding). When asked to show a sequence is increasing, Henry appeared to exhibit the Process conception stage of understanding. On the take home exam however, Henry exhibited the pre-Action conception of understanding.

Pertaining the limit of a sequence, Henry exhibited the Action conception stage of understanding on September 10th and appeared to exhibit the Process conception on September 17th. Like $\varepsilon - N$ definition of the limit of a sequence, boundedness, and convergence of a sequence, during the end of semester interview, Henry exhibited the pre-Action and Action conception stage of understanding. As for Cauchy Sequences, Henry exhibited the Action conception stage of understanding on September 17th and thereafter on September 24th. By exam one on September 28th, Henry exhibited the pre-Action conception of understanding. As the semester ended, on the final exam and during the end of semester interview, he exhibited the Action conception stage of understanding. In relation to the Ratio Test, Henry only exhibited the Action conception stage of understanding and this was on a homework problem assigned on October 17th. For the Comparison Test, Henry appeared to exhibit the Process conception stage of understanding but thereafter exhibited the Action conception stage of understanding on October 17th. Later in the semester on November 7th, exam two shoed that he exhibited the Pre-Action conception understanding.

Continuing with the tests for convergence, Henry only exhibited the Action conception stage of understanding for the Alternating Series Test on October 24 and the P-Series Test on November 7th. For the Root Test, he exhibited the pre-Action conception of understanding on November 7th. It is worth noting, the tests for convergence only required the Action conception stage of understanding. Next, for the $\varepsilon - \delta$ definition of the limit of a sequence, Henry exhibited the Action conception stage on November 7^{th}. On November 28^{th} and December fifth, for the take home exam and final exam, Henry appeared to exhibit the Process conception stage of understanding. The end of semester interview however, showed that he exhibited the Action conception stage of understanding. In relation to continuity at a point, Henry exhibited the Action conception stage of understanding on November 7^{th} and shortly after on November 26^{th}. On November 28^{th}, he appeared to exhibit the Process conception stage on soon after on December 5^{th}. However, like all the other concepts, the end of semester interview revealed that Henry did not exhibit the Process stage of conception but exhibited the pre-Action of conception of understanding. Lastly, for the concept metric space, Henry only exhibited the Action conception stage on November 26^{th}.

A summary of the conception stage for each concept that Henry ended the semester at is shown in Figure 5.2. Unlike the Bridge to Higher Math course, Henry was not able to go through the APO stages of APOS. He did not exhibit more than the Action conception stage. More explicitly, there appeared to be a non-presence of interiorization, coordination, reversal, encapsulation de-encapsulation by the end of the semester. Additionally, as the semester ended, Henry did not exhibit either the Process nor Object conception stage of understanding. The only concepts that Henry's level of understanding appeared to pass the Action conception stage when learning in the Analysis course were even and odd parities and modular arithmetic (from the proof out loud done in the end of semester interview). Note that Henry not being able to exhibit more than the Action stage of understanding for the concepts covered in the Analysis course may be an explanation as to why Henry got a C in the course.

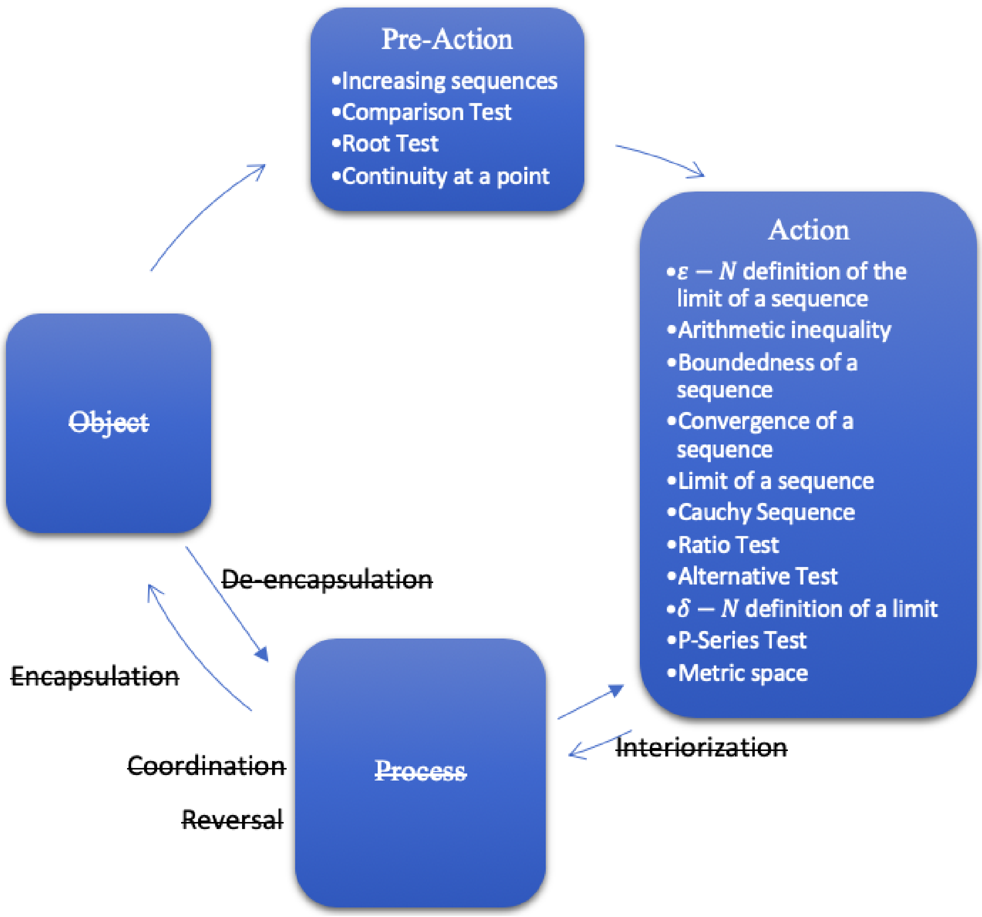


Figure 5.2 Summary of Henry's level of understanding in the Analysis course through the lens of APOS.

Specifically, he ended the semester at the pre-Action conception understanding for increasing sequences, Comparison Test for convergence, the Root Test and Continuity at a point. Moreover, at the end of the semester, he exhibited the Action conception stage of understanding for $\varepsilon - N$ definition of the limit of a sequence, arithmetic inequality, boundedness of a sequence, convergence of a sequence, limit of a sequence, Cauchy Sequence, the Ratio Test for convergence, Alternative Test for convergence, $\varepsilon - \delta$ definition of a limit, P-Series Test for convergence, and metric space. Observe that this was approximately 70 percent of the concepts covered in the course. This indicates that Henry had difficulties learning most of the concepts in the Analysis course.

When applying methods of proof, Henry used mostly the method of direct proof and indirect proof throughout the semester. Looking closely at the methods of proof used by Henry ([17]indirect proof, direct proof, proof by induction, and proof by contradiction), out of the 52 times Henry used a method of proof, he used the method of direct proof 75 percent of the times, indirect proof 21 percent of the times, induction two percent of the times and proof by contradiction two percent of the times.

Furthermore, Henry appeared to be a student who sought relational understanding. However, he had to adjust his study habits to meet the instructor's instrumental teaching style in the Analysis course. This too may have contributed to his doing so poorly in the course. Recall Skemp (1978) spoke about the "danger" of this mismatch for students. This mismatch may have affected Henry's conceptual understanding of the topics taught in the course.

In the Analysis course we saw, unlike the Bridge to Higher Math course, Henry did not appear to go through the APO stages of APOS theory in the Analysis course. In fact, he predominately exhibited the Action conception stage of understanding. A contribution to this was the mismatching in Henry's learning style and the instructor's teaching style. In relation to part b of research question one, Henry was able to go through the APO stages of APOS theory but was predominately at the Acton conception stage of understanding in the Analysis course.

5.2 Research question two

In this section I will discuss the results of my findings based on the data and answer the research question:

[17] In this instance, indirect proof refers to the backward implication method done by the instructor in the Analysis course.

How can we use the knowledge obtained about how a competent student learns and understands proof to help design pedagogical approaches?

5.2.1 Part a) What approaches to learning new concepts in proof courses, used by a competent mathematics student, could be used in teaching these concepts?

My analysis revealed that when Henry had to resort to studying the theory of a concept, he was able to develop his conceptual understanding at the Process stage. For instance, he mentioned that there was not a lot of examples on WOP, as he studied for test three in the Bridge to Higher Math course. However, on the exam, when he was asked a non-tradition proof question that required a Process stage of understanding of WOP or above, he was able to answer the question correctly. In fact, Henry exhibited the Process conception stage of understanding for majority of the course.

Additionally, in the Analysis course, Henry mentioned numerous times that the homework problems were not challenging because the instructor did similar problems in class and gave hints. He explicitly expressed that he was seeking tasks/problems that required more than the Action stage of conceptual understanding conception stage in the course but based on the instructor's style of teaching, he found it difficult to do so. We also saw that in the Bridge to Higher Math course, the instructor incorporated non-traditional questions that required more than the Action conception stage of understanding.

Comparing the two courses, we saw that Henry did better in the Bridge to Higher Math course than he did in the Analysis course based on how confident he was in his abilities to do the course work. This resulted in his earning a B in the Bridge to Higher Math course and a C in the analysis course. Based on the finding of this research and the existing knowledge from the literature, the following are suggestion for pedagogical suggestions:

- Instructors should use pedagogical strategies that incorporate ways for students to monitor and improve their self-regulated learning (Los & Schweinle, 2019; Zumbrunn, Tadlock, & Roberts, 2011; Li, Ye, Tang, & Zhou, 2018; Pintrich & De Groot, 1990). These strategies could include open ended tasks, projects and non-traditional proof questions as we saw in the Bridge to Higher Math course (Paris & Paris, 2001).
- It is desirable that the instructors with high self-efficacy teach proof courses as they will be confident enough to help struggling students (Los & Schweinle, 2019).
- At the beginning of the semester for a proof course, instructors should give students a SRL-questionnaire to determine the possibility of a student doing well or badly in the course (Huang & Fang, 2010). Based on the students' reported level of self-efficacy, the instructor may be able to determine which students need close attention. That is, a student who has a low percentage of success may be identified as an "at risk" student and be monitored throughout the course. It is important to note that the model developed in this study, in its current form, has been tested with Henry and it has been shown to be statistically significant. The model may be improved by doing the analysis with a larger number of students.
- Instructors should incorporate a feedback loop into their curriculum so students get constant feedback and can adjust their self-regulation accordingly (Xiao, Yao, & Wang, 2019).
- Instructors of proof courses should receive training on how to deliver material that encompasses SRL (Li, Ye, Tang, & Zhou, 2018).

- Instructors should incorporate questions that require students to reflect on the procedures used in the problem/proof to enhance student's development of understanding.
- Instructors of proof courses should consider providing students, with complete notes in electronic form and share with the class. In this era, this could be done by taking photos of the board and or compiling class notes. This could be done by instructors or students.

5.2.2 Part b) What challenges in learning new concepts in proof courses, encountered by a competent mathematics student, could be used in teaching these concepts?

One of the major challenges that we saw emerging for Henry was the mismatch in teaching and learning styles that affected him in the Analysis course (Skemp, 1978). The fact that he exhibited mostly the Action conceptions stage for concepts in Discrete Math, arithmetic manipulation, Fibonacci numbers, family of sets, and partition, indicates that he may have had difficulties learning these concepts. Particularly, we saw that the way in which the instructor presented the information in the Analysis course affected Henry's self-efficacy, level of conceptual understanding and thus grade in the course. Students who readily comprehend the teacher's instructions and explanations are apt to feel more efficacious for learning than those who experience less understanding (Li, Ye, Tang, & Zhou, 2018; Lindner & Harris, 1992; Xiao, Yao, & Wang, 2019; Skemp, 1978).

As already presented, Henry sought relational understanding while the instructor taught instrumentally. For instance, in the Analysis course, he sought to pass the Action conception stage in the course but based on the instructor's style of teaching, he found it difficult to do so.

Furthermore, it appeared as though Henry was only able to do familiar problems that he had done before or that the instructor covered in class. This is a characteristic of a student who exhibits the Action stage of conceptual understanding. I would like to point out that the reverse mismatch where the student sought to learn instrumentally and the instructor taught relationally is also dangerous (Skemp, 1978). It would be ideal if students could be screened and matched to the instructor who taught proof courses based on the way they sought to learn. Alternatively, the instructors of proof courses should incorporate both styles of teaching so that students with both learning styles can advance in their conceptual learning. I would like to point out that instrumental understanding only lasted for a short time. For this reason, I suggest having instructors incorporate the APOS teaching theory in their teaching method (Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007). That is, when using the APOS theory as a framework for instruction, all instructional activities that are aimed at the development of the Action stage of understanding would match the instrumental style learners while activities aimed at the development of higher stage of conceptual understanding would match the students with relational style of learning.

The last difficulty faced by Henry was the lack of conceptual understanding of some of the content covered in the Bridge to Higher Math course and most of the content covered in the Analysis course. My analysis revealed that Henry exhibited the Process conception stage of understanding for the methods of proof covered in the Bride to Higher Math course. However, he was still unable to come up with valid proofs for some of the content covered in the course. This implies that exhibiting the Process conception stage of the proof technique is not useful to a student who does not have any conceptual understanding of a mathematical concept. Furthermore, this was evident in the Analysis course where he was unable to exhibit more than the Action conception stage of understanding for the concepts covered and was thus unable to come up with valid proofs if he had not see them before.

In this section, we saw that the challenges that Henry faced when learning new concepts in the proof courses were a mismatch in his learning style and the Analysis instructor's teaching style, as well as a lack of conceptual understanding of some of the concepts covered in the Bridge to Higher Math course and most the concepts covered in the Analysis course. As a result, the following are suggestion for pedagogical suggestions:

- Instructors should monitor students' progress in a proof course and give positive reinforcements for students with reported low self-efficacy (Karabenick & Knapp, 1991; Ahmad, Hussain, & Azeem, 2012; Ganah, 2012).
- Instructors should implement a check point after each exam for students to acknowledge if the methods they are using to study are efficient or not. This could also provide an avenue to check if students have a conceptual understanding of the content being covered in the course.
- Instructors should guide students to evaluate their self-regulation (Li, Ye, Tang, & Zhou, 2018).
- Instructors of proof courses should use both instrumental and relational teaching styles so as to ensure that students who seek to learn from either or both are able to benefit from their instruction.

5.3 Limitations of the study

Some researchers may argue that a case study is not generalizable, especially, a single-case design. In this study, we are not particularly trying to generalize what we found to every student. Instead, we used data from previous studies that have reported on students' weakness and misconceptions relating to proof in conjunction with the findings of this study to propose effective teaching strategies for proof concept. With that said, it will be up to the reader to make their own conclusions concerning generalizability (Bogdan & Biklen, 2007).

One of the limitations include the nature of the reporting for the study. Being that I made the observations, wrote the fieldnotes and interpreted all of the data, this research is subjected to my bias, frame of mind and thoughts. However, since I triangulated the data using multiple sources, this bias was minimized, if not eliminated. This bias was also considered and noted as the findings were recorded and interpreted. Lastly, it should be pointed that this study was conducted for two semesters. A longer time period with Henry in his advanced math courses beyond the Analysis course may reveal more pertinent information as he transitions to higher lever courses. This might be considered for future research. Additionally, observing students before they enroll in proof courses, in a similar way to what was done in the Bridge to Higher Math course and the Analysis course, could add a better starting point for analysis as opposed to the criteria of number of A's in prior courses.

5.4 Summary of chapter

In this chapter, I discussed the results of my findings in relation to the research questions. For question one, *What learning strategies does a competent student in mathematics use when learning about proof and proof techniques in proof-based courses?* we saw that a student who self-regulates as he/she learns the concept of proof has a good chance of succeeding in the course. I also discussed the results of Henry's development of the proof concepts taught in both courses. We saw that Henry used the method of direct proof substantially more than he did the other methods of proof he learned. Henry's use of the different methods of proof in conjuction with the number of times he proved the converse of a statement is summarized in the pie chart shown in Figure 5.3.

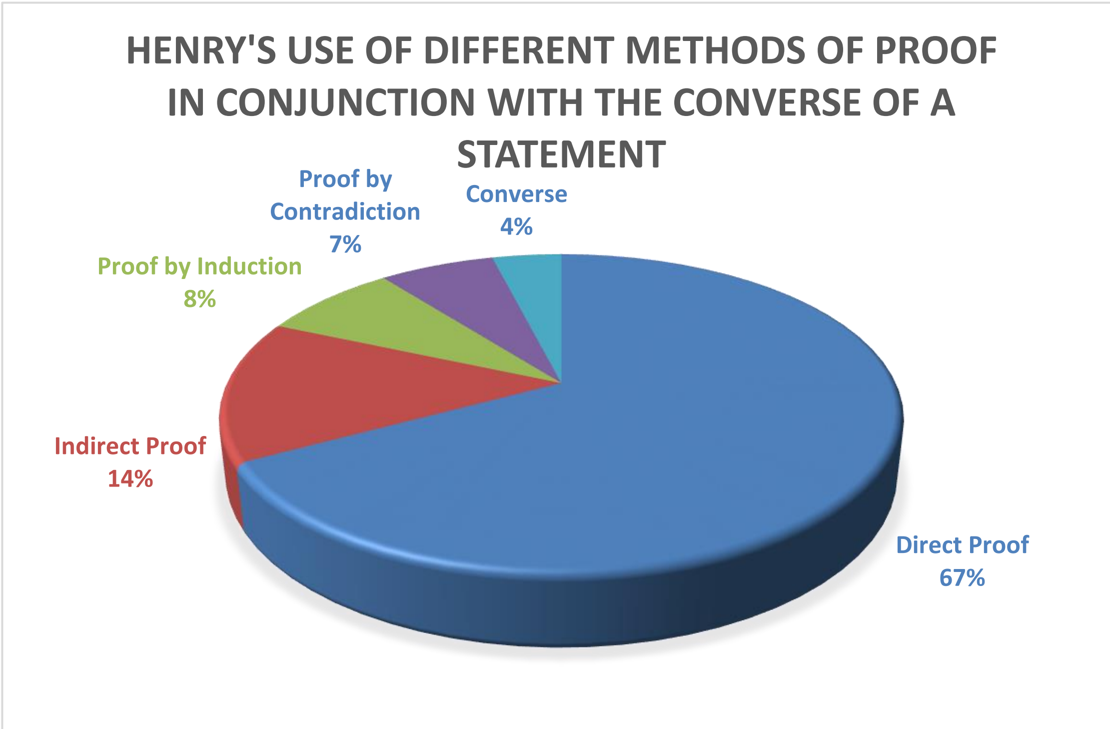


Figure 5.3 Henry's use of different methods of proof in conjunction with the converse of a statement in the Bridge to Higher Math course and the Analysis course.

For the second research question, *How can we use the knowledge obtained about how a competent student learns and understands proof to help design pedagogical approaches?* we saw that a pedagogical approach that incorporates SRL and that uses the APOS teaching theory are needed for students to succeed in proof courses.

6 CONCLUSION

In short, it was when Henry did most self-regulating that he scored the highest on a test (Duckworth & Carlson, 2013; Li, Ye, Tang, & Zhou, 2018; Los & Schweinle, 2019). Based on the data we note that there is a correlation in the level of self-regulation and academic success (Duckworth & Carlson, 2013; Li, Ye, Tang, & Zhou, 2018; Los & Schweinle, 2019; Sahranavard, Miri, & Salehiniya, 2018; Xiao, Yao, & Wang, 2019; Lindner & Harris, 1992). Also, it was when he was required to study in such a way that he had to understand the theory of a concept that he was able to arrive at the Process conception stage of understanding. This suggests that instructors should incorporate pedagogical strategies, such as the SRL and APOS teaching theory, into their teaching approach for proof courses (Arnawra, Sumarno, Kartasasmita, & Baskoro, 2007). We saw that the mismatch in teaching and learning styles hampered Henry in the Analysis course causing him to have low self-efficacy, a decrease in motivation leading to a decrease in self-regulation and thus causing him to be stuck at the Action conception stage. Additionally, a lack of conceptual understanding of the content covered in the proof courses affected Henry's performance in each course.

Future work may include gathering more data from a larger number of students to improve the SRL model. More work needs to be done to match instructors and students based on their teaching and learning styles. Additionally, more research needs to be conducted to see which phase of SRL is most important. A further look into the local dimensions and holistic dimensions (Mejia-Ramos, Fuller, Weber, Rhoads, & Samkoff, 2012) of the questions asked in both proof courses may be done to better assess Henry's understanding of proof. Lastly, ways to have instructors incorporate self-regulated activities for students, as well as the APOS teaching theory need to be developed and explored further.

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APPENDICES

Appendix A SRL questionnaire

This questionnaire is asking you about your study habits, your learning skills, and your motivation for work in your Bridge to Higher Math and math Analysis I classes. There are not right or wrong answers to this questionnaire. This is not a test. You should respond to the questions as accurately as possible, reflecting your own attitudes and behaviors. Use the scale below each question to answer the questions. If you think the statement is very true of you, circle the number 7; if a statement is not at all true of you, circle the number 1. If the statement is more or less true of you, circle the number between 1 and 7 that best describes you.

1. I prefer class work that is challenging so I can learn new things.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

2. Compared with other students in this class I expect to do well.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

3. I am so nervous during a test that I cannot remember facts I have learned.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

4. It is important for me to learn what is being taught in this class.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

5. I like what I am learning in this class.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

6. I'm certain I can understand the ideas taught in this course.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

7. I think I will be able to use what I learn in this class in other classes.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

8. I expect to do very well in this class.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

9. Compared with others in this class, I think I'm a good student.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

10. I often choose paper topics I will learn something from even if they require more work.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

11. I am sure I can do an excellent job on the problems and tasks assigned for this class.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

12. I have an uneasy, upset feeling when I take a test.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

13. I think I will receive a good grade in this class.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

14. Even when I do poorly on a test I try to learn from my mistakes.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

15. I think that what I am learning in this class is useful for me to know.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

16. My study skills are excellent compared with others in this class.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

17. I think that what we are learning in this class is interesting.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

18. Compared with other students in this class I think I know a great deal about the subject.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

19. I know that I will be able to learn the material for this class.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

20. I worry a great deal about tests.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

21. Understanding this subject is important to me.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

22. When I take a test, I think about how poorly I am doing.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

23. When I study for a test, I try to put together the information from class and from the book.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

24. When I do homework, I try to remember what the teacher said in class so I can answer the questions correctly.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

25. I ask myself questions to make sure I know the material I have been studying.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

26. It is hard for me to decide what the main ideas are in what I read.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

27. When work is hard I either give up or study only the easy parts.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

28. When I study I put important ideas into my own words.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

29. I always try to understand what the teacher is saying even if it doesn't make sense.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

30. When I study for a test I try to remember as many facts as I can.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

31. When studying, I copy my notes over to help me remember material.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

32. I work on practice exercises and answer end of chapter questions even when I don't have to.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

33. Even when study materials are dull and uninteresting, I keep working until I finish.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

34. When I study for a test I practice saying the important facts over and over to myself.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

35. Before I begin studying I think about the things I will need to do to learn.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

36. I use what I have learned from old homework assignments and the textbook to do new assignments.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

37. I often find that I have been reading for class but don't know what it is all about.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

38. I find that when the teacher is talking I think of other things and don't really listen to what is being said.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

39. When I am studying a topic, I try to make everything fit together.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

40. When I'm reading I stop once in a while and go over what I have read.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

41. When I read material for this class, I say the words over and over to myself to help me remember.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

42. I outline the chapters in my book to help me study.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

43. I work hard to get a good grade even when I don't like a class.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

44. When reading I try to connect the things I am reading about with what I already know.

1 Not at all true for me	2	3	4	5	6	7 Very true for me

SRL Key**

Intrinsic values: #1, 4, 5, 7, 10, 14, 15, 17, 21

Self-efficacy: #2, 6, 8, 9, 11, 13, 16, 18, 19

Test anxiety: #3, 12, 20, 22

Cognitive strategy use: #23, 24, 26, 28, 29, 30, 31, 34, 36, 39, 41, 42, 44

Self-regulation: #25, 27, 32, 33, 35, 37, 38, 40, 43

Appendix B Quiz and exam questions from Bridge to Higher Math course

Appendix B.1 Quiz one: Bridge to Higher Math course

1. Complete the following truth table to determine whether the following is a tautology, contradiction or neither.

$$(P \wedge Q) \vee (P \wedge \sim Q) \vee (\sim P \wedge Q) \vee (\sim P \wedge \sim Q)$$

P	Q	~P	~Q	$P \wedge Q$	$P \wedge \sim Q$	$\sim P \wedge Q$	$\sim P \wedge \sim Q$	Statement

Conclusion (and reason):

2. Consider the following statement: A sequence a_n is bounded whenever a_n is convergent.

   a. Write this statement in "if-then" form.

   b. Write the converse of your statement in part (a).

   c. Write the contrapositive of your statement in part (a).

3. Let the universe be the set of all real numbers and consider the following open sentence:

$$(\forall x)\left[x > 0 \Rightarrow (\exists y)(y < 0 \wedge xy > 0)\right].$$

   a. Translate the open sentence into English.

   b. Is the open sentence true or false? Why?

4. Prove the following statement.

   For integers m and n, one of which is even and the other odd, $m^2 + n^2$ has the form 4k + 1 for some integer k.

5. Read the following "proof" and identify what is incorrect about it and why it is incorrect.

Suppose m is an integer.

Claim: If m^2 is odd, then m is odd.

"**Proof**": Assume m is odd. Then m = 2k + 1 for some integer k. Therefore,

$$m^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2\left(2k^2 + 2k\right) + 1,$$

which is odd. Therefore, if m^2 is odd, then m is odd.

Appendix B.2 Quiz two: Bridge to Higher Math course

1.List the ordered pairs in $A \times B$ if A = {1, 3, 5} and B = {a, b, c}.

2. a. Prove that P(A) – P(B) $\subseteq$ P(A – B).

   b. Give an example of P(A) – P(B) $\neq$ P(A – B).

3. Let A = $\{A_n : n \geq 3\}$, where $A_n = \left[\frac{1}{n}, 2 + \frac{1}{n}\right] \ \forall n \ \in \mathbb{N} - \{1, 2\}$.

   a. Find the union of this family of sets.

   b. Find the intersection of this family of sets.

4. If A = $\{A_\alpha : \alpha \in \Delta\}$ is a family of sets and B is an arbitrary set, prove

$$B \cap \bigcup_{\alpha \in \Delta} A_\alpha = \bigcup_{\alpha \in \Delta} \left(B \cap A_\alpha\right)$$

5. Let A = $\{A_\alpha : \alpha \in \Delta, \Delta \neq \varnothing\}$ is a family of sets and B an arbitrary set. Give a nontrivial

   example showing that $B - \left(\bigcap_{\alpha \in \Delta} A_\alpha\right) \neq \bigcap_{\alpha \in \Delta} \left(B - A_\alpha\right)$.

Appendix B.3 Quiz three: Bridge to Higher Math course

1. Let T = {(3,1), (2,3), (3,5), (2,2), (1,6), (2,6), (1,2)}.

   a. List the elements in the domain of T.

   b. List the elements in the range of T.

c. List the elements in T^{-1}.

d. List the elements in $T \circ T$.

e. Draw the digraphs representing T and T^{-1}.

2. Let R be a relation from A to B and S a relation from B to C. One of the following statements is true, while the other is false. Prove the true one. Recall that Rng means range.

$$\text{Rng}(S) \subseteq \text{Rng}(S \circ R) \qquad\qquad \text{Rng}(S \circ R) \subseteq \text{Rng}(S)$$

Appendix B.4 Quiz Four: Bridge to Higher Math course

1. Define each of the following.

a. An equivalence relation R on a set A

b. A partition P of a non-empty set A

2. Consider the relation R on $\mathbb{N} - \{1\}$ defined by a R b if the prime factorizations of a and b have the same number of 2's. For example, 48 R 80 since $48 = 2^4 * 3$ and $80 = 2^4 * 5$.

a. Show R is an equivalence relation

b. Name (and verify) 2 elements in the equivalence class $\overline{72}$, other than 72.

3. Suppose P is a partition of a non-empty set A and suppose that x Q y if there exists C $\in$ P such that $x \in C$ and $y \in C$.

a. Prove that Q is symmetric.

b. Prove that Q is reflexive.

Appendix B.5 Test one: Bridge to Higher Math course

1. Define each of the following. Be sure to use correct grammar and symbolism (when appropriate). Also, give an example of each.

a. A proposition.

b. A tautology.

2. Give a useful denial (in words) of each statement.

   a. Roses are red and violets are blue.

   b. Neither $z < s$ *nor* $z \leq t$ is true.

3. Construct a truth table for the statement: $P \Longrightarrow Q \wedge R$.

4. Consider the following statement:

   If I receive an A in both Calculus I and Discrete Mathematics then I will take either Calculus II or Computer Programming next semester.

   a. Write this statement in a "$P \Longrightarrow Q$" format. Be sure to identify all statements.

   b. Write the contrapositive of this statement symbolically.

   c. Write the contrapositive statement in words.

5. Let $n \in \mathbb{Z}$. If $2n^2 + 3n + 4$ is odd then $5n + 1$ is even.

   a. State the converse of this statement.

   b. Prove the statement you gave in (a).

6. Translate each of the following into symbolic sentences with quantifiers. The universe for each is given in parentheses.

   a. No right triangle is isosceles. (all triangles)

   b. Every integer is greater than – 4 or less than 6. (real numbers)

7. Using appropriate symbols, give a useful, simplified denial of each statement in problem (6).

8. <u>True or False</u>: If the statement is true, prove it. If the statement is false provide appropriate justification or counterexample. (5 points each)

   a. P: $\sqrt{2}$ is rational, Q: 22/7 is rational. $(\sim P) \Longrightarrow (\sim Q)$.

b. If x and y are integers of the same parity then x – y is even.

c. If $n \in \mathbb{Z}$ and $\dfrac{n^2+n-6}{2}$ is odd then $\dfrac{2n^3+3n^2+n}{6}$ is even.

9. Proof evaluation: Consider the following “proof.” Determine whether or not this is a valid proof. If it is valid, simply say so; if it is not, indicate what is wrong with the “proof” and provide a valid proof. (7 points)

Statement: Let $n \in \mathbb{Z}$. If $3n - 8$ is odd, then n is odd.

Proof: Assume n is odd. Then n = 2k + 1 for some integer k. Then

$3n - 8 = 3(2k + 1) - 8 = 6k + 3 - 8 = 6k - 5 = 2(3k - 3) + 1$.

Since $3k - 3$ is an integer $3n - 8$ is odd.

10. Give an example of sets A, B, and C such that $A \subseteq B,\quad B \not\subseteq C,$ and $A \subseteq C$.

Appendix B.6 Test two: Bridge to Higher Math course

1. Consider the following collection of sets (open intervals).

$$\left\{(-1,2),\left(-\frac{3}{2},4\right),\left(-\frac{5}{3},6\right),\left(-\frac{7}{4},8\right),\ldots\right\}$$

a. Define a set A_n for each n ∈ ℕ such that the indexed collection of sets $\mathcal{A} = \{A_n : n \in \mathbb{N}\}$ is precisely this given set of sets.

b. Determine $\bigcup_{n\in\mathbb{N}} A_n$.

c. Determine $\bigcap_{n\in\mathbb{N}} A_n$.

2. a. Define what it means for a set S to be inductive.

b. Indicate which of the following sets are inductive by **circling** yes or no for each.

{20, 21, 22, 23,…}	Yes	No
{1, 2, 3, 4, 5, 6}	Yes	No

3. Suppose $\overline{\overline{A}} = 24, \overline{\overline{B}} = 21, \overline{\overline{A \cup B}} = 37, \overline{\overline{A \cap C}} = 11, \overline{\overline{B - C}} = 10, \overline{\overline{C - B}} = 12$. Find each of the following.

   a. $\overline{\overline{A \cap B}}$

   b. $\overline{\overline{B \cup C}}$

4. Prove each of the following.

   a. Let A be a family of sets. Then $\left(\bigcup_{A \in \mathcal{A}} A \right)^c = \bigcap_{A \in \mathcal{A}} A^c$.

   b. $\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2} \right]^2, \forall n \in \mathbb{N}$

   c. Let f_n denote the nth Fibonacci number (n ∈ ℕ). Then $f_2 + f_4 + \ldots + f_{2n} = f_{2n+1} - 1$.

   d. If A and B are disjoint sets and C is any other set then

$$\overline{\overline{A \cup B \cup C}} = \overline{\overline{A}} + \overline{\overline{B}} + \overline{\overline{C}} - \overline{\overline{A \cap C}} - \overline{\overline{B \cap C}}.$$

   e. For any two sets A and B, $A - B = A \cap B^c$.

5. Let A = {A_i: i ∈ ℕ} be a family of sets such that for all i, j ∈ ℕ, if i ≤ j then $A_j \subseteq A_i$. Such a family is called a nested family of sets.

   a. Prove that for such a family of sets that ∀k ∈ ℕ, $\bigcap_{i=1}^{k} A_i = A_k$.

b. Give a non-trivial example (that is, they are not all the same set) of a nested family $\{A_i : i \in \mathbb{N}\}$ such that $\bigcap_{i=1}^{\infty} A_i = [0,1]$.

6. Statement: For every positive integer n, $6 \mid (n^3 - n)$.

Proof:

a. Assume, to the contrary, that there is some positive integer n > 1 such that $6 \nmid (n^3 - n)$ and let T be the set of all such numbers.

b. Then T has a least element, call it m. Note that m $\geq$ 3.

c. Thus, m = k + 2, 1 $\leq$ k < m.

d. Algebra shows that $m^3 - m = (k^3 - k) + (6k^2 + 12k + 6)$.

e. Thus, since $6 \mid (k^3 - k)$, we have $6 \mid (m^3 - m)$.

f. Thus, no such m exists, and the statement is true.

a. What is the purpose of statement (i)?

b. Why is statement (ii) true? In particular, how do we know m $\geq$ 3?

c. In statement (v), why is $6 \mid (k^3 - k)$ true?

d. How is the conclusion, statement (vi), derived?

Appendix B.7 Test 3: Bridge to Higher Math course

1. Prove that for every integer k, 5 divides $k^5 - 5k^3 + 4k$.

2. Prove that $\binom{2n}{n} + \binom{2n}{n+1} = \frac{1}{2}\binom{2n+2}{n+1}$.

3. Let A = {a, b, c, d}.

a. Give a nontrivial example of relations R and S on A so that $R \circ S \neq S \circ R$, and verify this is true. Each of your relations should have at least 3 elements.

b. Find the inverse of one of your relations in part (a).

4. Let A = {a, b, c, d}.

   a. Give an example of a relation R with at least 4 ordered pairs on A that is symmetric, transitive, but not reflexive. Verify that your relation actually satisfies the requirements stated.

   b. Give a digraph of your relation.

   c. List the elements that need to be added to R to make it an equivalence relation.

5. List the ordered pairs in the equivalence relation on A = {1, 2, 3, 4, 5} that is associated with the partition { {1, 2}, {3, 4, 5} }.

6. <u>Describe</u> an equivalence relation on $\mathbb{N}$ that has the following partition:

   { {1, 2, …, 9}, {10, 11, …, 99}, {100, 101, …, 999}, …}

7. a. Calculate $2 \cdot 4 + 3 \cdot 5$ in $\mathbb{Z}_7$.

   b. Calculate 2^{26} in $\mathbb{Z}_7$.

   c. Solve 4x = 6 in $\mathbb{Z}_7$.

8. Give two definitions of a function $f : A \rightarrow B$, one that uses the idea of a relation and one that does not.

9. For the canonical map $f : \mathbb{Z} \rightarrow \mathbb{Z}_8,\ f(n) = \overline{n}, \mathbb{Z}_8 = \{\overline{0}, \overline{1}, ..., \overline{7}\}$, find each of the following.

   a. f (-38)

   b. the image of 1265

   c. two pre-images of $\overline{5}$

d. all pre-images of $\overline{1}$

10. Prove or give a counterexample.

a. If f and g are real functions that are increasing on the real numbers, then $f \circ g$ is increasing.

b. If f and g are real functions that are decreasing on the real numbers, then $f \circ g$ is increasing on the reals.

c. If A is a partition on a non-empty set A and B is a partition on a non-empty set B, then A $\cup$ B is a partition on $A \cup B$.

11. After reading the following statement and proof, write a brief (30 words or less) summary of the proof. Do NOT just rewrite the proof. Recall, if D is a subset of the domain of a function f, then $f|_D = \{(x, y): y = f(x), x \in D\}$.

Statement: If f and g are functions, then $f \cap g$ is a function.

Proof: Suppose f and g are functions and suppose $(x, y) \in f \cap g$. Then $(x, y) \in f$ and $(x, y) \in g$ so that we have $f(x) = y = g(x)$. Let $A = \{x : g(x) = f(x)\}$. Then, $x \in A \Rightarrow (x, y) \in g|_A$. Now, let $(x, y) \in g|_A$. Then, in particular, $(x, y) \in g$ and $f(x) = y = g(x)$, so that $(x, y) \in f$. Thus, $(x, y) \in f \cap g$. Therefore, $f \cap g = g|_A$, and so is a function.

Appendix B.8 Final Exam: Bridge to Higher Math course

1. Use a truth table to determine whether the following is a tautology, a contradiction, or neither.

$$P \vee [(\sim Q \wedge P) \wedge (R \vee Q)]$$

2. For the universe of all real numbers,

a. Translate "Every positive real number has a multiplicative inverse" using appropriate quantifier(s).

b. Find a useful denial of "Every positive real number has a multiplicative inverse."

3. Prove: If n ∈ ℕthen $n^2 + n + 3$is odd.

4. Prove: Let $x \in \mathbb{R}$. If $x^2 + 2x < 0$, then $x < 0$.

5. Give an example using a Venn Diagram, if there is one, of sets A, B, and C so that $A \subseteq B, B \not\subseteq C$ and $A \subseteq C$. If no such example is possible explain why.

6. Let A = {1, 3, 5, 7, 9}, B = {0, 2, 4, 6, 8}, C = {1, 2, 4, 5, 7, 8} and D = {1, 2, 3, 4, 5, 6, 7, 8, 9, 10}. Find each of the following.

a. $A \cup B$

b. $C \cap D$

c. $(A \cup B) - (C \cap D)$

7. Prove: $\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{1}{1*2} + \frac{1}{2*3} + \cdots + \cdots \frac{1}{n(n+1)} = \frac{n}{n+1}$ $\forall n \in \mathbb{N}$.

8. Consider the relation R defined on □ by x R y if x + 3y is even.

a. Show R is an equivalence relation.

b. Identify the 2 equivalence classes of R.

9. Let A = {1, 2, 3, 4}.

a. Give an example of a relation (with at least 6 ordered pairs) on A that is reflexive and symmetric, but not transitive. Verify your relation satisfies these requirements.

b. What elements have to be added to your relation in part (a) so that it is transitive, and hence an equivalence relation?

10. Find the union and intersection of the indexed family of sets given by

$A = \{A_r : n \in \mathbb{R}\} = A_r = [|r|, 2|r| + 1), \forall r \in \mathbb{R}.$

11. Let f and g be two 1-1 functions from ℝ into ℝ. Prove their composition is 1-1.

12. Suppose $f : A \to B$ and let $D \subseteq A$. Then $f(D) = \{f(x) : x \in D\}$. Let

    $f : \square_{12} \to \square_{12}$ be given by $f(\overline{x}) = \overline{3x+2}$. Find each of the following. (3 points each)

    a. $f(\{\overline{3}, \overline{4}\})$

    b. Is f onto? Why or why not?

13. Prove: If A and B are finite sets, then $\overline{\overline{A \cup B}} = \overline{\overline{A}} + \overline{\overline{B}} - \overline{\overline{A \cap B}}$. The use of a Venn Diagram argument will not be accepted.

14. Prove: For $n \in \mathbb{N}$,

$$\sum_{j=0}^{n} (-1)^j \binom{n}{j} = \binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots + (-1)^k \binom{n}{k} + \cdots + (-1)^n \binom{n}{n} = 0.$$

15. Prove the following property of the Fibonacci numbers: For every natural number n,

$$\sum_{i=1}^{n} f_{2i} = f_2 + f_4 + f_6 + \ldots + f_{2n} = f_{2n+1} - 1.$$

16. Give a description of each of the following types of proof for a statement of the form $P \Rightarrow Q$ or $P(n) \Rightarrow Q(n)$, as appropriate.

    a. Proof by contradiction

    b Proof by contraposition

    c. Proof by induction

Extra Credit

Prove: There is no integer a such that $a \equiv 5 \pmod{14}$ and $a \equiv 3 \pmod{21}$

Appendix C Homework problems and exams from Analysis course

Appendix C.1 Homework problems: Analysis course

1. Prove (use $\varepsilon - N$) that $\lim_{j\to\infty} a_j = 3$, where $a_j = \frac{1}{j} + 3, j = 1, 2, 3, \dots$

2. Prove for any given $x \in \mathbb{R}$, show that $[x] + 1 > x$.

3. Let $C_j = \frac{2j+1}{4j-5}, j = 1, 2, 3, \dots$

    a. Write out the first three terms.

    b. Guess what is $\lim_{j\longrightarrow\infty} C_j$

    c. Use $\varepsilon - N$ to prove the guess.

4. Let $a_j = \frac{1}{2^j}, j = 1, 2, 3, \dots$

    a. Write out the first three terms.

    b. Guess what is $\lim_{j\longrightarrow\infty} a_j$

    c. Use $\varepsilon - N$ to prove the guess.

5. Let $a_j = \frac{j^2+3}{2j^2-j+7}, j = 1, 2, 3, \dots$

    a. Write out the first three terms.

    b. Guess what is $\lim_{j\longrightarrow\infty} a_j$

    c. Use $\varepsilon - N$ to prove the guess.

6. If $\lim_{j\longrightarrow\infty} a_j = \alpha$ then $\lim_{j\to\infty}\left(c * a_j\right) = c * \alpha$, where is a scalar.

7. Prove that sequence $\{a_j\}, a_j = j, j \in \mathbb{N}$ is not bounded.

8. Show that:

    $If \lim_{j\longrightarrow\infty} b_j = \beta\ and\ \beta < 0, then\ \exists\ N \in \mathbb{N}\ s.t\ j > N \Longrightarrow b_j < 0.$

$If \lim_{j\to\infty} b_j = \beta \; and \; \beta > 0, then \; \exists \; N \in \mathbb{N} \; s.t \; j > N \Longrightarrow b_j > 0.$

9. Given $E \neq \emptyset, E \subseteq \mathbb{R}$, suppose $e \in E,\ e = max\, E\ (\forall\ x\ \in E, x\ \leq e)$. Show that $e = sup\, E$.
10. Given $E \neq \emptyset, E \subseteq \mathbb{R}$, and $m = min\, E, m\ \in \mathbb{R}\ \ (m\ \in E, \forall\ x\ \in E, x\ \geq m)$. Show that $m = inf\ E$.
11. Given $E \neq \emptyset, E \subseteq \mathbb{R}$, E is bounded above and, let $-E = \{y: y = -x, x\ \in E\}$. Show that $-E$ is bounded below and $sup\, E = inf(-E)$.
12. If $\{b_j\}$ is a decreasing sequence $(b_1 \geq b_2 \geq b_3 \geq \cdots)$ and bounded below, show that $\{b_j\}$ is convergent.
13. If $x_1 = \sqrt{2}$ and $x_{j+1} = \sqrt{2 + x_j},\ j = 1, 2, 3, ...$ Show that $\{x_j\}$ is increasing and bounded above.
14. If $x_1 = 2$, and $x_{j+1} = x_j - \frac{x^2-2}{2x_j}, j = 1, 2, 3 ...$ Show that $\{x_j\}$ is decreasing and bounded below.
15. Show that the sequence $\{a_j\}$ defined by $a_j = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{j}, j = 1, 2, 3 ...$ is divergent.
16. Prove the Pinching Theorem
17. Prove that $\lim_{j\to\infty}\left(4 + \frac{sin\, j}{j}\right) = 4$.
18. Show that the sequence $\{x_j\}$ is Cauchy, where $x_j = \int_1^j \frac{sin\, x}{x^2}\,dx, j = 1, 2, 3, ...$
19. Show that the sequence $\{y_j\}$ is Cauchy, where $y_j = \int_1^j \frac{sin\, x}{x}\,dx$.
20. Suppose $\lim_{j\to\infty} a_j = L, L\ \in \mathbb{R}$. Show that $\lim_{j\to\infty} a_j = \frac{a_1+a_2+\cdots+a_j}{j} = L$.
21. $F \subset \mathbb{R}$ is a set which is bounded. $E \neq \emptyset, E \subset F$. Prove i) $inf\ E\ \geq inf\ F;\ ii)\ sup\ E\ \leq sup\ F$.

22. Let $a_j = \frac{(-1)^j}{j}, j = 1, 2, 3, \ldots$ List A_j *and* B_j and the *lim sup* and *lim inf* of a_j.

23. If $\varliminf_{j\to\infty} a_j = L$, show that $\varlimsup_{j\to\infty} a_j = L$.

24. For a given bounded sequence $\{a_j\}$, $\alpha = \varliminf_{j\to\infty} a_j$. Prove that for $\varepsilon > 0$, $\exists\ N \in \mathbb{N}$ such that

$$j > N \Longrightarrow a_j > \alpha - \varepsilon.$$

25. (Optional) Show $\lim_{j\to\infty} x^{\frac{1}{x}}$ is positive infinity.

26. Determine if $\{(\frac{1}{j})^{\frac{1}{j}}\}$, j from 1 to infinity converges or diverges.

27. Prove that if $\sum b_j$ converges, then $\sum (b_j)^2$ converges.

28. Let $b_j = \frac{1}{j+1}$ converges or diverges?

29. Let $b_j > 0, j = 1, 2, 3$. If $\sum b_j$ converges, what about $\sum \frac{b_j}{1+b_j}$?

30. $\sum \frac{2j+1}{4j^3-3}$ convergent or divergent?

31. $\sum \frac{2j-1}{3j^2-2}$ convergent or divergent?

32. Prove $\sum j * (\frac{2}{3})^j$ and $\sum \frac{3^j}{j!}$ converges using the Ratio Test.

33. Test the convergence of the series $\sum \frac{3^j}{j!}$.

34. Determine if $\sum_{j=2}^{\infty} \frac{1}{j(\log j)}$ converges or diverges.

35. Determine if $\sum_{j=2}^{\infty} \frac{1}{j(\log j)^2}$ converges or diverges.

36. Determine if the series $\sum \frac{\sqrt{j+1}-\sqrt{j}}{j+1}$ converges or diverges.

37. If $0 < b_j < \frac{1}{2}, j = 1, 2, 3$ and $\sum b_j$ converges. Show that $\sum \frac{b_j}{1-b_j}$ converges.

38. Determine if the series $\sum \frac{\cos j}{j^3}$ converges or diverges.

39. Determine if $\sum_{j}^{\infty} \frac{(-1)^j}{\sqrt{j}}$ converges or diverges.

40. (Optional) $\sum_{j=1}^{\infty} \frac{\sin j}{j}$ converge or diverge?

41. Use the $\varepsilon - \delta$ definition to prove $\lim_{x \to 1}(4x - 1) = 3$.

42. Use the $\varepsilon - \delta$ definition to prove $\lim_{x \to 1}(4x^2 - 2) = 2$.

43. Use the $\varepsilon - \delta$ definition to prove $\lim_{x \to 1} \frac{1}{x} = 1$.

44. Recall in homework number 13 $x_1 = \sqrt{2}$ and $x_{j+1} = \sqrt{2 + x_j},\ j = 1, 2, 3$. We showed that $\lim_{j \to \infty} x_j = L, L > 0$. Find the value of L.

45. Suppose (X, ρ) is a metric space, let $d(x, y) = \frac{\rho(x,y)}{1+\rho(x,y)}$ $\forall x, y \in X$. Show that d is a metric.

46. Let (X, ρ) be a metric space $p \in X$ is given. Define: $X \longrightarrow \mathbb{R}$, $x \longmapsto f(x) = \rho(x, p)$. $\forall x \in X$, show that f is continuous on X.

Appendix C.2 Test one: Analysis course

1. (a) Find the limit of $a_j = \frac{j}{3j-1}$ and use $\varepsilon - N$ to prove it.

   (b) Find the limit of $b_j = \frac{j^2+2j-3}{j^2-j+2}$ and use $\varepsilon - N$ to prove it.

2. If a sequence $\{a_j\}$ is decreasing and bounded below then $\{a_j\}$ is convergent.

3. Using the $\varepsilon - N$ definition, that $\lim_{j \to \infty} \frac{x_1+x_2+\cdots+x_j}{j} = 4$.

4. Prove that the sequence $\{a_j\}$ converges, where $a_1 > 2$, $j \geq 1$, and $a_{j+1} = \frac{1}{2}\left(\frac{4}{a_j} + a_j\right)$.

5. Prove that the sequence $y_i = \int_1^j \frac{\cos x}{x^2} dx, for\ j = 1, 2, 3$ ... is Cauchy.

Appendix C.3 Test two: Analysis course

1. Determine the convergence (or divergence) of:

a. $\sum_{j=1}^{\infty} \frac{(j+1)^{1/2}+j^{1/2}}{j^2}$

b. $\sum_{j=2}^{\infty} \frac{1}{j(log j)}$

c. $\sum_{j=1}^{\infty} \frac{3^j j!}{j^j}$

2. Prove if $a_j > 0$ for every j and if $\sum_{j=1}^{\infty} a_j$ converges, then prove that $\sum_{j=1}^{\infty} \frac{a_j}{j^{3/4}}$ converges.

3. (a) Use $\varepsilon - \delta$ to prove that $\lim_{x\to 1} \frac{3}{x^2} = 3$.

   (b) Use $\varepsilon - \delta$ definition to prove that $\lim_{x\to 2} \frac{4x^2-16}{x-2} = 16$.

4. Use the $\varepsilon - \delta$ definition to prove that the function $f(x) = \frac{x^2+1}{x^2+3}$ is continuous at $x = 1$.

5. Suppose that $\{a_j\}$ is a bounded sequence, $\lim_{j\to\infty} inf\ a_j = \alpha$ and $\lim_{j\to\infty} sup\ a_j = \beta$.

   (a) Prove that for any $\varepsilon > 0$ there exists a positive integer N such that $a_j > \alpha - \varepsilon$ when $j > N$.

   (b) Prove that for any $\varepsilon > 0$ there exists a positive integer K such that $a_j < \beta + \varepsilon$ when $j > K$.

Appendix C.4 Test three (take-home exam): Analysis course

1. (a) Use $\varepsilon - N$ to prove that $\lim_{j\to\infty} \frac{j}{2j-9} = \frac{1}{2}$.

   (b) Use $\varepsilon - N$ to prove that $\lim_{j\to\infty} \frac{2j^2}{j^2-5} = 2$.

   (c) Use $\varepsilon - \delta$ to prove that $\lim_{j\to 1} \frac{3x}{2x^2-1} = 3$.

   (d) Use $\varepsilon - \delta$ definition to prove that the function $f(x) = 3x^2$ is continuous at $x = 4$.

2. Prove that the sequence $\{(1+\frac{1}{j})^j\}$ is increasing and bounded.

3. Show that the sequence $a_j = \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{2}\right) - Log(j), j = 1, 2, 3, \ldots$, where $Log(j) = Ln(j)$ is convergent.

4. (a) For any given $a, b \geq 0$, show that $\sqrt{ab} \leq \frac{a+b}{2}$.

   (c) Suppose that both $\sum_{j=1}^{\infty} a_j{}^2$ and $\sum_{j=1}^{\infty} b_j{}^2$ are convergent. Show that $\sum_{j=1}^{\infty} a_j b_j$ converges absolutely.

   (c)If $b_j > 0, j = 1, 2, 3, \ldots$ and $\sum_{j=1}^{\infty} b_j$ converges, then prove that $\sum_{j=1}^{\infty}(b_j{}^{1/2} \frac{1}{j^{\lambda}})$ converges for $\lambda > \frac{1}{2}$.

Appendix C.5 Final Exam: Analysis course

1. Use $\varepsilon - \delta$ to prove that $\lim\limits_{x \to 1} \frac{3x}{2x^2-1} = 3$.

2. (a) State the definition that a sequence $\{a_j\}$ in $\mathbb{R}$ is a Cauchy Sequence.

   (b) Prove that the sequence $\{x_j\}$ defined by $x_j = \int_1^j \frac{sinx}{x} dx$, $j = 1,2,3, \ldots$, is a Cauchy sequence.

3. Calculate the limit (or prove that the limit does not exist. No need to use $\varepsilon - N$).

   $\lim\limits_{j \to \infty}[j - \sqrt{(j^2 + 2j)}]$.

4. Let $\{x_j\}$ be a sequence defined by $x_1 = 1$, and $x_{j+1} = \sqrt{(x_j)^2 + (\frac{1}{j^2})}, j = 1, 2, 3, \ldots$ Prove that $\{x_1\}$ is convergent.

5. (a) Determine the convergence or divergence of $\sum_{j=1}^{\infty} \frac{1}{j((j+1)^{1/2} - (j)^{1/2})}$.

   (b) Determine the convergence or divergence of $\sum_{j=1}^{\infty} (\frac{j}{j+2})^j$.

   (c) Prove or disprove that if both $\sum_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ are both convergent, then $\sum_{j=1}^{\infty} a_j\, b_j$ is convergent

6. Define $f(x) = \begin{cases} 3x \ \ if \ \ x \ \ is \ \ rational \\ \frac{3}{x} \ \ \ if \ \ x \ \ is \ irrational. \end{cases}$ Prove (use $\varepsilon - \delta$) that $f(x)$ is continuous at

   $x = 1$ for the function

7. Show that the sequence $\{x_n\}$ defined by $x_n = (1 + \frac{1}{n})^n, n = 1, 2, 3, \ldots$ is increasing and bounded.

Appendix C.6 Read aloud proofs: Analysis course

1. Prove that if x is odd, then $x + 1$ is even.
2. Proof evaluation: Consider the following "proof." Determine whether or not this is a valid proof. If it is valid, simply say so; if it is not, indicate what is wrong with the "proof" and provide a valid proof.

   Statement: Let n be an integer. If *3n – 8* is odd, then n is odd.

   Proof: Assume n is odd. Then *n = 2k + 1* for some integer *k*. Then

   3n – 8 = 3(2k + 1) – 8 = 6k + 3 – 8 = 6k – 5 = 2(3k – 3) + 1. Since *3k – 3* is an integer

   3n – 8 is odd.

3. For any positive integers m & n, if m^2 and n^2 are divisible by 3, then $\mathrm{m} + \mathrm{n}$ is divisible by 3
4. (a) State the definition that a sequence $\{\mathrm{a_j}\}$ in $\mathbb{R}$ is a Cauchy Sequence.

   (b) Prove that the sequence $\{x_j\}$ defined by $x_j = \int_1^j \frac{sinx}{x} dx, j = 1,2,3, \ldots$, is a Cauchy sequence.

Appendix D End of semester interview questions for the Bridge to Higher Math course

1. What previous knowledge or concepts do you think a student needs to be successful in this class?

2. How would you describe your study habits in the class? What would you do more of if you could?
3. Is there anything you would have done differently in the class to be more successful?
4. If you were to give advice to a student who is about to take this class in the fall, how would you advise them on
    a. How to study
    b. How to approach proofs
    c. What would you tell he or she to pay attention to?
    d. How often, how many times per week do they need to commit to the course?
    e. What should they spend the most time studying?
5. What topic was trickier for you?
    a. What did you do to overcome it? If didn't, why not?
    b. What type of proof required more attention and why?
6. Where do you place yourself with respect to others in the class?
7. Are you willing to continue this research for the fall in real analysis? This will be for my dissertation work.
8. Go over final exam.

Appendix E End of semester interview questions for the Analysis course

1. What previous knowledge or concepts do you think a student needs to be successful in this class?
2. How would you describe your study habits in the class?
    a. What would you do more of if you could?
    b. Did you do anything different from bridge to advanced math?

3. Is there anything you would have done differently in the class to be more successful?
4. Explain a little further your note taking technique in the class. What do you jot down and why?
5. If you were to give advice to a student who is about to take this class in the fall, how would you advise them on
6. How to study
7. How to approach proofs
8. What would you tell he or she to pay attention to?
9. How often, how many times per week do they need to commit to the course?
10. What should they spend the most time studying?
11. What topic was trickier for you?
    a. What did you do to overcome it? If didn't, why not?
    b. What type of proof required more attention and why?
12. Where do you place yourself with respect to others in the class?
13. What is the connection between Math 3000 and Math 4661?
    a. I would like for you to go over the following proofs out loud.