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Berge - Fulkerson Conjecture And Mean Subtree Order

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by

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ABSTRACT

Let G be a graph, $V(G)$ and $E(G)$ be the vertex set and edge set of G, respectively. A perfect matching of G is a set of edges, $M \subseteq E(G)$, such that each vertex in G is incident with exactly one edge in M. An r-regular graph is said to be an r-graph if $|\partial(X)| \geq r$ for each odd set $X \subseteq V(G)$, where $|\partial(X)|$ denotes the set of edges with precisely one end in X. One of the most famous conjectures in Matching Theory, due to Berge, states that every 3-graph G has five perfect matchings such that each edge of G is contained in at least one of them. Likewise, generalization of the Berge Conjecture given, by Seymour, asserts that every r-graph G has $2r-1$ perfect matchings that covers each $e \in E(G)$ at least once. In the first part of this thesis, I will provide a lower bound to number of perfect matchings needed to cover the edge set of an r-graph. I will also present some new conjectures that might shade a light towards the generalized Berge conjecture. In the second part, I will present a proof of a conjecture stating that there exists a pair of graphs G and H with $H \supset G$, $V(H) = V(G)$ and $|E(H)| = |E(G)| + k$ such that mean subtree order of H is smaller then mean subtree order of G.

INDEX WORDS: Cubic graphs, r-graphs, Generalized Berge and Fulkerson Conjectures, Perfect Matching Polytope, Subtree, Mean subtree order

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CHAPTER 1
INTRODUCTION

Any definitions presented in this chapter can be found in any textbook about Graph Theory.

A graph G is an ordered triple $(V(G), E(G), \Psi(G))$ composed of a nonempty set of vertices $V(G)$, a set of edges $E(G)$ disjoint from $V(G)$, and an incidence function $\Psi(G)$ that associates each edge with an unordered pair of not necessarily distinct vertices. In this thesis, we are concerned only with finite graphs, namely $V(G)$ and $E(G)$ are finite. Although we permit multiple edges, graphs in this text are without loops. A graph H is a *subgraph* of G, denoted by $H \subseteq G$, if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$, and Ψ_H is the restriction of Ψ_G to $E(H)$. This thesis consists of two chapters concerning about subgraphs for which I will introduce the results from my last three years of research.

Let v be a vertex of a graph G. The *degree* of v, denoted by $d(v)$, is the number of edges incident with v. The *maximum degree* $\Delta(G)$ of a graph G is defined as $\Delta(G) = max\{d(v) : v \in V(G)\}$. A graph is said to be *k-regular* if $d(v) = k$ for all $v \in V(G)$. Specifically a 3-regular graph is called a *cubic* graph.

The first chapter of this thesis is about matchings of some specific families of graphs where a *matching* or an *independent edge set* in a graph G is a set of edges without common vertices. A *perfect matching* in a graph G is a matching such that every vertex of the graph is incident to an edge of the matching. For convenience an empty subset M of $E(G)$ of a graph G is also considered as a matching. Problems related to matchings have received continuous attention in Graph Theory. One of the earliest results in this area, due to Petersen (21), states that every bridgeless cubic graph has a perfect matching.

Although every graph has a matching, not every graph has a perfect matching. If a graph

G has a perfect matching, $|V(G)|$ must be even. Clearly if $|V(G)|$ is odd then any matching leaves at least one vertex that is not matched. Generally if a disconnected graph G has at least one component which has odd number of vertices, then G can not have a perfect matching.

Moreover consider a graph G with a vertex v such that removal of v results at least two components with odd number of vertices in each of them. Since any matching leaves at least one vertex that is not matched in these odd components, say v_1 and v_2, they can only be matched to v. Since there are at least two odd components, there must be at least one vertex, say v_2, that is not matched by any of the matchings of G.

The same idea can also be applied to any set of vertices U such that if we remove U, the resulting graph has at least $|U| + 1$ components with odd number of vertices in each of them. Hence we can conclude that, if a graph G has a perfect matching, then for every vertex set $U \subseteq V(G)$, the graph $G - U$ has at most $|U|$ components with odd number of vertices. The well-known Tutte's theorem (25) states that this necessary condition is also sufficient for existence of a perfect matching in a graph. In Chapter 1, we mainly discuss to cover the edge set of a graph by matchings, especially perfect matchings.

A k-edge-coloring of a graph G is an assignment of k colors to the edges of G such that every two adjacent edges receive distinct colors. The least number k for which G can be colored by k edges is called the *chromatic index* of G, written $\chi'(G)$. It is obvious that $\chi'(G) \geq \Delta(G)$. One of the most important results on edge coloring was given by Vizing (27) for the chromatic index of a graph: if G is a simple graph, then $\chi'(G) = \Delta(G)$ or $\Delta(G)+1$; and if G is a multigraph, then $\chi'(G) \leq \Delta(G)+\mu(G)$, where $\mu(G)$ is the maximum multiplicity of edges of G.

It is clear that, in a coloring of a graph G, a set of edges that receive the same color is a matching. Therefore the edge set of a graph can be covered by at least $\chi'(G)$ matchings. The problem I have interested in is that what is the minimum number of perfect matchings to cover the edge set of a cubic bridgeless graph and more generally an r-graph which I will define later. Although the answer to this question is not known, we do know that, as an implication of the Tutte's theorem, every edge of a bridgeless cubic graph belongs to a perfect matching. This immediately gives a trivial upper bound, $|E(G)|$, for the number of perfect matching that covers the edge set of a bridgeless cubic graph. It was conjectured by Claude Berge that every bridgeless cubic graph G has five perfect matchings that covers $E(G)$. In 1979 Seymour (23) generalized the Berge Conjecture (Generalized Berge Conjecture) to the class of r-graphs which defined as r-regular graphs for which $|\partial(X)| \geq r$ for all $X \subseteq V(G)$ with $|X|$ odd. Seymour conjectured that every r-graph has $2r-1$ perfect matchings such that each edge is contained at least one of them. In the first chapter, our aim is to establish a lower bound for the maximum proportion of edges of an r-graph that is covered by k perfect matchings over all r graphs, where $k > 0$.

In second chapter of this thesis I will present results about subgraphs like in first chapter. A graph is a *path* if its vertices can be arranged in a linear sequence such that two vertices are adjacent if and only if they are consecutive in the sequence. The first and the last vertices of a path are called the *ends* of the path. If the vertices u and v are the ends of a path P, then P is called an uv-path. A *tree* is a graph in which every pair of distinct vertices is connected by exactly one path. A subtree of a graph G is a subgraph of G that is a tree. By convention, the empty graph is not considered as a subtree of any graph. The *mean subtree order* of a graph G, denoted $\mu(G)$, is the average order of a subtree of G. Jamison, in (12), (13), initiated the study of the mean subtree order in the 1980s, considering only the case when G is a tree. Jamison in (12) proved that $\mu(T) \geq \frac{n+2}{3}$ for any tree T of order n. He further showed that this minimum is achieved if and only if when T is a path.

Recently, Chin, Gordon, MacPhee, and Vincent (4) initiated the study of subtrees of graphs not only for trees but in general. They believed that the parameter μ is monotonic with respect to the inclusion relationship of subgraphs. More specifically, they conjectured that for any simple connected graph G, adding any edge to G will increase the mean subtree order. Clearly, the truth of this conjecture implies that mean subtree order of the complete graph on n vertices is the maximum among all connected simple graphs of order n. We have compared the mean subtree order of pairs of graphs G and H such that $G \subset H$, $V(G) = V(H)$ and $|E(H) \setminus E(G)| = k$. More specifically we confirmed that there are infinitely many pairs of graphs G and H where the graph H is obtained from G by adding k edges to G such that $\mu(H) < \mu(G)$.

CHAPTER 2
GENERALIZED BERGE - FULKERSON CONJECTURE

2.1 Introduction

In graph theory, one of the most well-known problems is the four color problem, which states that given any separation of a plane into regions that sharing common borders, four colors would be enough to color all the regions so that no two neighboring regions receive the same color. This problem was first proposed by Guthrie (9) in 1852. Despite its very simple statement, the history of the four color problem has made clear its great difficulty, with many attempts to solve it and several incorrect proofs. It was finally solved by Appel and Haken (1)(2) in 1976 with the assistance of computer. Hence it is a theorem now. In the language of graph theory, the Four Color Theorem (the 4CT) simply states that every loopless planar graph admits a 4-vertex-coloring. Later it was proved by Tait that the 4CT is equivalent to the statement that every bridgeless cubic planar graph admits a 3-edge-coloring. Tait's work gave the origins of the term Tait-colouring (3-edge-colouring) for cubic graphs. This reduction motivated the search for bridgeless cubic graphs that do not admit a 3-edge-colouring, which we will call later as snarks.

The first snark was discovered by Petersen in 1898 which named after him, the Petersen Graph. It was the first counter-examaple to the conjecture of every bridgeless cubic graph is 3-edge-colorable. After the first discovery of the Petersen Graph, history of the "snark hunting" started. To this date several snark families discovered such as Blanuśa snarks (19), Descartes snarks (5), Isaacs snarks (11) and Szekeres snarks (24). For any snark G, although G is not a class 1 graph, it was conjectured by Berge (unpublished) but first published in (7) by Fulkerson that if one double every edge of any bridgeless cubic graph then the resulting graph, denoted by $2G$, is of class 1. Another way of stating that is for any bridgeless cubic graph G, $\chi'(2G) = 6$, or G has six perfect matchings such that every edge of G is covered exactly twice by them. We will call this conjecture as Fulkerson conjecture.

As mentioned in the Chapter 1, by the Petersen theorem (21) we know that every bridgeless cubic graph has a perfect matching. Another well known theorem due to Tutte (25) states that a graph G has a perfect matching if and only if for every $X \subseteq V(G)$, $G - X$ has at most $|X|$ components with odd number of vertices. As a corollary of the Petersen theorem Schönberger proved in (22) that every edge of a bridgeless cubic graph G is contained in a perfect matching of G. This yields the trivial case that $|E(G)|$ many perfect matchings covers the edge set of a bridgeless cubic graph G. Can we do better? Or more precise question would be: what is the minimum number of perfect matchings so that every edge of a bridgeless cubic graph is covered by the union of them? In the early seventies Berge conjectured that this number is 5.

A natural generalizations of the Fulkerson and the Berge conjectures to r-graphs given by Seymour in (23) are as follows: Every r-graph has $2r$ perfect matchings such that each edge is contained in exactly two of them, and every r-graph has $2r - 1$ perfect matchings such that each edge is contained at least one of them.

Although it is still unknown the existence of an integer t such that for any r-graph G having t perfect matchings covering the edge set of G. In this thesis we mainly discuss maximum fraction of edges that can be covered by t perfect matchings. In the first section of this chapter, we will provide some conjectures and partial results about union of perfect matchings in bridgeless cubic graphs, and in section 2.3 we will generalize these results to

r-graphs.

2.2 Cubic Graphs

For a cubic graph G with $\chi'(G) = 3$, each color class induces a perfect matching. Therefore edge set $E(G)$ of G can be covered by three perfect matchings. Apart from this trivial case, we turn our attention to cubic graphs that are not 3-edge-colorable. A *snark* is a bridgeless cubic graph that is not 3-edge-colorable. The Petersen graph is the first known snark. It is also the smallest snark and serves as a useful example or counter-example to many problems.

The Petersen graph has six distinct perfect matchings $M_1, M_2, \ldots, M_6$ such that $|M_i \cap M_j| = 1$ for $i \neq j$ and these six perfect matchings cover each edge precisely twice (see figure 2.1). Clearly union of any five of the six perfect matchings of the Petersen graph covers $E(P)$.

Although the family of snarks are of class 2, it was conjectured by Berge but published in 1971 by Fulkerson (8) states that if G is a bridgeless cubic graph then the graph $2G$ obtained from G by doubling each edge is class 1.

Conjecture 2.2.1 (Fulkerson)**.** *For every bridgeless cubic graph G, $\chi'(2G) = 6$.*

Another way of stating conjecture 2.2.1 is that every bridgeless cubic graph G has a family of six perfect matchings such that each edge of the graph contained in precisely two of them. A set six perfect matching satisfying the Conjecture 2.2.1 is called a Fulkerson-cover. One of the immediate implications of Conjecture 2.2.1 is that any five perfect matchings of a Fulkerson-cover of a bridgeless cubic graph G covers $E(G)$. That naturally yields the following conjecture attributed to Berge:

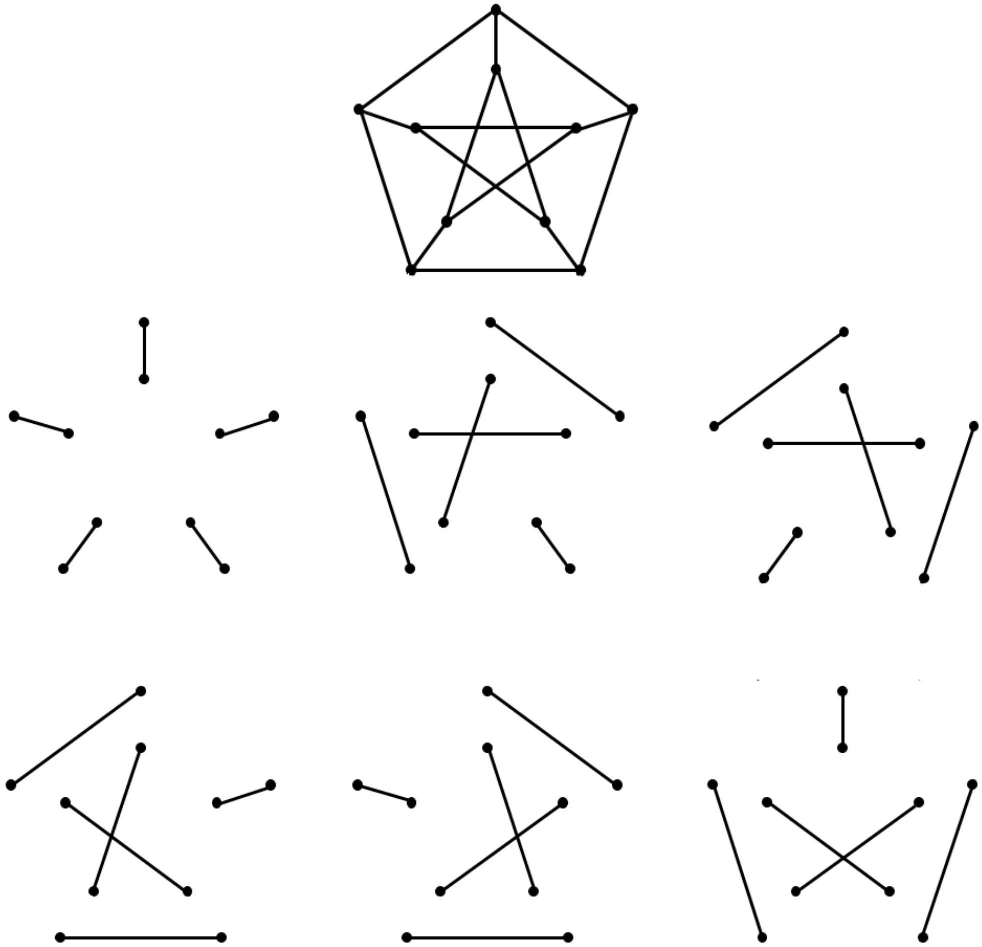

Figure 2.1 The Petersen Graph and its six perfect matchings

Conjecture 2.2.2 (Berge)**.** *Every bridgeless cubic graph G has five perfect matchings such that each edge of G is contained in at least one them.*

By the observation above Conjecture 2.2.1 implies Conjecture 2.2.2. It was proved by Mazzuoccolo in (15) that Conjecture 2.2.1 and Conjecture 2.2.2 are actually equivalent.

Although the truth of Conjecture 2.2.2 is unknown to this date, there is a more general approach, namely what is the maximum proportion of edges of a bridgeless cubic graph that can be covered by $t > 0$ perfect matchings. Let $m_t^3(G)$ be this number for a bridgeless cubic graph G. Note that the superscript 3 is due to the fact that we are concerning 3 regular graphs in this section. In the next section, when we generalize the ideas presented here, we will use superscript $r \geq 3$. Given any bridgeless cubic graph G, since any perfect matching covers $\frac{1}{3}|E(G)|$ many edges of G, then $m_1^3(G) = \frac{1}{3}$. Another way of stating the Conjecture 2.2.2 is that there exists 5 perfect matchings in every bridgeless cubic graph G so that $m_5^3(G) = 1$. We further define (actually first presented in (14)) the parameter m_t^r to be the infimum of all $m_t^r(G)$ over all bridgeless cubic graphs, that is

$$m_t^3(G) = \max_{M_1,\ldots,M_t} \frac{\left|\bigcup_{i=1}^{t} M_i\right|}{|E(G)|}, \text{and} \quad m_t^3 = \inf_G m_t^3(G).$$

As mentioned earlier, as a corollary of Petersen theorem, we have every edge e of a bridgeless cubic graph G is contained in a perfect matching. That trivially yields $m^3_{|E(G)|}(G) = m^3_{\frac{3n}{2}}(G) = 1$, where $n = |V(G)|$. The goal here is to find a universal constant t_0 such that $m^3_{t_0} = 1$.

Let P denote the Petersen Graph. We observed earlier that $m_1^3(G) = \frac{1}{3}$, for any bridgeless cubic graph, that is true for the Petersen Graph as well, so we have $m_1^3(P) = \frac{1}{3}$. Since the Petersen graph has exactly six distinct perfect matchings, every edge being contained in exactly two of them, and every pair of distinct perfect matchings intersecting in one edge, one can easily observe that $m_2^3(P) = \frac{3}{5}$, $m_3^3(P) = \frac{4}{5}$, $m_4^3(P) = \frac{14}{15}$, $m_5^3(P) = 1$. By the definition of m_t^3 and the fact that the Petersen Graph is a bridgeless cubic graph, we can easily see that $m_t^3 \leq m_t^3(P)$ for every integer $t \geq 1$. The following conjecture given by Patel in (20) is a refinement of Conjecture 2.2.2 and states that the converse $m_t^3 \geq m_t^3(P)$ is also true.

Conjecture 2.2.3 ((20), Patel)**.** *$m_t^3 = m_t^3(P)$ for $1 \leq t \leq 5$, where P is the Petersen Graph.*

Patel in (20) proved that Conjecture 2.2.1 implies Conjecture 2.2.3 and clearly Conjecture 2.2.3 implies Conjecture 2.2.2. Conjecture 2.2.3 was proved by Kaiser, Král, and Norine in (14) for the case $t = 2$. They also showed that $\frac{27}{35} \leq m_3^3 \leq \frac{4}{5}$ and predicted that $\frac{27}{35} \leq m_3^4 \leq \frac{4}{5}$ by using the Edmonds' perfect matching polytope theorem which we will introduce in section 2.4. Mazzuoccolo confirmed the lower bound $\frac{27}{35}$ in (17). The exact values for m_3^3, m_4^3 and m_5^3 are still unknown. The following lower bound for m_t^3 was given by Mazzuoccolo in (17), for any integer index $t \geq 0$: $m_t^3 \geq a_t^3$, where $a_0^3 = 0$ and a_t^3 satisfies the following recursive formula;

$$a_t^3 = a_{t-1}^3 + (1 - a_{t-1}^3)\frac{2 + 2(t-1)}{6 + 4(t-1)} = a_{t-1}^3 + (1 - a_{t-1}^3)\frac{t}{2t+1}.$$

2.3 A Generalization of Bridgeless Cubic Graphs

In this section we mainly focus on r-regular graphs with $r \geq 3$ that has perfect matchings. Note that not all r- regular graphs have perfect matchings. For instance the complete graphs with odd number of vertices do not have any perfect matching. Similar to cubic case, it is difficult to determine which graphs, regular of degree r, are r edge colorable. Let G be an r-regular graph that have perfect matching with $\chi'(G) = r$, then for any odd cardinality $X \subseteq V(G)$, every perfect matching in G intersects with $\partial(X)$ at least once, where $\partial(X)$ is the set of edges with precisely one end in X. Hence $|\partial(X)| \geq r$. However this condition is not sufficient.

An r-regular graph G is said to be an r-graph if $|\partial(X)| \geq r$ for each odd set $X \subseteq V(G)$.

First note that every r-graph G has even number of vertices; for otherwise since one can simply take $X = V(G)$ in the definition that yields $\partial(X) = \emptyset$ which results a contradiction. Further notice that every bridgeless cubic graph G is a 3-graph, since G is 3-regular and for any odd cardinality $X \subseteq V(G)$, $3 \cdot |X| = \sum_{v \in X} d_G(v) = 2|E(G[X])| + |\partial(X)|$, resulting $|\partial(X)|$ must be odd. Since G is bridgeless, $|\partial(X)| \geq 2$, which in turn gives $|\partial(X)| \geq 3$ and hence G is a 3-graph.

Recall the well-known Petersen theorem for bridgeless cubic graphs that guaranteeing the existence of a perfect matching. That can be generalized to the class of r-graphs where $r \neq 0$; Let G be an r-graph. By the Tutte's theorem, G has a perfect matching if and only if for any $X \subseteq V(G)$, $G - X$ has at most $|X|$ components with an odd number of vertices. Let $X_1, \ldots, X_t$ be the vertex sets of the components of $G - X$ with odd number of vertices. Note that $\partial(X_i) \subseteq \partial(X)$ and because G is an r-graph we have $|\partial(X_i)| \geq r$. Hence $|\partial(X)| \geq rt$. On the other hand trivially $|\partial(X)| \leq r|X|$, that results $t \leq |X|$. Applying the Tutte's theorem to any r-graph, we see that every edge $e \in E(G)$ is contained in a perfect matching in G. In addition to this Seymour in (23), as a corollary of Edmonds' perfect matching polytope theorem, which we will discuss in the next section, proved that for every r-graph G there exists an integer $p := p(G)$ such that $\chi'(tG) = tr$. In other words, every r-graph G has a set of pr perfect matchings such that each edge of G is covered exactly p times. The fact that every edge of an r-graph contained in a perfect matching is a immediate consequence of this theorem of Seymour. Furthermore, Seymour conjectured that $p = 2$ or 1 for any r-graph. Namely every r-graph has $2r$ perfect matchings such that each edge is contained in exactly two of them.

Conjecture 2.3.1 ((23), Generalized Fulkerson Conjecture)**.** *For every r-graph G, $\chi'(2G) = 2r$.*

Note that since every bridgeless cubic graph is 3-graph, then Conjecture 2.3.1 is simply Berge Fulkerson conjecture for $r = 3$. In order to support Conjecture 2.3.1, we would like to give some examples. Let G be a 2-graph, that is disjoint union of even cycles. Then clearly G is of class 1 or simply $\chi'(G) = 2$ and $\chi'(2G) = 4$. For another example, take $G = K_{r+1}$ with r is odd or $G = K_{r,r}$, then it is easy to observe that $\chi'(2K_{r+1}) = 2r = \chi'(2K_{r,r})$.

In (23) Seymour conjectured that if $r \geq 4$, any r-graph has a perfect matching such that its deletion gives an $(r-1)$-graph. Note that the condition $r \geq 4$ is necessary here, since for instance removal of any perfect matching of the Petersen graph does not result in a 2-graph. In the light of this conjecture, we make the following stronger version of it.

Conjecture 2.3.2. *For any $r \geq 4$ and every $r_1, r_2 \geq 1$ with $r_1 + r_2 = r$, any r- graph $G = G_1 \cup G_2$ where G_i is an r_i graph for $i = 1, 2$.*

We will now extend the definition discussed in the previous subsection for m_t^3 to the class of r-graphs. For a fixed positive integer r, let $m_t^r(G)$ be the maximum fraction of the edges in an r-graph G that can be covered by t perfect matchings, and m_t^r be the infimum of all $m_t^r(G)$ over all r-graphs, that is

$$m_t^r(G) = \max_{M_1,\ldots,M_t} \frac{\left|\bigcup_{i=1}^{t} M_i\right|}{|E(G)|}, \text{and} \quad m_t^r = \inf_G m_t^r(G),$$

where the infimum is taken over all r-graphs.

Notice that taking any $2r-1$ of the $2r$ perfect matchings in Conjecture 2.3.1 covers each edge of an r-graph G at least once. Mazzuoccolo in (16) showed that Conjecture 2.3.1 and having such $2r-1$ perfect matchings are equivalent and the value $2r-1$ is best possible.

Conjecture 2.3.3 (Generalized Berge Conjecture)**.** *Every r-graph has $2r-1$ perfect matchings such that each edge is contained at least one of them.That is $m^r_{2r-1}(G) = 1$.*

A natural question to ask in the light of the Generalized Berge Conjecture is that what can we say about the proportion of edges of an r-graph that can be covered by union of t perfect matchings? In section 2.5 we provide a recursive lower bound to this question by proving the following theorem:

Theorem 2.3.4. *For any fixed integer $r \geq 3$, let $a^r_0, a^r_1, \ldots$ be a sequence of rational numbers satisfying $a^r_0 = 0$ and*

$$a^r_t = a^r_{t-1} + (1 - a^r_{t-1})\frac{2 + (t-1)(r^2 - r - 4)}{2r + (t-1)(r^3 - r^2 - 6r + 4)}.$$

Then for any positive integer t, we have $m^r_t \geq a^r_t$.

Recall that Seymour conjectured that $m^r_{2r-1} = 1$ for any $r \geq 3$. By direct calculation using Theorem 2.3.4, we have the following lower bounds for m^r_{2r-1} for some values of r;

r	a^r_{2r-1}
3	0.930736
4	0.897367
5	0.885256
6	0.878973
7	0.875227
100	0.864721
1000	0.864665

Table 2.1 Some lower bounds for m^r_{2r-1}

2.4 The perfect matching polytope

Let $G = (V, E)$ be a graph which may contain multiple edges. For any set $C \subseteq E(G)$, if $G - C$ has more components than G, then C is said to be a *cut* in G. A k-cut is a cut consists of k edges. Recall that $\partial(X)$ is defined as the set of edges with precisely one end in $X \subseteq V(G)$. A cut C is *odd* if there exists $X \subseteq V(G)$ of odd cardinality such that $C = \partial(X)$. Notice that if G is an r-graph and $X \subseteq V(G)$ is an odd cardinality set, then r and $|\partial(X)|$ have the same parity.

Let w be a vector in $\mathbb{R}^{E(G)}$. The entry of w corresponding to an edge e is denoted by $w(e)$, and for $A \subseteq E(G)$, we define $w(A) = \sum\limits_{e \in A} w(e)$. The vector w is a fractional perfect matching of G if it satisfies the following properties:

(1) $0 \leq w(e) \leq 1$ for each $e \in E(G)$,

(2) $w(\partial(v)) = 1$ for each vertex $v \in V(G)$,

(3) $w(\partial(X)) \geq 1$ for each $X \subseteq V(G)$ of odd cardinality.

Note that $w = (\frac{1}{r}, \frac{1}{r}, \ldots, \frac{1}{r})$ is a fractional perfect matching for any r-graph G.

Let $P(G)$ denote the set of all fractional perfect matchings of G. Clearly, if M is a perfect matching, then the characteristic vector χ^M of M is contained in $P(G)$. Also, if $w_1, \ldots, w_n \in P(G)$, then any convex combination, $\sum_i \lambda_i w_i$ with $0 \leq \lambda_i \leq 1$ such that $\sum_i \lambda_i = 1$, of them belongs to $P(G)$. So $P(G)$ contains the convex hull of all vectors χ^M such that M is a perfect matching of G. The Perfect Matching Polytope Theorem of Edmonds (6) asserts that the converse inclusion also holds:

Theorem 2.4.1 (Perfect Matching Polytope Theorem)**.** *For any graph G, the set $P(G)$ is precisely the convex hull of the characteristic vectors of perfect matchings of G.*

The following lemma deducted from Edmonds' Perfect Matching Polytope Theorem, which introduced by Kaiser, Král, and Norine in (14), plays a crucial role in our result.

Lemma 2.4.2. *(14) If w is a fractional perfect matching in a graph G and $c \in \mathbb{R}^E$, then G has a perfect matching M such that $c \cdot \chi^M \geq c \cdot w$, where $\cdot$ denotes the dot product . Moreover M contains exactly one edge of each odd cut C with $w(C) = 1$.*

2.5 Proof of Theorem 2.3.4

Let $\mathcal{M}^t = \{M_1, \ldots, M_t\}$ be a set of $t \geq 0$ perfect matchings in an r-graph G. Recall that for any positive integers $r \geq 3$ and $t \geq 0$, we defined

$$m_t^r = \inf_G \max_{M_1,\ldots,M_t} \frac{\left|\bigcup_{i=1}^{t} M_i\right|}{|E(G)|}$$

where the infimum is taken over all r-graphs. It is clear that $m_0^r = 0$ and $m_1^r = \frac{1}{r}$. For any fixed integer r we define $a_0^r = 0$ and for $t \geq 1$,

$$a_t^r = a_{t-1}^r + (1 - a_{t-1}^r) \frac{2 + (t-1)(r^2 - r - 4)}{2r + (t-1)(r^3 - r^2 - 6r + 4)}.$$

We will show that $m_t^r \geq a_t^r$ for each index t and fixed $r \geq 3$. Before presenting the proof let us give some definitions. Let G be an r-graph and $\mathcal{M}^t = \{M_1, \ldots, M_t\}$ be a set of t perfect matchings in G for $t \geq 0$. For each subset $A \subseteq E(G)$, we define

$$\Phi(A, \mathcal{M}^t) = \sum_{i=1}^{t} |A \cap M_i|.$$

We further define the function $w_t^r : E(G) \to \mathbb{R}$ for any fixed integers r and t as;

$$w_t^r(e) = \frac{2 + t\,(r^2 - r - 4) - 2\,(r-2)\,\Phi(e, \mathcal{M}^t)}{2r + t\,(r^3 - r^2 - 6r + 4)}$$

Observe that when $t = 0$, we have $\mathcal{M}^0 = \emptyset$ and $\Phi(\{e\}, \mathcal{M}^0) = 0$ for each $e \in E(G)$. Hence $w_0^r(e) = \frac{1}{r}$ which is a fractional perfect matching. We further note that Kaiser, Král, and Norine (14), used w_1^3 and w_2^3 in their proof for $m_2^3 = \frac{3}{5}$ and $\frac{27}{35} \leq m_3^3 \leq \frac{4}{5}$. We would like to remark that the recursive formula a_t^r and the function $w_t^r(e)$ defined above are exactly the generalizations of the ones given in (17).

A natural extension of the function w_t^r for a set $A \subseteq E(G)$ is defined as;

$$w_t^r(A) = \sum_{e \in A} w_t^r(e) = \frac{|A| \cdot [2 + t\,(r^2 - r - 4)] - 2\,(r-2)\,\Phi(A, \mathcal{M}^t)}{2r + t\,(r^3 - r^2 - 6r + 4)}.$$

Instead of proving Theorem 2.3.4 directly, we prove the following technical theorem which is slightly stronger.

Theorem 2.5.1. *For any r-graph G with $r \geq 3$ and any integer $t \geq 0$, there exists a set of t perfect matchings $\mathcal{M}^t = \{M_1, M_2, \ldots, M_t\}$ such that*

(i) the function $w_t^r : E(G) \to \mathbb{R}$ defined as

$$w_t^r(e) = \frac{2 + t\,(r^2 - r - 4) - 2\,(r-2)\,\Phi(e, \mathcal{M}^t)}{2r + t\,(r^3 - r^2 - 6r + 4)}$$

is a fractional perfect matching, and

(ii)

$$\frac{\left|\bigcup_{i=1}^{t} M_i\right|}{|E(G)|} \geq a_t^r,$$

which consequently yields $m_t^r \geq a_t^r$.

Proof: Let G be an r-graph. We prove (i) and (ii) simultaneously by induction on t.

For $t = 0$, let $\mathcal{M}^0 = \emptyset$. Then by the definition of $w_t^r(e)$, we have $w_0^r(e) = \frac{2}{2r} = \frac{1}{r}$ for any $e \in E(G)$ and as observed earlier w_0^r is a fractional perfect matching. Therefore (i) follows. Since by definition $a_0^r = 0$, (ii) holds trivially. Now suppose that $t \geq 1$ and let $\mathcal{M}^{t-1} = \{M_1, \ldots, M_{t-1}\}$ be a set of $t-1$ perfect matchings in G such that w_{t-1}^r is a fractional perfect matching. Let $c = 1 - \chi^{\bigcup_{i=1}^{t-1} M_i}$. By Lemma 2.4.2 there exists a perfect matching, M_t, in G such that $c \cdot \chi^{M_t} \geq c \cdot w_{t-1}^r$ and M_t contains exactly one edge of each odd cut $\partial(X)$ with $|X|$ odd and $w_{t-1}^r(C) = 1$.

In order to prove that $w_t^r(e)$ is a fractional perfect matching, we need to verify the three conditions in the definition of fractional perfect matching.

(1) for each edge $e \in E(G)$, it is clear that $\Phi(e, \mathcal{M}^t) \geq 0$ and therefore

$$w_t^r(e) = \frac{2 + t\,(r^2 - r - 4) - 2\,(r-2)\,\Phi(e, \mathcal{M}^t)}{2r + t\,(r^3 - r^2 - 6r + 4)} \leq \frac{2 + t\,(r^2 - r - 4)}{2r + t\,(r^3 - r^2 - 6r + 4)}.$$

It is easy to verify that $r^3 - r^2 - 6r + 4 \geq r^2 - r - 4 > 0$ for all $r \geq 3$, so we have $w_t^r(e) \leq 1$. Moreover, since $\Phi(e, \mathcal{M}^t) = \sum_{i=1}^{t} |\{e\} \cap M_i| \leq t$, we have

$$w_t^r(e) \geq \frac{2 + t\,(r^2 - r - 4) - 2\,(r-2)\,t}{2r + t\,(r^3 - r^2 - 6r + 4)} = \frac{2 + t(r^2 - 3r)}{2r + t\,(r^3 - r^2 - 6r + 4)}.$$

Note that $r^2 - 3r$ and $r^3 - r^2 - 6r + 4$ are positive for all $r \geq 3$. Hence $w_t^r(e) \geq 0$. Therefore $0 \leq w_t^r(e) \leq 1$.

(2) Let $v \in V(G)$ be a vertex. It is clear that $|\partial(v) \cap M| = 1$ for any perfect matching M. Therefore $\Phi(\partial(v), \mathcal{M}^t) = \sum_{i=1}^{t} |\partial(v) \cap M_i| = t$. This together with $|\partial(v)| = r$ gives us

$$\begin{aligned} w_t^r(\partial(v)) &= \frac{2r + tr\,(r^2 - r - 4) - 2\,(r-2)\,\Phi(\partial(v), \mathcal{M}^t)}{2r + t\,(r^3 - r^2 - 6r + 4)} \\ &= \frac{2r + t\,(r^3 - r^2 - 4r) - 2\,(r-2)\,t}{2r + t\,(r^3 - r^2 - 6r + 4)} = 1 \end{aligned}$$

as required.

(3) Let X be an odd subset of $V(G)$ with $|\partial(X)| = k$. Since G is an r-graph, we have $k \geq r$ and note that k and r have the same parity. By induction hypothesis we have,

$$w_{t-1}^r(\partial(X)) = \frac{2k + (t-1)k\,(r^2 - r - 4) - 2\,(r-2)\,\Phi(\partial(X), \mathcal{M}^{t-1})}{2r + (t-1)\,(r^3 - r^2 - 6r + 4)} \geq 1. \qquad (\mathbf{1})$$

We will show that $w_t^r(\partial(X)) \geq 1$ by considering three cases.

Case 1: $k = r$

From inequality ($\mathbf{1}$) we get $\Phi(\partial(X), \mathcal{M}^{t-1}) \leq t - 1$. On the other hand, each r- cut intersects each of the $t-1$ perfect matchings $M_1, M_2, \ldots M_{t-1}$ at least once, which yields $\Phi(\partial(X), \mathcal{M}^{t-1}) \geq t-1$. Hence $\Phi(\partial(X), \mathcal{M}^{t-1}) = t-1$, and so

$$w_{t-1}^{r}(\partial(X)) = \frac{2r + (t-1)\,(r^3 - r^2 - 4r) - 2\,(r-2)\,(t-1)}{2r + (t-1)\,(r^3 - r^2 - 6r + 4)} = 1.$$

Then by the choice of M_t, we conclude that $|\partial(X) \cap M_t| = 1$ from Lemma 2.4.2. Therefore

$$\begin{aligned} w_t^r(\partial(v)) &= \frac{2r + tr\,(r^2 - r - 4) - 2\,(r-2)\,\Phi(\partial(X), \mathcal{M}^t)}{2r + t\,(r^3 - r^2 - 6r + 4)} \\ &= \frac{2r + t\,(r^3 - r^2 - 4r) - 2\,(r-2)\,[\Phi(\partial(X), \mathcal{M}^{t-1}) + |\partial(X) \cap M_t|]}{2r + t\,(r^3 - r^2 - 6r + 4)} \\ &= \frac{2r + t\,(r^3 - r^2 - 4r) - 2\,(r-2)\,t}{2r + t\,(r^3 - r^2 - 6r + 4)} = 1. \end{aligned}$$

Case 2: $k > r$, $w_{t-1}^{r}(\partial(X)) = 1$

First, note that since $w_{t-1}^{r}(\partial(X)) = 1$, by Lemma 2.4.2 we have $|\partial(X) \cap M_t| = 1$.

We will show that

$$\begin{aligned} w_t^r(\partial(X)) &= \frac{2k + tk\,(r^2 - r - 4) - 2\,(r-2)\,\Phi(\partial(X), \mathcal{M}^t)}{2r + t\,(r^3 - r^2 - 6r + 4)} \\ &= \frac{2k + tk\,(r^2 - r - 4) - 2(r-2)\,[\Phi(\partial(X), \mathcal{M}^{t-1}) + |\partial(X) \cap M_t|]}{2r + t\,(r^3 - r^2 - 6r + 4)} \\ &= \frac{2k + tk\,(r^2 - r - 4) - 2(r-2)\,[\Phi(\partial(X), \mathcal{M}^{t-1}) + 1]}{2r + t\,(r^3 - r^2 - 6r + 4)} \geq 1, \end{aligned}$$

Since $w_{t-1}^{r}(\partial(X)) = \frac{2k + (t-1)k\left(r^2 - r - 4\right) - 2(r-2)\Phi(\partial(X), \mathcal{M}^{t-1})}{2r + (t-1)(r^3 - r^2 - 6r + 4)} = 1$, we have

$$2(r-2)\Phi(\partial(X),\mathcal{M}^{t-1}) = 2k-2r+(t-1)k\left(r^2-r-4\right)-(t-1)\left(r^3-r^2-6r+4\right).$$

Substituting and simplifying gives,

$$w_t^r(\partial(X)) \;=\; \frac{2r+t\,(r^3-r^2-6r+4)+A}{2r+t\,(r^3-r^2-6r+4)},$$

where $A = k\,(r^2-r-4)-(r^3-r^2-6r+4)-2\,(r-2)$. In order to see that $w_t^r(\partial(X)) \geq 1$, it is enough to show that $A \geq 0$. This holds because

$$A = k\left(r^2-r-4\right)-(r^3-r^2-6r+4)-2(r-2) = (k-r)(r^2-r-4),$$

and $r^2-r-4>0$ for all $r \geq 3$. This completes the proof of the case 2.

Case 3: $k>r$, $w_{t-1}^r(\partial(X))>1$

Since $w_{t-1}^r(\partial(X)) = \frac{2k+(t-1)k\left(r^2-r-4\right)-2(r-2)\Phi(\partial(X),\mathcal{M}^{t-1})}{2r+(t-1)(r^3-r^2-6r+4)} > 1$, we have

$$\begin{aligned}
2(r-2)\cdot\Phi(\partial(X),\mathcal{M}^{t-1}) \;&<\; 2k-2r+(t-1)\left[k(r^2-r-4)-(r^3-r^2-6r+4)\right] \\
&=\; 2(k-r)+(t-1)\left[(k-r)(r^2-r-4)+2(r-2)\right].
\end{aligned}$$

Notice that because $k-r$ is even, both sides of inequality (**2**) is even. Hence we have

$$2(r-2)\Phi(\partial(X),\mathcal{M}^{t-1}) \le 2(k-r)+(t-1)\left[(k-r)(r^2-r-4)+2(r-2)\right]-2. \quad (\mathbf{2})$$

Now we will show that

$$w_t^r(\partial(X)) = \frac{2k+tk\,(r^2-r-4)-2(r-2)\left[\Phi(\partial(X),\mathcal{M}^{t-1})+|\partial(X)\cap M_t|\right]}{2r+t\,(r^3-r^2-6r+4)} \ge 1.$$

Applying $|\partial(X)\cap M_t|\le k$ and by inequality (**2**), we obtain

$$w_t^r(\partial(X)) \ge \frac{2r+t\,(r^3-r^2-6r+4)+B}{2r+t\,(r^3-r^2-6r+4)},$$

where $B=k\,(r^2-r-4)-(r^3-r^2-6r+4)+2-2(r-2)k=(k-r)(r^3-3r)+2(r-1)$. Since $r\ge 3$ and $k>r$, we have $B>0$. Hence $w_t^r(\partial(X)\ge 1$, and we are done with the last case. Therefore the function $w_t^r(e)$ is a fractional perfect matching for any integer $t\ge 0$ and fixed $r\ge 3$.

We now complete the prove of (ii). The assertion is clearly true for $t=1$ as $a_1^r=\frac{1}{r}$. So we may assume $t\ge 2$. By induction hypothesis, we have

$$\frac{\left|\bigcup_{i=1}^{t-1} M_i\right|}{E(G)} \ge a_{t-1}^r.$$

Recall $c=1-\chi^{\bigcup_{i=1}^{t-1} M_i}$. By the choice of M_t, we have

$$c \cdot \chi^{M_t} \geq c \cdot w_{t-1}^r. \tag{3}$$

Here the left hand side of (**3**) is $c \cdot \chi^{M_t} = |M_t \setminus \bigcup_{i=1}^{t-1} M_i|$. Since for each edge $e \notin \bigcup_{i=1}^{t-1} M_i$, we have $w_{t-1}^r(e) = \frac{2+(t-1)\left(r^2-r-4\right)}{2r+(t-1)(r^3-r^2-6r+4)}$. So the right hand side of inequality (**3**) is the number of edges not covered by $\mathcal{M}^{t-1}$ multiplied by $\frac{2+(t-1)\left(r^2-r-4\right)}{2r+(t-1)(r^3-r^2-6r+4)}$ which gives

$$|M_t \setminus \bigcup_{i=1}^{t-1} M_i| \geq \left(|E(G)| - |\bigcup_{i=1}^{t-1} M_i|\right) \cdot \frac{2+(t-1)\left(r^2-r-4\right)}{2r+(t-1)\left(r^3-r^2-6r+4\right)}.$$

Hence

$$\begin{aligned} |\bigcup_{i=1}^{t} M_i| &= |M_t \setminus \bigcup_{i=1}^{t-1} M_i| + |\bigcup_{i=1}^{t-1} M_i| \\ &\geq \left(|E(G)| - |\bigcup_{i=1}^{t-1} M_i|\right) \cdot \frac{2+(t-1)\left(r^2-r-4\right)}{2r+(t-1)\left(r^3-r^2-6r+4\right)} + |\bigcup_{i=1}^{t-1} M_i|. \end{aligned}$$

Dividing by $|E(G)|$ gives

$$\frac{|\bigcup\limits_{i=1}^{t} M_i|}{|E(G)|} \geq \left(1 - \frac{|\bigcup\limits_{i=1}^{t-1} M_i|}{|E(G)|}\right) \cdot \frac{2+(t-1)\left(r^2-r-4\right)}{2r+(t-1)\left(r^3-r^2-6r+4\right)} + \frac{|\bigcup\limits_{i=1}^{t-1} M_i|}{|E(G)|}.$$

With the assumption that $\frac{|\bigcup\limits_{i=1}^{t-1} M_i|}{|E(G)|} \geq a_{t-1}$, we conclude that

$$\frac{|\bigcup\limits_{i=1}^{t} M_i|}{|E(G)|} \geq a_{t-1}^r + (1 - a_{t-1}^r) \cdot \frac{2+(t-1)\left(r^2-r-4\right)}{2r+(t-1)\left(r^3-r^2-6r+4\right)} = a_t^r.$$

Since by definition, $m_t^r = \inf\limits_{G} \max\limits_{M_1,\ldots,M_t} \frac{\left|\bigcup\limits_{i=1}^{t} M_i\right|}{|E(G)|}$, we have

$$m_t^r \geq a_t^r,$$

for any integer $t \geq 0$ and fixed $r \geq 3$. $\square$

2.6 Covering an r- graph with t Perfect Matchings

It is still unknown whether $m_t^r = 1$ for any $r \geq 3$ and $t \geq 2r - 1$. The best known result for $r = 3$ is given by Mazzuoccolo which states that: if a cubic bridgeless graph G has fewer than $\lfloor \frac{2^t}{\sqrt{t}} \rfloor$ edges, then there is a covering of G by t perfect matchings. Now we will generalize his result and provide an upper bound for the size of an r-graph G so that G can be covered by t perfect matchings by using Theorem 2.5.1.

Theorem 2.6.1. *Let G is an r-graph and t be a positive integer. If $|E(G)| < \frac{1}{\sqrt{t}} \left(\frac{r^3 - r^2 - 6r + 4}{r^3 - 2r^2 - 5r + 8} \right)^t$, then G can be covered by t perfect matchings.*

Proof. As mentioned earlier, the special case $r = 3$ in Theorem 2.6.1 was proved in (17). Therefore in the proof it is enough for us to consider the case $r \geq 4$ and $t \geq 2$. Fix $r \geq 4$. Note that if $|E(G)| \cdot m_t^r > |E(G)| - 1$, then there exists a covering of G by t perfect matchings. In other words, if $|E(G)| < \frac{1}{1 - m_t^r}$ then we have a covering of $E(G)$ by t perfect matchings. By theorem 2.5.1, we know that $m_t^r \geq a_t^r$, that is $\frac{1}{1 - m_t^r} \geq \frac{1}{1 - a_t^r}$. So it is enough to show that $\frac{1}{\sqrt{t}} \left(\frac{r^3 - r^2 - 6r + 4}{r^3 - 2r^2 - 5r + 8} \right)^t \leq \frac{1}{1 - a_t^r}$, or equivalently $a_t^r \geq 1 - \sqrt{t} \left(\frac{r^3 - 2r^2 - 5r + 8}{r^3 - r^2 - 6r + 4} \right)^t$ for each $t \geq 2$. We prove by induction on t. For the base case, when $t = 2$,

$$\begin{aligned} a_2^r &= \frac{r+1}{r^2+r-2} + \frac{1}{r}\left(1 - \frac{r+1}{r^2+r-2}\right) \\ &= \frac{2r+3}{r(r+2)}. \end{aligned}$$

Now we want to show that the following inequality holds for all $r \geq 4$:

$$\frac{2r+3}{r(r+2)} \geq 1 - \sqrt{2}\left(\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\right)^2. \tag{4}$$

First note that $a_2^r \geq 0$, for all $r \geq 4$ and one can easily check that inequality **4** holds for $4 \leq r \leq 6$. Let $f(r) := 1 - \sqrt{2}\left(\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\right)^2$. Then

$$f'(r) = -\frac{2^{\frac{3}{2}}\left(r^3-2r^2-5r+8\right)\left(r^4-2r^3-5r^2+28\right)}{\left(r^3-r^2-6r+4\right)^3}.$$

It is easy to see that $f'(r) \leq 0$ for all $r \geq 6$. Therefore $f(r)$ is decreasing and $f(7) \leq 0$. Hence we conclude inequality (**4**) holds for all $r \geq 4$ and so the result follows for $t = 2$.

Now suppose $t \geq 3$ and $a_t^r \geq 1 - \sqrt{t}\left(\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\right)^t$ for each $r \geq 4$. We will show that

$$a_{t+1}^r \geq 1 - \sqrt{t+1}\left(\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\right)^{t+1}.$$

For the left hand side, we have

$$a_{t+1}^r = \frac{2+t\left(r^2-r-4\right)}{2r+t\left(r^3-r^2-6r+4\right)} + a_t^r \cdot \left(1 - \frac{2+t\left(r^2-r-4\right)}{2r+t\left(r^3-r^2-6r+4\right)}\right).$$

Applying the induction hypothesis gives,

$$\begin{aligned} a^r_{t+1} &\geq \frac{2+t\left(r^2-r-4\right)}{2r+t\left(r^3-r^2-6r+4\right)} + \\ &\left(1-\sqrt{t}\left(\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\right)^t\right)\cdot\left(1-\frac{2+t\left(r^2-r-4\right)}{2r+t\left(r^3-r^2-6r+4\right)}\right) \\ &= 1-\sqrt{t}\left(\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\right)^t\cdot\frac{2r-2+t\left(r^3-2r^2-5r+8\right)}{2r+t\left(r^3-r^2-6r+4\right)} = D. \end{aligned}$$

Now we are done if we can show that

$$D \geq 1-\sqrt{t+1}\left(\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\right)^{t+1},$$

or simply

$$\frac{2r-2+t\left(r^3-2r^2-5r+8\right)}{2r+t\left(r^3-r^2-6r+4\right)} \leq \frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\cdot\sqrt{1+\frac{1}{t}}. \tag{$\mathbf{5}$}$$

For the left hand side of (**5**) we have

$$\begin{aligned} \frac{2r-2+t\left(r^3-2r^2-5r+8\right)}{2r+t\left(r^3-r^2-6r+4\right)} &\leq \frac{2r+t\left(r^3-2r^2-5r+8\right)}{t\left(r^3-r^2-6r+4\right)} \\ &= \frac{2r}{t\left(r^3-r^2-6r+4\right)}+\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4} \end{aligned}$$

For the right hand side of (**5**), we have

$$\sqrt{1+\frac{1}{t}} = 1+\frac{1}{2t}-\frac{1}{8t^2}+\frac{1}{16t^3}-\frac{5}{128t^4}+\ldots$$

from the binomial expansion, which leads to

$$\sqrt{1+\frac{1}{t}} \geq 1+\frac{1}{2t}-\frac{1}{8t^2}.$$

Hence the right hand side of inequality (**5**) has the following lower bound:

$$\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\cdot\sqrt{1+\frac{1}{t}} \geq \frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\cdot\left(1+\frac{1}{2t}-\frac{1}{8t^2}\right),$$

So for the inequality (**5**), it is enough to show that

$$\frac{2r}{t\left(r^3-r^2-6r+4\right)}+\frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4} \leq \frac{r^3-2r^2-5r+8}{r^3-r^2-6r+4}\cdot\left(1+\frac{1}{2t}-\frac{1}{8t^2}\right),$$

which can be simplified to

$$\frac{16t}{4t-1} \leq r^2-2r-5+\frac{8}{r}.$$

One can easily check that the last inequality holds for any $r \geq 4$ and $t \geq 2$. Therefore we proved $a_t^r \geq 1-\sqrt{t}\left(\frac{r^2-3r+1}{r^2-2r-1}\right)^t$ for each $t \geq 2$ and we are done. $\square$

Theorem 2.6.1 gives an upper bound, in terms of t, for the number of edges of an r-graph G so that G can be covered by t perfect matchings. Here we want to note a trivial upper bound for t such that any r-graph G has a covering by t perfect matching.

As we discussed earlier every edge of an r-graph G is contained in at least one perfect matching. Since $|E(G)| = \frac{nr}{2}$, trivially there is a family of $\frac{nr}{2}$ perfect matchings of G (not necessarily distinct) that covers $E(G)$. That is $m^r_{\frac{nr}{2}}(G) = 1$, for any $r \geq 3$.

Now we turn our attention to r-graphs where $r \geq \frac{|V(G)|}{2}$. We know now by Theorem 2.6.1 if

$$\frac{nr}{2} < \frac{1}{\sqrt{t}} \left(\frac{r^3 - r^2 - 6r + 4}{r^3 - 2r^2 - 5r + 8} \right)^t, \tag{6}$$

then G can be covered by t perfect matchings.

Taking the log of both sides in (**6**) gives

$$\log(n) + \log(r) - \log(2) < t \log \left(1 + \frac{r^2 - r - 4}{r^3 - 2r^2 - 5r + 8} \right) - \frac{1}{2} \log(t),$$

which yield

$$t > \frac{\log(n) + \log(r) - \log(2) + \frac{1}{2}\log(t)}{\log\left(1 + \frac{r^2 - r - 4}{r^3 - 2r^2 - 5r + 8}\right)}. \tag{7}$$

Here we note that $f(r) := \log\left(1 + \frac{r^2-r-4}{r^3-2r^2-5r+8}\right) - 1/r > 0$. Indeed

$$f'(r) = -\frac{r^5 + 4r^4 - 29r^3 + 14r^2 + 68r - 32}{r^2 \cdot (r^3 - 2r^2 - 5r + 8)\,(r^3 - r^2 - 6r + 4)} < 0,$$

for all $r \geq 3$. Moreover $f(3) = 0.3598 > 0$, and $\lim_{r \to \infty} f(r) = 0$.

Therefore if

$$t > \frac{\log n + \log r - \log 2 + \frac{1}{2}t}{\frac{1}{r}},$$

then G can be covered by t perfect matchings. Since $n \leq 2r$ by assumption and $t \leq \frac{nr}{2}$, taking

$$t = \lceil 3r \log r \rceil$$

gives $m_t^r(G) = 1$.

Corollary 2.6.2. *Let G be an r-graph of order $n \leq 2r$. Then $m_t^r(G) = 1$ when $t = \lceil 3r \log r \rceil$.*

2.7 A New Conjecture

In this section we will give a natural generalization of Conjecture 2.2.3 for any $r \geq 3$, and using that we generalize the results given in (20) by Patel. We further present a new conjecture that may help in the proof of Generalized Fulkerson Conjecture.

First we define

$$\tau_t^r = \frac{t(4r - t - 1)}{2r(2r - 1)},$$

for any $r \geq 3$ and $t \geq 0$. Note that $\tau_t^3 = m_t^r(P)$, where P is the Petersen Graph.

Remark 2.7.1. *We note that an r-graph G satisfies $m_t^r(G) = \tau_t^r$ for any fixed t with $1 \leq t \leq 2r - 1$ if G contains $2r$ perfect matchings $M_1, \ldots, M_{2r}$ having the following properties; $|M_i \cap M_j| = 1$ for each $i \neq j$ and for each $e \in E(G)$ there is a unique pair of perfect matchings M_i and M_j so that $e \in M_i \cap M_j$.*

Conjecture 2.7.1. $m_t^r \geq \tau_t^r$ *for* $1 \leq t \leq 2r - 1$*, specifically,* $m_{2r-1}^r = 1$.

The property explained in Remark 2.7.1 clearly holds for the Petersen graph. However there is no r-graph known satisfying that property for $r > 3$ as far as we know to this date.

If one can find such an r-graph G among all r-graphs, then $\tau_t^r = m_t^r(G) \geq m_t^r$. Hence Conjecture 2.7.1 implies $m_t^r = \tau_t^r$ and the following theorem still holds. Now we show that Conjecture 2.3.1 implies Conjecture 2.7.1.

Theorem 2.7.2. *Generalized Fulkerson Conjecture (GFC) implies Conjecture 2.7.1.*

Proof. It suffices to show that for any r-graph G and each $1 \leq t \leq 2r-1$, $m_t(G) \geq \tau_t^r$. Fix $1 \leq t \leq 2r-1$. Given GFC holds for G, we can find a set of $2r$ perfect matchings, $\mathcal{M} = \{M_1, \ldots, M_{2r}\}$, such that each edge of G is contained in exactly two elements in $\mathcal{M}$.

Let S_t be a set of t elements chosen uniformly and randomly from $[2r]$. Fix $e \in E(G)$. Since GFC holds for G, there exists two perfect matchings, say M_a and M_b, in $\mathcal{M}$ that contains e. Then

$$
\begin{aligned}
\mathbb{P}(e \in \bigcup_{i \in S_t} M_i) &= \mathbb{P}(a \in S_t \text{ or } b \in S_t) \\
&= 1 - \mathbb{P}(a \notin S_t \text{ and } b \notin S_t) \\
&= 1 - \frac{\binom{2r-2}{t}}{\binom{2r}{t}} \\
&= \tau_t^r.
\end{aligned}
$$

Further we have the expectation,

$$\mathbb{E}(|\bigcup_{i \in S_t} M_i|) = \sum_{e \in E(G)} \mathbb{P}(e \in \bigcup_{i \in S_t} M_i) = |E(G)| \cdot \tau_t^r.$$

Therefore, there exists some t-element subset of $[2r]$, say S_t^*, satisfying

$$|\bigcup_{i\in S_t^*} M_i| \geq |E(G)| \cdot \tau_t^r.$$

Hence $m_t(G) \geq \tau_t^r$.

□

Now we will give a conjecture that is stronger than Conjecture 2.7.1.

Conjecture 2.7.3. *Let G be an r-graph. For each $t \in \{1, \dots 2r-1\}$, G has t perfect matchings, $M_1, \dots, M_t$, satisfying:*

(1) no edge of G is contained in more than two of the M_i's,

(2) $|\bigcup_{i=1}^t M_i| \geq \tau_t^r \cdot |E(G)|$, and

(3) for every odd cut C of G, if $|C| = k$ then $\sum_{i=1}^t |M_i \cap C| \leq 2(k-r)+t$.

We will show later that GFC implies Conjecture 2.7.3, but let us first present the reason why Conjecture 2.7.3 could be useful for proving Conjecture 2.7.1.

Theorem 2.7.4. *If Conjecture 2.7.3 holds for a given $t \in \{2, \dots 2r-2\}$, then Conjecture 2.7.1 holds for $t+1$. If Conjecture 2.7.3 holds for $t = 2r-1$, then GFC holds.*

Proof. Let G be an r-graph. Suppose G has t perfect matchings, $M_1, \dots, M_t$ satisfying Conjecture 2.7.3 for $t \in \{2 \dots, 2r-2\}$. Then set

$$w_t(e) = \begin{cases} 0 & \textit{if} \quad e \text{ is in exactly two of } M_1, \dots M_t; \\ \frac{1}{2r-t} & \textit{if} \quad e \text{ is in exactly one of } M_1, \dots M_t; \\ \frac{2}{2r-t} & \textit{if} \quad e \text{ is not in any of } M_1, \dots M_t. \end{cases}$$

Now we will check $w_t(e)$ is a fractional perfect matching for any $t \in \{2, \dots 2r-2\}$ by checking the three condition given in the definition of fractional perfect matching.

(1) Since $2 \leq t \leq 2r-2$, clearly $0 \leq w_t(e) \leq 1$.

(2) For any $v \in V(G)$, let a_0, a_1 and a_2 denote the number of edges of $\partial(v)$ that are covered by no, exactly one and exactly 2 perfect matchings respectively. Note that

$$a_0 + a_1 + a_2 = r \tag{8}$$

Also since $|M_i \cap \partial(v)| = 1$ for all $1 \leq i \leq t$, we have

$$a_1 + 2a_2 = t \tag{9}$$

Taking $\frac{2}{2r-t}$ times (relation (**8**)) $-$ $\frac{1}{2r-t}$ times (relation (**9**)) gives

$$w_t(\partial(v)) = \frac{2}{2r-t}a_0 + \frac{1}{2r-t}a_1 = 1.$$

So this condition is satisfied.

(3) Let $X \subseteq V(G)$ be an odd cardinality set with $|\partial(X)| = k$. Since G is an r-graph, it follows that $k \geq r$. Let b_0, b_1 and b_2 denote the number of edges of $\partial(X)$ that are covered by no, exactly one and exactly 2 perfect matchings respectively. Note that

$$b_0 + b_1 + b_2 = k, \tag{10}$$

and by Conjecture 2.7.3 we have

$$b_1 + 2b_2 \leq 2(k-r) + t. \qquad \textbf{(11)}$$

Taking $\frac{2}{2r-t}$ times (relation (**10**)) $-$ $\frac{1}{2r-t}$ times (relation (**11**)) gives

$$w_t(\partial(X)) \geq \frac{2}{2r-t} b_0 + \frac{1}{2r-t} b_1 \geq 1.$$

as we wanted and third condition is also satisfied. Hence $w_t(e)$ is a fractional perfect matching. By Lemma 1, there exists a perfect matching, say M_{t+1}, such that c $\cdot \chi^{M_{t+1}} \geq c \cdot w_t(e)$. Setting $c = \chi^{(\bigcup\limits_{i=1}^{t} M_i)^c}$ yields

$$|M_{t+1} \setminus \bigcup_{i=1}^{t} M_i)| = \chi^{(\bigcup_{i=1}^{t} M_i)^c} \cdot \chi^{M_{t+1}} \geq \chi^{(\bigcup_{i=1}^{t} M_i)^c} \cdot w_(e) = \frac{2}{2r-t} \cdot |E(G) \setminus \bigcup_{i=1}^{t} M_i|.$$

Therefore

$$\begin{aligned}
|\bigcup_{i=1}^{t+1} M_i| &= |\bigcup_{i=1}^{t} M_i| + |M_{t+1} \setminus \bigcup_{i=1}^{t} M_i| \\
&\geq |\bigcup_{i=1}^{t} M_i| + \frac{2}{2r-t} \cdot |E(G) \setminus \bigcup_{i=1}^{t} M_i| \\
&= \frac{2r-t-2}{2r-t} \cdot |\bigcup_{i=1}^{t} M_i| + \frac{2}{2r-t} \cdot |E(G)| \\
&\geq \frac{2r-t-2}{2r-t} \cdot m_t^r |E(G)| + \frac{2}{2r-t} \cdot |E(G)| \\
&= (\frac{2r-t-2}{2r-t} \cdot \frac{t(4r-t-1)}{2r(2r-1)} + \frac{2}{2r-t}) \cdot |E(G)| \\
&= \tau_{t+1}^r \cdot |E(G)|
\end{aligned}$$

Thus G satisfies Conjecture 2.7.1 for $2 \leq t \leq 2r-1$. Note that when $t = 2r-1$, then GFC holds.

$\square$

Theorem 2.7.5. *The GFC implies Conjecture 2.7.3.*

Proof. Let G be an r-graph satisfying GFC, that is G has $2r$ perfect matchings, $M_1, \ldots, M_{2r}$ with each edge of G are in exactly two of them. Clearly, for each $t \in \{2, \ldots, 2r-1\}$, any t-subset of $\{M_1, \ldots, M_{2r}\}$ satisfy the first condition of the Conjecture 2.7.3.

By the Theorem 2.7.2, we know that GFC implies $m_t^r \geq \tau_t^r$. Since $m_t^r = \inf_{G} \max_{M_1,\ldots,M_t} \frac{\left|\bigcup_{i=1}^{t} M_i\right|}{|E(G)|}$ where the infimum taken over all r-graphs, we have

$$\tau_t^r \cdot |E(G)| \leq m_t^r \cdot |E(G)| \leq |\bigcup_{i=1}^{t} M_i|.$$

So the second condition of Conjecture 2.7.3 is also satisfied.

For the third condition, first note that for any perfect matching M and any odd cut C then $|C \cap M| \geq 1$. Let $|C| = k$. Since $\sum_{i=1}^{2r} |C \cap M_i| = 2k$, then for $S \subseteq [2r]$ with $|S| = t$, we have

$$\begin{aligned}
\sum_{i \in S} |M_i \cap C| &= 2k - \sum_{i \notin S} |M_i \cap C| \\
&\leq 2k - |[2r] \setminus S| \\
&= 2k - (2r - t) \\
&= 2(k - r) + t
\end{aligned}$$

as required. $\square$

CHAPTER 3
MEAN SUBTREE ORDER OF GRAPHS

3.1 Introduction

Graphs in this chapter are simple unless otherwise specified. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. The *order* of G, denoted by $|G|$, is the number of vertices in G, that is, $|G| = |V(G)|$. The *complement* of G, denoted by $\overline{G}$, is the graph on the same vertex set as G such that two distinct vertices of $\overline{G}$ are adjacent if and only if they are not adjacent in G. For an edge subset $F \subseteq E(\overline{G})$, denote by $G + F$ the graph obtained from G by adding the edges of F. For a vertex subset $U \subseteq V(G)$, denote by $G - U$ the graph obtained from G by deleting the vertices of U and all edges incident with them.

A tree is a graph in which every pair of distinct vertices is connected by exactly one path. A subtree of a graph G is a subgraph of G that is a tree. By convention, the empty graph is not regarded as a subtree of any graph. The *mean subtree order* of G, denoted $\mu(G)$, is the average order of a subtree of G. Jamison (12; 13) initiated the study of the mean subtree order in the 1980s, considering only the case that G is a tree. In (12), he proved that $\mu(T) \geq \frac{n+2}{3}$ for any tree T of order n, with this minimum achieved if and only if T is a path; and $\mu(T)$ could be very close to its order n. Jamison's work on the mean order of the subtrees of a tree has received considerable attention (10; 18; 26; 28; 29), and many of them have been solved in the last decade. At the 2019 Spring Section AMS meeting in Auburn, Jamison gave a survey on the state of the art of the remaining open questions on the mean subtree order of a tree.

Recently, Chin, Gordon, MacPhee, and Vincent (4) initiated the study of subtrees of

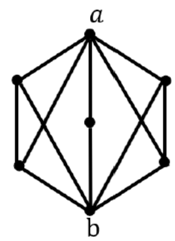

Figure 3.1 Adding the edges between a and b decreases the mean subtree order

graphs in general. They believed that the parameter μ is monotonic with respect to the inclusion relationship of subgraphs. More specifically, they (4, Conjecture 7.4) conjectured that for any simple connected graph G, adding any edge to G will increase the mean subtree order. Clearly, the truth of this conjecture implies that $\mu(K_n)$ is the maximum among all connected simple graphs of order n. Cameron and Mol (3) constructed some counterexamples to this conjecture by a computer search. Moreover, they found that the graph depicted in Figure 3.1 is the smallest counterexample to this conjecture and there are infinitely many graphs G with $xy \in E(\overline{G})$ such that $\mu(G + xy) < \mu(G)$. Although they had focused on adding of a single edge, they made the following conjecture regarding the adding several edges.

Conjecture 3.1.1. *For every positive integer k, there are two connected graphs G and H with $G \subset H$, $V(G) = V(H)$ and $|E(H) \backslash E(G)| = k$ such that $\mu(H) < \mu(G)$.*

In this chapter, we confirm Conjecture 3.1.1 by showing the following theorem.

Theorem 3.1.2. *For every positive integer k, there exist infinitely many pairs of connected graphs G and H with $G \subset H$, $V(G) = V(H)$ and $|E(H) \backslash E(G)| = k$ such that $\mu(H) < \mu(G)$.*

3.2 Preliminary and Construction

Let G be a graph of order n, and let $\mathcal{T}_G$ be the family of subtrees of G. By definition, we have $\mu(G) = (\sum_{T\in\mathcal{T}_G} |T|)/|\mathcal{T}_G|$. The *density* of G is defined by $\sigma(G) = \mu(G)/n$. More generally, for any subfamily $\mathcal{T} \subseteq \mathcal{T}_G$, we define $\mu(\mathcal{T}) = (\sum_{T\in\mathcal{T}} |T|)/|\mathcal{T}|$ and $\sigma(\mathcal{T}) = \mu(\mathcal{T})/n$. Clearly, $1 \le \mu(G) \le n$ and $0 < \sigma(G) \le 1$.

Fix a positive integer k. For some integer m, let $\{s_n\}_{n\ge m}$ be a sequence of non-negative integers satisfying: (1) $2s_n \le n-k-1$ for all $n \ge m$; (2) $s_n = o(n)$, i.e., $\lim_{n\to\infty} s_n/n = 0$; and (3) $2^{s_n} \ge n^2$ for all $n \ge m$. Notice that many such sequences exist. Take, for instance, the sequence $\{\lceil 2\log_2(n)\rceil\}_{n\ge m}$ similar as in (3), where m is the least positive integer such that $m - 2\lceil 2\log_2(m)\rceil \ge k+1$.

In the remainder of this paper, we fix P for a path $v_1v_2\cdots v_{n-2s_n}$ of order $n-2s_n$. Clearly, $|P| \ge k+1$. Furthermore, let $P^* := P - \{v_1,\dots,v_{k-1}\} = v_k\cdots v_{n-2s_n}$.

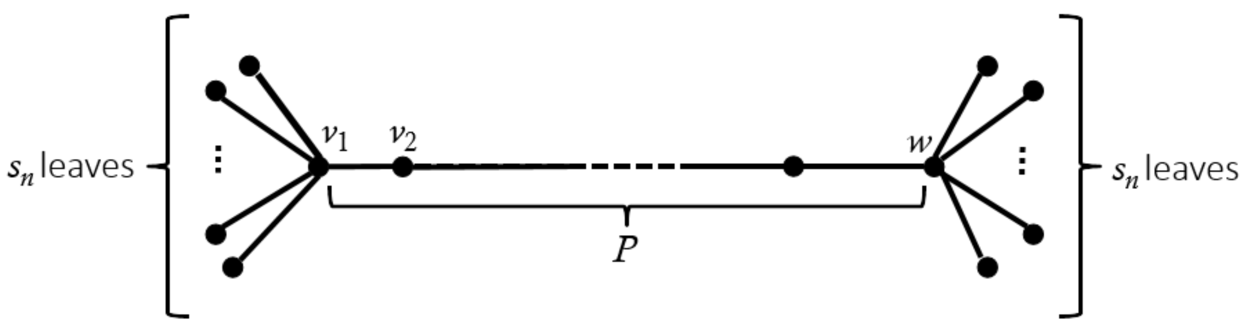

Figure 3.2 The double broom graph G_n

Let G_n be the graph obtained from the path P by joining s_n leaves to each of the two end-points v_1 and $w := v_{n-2s_n}$ of P (see Figure 3.2). Let $G_{n,k} := G_n + \{v_1w, v_2w, \dots, v_kw\}$, that is, $G_{n,k}$ is the graph obtained from G_n by adding k new edges $e_1 := v_1w, e_2 := v_2w, \dots, e_k := v_kw$ (see Figure 3.3).

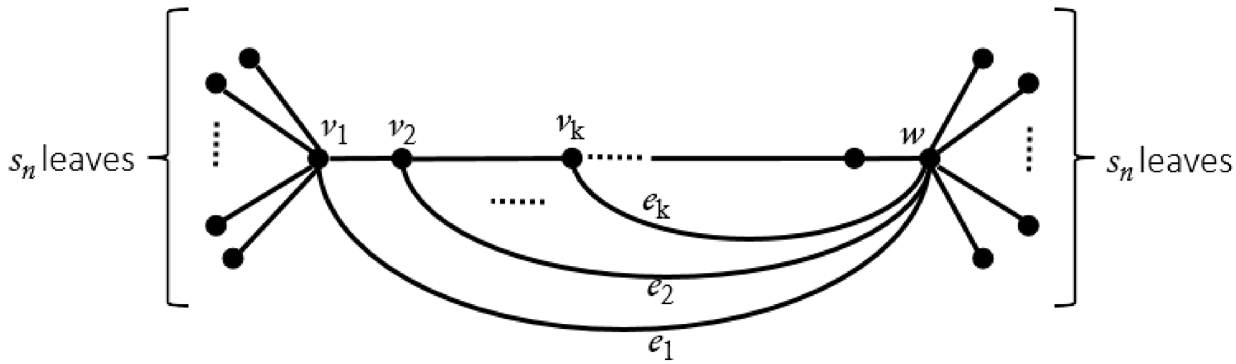

Figure 3.3 $G_{n,k}$; obtained from G_n by adding k edges

Let $\mathcal{T}_{n,k}$ be the family of subtrees of $G_{n,k}$ containing the vertex set $\{v_1, v_k, w\}$ but not containing the path $P^* = v_k \cdots w$. It is worth noting that $\mathcal{T}_{n,1}$ is the family of subtrees of $G_{n,1}$ containing edge $v_1 w$. Note that the graphs G_n and $G_{n,1}$ defined above are actually the graphs T_n and G_n constructed by Cameron and Mol in (3), respectively. From the proof of Theorem 3.1 in (3), we obtain the following two results regarding the density of $G_n, G_{n,1}, \mathcal{T}_{n,1}$.

Lemma 3.2.1. $\lim_{n\to\infty} \sigma(G_n) = 1$.

Lemma 3.2.2. $\lim_{n\to\infty} \sigma(G_{n,1}) = \lim_{n\to\infty} \sigma(\mathcal{T}_{n,1}) = \frac{2}{3}$.

In this paper, we prove the following two technical results regarding the density of $\mathcal{T}_{n,k}$, whose proofs will be presented in Section 3 and Section 4, respectively.

Lemma 3.2.3. *For any fixed positive integer k,* $\lim_{n\to\infty} \sigma(\mathcal{T}_{n,k}) = \lim_{n\to\infty} \sigma(\mathcal{T}_{n-k+1,1})$.

Lemma 3.2.4. *For any fixed positive integer k,* $\lim_{n\to\infty} \sigma(\mathcal{T}_{n,k}) = \lim_{n\to\infty} \sigma(G_{n,k})$.

The combination of Lemma 3.2.2, Lemma 3.2.3 and Lemma 3.2.4 gives immediately the following result.

Corollary 3.2.5. *For any fixed positive integer k,* $\lim_{n\to\infty} \sigma(G_{n,k}) = \frac{2}{3}$.

Combining Lemma 3.2.1 and Corollary 3.2.5, we have that $\lim_{n\to\infty} \sigma(G_{n,k}) = \frac{2}{3} < 1 = \lim_{n\to\infty} \sigma(G_n)$ for any fixed positive integer k. By definition, $\sigma(G_{n,k}) = \mu(G_{n,k})/|G_{n,k}|$ and $\sigma(G_n) = \mu(G_n)/|G_n|$. Since $|G_{n,k}| = |G_n|$, it follows that $\mu(G_{n,k}) < \mu(G_n)$ for n sufficiently large, which in turn gives Theorem 3.1.2.

The following result presented in (3, page 408, line -2) will be used in our proof.

Lemma 3.2.6. $|\mathcal{T}_{n,1}| = 2^{2s_n} \cdot \binom{n-2s_n}{2}$.

3.3 Proof of Lemma 3.2.3

Let H be the subgraph of $G_{n,k}$ induced by vertex set $\{v_1, \ldots, v_k, w\}$ (see Figure 3.4). Furthermore, set $n_1 = n - k + 1$, and let $G^+_{n_1}$ be the graph obtained from $G_{n,k}$ by contracting vertex set $\{v_1, \ldots, v_k\}$ into vertex v_1 and removing any resulting loops and multiple edges (see Figure 3.5). Clearly, $G^+_{n_1}$ is isomorphic to $G_{n_1,1}$.

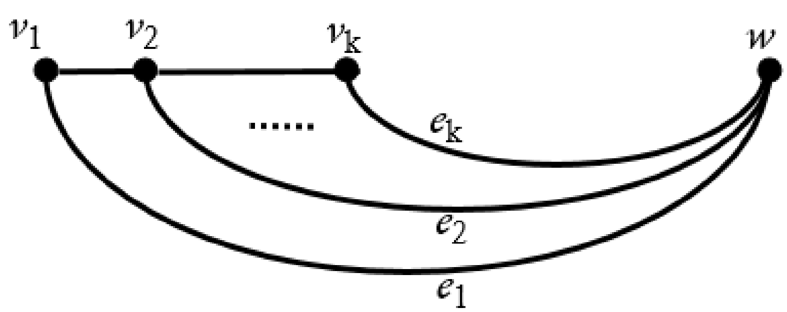

Figure 3.4 The subgraph H

Let $T \in \mathcal{T}_{n,k}$, that is, T is a subtree of $G_{n,k}$ containing the vertex set $\{v_1, v_k, w\}$ but not containing the path $P^* = v_k \cdots w$. Let T_1 be the subgraph of H induced by $E(H) \cap E(T)$. Since T does not contain the path P^*, we have that T_1 is connected, and so it is a subtree of H. Let T_2 be the graph obtained from T by contracting vertex set $\{v_1, \ldots, v_k\}$ into the vertex v_1 and removing any resulting loops and multiple edges. Since T_1 is connected and

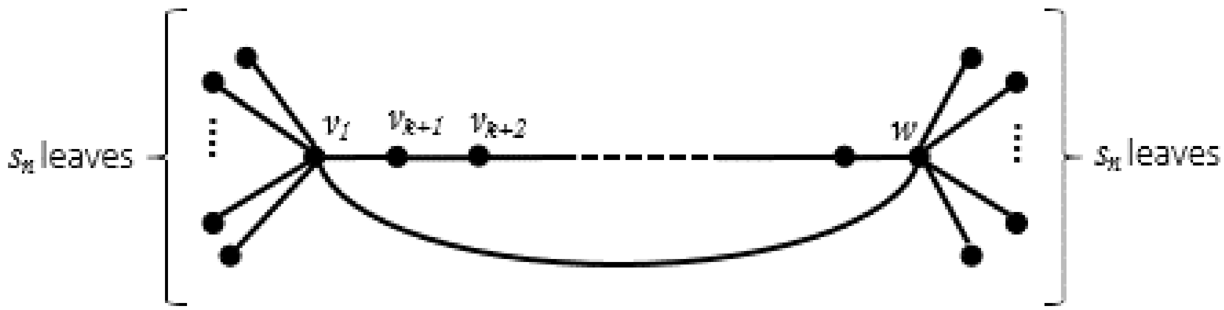

Figure 3.5 $G_{n_1}^{+}$ obtained from $G_{n,k}$ by contracting $\{v_1, \ldots, v_k\}$ into vertex v_1

contains vertex set $\{v_1, v_k, w\}$, it follows that T_2 is a subtree of $G_{n_1}^{+}$ containing edge v_1w. So, each $T \in \mathcal{T}_{n,k}$ corresponds to a unique pair (T_1, T_2) of trees, where T_1 is a subtree of H containing vertex set $\{v_1, v_k, w\}$, and $T_2 \in \mathcal{T}_{n_1,1}$. We also notice that $|T| = |T_1| + |T_2| - 2$, where the -2 arises due to the fact that T_1 and T_2 share exactly two vertices v_1 and w.

Let $\mathcal{T}'_H \subseteq \mathcal{T}_H$ be the family of subtrees of H containing vertex set $\{v_1, v_k, w\}$. By the corresponding relationship above, we have $|\mathcal{T}_{n,k}| = |\mathcal{T}'_H| \cdot |\mathcal{T}_{n_1,1}|$. Hence, we obtain that

$$\begin{aligned}
\mu(\mathcal{T}_{n,k}) &= \frac{\sum\limits_{T \in \mathcal{T}_{n,k}} |T|}{|\mathcal{T}_{n,k}|} = \frac{\sum\limits_{T_1 \in \mathcal{T}'_H} \sum\limits_{T_2 \in \mathcal{T}_{n_1,1}} (|T_1| + |T_2| - 2)}{|\mathcal{T}'_H| \cdot |\mathcal{T}_{n_1,1}|} \\
&= \frac{|\mathcal{T}'_H| \cdot \sum\limits_{T_2 \in \mathcal{T}_{n_1,1}} |T_2| + |\mathcal{T}_{n_1,1}| \cdot \sum\limits_{T_1 \in \mathcal{T}'_H} |T_1| - 2|\mathcal{T}_{n_1,1}| \cdot |\mathcal{T}'_H|}{|\mathcal{T}'_H| \cdot |\mathcal{T}_{n_1,1}|} \\
&= \mu(\mathcal{T}_{n_1,1}) + \mu(\mathcal{T}'_H) - 2.
\end{aligned}$$

Dividing through by n, we further gain that

$$\sigma(\mathcal{T}_{n,k}) = \frac{n_1}{n} \cdot \sigma(T_{n_1,1}) + \frac{k+1}{n} \cdot \sigma(\mathcal{T}'_H) - \frac{2}{n}.$$

Since $\sigma(\mathcal{T}'_H)$ is always bounded by 1, it follows that $\lim\limits_{n\to\infty} \frac{k+1}{n} \cdot \sigma(\mathcal{T}'_H) = 0$. Combining this with $\lim\limits_{n\to\infty} \frac{n_1}{n} = 1$ and $\lim\limits_{n\to\infty} \frac{2}{n} = 0$, we get $\lim\limits_{n\to\infty} \sigma(\mathcal{T}_{n,k}) = \lim\limits_{n\to\infty} \sigma(\mathcal{T}_{n_1,1}) = \frac{2}{3}$ (by Lemma 3.2.2), which completes the proof of Lemma 3.2.3. □

3.4 Proof of Lemma 3.2.4

Let $\mathcal{T}bar_{n,k} := \mathcal{T}_{G_{n,k}} \backslash \mathcal{T}_{n,k}$. If $\lim\limits_{n\to\infty} |\mathcal{T}bar_{n,k}|/|\mathcal{T}_{n,k}| = 0$, then $\lim\limits_{n\to\infty} \frac{|\mathcal{T}bar_{n,k}|}{|\mathcal{T}_{n,k}|+|\mathcal{T}bar_{n,k}|} = 0$ because $\frac{|\mathcal{T}bar_{n,k}|}{|\mathcal{T}_{n,k}|+|\mathcal{T}bar_{n,k}|} \leq |\mathcal{T}bar_{n,k}|/|\mathcal{T}_{n,k}|$, and so $\lim\limits_{n\to\infty} \frac{|\mathcal{T}_{n,k}|}{|\mathcal{T}_{n,k}|+|\mathcal{T}bar_{n,k}|} = 1$. Hence,

$$\begin{aligned}
\lim_{n\to\infty} \sigma(G_{n,k}) &= \lim_{n\to\infty} \frac{\mu(G_{n,k})}{n} = \lim_{n\to\infty} \frac{1}{n} \cdot \left(\frac{\sum\limits_{T\in\mathcal{T}_{n,k}} |T|}{|\mathcal{T}_{n,k}| + |\mathcal{T}bar_{n,k}|} + \frac{\sum\limits_{T\in\mathcal{T}bar_{n,k}} |T|}{|\mathcal{T}_{n,k}| + |\mathcal{T}bar_{n,k}|} \right) \\
&= \lim_{n\to\infty} \left(\sigma(\mathcal{T}_{n,k}) \cdot \frac{|\mathcal{T}_{n,k}|}{|\mathcal{T}_{n,k}| + |\mathcal{T}bar_{n,k}|} + \sigma(\mathcal{T}bar_{n,k}) \cdot \frac{|\mathcal{T}bar_{n,k}|}{|\mathcal{T}_{n,k}| + |\mathcal{T}bar_{n,k}|} \right) = \lim_{n\to\infty} \sigma(\mathcal{T}_{n,k}).
\end{aligned}$$

Thus, to complete the proof, it suffices to show that $\lim\limits_{n\to\infty} |\mathcal{T}bar_{n,k}|/|\mathcal{T}_{n,k}| = 0$. We now define the following two subfamilies of $\mathcal{T}_{G_{n,k}}$.

- $\mathcal{B}_1 = \{T \in \mathcal{T}_{G_{n,k}} \ : \ v_1 \notin V(T) \text{ or } w \notin V(T)\}$; and
- $\mathcal{B}_2 = \{T \in \mathcal{T}_{G_{n,k}} \ : \ T \cap P^* \text{ is a path, and } T \text{ contains } w\}$.

Recall that $\mathcal{T}_{n,k}$ is the family of subtrees of $G_{n,k}$ containing vertex set $\{v_1, v_k, w\}$ and not containing the path $P^* = v_k \cdots w$. For any $T \in \mathcal{T}bar_{n,k}$, by definition, we have the following scenarios: $v_1 \notin V(T)$, and so $T \in \mathcal{B}_1$ in this case; $w \notin V(T)$, and so $T \in \mathcal{B}_1$ in this case; $v_k \notin V(T)$ and $w \in V(T)$, then $T \cap P^*$ is a path, and so $T \in \mathcal{B}_2$ in this case; $P^* \subseteq T$, and so $T \in \mathcal{B}_2$ in this case. Consequently, $\mathcal{T}bar_{n,k} \subseteq \mathcal{B}_1 \cup \mathcal{B}_2$, which in turn gives that

$$|\mathcal{T}bar_{n,k}| \leq |\mathcal{B}_1| + |\mathcal{B}_2|. \tag{1}$$

Let S_{v_1} denote the star centered at v_1 with the s_n leaves attached to it and S_w denote the star centered at w with the s_n leaves attached to it. Then $G_{n,k}$ is the union of four subgraphs S_{v_1}, S_w, H, and P^*.

- Considering the subtrees of S_{v_1} with at least two vertices and the subtrees of S_{v_1} with a single vertex, we get $|\mathcal{T}_{S_{v_1}}| = (2^{s_n} - 1) + (s_n + 1) = 2^{s_n} + s_n = 2^{s_n} + o(2^{s_n})$.

- Considering the subtrees of S_w with at least two vertices and the subtrees of S_w with a single vertex, we get $|\mathcal{T}_{S_w}| = (2^{s_n} - 1) + (s_n + 1) = 2^{s_n} + s_n = 2^{s_n} + o(2^{s_n})$.

- Considering the subpaths of P^* with at least two vertices and the subpaths of P^* with a single vertex, we get $|\mathcal{T}_{P^*}| = \binom{|P^*|}{2} + |P^*| = \binom{|P^*|+1}{2} = \binom{n-2s_n-k+2}{2} \leq \frac{n^2}{2}$.

- The number of subpaths of P^* containing w is bounded above by $|P^*| = n-2s_n-k+1 \leq n$.

Since $s_n = o(n)$, we have the following two inequalities

$$\begin{aligned}
|\mathcal{B}_1| &\leq (s_n + |\mathcal{T}_H| \cdot |\mathcal{T}_{P^*}| \cdot |\mathcal{T}_{S_w}|) + (s_n + |\mathcal{T}_H| \cdot |\mathcal{T}_{P^*}| \cdot |\mathcal{T}_{S_{v_1}}|) \\
&\leq 2\left[s_n + |\mathcal{T}_H| \cdot (2^{s_n} + o(2^{s_n})) \cdot \frac{n^2}{2}\right] = |\mathcal{T}_H| \cdot \left(2^{s_n} \cdot n^2 + o(2^{s_n} \cdot n^2)\right) \\
|\mathcal{B}_2| &\leq |\mathcal{T}_{S_{v_1}}| \cdot |\mathcal{T}_{S_w}| \cdot |P^*| \cdot |\mathcal{T}_H| = \left(2^{2s_n} \cdot n + o(2^{2s_n} \cdot n)\right) \cdot |\mathcal{T}_H|.
\end{aligned}$$

Recall that $n_1 = n - k + 1$. Applying Lemma 3.2.6, we have

$$|\mathcal{T}_{n,k}| = |\mathcal{T}'_H| \cdot |\mathcal{T}_{n_1,1}| = |\mathcal{T}'_H| \cdot 2^{2s_n} \binom{n_1 - 2s_n}{2} = |\mathcal{T}'_H| \cdot 2^{2s_n} \cdot \left(\frac{n^2}{2} - o(n^2)\right).$$

Recall that $2^{s_n} \geq n^2$. Since $|\mathcal{T}_H|$ is bounded by a function of k because $|H| = k+1$, we have the following two inequalities.

$$\lim_{n\to\infty} \frac{|\mathcal{B}_1|}{|\mathcal{T}_{n,k}|} \quad = \quad \lim_{n\to\infty} \frac{|\mathcal{T}_H| \cdot 2^{s_n} \cdot n^2}{|\mathcal{T}'_H| \cdot 2^{2s_n} \cdot \frac{n^2}{2}} = \lim_{n\to\infty} \frac{2|\mathcal{T}_H|}{|\mathcal{T}'_H| \cdot 2^{s_n}} = 0$$

and

$$\lim_{n\to\infty} \frac{|\mathcal{B}_2|}{|\mathcal{T}_{n,k}|} \quad = \quad \lim_{n\to\infty} \frac{2^{2s_n} \cdot n \cdot |\mathcal{T}_H|}{|\mathcal{T}'_H| \cdot 2^{2s_n} \cdot \frac{n^2}{2}} = \lim_{n\to\infty} \frac{2 \cdot |\mathcal{T}_H|}{|\mathcal{T}'_H| \cdot n} = 0.$$

Hence, we conclude that

$$\lim_{n\to\infty} \frac{|\mathcal{B}_1| + |\mathcal{B}_2|}{|\mathcal{T}_{n,k}|} = 0$$

Combining this with (**1**), we have that $\lim_{n\to\infty} |\mathcal{T}bar_{n,k}|/|\mathcal{T}_{n,k}| = 0$, which completes the proof of Lemma 3.2.4. $\square$

3.5 Concluding Remarks

The graphs we generated for Theorem 3.1.2 are certainly not unique. Apart from the k edges set $\{v_1w, v_2w, \ldots, v_kw\}$, several different k edges sets may be added to G_n to decrease the mean subtree order, such as $\{v_1v_{n-2s_n}, v_2v_{n-2s_n-1}, \ldots, v_kv_{n-2s_n-k+1}\}$.

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