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Counting Generating Sets in Frobenius Skew Polynomial Rings

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by

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Under the Direction of Florian Enescu, PhD

ABSTRACT

This dissertation takes a close look into a Frobenius skew polynomial ring where some of typical invariants from noncommutative algebra do not provide any useful information about the ring. Yoshino provides some nice results for a general Frobenius skew polynomial ring in [9], however, there is still significant potential to study and identify more aspects of these rings. Here, we apply standard techniques from noncommutative algebra taking a finitely-generated subspace and attempt to count the number of generators needed for powers of the subspace. We find that in certain cases where the base ring is the commutative polynomial ring or a semigroup ring, that a nonhomogeneous recurrence develops in the counting and an invariant arises naturally when solving this recurrence. We define this invariant as the Gk-base and show examples where it arises.

INDEX WORDS: Noncommutative algebra, Skew Polynomial Rings, Frobenius Skew Polynomial Rings, Gelfand-Kirillov Dimension, Growth of an Algebra

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DEDICATION

This dissertation is dedicated to my wife, Rebecca, my son, Flay, and

In loving memory of

Mary Ann ”Nana” Betts

(1933-2021)

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CHAPTER 1

PRELIMINARIES

1.1 Introduction

In this paper, we look closely at a skew polynomial ring that is defined by the Frobenius endomorphism, called a Frobenius skew polynomial rings. There is a lot in the commutative algebra literature studying the combination of rings of positive characteristic and the Frobenius endomorphism since this ring appears naturally. However, there has not been much investigation into the basic characteristics of the ring itself. Most study has been applied to taking the Frobenius endomorphism and applying it to different structures of the ring. So here we look to understand the interplay between the elements of the ring and the Frobenius endomorphism.

The initial idea was to study a well-developed invariant related to noncommutative algebras called the Gelfand-Kirillov dimension. It turns out that the examples of Frobenius skew polynomial rings studied in this paper have an infinite Gelfand-Kirillov dimension, and thus, other avenues were needed to gain useful information about the ring. There is another invariant closely related to the Gelfand-Kirillov dimension called the superdimension, but for the Frobenius skew polynomial rings studied here the superdimension is 1 and again this is not a meaningful value related to the rings.

However, the structure used to determine the Gelfand-Kirillov dimension produced another invariant that we refer to as the GK-base in this paper, that seems to provide an interesting invariant for the ring. It comes from counting the number of generators from various powers a K-subspace. So this value is closely associated to number of generators of these K-subspaces, yet it is still not clear what the value indicates. Another difficulty that is not resolved in this paper comes with the fact that the GK-base defined in Chapter 2 might depend on the choice of the K-subspace chosen for the Frobenius skew polynomial ring.

In Section 1.2 we provide the basic definitions and results related to a general skew polynomial ring. Once the definitions are established for a general skew polynomial ring, we transition to a focus on a Frobenius skew polynomial ring. In [9], Yoshino provides the necessary conditions on a ring R such that the Frobenius skew polynomial ring over R is left or right Noetherian. We present some of his results in Section 1.2, as well as provide a proof that the definition Yoshino uses for a Frobenius skew polynomial ring is indeed the same as our definition in this paper.

In Section 1.3, we provide the general structure used to determine the Gelfand-Kirillov dimension of a noncommutative algebra following the standard approach presented in [5]. In this process, we show that the Gelfand-Kirillov dimension agrees with the classic Krull dimension when the algebra is commutative. We also provide some basic definitions and results related to the superdimension, which is studied in more detail in [1].

Chapter 2 begins by taking a general Frobenius skew polynomial ring and applying the methods described in Section 1.3 related to the Gelfand-Kirillov dimension. It is here that we see that the Gelfand-Kirillov dimension is infinite while the superdimension is one. Thus, we define a new invariant that appears in the process of computing the Gelfand-Kirillov dimension refered to as the GK-base. Section 2.2 then moves to compute this invariant for the Frobenius skew polynomial ring when the base ring is a commutative polynomial ring, while section 2.3 takes the underlying ring to be the semigroup ring $K[t^2, t^3]$.

1.2 Skew Polynomial Rings

It is standard to define a polynomial ring in a modern algebra course by adjoining an indeterminate, x, to a commutative ring, R. This gives the ring $R[x]$ the structure of a free R-module with basis $1, x, x^2, x^3, \cdots$. Here, the multiplication with elements from R and the indeterminate x is commutative. Hence, $rx = xr$ for all $r \in R$.

In this section, we define a skew polynomial ring. The construction given here follows the standard approach given in both [4] and [7]. A skew polynomial has a similar structure as the standard polynomial ring, but the elements from the base ring are not assumed to commute with the indeterminate. In particular, let R be a ring and f an indeterminate that does not commute with the elements of R. We construct a ring $S = \sum_i Rf^i$ where elements of R act as left coefficients on powers of f and S is a free left R-module with basis $1, f, f^2, \cdots$. The addition in S is the same as the standard polynomial ring. We define multiplication in S so that all elements of S can be written as $\sum_{i=0}^{n} r_i f^i$ where $r_i \in R$ for all i, as in the standard polynomial ring. However, since f is not assumed to commute with elements of R, we need to determine the multiplication fr for $r \in R$.

Recall that in the standard polynomial ring $R[x]$, we have $\deg(g(x)h(x)) \leqslant \deg(g(x)) + \deg(h(x))$. To achieve a similar relation in S, we first define degrees in S. For all $s \in S$, we have $s = \sum_{i=0}^{n} r_i f^i$, where $r_i \in R$ for all i. We define $\deg(s) = j$ where $r_j \neq 0$ and $j \geqslant i$ for all i such that $r_i \neq 0$. Now consider the product fr for any $r \in R$. To maintain the degree relation we want $\deg(fr) \leqslant \deg(f)\deg(r)$, but $\deg(f) = 1$ and $\deg(r) = 0$, so $\deg(fr) \leqslant 1$. From this, we see that the product fr needs to be linear for every $r \in R$. Hence, we should have $fr \in Rf + R$ and $fr = \sigma(r)f + \delta(r) \in Rf + R$ for all $r \in R$ where σ and δ are maps from R to R.

We can conclude that the maps σ and δ need to have certain properties in order for the multiplication in S to satisfy the ring axioms. In fact, σ must be a ring endomorphism of R and δ must be a σ-derivation. Before demonstrating why this is the case, we first remind the reader of the definition of a σ-derivation.

Definition 1.2.1. Let R be a ring and σ an endomorphism. A **σ-derivation** of R is an additive map $\delta : R \to R$ such that $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$ for any $r, s \in R$. If σ is the identity map on R, then δ is just referred to as a **derivation**.

We now consider the distributive and associative properties to show the implications of certain ring axioms on σ and δ.

$$\text{If } f * (r+s) = \sigma(r+s)f + \delta(r+s)$$
$$\text{and } f * r + f * s = (\sigma(r) + \sigma(s)) * f + \delta(r) + \delta(s)$$

Then combining these two equations, we obtain

$$\sigma(r+s) * f + \delta(r+s) = (\sigma(r) + \sigma(s)) * f + \delta(r) + \delta(s)$$
$$\implies \sigma(r+s) = \sigma(r) + \sigma(s) \text{ and } \delta(r+s) = \delta(r) + \delta(s)$$

Hence, σ and δ are endomorphisms of the underlying additive group of R.

$$\begin{aligned} f * (rs) &= \sigma(rs) * f + \delta(rs) \\ \text{and } (f * r)s &= (\sigma(r) * f + \delta(r))s \\ &= \sigma(r) * f * s + \delta(r)s \\ &= \sigma(r)(\sigma(s) * f + \delta(s)) + \delta(r)s \\ &= \sigma(r)\sigma(s) * f + \sigma(r)\delta(s) + \delta(r)s \end{aligned}$$

Hence we have $\sigma(rs) = \sigma(r)\sigma(s)$ and $\delta(rs) = \sigma(r)\delta(s) + \delta(r)s$.
Finally, we see that $f * 1_R = \sigma(1_R) * f + \delta(1_R)$ which implies $\sigma(1_R) = 1_R$ and $\delta(1_R) = 0$.

Now assume that we are given a ring R, ring endomorphism σ, and a σ-derivation δ. We will now show that there exists a well-defined skew polynomial ring S, that is a free left R-module with the multiplication $fr = \sigma(r)f + \delta(r)$. To do this, we will follow the construction given by Goodearl and Warfield in [4] to avoid the tedious calculation required to check the ring axioms individually. This relies on constructing $S = \sum_i Rf^i$ as a subring of the endomorphism ring $\mathrm{End}_{\mathbb{Z}} R[x]$, where $R[x]$ is the standard polynomial ring. We first provide a lemma to show that σ and δ as described above can be extended to $R[x]$.

Lemma 1.2.2. *Let R be a ring, σ an endomorphism of R, and δ a σ-derivation of R. Then σ can be extended to a ring endomorphism of $R[x]$ and δ can be extended to a σ-derivation over $R[x]$.*

Proof. We first extend σ and δ to map $R[x]$ to $R[x]$ by $\sigma(rx^i) = \sigma(r)x^i$ and $\delta(rx^i) = \delta(r)x^i$ for all $r \in R$ and $i = 0, 1, 2, \cdots$. We now show that σ is a ring endomorphism of $R[x]$ by showing that multiplication is preserved by σ, and that δ meets the definition of a σ-derivation over $R[x]$. Let $f(x), g(x) \in R[x]$ with $f(x) = \sum_{i=0}^{n} r_i x^i$ and $g(x) = \sum_{j=0}^{m} r_j x^j$.

$$\begin{aligned}
\sigma(f(x)g(x)) &= \sigma\Big(\sum_{i=0}^{n} r_i x^i \sum_{j=0}^{m} r_j x^j\Big) \\
&= \sigma\Big(\sum_{i=0}^{n}\sum_{j=0}^{m} r_i r_j x^{i+j}\Big) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} \sigma(r_i r_j x^{i+j}) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} \sigma(r_i r_j) x^{i+j} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} \sigma(r_i)\sigma(r_j) x^{i+j} \\
&= \sum_{i=0}^{n} \sigma(r_i) x^i \sum_{j=0}^{m} \sigma(r_j) x^j \\
&= \sigma(f(x))\sigma(g(x))
\end{aligned}$$

$$
\begin{aligned}
\delta(f(x)g(x)) &= \delta\Big(\sum_{i=0}^{n} r_i x^i \sum_{j=0}^{m} r_j x^j\Big) \\
&= \delta\Big(\sum_{i=0}^{n}\sum_{j=0}^{m} r_i r_j x^{i+j}\Big) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} \delta(r_i r_j x^{i+j}) \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} \delta(r_i r_j) x^{i+j} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} (\sigma(r_i)\delta(r_j) + \delta(r_i) r_j) x^{i+j} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} \sigma(r_i)\delta(r_j) x^{i+j} + \delta(r_i) r_j x^{i+j} \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} \sigma(r_i) x^i \delta(r_j) x^j + \delta(r_i) x^i r_j x^j \\
&= \sum_{i=0}^{n}\sum_{j=0}^{m} \sigma(r_i x^i)\delta(r_j x^j) + r_j x^j \delta(r_i x^i) \\
&= \sigma(f(x))\delta(g(x)) + \delta(f(x))g(x)
\end{aligned}
$$

Thus, σ is a ring endomorphism and δ is a σ-derivation over $R[x]$. □

The next proposition follows Proposition 1.10 from [4]. We provide the result with the proof as reference for the reader and adjustments to match the desired notations and details for this paper.

Proposition 1.2.3. *Let R be a ring, σ an endomorphism of R, and δ a σ-derivation of R. Then there exists a ring S, containing R as a subring, such that S is a free left R-module with a basis of the form $1, f, f^2, \cdots$ and $fr = \sigma(r)f + \delta(r)$ for all $r \in R$.*

Proof. Let $E = \text{End}_{\mathbb{Z}}(R[x])$ where x is an indeterminate. We can view R as a subring of this endomorphism ring by associating R with the image of the injective map $\varphi : R \to E$ defined by sending $r \to \varphi_r$ for all $r \in R$ where $\varphi_r : f(x) \to r * f(x)$.

Now we extend σ and δ to E as in Lemma 1.2.2 and define f such that $f(g) = \sigma(g)x + \delta(g)$ for all $g \in R[x]$. We can see that $f \in E$ since for all $g(x), h(x) \in R[x]$,

$$\begin{aligned} f(g(x) + h(x)) &= \sigma(g(x) + h(x)) + \delta(g(x) + h(x)) \\ &= \sigma(g(x)) + \sigma(h(x)) + \delta(g(x)) + \delta(h(x)) \\ &= f(g(x)) + f(h(x)) \end{aligned}$$

Also, for all $z \in \mathbb{Z}$ and $g(x) \in R[x]$,

$$\begin{aligned} f(zg(x)) &= \sigma(zg(x)) + \delta(zg(x)) \\ &= z\sigma(g(x)) + z\delta(g(x)) \\ &= zf(g(x) \end{aligned}$$

It will be shown that $S = \sum_{i=0}^{\infty} Rf^i$ is a subring of E.
We first show that $f^i R \subseteq Rf^i + Rf^{i-1} + \cdots + Rf + R$ by induction on i. For $i = 1$, notice that for any $r \in R$ and $g \in R[x]$ we have

$$(f \circ \varphi_r)(g) = f(\varphi_r(g)) = f(rg) = \sigma(rg)x + \delta(rg) = \sigma(r)\sigma(g)x + \sigma(r)\delta(f) + \delta(r)f = \sigma(r)f(g) + \delta(r)g$$

Hence, $fr = \sigma(r)f + \delta(r)$ for all $r \in R$, and $fR \subseteq Rf + R$. Now assume $f^k R \subseteq Rf^k + Rf^{k-1} + \cdots + Rf + R$ for some $k \geqslant 1$ and consider $f^{k+1}R$. We have

$$f^{k+1}R = f(f^k R) \subseteq fRf^k + fRf^{k-1} + \cdots + fRf + fR$$

Applying $fR \subseteq Rf + R$, we have

$$\begin{aligned} f^{k+1}R &\subseteq (Rf+R)f^k + (Rf+R)f^{k-1} + \cdots + (Rf+R)f + Rf + R \\ &= Rf^{k+1} + Rf^k + Rf^k + \cdots + Rf^2 + Rf^2 + Rf + R \\ &= Rf^{k+1} + Rf^k + Rf^{k-1} + \cdots + Rf + R \end{aligned}$$

Thus, $f^iR \subseteq Rf^i + Rf^{i-1} + \cdots + Rf + R$ for all $i \in \{0, 1, 2, \ldots\}$.
As a consequence, $(f^iR)(f^jR) \subseteq Rf^{i+j} + Rf^{i+j-1} + \cdots + Rf + R$ for all $i, j \in \{0, 1, 2, \ldots\}$, which shows that S is closed under multiplication, and hence, S is a subring of $\mathrm{End}_{\mathbb{Z}}(R[x])$. We can also see that, by definition, S is generated by $1, f, f^2, \cdots$.

Now if $1, f, f^2, \cdots$ are all linearly independent over R, then S is a free left R-module with basis $1, f, f^2, \cdots$ as desired. Now take $s \in S$. Then $s = r_0 + r_1f + \cdots + r_nf^n$ with $r_i \in R$ for all i. We have that $s = 0$ if and only if $s(g(x)) = 0$ for all $g(x) \in R[x]$. Notice that by the definition of f, we have $f(x) = \sigma(x)x + \delta(x)$ and with $\sigma(1) = 1$ and $\delta(1) = 0$ this gives $f(x) = x^2$. It can be seen by induction that $f^i(x) = x^{i+1}$. Now $s(x) = r_0 + r_1f(x) + \cdots + r_nf^n(x) = r_o + r_1x^2 + \cdots + r_nx^n$. Since $1, x, x^2, \cdots$ are linearly independent over R, we must have $r_0, r_1, \cdots, r_n = 0$. This shows that $s = 0$ if and only if $r_0, r_1, \cdots, r_n = 0$, and hence, $1, f, f^2, \cdots$ are linearly independent over R and S is a free left R-module. □

The next proposition is given as an exercise in [4]. We provide it as a proposition here with proof as it will be useful to show that two different formulations of a particular skew polynomial ring are isomorphic.

Proposition 1.2.4. *Let R be a ring, σ a ring endomorphism, and δ a σ-derivation. Let S_1 and S_2 be ring extensions of R such that each S_i is a free left R-module with basis elements $1, f_i, f_i^2, \cdots$ and $f_ir = \sigma(r)f_i + \delta(r)$ for all $r \in R$. Then there exists an isomorphism $\Phi : S_1 \to S_2$ such that $\Phi(f_1) = f_2$.*

Proof. We first define the map $\Phi : S_1 \to S_2$ by $\Phi(r) = r$ for all $r \in R$ and $\Phi(f_1) = f_2$. Then for any $s \in S_1$ we have $s = \sum_{i=0}^{n} r_i f_1^i$ and

$$\Phi(s) = \Phi\Big(\sum_{i=0}^{n} r_i f_1^i\Big) = \sum_{i=0}^{n} \Phi(r_i)\Phi(f_1)^i = \sum_{i=0}^{n} r_i f_2^i$$

We now show by induction that $f_1^i r = \sum_{k=0}^{i} \alpha_k(r) f_1^k$ for all $i \in \mathbb{N}$ and $r \in R$, where $\alpha_{k,i} : R \to R$ determined by compositions of σ and δ for all k, i. For $i = 1$ we have

$$f_1 r = \sigma(r) f_1 + \delta(r)$$

Taking $\alpha_{0,1}$ to be δ and $\alpha_{0,1}$ to be σ, the result holds. Now assume true for $i \geqslant 1$ and consider $i + 1$. We have

$$\begin{aligned}
f_1^{i+1} r &= f_1 f_1^i r \\
&= f_1\Big(\sum_{k=0}^{i} \alpha_{k,i}(r) f_1^k\Big) \text{ by induction hypothesis} \\
&= \sum_{k=0}^{i} f_1 \alpha_{k,i}(r) f_1^k \\
&= \sum_{k=0}^{i} \big(\sigma(\alpha_{k,i}(r)) f_1^{k+1} + \delta(\alpha_{k,i}(r) f_1^k\big) \\
&= \sum_{k=0}^{i+1} \alpha_{k,i+1}(r) f_1^k \text{ by a relabeling to } \alpha_{k,i+1}
\end{aligned}$$

Notice that we obtain a similar expression in S_2 for $f_2^i r = \sum_{k=0}^{i} \alpha_{k,i}(r) f_2^i$ where the maps $\alpha_{k,i}$ are the same as in the expression for $f_1^i r$ since σ and δ are the same maps $R \to R$ in S_1 and S_2.

Now consider $\Phi(r_i f_1^i r_j f_1^j)$.

$$\begin{aligned}
\Phi(r_i f_1^i r_j f_1^j) &= \Phi\left(r_i\left(\sum_{k=0}^{i} \alpha_{k,i}(r_j) f_1^i\right) f_1^j\right)\\
&= \Phi\left(\sum_{k=0}^{i} r_i \alpha_{k,i}(r_j) f_1^{i+j}\right)\\
&= \sum_{k=0}^{i} r_i \alpha_{k,i}(r_j) f_2^{i+j}\\
&= r_i f_2^i r_j f_2^j\\
&= \Phi(r_i f^i)\Phi(r_j f^j)
\end{aligned}$$

Hence, Φ is a ring homomorphism. To see that Φ is an isomorphism, consider $\ker(\Phi) = \{s \in S_1 | \Phi(s) = 0\}$. Let $s \in S_1$ such that $\Phi(s) = 0$. Since $s \in S_1$, we have $s = \sum_{i=0}^{n} r_i f_1^i$ where $n \in \mathbb{Z}_{\geqslant 0}$ and $r_i \in R$ for all i. Then

$$\Phi(s) = \Phi\left(\sum_{i=0}^{n} r_i f_1^i\right) = \sum_{i=0}^{n} r_i f_2^i = 0$$

Since S_2 is a free left R-module with basis elements $1, f_2, f_2^2, \cdots$, we have $r_i = 0$ for all i, and hence, $s = 0$. Thus, $\ker(\Phi) = 0$ and Φ is an isomorphism. $\square$

We now give an example that shows that a skew polynomial ring will not generally be left or right Noetherian even when the base ring is Noetherian. This particular example was motivated by exercises written by A. Hubery created for lectures given by W. Crawley-Boevey at Bielefeld University. These exercises can be found at https://www.math.uni-bielefeld.de/ hubery/pdf-files/non-comm-alg-1/exercises6.pdf.

Example 1.2.5. Let $R = K[x]$ be a polynomial ring over a field K, and let σ be the K-algebra endomorphism of R sending $x \mapsto x^2$ and consider the skew polynomial ring $S = R[f; \sigma]$.

Proposition 1.2.6. *S in example 1.2.5 is not left Noetherian.*

Proof. Let I_n be the left ideal generated by $x, xf, \ldots, xf^n$. Consider xf^{n+1}.We want to show that $xf^{n+1} \notin I_n$ to see that we can obtain an ascending chain of left ideals $I \subsetneq I_2 \subsetneq I_3 \subsetneq \ldots$, and hence, that S is not left Noetherian. Assume that $xf^{n+1} \in I_n$. Then we have $xf^{n+1} = \sum_{i=0}^{n} s_i x f^i$ where $s_i \in S$ for all i. First, we show that xf^{n+1} does not appear in any of the terms in the sum. Since S is a free left R-module, we have that $s_i = \sum_{j=0}^{m} r_j f^j$ where $r_j \in R$ for all j. Now for any $i \in \{0, 1, 2, \ldots, n\}$ we have

$$s_i x f^i = \left(\sum_{j=0}^{m} r_j f^j \right) x f^i = \sum_{j=0}^{m} (r_j f^j x f^i) = \sum_{j=0}^{m} r_j x^{2^j} f^{i+j}$$

Considering the degree on x, we have $j = 0$ and $r_0 = 1$. However, this gives xf^i where $i \in \{0, 1, 2, \ldots, n\}$, a contradiction. Thus, xf^{n+1} does not appear in any $s_i x f^i$.
Since xf^{n+1} does not appear in any term of the sum, we have two distinct representations for the same element as an R-linear combination of powers of f, all different from f^{n+1}, but this is impossible since S is a free left R-module with basis $1, f, f^2, \cdots$. Thus, $xf^{n+1} \notin I_n$ and the chain of ideals, $I \subsetneq I_2 \subsetneq I_3 \subsetneq \ldots$ is strictly ascending. □

Lemma 1.2.7. *Let $R = K[x_1, x_2, \cdots, x_n]$ be the polynomial ring over a field K in indeterminates $x_1, x_2, \cdots, x_n$, σ be any ring endomorphism of R, and $S = R[f, \sigma]$ the skew polynomial ring. Then the terms $\underline{x}^{\alpha_i} f^j$ are linearly independent over K for all $i, j \in \mathbb{Z}_{\geqslant 0}$ where $\underline{x}^{\alpha_i} = x_1^{a_1}, x_2^{a_2}, \cdots, x_n^{a_n}$.*

Proof. First, we have that the f^j are linearly independent over R since S is a free left R-module with basis $1, f, f^2, \cdots$. Now consider a sum of the form $\sum_{m=0}^{n} k_{i_m, j_m} \underline{x}^{\alpha_{i_m}} f^{j_m} = 0$ where $n, i_m, j_m \in \mathbb{Z}_{\geqslant 0}$ and $k_{i_m, j_m} \in K$. Notice that $k_{i_m, j_m} \underline{x}^{\alpha_{i_m}} \in R$ for all m and let $r_{j_m} = \sum_{i_m} k_{i_m, j_m} \underline{x}^{\alpha_{i_m}}$. Hence, we have $\sum_{m=0}^{n} k_{i_m, j_m} \underline{x}^{\alpha_{i_m}} f^{j_m} = \sum_{m=0}^{n} r_{j_m} f^{j_m} = 0$ for $r_{j_m} \in R$. Since the f^i are linearly independent over R, we have $r_{j_m} = 0$ for all j_m. This shows that $\sum_{i_m} k_{i_m, j_m} \underline{x}^{\alpha_{i_m}} = 0$ for all i_m. Since $1, x, x^2, \cdots$ are linearly independent over k, we have $k_{i_m, j_m} = 0$ for all i_m, j_m. Thus, the terms $\underline{x}^{\alpha_i} f^j$ are linearly independent over K. □

Proposition 1.2.8. *S in example 1.2.5 is not right Noetherian.*

Proof. Let J_n be the right ideal generated by $x^n f^n$. WTS that J_n has a K-basis $x^i f^j$ where $i \equiv n \pmod{2^n}$ and $j \geqslant n$. Take $g \in J_n$. Then $g = x^n f^n s$ for some $s \in S$. Now $s = \Sigma r_m f^m$ where $r_m \in R$ and we have $g = x^n f^n(\Sigma r_m f^m)$. Now consider one term of this expression, say $x^n f^n r_m f^m$. Since $r_m \in R$ we have $r_m = \Sigma c_l x^l$ where $c_l \in K$ for all l. Then $x^n f^n r_m f^m = x^n f^n(\Sigma c_l x^l) f^m$. Notice that

$$x^n f^n c_l x^l f^m = c_l x^n f^n x^l f^m = c_l x^n x^{2^n l} f^n f^m = c_l x^{2^n l+n} f^{n+m}$$

for all $l, m, n \in \mathbb{N}$. We have $2^n l + n \equiv n \pmod{2^n}$ and $n + m \geqslant n$. Thus, g is a linear combination of terms of the form $c_l x^i f^j$ where $c_l \in K$ for all l, $i \equiv n \pmod{2^n}$ and $j \geqslant n$. The lemma above shows that the terms $x^i f^j$ are linearly independent over K, and thus, the $x^i f^j$ form a K-basis of J_n. Denote the K-basis for J_n by X_n for all n.

Now consider the right ideal $J = \sum_{n \geqslant 1} J_n$. Then the set $\bigcup_{n \geqslant 1} X_n$ forms a K-basis for J, since $x^i f^j$ are linearly independent over k for all $i, j \in \mathbb{Z}_{\geqslant 0}$ by the Lemma 1.2.7 and every element in J is a sum of elements from some of the J_i which are K-linear combinations of $x^i f^j$ for some i, j as above.

If S is right Noetherian, then J is finitely generated. Suppose J is finitely generated by generators $a_1, a_2, \ldots, a_n$ where $a_i \in J$ for all i. Then each a_i is a finite sum of elements from some of the J_i, which are K-linear combinations of elements of the form $x^i f^j$ as described above. Hence, we can write a finite generating set as elements of the form $x^i f^j$. Notice that each of these elements comes from a set X_i. Let X_n be the set containing a generator where $n \geqslant m$ for all m with X_m containing a generator.

Now consider $x^{2^{n+1}}f^{2^{n+1}}$. Since $x^{2^{n+1}}f^{2^{n+1}} \in J$, $x^{2^{n+1}}f^{2^{n+1}}$ is a K-linear combination of elements from the X_i containing the generators of J. Since the x^if^j are K-linearly independent, $x^{2^{n+1}}f^{2^{n+1}}$ must be a generator and in one of the sets X_m where $m \leqslant n$. We have $2^{n+1} \equiv 0 \pmod{2^m}$ for all $1 \leqslant m \leqslant n$. However, all elements in the X_m are of the form x^if^j where $i \equiv m \pmod{2^m}$ for all $1 \leqslant m \leqslant n$, and $i \not\equiv 0 \pmod{2^m}$ for any $1 \leqslant m \leqslant n$, a contradiction. Thus, J is not finitely generated and S is not right Noetherian. $\square$

Corollary 1.2.9. *Let $R = K[x]$ where K is a field and char$(K) = 2$. Then the skew polynomial ring $R[f;\sigma]$ where $\sigma : r \mapsto r^2$ for all $r \in R$ is neither left nor right Noetherian.*

Now that we have shown the existence and uniqueness of a skew polynomial ring over a ring R given an endomorphism σ and σ-derivation δ, we will focus on a skew polynomial ring where $\delta = 0$ and σ is the Frobenius endomorphism.

Definition 1.2.10. Let R be a commutative Noetherian ring of characteristic $p > 0$. The skew polynomial ring $R[f;F,\delta]$ where F is the Frobenius endomorphism $F : R \to R$ by $F(r) = r^p$ for all $r \in R$ and $\delta = 0$ is called the **Frobenius skew polynomial ring** over R, and is denoted $R[f;F]$.

We will present results given by Yoshino in [9] that give conditions on the base ring so that the Frobenius skew polynomial ring is left or right Noetherian. In this paper, Yoshino defines a Frobenius skew polynomial ring as the residue ring of $R\langle f\rangle$ by the two-sided ideal $\langle r^pf - fr \mid r \in R\rangle$, where $R\langle f\rangle$ is the free algebra generated by f. We first show that this agrees with our definition of a Frobenius skew polynomial ring.

Lemma 1.2.11. *Let R be a commutative Noetherian ring of characteristic $p > 0$, with p prime. If $S = \frac{R\langle f\rangle}{\langle r^pf-fr \mid r\in R\rangle}$ is the residue ring of the free algebra $R\langle f\rangle$ and the two-sided ideal $\langle r^pf - fr|r \in R\rangle$, then S is a free left R-module with basis $1, f, f^2, \cdots$.*

Proof. Let $x \in R\langle f\rangle$. Then x is a finite sum of products from elements of R and powers of f. Then we can write $\overline{x}$ as $\sum_{i=0}^{n} r_if^i$ by applying the relation $fr = r^pf$ in the residue ring. Also, since $R\langle f\rangle$ if the free algebra generated by f, we have that $1, f, f^2, \cdots$ are linearly independent over R in $R\langle f\rangle$. Now viewing S as a left R-module, we want to show that the elements $1, f, f^2, \cdots$ form a basis of S. To see that $1, f, f^2, \cdots$ are R-linearly independent in S, we consider an element $\sum_{i=0}^{n} r_i f^i = 0 \in S$. We know that $\sum_{i=0}^{n} r_i f^i = 0 \in S$ implies that $\sum_{i=0}^{n} r_i f^i \in \langle r^p f - fr \mid r \in R\rangle$. We show that this implies $r_i = 0$ for all i, and hence, $1, f, f^2, \cdots$ are linearly independent over S.

Assume $\sum_{i=0}^{n} r_i f^i \in \langle r^p f - fr \mid r \in R\rangle$. Then $\sum_{i=0}^{n} r_i f^i = \sum_{j=0}^{m} s_j(r_j^p f - fr_j)t_j$ where $r_j \in R$ and $s_j, t_j \in R\langle f\rangle$ for all j. We have

$$\sum_{j=0}^{m} s_j(r_j^p f - fr_j)t_j = \sum_{j=0}^{m} s_j r_j^p f t_j - \sum_{j=0}^{m} s_j f r_j t_j$$

We can conclude that any term where $s_j, t_j \in S\backslash R$ must cancel since there is no term on the left-hand side with elements to the right of f. After cancellation, we have

$$\sum_{j=0}^{m} s_j r_j^p f t_j - \sum_{j=0}^{m'} s_j f r_j t_j$$

where $s_j, t_j \in R$ for all j. Notice that the degree of f in this sum is at most 1. Hence, in $\sum_{i=0}^{n} r_i f^i$, we have $r_i = 0$ for all $i \neq 1$. So we have $rf = \sum_{j=0}^{m} s_j r_j^p f t_j - \sum_{j=0}^{m'} s_j f r_j t_j$. It follows that all t_j remaining must be 1, which in turn, implies that $r_j = 1$ for the remaining j. Hence, the entire right hand side is zero since we began with $r_j^p f - fr_j$. Thus, $r_i = 0$ and we have shown that $1, f, f^2, \cdots$ are linearly independent over S. This shows that S is a free left R-module with basis $1, f, f^2, \cdots$ $\square$

Proposition 1.2.12. *Let R be a commutative Noetherian Ring of characteristic $p > 0$, and let $S_1 = R[f_1; F]$ be the Frobenius skew polynomial ring over R with indeterminate f_1. Then let $S_2 = \frac{R\langle f_2\rangle}{\langle r^p f_2 - f_2 r \mid r \in R\rangle}$, where $R\langle f_2\rangle$ is the free algebra over R generated by the indeterminate f_2. Then $S_1 \cong S_2$.*

Proof. By definition, S_1 is a free left R-module with basis $1, f_1, f_1^2$, and by Lemma 1.2.11, S_2 is also a free left R-module with basis $1, f_2, f_2^2, \cdots$. Hence, we can apply Proposition 1.2.4

with F the Frobenius endomorphism and the F-derivation, $\delta = 0$ because $f_i r = r^p f_i$ for all $r \in R$ and $i \in \{1, 2\}$. Thus, there exists an isomorphism $\Phi : S_1 \to S_2$ such that $\Phi(f_1) = f_2$ and we have $S_1 \cong S_2$. □

We will now present the results from [9] showing when a Frobenius skew polynomial ring is left Noetherian. The presentation of these results consists of adjustments to notations to match the objects in this paper and added details. Otherwise, they follow closely to the presentation in [9]. Thus, references will also be made so that the results can be found in Yoshino's paper.

Lemma 1.2.13. *(1.4 in [9]) Let R be a commutative Noetherian ring with characteristic $p > 0$. Then*

1. *If R is a field, then $R[f; F]$ is left Noetherian.*
2. *If $R[f; F]$ is left Noetherian, then any ideal I of R satisfies $I^p = I$.*

Proof. (1) Let R be a field and let I be a left ideal of $S = R[\sigma; F]$. To show that S is left Noetherian, we will show that every left ideal of S is in fact principal. Consider an element $a_0 \in I$. Then $a_0 = \sum_{i=0}^{n} r_i f^i$ where $r_i \in R$ for all i. Choose a_0 such that n is minimal among elements in I with $r_n \neq 0$. We can assume without loss of generality that $r_n = 1$ since R is a field. We want to show that $I = Sa_0$, where Sa_0 is the left ideal generated by a_0. It is clear that $Sa_0 \subseteq I$ since $a_0 \in I$, so it is enough to show that $I \subseteq Sa_0$. Now take an arbitrary element $a \in I$ such that $a = \sum_{i=0}^{m} s_i f^i$ where $s_i \in R$ for all i. We will show by induction that $a \in Sa_0$. Since a_0 is chosen to be minimal, we have $m \geqslant n$. Assume that $m = n$. Then since $a - s_n a_0 \in I$ and has degree less than n, we must have that $a - s_n a_0 = 0 \in I$. Hence, since $s_n a_0 \in Sa_0$, we have $a \in Sa_0$.
Now assume that $m > n$ and that the result holds for all k such that $n \leqslant k < m$. Then $a - s_m f^{m-n} a_0 \in I$ has degree less than m. By the induction hypothesis, we have $a - s_m f^{m-n} a_0 \in Sa_0$. Since $s_m f^{m-n} a_0 \in Sa_0$, it follows that $a \in Sa_0$. Hence, $I \subseteq Sa_0$ and $I = Sa_0$. Thus, S is left Noetherian.

(2) Let I be an ideal of R. Then IS is a left ideal of S. To see this, notice that $rIS \in IS$ for all $r \in R$. So it remains to show that $fIS \subset IS$. By the multiplication in S, we have $fI = I^p f$. So $fIS = I^p fS \subset IS$, and IS is a left ideal of S. Then IS can be viewed as a left S-module generated by the elements $\{rf^i | r \in I, i \in \mathbb{Z}_{\geqslant 0}\}$. Let $J_n = S\{rf^i | r \in I, 0 \leqslant i \leqslant n\}$ and consider the left ideal $J = \cup_{i=0}^{\infty} J_n$. By assumption, S is left Noetherian, and hence, J is finitely generated. Hence, there exists a k such that we can write rf^{k+1} as an linear combination of terms in degree lower than $k+1$. This gives an equality

$$rf^{k+1} = a_0 r_0 + a_1 r_1 f + \cdots + a_k r_k f^k$$

for some $a_j \in S$ and some $r_i \in I$. Since $a_j \in S$, we can write each as $\sum_{i_j=0}^{m_j} s_{i_j} f^{i_j}$ where $s_{i_j} \in R$ for all i_j. Substituting these relations into the equation gives:

$$\begin{aligned} rf^{k+1} &= a_0 r_0 + a_1 r_1 f + \cdots + a_k r_k f^k \\ &= \Big(\sum_{i_0=0}^{m_0} s_{i_0} f^{i_0}\Big) r_0 + \Big(\sum_{i_1=0}^{m_1} s_{i_1} f^{i_1}\Big) r_1 f + \cdots + \Big(\sum_{i_k=0}^{m_k} s_{i_k} f^{i_k}\Big) r_k f^k \\ &= \sum_{i_0=0}^{m_0} s_{i_0} r_0^{p^{i_0}} f^{i_0} + \sum_{i_1=0}^{m_1} s_{i_1} r_1^{p^{i_1}} f^{i_1+1} + \cdots + \sum_{i_k=0}^{m_k} s_{i_k} r_k^{p^{i_k}} f^{i_k+k} \end{aligned}$$

Now by collecting all terms with degree $k+1$ on f from the right had side, we get that r is equal to the sum of elements of the form $s_{i_j} r_j^{p^{i_j}}$. Since $r \in R$ we have $s_{i_j} \in R$ for all i_j. Hence, $r \in I^p$ and $I \subset I^p$. It is clear that $I^p \subset I$, and we have $I = I^p$. $\square$

Theorem 1.2.14. *(1.3 in [9]) $S = R[f;F]$ is left Noetherian if and only if R is a direct product of a finite number of fields.*

Proof. Suppose that $R = K_1 \times K_2 \times \cdots \times K_n$ with K_i a field for all i. Then for every $s \in S$ we have $s = \sum_{i=0}^{k}(r_{1i}, r_{2i}, \ldots, r_{ni})f^i$. It follows that

$$S \cong K_1[f;F] \times K_2[f;F] \times \cdots \times K_n[f;F]$$

by the map $\varphi : S \to K_1[f;F] \times K_2[f;F] \times \cdots \times K_n[f;F]$ by

$$\varphi(s) = \left(\sum_{i=0}^{k} r_{1i}f^i, \sum_{i=0}^{k} r_{2i}f^i, \ldots, \sum_{i=0}^{k} r_{ni}f^i \right)$$

By Lemma 1.2.13, each term on the right-hand side is a left Noetherian ring, and by the isomorphism, we have S is left Noetherian.

Now assume that S is left Noetherian. Then by Lemma 1.2.13, for any maximal ideal m of R we have $m^p = m$. Hence, by Nakayama's Lemma $mR_m = (0)$, and R_m is a field. This shows that R is Artinian. Now consider $\mathrm{Jac}(R)$, the Jacobson radical of R. Since R is Artinian, we know that $\mathrm{Jac}(R)$ is nilpotent, say $\mathrm{Jac}(R)^n = 0$. We also have that $\mathrm{Jac}(R)^p = \mathrm{Jac}(R)$ by Lemma 1.2.13. The only way this is possible is to have $\mathrm{Jac}(R) = 0$. Hence, we have $\frac{R}{\mathrm{Jac}(R)}$ is semisimple since R is Artinian, and it follows that R is semisimple because $\mathrm{Jac}(R)$ is trivial. Since R is commutative and semisimple, we have that R is the finite product of fields. $\square$

Before providing the necessary conditions for a Frobenius skew polynomial ring to be right Noetherian, we give a result that shows when a general skew polynomial ring is right Noetherian if the base ring is right Noetherian. The following lemma associates an ideal I in S with an ideal in R by taking the leading coefficients from the elements in I of a certain degree, and will be useful in proving Proposition 1.2.16.

Lemma 1.2.15. *Let R be a right Noetherian ring, $S = R[f;\sigma,\delta]$ be a skew polynomial ring with σ an automorphism and δ a σ-derivation, and I, I' be nonzero right ideals of S. If I_k is the set of leading coefficients from elements in I with degree at most k, then*

1. *I_k is a right ideal of R for all $k \geqslant 0$.*

2. *If $I \subseteq I'$ and $I_k = I'_k$ for all $k \geqslant 0$, then $I = I'$.*

Proof. 1) Let I be an ideal of S and I_k be the set of all leading coefficients in I with degree at most k. Since I is an ideal, we have $0 \in I$ and it follows that $0 \in I_k$. Now for any $r_{1_m}, r_{2_n} \in I_k$ with $0 \leqslant m, n \leqslant k$, there exists elements of the form $s_1 = \sum_{i=0}^{m} r_{1_i} f^j$ and $s_2 = \sum_{j=0}^{m} r_{2_j} f^j$ in I. Notice that if $r_{1_m} + r_{2_n} = 0$ then $r_{1_m} + r_{2_n} \in I_k$. So now assume that $r_{1_m} + r_{2_n} \neq 0$. If $m \leqslant n$, then $s_1 f^{n-m} + s_2$ is in I and has degree $n \leqslant k$ and leading coefficient $r_{1_m} + r_{2_n}$. Hence, $r_{1_m} + r_{2_n} \in I_k$. Similarly we get $r_{1_m} + r_{2_n}$ in I_k when $m \geqslant n$ by considering the element $s_1 + s_2 f^{m-n}$. So we have $r + s \in I_k$ for all $r, s \in I_k$.
Now let $r \in R$ such that $r_{1_m} r \neq 0$ for $r_{1_m} \in I_k$. We want to show that $r_{1_m} r \in I_k$. Since I is an ideal and σ is an automorphism, we have $s_1 \sigma^{-m}(r) \in I$. The leading term is $r_{1_m} f^m \sigma^{-m}(r)$. By the multiplication in S, we have $r_{1_m} f^m r = r_{1_m} \sigma^m(\sigma^{-m}(r)) f^m +$ terms in lower degree. Hence, the leading coefficient is $r_{1_m} \sigma^m(\sigma^{-m}(r)) = r_{1_m} r$ and it follows that $r_{1_m} r \in I_k$. Thus, I_k is a right ideal of R for all $k \geqslant 0$.
2) Let I, I' be right ideals of S such that $I \subseteq I'$ and $I_k = I'_k$ for all $k \geqslant 0$. Assume $I \subsetneq I'$ and choose an element $s' = \sum_{i=0}^{m} r_i f^i \in I'$ such that m is minimal among all $s \notin I$. Then $r_m \in I'_m$. Since $I_m = I'_m$ we must have $r_m \in I_m$. Hence, there exist a skew polynomial in I, say s', such that the leading coefficient is r_m. Let $s = r_m f^k + \sum_{j=0}^{k-1} r_j f^j$ where $k \leqslant m$. It follows that $r_m f^k \in I$ since $\sum_{j=0}^{k-1} r_j f^j \in I$ because it has degree less than m and m was chosen to be minimal. However, this implies $r_m f^k (f^{m-k}) + (s - r_m f^m) = s \in I$, a contradiction. Thus, $I = I'$. □

Proposition 1.2.16 can be found in Theorem 2.9 part IV in [7].

Proposition 1.2.16. *Let $S = R[f;\sigma,\delta]$ be the skew polynomial ring over R with indeterminate f. If R is right Noetherian and σ is an automorphism, then S is right Noetherian.*

Proof. First, assume that R is right Noetherian and that σ is an automorphism. Consider a chain of right ideals in S,

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots .$$

Let $I_{i,n}$ be the right ideal of R consisting of the leading coefficients of elements in I_i of degree at most n. Then we have the following array of chains of ideals in R.

$$\begin{array}{ccc}
I_{0,0} \subseteq & I_{1,0} \subseteq & I_{2,0} \subseteq \cdots \\
I_{0,1} \subseteq & I_{1,1} \subseteq & I_{2,1} \subseteq \cdots \\
I_{0,2} \subseteq & I_{1,2} \subseteq & I_{2,2} \subseteq \cdots \\
\vdots & \vdots & \ddots
\end{array}$$

We claim that $I_{i,m} \subseteq I_{j,n}$ for any $i \leqslant j$ and $m \leqslant n$. To see this, first consider $I_{i,m}$ and $I_{j,m}$ for some n and $i \leqslant j$. By definition, $I_i \subseteq I_j$, and hence, any element of degree at most m that is contained in I_i is also contained in I_j. It follows that $I_{i,m} \subseteq I_{j,m}$. Also, we have $I_{j,m} \subseteq I_{j,n}$ for any j and $m \leqslant n$ since $I_{j,m}$ contains all coefficients at most m, which includes those of degree at most n. Thus, $I_{i,m} \subseteq I_{j,m} \subseteq I_{j,n}$ and $I_{i,m} \subseteq I_{j,n}$ for all $i \leqslant j$ and $m \leqslant n$.

Based on this claim, we can construct the following ascending chain of right ideals in R corresponding to the diagonal of the array above.

$$I_{0,0} \subseteq I_{1,1} \subseteq I_{2,2} \subseteq \cdots$$

Now since R is right Noetherian, there exists a j such that this chain stabilizes at $I_{j,j}$, and hence, all ascending chains below the diagonal will stabilize at $I_{j,j}$ as well. Also, each row of this array above the diagonal must stabilize since R is right Noetherian. That is, each chain of the form

$$I_{0,n} \subseteq I_{1,n} \subseteq I_{2,n} \subseteq \cdots$$

will stabilize, say at k_n, for all $0 \leqslant n \leqslant j-1$. Let $m = \max\{k_0, k_1, \cdots, k_{n-1}, j\}$. Then we have $I_{i,n} = I_{m,n}$ for all $i \geqslant m$ and $n \geqslant 0$. Thus by Lemma 1.2.15, we have $I_i = I_m$ for all $i \geqslant m$ and the chain

$$I_0 \subseteq I_1 \subseteq I_2 \subseteq \cdots$$

stabilizes in S and S is right Noetherian. □

Corollary 1.2.17. *Let R be a perfect field. Then the Frobenius skew polynomial ring, $S = R[f;\sigma]$, is right Noetherian.*

Proof. Let R be a perfect field. Then R is right Noetherian and σ, the Frobenius endomorphism, is an automorphism. Thus, S is right Noetherian by Proposition 1.2.16. □

The following lemma is well known and even given as Exercise 13.35 in [8]. We provide it as a lemma here with proof as it will be useful in the proof of Proposition 1.2.20. The first implication comes as an immediate result from the more general result for skew polynomial rings. The other implications will be useful in refining the requirements for the base ring R of a Frobenius skew polynomial ring, and provide necessary and sufficient conditions for a Frobenius skew polynomial ring to be right Noetherian.

Lemma 1.2.18. *Let R be a subring of the commutative ring S, and suppose that S is integral over R. Then if $r \in R$ is a unit in S, then*

1. *If $r \in R$ is a unit in S, then r is a unit in R*

2. *$Jac(R) = Jac(S) \cap R$*

Proof. 1) Let $r \in R$ be a unit in S. Then there exists $s \in S$ such that $rs = 1$ in S. Since S is integral over R, we have $s^n + r_{n-1}s^{n-1} + \cdots + r_0 = 0$ for $r_i \in R$ and $n \in \mathbb{N}$. Hence, $s^n = -r_{n-1}s^{n-1} - \cdots - r_0$. Multiplying by r^{n-1} gives $s = -r_{n-1} - r_{n-2}r - \cdots - r_0r^{n-1} \in R$. So $s \in R$ and it follows that r is a unit in R.

2) To see that $\mathrm{Jac}(R) \supseteq \mathrm{Jac}(S) \cap R$, let $r \in \mathrm{Jac}(S) \cap R$. Then $r \in R$ and $1 - rs$ is a unit for all $s \in S$. Since $R \subseteq S$, we have $1 - rr_i$ is a unit in S for all $r_i \in R$. By part 1, this shows that $1 - rr_i$ is a unit in R for all $r_i \in R$ and $r \in$ Jac(R).

We now apply the Lying over property of integral extensions to demonstrate the reverse inclusion. For every maximal ideal $\mathfrak{m}_i$ of R, there exists a prime ideal P_i in S such that $\mathfrak{m}_i = P_i \cap R$. Let $\Omega = \{\mathfrak{m}' | \mathfrak{m}' \in \mathrm{Max}(R)$ and $\mathfrak{m}' = \mathfrak{n}_i \cap R$ for $n_i \in \mathrm{Max}(S)\}$. Then

$$\mathrm{Jac}(R) = \bigcap_{\mathfrak{m} \in \mathrm{Max}(R)} \mathfrak{m} \subseteq \bigcap_{\mathfrak{m}' \in \Omega} \mathfrak{m}' = \bigcap_{\mathfrak{n}_i \in \mathrm{Max}(S)} (\mathfrak{n}_i \cap R) = \left(\bigcap_{\mathfrak{n}_i \in \mathrm{Max}(S)} \mathfrak{n}_i \right) \cap R = \mathrm{Jac}(S) \cap R$$

□

Lemma 1.2.19. *Let A and B be domains such that $\varphi : A \hookrightarrow B$ is a faithfully-flat extension and $Q(A) = Q(B)$. Then $A = B$.*

Proof. Consider the short exact sequence $0 \to A \to B \to B/\mathrm{Im}(\varphi) \to 0$. Since this extension is flat, we maintain an exact sequence after tensoring with $Q(A)$. Hence, we have $0 \to Q(A) \otimes A \to Q(A) \otimes B \to Q(A) \otimes B/\mathrm{Im}(\varphi) \to 0$. Then $Q(A) \otimes A \cong Q(A)$, and since $Q(A) = Q(B)$, we have $Q(A) \otimes B = Q(B) \otimes B \cong Q(B)$. Hence the map $Q(A) \otimes A \to Q(A) \otimes B$ is the identity. Thus, the sequence $0 \to Q(A) \otimes A \to Q(A) \otimes B \to 0$ is exact, and since $A \hookrightarrow B$ is faithfully-flat, we have $0 \to A \to B \to 0$ is exact and $A = B$. □

We will now present the results of Yoshino that show when a Frobenius skew polynomial ring is right Noetherian.

Proposition 1.2.20. *(1.5 in [9]) Suppose that R is a local domain. Then the following conditions are equivalent.*

1. *$S = R[f; F]$ is right Noetherian.*

2. *R is a perfect field.*

3. *R is contained in $Q(R^p)$, the field of quotients of R^p.*

Proof. 2) $\Rightarrow$ 1) Done by Corollary 1.2.17.

1) $\Rightarrow$ 3) Assume that S is right Noetherian and that $R \not\subset Q(R^p)$ and take an element $r \in R$ such that $r \notin Q(R^p)$. Let J be the ideal of S generated by elements of the form $r^{p^i} f^{i+1}$ where $i \in \mathbb{N}$. Assume that J is finitely generated as a right S-module. Then there exists an n such that

$$r^{p^n} f^{n+1} = r f g_0 + r^p f^2 g_1 + \cdots + r^{p^{n-1}} f^n g_{n-1}$$

for some $g_i \in S$. Now the right hand side can be rearranged as a linear combination of elements from R on the left and powers of f. After rearranging, we can consider the coefficient from R on f^{n+1} on each side to get the following equality:

$$r^{p^n} = r r_0^p + r^p r_1^{p^2} + \cdots + r^{p^{n-1}} r_{n-1}^{p^n}$$

Since $r \neq 0$, there exists an $0 \leqslant i \leqslant n-1$ such that $r_i \neq 0$. Hence the term $r^{p^i} r_i^{p^{i+1}} \neq 0$. Now assume that i is the smallest integer with $r_i \neq 0$. Then we have

$$r^{p^n} = r^{p^i} r_i^{p^{i+1}} + r^{p^{i+1}} r_{i+1}^{p^{i+2}} + \cdots + r^{p^{n-1}} r_{n-1}^{p^n}$$

Since R is a domain we can take the p^i-th root, which yields

$$\begin{aligned} r^{p^{n-i}} &= r r_i^p + r^p r_{i+1}^{p^2} + \cdots + r^{p^{n-i-1}} r_{n-1}^{p^{n-i}} \\ r r_i^p &= r^{p^{n-i}} - r^p r_{i+1}^{p^2} - \cdots - r^{p^{n-i-1}} r_{n-1}^{p^{n-i}} \\ r &= \frac{r^{p^{n-i}} - r^p r_{i+1}^{p^2} - \cdots - r^{p^{n-i-1}} r_{n-1}^{p^{n-i}}}{r_i^p} \end{aligned}$$

However, since $r^{p^{n-i}} - r^p r_{i+1}^{p^2} - \cdots - r^{p^{n-i-1}} r_{n-1}^{p^{n-i}}$ and r_i^p are elements in R^p, this shows that $r \in Q(R^p)$, contradicting the choice of r. Thus, $R \subseteq Q(R^p)$.

3) $\Rightarrow$ 2) Assume that R is contained in $Q(R^p)$.It is enough to show that R is a field, because if R is a field, then $R = Q(R)$ and we have

$$R^p \subseteq R \implies Q(R^p) \subseteq Q(R) \text{ and } R \subseteq Q(R^p) \subseteq Q(R) \implies Q(R^p) = Q(R) \implies R = R^p$$

Thus, R is a perfect field.

Since R is assumed to be a local domain, it is enough to show that R is Artinian to show that R is a field. Assume that R is not Artinian. Then there exists a height one prime ideal, P, in R. Let R' denote the localization R_P.Then we have

$$R' \subseteq Q(R') = Q(R) = Q(R^p) = Q(R'^p)$$

Now let $\overline{R}$ be the integral closure of R' in $Q(R')$. We can apply the Krull-Akizuki (Theorem 11.13 in [3]) to conclude that R' is Noetherian. Now let $\mathrm{Jac}(\overline{R})$ be the Jacobson radical of $\overline{R}$. By Lemma 1.2.18, we have that $\mathrm{Jac}(\overline{R}) \cap R' = \mathrm{Jac}(R')$ and since R' is local, $\mathrm{Jac}(R')$ is just the maximal ideal of R'. Hence, $\mathrm{Jac}(\overline{R}) \neq 0$. Also, since $\overline{R}$ is an integrally closed local domain of dimension 1, we have that $\overline{R}$ is regular, and hence, the Frobenius map on $\overline{R}$ is flat by the Kunz Theorem (see [6]). We also have that the extension $\overline{R}^p \hookrightarrow \overline{R}$ is faithfully flat. This is because if we have $\overline{R} = \mathfrak{m}\overline{R}$ for a maximal ideal $\mathfrak{m} \in \overline{R}^p$, we would have $\overline{R} = 0$ by Nakayama, a contradicton.

By applying Lemma 1.2.19 and the equalities

$Q(\overline{R}^p) = Q(R'^p) = Q(R') = Q(\overline{R})$, we can conclude that $\overline{R}^p = \overline{R}$. But it follows that $J^{[p]} = J$. Then since $J^{[p]} \subseteq J^p \subseteq J$ we have $J^p = J$. By Nakayama, this implies $J = 0$, a contradiction. Thus, R is Artinian. $\square$

Theorem 1.2.21. *(1.3 in [9]) $S = R[f; F]$ is right Noetherian if and only if R is Artinian and R/m is a perfect field for each maximal ideal $\mathfrak{m}$ of R.*

Proof. Suppose that S is right Noetherian. Then let P be a prime ideal of R. We have the natural surjective homomorphism $\varphi : R \to R/P$ that can be extended to a surjective homomorphism $\varphi : S \to R/P[f; F]$ where $\varphi(r) = r + P$ and $\varphi(f) = f$. Hence, $R/P[f; F]$ is right Noetherian as well and by Proposition 1.2.20, we have R/P is a perfect field. Since P is an arbitrary prime, we have R/P is a perfect field for any P, showing that R is Artinian. It also follows that $R/\mathfrak{m}$ is a perfect field for all maximal ideals $\mathfrak{m}$.

Now suppose that R is Artinian and each residue field is perfect. Then $\frac{R}{\mathrm{Jac}(R)} \cong \frac{R}{\mathfrak{m}_1} \times \frac{R}{\mathfrak{m}_2} \times \cdots \times \frac{R}{\mathfrak{m}_n}$, where $\mathfrak{m}_i$ are the maximal ideals of R and $\mathrm{Jac}(R)$ is the Jacobson radical of R. By assumption, $\frac{R}{\mathfrak{m}_i}$ is a perfect field for all i. Then $\frac{R}{\mathrm{Jac}(R)}[f; \sigma] \cong \frac{R}{\mathfrak{m}_1}[f; \sigma] \times \frac{R}{\mathfrak{m}_2}[f; \sigma] \times \cdots \times \frac{R}{\mathfrak{m}_n}[f; \sigma]$ and each $\frac{R}{\mathfrak{m}_i}[f; \sigma]$ is right Noetherian. Hence, $\frac{R}{\mathrm{Jac}(R)}[f; \sigma]$ is right Noetherian. $\square$

1.3 Growth of Algebras

In this section, we will introduce the notion of the growth of an algebra and provide some of the basic definitions and results. The growth will then be used to define the concept of the Gelfand-Kirillov dimension. We introduce these ideas here because they will provide the setup used for the examples in Chapter 2. This will all be based on an algebra over a field, where the algebra is not necessarily commutative. We first provide the definition of a K-algebra in the noncommutative setting as a reminder to the reader.

Definition 1.3.1. Let k be a field and A a ring. Then A is said to be a K**-algebra** when equipped with a ring homomorphism $\varphi : K \to Z(A)$, where

$Z(A) = \{a \in A \mid as = sa \text{ for all } s \in A\}$ is the center of A.

The construction developed here follows the standard setup for the growth of an algebra given in [5]. Let K be a field and A be a finitely generated K-algebra with 1_A, generated as an algebra by $a_1, a_2, \cdots, a_h \in A$. We take V to be a finite dimensional generating subspace of A, that is the K-subspace spanned by $a_1, \cdots, a_h$. Now in this construction, we have $V^0 = K$ and then for any $m \in \mathbb{N}$ we have V^m is the K-vector subspace spanned by products of m elements from $\{a_1, a_2, \cdots, a_h\}$. By defining the following sum $V^d + V^k = \{v + v' \mid v \in V^d \text{ and } v' \in V^k\}$ we can now establish a decomposition of A using V^c.

$$\text{Let } A_n = \sum_{i=0}^{n} V^i \text{ then } A = \bigcup_{n=0}^{\infty} A_n.$$

This notion of the growth of an algebra is determined by such a finite dimensional generating subspace, V, and computing then dimension of A_n over K. However, this dimension is dependent on the choice of V. We denote $\dim_K(A_n)$ by $d_V(n)$. To avoid this dependence, and define the growth for an algebra, we can define an equivalence relation on $d_V(n)$. It will then be shown that the dimension of A_n over K for any finite dimensional generating subspace will belong to the same equivalence class.

Definition 1.3.2. Let Φ be the set of all functions $\varphi : \mathbb{N} \to \mathbb{R}$ in which there exists an $n_0 \in \mathbb{N}$ such that $\varphi(n) \leqslant \varphi(n+1) \in \mathbb{R}$ for all $n \geqslant n_0$ and $\varphi \in \Phi$. Then for any $\varphi, \gamma \in \Phi$ we say $\varphi \leqslant^* \gamma$ if and only if there exist $a, b \in \mathbb{N}$ such that

$$\varphi(n) \leqslant a\gamma(bn)$$

for all but finitely many $n \in \mathbb{N}$.

Then we define the equivalence relation $\sim$ by $\varphi \sim \gamma$ if and only if $\varphi \leqslant^* \gamma$ and $\gamma \leqslant^* \varphi$. We denote the equivalence class for any element $\varphi \in \Phi$ by $\mathcal{G}(\varphi) \in \Phi/\sim$. This equivalence class is called the **growth** of φ. Now we can induce a partial order given by $\leqslant^*$ on the set $\Phi/\sim$, and will denoted this partial order by $\leqslant$.

Proposition 1.3.3. *Let $\sim$ be the relation given in definition 1.3.5. Then $\sim$ is an equivalence relation.*

Proof. It is clear that $\sim$ is reflexive since $\varphi(n) \leqslant \varphi(n)$ for all $\varphi \in \Phi$ and $n \in \mathbb{N}$.
Let $\varphi, \gamma \in \Phi$ such that $\varphi \sim \gamma$. Then there exists $a, b, \alpha, \beta \in \mathbb{N}$ such that $\varphi(n) \leqslant a\gamma(bn)$ and $\gamma(n) \leqslant \alpha\varphi(\beta n)$ for all but finitely many $n \in \mathbb{N}$. Hence, by definition we have that if $\varphi \sim \gamma$, then $\gamma \sim \varphi$ and $\sim$ is symmetric.
Now to see that $\sim$ is transitive, let $\varphi, \gamma, \tau \in \Phi$ such that $\varphi \sim \gamma$ and $\gamma \sim \tau$. Then there exists $a, b, \alpha, \beta, c, d, e, f \in \mathbb{N}$ such that $\varphi(n) \leqslant a\gamma(bn), \gamma \leqslant \alpha\varphi(\beta n), \gamma(n) \leqslant c\tau(dn), \tau(n) \leqslant e\gamma(fn)$ for almost all n. Hence, $\varphi \leqslant ac\tau(bdn)$ for almost all n, and $\tau(n \leqslant \alpha e\varphi(\beta fn)$ for almost all n. Thus, $\varphi \sim \tau$. $\square$

We now prove a lemma that shows that the growth of an algebra is independent of the choice of the finite dimensional generating subspace by showing that any finite dimensional generating subspace of A belongs to the same equivalence class. This will allow us to define the growth of a K-algebra for an arbitrary finite dimensional generating subspace. This can be found as Lemma 1.1 in [5].

Lemma 1.3.4. *Let A be a finitely generated K-algebra for a field K with finite dimensional generating subspaces V and W. If $d_V(n)$ and $d_W(n)$ denote the dimensions of $\sum_{i=0}^{n} V^i$ and $\sum_{i=0}^{n} W^i$ over k, respectively, then $\mathcal{G}(d_V) = \mathcal{G}(d_W)$.*

Proof. Since V and W are finite dimensional subspaces, we have the following equality:

$$A = \bigcup_{n=0}^{\infty} \Big(\sum_{i=0}^{n} V_i \Big) = \bigcup_{n=0}^{\infty} \Big(\sum_{i=0}^{n} W_i \Big)$$

Hence, there exist $s, t \in \mathbb{N}$ such that $W \subseteq \sum_{i=0}^{s} V^i$ and $V \subseteq \sum_{i=0}^{t} W^i$. From here we can see that $d_V(n) \leqslant d_W(tn)$ and $d_W(n) \leqslant d_V(sn)$ for all $n \in \mathbb{N}$ and it follows that $d_V \sim d_W$. $\square$

Definition 1.3.5. Let A be a finitely-generated K-algebra and V be a finite dimensional generating subspace of A. Then the **growth of** A is defined to be the growth of $d_V(n)$ and is denoted by $\mathcal{G}(A)$.

If f, g are two polynomials then $f \sim g$ if and only if deg(f)=deg(g). For any function p where $p : n \to n^d$ for some real number $d \geqslant 0$, the growth of the function is denoted by $\mathcal{P}_d$. A finitely-generated k-algebra, A is said to have polynomial growth if $\mathcal{G}(A) = \mathcal{P}_d$ for some $d \in \mathbb{N}$.

Likewise, for any function $q_\epsilon : n \to e^{n^\epsilon}$ we denote the growth of q_ϵ by $\mathcal{E}_\epsilon$. A finitely generated K-algebra is said to have exponential growth if $\mathcal{G}(A) = \mathcal{E}_1$.

We now provide a proposition that shows that a finitely generated K-algebra will have growth between $\mathcal{P}_1$ and $\mathcal{E}_1$ (Proposition 1.4 in [5]).

Proposition 1.3.6. *If A is a finitely generated K-algebra that is not finite dimensional, then $\mathcal{P}_1 \leqslant \mathcal{G}(A) \leqslant \mathcal{E}_1$.*

Proof. Let V be a finite dimensional generating subspace of A such that $1 \in V$. Then we have that $\sum_{i=0}^{n} V^i = V^n$ for all $n \in \mathbb{N}$. Hence, $d_V(n) = \dim_K(V^n)$ for all $n \in \mathbb{N}$. Since we have $\dim_K(V^n) \leqslant \dim_K(V \otimes V \otimes \cdots \otimes V)$ where there are n terms in the product, and $\dim_K(V \otimes V \otimes \cdots \otimes V) = (\dim_K(V))^n$, it follows that $\mathcal{G}(d_V) \leqslant \mathcal{G}((\dim_K(V)^n) = \mathcal{E}_1$. Hence, we have $\mathcal{G}(A) \leqslant \mathcal{E}_1$.
Now since A is not finite dimensional, $V^n \neq A$ for all $n \in \mathbb{N}$. It follows that the sequence $K \subseteq V \subseteq V^2 \subseteq \cdots$ is strictly increasing. Otherwise, we would have $V^n = V^{n+1}$ for some $n \in \mathbb{N}$ and $V^{n+1} = A$, a contradiction. From this we can see that $d_V(n) \geqslant n$ for all $n \in \mathbb{N}$, and thus, $\mathcal{G}(A) \geqslant \mathcal{G}(n) = \mathcal{P}_1$. □

Example 1.3.7. Let $A = K\langle x_1, x_2, \cdots, x_m\rangle$ be the free K-algebra on m generators. Then take $V = Kx_1 + Kx_2 + \cdots + Kx_m$ as a finite dimensional generating subspace of A. Let $A_n = V^0 + V + V^2 + \cdots + V^n$ so that we have $A = \cup_{i=0}^{\infty} A_n$.

$$d_V(n) = \dim_K(A_n) = 1 + m + m^2 + \cdots + m^n \text{ and } \mathcal{G}(A) = \mathcal{E}_1$$

The following two lemmas will be used to show that the commutative polynomial ring over a field has polynomial growth. We have separated Lemma 1.5 from [5] for presentation purposes. We illustrate parts a and b from that lemma here.

Lemma 1.3.8. *Let f be a polynomial of degree d in $\mathbb{Q}[x]$ where $\mathbb{Q}$ is the field of rational numbers. Then there exist $a_i \in \mathbb{Q}$ for $i \in \{0, 1, 2, \cdots, d\}$ such that*

$$f(n) = a_d \binom{n}{d} + a_{d-1} \binom{n}{d-1} + \cdots + a_1 \binom{n}{1} + a_0$$

for all $n \in \mathbb{N}$.

Proof. We proceed by induction on d. If $d = 0$, then f is constant in $\mathbb{Q}$ and $f = a_0$ for some $a_0 \in \mathbb{Q}$.

Now assume the result holds for all degrees i such that $0 \leqslant i \leqslant d-1$. Consider $f(n) \in \mathbb{Q}[x]$ of degree d. We have that

$$\begin{aligned} \binom{n}{d} &= \frac{n!}{(n-d)!d!} \\ &= \frac{n(n-1)\cdots(n-(d-1))}{d!} \end{aligned}$$

Since the numerator has d terms, this is a polynomial in n of degree d with leading coefficient $\frac{1}{d!}$. Let $\alpha_d \in \mathbb{Q}$ be the leading coefficient of f. Then $f(n) - \alpha_d d! \binom{n}{d}$ is a polynomial of degree $d-1$. By the induction hypothesis, it follows that $f(n) - \alpha_d d! \binom{n}{d} = a_{d-1}\binom{n}{d-1} + \cdots + a_1 \binom{n}{1} + a_0$ for some $a_i \in \mathbb{Q}$. Hence, $f(n) = \alpha_d d! \binom{n}{d} + a_{d-1}\binom{n}{d-1} + \cdots + a_1\binom{n}{1} + a_0$. Since $\alpha_d d! \in \mathbb{Q}$, this completes the proof. $\square$

The next lemma will be useful in Proposition 1.3.10.

Lemma 1.3.9. *Let $a, b \in \mathbb{N}$ such that $a > b$. Then* $\binom{a+1}{b} - \binom{a}{b} = \binom{a}{b-1}$.

Proof. We have the following:

$$\begin{aligned}
\binom{a+1}{b} - \binom{a}{b} &= \frac{(a+1)!}{(a+1-b)!b!} - \frac{a!}{(a-b)!b!} \\
&= \frac{(a+1)(a)\cdots(a+2-b)}{b!} - \frac{a(a-1)\cdots(a+1-b)}{b!} \\
&= \frac{(a+1-(a+1-b))[a(a-1)\cdots(a+2-b)]}{b!} \\
&= \frac{a(a-1)\cdots(a+2-b)}{(b-1)!} \\
&= \frac{a(a-1)\cdots(a+2-b)(a+1-b)!}{(b-1)!(a+1-b)!} \\
&= \frac{a!}{(a-(b-1))!(b-1)!} \\
&= \binom{a}{b-1}
\end{aligned}$$

□

Proposition 1.3.10. *Let $f : \mathbb{N} \to \mathbb{Q}$. Then the following are equivalent:*

1. *There exist $a_i \in \mathbb{Q}$ for $i \in \{0, 1, 2, \cdots, d\}$ and $m \in \mathbb{Z}_{\geqslant 0}$ such that for all $n \geqslant m$*

$$f(n) = a_d\binom{n}{d} + a_{d-1}\binom{n}{d-1} + \cdots + a_1\binom{n}{1} + a_0$$

2. *There exist $a_i \in \mathbb{Q}$ for $i \in \{1, 2, \cdots, d\}$ and and $m \in \mathbb{Z}_{\geqslant 0}$ such that for all $n \geqslant m$ such that*

$$f(n+1) - f(n) = a_d\binom{n}{d-1} + \cdots + a_2\binom{n}{1} + a_1$$

Proof. 1) $\Rightarrow$ 2) Let $f(n) = a_d\binom{n}{d} + a_{d-1}\binom{n}{d-1} + \cdots + a_1\binom{n}{1} + a_0$. Then $f(n+1) = a_d\binom{n+1}{d} + a_{d-1}\binom{n+1}{d-1} + \cdots + a_1\binom{n+1}{1} + a_0$. We now compute $f(n+1) - f(n)$:

$$\begin{aligned} f(n+1) - f(n) =& a_d\binom{n+1}{d} + \cdots + a_1\binom{n+1}{1} + a_0 - \left(a_d\binom{n}{d} + \cdots + a_1\binom{n}{1} + a_0\right) \\ =& a_d\left(\binom{n+1}{d} - \binom{n}{d}\right) + a_{d-1}\left(\binom{n+1}{d-1} - \binom{n}{d-1}\right) \\ &+ \cdots + a_1\left(\binom{n+1}{1} - \binom{n}{1}\right) \end{aligned}$$

Now by applying Lemma 1.3.8 to each of the differences, we obtain:

$$f(n+1) - f(n) = a_d\binom{n}{d-1} + \cdots + a_2\binom{n}{1} + a_1$$

2) $\Rightarrow$ 1) Assume $f(n+1) - f(n) = a_d\binom{n}{d-1} + \cdots + a_2\binom{n}{1} + a_1$ for every $n \geqslant m$. Then by applying the identity from Lemma 1.3.9, and then reversing the steps from the first part of the proof, we obtain:

$$f(n+1) - f(n) = a_d\binom{n+1}{d} + \cdots + a_1\binom{n+1}{1} + a_0 - \left(a_d\binom{n}{d} + \cdots + a_1\binom{n}{1} + a_0\right)$$

Hence, $f(n) = a_d\binom{n}{d} + a_{d-1}\binom{n}{d-1} + \cdots + a_1\binom{n}{1} + a_0$. $\square$

Corollary 1.3.11. *Let K be a field and $A = K[x_1, \cdots, x_d]$ be the polynomial algebra over K. Then $\mathcal{G}(A) = \mathcal{P}_d$.*

Proof. Since A is a finitely generated K-algebra we can choose any finite dimensional generating subspace of A by Lemma 1.3.4. Let $V = \langle x_1, x_2, \cdots, x_d\rangle$, be a finite dimensional generating subspace of A. To compute $\dim_K(V^{n+1})$, we just have to determine the number of monomials of total degree d that can be formed by $x_1.x_2, \cdots, x_d$.

Hence, $\dim_K(V^{n+1}) = \binom{n+d}{d-1}$, which is a polynomial in n of degree $d-1$. Notice that $\dim_K(V^{n+1}) = d_V(n+1) - d_v(n)$. Hence by Proposition 1.3.10, $d_V(n)$ is a polynomial in n of degree d. Thus, $\mathcal{G}(A) = \mathcal{P}_d$. □

We now define the invariants known as the Gelfand-Kirillov dimension and the Gelfand-Kirillov superdimension. The Gelfand-Kirillov dimension is useful for K-algebras whose growth is polynomial. The superdimension is useful for an algebra with exponential growth. The definitions are given in [1] and [5].

Definition 1.3.12. Let A be a K-algebra. The **Gelfand-Kirillov dimension** of A is defined by

$$\text{GKdim}(A) = \sup_V \overline{\lim} \frac{\log\, d_V(n)}{\log\, n}$$

where the supremum is taken over all finite dimensional generating subspaces V of A, and $\overline{\lim}$ denotes the limit superior.

In a similar way, the **Gelfand-Kirillov superdimension** of A is defined by

$$\text{GKsdim}(A) = \sup_V \overline{\lim} \frac{\log\, \log\, d_V(n)}{\log\, n}$$

where the supremum is taken over all finite dimensional generating subspaces V of A.

The next lemma will provide some connections between the limit superior as given in the definition of the Gelfand-Kirillov dimension and the equivalence classes determined by the growth in Definition 1.3.5. This lemma is a combination of Lemma 2.1 in [5] and Lemma 1.5 in [1].

Lemma 1.3.13. *Let Φ be the set of all functions $\varphi : \mathbb{N} \to \mathbb{R}$ in which there exists an $n_0 \in \mathbb{N}$ such that $\varphi(n) \leqslant \varphi(n+1) \in \mathbb{R}$ for all $n \geqslant n_0$ and let $f, g \in \Phi$. Then*

1. $$\overline{lim}\frac{log f(n)}{log\ n} = inf\{\rho \in \mathbb{R} | f(n) \leqslant n^\rho \text{ for almost all } n\}$$
$$= inf\{\rho \in \mathbb{R} | \mathcal{G}(f) \leqslant \mathcal{P}_\rho\}$$

2. $$\overline{lim}\frac{log\ log f(n)}{log\ n} = inf\{\epsilon \in \mathbb{R} | f(n) \leqslant e^{n^\epsilon} \text{ for almost all } n\}$$
$$= inf\{\epsilon \in \mathbb{R} | \mathcal{G}(f) \leqslant \mathcal{E}_\epsilon\}$$

3. *If $\mathcal{G}(f) = \mathcal{G}(g)$, then $\overline{lim}\frac{log\ f(n)}{log\ n} = \overline{lim}\frac{log\ g(n)}{log\ n}$ and $\overline{lim}\frac{loglog f(n)}{log\ n} = \overline{lim}\frac{loglog\ g(n)}{log\ n}$*

Proof. 1. Let $r = \overline{\lim}\frac{\log f(n)}{\log n}$, $s = \inf\{\rho \in \mathbb{R} | f(n) \leqslant n^\rho$ for almost all $n\}$, and $t = \inf\{\rho \in \mathbb{R} | \mathcal{G}(f) \leqslant \mathcal{P}_\rho\}$. If we have that $f(n) \leqslant n^\rho$ for almost all n, then $\mathcal{G}(f) \leqslant \mathcal{P}_\rho$ and $\{\rho \in \mathbb{R} | f(n) \leqslant n^\rho$ for almost all $n\} \subseteq \{\rho \in \mathbb{R} | \mathcal{G}(f) \leqslant \mathcal{P}_\rho\}$. This shows that $t \leqslant s$. For any $\epsilon > 0$, we have $\frac{\log f(n)}{\log n} \leqslant r + \epsilon$ for almost all n. It follows that $f(n) \leqslant n^{r+\epsilon}$ for almost all n. Hence, $s \leqslant \inf\{r + \epsilon | \epsilon > 0\} = r$.
Now assume that $r > t$. Let $\epsilon = \frac{r-t}{3}$. Then $\mathcal{G}(f) \leqslant \mathcal{P}_{t+\epsilon}$, and $f(n) \leqslant (mn)^{t+\epsilon}$ for some $m \in \mathbb{N}$ and for almost all n. Take n sufficiently large so that $m^{t+\epsilon} \leqslant n^\epsilon$, then $f(n) \leqslant n^{t+2\epsilon}$ for almost all n. This contradicts the fact that there are infinitely many values of $\frac{\log f(n)}{\log n}$ that are greater than $\frac{\log f(n)}{\log n} - \epsilon = r - \epsilon = t + 2\epsilon$. Hence, $r = t$ and by $t \leqslant s \leqslant r$, we have $t = s = r$.
2. Now let r, s, t be the values shown in 2. in the respective order. If $f(n) \leqslant e^{n^\epsilon}$ for almost all n, then $\mathcal{G}(f) \leqslant \mathcal{E}_\epsilon$, and we can conclude $t \leqslant s$ since $\{\epsilon \in \mathbb{R} | f(n) \leqslant e^{n^\epsilon}\} \subseteq \{\epsilon \in \mathbb{R} | \mathcal{G}(f) \leqslant \mathcal{E}_\epsilon\}$. Now for any $\epsilon > 0$, we have $\frac{\log\ \log f(n)}{\log n} \leqslant r + \epsilon$. Then $f(n) \leqslant e^{n^{r+\epsilon}}$ for almost all n. Hence, $s \leqslant \inf\{r + \epsilon | \epsilon > 0\} = r$. Hence, we have $t \leqslant s \leqslant r$.
3. This follows directly from 1. and 2. $\square$

We recall that by Lemma 1.3.4, if A is finitely generated as a K-algebra, then any finite dimensional generating subspace will have the same growth, and the definitions above can be simplified. Thus, we have $\text{GKdim}(A) = \overline{\lim}\frac{\log d_V(n)}{\log n}$ and $\text{GKsdim}(A) = \overline{\lim}\frac{\log\log d_V(n)}{\log n}$ when A is a finitely generated K-algebra.

Proposition 1.3.14. *Let K be a field and $A = K[x_1, \cdots, x_d]$ be the polynomial algebra over K. Then GKdim$(A) = d$.*

Proof. Since A is finitely generated as a K-algebra, we have GKdim(A) $= \overline{\lim}\frac{\log d_V(n)}{\log n}$. By Corollary 1.3.11, $\mathcal{G}(A) = \mathcal{P}_d$, and by part 1 of Lemma 1.3.13 we have $\overline{\lim}\frac{\log d_V(n)}{\log n} = \inf\{\rho \mid \mathcal{G}(d_V(n)) \leqslant \mathcal{P}_\rho\} = d$. Thus, GKdim$(A) = d$. □

The next results are provided to show some basic results concerning the Gelfand-Kirillov dimension. The results will also be used to show that the Gelfand-Kirillov dimension agrees with the standard Krull dimension in the commutative setting. The next lemma can be found as Lemma 3.1 in [5].

Lemma 1.3.15. *Let K be a field and A be a K-algebra. If B is a subalgebra of A, then GKdim(B) $\leqslant$GKdim(A).*

Proof. By definition, GKdim(A) $= \sup_V \overline{lim}\frac{\log d_V(n)}{\log n}$, where the supremum is taken over all finite dimensional generating subspaces. Now since B is a subalgebra of A, any finite dimensional generating subspace of B will be contained in a finite dimensional generating subspace of A. Hence, for every finite dimensional generating subspace V' of B, there exists a finite dimensional generating subspace V of A, such that V' is contained in V, and $d_{V'}(n) \leqslant d_V(n)$. Thus, GKdim$(B)$ $\leqslant$GKdim(A). □

The next proposition shows when equality is achieved in Lemma 1.3.15. The next proposition is given in Lemma 4.3 in [5].

Proposition 1.3.16. *Let K be a field and A a commutative K-algebra. If B is a subalgebra such that A is finitely generated as a B-module, then GKdim(A) =GKdim(B).*

Proof. We have that GKdim(B) $\leqslant$GKdim(A) by Lemma 1.3.15. So it is enough to show GKdim(B) $\geqslant$GKdim(A). Say A is generated as a B-module by $a_1, a_2, \cdots, a_h$. Let V be a finite dimensional subspace of A with spanning set $v_1, v_2, \cdots, v_s$. Without loss of generality, we can assume that $a_1, a_2, \cdots, a_h$ are contained in the spanning set for V. Then every element in the spanning set for V can be expressed as a linear combination of elements from

B and elements from $\{a_1, a_2, \cdots, a_h\}$. That is, $v_i = \sum_{k=1}^{h} a_k b_{ik}$ for all $1 \leqslant i \leqslant s$, and hence, $v_i v_j = \sum_{k=1}^{h} a_k b_{ijk}$ for all $1 \leqslant i, j \leqslant s$, where $b_{ik}, b_{ijk} \in B$. We can now take W to be the finite dimensional subspace of B that is spanned by all the b_{ik} and b_{ijk}. We immediately have that $V \subseteq a_1W + \cdots + a_hW$ and $V^2 \subseteq a_1W + \cdots + a_hW$. We want to show that $V^n \subseteq a_1W^{2n-1} + \cdots + a_hW^{2n-1}$ by induction on n. The case for $n = 1$ is done. Assume true for any k such that $1 \leqslant k \leqslant n$ and consider V^{n+1}. We have $V^{n+1} = VV^n$ and by the induction hypothesis,

$$\begin{aligned} VV^n &\subseteq V\Big(a_1W^{2n-1} + \cdots + a_hW^{2n-1}\Big) \\ &\subseteq \Big(a_1W + \cdots + a_hW\Big)\Big(a_1W^{2n-1} + \cdots + a_hW^{2n-1}\Big) \\ &\subseteq a_1W^{2n11} + \cdots + a_hW^{2n+1} \end{aligned}$$

Hence, $V^n \subseteq a_1W^{2n-1} + \cdots + a_hW^{2n-1}$ for any $n \in \mathbb{Z}$, such that $n \geqslant 1$. It follows that $d_V(n) \leqslant m \cdot d_W(2n-1)$. Since V was taken to be any finite dimensional subspace of A, we have GKdim$(A) \leqslant$ GKdim(B). Thus, GKdim$(A) =$ GKdim(B). □

Now we have all of the necessary pieces to demonstrate that the classical Krull dimension agrees with the Gelfand-Kirillov dimension in the commutative case. This is Theorem 4.5 in [5].

Theorem 1.3.17. *Let K be a field and let A be a commutative finitely generated K-algebra. Then GKdim(A) =dim(A), where dim(A) is the classical Krull dimension of A.*

Proof. By the Noether Normalization Theorem, we have that A is finitely generated as a B-module where $B = K[x_1, x_2, \cdots, x_n]$ is a polynomial algebra over K and dim$(A) = r$. By Proposition 1.3.14, GKdim$(B) = r$ and by Proposition 1.3.16, GKdim(A) =GKdim(B). Thus, GKdim(A) =dim(A). □

We end this section by stating without proof, an interesting theorem concerning possible values of the Gelfand-Kirillov dimension. This result and its proof can be found in Theorem 2.5 in [5].

Theorem 1.3.18. *No algebra has Gelfand-Kirillov dimension between 1 and 2.*

1.4 General Recurrence Results

The method used for the main results of this paper require working with nonhomogeneous linear recurrences. In particular, we will need to be able to take the sum of many nonhomogeneous linear recurrences, and say something about this sum. In this section, we show that the sum of many nonhomogeneous linear recurrences is also a nonhomogeneous linear recurrence of the same form. Then we provide the standard process of finding a general solution for a nonhomogeneous linear recurrence along with a few well-known results about polynomials since the general solutions are dependent on roots of a characteristic polynomial.

Proposition 1.4.1. *Let $a_n = \beta_1 a_{n-1} + \beta_2 a_{n-2} + \cdots + \beta_l a_{n-l} + T(n)$ be a nonhomogeneous linear recurrence with $\beta_i \in \mathbb{Z}_{\geqslant 0}$ for all i and $T(n)$ a function of n. If $s_n = \sum_{i=0}^{n} a_i$, then $s_n = \beta_1 s_{n-1} + \beta_2 s_{n-2} + \cdots + \beta_l s_{n-l} + \widetilde{T}(n)$, where $\widetilde{T}(n) = \sum_{i=0}^{n} T(i) - B$ where B is a constant.*

Proof. Let $a_n = \beta_1 a_{n-1} + \beta_2 a_{n-2} + \cdots + \beta_l a_{n-l} + T(n)$ and $s_n = \sum_{i=0}^{n} a_i$. We can expand the sum for s_n and regroup the terms in the following way:

$$
\begin{aligned}
s_n =& \beta_1(a_{n-1} + a_{n-3} + \cdots + a_{l-1}) \\
&+\beta_2(a_{n-2} + a_{n-3} + \cdots + a_{l-2}) \\
&\ \vdots \\
&+\beta_{l-1}(a_{n-(l-1)} + a_{n-l} + \cdots + a_1) \\
&+\beta_l(a_{n-l} + a_{n-(l+1)} + \cdots + a_0) \\
&+\sum_{i=0}^{l-1} a_i \\
&+\sum_{i=0}^{n} T(i)
\end{aligned}
$$

Then for each β_i with $1 \leqslant i \leqslant l-1$ we can add the term $\beta_i(a_0+a_1+\cdots+a_{l-(i+1)})$ to complete the sum. Let $\sum_{i=1}^{l-1}\left(\beta_i(a_0+a_1+\cdots+a_{l-(i+1)})\right) - \sum_{i=0}^{l-1} a_i = B$. Since $a_0, a_1, \cdots, a_{l-1}$ are the initial conditions, this is a constant. So now consider $s_n + B - B$ to obtain

$$s_n = \beta_1 \sum_{i=0}^{n-1} a_i + \beta_2 \sum_{i=0}^{n-2} a_i + \cdots + \beta_l \sum_{i=0}^{n-l} a_i + \sum_{i=0}^{n} T(i) - B$$

Let $\widetilde{T}(n) = \sum_{i=0}^{n} T(i) - B$ we can substitute into the equation to get

$$s_n = \beta_1 s_{n-1} + \beta_2 s_{n-2} + \cdots + \beta_l s_{n-l} + \widetilde{T}(n).$$

□

Once we can determine the form of the sum of nonhomogeneous linear recurrences using Proposition 1.4.1, we will then need to apply methods to find a general solution for the recurrence. To do this, we will follow the standard technique for finding a general solution for a nonhomogeneous linear recurrence. First, we take the associated homogeneous recurrence and find a general solution. The form for the general solution of a homogeneous recurrence is well-known. We provide this as a Theorem as stated in Theorem 7.2.2 in [2] for reference.

Theorem 1.4.2. *Let q be a nonzero number. Then $h_n = q^n$ is a solution of the linear homogeneous recurrence relation*

$$h_n - a_1 h_{n-1} - a_2 h_{n-2} - \cdots - a_k h_{n-k} = 0 \text{ for } a_k \neq 0, n \geqslant k$$

with constant coefficients if and only if q is a root of the characteristic polynomial equation

$$x^k - a_1 x^{k-1} - \cdots - a_k = 0$$

1. *If the roots $q_1, \cdots, q_k$ for the characteristic polynomial are all simple, then*

$$h_n = c_1 q_1^n + c_2 q_2^n + \cdots + c_k q_k^n \tag{1.1}$$

is the general solution of the linear homogeneous recurrence relation.

2. *If the distinct roots of the characteristic polynomial are $q_1, \cdots, q_t$ for some $t \leqslant k$ are not all simple, then for a root q_i with multiplicity s_i, the part of the general solution corresponding to q_i is*

$$h_n^{(i)} = c_1 q_i^n + c_2 n q_i^n + \cdots + c_{s_i} n^{s_i - 1} q_i^n$$

and the general solution of the recurrence relation is

$$h_n = h_n^{(1)} + \cdots + h_n^{(t)}$$

Notice that by Theorem 1.4.2, the general solution for the associated homogeneous component of the recurrence with be determined by roots of the characteristic polynomial. We now provide two well known polynomial results with proof that will be useful for the characteristic polynomials of the examples in Chapter 2.

Theorem 1.4.3. *[Cauchy] Let $f(x) = x^n - a_1 x^{n-1} - \cdots - a_{n-1} x - a_n$, where $a_i \in \mathbb{R}_{\geqslant 0}$, for all $i \in \{1, 2, \cdots, n\}$ and at least one is nonzero. Then $f(x)$ has exactly one positive root λ that is simple, and such that the absolute value of all remaining roots is less than or equal to λ.*

Proof. We first define the following function:

$$F(x) = -\frac{f(x)}{x^n} = -1 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots + \frac{a_n}{x^n}$$

Notice that by defining F in this way, any nonzero root of $F(x)$ is also a root of $f(x)$. Also, $F(x)$ is monotone decreasing on the interval $(0, \infty)$ with $\lim_{x \to 0} F(x) = \infty$ and

$\lim_{x\to\infty} F(x) = -1$. Hence, $F(x)$ has only one positive root, and it follows that any positive root of $F(x)$ is also a positive root of $f(x)$. Thus, $f(x)$ has a unique positive root. We denote this root by λ.

Now assume that λ is a multiple root. This is only the case when $f(\lambda) = 0 \implies f'(\lambda) = 0$. We first consider $F'(x)$. We have $F'(x) = -\frac{f'(x)}{x^n} - \frac{f(x)}{x^{n+1}}$, and $F'(\lambda) = \frac{f'(\lambda)}{\lambda^n} = -\frac{a_1}{\lambda^2} - \cdots - \frac{na_n}{\lambda^{n+1}}$. Now since not all a_i are zero and every nonzero term is less than zero, it follows that $F'(\lambda) < 0$. Hence, λ is a simple root.

It now needs to be shown that $|\lambda_i| \leqslant \lambda$ where λ_i is any other root of $f(x)$. Assume $|\lambda_i| > \lambda$. We know that $F(x)$ is monotone decreasing on $(0, \infty)$, and hence, $F(|\lambda_i|) < F(\lambda) = 0$. Hence, $F(|\lambda_i|) = -\frac{f(|\lambda_i|)}{|\lambda_i|^n}$ and we can deduce that $f(|\lambda_i|) > 0$. Now we have

$$|\lambda_i|^n > a_1|\lambda_i|^{n-1} + a_2|\lambda_i|^{n-2} + \cdots + a_n$$

Since λ_i is a root of $f(x)$, we also have

$$\lambda_i^n = a_1\lambda_i^{n-1} + a_2\lambda_i^{n-2} + \cdots + a_n$$

By applying the triangle inequality, this becomes

$$|\lambda_i|^n \leqslant a_1|\lambda_i|^{n-1} + a_2|\lambda_i|^{n-2} + \cdots + a_n$$

However, this contradicts the first inequality. Thus, it follows that $|\lambda_i| < \lambda$, and since λ_i was taken to be any root of $f(x)$, this holds for all other roots. $\square$

Theorem 1.4.4. *[Ostrovsky] Let $f(x) = x^n - a_1x^{n-1} - \cdots - a_{n-1}x - a_n$, where $a_i \in \mathbb{R}_{\geqslant 0}$ for all $i \in \{1, 2, \cdots, n\}$, and at least one nonzero. If the greatest common divisor of the indices of the negative coefficients equals 1, then f has a unique positive root λ, that is simple and such that the absolute value of all other roots of $f(x)$ is strictly less than λ.*

Proof. By Theorem 1.4.3, we know that $f(x)$ has a unique simple positive root, say λ. We also have that $|\lambda_i| \leqslant \lambda$ for any root λ_i of $f(x)$. Now assume that $a_{k_1}, a_{k_2}, \cdots, a_{k_s}$ are the nonzero coefficients of $f(x)$. By assumption, we have that $gcd(k_1, k_2, \cdots, k_s = 1)$, and hence, there exist $\alpha_1, \alpha_2, \cdots, \alpha_s \in \mathbb{Z}$ such that $\alpha_1 k_1 + \alpha_2 k_2 + \cdots + \alpha_s k_s = 1$. We define $F(x)$ as in Theorem 1.4.3. Let λ_i be any root of $F(x)$ different than λ.We know by definition of $F(x)$, that λ is also a root of $f(x)$. Denote $|\lambda_i|$ by q. Then

$$F(\lambda_i) = -1 + \frac{a_{k_1}}{\lambda_i} + \cdots + \frac{a_{k_s}}{\lambda_i^{k_s}} = 0$$

Then by application of the triangle inequality we obtain the following:

$$\begin{aligned} 1 &= \frac{a_{k_1}}{\lambda_i} + \cdots + \frac{a_{k_s}}{\lambda_i^{k_s}} \\ &\leqslant |\frac{a_{k_1}}{\lambda_i}| + \cdots + |\frac{a_{k_s}}{\lambda_i^{k_s}}| \\ &= \frac{a_{k_1}}{q} + \cdots + \frac{a_{k_s}}{q^{k_s}} \end{aligned}$$

So we have that $1 \leqslant \frac{a_{k_1}}{q} + \cdots + \frac{a_{k_s}}{q^{k_s}}$ and can conclude that $F(q) \geqslant 0$. To have $F(q) = 0$ we must have $1 = \frac{a_{k_1}}{q} + \cdots + \frac{a_{k_s}}{q^{k_s}}$, which requires the triangle inequality to be an equality. If the triangle inequality is an equality, then we can find values $b_j > 0$ such that $\frac{a_{k_j}}{\lambda_i^{k_j}} = b_i \frac{a_{k_1}}{\lambda_i^{k_1}}$ for all $j \in \{2, \cdots, m\}$. Then $1 = \frac{a_{k_1}}{\lambda_i} + \cdots + \frac{a_{k_s}}{\lambda_i^{k_s}} = \frac{a_{k_1}}{\lambda_i^{k_1}}(1 + t_1 + \cdots + t_m)$. Hence, $\frac{a_{k_j}}{\lambda_i^{k_j}} > 0$ and it follows that $\lambda_i^{k_j} > 0$ for all j. Now

$$\left(\frac{a_{k_1}}{\lambda_i^{k_1}}\right)^{\alpha_1} \cdots \left(\frac{a_{k_s}}{\lambda_i^{k_s}}\right)^{\alpha_s} > 0$$

Now since $\alpha_1 k_1 + \alpha_2 k_2 + \cdots + \alpha_s k_s = 1$ we get

$$0 < \left(\frac{a_{k_1}}{\lambda_i^{k_1}}\right)^{\alpha_1} \cdots \left(\frac{a_{k_s}}{\lambda_i^{k_s}}\right)^{\alpha_s} = \frac{a_{k_1}^{\alpha_1} \cdots a_{k_m}^{\alpha_m}}{\lambda_i}$$

However, $a_j > 0$ for all j, and this implies $\lambda_i > 0$. This contradicts the fact that λ is the unique positive root of $F(x)$. Hence, $F(q) > 0$. Now since $F(x)$ is monotone decreasing on

$(0, \infty)$ and $q > 0$, $F(\lambda) = 0 < F(q)$ implies that $\lambda > q$. Thus, $\lambda > q = \lambda_i$ and since λ_i was taken as any other root of $F(x)$, this holds for all roots of $F(x)$, and in turn all roots of $f(x)$. $\square$

Once a general solution is found for the associated homogeneous recurrence, we then find a particular solution for the nonhomogeneous recurrence using the initial conditions and guessing a function of the same type as the nonhomogeneous part of the recurrence. For instance, if the nonhomogeneous component of the recurrence is polynomial, the guess for the particular solution should be polynomial. Then the general solution and particular solution are combined to determine the constants of the general solution that satisfy the particular solution.

CHAPTER 2

MAIN RESULTS

2.1 Growth Recurrence for a Commutative Ring

In this chapter, we let R be a finitely generated K-algebra where $K = \mathbb{F}_p$ with p prime, and hence, $k^p = k$ for all $k \in K$. Say R is generated as a K-algebra by the set $U = \{r_1, r_2, \cdots, r_m\}$ of elements from R. We consider $A = R[f; F]$, the Frobenius skew polynomial ring over R which is a finitely generated K-algebra generated by the set $U \cup \{f\}$. We now follow the structure used for the growth of the algebra and the Gelfand-Kirillov dimension. Let V be the K-subsapce spanned by the set $U \cup \{f\}$. This makes V a finite dimensional generating subspace of A such that every element of V is a K-linear combination of elements from the set $U \cup \{f\}$. We let $V^0 = K$ and V^i be the K-subspace spanned by products of i elements from $U \cup \{f\}$. Then we have $A_n = \sum_{i=0}^{n} V^i$ and $A = \cup_{i=0}^{\infty} A_n$. To emphasize the algebra generators chosen for R, we will denote the growth function by $d_{U,F}(n)$.

In the following sections, we will look at examples for R where A has a direct sum decomposition by a suitable subspace of V^i, denoted W_i. We will be able to develop a nonhomogeneous linear recurrence that gives the number of generators for each W_i, which is to say a recurrence for $\dim_K(W_i)$. Since the W_i form a direct sum decomposition of A, we can compute $d_{U,F}(n)$ by taking the sum of the recurrences for $\dim_K(W_i)$ where i goes from 0 to n. By Proposition 1.4.1, we can translate the sum of these recurrences to a recurrence for $d_{U,F}(n) = \dim_K(A_n)$.

For the remainder of this section, we assume that there exists a, possibly nonhomogeneous, linear recurrence s_n for $d_{U,F}(n)$ and provide some general results and definitions that arise naturally when such a recurrence exists.

Definition 2.1.1. Let $d_{U,F}(n)$ =$\dim_K(A_n)$ and assume there exists a recurrence, s_n, such that

$$d_{U,F}(n) = s_n = \beta_1 s_{n-1} + \beta_2 s_{n-2} + \cdots + \beta_l s_{n-l} + \widetilde{T}(n)$$

We will refer to this recurrence as the **growth recurrence** of R with respect to U and F. Also, the associated homogeneous recurrence for s_n is $\beta_0 s_{n-1} + \beta_1 s_{n-2} + \cdots + \beta_{l-1} s_{n-l}$, which will be called the **homogeneous growth recurrence** of R with respect to U and F.

Now considering the homogeneous growth recurrence of R with respect to U and F along with Theorem 1.4.2, we can determine the characteristic polynomial equation.

Definition 2.1.2. Let $\overline{s}_n$ be the homogeneous growth recurrence of R with respect to U and F. The the characteristic polynomial equation for $\overline{s}_n$ is $x^l - \beta_1 x^{l-1} - \cdots - \beta_l = 0$. We will call this characteristic polynomial equation the **growth equation** of R with respect to U and F.

By Theorem 1.4.2 we can determine the general solution for the homogeneous growth recurrence by finding the roots of the growth equation. Notice that this solution determined by the roots of the homogeneous growth recurrence has exponential form. We can make a few conclusions from this. First, if the nonhomogeneous portion of the growth recurrence is polynomial or constant, then as n goes to infinity, $d_{U,F}(n)$ will be determined by the exponential general solution of the homogeneous growth recurrence. This also shows that the growth of A is exponential and GKdim(A) is infinite. We provide these conclusions in a proposition.

Proposition 2.1.3. *Let R be a finitely generated K-algebra, and $A = R[f;F]$ the Frobenius skew polynomial ring over R. If the growth recurrence of R with respect to U and F exists and is of the form $s_n = \beta_1 s_{n-1} + \beta_2 s_{n-2} + \cdots + \beta_l s_{n-l} + \widetilde{T}(n)$ such that the gcd(β_i) $= 1$ and $\widetilde{T}(n)$ is polynomial or constant, then $\mathcal{G}(A) = \mathcal{E}_1$ and GKdim(A) $= \infty$ while GKsdim(A) $= 1$.*

Proof. First, since the growth recurrence of R with respect to U and F is of the form $s_n = \beta_1 s_{n-1} + \beta_2 s_{n-2} + \cdots + \beta_l s_{n-l} + \widetilde{T}(n)$, we have the growth equation of R as $x^l - \beta_1 x^{l-1} - \cdots - \beta_l = 0$. Since the $\gcd(\beta_i) = 1$, Theorem 1.4.4 guarantees that the growth equation of R has a unique positive root, λ_p, such that λ_p is larger than the absolute value of all other roots.

Now using Theorem 1.4.2, we can conclude that the general solution of the homogeneous growth recurrence of R is of the form $c_p\lambda_p^n + \sum c_i(\lambda_i)^n$ where the sum includes all other roots of the growth equation and c_i are constants. Then since the nonhomogeneous part of the growth recurrence is polynomial or constant, the particular solution will be polynomial or constant. Hence, we get that the combination of the particular solution and general solution has the form $c_p\lambda_p^n + \sum c_i(\lambda_i)^n + g(n)$ where $g(n)$ is the particular solution of the nonhomogeneous component of the growth recurrence, which has the same form as $\widetilde{T}(n)$. Thus, $\mathcal{G}(A) = \mathcal{E}_1$ and it follows by Lemma 1.3.13 that $\text{GKdim}(A) = \infty$ while $\text{GKsdim}(A) = 1$. $\square$

Thus, the Gelfand-Kirillov dimension and superdimension do not provide any useful information about the ring and we need a more refined value to consider for A that comes particularly from $d_{U,F}(n)$. We choose to isolate the unique positive root of the growth equation for this refined value as it is the determining value for the solution of the growth recurrence.

Definition 2.1.4. We define the **Gelfand-Kirillov base** of R with respect to U, denoted $\text{GKbase}_{U,F}(R)$, to be $\text{GKbase}_{U,F}(R) = \inf\{\lambda \in \mathbb{R}_{>0} \mid d_{U,F}(n) = O(\lambda^n)\}$ taken over all n.

2.2 Growth Recurrence for a Polynomial Ring

We now focus on the Frobenius skew polynomial ring $A = R[f;F]$ where $R = K[x_1, x_2, \cdots, x_m]$ is a commutative polynomial ring over a field, $K = \mathbb{F}_p$, with p prime. We choose U to be the set $\{x_1, x_2, \cdots, x_m\}$ which generate R as a K-algebra and V to be the finite dimensional K-subspace to be the subspace spanned by the set $U \cup \{f\}$. To be able to apply the results from Section 1.3, we must first show that a recurrence exists for $d_{U,F}(n)$.

As described before, V^i is the K-subspace spanned by elements that are products of i elements from $x_1, x_2, \cdots, x_m, f$. We ultimately want to be able to count how many unique generators there are for a given value of i. To do this, we must first understand these generators better and develop a unique representation.

2.2.1 Skew Monomials

Here we look specifically at the elements that generate the K-subspaces V^i and develop a unique representation for these elements.

Definition 2.2.1. We refer to an element $z \in A$ as a **skew monomial** when z is a product of variables from $\{x_1, x_2, \cdots, x_m, f\}$. Note that elements that are products of variables from $\{x_1, x_2, \cdots, x_m\}$ are also considered skew monomials when viewed as elements of A.

We would like to define a notion of degree for a skew monomial that associates the skew monomial as a generator for a particular subspace V^i. Since V^i is generated by skew monomials that are products of i elements from $x_1, x_2, \cdots, x_m, f$, a potential way to define degree is by taking the sum of the exponents of the elements $x_1, x_2, \cdots, x_m, f$ that appear in the product. However, this is not a well-defined value since the multiplication in A can produce multiple representations for a skew monomial that could associate the skew monomial to multiple subspaces.

For instance, the sum of powers for the skew monomial $x_1x_2^pf$ is $p+2$. We can then give another representation of this element by applying the ring multiplication. Hence, $x_1x_2^pf$ becomes x_1fx_2 and now the sum of powers is 3. Thus, to provide a notion of degree for a skew monomial, we need to determine a unique representation. There could be many possibilities for the choice of this unique representation. For instance, we could take the representation with the largest possible sum of powers or the representation with smallest possible sum of powers. In this paper, we choose to define a notion of degree by using the representation with the lowest possible sum of powers. This decision essentially views the skew monomial in the representation with any f in the product appearing as far to the left as possible. For example, x_i^pf has the largest possible sum of powers for this skew monomial. When we apply the ring multiplication, the f moves to the left and the sum of powers drops giving fx_i. The motivation for this choice comes from methods used in counting the skew monomials. We describe this representation in a definition.

Definition 2.2.2. A skew monomial $z \in A$ is in **irreducible form** when no x_i^j with $j \geqslant p$ appears to the left of an f. Equivalently, a skew monomial is in irreducible form when it is written as a product with the fewest possible elements from $x_1, x_2, \cdots, x_m, f$.

Example 2.2.3. Let $R = K[x_1, x_2, x_3]$ where K is a field of characteristic $p = 3$, and let $A = R[f; F]$. Consider the skew monomial $x_1^3x_2^2fx_3^3$. Since x_1 has a power greater than the characteristic of K and appears to the left of f in the product, this skew monomial is not in irreducible form. Note that the sum of the powers in this representation is 9. We now convert this to irreducible form. We have $x_1^3x_2^2fx_3^3 = x_2^2x_1^3fx_3^3$ and then by making the substitution $x_1^3f = fx_1$, this becomes $x_2^2x_1^3fx_3^3 = x_2^2fx_1x_3^3$. This is now in irreducible form since there are no other substitutions that can be made. In other words, there is no x_i with power at least 3 to the left of f. Now the sum of the powers is 7. Since the irreducible form is unique, we can now use the sum of the powers appearing in the irreducible form to define the notion of degree for a skew monomial.

Definition 2.2.4. We define the **degree** of a skew monomial to be the sum of the powers of $x_1, x_2, \cdots, x_m, f$ when the skew monomial is written in irreducible form. For the remainder of the section, when the degree of a skew monomial is referred to, it is assumed that the skew monomial is written in irreducible form.

Example 2.2.5. Let A be as in Example 2.2.3. The skew monomial $x_1^3x_2^2fx_3^3 = x_2^2fx_1x_3^3$ has degree 7.

We can now determine a more refined K-subspace than V^i that is spanned only by skew monomials that are in irreducible form. This will allow us to determine a direct sum decomposition of A.

2.2.2 Direct sum decomposition for the Growth Recurrence

We can now view the K-subspaces, V^i, as the subspaces spanned by skew monomials that are products of i elements. However, these are not necessarily skew monomials of degree i since the definition of V^i does not require the skew monomials to be in irreducible form.

There are a few key aspects to notice here. First, the construction given here is not a graded object. Confusion could arise after the discussion of degrees of the skew monomials that could lead to the expectation of a graded object. However, due to the multiplication in A, this is not the case.

Second, the K-subspaces V^i do not for a direct sum decomposition of A_n because $V^i \cap V^j$ is not necessarily trivial. For example, $x^p f = fp$ appears in V^{p+1} and V^2. If we hope to be able to determine $d_{U,F}(n)$ by a counting technique applied to the generating skew monomials, we must determine a more suitable K-subspace related to V^i that avoids this issue. This is the motivation for the next definition.

Definition 2.2.6. Let A be defined as above. We define the set W_i to be the K-subspace spanned by irreducible skew monomials of degree i. Hence, W_i is spanned by the subset of generators for V^i that are in irreducible form.

Proposition 2.2.7. *Let $R = K[x_1, x_2, \cdots, x_m]$ and $A = R[f;F]$ be the Frobenius skew polynomial ring over R. If W_i is the K-subspace spanned by skew monomials from A in irreducible form of degree i, then $A_n = \bigoplus_{i=0}^{n} W_i$ and the decomposition of A_n given by $A_n = \bigoplus_{i=0}^{n} W_i$ is a direct sum decomposition.*

Proof. By definition, every element in A_n is the sum of elements from the subspaces V^i where $0 \leqslant i \leqslant n$. Also, every generating element from V^i can be written in irreducible form, and hence every generating element of V^i is in some W_j where $j \leqslant i$. Hence, $V^j \subset \sum_{i=0}^{n} W_i$ for all $0 \leqslant j \leqslant n$. It follows that $\sum_{i=0}^{n} V^i = \sum_{i=0}^{n} W_i$, and hence, $A_n = \sum_{i=0}^{n} W_i$.
Now to show that this is a direct sum decomposition, we only need to show that $W_i \cap \sum_{j \neq i} W_j = 0$ for all $0 \leqslant i, j \leqslant n$. This follows immediately from the definition of W_i since each subspace is spanned by skew monomials in irreducible form of degree i and elements in $\sum_{j \neq i} W_j$ are linear combinations of skew monomials in irreducible form of degree different than i and coefficients from K. Thus, let $z \neq 0 \in W_i$. Then z is a linear combination of degree i skew monomials and coefficients from K. Then z is not in $\sum_{j \neq i} W_j$. Likewise, let $z' \in \sum_{j \neq i} W_j$, then z' is a linear combination of skew monomials in irreducible form of degree different than i with coefficients from K, and hence, is not in W_i. $\square$

Now let us consider the function $d_{U,F}(n) = \dim_K(A_n)$. Since the decomposition of A_n for any n by $\bigoplus_{i=0}^{n} W_i$ is a direct sum decomposition, we can compute $d_{U,F}(n)$ by the following sum:

$$d_{U,F}(n) = \dim_K \left(\sum_{i=0}^{n} V^i \right) = \sum_{i=0}^{n} \dim_K(W_i)$$

Hence, to compute $d_{U,F}(n)$ we must first develop a method to compute $\dim_K(W_i)$ for any $0 \leqslant i \leqslant n$. This can be done by counting the skew monomials in irreducible form of degree i for any $i \leqslant n$. We can then compute $d_{U,F}(n)$ by taking the sum of $\dim_K(V^i)$ for all $0 \leqslant i \leqslant n$. This is the focus of the next section.

2.2.3 Computing the Growth Recurrence

We now take the direct sum decomposition constructed in Section 2.2.2 and use it find a recurrence for $d_{U,F}(n)$. By Proposition 1.4.1, we can do this by first finding a recurrence for $\dim_K(W_i)$ for all i. By Proposition 2.2.7, we can determine $\dim_K(W_i)$ by determining the number of generators for W_i.

Remark 2.2.8. A key observation is that any skew monomial in irreducible form containing f in the product must lead with a skew monomial in variables from $\{x_1, x_2, \cdots, x_m\}$ with an exponent vector from the set $\{(a_1, a_2, \ldots, a_m) | a_i \in \{0, 1, 2, \ldots, p-1\}\}$. This is the set of m-tuples whose entries are restricted to elements of the set $\{0, 1, 2, \cdots, p-1\}$. The cardinality of this set will allow us to count the skew monomials in irreducible form containing an f for any degree d. Hence, it is necessary to define this set and develop notations to represent the components needed for the calculation.

Definition 2.2.9. We define the set

$$X = \{(a_1, a_2, \cdots, a_m) \mid a_i \in \mathbb{Z} \text{ and } 0 \leqslant a_i \leqslant p-1 \text{ for all } i\}$$

to be the set of all m-tuples with entries from the set $\{0, 1, 2, \cdots, p-1\}$. Let $\underline{a} \in X$ such that $\underline{a} = (a_1, a_2, \cdots, a_m)$. We denote the sum of the entries of $\underline{a}$ by $|\underline{a}| = \sum_{i=1}^{m} a_i$. Further, we define X_j with $0 \leqslant j \leqslant m(p-1)$ to be the subset of X whose elements are partitions of the natural number j. That is,

$$X_j = \{\underline{a} \in X \mid |\underline{a}| = j\}$$

Finally, we denote the cardinality of X_j by β_j.

Lemma 2.2.10. *Let the set X and subsets X_j be as in Definition 2.2.9. Then*

1. $X_i \cap X_j = \emptyset$ *for all* $i \neq j$

2. $X = \bigcup_{j=0}^{m(p-1)} X_j$

Proof. (1) This follows immediately from the definition.

(2) Since each X_j is a subset of X, it follows that $X \supseteq \cup_{j=0}^{m(p-1)} X_j$. Now let $\underline{a} \in X$. We have that $0 \leqslant |\underline{a}| \leqslant m(p-1)$, and hence, $\underline{a} \in X_j$ for some $0 \leqslant j \leqslant m(p-1)$. Thus, $X \subseteq \cup_{j=0}^{m(p-1)} X_j$ and we have equality. $\square$

Example 2.2.11. Let $m = 2$ and $p = 3$. Then X is the set of all pairs (a_1, a_2) where $a_1, a_2 \in \{0, 1, 2\}$. Now consider X_2. This is the subset of X containing all pairs (a_1, a_2) such that $a_1 + a_2 = 2$. Hence, $X_j = \{(1, 1), (0, 2), (2, 0)\}$. Then it follows that $\beta_2 = |X_2| = 3$.

We define the set X in this way because the exponent vectors given by the product of variables in $\{x_1, x_2, \cdots, x_m\}$ that appear at the beginning of a skew polynomial containing an f in the product will come from this set when the skew monomial is written in irreducible form. We cannot have a value greater than $p-1$ because the skew monomial will no longer be in irreducible form. Then by defining the subsets X_j of X, we partition X into disjoint subsets. That is, $X = \cup_{i=0}^{m(p-1)} X_i$. We now give a definition that relates these sets to the skew monomials that we will count.

Definition 2.2.12. The following definitions and notations provide the building blocks to determine $\dim_K(W_d)$.

- We will now denote $\dim_K(W_d)$ by C_d. Hence, C_d is equal to the number of skew monomials of degree d.

- We denote the skew monomial $x_1^{a_1} x_2^{a_2} \cdot \ldots \cdot x_m^{a_m}$ by $\underline{x}^{\underline{a}}$ where $\underline{a} \in X$.

- Let z be a skew monomial of degree d written in irreducible form. We can view z in the form $\underline{x}^{\underline{a}} f z'$ where z' is a skew monomial of degree $d - (|\underline{a}| + 1)$, also in irreducible form . We refer to $\underline{x}^{\underline{a}}$ as the **leading product** of z and $\underline{a}$ as the **leading exponent vector** of z.

- Let $c_{\underline{a},d}$ denote the number of skew monomials of degree d of the form $\underline{x}^{\underline{a}} f \cdot z'$.

The next proposition shows how the values of $c_{\underline{a},d}$ provide the building blocks to develop a recurrence to count the skew monomials of degree d. However, before giving the general result, we provide an example to help present the intuition.

Example 2.2.13. Let $R = K[x_1, x_2]$ where K is a field of characteristic $p = 5$, and $A = R[f; F]$ the Frobenius skew polynomial ring over R. Let $\underline{a} = (1, 4)$ and consider $c_{\underline{a},10}$. Then every skew monomial counted in $c_{\underline{a},10}$ is of the form $x_1x_2^4fz$ where z is a skew monomial in irreducible form of degree $10-(1+4+1) = 4$. Since we are only counting skew monomials in irreducible form, z must also be irreducible. Hence, z is counted in C_4. Now we can attach $x_1x_2^4f$ to any skew monomial counted in C_4, and the resulting skew monomial will still be in irreducible form, and hence, is counted in $c_{\underline{a},10}$. Thus, $c_{\underline{a},10} = C_4$.

The conclusion of the example above holds in general and is shown in the next proposition. This result will be useful in developing a recurrence to compute C_d. Note that we are only counting skew monomials that contain an f in the product with the values $c_{\underline{a},d}$. The skew monomials that do not contain an f are counted in the same manner as counting monomials of degree d in R.

Proposition 2.2.14. *Let $C_d =\dim_K(W_d)$ and $c_{\underline{a},d}$ be defined as in 2.2.12. Then we have*

1. $C_d = \sum_{\underline{a}\in X} c_{\underline{a},d} + \binom{m+d-1}{d}$*,where the sum is taken over* $|X| = p^m$ *terms*
2. *For all* $\underline{a} \in X$*,* $c_{\underline{a},d} = C_{d-(|\underline{a}|+1)}$*.*

Proof. 1) First, recall that all monomials $z \in R$ of degree d are in irreducible form in V^d in $x_1, x_2, \ldots, x_m$, and hence are part of the spanning set for W_d. The number of skew monomials of this form is given by $\binom{m+d-1}{d}$. By definition, each $c_{\underline{a},d}$, counts skew monomials in irreducible form of degree d. Thus, we have $C_d \geqslant \sum_{\underline{a}\in X} c_{\underline{a},d} + \binom{m+d-1}{d}$.

Now assume that $C_d > \sum_{\underline{a}\in X} c_{\underline{a},d} + \binom{m+d-1}{d}$. Then there exists a skew monomial in irreducible form, $z \in V^d$ counted in C_d, but not counted in $\sum_{\underline{a}\in X} c_{\underline{a},d} + \binom{m+d-1}{d}$. It follows that $z = \underline{x}^{\underline{a}}f \cdot z'$ where $\underline{a} \notin X$ and z' is an skew monomial in irreducible form of degree $d-\deg(\underline{x}^{\underline{a}}f)$. Since $\underline{a} \notin X$, we have x_i^j for some $i \in \{1, 2, \ldots, m\}$ and $j \geqslant p$ appearing in $\underline{x}^{\underline{a}}$.

Thus, z is not in irreducible form, contradicting the assumption that z is counted in C_d and it follows that $C_d = \sum_{\underline{a}\in X} c_{\underline{a},d} + \binom{m+d-1}{d}$.

2) Let z be an skew monomial in irreducible form counted in $c_{\underline{a},d}$. Then $z = \underline{x}^{\underline{a}} \cdot fz'$ where $\deg(z') = d - (|\underline{a}| + 1))$. Thus, every skew monomial counted in $c_{\underline{a},d}$ comes from a skew monomial counted in $C_{d-(|\underline{a}|+1)}$ and $c_{\underline{a},d} \leqslant C_{d-(|\underline{a}|+1)}$.

Now let z be an skew monomial counted in $C_{d-(|\underline{a}|+1)}$. Then $\deg(z) = d - (|\underline{a}| + 1)$ and $\deg(f \cdot z) = d - |\underline{a}|$. We can now multiply fz by $\underline{x}^{\underline{a}}$ and the skew monomial will remain in irreducible form. Now $\deg(\underline{x}^{\underline{a}} \cdot fz) = d$. This implies that $\underline{x}^{\underline{a}} \cdot fz$ is counted in $c_{\underline{a},d}$ and $c_{\underline{a},d} \geqslant C_{d-(\sum_{i=1}^{m} a_i+1)}$. Thus, $c_{\underline{a},d} = C_{d-(|\underline{a}|+1)}$. □

Corollary 2.2.15. *Let C_d =$\dim_K(W_d)$ and $c_{\underline{a},d}$ be defined as above. Then we have*

$$C_d = \sum_{\underline{a}\in X} C_{d-(|\underline{a}|+1)} + \binom{m+d-1}{d}$$

. where the sum is taken over all $\underline{a} \in X$.

Corollary 2.2.15 shows how we can construct a recurrence for C_d by using terms C_i where $i < d$. Notice that the sum is taken over all elements of X, but the C_i that appear in the recurrence are determined by the sum of the components of each $\underline{a} \in X$. Hence, we can have multiple elements $\underline{a}$ that produce the same C_i in the recurrence. For example, consider $c_{\underline{a},d}$ and $c_{\underline{a}',d}$ where $\underline{a} = (1, 4)$ and $\underline{a}' = (4, 1)$. Then by Proposition 2.2.14, $c_{\underline{a},d} = C_{d-6} = c_{\underline{a}',d}$. It follows that C_{d-6} would have a coefficient of at least two in the recurrence. The set X_j gives all of the possible leading exponent vectors that will sum to j, and hence, the cardinality β_j of this set will determine the coefficient of $C_{d-(j+1)}$. Thus, we can refine this sum by using the partitions of X created by the set X_j and the cardinality β_j of X_j. The next theorem makes this refinement and provides the recurrence relation for C_d.

Theorem 2.2.16. *Let C_d =$\dim_K(W_d)$. Then*

$$C_d = \sum_{i=1}^{m(p-1)+1} \beta_{i-1} C_{d-i} + \binom{m+d-1}{d}$$

Proof. By Corollary 2.2.15, we have $C_d = \sum_{\underline{a}\in X} C_{d-(|\underline{a}|+1)} + \binom{m+d-1}{d}$, where the sum is taken over all $\underline{a} \in X$ and $|\underline{a}|$ denotes the sum of the components of the element $\underline{a} \in X$. Hence, the sum $\sum_{\underline{a}\in X} C_{d-(|\underline{a}|+1)}$ has $|X| = p^m$ terms. Let $j < d$ and consider an element $\underline{a'} \in X_j$. Then $|\underline{a'}| = j$ and the term $c_{\underline{a'},d}$ in the recurrence for C_d can be written as $C_{d-(j+1)}$ by Proposition 2.2.14. Hence, for every $\underline{a} \in X_j$ we get $C_{d-(j+1)}$ in the recurrence and the coefficient of $C_{d-(j+1)}$ is $|X_j| = \beta_j$. Now since $X = \cup_{i=0}^{m(p-1)} X_i$, we have $\sum_{i=0}^{m(p-1)} \beta_j = |X| = p^m$. It follows that the values $0, 1, \cdots, m(p-1)$ determine every term of the recurrence. That is the terms of the recurrence are $C_{d-(i+1)}$ for all $i \in \{0, 1, 2, \cdots, m(p-1)\}$, and the coefficient for each term is β_i. Thus, $\sum_{\underline{a}\in X} C_{d-(|\underline{a}|+1)} = \sum_{i=0}^{m(p-1)} \beta_i C_{d-(i+1)} = \sum_{i=1}^{m(p-1)+1} \beta_{i-1} C_{d-i}$, and the result holds. $\square$

Corollary 2.2.17. *Let $R = \mathbb{F}_p[x_1, x_2, \cdots, x_m]$, and let $A = R[f; F]$ be the Frobenius skew polynomial ring of R. If we take the set $U = \{x_1, x_2, \cdots, x_m\}$ as the algebra generating set of R and V as the K-subspace of A spanned by the set $U \cup \{f\}$, then the growth recurrence of R with respect to U and F is*

$$d_{U,F}(n) = \sum_{i=1}^{m(p-1)+1} \beta_{i-1} d_{U,F}(n-i) + \widetilde{T}(n), \text{ where}$$

$\widetilde{T}(n) = \sum_{i=0}^{n} \binom{m+i-1}{i} - \left[\sum_{i=1}^{m(p-1)} (\beta_{i-1}(d_{U,F}(0) + \cdots + d_{U,F}(m(p-1)-i)) - \sum_{i=1}^{m(p-1)} d_{U,F}(i) \right].$

Proof. By Proposition 2.2.7, we have that $d_{U,F}(n) = \sum_{i=0}^{n} \dim_K(W_i)$. Thus, we can apply Proposition 1.4.1 to the recurrence in Theorem 2.2.16 and the result follows. $\square$

In the next section, the focus will be on finding the growth equation for $d_{U,F}(n)$ and using it to determine the $\mathrm{GKbase}_{U,F}(R)$.

2.2.4 Computing the GK base

To determine the $\mathrm{GKbase}_{U,F}(R)$ we first need to gain some information about the β_i that appear in the recurrence. The following proposition shows that there is a nice symmetry with the coefficients β_i.

Proposition 2.2.18. *If* $\beta_i = |X_i|$, *then* $\beta_i = \beta_{m(p-1)-i}$ *for all* $0 \leqslant i \leqslant \frac{m}{2}(p-1)$. *Further, we have* $\beta_0 = 1 = \beta_{m(p-1)}$

Proof. Let $\phi : X_i \to X_{m(p-1)-i}$ for $0 \leqslant i \leqslant \frac{m}{2}(p-1)$ be defined by
$\phi((a_1, a_2, \ldots, a_m)) = (p-1-a_1, p-1-a_2, \ldots, p-1-a_m)\ \forall (a_1, a_2, \ldots, a_m) \in X_i$.
Note that $(p-1-a_1, p-1-a_2, \ldots, p-1-a_m) \in X_{m(p-1)-i}$ since

$$\sum_1^m p-1-a_i = m(p-1) - \sum_1^m a_i = m(p-1) - i$$

Also, if $\phi((a_1, a_2, \ldots, a_m)) = \phi((a_1', a_2', \ldots, a_m'))$ we have, $p-1-a_j = p-1-a_j'$ and it follows that $a_j = a_j'\ \forall j \in \{1, 2, \ldots, m\}$. Hence, $(a_1, a_2, \ldots, a_m) = (a_1', a_2', \ldots, a_m')$, and ϕ is injective. To show that ϕ is surjective and thus a bijection, let $(b_1, b_2, \ldots, b_m) \in X_{m(p-1)-i}$
Then $\sum_{j=1}^m b_j = m(p-1) - i$. Now since $b_j \in B\ \forall j$ we have that $b_j = p-1-a_j$ for some $a_j \in B$. It follows that $\sum_{j=1}^m b_j = \sum_{j=1}^m p-1-a_j = m(p-1) - \sum_{j=1}^m a_j$ combining this with the fact that $\sum_{j=1}^m b_j = m(p-1) - i$ we get $m(p-1) - \sum_{j=1}^m a_j = m(p-1) - i$, and hence, $\sum_{j=1}^m a_j = i$. Thus, $(a_1, a_2, \ldots, a_m) \in X_i$ and ϕ is surjective. Since ϕ is a bijection we have $|X_i| = |X_{m(p-1)-i}|$ and $\beta_i = \beta_{m(p-1)-i}$. Note: for $p = 2$ we take the ceiling of $\frac{m}{2}$.
For the last part, it is clear that $\beta_0 = 1$ since the only element in X_0 is the m-tuple consisting of all zeros. The fact that $\beta_{m(p-1)} = 1$ follows from part 1, and can also be realized since the largest possible entry for an element in $X_{m(p-1)}$ is $p-1$, then the only way to have a sum of $m(p-1)$ is for every entry to be $p-1$. $\square$

Now before proceeding to the general solution, we provide a concrete example to illustrate the process.

Example 2.2.19. Let $R = \mathbb{F}_2[x_1, x_2]$, $U = \{x_1, x_2\}$ the chosen set of K-algebra generators of R, and let $A = R[f; F]$ be the skew polynomial ring over R. Then we can apply Corollary 2.2.17 to obtain the recurrence for $d_{U,F}(n)$.

$$d_{U,F}(n) = \beta_0 d_{U,F}(n-1) + \beta_1 d_{U,F}(n-2) + \beta_2 d_{U,F}(n-3) + \widetilde{T}(n)$$

where $\widetilde{T}(n) = \sum_{i=0}^{n} i + 1 - \beta_0(d_{U,F}(1) + d_{U,F}(0)) - \beta_1 d_{U,F}(0) - d_{U,F}(1) - d_{U,F}(2)$. We now compute the values for each β_j by using the following sets:

- $X = \{(a_1, a_2) \mid a_i \in \{0, 1\} \text{ for all } i\}$

- $X_j = \{(a_1, a_2) \in B \mid a_1 + a_2 = j\}$ for $0 \leqslant j \leqslant 2$

- $\beta_j = |X_j|$

Then we can compute the values for each beta. We have $\beta_0 = 1$ since the only way to have a sum of 0 is if the exponents of x_1 and x_2 are both zero, $\beta_1 = 2$ by $(0, 1)$ and $(1, 0)$, and $\beta_2 = 1$. We can also compute the nonhomogeneous term by using initial conditions. We have $\dim_K(W_0) = 1$ and hence $d_{U,F}(0) = 1$. Then $\dim_K(W_1) = 3$ and $d_{U,F}(1) = \dim_K(W_0) + \dim_K(W_1) = 4$. Finally, $\dim_K(W_2) = 8$ and it follows that $d_{U,F}(2) = 13$. We also have that $\sum_{i=0}^{n} i + 1 = \frac{1}{2}(n^2 + 3n + 2)$. By making all of these substitutions, the growth recurrence of R becomes

$$d_{U,F}(n) = d_{U,F}(n-1) + 2d_{U,F}(n-2) + d_{U,F}(n-3) + \frac{1}{2}n^2 + \frac{3}{2}n - 23$$

We first consider the homogeneous growth recurrence of R,

$$d_{U,F}(n-1) + 2d_{U,F}(n-2) + d_{U,F}(n-3)$$

and obtain the growth equation for R to be

$$x^3 - x^2 - 2x - 1 = 0$$

Using SAGE, we get the solutions $x = 2.1479, -0.57395 - 0.36899i, -0.57395 + 0.36899i$.

Since each root is simple, the general solution of the associated homogeneous recurrence is

$$d_{U,F}(n) = c_1(2.1479)^n + c_2(-.57395 - 0.36899i)^n + c_3(-.57395 + 0.36899i)^n$$

Putting this particular solution together with the general solution from before, we obtain

$$d_{U,F}(n) = c_1(2.1479)^d + c_2(-.57395 - 0.36899i)^d + c_3(-.57395 + 0.36899i)^d + an^2 + bn + c$$

Where the coefficients a, b, c are determined by the particular solution of the growth recurrence and the coefficients c_i are determined by the initial conditions of $d_{U,F}(n)$. However, we are only concerned with determining the Gelfand-Kirillov base of R and for that we need to consider $\inf\{\lambda \in \mathbb{R}_{>0} \mid d_{U,F}(n) = \mathcal{O}(\lambda^n)\}$ taken over all n. Since the general solution of the homogeneous growth recurrence is exponential with largest base 2.1479, we have $\text{GKbase}_{U,F}(R) = 2.1479$.

We now apply the same procedure to show the existence of the $\text{GKbase}_{U,F}(R)$ for the case when R is polynomial ring $K[x_1, x_2, \cdots, x_m]$ where K is a field of characteristic p and $A = R[f; F]$. First, we find the general solution for the homogeneous growth recurrence.

By Theorem 1.4.2, we can find a general solution to the homogeneous growth recurrence of V by determining the characteristic equation and finding the roots of this equation.

Proposition 2.2.20. *Let $R = \mathbb{F}_p[x_1, x_2, \cdots, x_m]$, and let $A = R[f; F]$ be the Frobenius skew polynomial ring of R. If $U = \{x_1, x_2, \cdots, x_m\}$, then the growth equation of R with respect to U and F is*

$$x^{m(p-1)+1} - x^{m(p-1)} - \beta_1 x^{m(p-1)-1} - \cdots - \beta_{m(p-1)-1} x - 1 = 0$$

Further, this growth equation has a unique positive root whose value is larger than the absolute value for any other root.

Proof. We can determine the growth equation by applying Theorem 1.4.2 to the homogeneous recurrence from the growth recurrence described in Corollary 2.2.17.
Then we know the growth equation has a unique positive root by Theorem 1.4.3 and Theorem 1.4.4. □

We will denote this unique positive root of the growth equation by λ_1. By Theorem 1.4.2, the term corresponding to λ_1 in the general solution for the homogeneous recurrence of $d_{U,F}(n)$ is of the form $c_1\lambda_1^d$.

Let $\lambda_2, \lambda_3, \cdots, \lambda_{m(p-1)+1}$ be the remaining roots of the growth equation of V. If all of these roots are simple, then the general solution of the homogeneous growth recurrence of R is

$$c_1\lambda_1^d + c_2\lambda_2^d + \cdots + c_{m(p-1)+1}\lambda_{m(p-1)+1}^d$$

for some constants c_i.

Otherwise, we have the general solution to the homogeneous growth recurrence of R as $d_V^{(i)} = \sum_{j=0}^{s_i} c_{i,j} d^j \lambda_i^d$, where s_i is the multiplicity of λ_i and the $c_{i,j}$ are constants, where $d_V^{(i)}$ corresponds to the root λ_i for all $1 < i \leqslant m(p-1)+1$. Hence, the general solution becomes

$$c_1\lambda_1^d + \sum_{i=2}^{m(p-1)+1} d_V^{(i)}$$

Now to find a particular solution for the nonhomogeneous recurrence we consider the nonhomogeneous term

$$\widetilde{T}(n) = \sum_{i=0}^{n} \binom{m+i-1}{i} - \left[\sum_{i=1}^{m(p-1)} (\beta_{i-1}(d_{U,F}(0) + \cdots + d_{U,F}(m(p-1)-i)) - \sum_{i=1}^{m(p-1)} d_{U,F}(i) \right]$$

To determine a suitable form for the particular solution, we need to determine a function to match this combination. Since

$$\begin{aligned}\binom{m+d-1}{d} &= \frac{(d+m-1)(d+m-2)\cdots(d+1)(d)!}{(m-1)!d!}\\ &= \frac{1}{(m-1)!}(m+d-1)(m+d-2)\cdots(d+1)\end{aligned}$$

We have

$$\sum_{i=0}^{n}\binom{m+i-1}{i} = \sum_{i=0}^{n}\frac{1}{(m-1)!}(m+i-1)(m+i-2)\cdots(i+1)$$

Hence, we have a polynomial in n of degree m and the particular solution will have the form

$$c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_{m(p-1)+1}\lambda_{m(p-1)+1}^n + \sum_{i=0}^{m} a_i n^i$$

Theorem 2.2.21. *Let $R = \mathbb{F}_p[x_1, x_2, \cdots, x_m]$ and let $A = R[f; F]$ be the Frobenius skew polynomial ring of R. If $U = \{x_1, x_2, \cdots, x_m\}$, then $GKbase_{U,F}(R) = \lambda_1$ where λ_1 is the unique positive root of the growth equation of R with respect to U and F.*

Proof. This follows immediately from the growth equation of R having the form

$$c_1\lambda_1^n + c_2\lambda_2^n + \cdots + c_{m(p-1)+1}\lambda_{m(p-1)+1}^n + \sum_{i=0}^{m} a_i n^i$$

where λ_1 is larger than the absolute value of all other λ_i. □

Example 2.2.22. In this example, we provide the computation of the Gelfand-Kirillov base for the case when $m = 1$. Let $R = \mathbb{F}_p[x]$ and $A = R[f; F]$ be the Frobenius skew polynomial ring over R. We choose $U = \{x\}$ as the K-algebra generating set of R. Hence, V is the K-subspace of A spanned by $U \cup \{f\}$. Then the growth recurrence of R with respect to U and F is

$$d_{U,F}(n) = \sum_{i=1}^{p} \beta_{i-1} d_{U,F}(n-i) + \widetilde{T}(n)$$

First notice that $\widetilde{T}(n)$ is a linear function since

$$\widetilde{T}(n) = \sum_{i=0}^{n} \binom{i}{i} - \left[\sum_{i=1}^{(p-1)} (\beta_{i-1}(d_{U,f}(0) + \cdots + d_{U,f}((p-1)-i)) - \sum_{i=1}^{(p-1)} d_{U,f}(i) \right] = n - B$$

where $B = \sum_{i=1}^{(p-1)} (\beta_{i-1}(d_{U,f}(0) + \cdots + d_{U,f}((p-1)-i)) - \sum_{i=1}^{(p-1)} d_{U,f}(i)$ is a constant since the $d_{U,f}(i)$ are initial conditions for $0 \leqslant i \leqslant p-1$. We can also determine that $\beta_i = 1$ for all $0 \leqslant i \leqslant p-1$ since X is the set $\{0, 1, \cdots, p-1\}$. Hence, the growth equation of R with respect to U and F is

$$x^p - x^{p-1} - \cdots - x - 1 = 0$$

Proposition 2.2.23. *Let $R = \mathbb{F}_p[x]$, with p prime, $U = \{x\}$, and $A = R[f; F]$ be the Frobenius skew polynomial Ring over R. If λ_1 is the unique positive root of the growth equation of R with respect to U and F, then $GKbase_{U,F}(R) = \lambda_1$.*

Proof. We know that the growth equation has a unique positive root, λ_1, with absolute value greater that the absolute value of all the other roots. Hence, the term $c_1\lambda_1^n$ will dominate the terms of the general solution. Also, the particular solution to the nonhomogeneous portion of the growth recurrence is linear, which will also be dominated by $c_1\lambda_1^n$. Thus, $\text{GKbase}_{U,F}(R) = \lambda_1$. □

The last few results for this example provide bounds for λ_1.

Proposition 2.2.24. *Let $f(x)$ denote the polynomial the defines the growth equation of R with respect to U and F, and λ_1 denote the unique positive root of $f(x)$ where char$(K) = p$. Then $2 - \frac{1}{p} < \lambda_1 < 2$ for all p.*

Proof. Let

$$F(x) = -\frac{f(x)}{x^p} = -1 + \frac{1}{x} + \frac{1}{x^2} + \cdots + \frac{1}{x^{p-1}} + \frac{1}{x^p}$$

Then any root of $F(x)$ is also a root of $f(x)$. Also, for any $x > 0$ we have that $F(x) > 0$ implies that $f(x) < 0$, and $F(x) < 0$ implies that $f(x) > 0$. It is clear that for $x > 0$, $F(x)$ is monotone decreasing and $F(x)$ goes to -1 as x goes to ∞. Now consider $F\left(2 - \frac{1}{p}\right)$.

$$\begin{aligned}
F\left(2 - \frac{1}{p}\right) &= \frac{-f(2-\frac{1}{p})}{(2-\frac{1}{p})^p} \\
&= -1 + \frac{1}{2-\frac{1}{p}} + \frac{1}{(2-\frac{1}{p})^2} + \cdots + \frac{1}{(2-\frac{1}{p})^{p-1}} + \frac{1}{(2-\frac{1}{p})^p} \\
&= -1 + \sum_{i=1}^{p} \left(2 - \frac{1}{p}\right)^{-i} \\
&= -1 + \frac{(1-(2-\frac{1}{p})^{-p})p}{p-1} \\
&= -1 + \frac{(1-(2-\frac{1}{p})^{-p})p}{(1-\frac{1}{p})p}
\end{aligned}$$

Now since $(2-\frac{1}{p})^p > p$ for all p, we have $(2-\frac{1}{p})^{-p} < \frac{1}{p}$. Hence,

$$\frac{(1-(2-\frac{1}{p})^{-p})p}{p-1} > 1 \text{ and } -1 + \frac{(1-(2-\frac{1}{p})^{-p})p}{p-1} > 0$$

Thus, $F\left(2 - \frac{1}{p}\right) > 0$ and it follows that $f\left(2 - \frac{1}{p}\right) < 0$. Now considering $F(2)$ we have

$$F(2) = -1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^{p-1}} + \frac{1}{2^p} < 0$$

It then follows that $f(2) > 0$ for all p. Hence, we can conclude that $f\left(2 - \frac{1}{p}\right) < 0$ and $f(2) > 0$. The intermediate value theorem guarantees a root between $2 - \frac{1}{p}$ and 2. Since λ_1 is the unique positive root, we can conclude that $2 - \frac{1}{p} < \lambda_1 < 2$ for all p. $\square$

Example 2.2.25. In this example, let $R = \mathbb{F}_p[x_1, x_2]$, p prime and let $U = \{x_1, x_2\}$. Let $A = R[f; F]$ be the Frobenius skew polynomial ring over R and V be the finite dimensional K-subspace of A spanned by $U \cup \{f\}$. Then the growth recurrence of R with respect to U and F is

$$d_{U,F}(n) = \sum_{i=1}^{2(p-1)+1} \beta_{i-1} d_{U,F}(n-i) + \widetilde{T}(n)$$

We first consider $\widetilde{T}(n)$. Let $B = \left[\sum_{i=1}^{2(p-1)} (\beta_{i-1}(d_V(0) + \cdots + d_V(2(p-1)-i)) - \sum_{i=1}^{2(p-1)} d_V(i)\right]$ which is some constant. Then

$$\begin{aligned}
\widetilde{T}(n) &= \sum_{i=0}^{n} \binom{i+1}{i} - B \\
&= \sum_{i=0}^{n} (i+1) - B \\
&= \frac{(n+1)(n+2)}{2} - B
\end{aligned}$$

Hence, the nonhomogeneous component of the growth recurrence of R has polynomial form of degree two. We now look at the homogeneous growth recurrence of R with respect to U and F.

The next lemma gives the coefficients, β_i, for the homogeneous growth recurrence of R with respect to U and F for any p when $m = 2$.

Lemma 2.2.26. *Let A be the Frobenius skew polynomial ring over the ring $\mathbb{F}_p[x_1, x_2]$ and U be defined as above. Then the coefficients, β_i with $0 \leqslant i \leqslant 2(p-1)$, for the homogeneous growth recurrence of R with respect to U and F have the following equalities*

1. $\beta_i = i + 1$ *for all* $0 \leqslant i \leqslant p - 1$

2. $\beta_j = 2p - (i + 1)$ *for all* $p \leqslant i \leqslant 2(p-1)$

Proof. First, since the homogeneous growth recurrence of R with respect to U and F is $d_{U,F}(n) = \sum_{i=1}^{2(p-1)+1} \beta_{i-1} d_{U,F}(n-i)$, we have the coefficients β_i for all $0 \leqslant i \leqslant 2(p-1)$. Now we have $\beta_i = |X_i|$ where X_i is the set of ordered pairs whose entries are from the set $\{0, 1, 2, \cdots, p-1\}$ and whose sum is i. To show 1), consider β_i for some $0 \leqslant i \leqslant p-1$. Since i is less than $p-1$, we have $0, 1, 2, \cdots, i \in \{0, 1, \cdots, p-1\}$. Then for an odd i, there are $\frac{i+1}{2}$ pairs that add to i and we have $2 * \frac{i+1}{2}$ order pairs. For an even i, there are $\frac{i}{2}$ pairs that add to i with distinct entries and one pair whose entries are $\frac{i}{2}$. Hence, there are $2 * \frac{i}{2} + 1$ ordered pairs whose components sum to i. Thus, $\beta_i = i + 1$ for all $0 \leqslant i \leqslant p-1$.
We can prove 2) by applying Proposition 2.2.18 to the values obtained in 1). □

Now we can substitute these values for β_i in the homogeneous growth recurrence on V to determine the growth equation. Hence, the growth equation of R is

$$x^{2(p-1)+1} - x^{2(p-1)} - 2x^{2(p-1)-1} - \cdots - px^{p-1} - (p-1)x^{p-2} - \cdots - 2x - 1 = 0$$

Proposition 2.2.27. *Let $R = \mathbb{F}_p[x_1, x_2]$, p prime and $U = \{x_1, x_2\}$. Let $A = R[f; F]$ be the Frobenius skew polynomial ring over R and V be the finite dimensional K-subspace of A spanned by $U \cup \{f\}$. If λ_1 is the unique positive root of the growth equation of R with respect to U and F, then $GKbase_{U,F}(R) = \lambda_1$.*

Proof. We know that the growth equation has a unique positive root, λ_1, with absolute value greater that the absolute value of all the other roots. Hence, the term $c_1\lambda_1^n$ will dominate the terms of the general solution. Also, the particular solution to the nonhomogeneous portion of the growth recurrence is polynomial of degree two, which will also be dominated by $c_1\lambda_1^n$. Thus, we can conclude that $\text{GKbase}_{U,F}(R) = \lambda_1$. □

Let λ_1 be the determining root for the growth equation of R. We will now show that λ_1 is bounded by 2 and $\frac{3+\sqrt{5}}{2}$.

Proposition 2.2.28. *Let λ_1 be the determining root for the growth equation of R with respect to U and F, where char$(K) = p$. Then we have*

$$2 < \lambda_1 < \frac{3+\sqrt{5}}{2}$$

Proof. Let $f(x)$ be the growth polynomial of V and consider the following polynomial

$$\begin{aligned}
F(x) &= \frac{-f(x)}{x^{2p-1}} \\
&= -1 + \frac{1}{x} + \frac{2}{x^2} + \cdots + \frac{p}{x^p} + \frac{p-1}{x^{p+1}} + \cdots + \frac{2}{x^{2p-2}} + \frac{1}{x^{2p-1}} \\
&= -1 + \sum_{i=1}^{p} \frac{i}{x^i} + \sum_{p+1}^{2p-1} \frac{2p-i}{x^i}
\end{aligned}$$

We now use the following equalities to substitute into $F(x)$ for any $x > 1$:

$$\sum_{i=1}^{p} \frac{i}{x^i} = \frac{x^{-p}(x^{p+1} - x(p+1) + p)}{(x-1)^2} \text{ and } \sum_{p+1}^{2p-1} \frac{2p-i}{x^i} = \frac{x^{-2p}(px^{p+1} - x^{p+1} - px^p + x)}{(x-1)^2}$$

Hence,

$$\sum_{i=1}^{p} \frac{i}{x^i} + \sum_{p+1}^{2p-1} \frac{2p-i}{x^i} = \frac{(x^p-1)^2}{x^{2p-1}(x-1)^2}$$

and

$$F(x) = -1 + \frac{(x^p-1)^2}{x^{2p-1}(x-1)^2}$$

Now any root of $F(x)$ is also a root of $f(x)$. Also, for $x > 0$ we have, $F(x) > 0 \implies f(x) < 0$ and $F(x) < 0 \implies f(x) > 0$. Combining the fact that $f(x)$ is of odd degree $\forall p$, and that λ_1 is the largest root of $f(x)$, we know that $\lambda_1 > x$ $\forall x$ such that $f(x) < 0$ and $\lambda_1 < x$ $\forall x$ such that $f(x) > 0$.

To show $\lambda_1 > 2$, consider

$$F(2) = -1 + \frac{(2^p-1)^2}{2^{2p-1}(2-1)^2} = -1 + \frac{(2^p-1)^2}{2^{2p-1}}$$

Now since $\frac{(2^p-1)^2}{2^{2p-1}} = 2\left(\frac{2^p-1}{2^p}\right)^2 = 2\left(1-\frac{1}{2^p}\right)^2$, this expression is monotone increasing for all $p > 0$. Also, when $p = 2$ we have $\frac{(2^p-1)^2}{2^{2p-1}} = \frac{9}{8} > 1$, and hence, $\frac{(2^p-1)^2}{2^{2p-1}} > 1$ for all p. Thus, $F(2) > 0$ which implies that $f(2) < 0$, and hence, $\lambda_1 > 2$.

We now consider $F\left(\frac{3+\sqrt{5}}{2}\right)$. First, notice that we have the following relationship for $\frac{3+\sqrt{5}}{2}$ that will be applied in the computation:

$$\left(\frac{3+\sqrt{5}}{2} - 1\right)^2 = \frac{3+\sqrt{5}}{2}$$

$$\begin{aligned}
F\left(\frac{3+\sqrt{5}}{2}\right) &= -1 + \frac{\left(\left(\frac{3+\sqrt{5}}{2}\right)^p - 1\right)^2}{\left(\frac{3+\sqrt{5}}{2}\right)^{2p-1}\left(\left(\frac{3+\sqrt{5}}{2}\right) - 1\right)^2} \\
&= -1 + \frac{\left(\left(\frac{3+\sqrt{5}}{2}\right)^p - 1\right)^2}{\left(\frac{3+\sqrt{5}}{2}\right)^{2p-1}\left(\frac{3+\sqrt{5}}{2}\right)} \\
&= -1 + \frac{\left(\left(\frac{3+\sqrt{5}}{2}\right)^p - 1\right)^2}{\left(\frac{3+\sqrt{5}}{2}\right)^{2p}} \\
&= -1 + \frac{\left(\frac{3+\sqrt{5}}{2}\right)^{2p} - \left(2\left(\frac{3+\sqrt{5}}{2}\right)^p - 1\right)}{\left(\frac{3+\sqrt{5}}{2}\right)^{2p}}
\end{aligned}$$

Since $2\left(\frac{3+\sqrt{5}}{2}\right)^p - 1 > 0$ for all p, we have

$$F\left(\frac{3+\sqrt{5}}{2}\right) = -1 + \frac{\left(\frac{3+\sqrt{5}}{2}\right)^{2p} - \left(2\left(\frac{3+\sqrt{5}}{2}\right)^p - 1\right)}{\left(\frac{3+\sqrt{5}}{2}\right)^{2p}} < 0$$

Thus, we have $F\left(\frac{3+\sqrt{5}}{2}\right) < 0$ which implies $f\left(\frac{3+\sqrt{5}}{2}\right) > 0$. It follows by the Intermediate Value Theorem that there exists a root of $f(x)$ between 2 and $\frac{3+\sqrt{5}}{2}$. Since λ_1 is the unique positive root of $f(x)$, we conclude that $2 < \lambda_1 < \frac{3+\sqrt{5}}{2}$. $\square$

2.3 Growth Recurrence for a Numerical Semigroup Ring

We conclude this chapter by applying this method of determining a recurrence for $d_{U,F}(n)$ to find $\text{GKbase}_{U,F}(R)$ where R is a particular numerical semigroup ring. For this example, let $R = \mathbb{F}_p[t^2, t^3]$, p prime and $U = \{t^2, t^3\}$, which is a set of K-algebra generators of R. Let $A = R[f; F]$ be the Frobenius skew polynomial ring over R and V be the finite dimensional K-subspace of A spanned by $U \cup \{f\}$. Then $V^0 = K$ and V^i for $i \geqslant 0$ is the subspace spanned by skew monomials given by the product of i terms from the elements t^2, t^3, f. We define $A_n = \sum_{i=0}^{n} V^i$ and consider the function $d_{U,F}(n) = \dim_K(A_n)$.

2.3.1 Irreducible Skew Monomials

We will now determine a suitable definition for a minimal representation for skew monomials in $K[t^2, t^3]$. This is again necessary since the V^i are not disjoint, and it would not be possible to determine a recurrence for the V^i such that the sum would give $d_{U,F}(n)$. The method will be similar to what was done for the Frobenius skew polynomial ring over the commutative polynomial ring. However, more care has to be taken when considering what skew monomials are considered in irreducible form. We first give a definition for a skew monomial to be in irreducible form and provide necessary characterizations. Then we can define subspaces of the V^i spanned by these skew monomials in irreducible form.

Definition 2.3.1. Let $A = R[f; F]$ where $R = K[t^2, t^3]$. An element $z \in A$ is called a **skew monomial** if and only if z is the product of the elements t^2, t^3, f. A skew monomial $z \in A$ is said to be in **irreducible form** if and only if z is written as the product of the fewest possible elements from t^2, t^3, f.

This is a broad definition and there are a few possibilities that need to be addressed. Skew monomials in A can be written in multiple representations due to the multiplication rule in A as well as by using different powers of t^2 and t^3. Consider the case when $R = K[t^2, t^3]$ where the characteristic of K is p. Then the skew monomial $t^{2p} f = (t^2)^p f = f t^2$. The representation of this skew monomial $(t^2)^p f$ is the product of $p + 1$ elements from t^2, t^3 and the representation ft^2 is the product of 2 elements. Hence, ft^2 is the irreducible form of the skew monomial $(t^2)^p f$.

Now let $p = 7$ and consider a skew monomial in A of the form $t^8 f = (t^2)^4 f = t^2(t^3)^2 f$. Again, we have three representations of the same skew monomial. The representation $(t^2)^4 f$ is the product of 5 elements from A and the representation $t^2(t^3)^2 f$ is the product of 4 elements from A. Since $p = 7$, we cannot reduce the skew monomial any further using the multiplication rule of the ring. To see this, consider $t^8 f = tt^7 f = tft$, however, t is not an element of the ring and this is not a valid representation. Thus, $t^2(t^3)^2 f$ is the irreducible form of $t^8 f$. We will first focus on when a skew monomial is in irreducible form regarding the powers of t, and then address when a skew monomial is in irreducible form by the multiplication rule of A.

Lemma 2.3.2. *Let $A = R[f; F]$ where $R = K[t^2, t^3]$. Then products of the form $(t^2)^i (t^3)^j$ are irreducible if and only if $i \in \{0, 1, 2\}$. Further, the only skew monomials of the form $(t^2)^i (t^3)^j$ such that $i + j = d$ that are irreducible are $t^{3d}, t^{3d-1}, t^{3d-2}$.*

Proof. We first prove that if $(t^2)^i (t^3)^j$ is in irreducible form, then $i \in \{0, 1, 2\}$. To do this, assume that $i \geqslant 3$. Then we have $(t^2)^i (t^3)^j = (t^2)^{i-3} (t^2)^3 (t^3)^j = (t^2)^{i-3} (t^3)^2 (t^3)^j = (t^2)^{i-3} (t^3)^{j+2}$. The representation $(t^2)^i (t^3)^j$ is a product of $i + j$ terms and the representation $(t^2)^{i-3} (t^3)^{j+2}$ is the product of $i - 3 + j + 2 = i + j - 1$ terms. Hence, $(t^2)^i (t^3)^j$ is reducible. Now assume that $i \in \{0, 1, 2\}$ and consider $(t^2)^i (t^3)^j$. We can prove this skew monomial is in irreducible form by considering each of the cases. If $i = 0$, then $(t^2)^0 (t^3)^j = (t^3)^j$. If $j = 1$, then $(t^3)^j$ is the unique representation of the skew monomial. Assume that $j \geqslant 2$. We have $(t^3)^j = (t^3)^{j-2} (t^3)^2 = (t^3)^{j-2} (t^2)^3$, but this representation is the product of $j - 2 + 3 = j + 1$ elements. It follows that for each time we replace $(t^3)^2$ with $(t^2)^3$, the number of elements in the product increases by 1. Hence, $(t^3)^j$ is the irreducible form. Likewise for $i = 1$ or $i = 2$, it is not possible to write $(t^2)^i$ as (t^3) to some power less than i, and hence, any substitution would increase the number of elements in the product. Thus, $(t^2)^i(t^3)^j$ is in irreducible form when $i \in \{0, 1, 2\}$.

To prove the last assertion, we consider a skew monomial $(t^2)^i(t^3)^j$ in irreducible form such that $i + j = d$. By the first part of the proof, we have that $i \in \{0, 1, 2\}$. Hence, the only possibilities are $(t^2)^0(t^3)^d = t^{3d}$, $(t^2)^1(t^3)^{d-1} = t^{2+3(d-1)} = t^{3d-1}$, and $(t^2)^2(t^3)^{d-2} = t^{4+3(d-2)} = t^{3d-2}$. □

Lemma 2.3.2 gives a characterization of irreducible form for skew monomials that are products of t^2 and t^3. We now move to provide a characterization of irreducible form for skew monomials that are products of t^2, t^3 and f. We get separate characterizations for skew monomials involving an f in the product since the multiplication rule of A can only be applied with f. Hence there are pairs i, j such that $(t^2)^i(t^3)^j$ is in irreducible form but $(t^2)^i(t^3)^j f$ is reducible. For instance, consider the pair $(1, p)$. Then by Lemma 2.3.2, $(t^2)(t^3)^p$ is in irreducible form. However, $(t^2)(t^3)^p f$ has irreducible form $t^2 f t^3$. In the next proposition, we address skew monomials of this form and a characterization for when they are in irreducible form.

Proposition 2.3.3. *Let $A = R[f; F]$ where $R = K[t^2, t^3]$. Then products of the form $(t^2)^i(t^3)^j f$ are irreducible if and only if $i \in \{0, 1, 2\}$, $2i + 3j \leqslant 2p + 1$, and $2i + 3j \neq 2p$.*

Proof. We first prove that if a product of the form $(t^2)^i(t^3)^j f$ is in irreducible form, then $i \in \{0, 1, 2\}$, $2i+3j \leqslant 2p+1$, and $2i+3j \neq 2p$. We know by Lemma 2.7.2, that if $i \notin \{0, 1, 2\}$ then $(t^2)^i(t^3)^j f$ is reducible. Now assume that $2i+3j = 2p$. Then $(t^2)^i(t^3)^j f = t^{2p} f = f t^2$ and the product is reducible. Otherwise, assume that $2i+3j \geqslant 2p+2$. Then $2i+3j = 2p+2+k$ for some $k \geqslant 0$. If k is even, then $k = 2m$ for some m and we have $(t^2)^i(t^3)^j f = t^{2p+2+2m} f = (t^2)^{m+1}(t^2)^p f = (t^2)^{2(m+1)} f t^2$ and the product is reducible. Now if k is odd, then $k = 2m+1$ for some m and we have $(t^2)^i(t^3)^j f = t^{2p+2+2m+1} f = (t^2)^m t^3 (t^2)^p f = (t^2)^m t^3 f t^2$ and the product is reducible.

For the other implication, assume that $i \in \{0,1,2\}$, $2i+3j \leqslant 2p+1$, and $2i+3j \neq 2p$. By Lemma 2.3.2, the product $(t^2)^i(t^3)^j$ is in irreducible form. Hence, any reduction must come by the multiplication rule of A. First assume that $2i+3j = 2p+1$ and consider $(t^2)^i(t^3)^j f = t^{2p+1}f = t(t^2)^p f = tft^2$. However, t is not an element of A and this is not a valid representation. We now claim that $2p$ is the minimum power of t such that the multiplication rule for A is possible. First, $t^{2p}f = ft^2$ and is reducible. Now if we have $t^m f$ for some $m < p$, then it is clear that $t^m f$ is in irreducible form. Let $p \leqslant m \leqslant 2p$. Then $m = p+k$ for some k. Hence, $t^m f = t^k ft$ but t is not an element of A and this is again not a valid representation. Thus, $2p$ is the minimum power of t such that $t^m f$ is reducible. It follows from this that if $2i+3j \leqslant 2p-1$, then $(t^2)^i(t^3)^j f$ is in irreducible form. □

Corollary 2.3.4. *Let $(t^2)^i(t^3)^j f$ be a skew monomial in irreducible form. Then we have the following inequalities for j.*

1. *If $i = 0$, then $0 \leqslant j \leqslant \lfloor \frac{2p+1}{3} \rfloor$.*
2. *If $i = 1$, then $0 \leqslant j \leqslant \lfloor \frac{2p-1}{3} \rfloor$.*
3. *If $i = 2$, then $0 \leqslant j \leqslant \lfloor \frac{2p-3}{3} \rfloor$.*

Proof. By Proposition 2.3.3, we have that $2i+3j \neq 2p$ and $2i+3j \leqslant 2p+1$. We will use these relations to analyze each case. We already know that $j \leqslant p-1$, so it is necessary to first show that each of the characterizations are less than or equal to $p-1$. It suffices to show that $\lfloor \frac{2p+1}{3} \rfloor \leqslant p-1$ since it is the largest value in the claim. First, notice that $\frac{2p+1}{3} \leqslant p-1$ for all $p > 4$. Since $\lfloor \frac{2p+1}{3} \rfloor \leqslant \frac{2p+1}{3}$, we have that $\lfloor \frac{2p+1}{3} \rfloor \leqslant p-1$ for all $p > 4$ and we only need to check $p = 3$ and $p = 2$. This follows by evaluating $\lfloor \frac{2p+1}{3} \rfloor$ at these values. We get $\lfloor \frac{2p+1}{3} \rfloor = p-1$ for both $p = 2$ and $p = 3$. Hence, the claim holds for all p.

To show 1), we consider the inequality $2i+3j \leqslant 2p+1$ with $i = 0$. Then we have $j \leqslant \frac{2p+1}{3}$. Since $\frac{2p+1}{3}$ is not necessarily an integer, we get $j \leqslant \lfloor \frac{2p+1}{3} \rfloor$.

For 2), we have $2+3j \leqslant 2p+1$ and it follows that $j \leqslant \frac{2p-1}{3}$ and hence, $j \leqslant \lfloor \frac{2p-1}{3} \rfloor$. The result for 3) follows in the same way. □

Proposition 2.3.3 and Corollary 2.3.4 give a characterization for skew monomials of the form $(t^2)^i(t^3)^j f$ that are in irreducible form. However, to be able to define a recurrence, we need to know when a product of irreducible skew monomials remains in irreducible form. First, we give a definition of a leading product for any skew monomial in irreducible form. Then by separating the leading product from the skew monomial leaves a skew monomial that is the product of fewer elements.

Definition 2.3.5. Let $z \in A$ be a skew monomial in irreducible form that contains at least one f in the product. Then z is written as the product of the fewest possible elements from t^2, t^3, f. Then the **leading product** of z is $(t^2)^i(t^3)^j f$ where $z = (t^2)^i(t^3)^j f\cdot$ product of elements from t^2, t^3, f. We will denote the leading product of a skew monomial z by lead(z).

Notice that this definition is for skew monomials that contain an f in the product. If we consider a similar definition of "leading product" for a skew monomial that is only the product of t^2, t^3, then the skew monomial is equal to its leading product and the definition adds no valuable information. The purpose for defining the leading product up to the first occurence of an f in the product of z is simply because this is where commutativity is not applicable. Since $K[t^2, t^3]$ is a commutative ring, we can order a product of t^2 and t^3 in any way that we want. These powers of t^2 and t^3 can also be combined into some power of t. So to maintain clarity on what elements are used in the product of z, we identify the leading product by $(t^2)^i(t^3)^j f$ and consider the ordered pairs (i, j) where i is the power on t^2 and j is the power on t^3. The next proposition shows that every skew monomial in irreducible form can be identified by the leading product and a skew monomial in irreducible form.

Proposition 2.3.6. *Let $z \in A$ be a skew monomial in irreducible form with at least one f contained in the product. Then z can be written in the form $z = (t^2)^i(t^3)^j f z'$ where $(t^2)^i(t^3)^j f$ and z' are skew monomials in irreducible form and $(t^2)^i(t^3)^j f$ and z' are uniquely determined by z.*

Proof. We first prove that there is only one pair (i, j) such that the leading product of z is written in fewest terms. Let $(t^2)^i(t^3)^j = t^m$. Then $2i + 3j = m$ for some $m \neq 1$. If l is the minimal possible value of $i+j$ such that $2i+3j=m$, then we have a system of equations with a unique solution. Thus, (i,j) are uniquely determined for the leading product of any skew monomial in irreducible form.

Now we have $z=(t^2)^i(t^3)^j f\cdot z'$ where z' is a product of elements from t^2,t^3,f. It is clear that z' must be a skew monomial in irreducible form. Otherwise, z could be reduced by writing z' with fewer terms. To see that z' is also uniquely determined by z, assume $z=(t^2)^i(t^3)^j fz'=(t^2)^i(t^3)^j fz''$. It follows immediately that $z'=z''$ since A is a domain. □

The last piece that is needed here is to show when the product of two skew monomials in irreducible form is also in irreducible form. It is not always the case that the product of skew monomials in irreducible form is still in irreducible form. For instance, let $p=5$ and consider the skew monomials t^2t^3f and t^2f. Then the product is $t^2t^3ft^2f=t^5ft^2f=ft^3f$. Hence, the irreducible form of the product in ft^3f. The next proposition shows when we can multiply a skew monomial in irreducible form by $(t^2)^i(t^3)^j f$ and still have the product in irreducible form. This gives the ability to count all skew monomials in irreducible form by taking a skew monomial and multiplying by $(t^2)^i(t^3)^j f$ on the left.

Proposition 2.3.7. *Let $z\in A$ be a skew monomial in irreducible form and $(t^2)^i(t^3)^j f$ be irreducible. If lead$(z)=f$, then $(t^2)^i(t^3)^j fz$ is in irreducible form. Otherwise, $(t^2)^i(t^3)^j fz$ is in irreducible form if and only if $2i+3j\neq p$ and $2i+3j\leqslant p+1$.*

Proof. Let $z\in A$ such that z is in irreducible form and lead$(z)=f$. Then $z=f\cdot z'$ for some $z'\in A$ that is also in irreducible form. Now consider the product $(t^2)^i(t^3)^j fz$ where $(t^2)^i(t^3)^j f$ is in irreducible form. Then $2i+3j\neq 2p$ and $2i+3j\leqslant 2p+1$. Let $2i+3j=2p+1-k$ for some $k\neq 1$. We have $(t^2)^i(t^3)^j fz=t^{2p+1-k}ffz'=t^pt^{p+1-k}ffz'$. If $k=0$, then this becomes $t^pt^{p+1}ffz'=tt^2pffz'=tft^2fz'$ but this is not an element of A and hence the product $(t^2)^i(t^3)^j fz$ is already in irreducible form. Now let $k\geqslant 2$. Then we have $t^pt^{p+1-k}ffz'=t^{k'}t^pffz'=t^{k'}ftz'$ where $k'=p+1-k<p$. Again, this element is not in A and $(t^2)^i(t^3)^j fz$ is already in irreducible form.

Now let $z = t^m z'$ for some $m \geqslant 2$. We first show that if $2i+3j = p$ or $2i+3j \geqslant p+2$, then $(t^2)^i(t^3)^j fz$ is reducible. Let $2i+3j = p$. Then $(t^2)^i(t^3)^j fz = t^p f t^m z' = f t^{m+1} z'$. Hence, $(t^2)^i(t^3)^j fz$ is reducible since t^{m+1} is an element of A. Now let $2i+3j \geqslant p+2$, say $2i+3j = p+2+k$ for some $k \geqslant 0$. Then $(t^2)^i(t^3)^j fz = t^{2+k} t^p f t^m z' = t^{2+k} f t^{m+1} z'$ which is an element of A for all $k \geqslant 0$ and $(t^2)^i(t^3)^j fz$ is reducible.
Finally, we show that if $2i+3j \neq p$ and $2i+3j \leqslant p+1$, then $(t^2)^i(t^3)^j fz$ is in irreducible form. Let $2i+3j = p+1$. Then $(t^2)^i(t^3)^j fz = t^{p+1} f t^m z' = t f t^{m+1} z'$, but this is not an element of A and the product $(t^2)^i(t^3)^j fz$ is in irreducible form. Now let $2i+3j = p-1-k$ for some $k \geqslant 0$. Since $p-1-k < p$, it is not possible to apply the multiplication rule of A to commute t^{p-1-k} with f. Thus, the product $(t^2)^i(t^3)^j fz$ is in irreducible form. $\square$

2.3.2 Direct Sum Decomposition for the Growth Recurrence

Now that we have characterized skew monomials in irreducible form, we can define a K-subspace of the V^i that is spanned by only the skew monomials in irreducible form. Since the irreducible form is unique, this will allow us to determine a direct sum decomposition of A_n.

Definition 2.3.8. Let $A = R[f; F]$ where $R = \mathbb{F}_p[t^2, t^3]$ and V^i be the K-subspace spanned by elements that are products of i elements from the set $U \cup \{f\}$. We define the **growth subspace** to be the K-subspace of V^i spanned by all skew monomials in the spanning set for V^i that are in irreducible form. The growth subspace for V^i will be denoted W_i.

Defining the growth subspace in this way will allow us to determine $d_{U,F}(n)$ by summing up the cardinalities of the spanning sets of $W_0, W_1, \cdots, W_n$. This is not possible when only considering the subspaces V^i because different representations of skew monomials can be counted in different subspaces. For instance, the skew monomial $(t^2)^p f$ is in the spanning set for V^{p+1} and V^2 since $(t^2)^p f = f t^2$. The next proposition shows that the growth subspaces W_i are disjoint, and will use that to show that they form a direct sum decomposition.

Proposition 2.3.9. *Let $A = R[f;F]$ where $R = \mathbb{F}_p[t^2, t^3]$. Then $A_n = \sum_{i=0}^{n} V^i = \bigoplus_{i=0}^{n} W_i$ and $d_{U,F}(n) = \sum_{i=0}^{n} dim_K(W_i)$.*

Proof. First consider W_i and W_j for some $i < j$. The any skew monomial in the spanning of W_j is the product of i elements in t^2, t^3, f and is in irreducible form. Hence, no skew monomial in the spanning set of W_j can be written as the product of i elements and is not in the spanning set of W_i. Thus, we have that $W_i \cap \sum_{j \neq i} W_j = 0$ for all $i, j \in \mathbb{Z}_{\geqslant 0}$.
By definition, $W_i \subseteq V^i$ for all i and it follows that $\sum_{i=0}^{n} W_i \subseteq \sum_{i=0}^{n} V^i$ for all n. Now let z be a generator of V^i for some i that does not appear in W_i. Then z is reducible and the irreducible form of z is also contained in the spanning set of V^j for some $j < i$. Then z is either in W_j or it appears in V^k for some $k < j$. This process must necessarily terminate since $V^1 = \langle t^2, t^3, f \rangle = W_1$. Hence, every generator from V^i will appear in $\sum_{i=0}^{n} W_i$ for all i. Thus, $A_n = \sum_{i=0}^{n} V^i = \bigoplus_{i=0}^{n} W_i$ and $\sum_{i=0}^{n} W_i$ is a direct sum decomposition of A_n. Then it follows that $d_{U,F}(n) = \dim_K(A_n) = \dim_K \left(\sum_{i=0}^{n} V^i\right) = \sum_{i=0}^{n} \dim_K(W_i)$. $\square$

2.3.3 Describing $d_{U,F}(n)$

Proposition 2.3.9 shows that we need to be able to describe $\dim_K(W_i)$ for each $0 \leqslant i \leqslant n$ to be able to determine $d_{U,F}(n)$. In this section, we will develop a method to determine a recurrence $\dim_K(W_i)$. Before determining which W_j are necessary to do this, we first introduce some definitions to ease the cumbersome terminology and notation and to give a characteristic of skew monomials in the ring $A = R[f;F]$ over $R = \mathbb{F}_p[t^2, t^3]$ similar to the characteristic of degree when R was the commutative polynomial ring.

Definition 2.3.10. Let $A = R[f;F]$ where $R = \mathbb{F}_p[t^2, t^3]$ and let W_i be the growth subspace of V^i where V is the finite dimensional generating subspace spanned by $U \cup \{f\}$. Then we have the following definitions and notations

- We will denote $\dim_K(W_d)$ by C_d.
- Let z be a skew monomial in irreducible form that is the product of d elements from t^2, t^3, f. Then d is the **length** of z. We denote the length of z by $l(z)$.

To understand what is meant by length here, consider the skew monomial $t^8 f$. As shown previously, the irreducible form of $t^8 f$ is $t^2(t^3)^2 f$. Hence, $l(t^8 f) = 4$. From this point forward, anytime the length of a skew monomial is referred to, it will be assumed that the skew monomial is in irreducible form.

Now using the terminology in Defintion 2.3.10, we can view W_d as the K-subspace spanned by the skew monomials of length d. Hence, $C_d = \dim_K(W_d)$ is equal to the number of skew monomials of length d. By Proposition 2.3.6, every skew monomial of length d is uniquely determined by the leading product and a skew monomial of length $d - (i + j + 1)$.

The next definition will provide a notation for all skew monomials of length d with a common leading product. We can then determine which skew monomials of length $d - (i + j + 1)$ that the leading product can be multiplied to and remain in irreducible form to determine the number of skew monomials with the leading product given by (i, j).

Definition 2.3.11.

- Let $\overline{W}_d$ denote the set of skew monomials that generate W_d. That is, $\overline{W}_d$ is the set of all skew monomials in irreducible form of length d. Since $C_d = \dim_K(W_d)$, we have $C_d = |\overline{W}_d|$.
- Let $\overline{W}_{i,j,d}$ denote the subset of $\overline{W}_d$ such that for every $z \in \overline{W}_{i,j,d}$, we have $\text{lead}(z) = (t^2)^i (t^3)^j) f$.
- We will denote $|\overline{W}_{i,j,d}|$ by $c_{i,j,d}$. That is, $c_{i,j,d}$ counts the number of skew monomials in irreducible form of length d with leading product $(t^2)^i (t^3)^j f$.

Proposition 2.3.12. *Let X be the set of ordered pairs (i, j) such that $(t^2)^i (t^3)^j f$ is in irreducible form. Then $C_d = \sum_{(i,j) \in X} c_{i,j,d} + 3$, where the sum is taken over all elements of X.*

Proof. By Lemma 2.3.2, there are exactly 3 skew monomials that are products of only t^2 and t^3 of length d that are in irreducible form. This accounts for the constant term. Hence, all other skew monomials counted in C_d are uniquely determined by the leading product and a skew monomial of length $d - (i + j + 1)$. Since the sum is taken over all elements in X, and X represents the set of all possible pairs of (i,j) for a leading product, the sum $\sum_{(i,j)\in X} c_{i,j,d}$ counts all of the skew monomials that are products with at least one f. Thus, $C_d = \sum_{(i,j)\in X} c_{i,j,d} + 3$. $\square$

Lemma 2.3.13. *For any pair (i,j) such that $(t^2)^i(t^3)^j f$ is in irreducible form and $2i+3j \leqslant p+1$ with $2i+3j \neq p$ there is a one-to-one correspondence between $\overline{W}_{i,j,d}$ and $\overline{W}_{d-(i+j+1)}$. Hence, $c_{i,j,d} = C_{d-(i+j+1)}$.*

Proof. Let $\phi : \overline{W}_{d-(i+j+1)} \to \overline{W}_{i,j,d}$ by $\phi(z) = (t^2)^i(t^3)^j f z$ for all $z \in \overline{W}_{d-(i+j+1)}$. Since $2i+3j \leqslant p+1$ and $2i+3j \neq p$, we can apply Proposition 2.3.7 to see that $(t^2)^i(t^3)^j f z$ is in irreducible form for all $z \in \overline{W}_{d-(i+j+1)}$, and hence, is unique. This shows that $\phi(z) \in \overline{W}_{i,j,d}$ for all $z \in \overline{W}_{d-(i+j+1)}$ and ϕ is well-defined. To see that ϕ is injective, notice that if $\phi(z) = \phi(z')$ then $(t^2)^i(t^3)^j f z = (t^2)^i(t^3)^j f z'$. Since A is a domain we have $z = z'$.
We now show that ϕ is surjective. Let $z \in \overline{W}_{i,j,d}$. By Proposition 2.3.6, we can write z uniquely as $(t^2)^i(t^3)^j f z'$ for some z' such that $l(z') = d-(i+j+1)$. Hence, $z' \in \overline{W}_{d-(i+j+1)}$ and since this writing is unique, we have that ϕ is surjective. Thus, ϕ is bijective and it follows that $c_{i,j,d} = C_{d-(i+j+1)}$. $\square$

Lemma 2.3.14. *For any pair (i,j) such that $(t^2)^i(t^3)^j f$ is in irreducible form and $2i+3j = p$ or $2i+3j \geqslant p+2$ there is a one-to-one correspondence between $\overline{W}_{i,j,d}$ and $\overline{W}_{d-(i+j+2)}$. Hence, $c_{i,j,d} = C_{d-(i+j+2)}$.*

Proof. Let $\phi : \overline{W}_{d-(i+j+2)} \to \overline{W}_{i,j,d}$ by $\phi(z) = (t^2)^i(t^3)^j f^2 z$ for all $z \in \overline{W}_{d-(i+j+2)}$. Since $2i+3j \geqslant p+2$ or $2i+3j = p$, we can apply Proposition 2.3.7 to see that $(t^2)^i(t^3)^j f^2 z$ is in irreducible form for all $z \in \overline{W}_{d-(i+j+2)}$, and hence, is unique. This shows that $\phi(z) \in \overline{W}_{i,j,d}$ for all $z \in \overline{W}_{d-(i+j+2)}$ and ϕ is well-defined. To see that ϕ is injective, notice that if $\phi(z) = \phi(z')$ then $(t^2)^i(t^3)^j f^2 z = (t^2)^i(t^3)^j f^2 z'$. Since A is a domain we have $z = z'$.

We now show that ϕ is surjective. Let $z \in \overline{W}_{i,j,d}$. By Proposition 2.3.6, we can write z uniquely as $(t^2)^i(t^3)^j fz'$ for some z' with lead$(z') = f$ such that $l(z') = d - (i + j + 2)$. Since z' is uniquely written as fz'', the z'' is unique for z. Hence, $z'' \in \overline{W}_{d-(i+j+2)}$ and since this writing is unique, we have that ϕ is surjective. Thus, ϕ is bijective and it follows that $c_{i,j,d} = C_{d-(i+j+2)}$. □

By Lemma 2.3.13 and Lemma 2.3.14, we can now substitute values in for $c_{i,j,d}$ in the equation from Proposition 2.3.12 to determine a recurrence for C_d. However, it is possible to have pairs (i, j) and (i', j') where $i + j = i' + j'$, and hence, $c_{i,j,d} = c_{i',j',d}$. Then if $c_{i,j,d} = C_m$, we also have $c_{i',j',d} = C_m$. Thus, certain C_m will appear multiple times in the recurrence, and hence, have a coefficient different than one. We now define sets such that the cardinality will determine coefficients in the recurrence.

Definition 2.3.15. Let X be the set of all ordered pairs (i, j) such that the skew monomial $(t^2)^i(t^3)^j f$ is in irreducible form. Then we take X_m to be the subset of X containing all pairs (i, j) such that $i + j = m$, $0 \leqslant 2i + 3j \leqslant p + 1$ and $2i + 3j \neq p$. We will denote the cardinality of X_m by β_m.

Lemma 2.3.16. *The largest possible m such that X_m is nonempty is*

$$max\left\{\left\lfloor\frac{p+1}{3}\right\rfloor, 1 + \left\lfloor\frac{p-1}{3}\right\rfloor, 2 + \left\lfloor\frac{p-3}{3}\right\rfloor\right\}$$

Proof. We have $2i + 3j \leqslant p + 1$ for all $(i, j) \in X_m$ and by Lemma 2.3.2, $i \in \{0, 1, 2\}$. Hence, we only have the following pairs to consider:

$$\begin{aligned} i = 0 \;\; j &= \left\lfloor\frac{p+1}{3}\right\rfloor \\ i = 1 \;\; j &= \left\lfloor\frac{p-1}{3}\right\rfloor \\ i = 2 \;\; j &= \left\lfloor\frac{p-3}{3}\right\rfloor \end{aligned}$$

Thus, the maximum sum is $\max\{\lfloor\frac{p+1}{3}\rfloor, 1 + \lfloor\frac{p-1}{3}\rfloor, 2 + \lfloor\frac{p-3}{3}\rfloor\}$. □

Proposition 2.3.17. *Let* $m = max\{\lfloor\frac{p+1}{3}\rfloor, 1+\lfloor\frac{p-1}{3}\rfloor, 2+\lfloor\frac{p-3}{3}\rfloor\}$ *and consider some* $k \leqslant n$. *If* $\sum_{(i,j)\in X_k} c_{i,j,d}$ *is the sum taken over all elements in* X_k, *then* $\sum_{(i,j)\in X_k} c_{i,j,d} = \beta_k C_{d-(i+j+1)}$.

Proof. By Lemma 2.3.13, we have $c_{i,j,d} = C_{d-(i+j+1)}$ for all $(i,j) \in X_k$. Since β_k denotes the cardinality of X_k and no skew monomial counted in $c_{i,j,d}$ is counted in $c_{i'j',d}$ for some $i \neq i'$ or $j \neq j'$, the result follows. □

We now apply the same procedure to develop the portion of the recurrence for the skew monomials with leading product such that $2i + 3j = p$ or $2i + 3j \geqslant p + 2$.

Definition 2.3.18. Let $\overline{X}_m$ be the subset of X containing all pairs (i,j) with $i + j = m$ such that $2i + 3j = p$ or $2i + 3j \geqslant p + 2$. Notice that this is equivalent to taking $\overline{X}_m$ as the complement of X_m in X. We will denote the cardinality of $\overline{X}_m$ by $\overline{\beta}_m$.

Lemma 2.3.19. *The largest possible* q *such that* $\overline{X}_q$ *is nonempty is*

$$max\left\{\left\lfloor\frac{2p+1}{3}\right\rfloor, 1+\left\lfloor\frac{2p-1}{3}\right\rfloor, 2+\left\lfloor\frac{2p-3}{3}\right\rfloor\right\}$$

Proof. By Corollary 2.3.4, we have the following largest possible pairs for i and j such that $(t^2)^i(t^3)^j f$ is in irreducible form:

$$\begin{aligned} i = 0 \quad j &= \left\lfloor\frac{2p+1}{3}\right\rfloor \\ i = 1 \quad j &= \left\lfloor\frac{2p-1}{3}\right\rfloor \\ i = 2 \quad j &= \left\lfloor\frac{2p-3}{3}\right\rfloor \end{aligned}$$

Hence, the maximum sum possible is the maximal value from $\lfloor\frac{2p+1}{3}\rfloor, 1 + \lfloor\frac{2p-1}{3}\rfloor$, and $2 + \lfloor\frac{2p-3}{3}\rfloor$. □

Proposition 2.3.20. *Let* $q = max\{\lfloor\frac{2p+1}{3}\rfloor, 1+\lfloor\frac{2p-1}{3}\rfloor, 2+\lfloor\frac{2p-3}{3}\rfloor\}$ *and consider some* $k \leqslant n$. *If* $\sum_{(i,j)\in\overline{X}_k} c_{i,j,d}$ *is the sum taken over all elements in* $\overline{X}_k$, *then* $\sum_{(i,j)\in\overline{X}_k} c_{i,j,d} = \overline{\beta}_k C_{d-(i+j+2)}$.

Proof. By Lemma 2.3.14, we have $c_{i,j,d} = C_{d-(i+j+2)}$ for all $(i,j) \in \overline{X}_k$. Since $\overline{\beta}_k$ denotes the cardinality of $\overline{X}_k$ and no skew monomial counted in $c_{i,j,d}$ is counted in $c_{i'j',d}$ for some $i \neq i'$ or $j \neq j'$, the result follows. □

We are now ready to define the recurrence for C_d. The purpose for defining the sets X_k and $\overline{X}_k$ is to provide a partition of X. This allows us to determine how many $c_{i,j,d}$ are equal to $C_{d-(i+j+1)}$ and how many are equal to $C_{d-(i+j+2)}$.

Theorem 2.3.21. *Let $A = R[f;F]$ where $R = \mathbb{F}_p[t^2,t^3]$ and let W_d be the growth subspace of V^d where V is the finite dimensional generating subspace spanned by $U \cup \{f\}$. Let C_d =dim$_K(W_d)$, then*

$$C_d = \sum_{k=0}^{m} \beta_k C_{d-(k+1)} + \sum_{k=m'}^{q} \overline{\beta}_k C_{d-(k+2)} + 3$$

where $m = max\{\lfloor \frac{p+1}{3} \rfloor, 1 + \lfloor \frac{p-1}{3} \rfloor, 2 + \lfloor \frac{p-3}{3} \rfloor\}, m'$ =the minimum possible sum of $i+j$ such that $2i+3j = p$ or $2i+3j \geqslant p+2$, and q =max$\{\lfloor \frac{2p+1}{3} \rfloor, 1 + \lfloor \frac{2p-1}{3} \rfloor, 2 + \lfloor \frac{2p-3}{3} \rfloor\}$

Proof. We first show a relation between the sets X, X_k, and $\overline{X}_k$ for some value k. By definition, for any value k we have $X_k \cap \overline{X}_k = \emptyset$. Further, we have

$$X = \bigcup_{k=0}^{q} \left(X_k \cup \overline{X}_k\right) = \left(\bigcup_{k=0}^{m} X_k\right) \bigcup \left(\bigcup_{k=0}^{q} \overline{X}_k\right)$$

Now since this is a disjoint union, the carinality is the sum of each β_k and $\overline{\beta}_k$. Hence, β_k will determine how many times C_{k+1} is equal to some $c_{i,j,d}$. Likewise, $\overline{\beta}_k$ will determine how many times $C_{d-(k+2)}$ is equal to some $c_{i,j,d}$. Thus, the values β_k and $\overline{\beta}_k$ will determine the coefficients of the recurrence. Now we can take the equation in Proposition 2.3.12 and for each (i,j) in X_m, we make the substitution $c_{i,j,d} = C_{d-(i+j+1)}$ and for each (i,j) in $\overline{X}_m$ we make the substitution $c_{i,j,d} = C_{d-(i+j+2)}$. Then applying Propositions 2.3.17 and 2.3.20 we get the coefficients. □

Corollary 2.3.22. *The degree of the growth recurrence for $R = \mathbb{F}_p[t^2, t^3]$ is*

$$max\left\{ \left\lfloor \frac{2p+1}{3} \right\rfloor, 1 + \left\lfloor \frac{2p-1}{3} \right\rfloor, 2 + \left\lfloor \frac{2p-3}{3} \right\rfloor \right\} - 2$$

Proof. By Lemma 2.3.19, the maximum sum possible for $i + j$ is $q = \max\{\lfloor \frac{2p+1}{3} \rfloor, 1 + \lfloor \frac{2p-1}{3} \rfloor, 2 + \lfloor \frac{2p-3}{3} \rfloor\}$. Then by Lemma 2.3.14, the ordered pair that produces this sum is in correspondence to C_{d-q-2}. Hence, the recurrence includes the terms $C_{d-1}, \cdots, C_{d-q-2}$ and the degree of the recurrence is $q - 2$. □

Corollary 2.3.23. *Let $A = R[f; F]$ where $R = \mathbb{F}_p[t^2, t^3]$ and let W_d be the growth subspace of V^d where V is the finite dimensional generating subspace spanned by $U \cup \{f\}$. Then the growth recurrence of R with respect to U and F is*

$$d_{U,F}(n) = \sum_{k=0}^{m} \beta_k d_{U,F}(n - (k+1)) + \sum_{k=m'}^{q} \overline{\beta}_k d_{U,F}(n - (k+2)) + \widetilde{T}(n)$$

where $\widetilde{T}(n)$ is a linear function, and the growth equation of R with respect to U and F is

$$x^{q+2} - \Big(\sum_{k=0}^{m} \beta_k x^{(q+2)-(k+1)} + \sum_{k=m'}^{q} \overline{\beta}_k x^{(q+2)-(k+2)} \Big)$$

Proof. We obtain the growth recurrence of R by applying Proposition 1.4.1 to the recurrence in Theorem 2.3.21. Also, if we let B be the constant given in the nonhomogeneous portion of the recurrence by Proposition 1.4.1, then $\widetilde{T}(n) = \sum_{i=0}^{n} 3 - B = 3n - B$ and $\widetilde{T}(n)$ is linear.

The growth equation is defined to be the characteristic equation determined by the associated homogenous growth recurrence for R. The associated growth recurrence for R is $\sum_{k=0}^{m} \beta_k C_{d-(k+1)} + \sum_{k=m'}^{q} \overline{\beta}_k C_{d-(k+2)}$ and the result follows by applying Theorem 1.4.2. □

We know provide a result that shows the possible values for the β_k. This will allow us to apply Ostrovsky's theorem to guarantee the existence of a unique positive root for the growth equation of R with respect to U and f.

Lemma 2.3.24. *Let $\beta_k, \overline{\beta}_k$ denote the cardinalities of X_k and $\overline{X}_k$ respectively and $m = max\{\lfloor \frac{p+1}{3} \rfloor, 1 + \lfloor \frac{p-1}{3} \rfloor, 2 + \lfloor \frac{p-3}{3} \rfloor\}$. Then $0 \leqslant \beta_k, \overline{\beta}_k \leqslant 3$ for all $0 \leqslant k \leqslant m$. Further, we have that $\beta_0 = 1$, $\beta_1 = 2$.*

Proof. Let k be any value in $\{0, 1, 2, \cdots, m\}$. It is given by lemma 2.7.2 that $i = 0, 1, 2$ and we have the following possible pairs from X_k: $(0, k), (1, k-1)$, and $(2, k-2)$. Since the values for i are limited to these three values, these are the only possible options and $\beta_k \leqslant 3$. The only pair (i, j) with a sum of 0 is $(0, 0)$ and $\beta_0 = 1$. Also, for any prime p, the skew monomial $(t^2)^0(t^3)^0 f = f$ is in irreducible form. Thus, this holds for all p. We have two pairs (i, j) wth a sum of 1, $(1, 0)$ and $(0, 1)$. Hence, $\beta_1 = 2$. These pairs are associated to the skew monomials $t^2 f$ and $t^3 f$ which are both in irreducible form for all p. □

Corollary 2.3.25. *The growth equation of R with respect to U and F has a unique positive root, λ_1, and GKbase$_{U,F}(R) = \lambda_1$*

Proof. We can apply Theorem 1.4.3 and Theorem 1.4.4 to the growth equation of V since $\beta_0 = 1$ and $\beta_1 = 2$ by Lemma 2.3.24. Hence, the gcd of the negative coefficients is 1. Thus, the growth equation has a unique positive root, say λ_1 that is larger than the absolute value of all other roots. Then by Theorem 1.4.2, the term $c_1\lambda_1^n$ appears in the general solution for the homogeneous recurrence of V and will dominate all other terms. Also, since $\widetilde{T}(n)$ is a linear function, the particular solution for the growth recurrence of V is linear and will be dominated by the term $c_1\lambda_1^n$. Thus, we can conclude that GKbase$_{U,F}(R) = \lambda_1$. □

We now provide an example computation for a semigroup ring.

Example 2.3.26. Let $R = \mathbb{F}_5[t^2, t^3]$ with $U = \{t^2, t^3\}$. Let $A = R[f; F]$ be the Frobenius skew polynomial ring over R with V the finite dimensional K-subspace of A spanned by $U \cup \{f\}$.

Then by Theorem 2.3.21, we have the following growth recurrence for R with respect to U and F:

$$C_d = \sum_{k=0}^{m} \beta_k C_{d-(k+1)} + \sum_{k=m'}^{q} \overline{\beta}_k C_{d-(k+2)} + 3$$

where $m = 2, q = 4$ and $m' = 2$. Thus,

$$C_d = \beta_0 C_{d-1} + \beta_1 C_{d-2} + \beta_2 C_{d-3} + \overline{\beta}_2 C_{d-4} + \overline{\beta}_3 C_{d-5} + \overline{\beta}_4 C_{d-6}$$

We can determine the values of the coefficients by considering the sets they correspond to.

- $(0,0) \in X_0$ and $\beta_0 = 1$
- $(1,0),(0,1) \in X_1$ and $\beta_1 = 2$
- $(0,2),(2,0) \in X_2$ and $\beta_2 = 2$
- $(1,1) \in \overline{X}_2$ and $\overline{\beta}_2 = 1$
- $(0,3),(1,2),(2,1) \in \overline{X}_3$ and $\overline{\beta}_3 = 3$
- $(1,3) \in \overline{X}_4$ and $\overline{\beta}_4 = 1$

Now the growth recurrence becomes

$$C_d = C_{d-1} + 2C_{d-2} + 2C_{d-3} + 1C_{d-4} + 3C_{d-5} + C_{d-6}$$

and the growth equation is

$$x^6 - x^5 - 2x^4 - 2x^3 - x^2 - x - 1 = 0$$

We find the unique positive root using SAGE and obtain $\text{GKbase}_{U,F}(R) = 2.34301$.

We end with an interesting observation regarding the GKbase for early p values and how the rings $K[x], K[x_1, x_2]$ and $K[t^2, t^3]$ relate.

p	$K[x]$	$K[t^2, t^3]$	$K[x_1, x_2]$
2	1.61803	1.92756	2.14790
3	1.83929	2.09431	2.46536
5	1.96595	2.34301	2.59832

Hence, based on this small data set, it appears that the GKbase for the semi group ring falls between the GKbase for the polynomial ring in one variable and the polynomial ring in two variables. This matches a general intuition since the GKbase is a value derived from the growth recurrence which is counting generators from each ring.

CHAPTER 3

CONCLUSIONS

3.1 Discussion and Questions

We have now defined a new invariant for the Frobenius skew polynomial rings presented in this paper. It is clear that the invariant is closely related to the size of the generating sets necessary for a power of a finitely generated finite dimension K-subspace. However, it is not clear exactly what the GK base is determining. The combination of the fact that the Gelfand-Kirillov dimension and superdimension provided no useful information about the rings of interest in this paper and the fact that this GK base arose naturally in the process provides hope that this invariant is a meaningful aspect of the ring. The results given in this paper are enticing enough to give motivation to continue research in this area.

Now that it is established that this invariant defined as the GK-base exists in basic cases, there are key questions that could be topics for future research. First, we find the GK base by determining recurrences for subspaces determined by generating sets of the K-algebra. This is shown for the commutative polynomial ring and a basic semigroup ring. It is natural to then ask, for what rings is it possible to define a recurrence in this way?

Question 3.1.1. Do all commutative Noetherian rings which are K-algebras over finite field K have a set of generators U that admit a nonhomogeneous linear recurrence for $d_{U,F}(n)$?

The existence of a nonhomogeneous linear recurrence does not guarantee that the GK base will exist as the nonhomogeneous could cause issues in the solution. So we provide a follow up question.

Question 3.1.2. Is there a set of generators V of R such that the GK-base for the generating set exists?

Without a recurrence, it is not clear the GK base will arise naturally. The hope is that conditions can be determined for the base ring R that will guarantee the existence of a recurrence. Then the existence of a more general GK base can be investigated for the class of rings where a recurrence is possible.

The next pressing question on the results provided in this paper relates to the subspace V used for each of the examples. Throughout the paper, the results are dependent on the choice of V.

Question 3.1.3. If there exist two sets of generators V and W for R such that the GK base exists, does $\text{GKbase}_{V,F}(R) = \text{GKbase}_{W,F}(R)$?

It is not clear that the dependence can be removed. So the natural questions here are can the dependence on V be removed, or what does the GK base illuminate if the dependence on V cannot be removed?

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