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doi: https://doi.org/10.57709/20410419

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TOPOLOGICAL PROPERTIES OF J-ORTHOGONAL MATRICES

by

SEYEDEH SARA MOOSAVI MOTLAGHIAN

Under the Direction of Frank J. Hall

ABSTRACT

Let $\mathbf{M}_n$ be the set of all $n \times n$ real matrices. A matrix $J \in \mathbf{M}_n$ is said to be a signature matrix if J is diagonal and its diagonal entries are ± 1. If J is a signature matrix, a nonsingular matrix $A \in \mathbf{M}_n$ is said to be a J-orthogonal matrix if $A^\top J A = J$. Let Ω_n be the set of all $n \times n$, J-orthogonal matrices. Part 2 of this dissertation includes a straightforward proof of the known topological result that for $J \neq \pm I$, the set of all $n \times n$ J-orthogonal matrices has four connected components. An important tool in this analysis is Proposition 3.2.1 on the characterization of J-orthogonal matrices in the paper "J-orthogonal matrices: properties and generation", SIAM Review 45 (3) (2003), 504–519, by Higham. The expression of the four components allows formulation of some further noteworthy properties. For example, it is shown that the four components are homeomorphic and group isomorphic, and that each component has exactly 2^{n-2} signature matrices. In Part 3 of this dissertation, the standard linear operators $T : \mathbf{M}_n \to \mathbf{M}_n$ that strongly preserve J-orthogonal matrices, i.e. $T(A)$ is J-orthogonal if and only if A is J-orthogonal are characterized. The material in Part 2 of this dissertation is contained in the article "Topological properties of J-orthogonal matrices", Linear and Multilinear Algebra, 66 (2018), 2524-2533, by S. Motlaghian, A. Armandnrjad, F. J. Hall. The material in Part 3 of this dissertation is contained in the article "Topological properties of J-orthogonal matrices, Part II", Linear and Multilinear Algebra, doi: 10.1080/03081087.2019.1601667 by S. Motlaghian, A. Armandnrjad, F. J. Hall. In Part 4 of this dissertation some connections between J-orthogonal and G-matrices are investigated.

INDEX WORDS: Signature matrix, Signed permutation matrix, Linear preservers, J-orthogonal matrix, G-matrix.

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SEYEDEH SARA MOOSAVI MOTLAGHIAN

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

in the College of Arts and Sciences

Georgia State University

2020

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2020

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College of Arts and Sciences

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December 2020

DEDICATION

This dissertation is dedicated to Georgia State University.

ACKNOWLEDGEMENTS

This dissertation work would not have been possible without the support of many people. I want to express my gratitude to my advisor Professor Frank J. Hall, and to the other dissertation committee members, Professor Marina Arav, Professor Michael Stewart, Professor Zhongshan Li and Professor Hein van der Holst.

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- GSU - Georgia State University

PART 1

INTRODUCTION

Let $\mathbf{M}_n$ be the set of all $n \times n$ real matrices. A matrix $J \in \mathbf{M}_n$ is said to be a signature matrix if J is diagonal and its diagonal entries are ± 1. As in [1], if J is a signature matrix, a nonsingular matrix $A \in \mathbf{M}_n$ is said to be a J-orthogonal matrix if $A^\top JA = J$. Some properties of J-orthogonal matrices were investigated in [1]. J-orthogonal matrices were studied for example in the context of the group theory [2] or generalized eigenvalue problems [3]. Numerical properties of several orthogonalization techniques with respect to symmetric indefinite bilinear forms have been analyzed recently in . Although J-orthogonality has many numerical connections, the recent paper [4] has more of a combinatorial matrix theory point of view, in particular, the analysis of sign patterns of J-orthogonal matrices.

The following conventions will also be fixed throughout this dissertation:
The set of all real numbers is denoted as usual by $\mathbb{R}$; for $A \in \mathbf{M}_n$, $\sigma(A)$ and $\mathcal{S}(A)$ are the set of eigenvalues of A and the set of singular values of A respectively; E_{ij} is the $n \times n$ matrix whose (i, j) entry is one and all other entries are zero; $\mathcal{O}_n$ is the set of all $n \times n$ orthogonal matrices; $\mathcal{O}_n^+$ is the set of all $n \times n$ orthogonal matrices with determinant 1; $\mathcal{O}_n^-$ is the set of all $n \times n$ orthogonal matrices with determinant -1; the matrix norm used in this paper is the spectral norm $\| \ \|_2$; $\mathcal{S}_n$ is the set of all $n \times n$ signature matrices; $\mathcal{SP}_n$ is the set of all $n \times n$ signed permutation matrices, the $n \times n$ matrices with exactly one nonzero entry ± 1 in each row and in each column; for $J \in \mathcal{S}_n$,

$$\Gamma_n(J) = \{A \in \mathbf{M}_n : A^\top JA = J\};$$

Ω_n is the set of all $n \times n$ J-orthogonal matrices for all possible $n \times n$ matrices $J = \mathrm{diag}(\pm 1)$, that is,

$$\Omega_n = \cup_{J \in \mathcal{S}_n} \Gamma_n(J).$$

We consider the set of all $n \times n$ J-orthogonal matrices and in Part 2 we first find some interesting properties of these matrices. For example, we show that

$$\mathcal{S}_n = \cap_{J \in \mathcal{S}_n} \Gamma_n(J).$$

We also find that for $P \in \mathbf{M}_n$, $P \in \mathcal{SP}_n$ if and only if $P^\top \mathcal{S}_n P = \mathcal{S}_n$. The fact that for $J \neq \pm I$, $\Gamma_n(J)$ has four connected components is known. However, through the use of a characterization of J-orthogonal matrices contained in the article [1] by N. Higham, we give a straightforward matrix analysis proof of this topological result. We also show that Ω_n has two connected components. Our expression of the four components allows us to then formulate some interesting by-products. For example, we show that the four components are homeomorphic and group isomorphic. Also for $J \neq \pm I$, we prove that every component of $\Gamma_n(J)$ has exactly 2^{n-2} signature matrices. We also show that for non-scalar matrices J_1 and J_2, to determine whether $\Gamma_n(J_1) = \Gamma_n(J_2)$, it is sufficient to know whether they contain the same orthogonal matrices, or whether they contain the same matrices with spectral norm 1.

For $A \in \mathbf{M}_n$, the linear operator $T : \mathbf{M}_n \to \mathbf{M}_n$ defined by $T(X) = A^\top X A$ or $T(X) = A^\top X^\top A$ is called a standard linear operator on $\mathbf{M}_n$. In Part 3 we show that a standard linear operator $T : \mathbf{M}_n \to \mathbf{M}_n$ strongly preserves the set of J-orthogonal matrices if and only if A is a signed permutation matrix. In fact we show that for a (necessarily nonsingular) matrix B the following conditions are equivalent:

(i) $B \in \mathcal{SP}_n$,

(ii) $B^\top \mathcal{S}_n B = \mathcal{S}_n$,

(iii) $B^\top \Omega_n B = \Omega_n$.

The material in Part 2 of this dissertation is contained in the article "Topological properties of J-orthogonal matrices", Linear and Multilinear Algebra, 66 (2018), 2524-2533, by S. Motlaghian, A. Armandnrjad, F. J. Hall. The material in Part 3 of this dissertation is contained in the article "Topological properties of J-orthogonal matrices, Part II", Linear and

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PART 2

TOPOLOGICAL PROPERTIES OF J-ORTHOGONAL MATRICES I

2.1 Introduction

Let $\mathbf{M}_n$ be the set of all $n \times n$ real matrices. A matrix $J \in \mathbf{M}_n$ is said to be a signature matrix if J is diagonal and its diagonal entries are ± 1. As in [1], if J is a signature matrix, a nonsingular matrix $A \in \mathbf{M}_n$ is said to be a J-orthogonal matrix if $A^\top JA = J$. Some properties of J-orthogonal matrices were investigated in [1]. J-orthogonal matrices were studied for example in the context of the group theory [2] or generalized eigenvalue problems [3]. Numerical properties of several orthogonalization techniques with respect to symmetric indefinite bilinear forms have been analyzed recently in [5]. Although J-orthogonality has many numerical connections, the recent paper [4] has more of a combinatorial matrix theory point of view, in particular, the analysis of sign patterns of J-orthogonal matrices.

The following conventions will also be fixed throughout the paper:

The set of all real numbers is denoted as usual by $\mathbb{R}$; for $A \in \mathbf{M}_n$, $\sigma(A)$ is the spectrum of A, the set of eigenvalues of A; $\mathcal{O}_n$ is the set of all $n \times n$ orthogonal matrices; $\mathcal{O}_n^+$ is the set of all $n \times n$ orthogonal matrices with determinant 1; $\mathcal{O}_n^-$ is the set of all $n \times n$ orthogonal matrices with determinant -1; $\mathcal{S}_n$ is the set of all $n \times n$ signature matrices; $\mathcal{SP}_n$ is the set of all $n \times n$ signed permutation matrices, the $n \times n$ matrices with exactly one nonzero entry ± 1 in each row and in each column; for $J \in \mathcal{S}_n$,

$$\Gamma_n(J) = \{A \in \mathbf{M}_n : A^\top JA = J\};$$

Ω_n is the set of all $n \times n$ J-orthogonal matrices for all possible $n \times n$ matrices $J = \mathrm{diag}(\pm 1)$, that is,

$$\Omega_n = \cup_{J \in \mathcal{S}_n} \Gamma_n(J).$$

In this paper we consider the set of all $n \times n$ J-orthogonal matrices and we first find some interesting properties of these matrices. For example, we show that

$$\mathcal{S}_n = \cap_{J \in \mathcal{S}_n} \Gamma_n(J).$$

We also find that for $P \in \mathbf{M}_n$, $P \in \mathcal{SP}_n$ if and only if $P^\top \mathcal{S}_n P = \mathcal{S}_n$. The fact that for $J \neq \pm I$, $\Gamma_n(J)$ has four connected components is known. However, through the use of a characterization of J-orthogonal matrices contained in the article [1] by N. Higham, we give a straightforward matrix analysis proof of this topological result. We also show that Ω_n has two connected components. Our expression of the four components allows us to then formulate some interesting by-products. For example, we show that the four components are homeomorphic and group isomorphic. Also for $J \neq \pm I$, we prove that every component of $\Gamma_n(J)$ has exactly 2^{n-2} signature matrices. In the paper, we also show that for non-scalar matrices J_1 and J_2, to determine whether $\Gamma_n(J_1) = \Gamma_n(J_2)$, it is sufficient to know whether they contain the same orthogonal matrices, or whether they contain the same matrices with spectral norm 1.

2.2 Some properties of J-orthogonal matrices

In this section we discuss some general properties of J-orthogonal matrices. These include algebraic and topological properties of $\Gamma_n(J)$ and Ω_n.

Proposition 2.2.1. *The following statements are true.*

(i) $\Gamma_n(J)$ *is a Lie group for every* $J \in \mathcal{S}_n$.

(ii) $\boldsymbol{M}_n = \mathrm{Span}(\Omega_n) = \mathrm{Span}(\mathcal{SP}_n)$.

(iii) For every symmetric matrix norm $\| \; \|$ $(\|A\| = \|A^\top\|)$, $\|A\| \geq 1$ *for all* $A \in \Omega_n$.

(iv) $\mathcal{S}_n = \cap_{J \in \mathcal{S}_n} \Gamma_n(J)$.

Proof. The fact that $\Gamma_n(J)$ is a Lie group is contained for example in [6, page 7]. To prove (ii), for every $1 \le i, j \le n$ let P_{ij} be a permutation matrix with 1 in the (i, j) position and Q_{ij} be the signed permutation matrix such that $P_{ij} + Q_{ij} = 2E_{ij}$. So, $E_{ij} = \frac{1}{2}(P_{ij} + Q_{ij})$ and hence $\mathbf{M}_n = \mathrm{Span}(\mathcal{SP}_n)$. Then it is clear that $\mathbf{M}_n = \mathrm{Span}(\Omega_n)$, since $\mathcal{SP}_n \subseteq \Omega_n$. To prove (iii), assume that $A \in \Omega_n$, so that there exists a signature matrix J such that $A^\top JA = J$. Then

$$\|A\|\|J\|\|A^\top\| \ge \|J\|,$$

and hence $\|A\| \ge 1$. To prove (iv), since diagonal matrices commute under multiplication, we have that $\mathcal{S}_n \subseteq \cap_{J\in\mathcal{S}_n}\Gamma_n(J)$. Let $A = (a_{ij}) \in \cap_{J\in\mathcal{S}_n}\Gamma_n(J)$. Then $A \in \Gamma_n(I)$ (i.e. A is an orthogonal matrix) and hence the Euclidean norm of every row and every column of A is 1. Now consider the signature matrix $J = (-1) \oplus I_{n-1}$. Then $A^\top JA = J$ and hence $a_{11}^2 = 1$. Since the Euclidean norm of the first row and the first column of A is 1, we have $a_{12} = \cdots = a_{1n} = a_{21} = \cdots = a_{n1} = 0$. Similarly, for every $2 \le i \le n$, we can show that $a_{ii}^2 = 1$ and $a_{ij} = a_{ji} = 0$ for all $j \ne i$. Therefore A is a signature matrix.

□

The following proposition gives an equivalent condition for a matrix to be a J-orthogonal matrix.

Proposition 2.2.2. *Let $A \in \boldsymbol{M}_n$. Then $A \in \Omega_n$ if and only if there exists a signature matrix J such that $x^\top(A^\top JA)x = x^\top Jx$ for all $x \in \mathbb{R}^n$.*

Proof. If $A \in \Omega_n$ then for some $J \in \mathcal{S}_n$, $A \in \Gamma_n(J)$, so that $A^\top JA = J$. Hence $x^\top(A^\top JA)x = x^\top Jx$ for all $x \in \mathbb{R}^n$. Conversely, assume that there exists a signature matrix J such that $x^\top(A^\top JA)x = x^\top Jx$ for all $x \in \mathbb{R}^n$. Then $x^\top(A^\top JA - J)x = 0$ for all $x \in \mathbb{R}^n$. Since $A^\top JA - J$ is Hermitian, $A^\top JA = J$ and hence A is J-orthogonal, so that $A \in \Omega_n$. □

Remark 2.2.3. *It is known that in any Lie group G, the connected component H containing the identity also forms a Lie group. Furthermore, H is a normal subgroup of G and G/H is a discrete group [6]. In Section 3 we will specifically exhibit these facts for the Lie group $\Gamma_n(J)$.*

The following example, the Lorentz group, has particular interest in physics, [6, page 7].

Example 2.2.4. *Let* $J = 1 \oplus (-I_3)$. *Then*

$$\Gamma_4(J) = \{A \in \boldsymbol{M}_4 : d(Ax) = d(x),\ \forall\ x \in \mathbf{R}^4\},$$

where d *is the space-time metric* $d(x_1, x_2, x_3, x_4) = x_1^2 - x_2^2 - x_3^2 - x_4^2$.

The following proposition shows that every line in $\mathbf{M}_n$ passing from the origin, either does not meet Ω_n or meets Ω_n at exactly two points. Letting $\mathcal{T}_n(\epsilon)$ be the circle $\{A \in \mathbf{M}_n : \|A\| = \epsilon\}$ we also show that when $\epsilon \geq \|I\|$, $\mathcal{T}_n(\epsilon) \cap \Gamma_n(J) \neq \emptyset$.

Proposition 2.2.5. *Let* $J \in \mathcal{S}_n$. *Then the following hold:*

(i) For every $A \in \boldsymbol{M}_n$, *if* $L_A = \{rA : r \in \mathbb{R}\}$, *then* $L_A \cap \Omega_n = \emptyset$ *or* $L_A \cap \Omega_n = \{B, -B\}$ *for some* $B \in \Omega_n$.

(ii) If $n \geq 2$ *and* $J \in \mathcal{S}_n$ *with* $J \neq \pm I$, *then for every* $\epsilon \geq \|I\|$, $\mathcal{T}_n(\epsilon) \cap \Gamma_n(J) \neq \emptyset$, *where* $\| \ \|$ *is any norm on* $\boldsymbol{M}_n$.

Proof. (i). If $L_A \cap \Omega_n = \emptyset$, there is nothing to prove. So assume that $L_A \cap \Omega_n \neq \emptyset$, and hence there exists some $B \in L_A \cap \Omega_n$. Then $-B \in L_A \cap \Omega_n$ and so $\{B, -B\} \subseteq L_A \cap \Omega_n$. If $C \in L_A \cap \Omega_n$, then $C = rB$ for some $r \in \mathbb{R}$. Hence there exist $J_1, J_2 \in \mathcal{S}_n$ such that $B^\top J_1 B = J_1$ and $(rB^\top)J_2(rB) = J_2$. These imply that $B = J_1^{-1}B^{-\top}J_1$ and $r^2B = J_2^{-1}B^{-\top}J_2$ and hence $\sigma(B) = r^2\sigma(B)$. Since $\sigma(B) \neq \{0\}$, $r^2 = 1$, which implies that $r = \pm 1$ and hence $L_A \cap \Omega_n \subseteq \{B, -B\}$.

(ii). Without loss of generality we assume that the first and the second diagonal entries of J are 1 and -1 respectively (if the 1 and the -1 are in the i^{th} and j^{th} diagonal positions respectively, the construction is similar). For every $t \in [0, \infty)$,

$$C_t = \begin{pmatrix} \sqrt{1+t^2} & t \\ t & \sqrt{1+t^2} \end{pmatrix} \oplus I_{n-2}$$

is a J-orthogonal matrix. So we can choose $t \in [0, \infty)$ such that $\|C_t\| = \epsilon$. $\square$

The following example shows that $P^\top \Omega_n P \neq \Omega_n$ for some orthogonal matrix P.

Example 2.2.6. *Let* $P = \frac{\sqrt{2}}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \oplus I_{n-2}$. *For* $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \oplus I_{n-2}$, *a* J*-orthogonal matrix is* $A = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix} \oplus I_{n-2}$. *Now,*

$$P^\top AP = \begin{pmatrix} 1+\sqrt{2} & 0 \\ 0 & -1+\sqrt{2} \end{pmatrix} \oplus I_{n-2},$$

which is not J*-orthogonal for any* $J \in \mathcal{S}_n$.

However, for matrices $P \in \mathcal{SP}_n$, the following holds.

Theorem 2.2.7. *Let* $P \in \boldsymbol{M}_n$. *Then* $P \in \mathcal{SP}_n$ *if and only if* $P^\top \mathcal{S}_n P = \mathcal{S}_n$. *Furthermore, if* $P \in \mathcal{SP}_n$, *then* $P^\top \Omega_n P = \Omega_n$.

Proof. If $P \in \mathcal{SP}_n$, changing the signs of the rows of P is equivalent to changing the signs of the columns of P, that is $\mathcal{S}_n P = P\mathcal{S}_n$, and hence $P^\top \mathcal{S}_n P = \mathcal{S}_n$. To prove the converse, we use induction on n. If $n = 1$, the proof is clear. Let $n > 1$, $P \in \mathbf{M}_n$, and $P^\top \mathcal{S}_n P = \mathcal{S}_n$. By assumption, $P^\top IP$ is a signature matrix and by the Sylvester law of inertia, $PIP^\top = I$, which implies that P is an orthogonal matrix. Let $J_1 = (-1) \oplus I_{n-1}$. Then there exists $J_2 \in \mathcal{S}_n$ such that $P^\top J_1 P = J_2$. By again using the Sylvester law of inertia, J_2 has exactly one -1 entry and without loss of generality we may assume that this -1 is in the $(1,1)$-position, so that $J_1 = J_2$. Since P is an orthogonal matrix, $PJ_1 = J_1 P$ which implies that $P = (\pm 1) \oplus Q$ for some $Q \in \mathbf{M}_{n-1}$. Since $P^\top JP$ is a signature matrix for every $J \in \mathcal{S}_n$, we can obtain that $Q^\top \widehat{J} Q$ is a signature matrix, for every $\widehat{J} \in \mathcal{S}_{n-1}$. Now, by the induction assumption, $Q \in \mathcal{SP}_{n-1}$. Thus, $P \in \mathcal{SP}_n$, which completes the proof of the if and only if. Finally, since for $J \in \mathcal{S}_n$ and $P \in \mathcal{SP}_n$, $A^\top JA = J$ implies that

$$(P^\top AP)^\top (P^\top JP)(P^\top AP) = (P^\top JP),$$

the last statement of the theorem is seen to be true. □

An interesting open question is the following: if $P^\top \Omega_n P = \Omega_n$, is it necessarily the case that $P \in \mathcal{SP}_n$. If the answer is yes, then topologically this means the following: if the linear transformation $X \mapsto P^\top X P$ maps Ω_n to Ω_n, then the transformation must map the special points $\mathcal{S}_n$ to $\mathcal{S}_n$. Although geometrically this appears to be true, it is a nontrivial question. When $n = 2$, the result is true, see Example 2.3.7. However, for $n > 2$, it becomes much more complicated.

2.3 Connected components of J-orthogonal matrices

The following two propositions are well known results for orthogonal and J-orthogonal matrices.

Proposition 2.3.1. *[7, Theorem 3.67] For every $n \geq 1$, $\mathcal{O}_n$ has two connected components, $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$.*

The following characterization of J-orthogonal matrices is contained in the article [1] by N. Higham. As stated in [1], this decomposition was first derived in [8]; it is also mentioned in [1] that in a preliminary version of [9] (which was published later) the authors treat this decomposition in more depth.

Proposition 2.3.2. *[1, Theorem 3.2 (hyperbolic CS decomposition)] Let $q \geq p$ and $J = I_p \oplus (-I_q)$. Then every $A \in \Gamma_n(J)$ is of the form*

$$(U_1 \oplus U_2)(\begin{pmatrix} C & -S \\ -S & C \end{pmatrix} \oplus I_{q-p})(V_1 \oplus V_2), \tag{2.1}$$

where $U_1, V_1 \in \mathcal{O}_p$, $U_2, V_2 \in \mathcal{O}_q$ and $C, S \in \mathbf{M}_p$ are nonnegative diagonal matrices such that $C^2 - S^2 = I$. Also, any matrix of the form (3.1) is J-orthogonal.

Corollary 2.3.3. *Let $q \geq p$ and $J = I_p \oplus (-I_q)$. Then $A \in \Gamma_n(J)$ if and only if A is of the form*

$$(U_1 \oplus U_2)(\begin{pmatrix} \mathrm{diag}(\sqrt{1+\mathrm{b}_1^2}, \ldots, \sqrt{1+\mathrm{b}_\mathrm{p}^2}) & \mathrm{diag}(\mathrm{b}_1, \ldots, \mathrm{b}_\mathrm{p}) \\ \mathrm{diag}(\mathrm{b}_1, \ldots, \mathrm{b}_\mathrm{p}) & \mathrm{diag}(\sqrt{1+\mathrm{b}_1^2}, \ldots, \sqrt{1+\mathrm{b}_\mathrm{p}^2}) \end{pmatrix} \oplus I_{q-p})(V_1 \oplus V_2), \tag{2.2}$$

where $U_1, V_1 \in \mathcal{O}_p$, $U_2, V_2 \in \mathcal{O}_q$, and $b_1, \ldots, b_p \in \mathbb{R}$.

Proof. The proof of the necessity follows from Proposition 3.2.1. To prove the converse, observe that by using a suitable signature matrix D, we have

$$D\begin{pmatrix} \operatorname{diag}(\sqrt{1+b_1^2},\ldots,\sqrt{1+b_p^2}) & \operatorname{diag}(b_1,\ldots,b_p) \\ \operatorname{diag}(b_1,\ldots,b_p) & \operatorname{diag}(\sqrt{1+b_1^2},\ldots,\sqrt{1+b_p^2}) \end{pmatrix} D = \begin{pmatrix} C & -S \\ -S & C \end{pmatrix},$$

where $C = \operatorname{diag}(\sqrt{1+b_1^2},\ldots,\sqrt{1+b_p^2})$ and $S = \operatorname{diag}(|b_1|,\ldots,|b_p|)$. Hence, every matrix of the form (2.2) is of the form (3.1). □

Letting $\mathcal{T}_n$ be the set of $n \times n$ matrices A with spectral norm $\|A\|_2$ equal to one, we have the following result.

Corollary 2.3.4. *For every n, $\mathcal{O}_n = \Omega_n \cap \mathcal{T}_n$.*

Proof. If $A \in \mathcal{O}_n$, then clearly $A \in \Omega_n$ and $\|A\|_2 = 1$. On the other hand, let $A \in \Omega_n$. By the use of Corollary 2.3.3,
$A = (U_1 \oplus U_2)(\begin{pmatrix} \operatorname{diag}(\sqrt{1+b_1^2},\ldots,\sqrt{1+b_p^2}) & \operatorname{diag}(b_1,\ldots,b_p) \\ \operatorname{diag}(b_1,\ldots,b_p) & \operatorname{diag}(\sqrt{1+b_1^2},\ldots,\sqrt{1+b_p^2}) \end{pmatrix} \oplus I_{q-p})(V_1 \oplus V_2)$, for some $U_1, V_1 \in \mathcal{O}_p$, $U_2, V_2 \in \mathcal{O}_q$, and $b_1, \ldots, b_p \in \mathbb{R}$. If $\|A\|_2 = 1$, then

$$\|\begin{pmatrix} \operatorname{diag}(\sqrt{1+b_1^2},\ldots,\sqrt{1+b_p^2}) & \operatorname{diag}(b_1,\ldots,b_p) \\ \operatorname{diag}(b_1,\ldots,b_p) & \operatorname{diag}(\sqrt{1+b_1^2},\ldots,\sqrt{1+b_p^2}) \end{pmatrix}\|_2 \le 1.$$

Therefore $\|(\sqrt{1+b_j^2}, b_j)\|_2 \le 1$ for every j $(1 \le j \le p)$ and hence $b_1 = \cdots = b_p = 0$. This implies that A is orthogonal.

□

The fact that for $J \ne \pm I$, $\Gamma_n(J)$ has four connected components is known, see [10, page 345] (in this book $\Gamma_n(J)$ is referred to as an indefinite orthogonal group). However, we can give a straightforward matrix analysis proof of this topological result. We also show that Ω_n has two connected components.

Theorem 2.3.5. *The following assertions hold:*

(i) Let $J \in \mathcal{S}_n$. If $J \ne \pm I$ then $\Gamma_n(J)$ has four connected components.

(ii) Ω_n *has two connected components.*

Proof. (i). Let p and q be the numbers of $+1$ and -1 diagonal entries of J respectively. If $p > q$, we use the fact that $\Gamma_n(J) = \Gamma_n(-J)$, and so we can assume that $q \geq p$. Since for every $n \times n$ permutation matrix P we have

$$\Gamma_n(J) = P(\Gamma_n(P^\top JP))P^\top,$$

without loss of generality we may assume that $q \geq p$ and $J = I_p \oplus (-I_q)$. The function $\varphi : \mathbb{R}^p \to \mathbf{M}_n$ defined by

$$\varphi(b_1, \ldots, b_p) = \begin{pmatrix} \mathrm{diag}(\sqrt{1+b_1^2}, \ldots, \sqrt{1+b_p^2}) & \mathrm{diag}(b_1, \ldots, b_p) \\ \mathrm{diag}(b_1, \ldots, b_p) & \mathrm{diag}(\sqrt{1+b_1^2}, \ldots, \sqrt{1+b_p^2}) \end{pmatrix} \oplus I_{q-p},$$

is continuous and hence $\{\varphi(b_1, \ldots, b_p) : b_1, \ldots, b_p \in \mathbb{R}\}$ is a connected set in $\mathbf{M}_n$. From Proposition 4.3.1, we have $\mathcal{O}_p = \mathcal{O}_p^+ \cup \mathcal{O}_p^-$ and $\mathcal{O}_q = \mathcal{O}_q^+ \cup \mathcal{O}_q^-$. So we have 16 possible cases to choose $U_1, V_1 \in \mathcal{O}_p$, $U_2, V_2 \in \mathcal{O}_q$. Then by Corollary 2.3.3, $\Gamma_n(J) = \cup_{i=1}^{16} \Phi_i$ where

$$\Phi_1 = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) : \; U_1, V_1 \in \mathcal{O}_p^+, U_2, V_2 \in \mathcal{O}_q^+, b \in \mathbb{R}^p\},$$

$$\Phi_2 = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) : \; U_1, V_1 \in \mathcal{O}_p^+, U_2, V_2 \in \mathcal{O}_q^-, b \in \mathbb{R}^p\},$$

$$\Phi_3 = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) : \; U_1, V_1 \in \mathcal{O}_p^-, U_2, V_2 \in \mathcal{O}_q^+, b \in \mathbb{R}^p\},$$

$$\Phi_4 = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) : \; U_1, V_1 \in \mathcal{O}_p^-, U_2, V_2 \in \mathcal{O}_q^-, b \in \mathbb{R}^p\},$$

$$\Phi_5 = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) : \; U_1, V_1 \in \mathcal{O}_p^+, U_2 \in \mathcal{O}_q^+, V_2 \in \mathcal{O}_q^-, b \in \mathbb{R}^p\},$$

$$\Phi_6 = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) : \; U_1, V_1 \in \mathcal{O}_p^+, U_2 \in \mathcal{O}_q^-, V_2 \in \mathcal{O}_q^+, b \in \mathbb{R}^p\},$$

$$\Phi_7 = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) : \; U_1, V_1 \in \mathcal{O}_p^-, U_2 \in \mathcal{O}_q^+, V_2 \in \mathcal{O}_q^-, b \in \mathbb{R}^p\},$$

$$\Phi_8 = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) :\ U_1, V_1 \in \mathcal{O}_p^-, U_2 \in \mathcal{O}_q^-, V_2 \in \mathcal{O}_q^+, b \in \mathbb{R}^p\},$$

$$\Phi_9 = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) :\ U_1 \in \mathcal{O}_p^+, V_1 \in \mathcal{O}_p^-, U_2 \in \mathcal{O}_q^+, V_2 \in \mathcal{O}_q^-, b \in \mathbb{R}^p\},$$

$$\Phi_{10} = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) :\ U_1 \in \mathcal{O}_p^+, V_1 \in \mathcal{O}_p^-, U_2 \in \mathcal{O}_q^-, V_2 \in \mathcal{O}_q^+, b \in \mathbb{R}^p\},$$

$$\Phi_{11} = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) :\ U_1 \in \mathcal{O}_p^-, V_1 \in \mathcal{O}_p^+, U_2 \in \mathcal{O}_q^+, V_2 \in \mathcal{O}_q^-, b \in \mathbb{R}^p\},$$

$$\Phi_{12} = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) :\ U_1 \in \mathcal{O}_p^-, V_1 \in \mathcal{O}_p^+, U_2 \in \mathcal{O}_q^-, V_2 \in \mathcal{O}_q^+, b \in \mathbb{R}^p\},$$

$$\Phi_{13} = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) :\ U_1 \in \mathcal{O}_p^+, V_1 \in \mathcal{O}_p^-, U_2, V_2 \in \mathcal{O}_q^+, b \in \mathbb{R}^p\},$$

$$\Phi_{14} = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) :\ U_1 \in \mathcal{O}_p^+, V_1 \in \mathcal{O}_p^-, U_2, V_2 \in \mathcal{O}_q^-, b \in \mathbb{R}^p\},$$

$$\Phi_{15} = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) :\ U_1 \in \mathcal{O}_p^-, V_1 \in \mathcal{O}_p^+, U_2, V_2 \in \mathcal{O}_q^+, b \in \mathbb{R}^p\},$$

$$\Phi_{16} = \{(U_1 \oplus U_2)\varphi(b)(V_1 \oplus V_2) : U_1 \in \mathcal{O}_p^-, V_1 \in \mathcal{O}_p^+, U_2, V_2 \in \mathcal{O}_q^-, b \in \mathbb{R}^p\}.$$

Since in each of the 16 cases,

$$\{U_1 \oplus U_2\},\ \{\varphi(b)\},\ \{V_1 \oplus V_2\}$$

are all connected, we have that for every i $(1 \le i \le 16)$, Φ_i is a connected set. Let

$$\mathcal{C}_1 = \cup_{i=1}^4 \Phi_i,\ \mathcal{C}_2 = \cup_{i=5}^8 \Phi_i,\ \mathcal{C}_3 = \cup_{i=9}^{12} \Phi_i,\ \mathcal{C}_4 = \cup_{i=13}^{16} \Phi_i.$$

Now,

$$I_n \in \cap_{i=1}^4 \Phi_i,\ I_{n-1} \oplus (-1) \in \cap_{i=5}^8 \Phi_i,$$

$$(-1) \oplus I_{n-2} \oplus (-1) \in \cap_{i=9}^{12} \Phi_i, \ (-1) \oplus I_{n-1} \in \cap_{i=13}^{16} \Phi_i.$$

So, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ are connected sets. Then $\Gamma_n(J) = \cup_{i=1}^4 \mathcal{C}_i$ has at most four connected components. We show that $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{C}_3$ and $\mathcal{C}_4$ are mutually disjoint. Since the sign of the determinant is constant on a connected set of nonsingular matrices, for every $A \in \mathcal{C}_1 \cup \mathcal{C}_3$, $\det(A) > 0$ and for every $A \in \mathcal{C}_2 \cup \mathcal{C}_4$, $\det(A) < 0$. Hence, we have that $\mathcal{C}_1 \cap \mathcal{C}_2 = \mathcal{C}_1 \cap \mathcal{C}_4 = \mathcal{C}_2 \cap \mathcal{C}_3 = \mathcal{C}_3 \cap \mathcal{C}_4 = \emptyset$. Assume if possible that $A \in \mathcal{C}_1 \cap \mathcal{C}_3$; then by Proposition 3.2.1, there exist orthogonal matrices $U_1, U_1^{'}, U_2, U_2^{'}, V_1, V_1^{'}, V_2, V_2^{'}$ and diagonal matrices $C, C^{'}, S, S^{'}$ such that $U_1V_1 \in \mathcal{O}_p^+$, $U_1^{'}V_1^{'} \in \mathcal{O}_p^-$ and

$$\begin{aligned} A &= (U_1 \oplus U_2)(\begin{pmatrix} C & -S \\ -S & C \end{pmatrix} \oplus I_{q-p})(V_1 \oplus V_2) \\ &= (U_1^{'} \oplus U_2^{'})(\begin{pmatrix} C^{'} & -S^{'} \\ -S^{'} & C^{'} \end{pmatrix} \oplus I_{q-p})(V_1^{'} \oplus V_2^{'}). \end{aligned}$$

By a simple multiplication we obtain that $U_1CV_1 = U_1^{'}C^{'}U_1^{'}$ and hence $\det(C) = -\det(C^{'})$ which is a contradiction because $\det(C), \det(C^{'}) > 0$. Therefore $\mathcal{C}_1 \cap \mathcal{C}_3 = \emptyset$. Similarly we can show that $\mathcal{C}_2 \cap \mathcal{C}_4 = \emptyset$.

(ii). We know that for every $J \in \mathcal{S}_n$, $\mathcal{S}_n \subset \mathcal{O}_n \cap \Gamma_n(J)$. Then by (i) and the use of Proposition 4.3.1, for every $J \in \mathcal{S}_n$, $\mathcal{O}_n \cup \Gamma_n(J)$ has two connected components $\mathcal{C}_1^{'}$ and $\mathcal{C}_2^{'}$ such that for every $A \in \mathcal{C}_1^{'}$, $\det(A) > 0$ and for every $A \in \mathcal{C}_2^{'}$, $\det(A) < 0$. Since the sign of the determinant is constant on a connected set of nonsingular matrices, $\Omega_n = \cup_{J \in \mathcal{S}_n}(\mathcal{O}_n \cup \Gamma_n(J))$ has two connected components. □

By Remark 2.2.3, $\mathcal{C}_1$, the connected component of $\Gamma_n(J)$ containing the identity matrix is a normal subgroup of $\Gamma_n(J)$ and $\Gamma_n(J)/\mathcal{C}_1$ is a group isomorphic to the Klein four-group [6]. The following proposition shows that for every i $(2 \leq i \leq 4)$, there exists an operation $*_i$ such that $(\mathcal{C}_i, *_i)$ is a group that is isomorphic to $(\mathcal{C}_1, .)$.

Proposition 2.3.6. *Let $J \in \mathcal{S}_n$ and $J \neq \pm I$. Then for every i $(2 \leq i \leq 4)$, the component $\mathcal{C}_i$ of $\Gamma_n(J)$ is homeomorphic and group isomorphic to $\mathcal{C}_1$.*

Proof. Let $\mathcal{C}_i$ $(1 \leq i \leq 4)$ be as in the proof of Theorem 2.3.5. We define the binary operations $*_i$ $(2 \leq i \leq 4)$ on $\mathcal{C}_i$ as follows:

$$\begin{aligned} A *_2 B &= A(I_{n-1} \oplus (-1))B \text{ for all A, B} \in \mathcal{C}_2, \\ A *_3 B &= A((-1) \oplus I_{n-2} \oplus (-1))B \text{ for all A, B} \in \mathcal{C}_3, \\ A *_4 B &= A((-1) \oplus I_{n-1})B \text{ for all A, B} \in \mathcal{C}_4. \end{aligned}$$

We just show that $(\mathcal{C}_2, *_2)$ is group isomorphic to $(\mathcal{C}_1, .)$; the other cases are similar. Clearly, $*_2$ is associative. Let $A, B \in \mathcal{C}_2$. Since $\mathcal{C}_2$ is connected and $I_{n-1} \oplus (-1) \in \mathcal{C}_2$, there exist continuous functions $f, g : [0,1] \to \Gamma_n(J)$ such that $f(0) = A$, $g(0) = B$, $f(1) = g(1) = I_{n-1} \oplus (-1)$. Define the continuous functions $h, k : [0,1] \to \Gamma_n(J)$ by

$$h(x) = f(x)(I_{n-1} \oplus (-1))g(x)$$

and

$$k(x) = f(x)A^{-1}(I_{n-1} \oplus (-1))$$

respectively. Then $h(0) = A *_2 B$, $k(1) = (I_{n-1} \oplus (-1))A^{-1}(I_{n-1} \oplus (-1))$ and $h(1) = k(0) = I_{n-1} \oplus (-1)$. These imply that $A *_2 B \in \mathcal{C}_2$ (i.e. $\mathcal{C}_2$ is closed under the operation $*_2$) and that

$$\widetilde{A} := (I_{n-1} \oplus (-1))A^{-1}(I_{n-1} \oplus (-1)) \in \mathcal{C}_2.$$

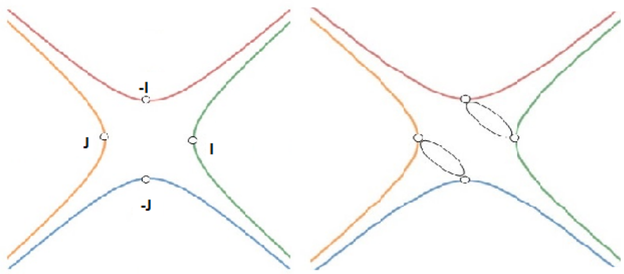
For every $A \in \mathcal{C}_2$, $A *_2 (I_{n-1} \oplus (-1)) = A$ and $A *_2 \widetilde{A} = \widetilde{A} *_2 A = I_{n-1} \oplus (-1)$. Therefore $(\mathcal{C}_2, *_2)$ is a group with $I_{n-1} \oplus (-1)$ as the identity element.

For every $A \in \mathcal{C}_2$, define the continuous function $l : [0,1] \to \Gamma_n(J)$ by $l(x) = f(x)(I_{n-1} \oplus (-1))$ where f is as the above. Then $l(0) = A(I_{n-1} \oplus (-1))$ and $l(1) = I_n$, and hence $A(I_{n-1} \oplus (-1)) \in \mathcal{C}_1$. To complete the proof, just consider the homeomorphism $\varphi : (\mathcal{C}_2, *_2) \to (\mathcal{C}_1, .)$ defined by $\varphi(A) = A(I_{n-1} \oplus (-1))$, which also is a group isomorphism.

□

Example 2.3.7. *Let* $J = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. *With the notation used in Theorem 2.3.5, the*

components of $\Gamma_2(J)$ *are as follows:*
$\mathcal{C}_1 = \{\begin{pmatrix} \sqrt{1+b^2} & b \\ b & \sqrt{1+b^2} \end{pmatrix} : b \in \mathbb{R}\}$, $\mathcal{C}_2 = \{\begin{pmatrix} -\sqrt{1+b^2} & b \\ b & -\sqrt{1+b^2} \end{pmatrix} : b \in \mathbb{R}\}$,
$\mathcal{C}_3 = \{\begin{pmatrix} \sqrt{1+b^2} & b \\ -b & -\sqrt{1+b^2} \end{pmatrix} : b \in \mathbb{R}\}$, $\mathcal{C}_4 = \{\begin{pmatrix} -\sqrt{1+b^2} & b \\ -b & \sqrt{1+b^2} \end{pmatrix} : b \in \mathbb{R}\}$. *These components are also mentioned in Exercise 7, page 24 of [6]. See the following pictures:*



The following proposition shows that $\mathcal{S}_n$ splits equally between the components of $\Gamma_n(J)$.

Proposition 2.3.8. *Let* $J \in \mathcal{S}_n$ *and* $J \neq \pm I$. *Then every component of* $\Gamma_n(J)$ *has exactly* 2^{n-2} *signature matrices.*

Proof. As in the proof of Theorem 2.3.5, we may assume that $q \geq p$ and $J = I_p \oplus (-I_q)$. In fact, we show that for every i $(1 \leq i \leq 16)$, Φ_i in the proof of Theorem 2.3.5 has 2^{n-2} signature matrices. In Φ_1, put $b_1 = \cdots = b_p = 0$, $V_1 = I_p$ and $V_2 = I_q$; then

$$\mathcal{F}_1 := \{U_1 \oplus U_2 : U_1 \in \mathcal{S}_p, U_2 \in \mathcal{S}_q, \det(U_1) = \det(U_2) = 1\} \subset \Phi_1.$$

Since $\mathcal{F}_1$ has 2^{n-2} elements, Φ_1 has at least 2^{n-2} signature matrices. With a similar argument we may show that for every i $(2 \leq i \leq 16)$, Φ_i has at least 2^{n-2} signature matrices and hence every component of $\Gamma_n(J)$ has at least 2^{n-2} signature matrices. On the other hand, $\Gamma_n(J)$ has four components and $\mathcal{S}_n$ has 2^n elements. This implies that every component of $\Gamma_n(J)$ has exactly 2^{n-2} signature matrices (and also each Φ_i has exactly 2^{n-2} signature matrices).

□

Lemma 2.3.9. *Let* $J \in \mathcal{S}_n$. *If* $J \neq \pm I$ *then* $\mathcal{O}_n \not\subseteq \Gamma_n(J)$.

Proof. Since $J \neq \pm I$, by Theorem 2.3.5, $\Gamma_n(J)$ has four connected components. Assume if possible that $\mathcal{O}_n \subseteq \Gamma_n(J)$, so that $\Gamma_n(J) = \mathcal{O}_n \cup \Gamma_n(J)$. As in the proof of Theorem 2.3.5, $\mathcal{O}_n \cup \Gamma_n(J)$ has two connected components, and then $\Gamma_n(J)$ has two connected components which is a contradiction. Therefore $\mathcal{O}_n \nsubseteq \Gamma_n(J)$. □

Our final result shows that to determine whether $\Gamma_n(J_1) = \Gamma_n(J_2)$, it is sufficient to know whether they contain the same orthogonal matrices, or whether they contain the same matrices with spectral norm 1.

Theorem 2.3.10. *Let $J_1, J_2 \in \mathcal{S}_n$. If J_1 and J_2 are non-scalar matrices then the following conditions are equivalent.*

(i) $J_1 = \pm J_2$,

(ii) $\Gamma_n(J_1) = \Gamma_n(J_2)$,

(iii) $\Gamma_n(J_1) \subseteq \Gamma_n(J_2)$,

(iv) $\Gamma_n(J_2) \subseteq \Gamma_n(J_1)$,

(v) $\mathcal{O}_n \cap \Gamma_n(J_1) \subseteq \mathcal{O}_n \cap \Gamma_n(J_2)$,

(vi) $\mathcal{O}_n \cap \Gamma_n(J_2) \subseteq \mathcal{O}_n \cap \Gamma_n(J_1)$,

(vii) $\mathcal{O}_n \cap \Gamma_n(J_1) = \mathcal{O}_n \cap \Gamma_n(J_2)$

(viii) $\mathcal{T}_n \cap \Gamma_n(J_1) \subseteq \mathcal{T}_n \cap \Gamma_n(J_2)$,

(ix) $\mathcal{T}_n \cap \Gamma_n(J_2) \subseteq \mathcal{T}_n \cap \Gamma_n(J_1)$,

(x) $\mathcal{T}_n \cap \Gamma_n(J_1) = \mathcal{T}_n \cap \Gamma_n(J_2)$.

Proof. Clearly we have $(i) \to (ii) \to (iii) \to (v)$, $(i) \to (ii) \to (iv) \to (vi)$ and $(ii) \to (vii)$. By the use of Corollary 2.3.4, we have $(v) \leftrightarrow (viii)$, $(vi) \leftrightarrow (ix)$ and $(vii) \leftrightarrow (x)$. So, it is enough to show that $(v) \to (i)$ and $(vi) \to (i)$. For this purpose we show that $\sim(i) \to \sim(v)$ and $\sim(i) \to \sim(vi)$.

$\sim(i)\to\sim(v)$. Let p and q be the numbers of $+1$ and -1 diagonal entries of J_1 respectively. Since for every $n \times n$ permutation matrix P we have $\Gamma_n(J_1) = P(\Gamma_n(P^\top J_1 P))P^\top$, without loss of generality, we may assume that $J_1 = I_p \oplus (-I_q)$. Let $J_2 = K_1 \oplus K_2$ with $K_1 \in \mathcal{S}_p$ and $K_2 \in \mathcal{S}_q$. Since $J_2 \neq \pm I$ and $J_2 \neq \pm J_1$, we have that $K_1 \neq \pm I_p$ or $K_2 \neq \pm I_q$. Without loss of generality we may assume that $K_1 \neq \pm I_p$ (the other case is similar). By the use of Lemma 2.3.9, $\mathcal{O}_p \nsubseteq \Gamma_p(K_1)$. Then there exists a $Q \in \mathcal{O}_p$ such that $Q \notin \Gamma_p(K_1)$. Let $A = Q \oplus I_q$. It is clear that $A \in \mathcal{O}_n \cap \Gamma_n(J_1)$. Assume if possible that $A \in \Gamma_n(J_2)$. Then $(Q \oplus I_q)^\top (K_1 \oplus K_2)(Q \oplus I_q) = K_1 \oplus K_2$ and hence $Q^\top K_1 Q = K_1$. This implies that $Q \in \Gamma_p(K_1)$ which is a contradiction. So $A \notin \Gamma_n(J_2)$ and therefore $\mathcal{O}_n \cap \Gamma_n(J_1) \nsubseteq \mathcal{O}_n \cap \Gamma_n(J_2)$.

To prove $\sim(i)\to\sim(vi)$, it is enough to exchange J_1 and J_2 in $\sim(i)\to\sim(v)$. $\square$

PART 3

TOPOLOGICAL PROPERTIES OF J-ORTHOGONAL MATRICES II

3.1 Introduction

Let $\mathbf{M}_n$ be the set of all $n \times n$ real matrices. A matrix $J \in \mathbf{M}_n$ is said to be a signature matrix if J is diagonal and its diagonal entries are ± 1. As in [1] and [11], if J is a signature matrix, a nonsingular matrix $B \in \mathbf{M}_n$ is said to be a J-orthogonal matrix if $B^\top JB = J$. Some properties of J-orthogonal matrices were investigated in [4], [1] and [11]. In this paper some further interesting properties of these matrices are obtained.

The following conventions will also be fixed throughout the paper:
E_{ij} is the $n \times n$ matrix whose (i, j) entry is one and all other entries are zero; $\mathcal{O}_n$ is the set of all $n \times n$ orthogonal matrices; $\mathcal{O}_n^+$ is the set of all $n \times n$ orthogonal matrices with determinant 1; $\mathcal{O}_n^-$ is the set of all $n \times n$ orthogonal matrices with determinant -1; $\mathcal{P}_n$ is the set of all $n \times n$ permutation matrices; $\mathcal{S}_n$ is the set of all $n \times n$ signature matrices; $\mathcal{SP}_n$ is the set of all $n \times n$ signed permutation matrices, the $n \times n$ matrices with exactly one nonzero entry ± 1 in each row and in each column; for $J \in \mathcal{S}_n$, $\Gamma_n(J) = \{A \in \mathbf{M}_n : A^\top JA = J\}$; Ω_n is the set of all $n \times n$ J-orthogonal matrices, i.e. $\Omega_n = \bigcup_{J \in \mathcal{S}_n} \Gamma_n(J)$; for $A \in \mathbf{M}_n$, $\sigma(A)$ and $\mathcal{S}(A)$ are the set of eigenvalues of A and the set of singular values of A respectively; the matrix norm used in this paper is the spectral norm $\| \ \|_2$.

In [11], the following conditions were considered for a (necessarily nonsingular) matrix B:

(i) $B \in \mathcal{SP}_n$,

(ii) $B^\top \mathcal{S}_n B = \mathcal{S}_n$,

(iii) $B^\top \Omega_n B = \Omega_n$.

In was shown in [11] that $(i) \leftrightarrow (ii)$ and $(ii) \rightarrow (iii)$. However, the major open question of whether $(iii) \rightarrow (i)$ or $(iii) \rightarrow (ii)$ was left unresolved in [11]. The main topic of this paper is to answer this question in the affirmative. This is done in two major steps: Theorem 3.2.4 for the case that B is orthogonal and Section 3.3 where we reduce the general case of B to the orthogonal case.

For $A \in \mathbf{M}_n$, the linear operator $T : \mathbf{M}_n \rightarrow \mathbf{M}_n$ defined by $T(X) = A^\top X A$ or $T(X) = A^\top X^\top A$ is called a standard linear operator on $\mathbf{M}_n$. In this paper we show that a standard linear operator $T : \mathbf{M}_n \rightarrow \mathbf{M}_n$ strongly preserves the set of J-orthogonal matrices if and only if A is a signed permutation matrix.

3.2 Other properties of J-orthogonal matrices

In this section we first collect some properties of J-orthogonal matrices that have been mentioned or proved in [1] and [11].

Proposition 3.2.1. *[1, Theorem 3.2 (hyperbolic CS decomposition)] Let $q \geq p$ and $J = I_p \oplus (-I_q)$. Then every $A \in \Gamma_n(J)$ is of the form*

$$(U_1 \oplus U_2)(\begin{pmatrix} C & -S \\ -S & C \end{pmatrix} \oplus I_{q-p})(V_1 \oplus V_2), \tag{3.1}$$

where $U_1, V_1 \in \mathcal{O}_p$, $U_2, V_2 \in \mathcal{O}_q$ and $C, S \in \mathbf{M}_p$ are nonnegative diagonal matrices such that $C^2 - S^2 = I_p$. Also, any matrix of the form (3.1) is J-orthogonal.

Corollary 3.2.2. *Let $q \geq p$ and $J = I_p \oplus (-I_q)$. Then for every $A \in \Gamma_n(J)$ the singular values of A are*

$$c_i + s_i \quad \text{and} \quad \frac{1}{c_i + s_i}, \quad 1 \leq i \leq p; \quad 1, \text{with multiplicity } q\text{-}p,$$

where $C = \operatorname{diag}(c_1, \ldots, c_p)$ and $S = \operatorname{diag}(s_1, \ldots, s_p)$ are as in (3.1).

Proof. Since the singular values are unitarily invariant, we may assume that $A =$

$\begin{pmatrix} C & -S \\ -S & C \end{pmatrix} \oplus I_{q-p}$. Then A is real symmetric positive definite and hence

$$\mathcal{S}(A) = \sigma(A) = \{c_i + s_i, \frac{1}{c_i + s_i} : 1 \leq i \leq p\} \cup \{1\}.$$

$\square$

Remark 3.2.3. *The following statements are true; see [11].*

(i) $\Gamma_n(J)$ *is a closed multiplicative group for every* $J \in \mathcal{S}_n$.

(ii) $\mathcal{S}_n = \bigcap_{J \in \mathcal{S}_n} \Gamma_n(J)$.

(iii) *If* $P \in \mathcal{SP}_n$*, then* $P^\top \Omega_n P = \Omega_n$.

(iv) $P \in \mathcal{SP}_n$ *if and only if* $P^\top \mathcal{S}_n P = \mathcal{S}_n$.

(v) *For every* $n \geq 1$, $\mathcal{O}_n$ *has 2 connected components. In fact,* $\mathcal{O}_n = \mathcal{O}_n^+ \bigcup \mathcal{O}_n^-$.

(vi) *If* $J \neq \pm I$ *then* $\Gamma_n(J)$ *has 4 connected components and hence* Ω_n *has 2 connected components.*

(vii) *Let* $q \geq p$ *and* $J = I_p \oplus (-I_q)$. *Then* $\Gamma_n(J)$ *is*

$$\{(U_1 \oplus U_2)(\begin{pmatrix} \mathrm{diag}(\sqrt{1+\mathrm{b}_1^2}, \ldots, \sqrt{1+\mathrm{b}_\mathrm{p}^2}) & \mathrm{diag}(\mathrm{b}_1, \ldots, \mathrm{b}_\mathrm{p}) \\ \mathrm{diag}(\mathrm{b}_1, \ldots, \mathrm{b}_\mathrm{p}) & \mathrm{diag}(\sqrt{1+\mathrm{b}_1^2}, \ldots, \sqrt{1+\mathrm{b}_\mathrm{p}^2}) \end{pmatrix} \oplus I_{q-p})(V_1 \oplus V_2)\},$$

where $U_1, V_1 \in \mathcal{O}_p$, $U_2, V_2 \in \mathcal{O}_q$ *and* $b_1, \ldots, b_p \in \mathbb{R}$.

(viii) *For* $A \in \Omega_n$, $\|A\|$*=1 if and only if* $A \in \mathcal{O}_n$.

Now, we need some preliminaries to prove the following theorem. The implications $(i) \leftrightarrow (ii)$ and $(ii) \rightarrow (iii)$ in Theorem 3.2.4 are in Remark 3.2.3. In fact we need to prove $(iii) \rightarrow (i)$ or $(iii) \rightarrow (ii)$.

Theorem 3.2.4. *Let* $U \in \mathcal{O}_n$. *The following conditions are equivalent.*

(i) $U \in \mathcal{SP}_n$,

(ii) $U^\top \mathcal{S}_n U = \mathcal{S}_n$,

(iii) $U^\top \Omega_n U = \Omega_n$.

To complete the proof of this theorem we need some other results. For every $n \in \mathbb{N}$ and every integer k $(0 \le k \le \frac{n}{2})$, let

$$J_k = \begin{cases} \begin{pmatrix} I_{\frac{n}{2}-k} & 0 \\ 0 & -I_{\frac{n}{2}+k} \end{pmatrix}, & \text{if n is even;} \\ \\ \begin{pmatrix} I_{\frac{n-1}{2}-k} & 0 \\ 0 & -I_{\frac{n+1}{2}+k} \end{pmatrix}, & \text{if n is odd,} \end{cases}$$

$$\mathcal{E}_k = \bigcup_{P \in \mathcal{P}_n} \Gamma_n(P^\top J_k P), \text{ and}$$

$$\mathcal{F}_k = \begin{cases} \{A \in \mathcal{E}_k : \ A \text{ has } 2k \ \text{ unit singular values}\}, & \text{if n is even;} \\ \\ \{A \in \mathcal{E}_k : \ A \text{ has } 2k+1 \ \text{ unit singular values}\}, & \text{if n is odd.} \end{cases}$$

The following proposition gives some properties of $\mathcal{E}_k$ and $\mathcal{F}_k$.

Proposition 3.2.5. *For every $n \in \mathbb{N}$ and every integer k $(0 \le k \le \frac{n}{2})$, the following properties hold:*

(i) *If $Q \in \mathcal{P}_n$, then $Q^\top \mathcal{E}_k Q = \mathcal{E}_k$.*

(ii) *$\mathcal{E}_k$ is a closed subset of Ω_n.*

(iii) *$\mathcal{F}_k$ is a dense subset of $\mathcal{E}_k$ and hence $\overline{\mathcal{F}_k} = \mathcal{E}_k$.*

(iv) *$\mathcal{F}_k \cap \mathcal{E}_j = \emptyset$, for every $j > k$.*

(v) $\Omega_n = \bigcup_{0\le k\le \frac{n}{2}} \mathcal{E}_k$.

(vi) *If* $A \in \mathcal{E}_k$ *then* $AJ \in \mathcal{E}_k$ *for every signature matrix* $J \in \mathcal{S}_n$.

Proof. (i). For every $Q \in \mathcal{P}_n$, $\Gamma_n(Q^\top J_k Q) = Q^\top \Gamma_n(J_k) Q$ and hence

$$\begin{aligned} Q^\top \mathcal{E}_k Q &= Q^\top \Big(\bigcup_{P\in\mathcal{P}_n} \Gamma_n(P^\top J_k P)\Big) Q \\ &= \bigcup_{P\in\mathcal{P}_n} \Gamma_n(Q^\top P^\top J_k P Q) = \bigcup_{\widetilde{P}\in\mathcal{P}_n} \Gamma_n(\widetilde{P}^\top J_k \widetilde{P}) = \mathcal{E}_k. \end{aligned}$$

(ii). For every $P \in \mathcal{P}_n$, $\Gamma_n(P^\top J_k P)$ is a closed multiplicative group and hence $\mathcal{E}_k$ is closed.
(iii). We prove the result when n is even (the odd case is similar). Let $A \in \Gamma_n(J_k)$. First assume that $A = \begin{pmatrix} C & -S \\ -S & C \end{pmatrix} \oplus I_{2k}$. Note that $c_i + s_i = 1$ if and only if $c_i = 1$ and $s_i = 0$. For every $\delta > 0$, let $A_\delta = \begin{pmatrix} D & -T \\ -T & D \end{pmatrix} \oplus I_{2k}$, where $D = \mathrm{diag}(d_1, \ldots, d_{\frac{n}{2}-k})$, $T = \mathrm{diag}(t_1, \ldots, t_{\frac{n}{2}-k})$ and

$$d_i = \begin{cases} c_i, & \text{if } c_i \ne 1; \\ 1+\delta, & \text{if } c_i = 1, \end{cases} \quad \text{and } t_i = \begin{cases} s_i, & \text{if } c_i \ne 1; \\ \sqrt{(1+\delta)^2 - 1}, & \text{if } c_i = 1. \end{cases}$$

By the use of Corollary 3.2.2, we have $A_\delta \in \mathcal{F}_k$. For every $\epsilon > 0$ we may choose sufficiently small δ such that $\|A - A_\delta\| < \epsilon$. Now, assume that $A = (U_1 \oplus U_2)(\begin{pmatrix} C & -S \\ -S & C \end{pmatrix} \oplus I_{q-p})(V_1 \oplus V_2)$. By substituting A_δ with $(U_1 \oplus U_2) A_\delta (V_1 \oplus V_2)$ in the above, we have $\|A - A_\delta\| < \epsilon$. Finally, if $B \in \mathcal{E}_k$, then $B \in \Gamma_n(P^\top J_k P)$ where $P \in \mathcal{P}_n$, so $P^\top B P \in \Gamma_n(J_k)$. From the above, if $\epsilon > 0$ then there exists $C \in \mathcal{F}_k$ such that $\|P^\top B P - C\| < \epsilon$. Hence $\|B - PCP^\top\| < \epsilon$. Observing that $\mathcal{F}_k$ is closed under permutational similarity, we have that $PCP^\top \in \mathcal{F}_k$. Thus, $\mathcal{F}_k$ is a dense subset of $\mathcal{E}_k$.
(iv). Every $A \in \mathcal{F}_k$ has at most $2k+1$ unit singular values and every $A \in \mathcal{E}_j$ has at least $2j$ unit singular values. Since $j > k$, $2j > 2k+1$ and this implies that $\mathcal{F}_k \cap \mathcal{E}_j = \emptyset$.

(v). This is clear from the definition of Ω_n since

$$\mathcal{S}_n = \{\pm P^\top J_k P : \ 0 \leq k \leq \frac{n}{2}, \ P \in \mathcal{P}_n\}.$$

(vi). If $A \in \mathcal{E}_k$, then there exists some permutation matrix $Q \in \mathcal{P}_n$ such that $A \in \Gamma_n(Q^\top J_k Q)$. Also $J \in \Gamma_n(Q^\top J_k Q)$. Since $\Gamma_n(Q^\top J_k Q)$ is a group, $AJ \in \Gamma_n(Q^\top J_k Q)$ and therefore $AJ \in \mathcal{E}_k$.

□

Let $J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. For $A \in \mathbf{M}_n$, we say that A is conformal to J if $A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ with $A_1 \in \mathbf{M}_p$ and $A_2 \in \mathbf{M}_q$, see [12].

Lemma 3.2.6. *Let* $J = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$. *If* $A \in \Gamma_n(J) \cap \mathcal{O}_n$ *then* A *is conformal to* J.

Proof. Since $A \in \Gamma_n(J) \cap \mathcal{O}_n$ we have $AJ = JA$ and consequently
$A = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$ with $A_1 \in \mathcal{O}_p$ and $A_2 \in \mathcal{O}_q$, which is conformal to J. □

Remark 3.2.7. *If* G *is a real positive semi-definite matrix, then there exist an orthogonal matrix* U *and a diagonal matrix* $D = diag(\lambda_1, \ldots, \lambda_n)$ *such that* $G = UDU^\top$. *So* $(I+G)^{\frac{-1}{2}} - I = U[(I+D)^{\frac{-1}{2}} - I]U^\top$ *and hence*

$$\|(I+G)^{\frac{-1}{2}} - I\| = 1 - \frac{1}{\sqrt{1+\lambda_{max}}} \leq \frac{\lambda_{max}}{2} = \frac{\|G\|}{2}.$$

Lemma 3.2.8. *There exits a sequence* $\{Q_m\}_{m=1}^{\infty} \subseteq \mathcal{O}_n$ *such that* Q_m *has no zero entry for every* $m \geq 1$ *and* $lim_{m\to\infty} Q_m = I$.

Proof. Let $F = \begin{pmatrix} 0 & -1 & -1 & \dots & -1 \\ 1 & 0 & -1 & \dots & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & \dots & 1 & 0 & -1 \\ 1 & 1 & \dots & 1 & 0 \end{pmatrix}$. For sufficiently large positive integer m, $I + \frac{1}{m^2}F^\top F$ is positive definite and we have $(I + \frac{1}{m}F)^\top(I + \frac{1}{m}F) = I + \frac{1}{m^2}F^\top F$. Let $Q_m = (I + \frac{1}{m}F)(I + \frac{1}{m^2}F^\top F)^{\frac{-1}{2}}$. It is easy to check that Q_m is orthogonal. If we put $G = \frac{1}{m^2}F^\top F$ in Remark 3.2.7, then we obtain that

$$\|(I + \frac{1}{m^2}F^\top F)^{\frac{-1}{2}} - I\| \le \frac{1}{2m^2}\|F^\top F\| \le \frac{1}{2m^2}\|F\|^2.$$

Let $H_m = (I + \frac{1}{m^2}F^\top F)^{\frac{-1}{2}} - I$. Then $(I + \frac{1}{m^2}F^\top F)^{\frac{-1}{2}} = H_m + I$ and

$$\|H_m\| \le \frac{1}{2m^2}\|F\|^2, \quad \|\frac{1}{m}FH_m\| \le \frac{1}{2m^3}\|F\|^3. \tag{3.2}$$

Now we have

$$Q_m = (I + \frac{1}{m}F)(H_m + I) = (I + \frac{1}{m}F) + H_m + \frac{1}{m}FH_m.$$

By the use of (3.2), we see that $\lim_{m\to\infty}Q_m = I$. Since $(I + \frac{1}{m}F)$ has no zero entry and $\frac{1}{m}F \in o(\frac{1}{m})$, $H_m \in o(\frac{1}{m^2})$ and $\frac{1}{m}FH_m \in o(\frac{1}{m^3})$ we see that Q_m has no zero entry for large enough m. □

Theorem 3.2.9. *Let $U \in \mathcal{O}_n$. If $U^\top\Omega_nU = \Omega_n$, then $U^\top\mathcal{E}_kU = \mathcal{E}_k$, for every integer k $(0 \le k \le \frac{n}{2})$.*

Proof. We prove the result when n is even (the odd case is similar). First assume that $k = 0$. By Proposition 3.2.1, we have

$$\Gamma_n(J_0) = \{(U_1 \oplus U_2)\begin{pmatrix} C & -S \\ -S & C \end{pmatrix}(V_1 \oplus V_2)\},$$

where $U_1, V_1, U_2, V_2 \in \mathcal{O}_{\frac{n}{2}}$ and $C, S \in \mathbf{M}_{\frac{n}{2}}$ are nonnegative diagonal matrices such that $C^2 - S^2 = I$. Observe that $\mathcal{F}_0$ is the set of $n \times n$ J-orthogonal matrices which have no unit singular values. By assumption, $U^\top \Omega_n U = \Omega_n$ and $U \in \mathcal{O}_n$, and since the singular values are invariant under an orthogonal similarity, we have that $U^\top \mathcal{F}_0 U = \mathcal{F}_0$. Therefore, using Proposition 3.2.5,

$$U^\top \mathcal{E}_0 U = U^\top \overline{\mathcal{F}_0} U = \overline{U^\top \mathcal{F}_0 U} = \overline{\mathcal{F}_0} = \mathcal{E}_0.$$

Now let $1 \leq k \leq \frac{n}{2}$ and assume that $U^\top \mathcal{E}_j U = \mathcal{E}_j$ for every $j < k$, and hence

$$U^\top (\bigcup_{j=0}^{k-1} \mathcal{E}_j) U = \bigcup_{j=0}^{k-1} \mathcal{E}_j.$$

We have

$$\mathcal{F}_k = [\mathcal{F}_k \cap \bigcup_{j=0}^{k-1} \mathcal{E}_j] \bigcup [\mathcal{F}_k \cap (\mathcal{E}_k \setminus \bigcup_{j=0}^{k-1} \mathcal{E}_j)].$$

Let $A \in \mathcal{F}_k$. Then $U^\top A U$ has $n - 2k$ non-unit singular values. Using Proposition 3.2.5 (iv), (v) and again the assumption that $U^\top \Omega_n U = \Omega_n$, $U \in \mathcal{O}_n$, we then have that $U^\top A U \in \bigcup_{j=0}^{k} \mathcal{E}_j$ and hence

$$U^\top \mathcal{F}_k U \subseteq \bigcup_{j=0}^{k} \mathcal{E}_j = (\bigcup_{j=0}^{k-1} \mathcal{E}_j) \bigcup (\mathcal{E}_k \setminus \bigcup_{j=0}^{k-1} \mathcal{E}_j),$$

which is a disjoint union. Since $U^\top (\bigcup_{j=0}^{k-1} \mathcal{E}_j) U = \bigcup_{j=0}^{k-1} \mathcal{E}_j$, we obtain that

$$U^\top [\mathcal{F}_k \cap (\mathcal{E}_k \setminus \bigcup_{j=0}^{k-1} \mathcal{E}_j)] U \subseteq \mathcal{E}_k \setminus \bigcup_{j=0}^{k-1} \mathcal{E}_j \subseteq \mathcal{E}_k. \tag{3.3}$$

Now, let $A \in \mathcal{F}_k \cap (\bigcup_{j=0}^{k-1} \mathcal{E}_j)$. By Lemma 3.2.8, there exists a sequence $\{Q_m\}_{m=1}^{\infty} = \{\begin{pmatrix} Q_{1m} & 0 \\ 0 & Q_{2m} \end{pmatrix}\}_{m=1}^{\infty}$ of orthogonal matrices such that Q_{1m} and Q_{2m} have no zero entry for every $m \geq 1$ and $\lim_{m\to\infty} Q_{1m} = I_{\frac{n}{2}-k}$ and $\lim_{m\to\infty} Q_{2m} = I_{\frac{n}{2}+k}$. Just from the form of Q_m, we see that $Q_m \in \Gamma_n(J_k)$; also, from Lemma 3.2.6, $Q_m \notin \Gamma_n(J_i)$ for every $i < k$.

So $Q_m \notin \bigcup_{j=0}^{k-1} \mathcal{E}_j$. Since $A \in \mathcal{F}_k$, there exists a permutation matrix P such that $A \in \Gamma_n(P^\top J_k P)$; also, $P^\top Q_m P \notin \Gamma_n(J_i)$ for every $i < k$. For the sequence $\{P^\top Q_m PA\}_{m=1}^\infty$, we have $P^\top Q_m PA \in \mathcal{E}_k$ and $P^\top Q_m PA \notin \bigcup_{j=0}^{k-1} \mathcal{E}_j$ which imply that $P^\top Q_m PA \in \mathcal{E}_k \setminus \bigcup_{j=0}^{k-1} \mathcal{E}_j$ and $P^\top Q_m PA$ has $n - 2k$ non-unit singular values. Then $P^\top Q_m PA \in \mathcal{F}_k$ and since $\mathcal{E}_k$ is closed, by (3.3), we have

$$U^\top AU = \lim_{m\to\infty} U^\top P^\top Q_m PAU \in \mathcal{E}_k,$$

and hence $U^\top(\mathcal{F}_k \cap (\bigcup_{j=0}^{k-1} \mathcal{E}_j))U \subseteq \mathcal{E}_k$. Therefore by the use of (3.3), we obtain that $U^\top \mathcal{F}_k U \subseteq \mathcal{E}_k$. Thus

$$\overline{U^\top \mathcal{F}_k U} \subseteq \mathcal{E}_k \Rightarrow U^\top \mathcal{E}_k U \subseteq \mathcal{E}_k.$$

Since $U^\top \Omega_n U = \Omega_n$ implies that $U \Omega_n U^\top = \Omega_n$, we can replace U by $U^\top$ in the above and similarly obtain that $U \mathcal{E}_k U^\top \subseteq \mathcal{E}_k$. Therefore $\mathcal{E}_k \subseteq U^\top \mathcal{E}_k U$ and hence $U^\top \mathcal{E}_k U = \mathcal{E}_k$.
For $k = \frac{n}{2}$, $\mathcal{E}_k = \mathcal{O}_n$ and trivially we have $U^\top \mathcal{E}_k U = \mathcal{E}_k$. □

Proof of Theorem 3.2.4. $(iii) \to (ii)$. We prove the result when n is even (the odd case is similar). Let J be an arbitrary signature matrix of order n. We show that $U^\top JU$ is a signature matrix.

Since $J \in \mathcal{E}_{\frac{n}{2}-1}$, by Theorem 3.2.9, $U^\top JU \in \mathcal{E}_{\frac{n}{2}-1}$. Then there exists a permutation matrix P such that $P(U^\top JU)P^\top \in \Gamma_n(J_{\frac{n}{2}-1})$. Put $K = P(U^\top JU)P^\top$ and observe that K is orthogonal so that by the use of Lemma 3.2.6, K is conformal to $J_{\frac{n}{2}-1}$ and hence K has at least one ± 1 on its main diagonal. Assume if possible that K is not a signature matrix; then K has exactly r ± 1s on its main diagonal with $1 \leq r < n$. Without loss of generality we may assume that $K = \begin{pmatrix} \pm 1 & & 0 \\ & \ddots & \\ 0 & & \pm 1 \end{pmatrix} \oplus V$, where $V \in \mathbf{M}_{n-r}$ is an orthogonal matrix with no ± 1 entries on its main diagonal. Note that $r \neq n-1$; otherwise $V = \pm 1$ and then K is a signature matrix. Also, we will discus the case $r = 1$ separately.

Let $A = \begin{pmatrix} B & 0 \\ 0 & I_{n-r} \end{pmatrix}$, where $B \in \mathbf{M}_r$ is an orthogonal matrix with no zero entries. Since $n - r \geq 1$, we have that $A \in \mathcal{E}_{\frac{n}{2}-1}$. By the use of equation $P(U^\top \mathcal{E}_{\frac{n}{2}-1} U)P^\top = \mathcal{E}_{\frac{n}{2}-1}$, there exists $D \in \mathcal{E}_{\frac{n}{2}-1}$ such that $P(U^\top DU)P^\top = A$. By Proposition 3.2.5 (vi), $DJ \in \mathcal{E}_{\frac{n}{2}-1}$ and hence

$$AK = PU^\top DUP^\top PU^\top JUP^\top = PU^\top (DJ)UP^\top \in \mathcal{E}_{\frac{n}{2}-1}.$$

On the other hand, when $2 \leq r \leq n-2$, by a simple computation one can show that AK has no ± 1 on its main diagonal and hence by Lemma 3.2.6, the orthogonal matrix $AK \notin \mathcal{E}_{\frac{n}{2}-1}$ which is a contradiction.

We now handle the case $r = 1$. Here, we may assume that $K = (\pm 1) \oplus V$, where $V \in \mathbf{M}_{n-1}$ is an orthogonal matrix with no ± 1 on its main diagonal. Let $A = B \oplus (1)$ where B is orthogonal with no zero entries in its first row. We have that $A \in \mathcal{E}_{\frac{n}{2}-1}$, so that by Theorem 3.2.9 there exists $D \in \mathcal{E}_{\frac{n}{2}-1}$ such that $U^\top DU = P^\top AP$. Then $JD \in \mathcal{E}_{\frac{n}{2}-1}$ and hence by Theorem 3.2.9 $U^\top JDU \in \mathcal{E}_{\frac{n}{2}-1}$. But

$$\begin{aligned} U^\top JDU &= (U^\top JU)(U^\top DU) \\ &= P^\top \begin{pmatrix} \pm 1 & 0 \\ 0 & V \end{pmatrix} PP^\top \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} P \\ &= P^\top \begin{pmatrix} \pm 1 & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} P = P^\top V_1 P, \end{aligned}$$

where $V_1 = \begin{pmatrix} \pm 1 & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}$. Since V_1 is orthogonal with no zero entries in the first $n-1$ positions of the first row and V specifically does not have ± 1 as its last diagonal entry, V_1 has no ± 1 on its main diagonal. Then $U^\top JDU$ has no ± 1 on its main diagonal. Since $U^\top JDU$ is orthogonal and $U^\top JDU \in \mathcal{E}_{\frac{n}{2}-1}$, by Lemma 3.2.6, $U^\top JDU$ has at least one ± 1 on its main diagonal, which is a contradiction.

Therefore K is a signature matrix and consequently $U^\top JU$ is a signature matrix. Thus, we have shown that $U^\top \mathcal{S}_n U \subseteq \mathcal{S}_n$. Since the transformation $X \mapsto U^\top X U$ is $1-1$ and $\mathcal{S}_n$ is finite, we conclude that $U^\top \mathcal{S}_n U = \mathcal{S}_n$, so that $(iii) \to (ii)$. $\square$

3.3 The general case of $U^\top \Omega_n U = \Omega_n$

The purpose of this section is to prove that if $B^\top \Omega_n B = \Omega_n$, then in fact $B \in \mathcal{O}_n$. First, if W is a nonempty set of $n \times n$ invertible matrices, by W^{-1} we shall mean the set $\{A^{-1} : A \in W\}$. Note that for $J \in \mathcal{S}_n$, since $\Gamma_n(J)$ is a group, $(\Gamma_n(J))^{-1} = \Gamma_n(J)$ and hence $\Omega_n^{-1} = \Omega_n$.

Lemma 3.3.1. *If $B \in \boldsymbol{M}_n$ and $B^\top \Omega_n B = \Omega_n$, then $B \Omega_n B^\top = \Omega_n$ and hence $B^\top B \Omega_n B^\top B = \Omega_n$.*

Proof. Since $B^\top \Omega_n B = \Omega_n = \Omega_n^{-1}$, we have $\Omega_n = (B^\top)^{-1} \Omega_n^{-1} B^{-1} = (B \Omega_n B^\top)^{-1}$. Then $B \Omega_n B^\top = \Omega_n^{-1} = \Omega_n$. $\square$

In what follows, we assume that $B \in \mathbf{M}_n$ and $B^\top \Omega_n B = \Omega_n$. In particular, $B^\top B = B^\top I B \in \Gamma_n(J)$ for at least one $J \in \mathcal{S}_n$. For any $A \in \Omega_n$, let $L_1(A) = \{J \in \mathcal{S}_n : A \in \Gamma_n(J)\}$ and $L_2(A) = \{J \in \mathcal{S}_n : A \notin \Gamma_n(J)\}$. Then $\mathcal{S}_n = L_1(A) \bigcup L_2(A)$, for any $A \in \Omega_n$.

Lemma 3.3.2. *Let $A \in \Omega_n \setminus \mathcal{O}_n$ and $A^\top \Omega_n A = \Omega_n$. Then $A^\top A \in \bigcup_{J \in L_2(A)} \Gamma_n(J)$.*

Proof. Define the isomorphism $T : \mathbf{M}_n \to \mathbf{M}_n$ by $T(X) = A^\top X A$. If $J \in L_1(A)$, then $A \in \Gamma_n(J)$ and since $\Gamma_n(J)$ is a group we have $A^\top \Gamma_n(J) A = \Gamma_n(J)$. This implies that $T(\bigcup_{J \in L_1(A)} \Gamma_n(J)) = \bigcup_{J \in L_1(A)} \Gamma_n(J)$ and hence

$$T(\Omega_n \setminus \bigcup_{J \in L_1(A)} \Gamma_n(J)) = (\Omega_n \setminus \bigcup_{J \in L_1(A)} \Gamma_n(J)) \subseteq \bigcup_{J \in L_2(A)} \Gamma_n(J).$$

By Lemma 3.2.8, there exits a sequence $\{Q_m\}_{m=1}^\infty \subseteq \mathcal{O}_n$ such that Q_m has no zero entry for every $m \geq 1$ and $\lim_{m \to \infty} Q_m = I$. Since $A \in \Omega_n \setminus \mathcal{O}_n$, $\pm I \notin L_1(A)$ and hence by Lemma 3.2.6, $Q_m \in (\Omega_n \setminus \bigcup_{J \in L_1(A)} \Gamma_n(J))$ for every $m \geq 1$. Then $T(Q_m) \in \bigcup_{J \in L_2(A)} \Gamma_n(J)$, and since $\bigcup_{J\in L_2(A)}\Gamma_n(J)$ is a closed set, we have

$$A^\top A = T(I) = \lim_{m\to\infty} T(Q_m) \in \bigcup_{J\in L_2(A)} \Gamma_n(J).$$

□

Theorem 3.3.3. *Let $B \in \boldsymbol{M}_n$. If $B^\top \Omega_n B = \Omega_n$, then $B \in \mathcal{SP}_n$.*

Proof. By Theorem 3.2.4, it is enough to show that $B \in \mathcal{O}_n$. Assume if possible that $B \notin \mathcal{O}_n$, so that $B^\top B \notin \mathcal{O}_n$. By the use of Lemma 3.3.1, $B^\top B\Omega_n B^\top B = \Omega_n$ and hence by Lemma 3.3.2,

$$(B^\top B)^2 \in \bigcup_{J\in L_2(B^\top B)} \Gamma_n(J). \tag{3.4}$$

If $B^\top B \in \Gamma_n(J)$, then $(B^\top B)^2 \in \Gamma_n(J)$ and hence $L_1(B^\top B) \subseteq L_1((B^\top B)^2)$. Now, by the use of (3.4), $L_1(B^\top B) \subsetneq L_1((B^\top B)^2)$. By a similar argument, for every positive integer k, we can obtain $L_1((B^\top B)^k) \subsetneq L_1((B^\top B)^{2k})$. Then $\{L_1((B^\top B)^{2k})\}_{k=1}^{\infty}$ is a strictly increasing sequence of subsets of $\mathcal{S}_n$, so that $\mathcal{S}_n$ is an infinite set which is a contradiction. Therefore $B \in \mathcal{O}_n$. □

We have now completely answered the open question raised in [11] in the affirmative. In fact, for a (necessarily nonsingular) matrix B we showed that the following conditions are equivalent:

(i) $B \in \mathcal{SP}_n$,

(ii) $B^\top \mathcal{S}_n B = \mathcal{S}_n$,

(iii) $B^\top \Omega_n B = \Omega_n$.

As an interesting consequence, we have the following theorem that characterizes the standard linear operators $T : \mathbf{M}_n \to \mathbf{M}_n$ strongly preserving the set of J-orthogonal matrices, i.e. $T(A)$ is J-orthogonal if and only if A is J-orthogonal.

Theorem 3.3.4. *Let $T : \boldsymbol{M}_n \to \boldsymbol{M}_n$ be a standard linear operator. Then $T(\Omega_n) = \Omega_n$ if and only if there exists a signed permutation matrix P such that*

$$T(X) = P^\top X P, \quad \forall\ X \in \boldsymbol{M}_n,$$

or

$$T(X) = P^\top X^\top P, \quad \forall\ X \in \boldsymbol{M}_n.$$

As mentioned earlier in this dissertation, the work in Part 3 will appear in the paper [13].

PART 4

SOME CONNECTIONS BETWEEN J-ORTHOGONAL MATRICES AND G-MATRICES

4.1 Introduction

Let $\mathbf{M}_n$ be the set of all $n \times n$ real matrices. A nonsingular matrix $A \in \mathbf{M}_n$ is called a G-matrix if there exist nonsingular diagonal matrices D_1 and D_2 such that $A^{-T} = D_1 A D_2$, where A^{-T} denotes the transpose of the inverse of A. These matrices form a rich class and were originally studied in [14] by Fiedler and Hall. Some properties of these matrices are as follows:

All orthogonal (J-orthogonal) matrices are G-matrices.

All nonsingular diagonal matrices are G-matrices.

Any n positive real numbers are the singular values and eigenvalues of a diagonal G-matrix D.

If A is a G-matrix, then both A^T and A^{-1} are G-matrices.

If A is an $n \times n$ G-matrix and D is an $n \times n$ nonsingular diagonal matrix, then both AD and DA are G-matrices.

If A is an $n \times n$ G-matrix and P is an $n \times n$ permutation matrix, then both AP and PA are G-matrices.

Cauchy matrices have the form $C = [c_{ij}]$, where $c_{ij} = \frac{1}{x_i + y_j}$ for some numbers x_i and y_j.

We shall restrict to square, say $n \times n$, Cauchy matrices - such matrices are defined only if $x_i + y_j \neq 0$ for all pairs of indices i, j, and it is well known that C is nonsingular if and only if all the numbers x_i are mutually distinct and all the numbers y_j are mutually distinct.

From M. Fiedler, Notes on Hilbert and Cauchy matrices, LAA, 2010:

Every nonsingular Cauchy matrix is a G-matrix.

The G-matrices were later studied in two papers [15] and [16].

Denote by $J = \text{diag}(\pm 1)$ a diagonal (signature) matrix, each of whose diagonal entries is $+1$ or -1. As in [1], a nonsingular real matrix Q is called J-orthogonal if

$$Q^T J Q = J,$$

or equivalently, if

$$Q^{-T} = JQJ.$$

Of course, every orthogonal matrix is a J-orthogonal matrix. And clearly, every J-orthogonal matrix is a G-matrix. Not every G-matrix is a J-orthogonal matrix. But, a G-matrix can always be "transformed" to a J-orthogonal matrix [4].

4.2 Classes of G-matrices

For nonsingular $n \times n$ diagonal matrices D_1 and D_2, the following known result from [14] shows that if $A^{-T} = D_1 A D_2$ then D_1 and D_2 have the same inertia matrix.

Proposition 4.2.1. *Suppose A is a G-matrix and $A^{-T} = D_1 A D_2$, where D_1 and D_2 are nonsingular diagonal matrices. Then the inertia of D_1 is equal to the inertia of D_2.*

Proof. We have $A^T D_1 A D_2 = I$ and so $A^T D_1 A = D_2^{-1}$. Since A is nonsingular, the result follows from Sylvester's Law of Inertia. □

For fixed nonsingular diagonal matrices D_1 and D_2, let the class of G-matrices

$$\mathbb{G}(D_1, D_2) = \{A \in \mathbf{M}_n : A^{-T} = D_1 A D_2\}.$$

In this section we find a characterization of $\mathbb{G}(D_1, D_2)$.

Let D be a nonsingular diagonal matrix with the inertia matrix J (a signature matrix having all its positive ones in the upper left corner). Then there exists a permutation matrix P such that $D = |D| P^T J P$, where $|D|$ is obtained by taking the absolute value on entries of D. Recall that for a fixed signature matrix J, $\Gamma_n(J) = \{A \in \mathbf{M}_n : A^\top JA = J\}$. In fact,

$$\Gamma_n(J) = \mathbb{G}(J, J).$$

Theorem 4.2.2. *Let D_1 and D_2 be nonsingular diagonal matrices with the inertia matrix J. Then there exist permutation matrices P and Q such that*

$$\mathbb{G}(D_1, D_2) = \{|D_1|^{-1/2} P^T AQ|D_2|^{-1/2} : A \in \Gamma_n(J)\}.$$

Proof. Since J is the inertia matrix for D_1 and D_2, there exist permutation matrices P and Q such that $D_1 = |D_1|P^T JP$ and $D_2 = |D_2|Q^T JQ$. Then $J = P|D_1|^{-1/2}D_1|D_1|^{-1/2}P^T = Q|D_2|^{-1/2}D_2|D_2|^{-1/2}Q^T$. These imply that

$$\begin{aligned}
A \in \Gamma_n(J) &\Leftrightarrow A^{-T} = JAJ \\
&\Leftrightarrow A^{-T} = P|D_1|^{-1/2}D_1|D_1|^{-1/2}P^T AQ|D_2|^{-1/2}D_2|D_2|^{-1/2}Q^T \\
&\Leftrightarrow (|D_1|^{-1/2}P^T AQ|D_2|^{-1/2})^{-T} = D_1(|D_1|^{-1/2}P^T AQ|D_2|^{-1/2})D_2 \\
&\Leftrightarrow |D_1|^{-1/2}P^T AQ|D_2|^{-1/2} \in \mathbb{G}(D_1, D_2).
\end{aligned}$$

Therefore

$$\mathbb{G}(D_1, D_2) = \{|D_1|^{-1/2} P^T AQ|D_2|^{-1/2} : A \in \Gamma_n(J)\}.$$

□

We will now incorporate the hyperbolic CS Decomposition for $\Gamma_n(J)$ into a simplified version of $\mathbb{G}(D_1, D_2)$. Assume $q \geq p$, the common inertia matrix J of D_1 and D_2 has the form

$$\begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$$

and that D_1 and D_2 have the form

$$\begin{pmatrix} +_p & 0 \\ 0 & -_q \end{pmatrix},$$

where $+_p$ $(-_q)$ denotes an order p (q) diagonal matrix with positive (negative) diagonal entries. Then $P = Q = I$, $D_1 = |D_1|J$, and $D_2 = |D_2|J$. Using Proposition 3.2.1 on the CS Decomposition, we then have the following result.

Corollary 4.2.3. *With the above notation,*

$$\mathbb{G}(D_1, D_2) = \{|D_1|^{-1/2}(U_1 \oplus U_2)(\begin{pmatrix} C & -S \\ -S & C \end{pmatrix} \oplus I_{q-p})(V_1 \oplus V_2)|D_2|^{-1/2}\}.$$

where $U_1, V_1 \in \mathcal{O}_p$, $U_2, V_2 \in \mathcal{O}_q$ and $C, S \in \mathbf{M}_p$ are nonnegative diagonal matrices such that $C^2 - S^2 = I_p$.

Let A be an $n \times n$ nonsingular matrix with Singular Value Decomposition

$$U\Sigma W$$

where U and W are orthogonal matrices. So, $AW^T = U\Sigma$. Now, it is easy to see that $U\Sigma \in \mathbb{G}(I, \Sigma^{-2})$. (Also: since U is an orthogonal matrix, U is a G-matrix; multiplying U by the nonsingular diagonal matrix Σ we still have a G-matrix.) Hence, $AW^T = B$, where $B \in \mathbb{G}(I, \Sigma^{-2})$, so that $A = BW$. We thus arrive at the following result.

Proposition 4.2.4. *Every $n \times n$ nonsingular matrix is a product of a G-matrix and an orthogonal matrix. In particular, if $U\Sigma W$ is a Singular Value Decomposition of a nonsingular matrix A, then $U\Sigma$ is a G-matrix.*

Note 4.2.5. *We can observe that by using Theorem 4.2.2 the above matrix $B \in \mathbb{G}(I, \Sigma^{-2})$*

simply equals to $C\Sigma$, where C is an orthogonal matrix. This yields that

$$A = BW = C\Sigma W,$$

a Singular Value Decomposition! In fact, the matrix C must be the same as the matrix U above.

Remark 4.2.6. *Given a fixed $n \times n$ inertia matrix J, we have various classes of G-matrices associated with J. We then have the following relation on the collection of the classes of $n \times n$ G-matrices:*

$$\mathbb{G}(D_1, D_2) \sim \mathbb{G}(D_3, D_4)$$

if and only if each class is associated with the same J, i.e. the inertia matrix of D_1, D_2, D_3, D_4 is J. (Note that $\mathbb{G}(J, J)$ is in the same equivalence class.) Then, it is clear that $\sim$ is an equivalence relation on the collection of the classes of $n \times n$ G-matrices.

4.3 The Connected Components

In this section we show that $\mathbb{G}(D_1, D_2)$ has two or four connected components in $\mathbf{M}_n$. Also we show that

$$\mathbb{G}_n = \bigcup_{D_1, D_2} \mathbb{G}(D_1, D_2),$$

the set of all $n \times n$ G-matrices, has two connected components in $\mathbf{M}_n$. Let $\mathcal{O}_n$ be the set of all $n \times n$ orthogonal matrices, $\mathcal{O}_n^+$ be the set of all $n \times n$ orthogonal matrices with determinant 1, and $\mathcal{O}_n^-$ be the set of all $n \times n$ orthogonal matrices with determinant -1.

Proposition 4.3.1. *[7, Theorem 3.67] For every $n \geq 1$, $\mathcal{O}_n$ has two connected components, $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$.*

Proposition 4.3.2. *[11, Theorem 3.5] Let J be an $n \times n$ signature matrix. If $J \neq \pm I$ then $\Gamma_n(J)$ has four connected components.*

Corollary 4.3.3. *For every $n \times n$ signature matrix J, $\mathcal{O}_n \cup \Gamma_n(J)$ has two connected components.*

Proof. Since every component of $\Gamma_n(J)$ has some orthogonal matrices (this is because each component has signature matrices, which in fact are orthogonal matrices), by the use of Proposition 4.3.1, the result is obtained. □

Theorem 4.3.4. *Let D_1 and D_2 be nonsingular diagonal matrices with the inertia matrix J.*

(*i*) *If $J \neq \pm I$, then $\mathbb{G}(D_1, D_2)$ has four connected components.*

(*ii*) *If $J = \pm I$, $\mathbb{G}(D_1, D_2)$ has two connected components.*

Proof. Let P and Q be as in the proof of Theorem 4.2.2. Consider the linear operator $T : \mathbf{M}_n \longrightarrow \mathbf{M}_n$ defined by

$$T(A) = |D_1|^{-1/2} P^T A Q |D_2|^{-1/2}.$$

Both T and T^{-1} are continuous and $T(\Gamma_n(J)) = \mathbb{G}(D_1, D_2)$ by Theorem 4.2.2. So the number of connected components of $\Gamma_n(J)$ and $\mathbb{G}(D_1, D_2)$ are the same. Now, by the use of Propositions 4.3.1, 4.3.2, the proof is complete. □

Theorem 4.3.5. *The set $\mathbb{G}_n$ of all $n \times n$ G-matrices has two connected components.*

Proof. We present the proof in two steps.

Step 1: First we show that $\mathbb{G}(D_1, D_2) \bigcup \mathbb{G}(|D_1|, |D_2|)$ has two connected components, where J is the inertia matrix of D_1 and D_2. By Theorem 4.2.2, there exist permutation matrices P and Q such that

$$\mathbb{G}(D_1, D_2) = \{|D_1|^{-1/2} P^T A Q |D_2|^{-1/2} : A \in \Gamma_n(J)\},$$

$$\mathbb{G}(|D_1|, |D_2|) = \{|D_1|^{-1/2} P^T A Q |D_2|^{-1/2} : A \in \mathcal{O}_n\},$$

where J is the inertia matrix of D_1 and D_2.

Then $\mathbb{G}(D_1, D_2) \cup \mathbb{G}(|D_1|, |D_2|) = T(\mathcal{O}_n \cup \Gamma_n(J))$, where T is the linear operator in the proof of Theorem 4.3.4. By the use of Corollary 4.3.3, $\mathbb{G}(D_1, D_2) \cup \mathbb{G}(|D_1|, |D_2|)$ has two connected components.

Step 2: Let $\mathbb{D}_n$ be the set of all $n \times n$ diagonal matrices with positive diagonal entries. It is clear that $\mathbb{D}_n$ is a connected set. For every $D_1, D_2 \in \mathbb{D}_n$, we have $\mathbb{G}(D_1, D_2) = D_1^{-1/2}(\mathcal{O}_n^+)D_2^{-1/2} \cup D_1^{-1/2}(\mathcal{O}_n^-)D_2^{-1/2}$. Then we have

$$\bigcup_{D_1, D_2 \in \mathbb{D}_n} \mathbb{G}(D_1, D_2) = [\bigcup_{D_1, D_2 \in \mathbb{D}_n} D_1^{-1/2}(\mathcal{O}_n^+)D_2^{-1/2}] \cup [\bigcup_{D_1, D_2 \in \mathbb{D}_n} D_1^{-1/2}(\mathcal{O}_n^-)D_2^{-1/2}].$$

Since $\mathbb{D}_n$, $\mathcal{O}_n^+$ and $\mathcal{O}_n^-$ are connected sets,

$$\bigcup_{D_1, D_2 \in \mathbb{D}_n} \mathbb{G}(D_1, D_2)$$

has two components.

Recalling that $\mathbb{G}_n = \bigcup_{D_1, D_2} \mathbb{G}(D_1, D_2)$, by use Step 1 and Step 2, the set of all $n \times n$ G-matrices has two connected components.

$\square$

The work in Part 4 of this dissertation will soon be submitted for publication [17].

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