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On the 4 by 4 Irreducible Sign Pattern Matrices that Require Four Distinct Eigenvalues

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ON THE 4×4 IRREDUCIBLE SIGN PATTERN MATRICES
THAT REQUIRE FOUR DISTINCT EIGENVALUES

by

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Under the Direction of Drs. Frank J. Hall and Zhongshan Li

ABSTRACT

A *sign pattern matrix* is a matrix whose entries are from the set $\{+, -, 0\}$. For a real matrix B, $\text{sgn}(B)$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by $+$ (respectively, $-$, 0). For a sign pattern matrix A, the *sign pattern class of* A, denoted $Q(A)$, is defined as $\{ B : \text{sgn}(B) = A \}$.

An $n \times n$ sign pattern matrix A requires all distinct eigenvalues if every real matrix whose sign pattern is represented by A has n distinct eigenvalues. In this thesis, a number of sufficient and/or necessary conditions for a sign pattern to require all distinct eigenvalues are reviewed. In addition, for $n = 2$ and 3, the $n \times n$ sign patterns that require all distinct eigenvalues are surveyed. We determine most of the 4×4 irreducible sign patterns that require four distinct eigenvalues.

INDEX WORDS: Sign pattern matrices, Distinct eigenvalues, Cycles, Permutational similarity, Signature similarity, Discriminant, Nonsingular matrix, Diagonal equivalence, Diagonalizability, Resultant

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1. Introduction and Preliminaries

In qualitative and combinatorial matrix theory, we study properties of a matrix based on combinatorial information, such as the signs of entries in the matrix. Such approach originated from the work in the 1940's of the Nobel Economics Prize winner P.A. Samuelson, as described in his original and creative book Foundations of Economic Analysis [12] in 1947. Due to its theoretical importance and applications in economics, biology, chemistry, sociology and computer science, qualitative and combinatorial matrix analysis flourished in the past few decades. R. Brualdi and B. Shader summarized and organized some of the research in this area in their 1995 book "Matrices of Sign-solvable Linear Systems" [2].

A matrix whose entries come from the set $\{+,-,0\}$ is called a *sign pattern matrix*. We denote the set of all $n \times n$ sign pattern matrices by Q_n, and more generally, the set of all $m \times n$ sign pattern matrices by $Q_{m,n}$. For a real matrix B, $sgn(B)$ is the sign pattern matrix obtained by replacing each positive (respectively, negative, zero) entry of B by $+$ (respectively, $-$, 0). If $A \in Q_{m,n}$, then the *sign pattern class of* A is defined by

$$Q(A) = \{B \ : \ sgn(B) = A\}.$$

For $A \in Q_{m,n}$, the *minimum rank* of A, denoted as $mr(A)$, is defined by

$$mr(A) = \min \{\text{rank } B : B \in Q(A)\}.$$

The *maximum rank* of A, $MR(A)$, is given by

$$MR(A) = \max \{\text{rank } B : B \in Q(A)\}.$$

The minimum rank of a sign pattern is not only of interest theoretically, it is also of practical value. For instance [3] is devoted to the question of constructing real $m \times n$ matrices of low rank under the constraint that each entry is nonzero and has a given sign. This problem arises from an interesting topic in neural networks or, more specifically, multilayer perceptrons. In this application, the rank of a realization matrix can be interpreted as the number of elements in a hidden layer, which motivates a search for low rank solutions.

As shown in [11], there exist sign patterns A such that $\mathrm{mr}(A)$ cannot be achieved by any rational matrix $B \in Q(A)$.

The characterization of the $\mathrm{mr}(A)$ (or finding $\mathrm{mr}(A)$) for a general $m \times n$ sign pattern matrix A is difficult and is a long outstanding problem. However, the $\mathrm{MR}(A)$ is easily described. Indeed, $\mathrm{MR}(A)$ is equal to the *term rank* of A, namely,the maximum number of nonzero entries of A in distinct rows and columns.

A sign pattern matrix S is called a *permutation pattern* if exactly one entry in each row and column is equal to $+$, and all the other entries are 0. A product of the form S^TAS, where S is a permutation pattern, is called a *permutational similarity*. We say that A and S^TAS are *permutationally similar*. Two sign pattern matrices A_1 and A_2 are said to be *permutationally equivalent* if there are permutation patterns S_1 and S_2 such that $A_1 = S_1A_2S_2$.

A diagonal sign pattern D is called a *signature sign pattern* if each of its diagonal entries is either $+$ or $-$. For a signature sign pattern D and a sign pattern A of the same order, we say that DAD and A are signature similar. Two sign patterns A_1 and A_2 are said to be *signature equivalent* if $A_1 = D_1A_2D_2$ for some signature sign patterns matrices D_1 and D_2.

If $A = (a_{ij})$ is an $n \times n$ sign pattern matrix, then a formal product of the form $\gamma = a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_k i_1}$, where each of the elements is nonzero and the index set $\{i_1, i_2, \ldots, i_k\}$ consists of distinct indices, is called a *simple cycle* of length k, or a k-cycle, in A. A *composite cycle* γ in A is a product of simple cycles, say $\gamma = \gamma_1 \gamma_2 \ldots \gamma_m$, where the index sets of the γ_i's are mutually disjoint. If the length of γ_i is l_i, then the length of γ is $\sum_{i=1}^{m} l_i$. If we say a cycle γ is an *odd* (respectively *even) cycle*, we mean that the length of the simple or composite cycle γ is odd (even). In this thesis, the term cycle always refers to a composite cycle (which, as a special case, could be a simple cycle).

Let $A = (a_{ij})$ be an $n \times n$ sign pattern matrix. The digraph of A, denoted $D(A)$, is the directed graph with vertex set $\{1, 2, \ldots, n\}$ such that (i, j) is an arc of $D(A)$ iff $a_{ij} \neq 0$. The (undirected) graph of A, denoted $G(A)$, is the graph with vertex set $\{1, 2, \ldots, n\}$ such that $\{i, j\}$ is an edge of $G(A)$ iff at least one of the entries a_{ij} and a_{ji} is nonzero.

An undirected graph G is a tree if it is connected and has no cycles (thus G is minimally connected). For a symmetric $n \times n$ sign pattern A, by $G(A)$ we mean the undirected graph of A, with vertex set $\{1, \ldots, n\}$ and $\{i, j\}$ is an edge if and only if $a_{ij} \neq 0$. A sign pattern A is a *symmetric tree sign pattern* if A is symmetric and $G(A)$ is a tree, possibly with loops.

Suppose P is a property referring to a real matrix. A sign pattern A is said to *require* P if every matrix in $Q(A)$ has property P; A is said to *allow* P if some real matrix in $Q(A)$ has property P. A sign pattern $A \in Q_n$ is said to be *sign nonsingular* (SNS for short) if every matrix $B \in Q(A)$ is nonsingular. It is well known that, A is sign nonsingular if and only if det A $= +$ or $\det A = -$, that is, in the standard expansion of $\det A$ into $n!$ terms, there is at least one nonzero term, and all the nonzero terms have the same sign. Note that a nonzero term in such expansion of det A corresponds to a cycle of length n in A. It is also known that if all the diagonal entries of an $n \times n$ sign pattern A are negative, then A is sign nonsingular if and only if every simple cycle in A has negative weight (namely, the product of the entries of every simple cycle in A is negative.)

For $n \geq 3$, every $n \times n$ SNS sign pattern has at least $\binom{n-1}{2}$ zero entries.

If $D(A)$ of a sign pattern A is a k-cycle, then A is called a *k-cycle sign pattern.*

Matrices all of whose eigenvalues are distinct have some nice properties, such as diagonalizability; those matrices have been studied in a number of papers, for instance [10],[14], and [15].

Sign patterns that allow all eigenvalues to be distinct are easily characterized as follows.

A sign pattern A of order n allows all eigenvalues to be distinct if and only if the maximal composite cycle length of A is $n-1$ or n ; refer to [7].

A square sign pattern matrix A is said to be *reducible* if

$$P^T AP = \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}$$

for some permutation sign pattern matrix P, where A_{11} and A_{22} are nonempty square matrices. A square sign pattern that is not reducible is said to be *irreducible.* It is well known that A is irreducible if and only if the digraph of A is strongly connected.

Let A be an $n \times n$ sign pattern and let A_I denote the sign pattern obtained from A by replacing all the diagonal entries by $+$. Then A_I requires n distinct eigenvalues if and only if A is permutationally similar to a symmetric irreducible tridiagonal sign pattern; refer to [5].

Let $\mathcal{DE}$ represent the set of all sign patterns that require the property of all distinct eigenvalues. In this paper, we determine all 4×4 irreducible sign patterns in $\mathcal{DE}$.

Since matrices with all distinct eigenvalues have many nice properties, such as diagonalizability, it is quite useful to be able to predict if a certain matrix has all distinct eigenvalues by inspecting the sign pattern of the matrix.

Let $\mathbb{F}$ be a field and let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$ be two polynomials in $\mathbb{F}[x]$.

A necessary and sufficient condition for $f(x)$ and $g(x)$ to have a common root (or, equivalently, a common divisor of degree at least 1 in $\mathbb{F}[x]$) is the existence of a polynomial $a(x)$ in $\mathbb{F}[x]$ of degree at most $m-1$ and a polynomial $b(x)$ in $\mathbb{F}[x]$ of degree at most $n-1$ with $a(x)f(x) = b(x)g(x)$. Writing $a(x)$ and $b(x)$ explicitly as polynomials and then equating coefficients in the equation $a(x)f(x) = b(x)g(x)$ gives a system of $n+m$ linear equations for the coefficients of $a(x)$ and $b(x)$. This system has a nontrivial solution (hence $f(x)$ and $g(x)$ have a common zero) if and only if the determinant of the matrix of order $m + n$ displayed below

$$\mathrm{Res}(f,g) = \det \begin{bmatrix} a_n & a_{n-1} & \cdots & a_0 & & & \\ & a_n & a_{n-1} & \cdots & a_0 & & \\ & & \ddots & \ddots & \cdots & \ddots & \\ & & & a_n & a_{n-1} & \cdots & a_0 \\ b_m & b_{m-1} & \cdots & b_0 & & & \\ & b_m & b_{m-1} & \cdots & b_0 & & \\ & & \ddots & \ddots & \cdots & \ddots & \\ & & & b_m & b_{m-1} & \cdots & b_0 \end{bmatrix}$$

is zero. Here $\mathrm{Res}(f,g)$, called the *resultant* of the two polynomials f and g.

Let $f(x)$ and $g(x)$ be two polynomials over a field $\mathbb{F}$ with roots of $f(x)$ being

$x_1, \ldots, x_n$ and the roots of $g(x)$ being $y_1, \ldots, y_m$. The coefficients of $f(x)$ are a_n times the elementary symmetric functions in $x_1, x_2, \ldots, x_n$ properly signed, and the coefficients of $g(x)$ are b_m times the elementary symmetric functions in $y_1, y_2, \ldots, y_m$ properly signed.

By expanding the determinant, it can be seen that $\text{Res}(f, g)$ is homogeneous of degree m in the coefficients a_i and homogeneous of degree n in the coefficients b_j. Furthermore,

$$\text{Res}(f, g) = a_n^m b_m^n \prod_{1 \le i \le n, 1 \le j \le m} (x_i - y_j).$$

The above formula for $\text{Res}(f, g)$ can also be written as

$$\text{Res}(f, g) = a_n^m \prod_{i=1}^{n} g(x_i) = (-1)^{nm} b_m^n \prod_{j=1}^{m} f(y_j).$$

This gives an interesting reciprocity between the product of g evaluated at the roots of f and the product of f evaluated at the roots of g.

Consider now the special case where $g(x) = f'(x)$ is the formal derivative of the polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and suppose the roots of $f(x)$ are $r_1, r_2, \ldots, r_n$. Using the formula $\text{Res}(f, f') = \prod_{i=1}^n f'(r_i)$ of the previous paragraph, one gets that

$$D = (-1)^{n(n-1)/2} \text{Res}(f, f'),$$

where $D = \prod\limits_{i<j} (r_i - r_j)^2$ is known as the *discriminant* of $f(x)$, denoted $\text{disc}(f(x))$.

It is clear that $f(x)$ and $g(x)$ have no common zero if and only if $\text{Res}(f(x), g(x)) \neq 0$. It is well known that $f(x)$ does not have a multiple root if and only if $\text{Res}(f(x), f'(x)) \neq 0$, if and only if $\text{disc}(f(x)) \neq 0$.

Lemma 1.1. ([15]) The set $\mathcal{DE}$ is closed under the following operations:

1. Negation,

2. Transposition,

3. Permutational similarity, and

4. Signature similarity.

We say two sign patterns are equivalent if one can be obtained from the other by performing a sequence of operations listed on Lemma 1.1. The following three lemmas are useful mechanisms.

Lemma 1.2. ([5], [15]) If a sign pattern A is in $\mathcal{DE}$, then A requires a fixed number of real eigenvalues.

Lemma 1.3. ([15]) Let $A \in Q_n$. Then $A \in \mathcal{DE}$ if and only if for all permissible values of the entries of a general matrix $B \in Q(A)$, $\mathrm{Res}(P_B(x), P'_B(x)) \neq 0$, where $P_B(x) = \det(xI - B)$ is the characteristic polynomial of B.

Lemma 1.4. ([15]) Let A be an n-cycle sign pattern. Then for each $B \in Q(A)$, the eigenvalues of B are evenly distributed on a circle in the complex plane centered at the origin, and the arguments of the eigenvalues of B are $2k\pi/n\,(0 \leq k \leq n-1)$ or $(2k+1)\pi/n, (0 \leq k \leq n-1)$, depending on whether the n-cycle in A is positive or negative.

Note that for a real matrix B, the nonreal eigenvalues occur in complex conjugate pairs. By Lemma 1.2, every 4×4 sign pattern $A \in \mathcal{DE}$ falls into exactly one of the following three cases:

1. A requires four distinct real eigenvalues.
2. A requires two pairs distinct conjugate non-real eigenvalues.

3. A requires a pair of conjugate non-real eigenvalues and two distinct real eigenvalues.

We shall consider each of these cases separately in three subsequent sections.

2. 2×2 and 3×3 Sign Patterns Requiring All Distinct Eigenvalues

The following theorems provide several sufficient conditions, and one necessary and sufficient condition, for a sign pattern matrix to require all distinct eigenvalues. We begin with three fundamental results. A square sign pattern $A=(a_{ij})$ is said to be symmetric (respectively, skew symmetric) if $a_{ji}=a_{ij}$ (respectively, $a_{ji}=-a_{ij}$) for all i and j. For the sake of completeness, we also include some basic proofs.

Theorem 2.1. ([5]) If A is an $n\times n$ symmetric irreducible tridiagonal sign pattern matrix, then A requires n distinct eigenvalues.

Proof: Assume that A is an $n\times n$ symmetric irreducible tridiagonal sign pattern matrix. Let B be a matrix in $Q(A)$,

$$B=\begin{bmatrix} c_1 & a_1 & & & \\ b_1 & c_2 & a_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & a_{n-1} \\ & & & b_{n-1} & c_n \end{bmatrix}.$$

Since B and $S^{-1}BS$ have the same spectrum for any signature matrix S (namely, a diagonal matrix with diagonal entries equal to 1 or -1), we may assume that $a_i>0$ and $b_i>0$ for all $i=1,2,\ldots,n-1$. Let

$$D=\ \text{diag}\left(1,\sqrt{\frac{b_1}{a_1}},\sqrt{\frac{b_1b_2}{a_1a_2}},\ldots,\sqrt{\frac{b_1b_2\ldots b_{n-1}}{a_1a_2\ldots a_{n-1}}}\right).$$

Then

$$B_1=D^{-1}BD=\begin{bmatrix} c_1 & \gamma_1 & & & \\ \gamma_1 & c_2 & \gamma_2 & & \\ & \gamma_2 & c_3 & \ddots & \\ & & \ddots & \ddots & \gamma_{n-1} \\ & & & \gamma_{n-1} & c_n \end{bmatrix},$$

where $\gamma_i = \sqrt{a_i b_i}$. Since B_1 is a real symmetric matrix, it is normal and hence diagonalizable, and all eigenvalues of B_1 are real. For each λ in $\sigma(B_1)$, $B_1 - \lambda I$ has a nonsingular submatrix of order $n-1$ in the upper-right corner, so that rank $(B_1 - \lambda I) =$ rank $(B - \lambda I) = n-1$. Hence the geometric multiplicity of λ is 1. Since B_1 is diagonalizable, for each eigenvalue the geometric multiplicity is equal to the algebraic multiplicity. Thus, each eigenvalue of B is algebraically simple, and it follows that A requires all distinct real eigenvalues. $\square$

Since the proof of the following theorem is very similar to the previous proof, Theorem 2.2 is stated here without proof.

Theorem 2.2. ([5]) If A is an $n \times n$ skew-symmetric irreducible tridiagonal sign pattern matrix, then A requires n distinct pure imaginary (possibly including zero) eigenvalues.

Theorem 2.3. ([15]) If A is an $n \times n$ sign pattern matrix such that $D(A)$ is an n-cycle, then $A \in \mathcal{DE}$.

Proof: Let A be a sign pattern matrix such that $D(A)$ is an n-cycle. Consider the characteristic polynomial of any matrix $B \in Q(A)$. Since for $k \leq n-1$, all the $k \times k$ principal minors of B are zero, we see that the characteristic polynomial of B is of the form $x^n - c$ where $c \neq 0$ is a constant. It follows that B has n distinct eigenvalues, namely, the n distinct n-th roots of c. $\square$

An $n \times n$ sign pattern matrix such that $D(A)$ is an n-cycle is called an n-cycle sign pattern. It is clear from the proof of Theorem 2.3 that an n-cycle sign pattern A requires the property that the eigenvalues of every $B \in Q(A)$ are evenly distributed on a circle in the complex plane centered at the origin, and the arguments of the eigenvalues of B are $2k\pi/n$ $(0 \leq k \leq n-1)$ or

$(2k+1)\pi/n$ $(0 \le k \le n-1)$, depending on whether the n-cycle in A is positive or negative. This observation may be used to construct sign patterns in $\mathcal{DE}$ whose digraphs are unions of simple cycles with disjoint index sets.

Proposition 2.4. For $n = 3$, if every $B \in Q(A)$ has precisely one real eigenvalue, then $A \in \mathcal{DE}$.

Proof: Assume that A is a 3×3 sign pattern matrix such that every $B \in Q(A)$ has precisely one real eigenvalue. Let B be an arbitrary matrix in $Q(A)$. Then B has precisely one real eigenvalue. Therefore, B has precisely two imaginary eigenvalues. Since B is a real matrix, the two non-real eigenvalues of B are conjugates of each other and are distinct. They are certainly distinct from the real eigenvalue. Therefore, B has 3 distinct eigenvalues. Thus $A \in \mathcal{DE}$. $\square$

For a sign pattern matrix A, we may represent the non-zero entries of a general matrix $B \in Q(A)$ by variables which are permitted to assume any positive real values, or negations of such variables. For instance, let

$$A = \begin{bmatrix} 0 & + & 0 \\ - & 0 & - \\ - & 0 & + \end{bmatrix}.$$

Then a general matrix $B \in Q(A)$ may be written as

$$B = \begin{bmatrix} 0 & a & 0 \\ -b & 0 & -c \\ -d & 0 & e \end{bmatrix},$$

where $a, b, c, d, e > 0$. Thus, the characteristic polynomial of B is

$$P_B(x) = \det(xI - B) = x^3 - ex^2 + abx - a(cd + be).$$

Since a polynomial $f(x)$ has no multiple root if and only if $\mathrm{Res}(f(x), f'(x)) \neq 0$, we obtain the following characterization of sign patterns in $\mathcal{DE}$.

Theorem 2.5. Let A be an $n \times n$ sign pattern. Then $A \in \mathcal{DE}$ if and only if $\mathrm{Res}(P_B(x), P'_B(x)) \neq 0$ for all permissible values of the entries of a general matrix $B \in Q(A)$.

We remark that the coefficients of the polynomial $P_B(x)$ are polynomials with integer coefficients over the positive variables that represent the nonzero entries of B. Thus, $(-1)^{n(n-1)/2}\mathrm{Res}(P_B(x), P'_B(x))$, known as the discriminant of $P_B(x)$, is also a polynomial with integer coefficients over the variables that represent the nonzero entries of B. Therefore, the condition in Theorem 2.5. amounts to testing if an integer coefficient polynomial equation has a positive solution. Even though it is theoretically possible to determine whether a multi-variable polynomial equation has a positive solution (refer to [13], 5.6), there does not seem to be an efficient algorithm for this purpose.

The following theorems establish several necessary conditions for a sign pattern matrix to be an element of $\mathcal{DE}$.

Theorem 2.6. ([5], [15]) If a sign pattern matrix A is in $\mathcal{DE}$, then A requires a fixed number of real eigenvalues.

Proof: Suppose that $A \in \mathcal{DE}$. For any B_1 and B_2 in $Q(A)$, define $B(t) = B_1 + t(B_2 - B_1) = (1-t)B_1 + B_2$ and $S_{B(t)} = (k(t), n-k(t))$, where $k(t)$ is the number of real eigenvalues of $B(t)$.

Note that if $b_1 = b_2$, then $b_1 + t(b_2 - b_1) = b_1$ for all t; and if $b_1 \neq b_2$ and $sgn\ b_1 = sgn\ b_2$, then $b_1 + t(b_2 - b_1)$ has the same sign for all $t \in [0-\delta, 1+\delta]$, where

$$0 < \delta < \frac{1}{|b_2 - b_1|}\min\{|b_1|, |b_2|\}.$$

Consequently, for sufficiently small $\delta_1 > 0, B(t) \in Q(A)$ for all t in $[0-\delta_1, 1+\delta_1]$. Let c be in the interval $[0,1]$. Then by hypothesis, $B(c)$ has n distinct eigenvalues, and without loss of generality, we may assume the real eigenvalues are $\lambda_1, \lambda_2, \ldots, \lambda_{k(c)}$ and the nonreal eigenvalues are $\lambda_{k(c)+1}, \lambda_{k(c)+2}, \ldots, \lambda_n$. Let

$$\begin{aligned}
\epsilon_1 &= \min \left\{ \frac{|\lambda_i - \lambda_j|}{2} \mid 1 \leq i < j \leq n \right\}, \\
\epsilon_2 &= \min \{|\mathrm{Im}(\lambda_j)| \;\mid\; k(c)+1 \leq j \leq n\}, \\
\epsilon_c &= \min \{\epsilon_1, \epsilon_2\}, \quad \text{and} \\
D_i &= \{x \in C| \;\; |x - \lambda_i| < \epsilon_c\}.
\end{aligned}$$

Note that $D_i \bigcap D_j = \phi$ if $i \neq j$.

At this point, we make use of the fact that the eigenvalues of $B(t)$ are continuous functions of t. We know that $B(c)$ has n distinct eigenvalues. Consequently, there exists a $\delta_c > 0$ such that for each $i, 1 \leq i \leq n$, the disc D_i contains precisely one eigenvalue of $B(t)$ whenever $|t-c| < \delta_c$. Since $B(t)$ is a real matrix for any real number t, we know that the nonreal eigenvalues occur in complex conjugate pairs, and it follows that each disc $D_i,\ 1 \leq i \leq k(c)$, contains exactly one real eigenvalue of $B(t)$. The other $n-k(c)$ eigenvalues are contained in the discs $D_j,\ k(c)+1 \leq j \leq n$.

However, since $\epsilon_c \leq \epsilon_2$, we have $D_j \bigcap R = \phi$ for $k(c)+1 \leq j \leq n$, and we conclude that the $n-k(c)$ eigenvalues contained in these discs are nonreal. Thus, $k(t) = k(c)$ for all $t \in (c-\delta_c, c+\delta_c)$.

As c ranges over the interval $[0,1]$, $\{(c-\delta_c, c+\delta_c)\}_c$ is an open cover of the compact set $[0,1]$, and it follows that there is a finite subcover of

$$\{(c-\delta_c, c+\delta_c)\}_c.$$

Moreover, for any $c \in [0,1], k(t)$ is a constant function on $(c-\delta_c, c+\delta_c)$, and it follows that $k(t) = k(0)$ on $[0,1]$. Hence, $S_{B_1} = S_{B_2}$ for every B_1 and B_2 in $Q(A)$, that is, A requires a fixed number of real eigenvalues. $\square$

Theorem 2.7. If an $n \times n$ sign pattern matrix A is in $\mathcal{DE}$, then $mr(A) \geq n-1$.

Proof: Suppose A is in $\mathcal{DE}$. Then every $B \in Q(A)$ has n distinct eigenvalues and hence is diagonalizable. Thus, the rank of B is the same as the number of nonzero eigenvalues of B, which is at least $n-1$ as B has n distinct eigenvalues. It follows that $\text{rank}(B) \geq n-1$. Therefore, $mr(A) \geq n-1$. $\square$

Theorem 2.8. If an $n \times n$ sign pattern matrix A has a simple odd cycle of length $k > 1$ and A has at most one zero diagonal entry, then $A \notin DE$.

Proof: Suppose that A has a simple odd cycle γ of length $k > 1$ and at most one zero diagonal entry. By emphasizing the cycle γ (namely, by choosing a matrix $B \in Q(A)$ such that the entries of B in the positions indicated by γ have absolute value 1, while all the other entries of B have absolute values equal to 0 or equal to a sufficiently small $\epsilon > 0$, refer to [5]), we get a matrix $B \in Q(A)$ with at least $k > 1$ nonreal eigenvalues. However, in view of Geršgorin disc theorem, by suitably emphasizing all the nonzero diagonal entries, we can get another matrix $B \in Q(A)$ such that B has n real eigenvalues (and hence B has no nonreal eigenvalues). By Theorem 2.6, we get that $A \notin \mathcal{DE}$. $\square$

Let $\gamma = \gamma_1\gamma_2 \ldots \gamma_m$ be a composite cycle in a sign pattern matrix A of order n, where $\gamma_1, \gamma_2, \ldots, \gamma_m$ are simple cycles in A with disjoint index sets. Let m_1 be the number of odd simple cycles in γ, let m_2 be the number of positive even simple cycles in γ. We define $r(\gamma)$ ([15]) as follows

$$r(\gamma) = \begin{cases} m_1 + 2m_2, & \text{if the length of } \gamma \text{ is not } \ n-1; \\ 1 + m_1 + 2m_2, & \text{if the length of } \ \gamma \text{ is } n-1. \end{cases}$$

By emphasizing the entries contained in γ appropriately (refer to [5]), it can be seen that if γ is a cycle of length $n-1$ or n, then there is a matrix $B \in Q(A)$

that has precisely $r(\gamma)$ real eigenvalues. In view of Theorem 2.6, we arrive at the following result.

Theorem 2.9. Suppose that an $n \times n$ sign pattern A has two composite cycles γ_1 and γ_2 of lengths greater than or equal to $n-1$ and $r(\gamma_1) \neq r(\gamma_2)$. Then $A \notin \mathcal{DE}$. $\square$

If the length of γ is less than $n-1$, then by emphasizing γ, we can find a matrix $B \in Q(A)$ with at least $r(\gamma)$ real eigenvalues. Thus, we have

Theorem 2.10. Suppose that an $n \times n$ sign pattern A has two composite cycles γ_1 and γ_2. If the length of γ_2 is $n-1$ or n and $r(\gamma_1) > r(\gamma_2)$, then $A \notin \mathcal{DE}$. $\square$

We note that for $n \geq 3$, $A \in \mathcal{DE}$ implies that A has a certain degree of sparsity. For instance, if an $n \times n$ irreducible sign pattern A is in $\mathcal{DE}$ and A has n positive diagonal entries, then A requires n distinct real eigenvalues, and A is permutationally similar to a symmetric tridiagonal sign pattern (refer to [5], Corollary 2.3). More generally, if an $n \times n$ irreducible sign pattern A is in $\mathcal{DE}$ and A has a cycle γ with $r(\gamma) = n$, then A requires n distinct real eigenvalues, and hence (refer to [5]), A is a symmetric sign pattern whose graph is a tree (possibly with loops).

Using the results in the previous paragraphs, we will now consider specific sign pattern matrices of orders 2 or 3 that are elements of $\mathcal{DE}$.

Lemma 2.11 The set $\mathcal{DE}$ is closed under the following operations:

1. Negation,
2. Transposition,
3. Permutational similarity,

4. Signature similarity.

In this paper, we say two sign patterns are equivalent if one can be obtained from the other by performing a sequence of operations listed on Lemma 2.11. This is indeed an equivalence relation. The above lemma says that a sign pattern $A \in \mathcal{DE}$ if and only if all the sign patterns equivalent to A are in $\mathcal{DE}$. Thus, to determine the sign patterns in $\mathcal{DE}$ for a specified n, it suffices to consider the $n \times n$ sign patterns up to equivalence.

To illustrate the terminology in the above lemma, we display some sign patterns equivalent to $A = \begin{bmatrix} + & - \\ + & 0 \end{bmatrix}$:

$$-A = \begin{bmatrix} - & + \\ - & 0 \end{bmatrix}$$

(by negation),

$$A^T = \begin{bmatrix} + & + \\ - & 0 \end{bmatrix}$$

(by transposition),

$$P^T AP = \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix} \begin{bmatrix} + & - \\ + & 0 \end{bmatrix} \begin{bmatrix} 0 & + \\ + & 0 \end{bmatrix} = \begin{bmatrix} 0 & + \\ - & + \end{bmatrix}$$

(by permutational similarity),

$$SAS = \begin{bmatrix} - & 0 \\ 0 & + \end{bmatrix} \begin{bmatrix} + & - \\ + & 0 \end{bmatrix} \begin{bmatrix} - & 0 \\ 0 & + \end{bmatrix} = \begin{bmatrix} + & + \\ - & 0 \end{bmatrix}$$

(by signature similarity).

Since a reducible 2×2 sign pattern may be assumed to be upper triangular, the following example is immediate.

Example 2.12. Up to equivalence, the 2×2 reducible matrices in $\mathcal{DE}$ are

$$\begin{bmatrix} + & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} + & + \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} + & 0 \\ 0 & - \end{bmatrix}, \begin{bmatrix} + & + \\ 0 & - \end{bmatrix}.$$

Example 2.13. By Theorems 2.1 and 2.2 we get

$$\begin{bmatrix} * & + \\ + & * \end{bmatrix} \in \mathcal{DE},$$

and

$$\begin{bmatrix} 0 & + \\ - & 0 \end{bmatrix} \in \mathcal{DE},$$

where (and henceforth) * indicates an arbitrary entry ($+, -,$ or 0).

Example 2.14.

$$\begin{bmatrix} + & + \\ - & * \end{bmatrix} \notin \mathcal{DE}.$$

Proof: Let $\gamma_1 = a_{11}$ and $\gamma_2 = a_{12}a_{21}$. Then $r(\gamma_1) = 2$ and $r(\gamma_2) = 0$. Therefore, by Theorem 2.9, the sign pattern is not in $\mathcal{DE}$. $\Box$

Examples 2.13 and 2.14 are consequences of the following result.

Proposition 2.15. ([15]) A 2×2 irreducible sign pattern requires two distinct eigenvalues if and only if it is symmetric or skew-symmetric.

Proof: Since A is irreducible, A has the cycle $\gamma = a_{12}a_{21}$. If γ is a positive cycle, then A is symmetric and $A \in \mathcal{DE}$ by Theorem 2.1. If γ is a negative cycle and both diagonal entries are zero, then A is skew-symmetric and $A \in \mathcal{DE}$ by Theorem 2.2. If γ is a negative cycle, and one or both diagonal entries are nonzero, then by Theorem 2.9, we get $A \notin \mathcal{DE}$. $\Box$

We now consider 3×3 sign patterns. A reducible 3×3 sign pattern A is permutationally similar to a sign pattern of the form

$$\begin{bmatrix} A_1 & * \\ 0 & a \end{bmatrix} \text{ or } \begin{bmatrix} a & * \\ 0 & A_1 \end{bmatrix},$$

where A_1 is 2×2. It is clear that $A \in \mathcal{DE}$ if and only if $A_1 \in \mathcal{DE}$ and A_1 does not allow any eigenvalue of the sign of a. Thus, the 3×3 reducible sign patterns in $\mathcal{DE}$ can be easily determined. The following example displays some 3×3 reducible sign patterns in $\mathcal{DE}$.

Example 2.16.

$$\begin{bmatrix} + & + & * \\ + & - & * \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{DE},$$

$$\begin{bmatrix} 0 & + & * \\ - & 0 & * \\ 0 & 0 & + \end{bmatrix} \in \mathcal{DE},$$

$$\begin{bmatrix} + & * & * \\ 0 & - & * \\ 0 & 0 & 0 \end{bmatrix} \in \mathcal{DE}.$$

Then, we investigate all the 3×3 irreducible sign patterns (up to equivalence) and identify those in $\mathcal{DE}$. It is possible to generate the 3×3 irreducible sign patterns, up to equivalence as defined immediately after Lemma 2.11. This yields a list of 210 sign patterns. Through extensive amount of work involving careful constructions and applications of Theorems 2.1-2.9, all the 3×3 irreducible sign patterns that are in $\mathcal{DE}$ are determined in [15].

The main theorem of this section (Theorem 2.22) states that, up to equivalence, the following sign patterns contained in Examples 2.17-2.21 are the only 3×3 irreducible sign patterns in $\mathcal{DE}$ that cannot be obtained by using Theorems 2.1-2.3.

Example 2.17.

$$\begin{bmatrix} + & + & 0 \\ 0 & 0 & + \\ + & 0 & 0 \end{bmatrix} \in \mathcal{DE}.$$

Proof: By performing a diagonal similarity on a general matrix B of the given sign pattern if necessary, we may assume that the matrix B has the form

$$B = \begin{bmatrix} a & 1 & 0 \\ 0 & 0 & 1 \\ b & 0 & 0 \end{bmatrix},$$

where a and b are some positive numbers.

The characteristic polynomial of B is $p(x) = x^3 - ax - b$. The discriminant of $p(x)$ is $\mathrm{Res}(p(x), p'(x)) = b(27b + 4a^3)$, which is clearly positive for all positive values of a and b. Therefore, the sign pattern of B is in $\mathcal{DE}$. $\square$

Example 2.18.

$$\begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ + & - & 0 \end{bmatrix} \in \mathcal{DE}.$$

Proof: By performing a scalar multiplication and a diagonal similarity on a general matrix B of the given sign pattern if necessary, we may assume that the matrix B has the form

$$B = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ a & -b & 0 \end{bmatrix},$$

where a and b are some positive numbers. The characteristic polynomial of this matrix is $p(x) = x^3 + (1+b)x - a$. Hence, $p'(x) = 3x^2 + (1+b)$ has no real zero. If $p(x) = 0$ has a repeated solution, then it must be a solution of $p'(x) = 0$, which has only pure imaginary roots. However, $p(x)$ has no pure imaginary roots, since if x is pure imaginary, then $x^3 + (1+b)x$ is also pure imaginary and so $x^3 + (1+b)x \neq a$. This is a contradiction. Thus,

$$\begin{bmatrix} 0 & + & 0 \\ - & 0 & - \\ - & + & 0 \end{bmatrix} \in \mathcal{DE}.$$

Alternatively, we may show that the discriminant of $p(x)$, namely, $\mathrm{Res}\ (p(x), p'(x))$, is always nonzero. In fact, $\mathrm{Res}\ (p(x), p'(x)) = 27a^2 + 4b^3 + 12b^2 + 12b + 4$, which is clearly positive for all $a, b > 0$. $\square$

By a similar argument analogous to Example 2.18 (by setting $b = 0$ on the above), we can prove the following result.

Example 2.19.

$$\begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ + & 0 & 0 \end{bmatrix} \in \mathcal{DE}.$$

Example 2.20.

$$\begin{bmatrix} 0 & + & - \\ - & 0 & + \\ + & - & 0 \end{bmatrix} \in \mathcal{DE}.$$

Proof: By performing a diagonal similarity on a general matrix B of the given sign pattern if necessary, we may assume that the matrix B has the form

$$B = \begin{bmatrix} 0 & 1 & -1 \\ -a & 0 & b \\ c & -d & 0 \end{bmatrix},$$

where a, b, c, and d are some positive real numbers. The characteristic polynomial of B and its derivative are:

$$p(x) = x^3 + (a + c + bd)x + (ad - bc),$$

$$p'(x) = 3x^2 + (a + c + bd).$$

Because $p(x)$ is a real polynomial of degree 3, it has no repeated nonreal zero. Assume that $p(x)$ has a repeated real zero. Then $p'(x)$ has a real zero. But, $p'(x)$ clearly has no real zero, a contradiction. Therefore, B has 3 distinct eigenvalues. $\square$

Example 2.21.

$$\begin{bmatrix} + & + & 0 \\ 0 & 0 & + \\ + & - & 0 \end{bmatrix} \in \mathcal{DE}.$$

Proof: By performing a diagonal similarity and a scalar multiplication on a general matrix B of the given sign pattern if necessary, we may assume that the matrix B has the form

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ a & -b & 0 \end{bmatrix},$$

where a and b are some positive real numbers. The characteristic polynomial of B, and its derivative are:

$$p(x) = x^3 - x^2 + bx - (a + b),$$

$$p'(x) = 3x^2 - 2x + b.$$

MAPLE shows that the discriminant of $p(x)$ is

$$27a^2 + 36ab + 8b^2 + 4a + 4b + 4b^3,$$

which is clearly positive whenever $a, b > 0$. So it is nonzero. An application of Theorem 2.5 completes the proof. $\square$

We are now ready to state and prove our main result on 3×3 sign patterns $\in \mathcal{DE}$.

Theorem 2.22 ([15]) Up to equivalence, the 3×3 irreducible sign patterns that require 3 distinct eigenvalues are the irreducible tridiagonal symmetric sign patterns, the irreducible tridiagonal skew-symmetric sign patterns, and the 3-cycle sign patterns, together with the following:

$$\begin{bmatrix} + & + & 0 \\ 0 & 0 & + \\ + & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ + & - & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ + & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & - \\ - & 0 & + \\ + & - & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} + & + & 0 \\ 0 & 0 & + \\ + & - & 0 \end{bmatrix}.$$

Proof: We determine the 3×3 irreducible sign patterns $A \in \mathcal{DE}$ by systematically reviewing a few cases (and subcases).

Case 1: A has no 3-cycle.

Since A is irreducible, $D(A)$ is strongly connected. By performing a permutational similarity on A if necessary, we may assume that $D(A)$ has the directed path of length 2: $1 \to 2 \to 3$. If $a_{31} \neq 0$, then A would have the 3-cycle $a_{12}a_{23}a_{31}$, contradicting the assumption that A has no 3-cycle. Thus, $a_{31} = 0$ and it follows that a directed path in $D(A)$ from 3 to 1 must be $3 \to 2 \to 1$. Since A has no 3-cycle, we also get that $a_{13} = 0$. Therefore, A can be assumed to be tridiagonal.

Subcase1.1: Suppose that both 2-cycles are positive. Then A is symmetric and $A \in \mathcal{DE}$ by Theorem 2.1.

Subcase 1.2: Suppose that there is a positive 2-cycle γ_1 and a negative 2-cycle γ_2. Then $r(\gamma_1) = 3$ and $r(\gamma_2) = 1$. Hence, $A \notin \mathcal{DE}$ by Theorem 2.9.

Subcase 1.3: Suppose both 2-cycles are negative. If all the diagonal entries are zero, then by Theorem 2.2, $A \in \mathcal{DE}$. If there are at least two nonzero diagonal entries, then an application of Theorem 2.9 with γ_1 being a negative 2-cycle and γ_2 being the product of two 1-cycles proves that $A \notin \mathcal{DE}$. We now assume that A has precisely one nonzero diagonal entry. Replacing A with $-A$ if necessary, we may assume that the nonzero diagonal entry is $+$. Up to equivalence, there are two sign patterns satisfying such conditions:

$$\begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ 0 & - & + \end{bmatrix} \text{ and } \begin{bmatrix} 0 & + & 0 \\ - & + & + \\ 0 & - & 0 \end{bmatrix}.$$

The first sign pattern is not in $\mathcal{DE}$ because the matrix

$$B = \begin{bmatrix} 0 & 1 & 0 \\ \frac{-1(5\sqrt{5}-11)}{8} & 0 & 1 \\ 0 & \frac{-1}{4} & 1 \end{bmatrix}$$

has an eigenvalue $\frac{(3-\sqrt{5})}{4}$ of multiplicity 2. This matrix B is constructed through detailed analysis of the discriminant of the characteristic polynomial of a matrix of the form

$$\begin{bmatrix} 0 & 1 & 0 \\ -a & 0 & 1 \\ 0 & -b & 1 \end{bmatrix},$$

where $a > 0, b > 0$. For the second sign pattern $A = \begin{bmatrix} 0 & + & 0 \\ - & + & + \\ 0 & - & 0 \end{bmatrix}$, we first emphasize the 2-cycle $a_{12}a_{21}$ to get a matrix $B_1 \in Q(A)$ with two nonreal eigenvalues. Note that det $A = 0$, so that A requires 0 to be an eigenvalue. By emphasizing the 1-cycle $\gamma_2 = a_{22}$, we get a matrix $B_2 \in Q(A)$ with a real eigenvalue close to 1. In view of 0 as a required eigenvalue, we get that B_2 has 3 real eigenvalues (since a real matrix cannot have just one nonreal eigenvalue). By Theorem 2.6, $A \notin \mathcal{DE}$.

To summarize this case, we have shown that if a 3×3 irreducible sign pattern is in $\mathcal{DE}$ and it does not contain a 3-cycle, then (up to equivalence) it is either tridiagonal symmetric or tridiagonal skew-symmetric.

Case 2: A has a 3-cycle. In view of equivalence, we may assume that A has the 3-cycle $\gamma_1 = a_{12}a_{23}a_{31}$, with $a_{12} = a_{23} = a_{31} = +$.

Subcase 2.1: A has no 2-cycles. If A has at least two 1-cycles, then, by Theorem 2.9, $A \notin \mathcal{DE}$. Now assume that A has precisely one 1-cycle. Then A is equivalent to one of the following two sign patterns:

$$\begin{bmatrix} + & + & 0 \\ 0 & 0 & + \\ + & 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} - & + & 0 \\ 0 & 0 & + \\ + & 0 & 0 \end{bmatrix}.$$

The first one is in $\mathcal{DE}$ by Example 2.17. By negating the (1,1) entry, it can be seen from the proof of Example 2.17 that the second one is not in $\mathcal{DE}$. Indeed, the matrix $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{4}{27} & 0 & 0 \end{bmatrix}$ has $\frac{-2}{3}$ as an eigenvalue of multiplicity 2.

Subcase 2.2: A has at least one 2-cycle.

If A has a positive 2-cycle γ_2, then $r(\gamma_1) = 1$ (where γ_1 is the 3-cycle mentioned in the beginning of Case 2) and $r(\gamma_2) = 3$. Hence, by Theorem 2.9, $A \notin \mathcal{DE}$. Now suppose that every 2-cycle in A is negative. If A has at least two 1-cycles, then, by Theorem 2.9, $A \notin \mathcal{DE}$. Thus, we may assume that A has at most one 1-cycle. This results in the following two subcases.

Subcase 2.2.1: A has no 1-cycle.

Because A has the 3-cycle $\gamma_1 = a_{12}a_{23}a_{31}$ with $a_{12} = a_{23} = a_{31} = +$ and every 2-cycle of A is negative, up to equivalence, there are three possibilities:

$$\begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ + & - & 0 \end{bmatrix}, \begin{bmatrix} 0 & + & 0 \\ - & 0 & + \\ + & 0 & 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 & + & - \\ - & 0 & + \\ + & - & 0 \end{bmatrix}.$$

These sign patterns require all distinct eigenvalues by Examples 2.18-2.20.

Subcase 2.2.2: A has precisely one 1-cycle. Without loss of generality, we may assume that $a_{11} \neq 0$. Because A has the 3-cycle $\gamma_1 = a_{12}a_{23}a_{31}$ with $a_{12} = a_{23} = a_{31} = +$ and every 2-cycle of A is negative, the sign pattern A has the form

$$A = \begin{bmatrix} \pm & + & ? \\ ? & 0 & + \\ + & ? & 0 \end{bmatrix},$$

where each "?" entry can be either $-$ or 0. If $a_{21} = -$, then the leading 2×2 principal submatrix $\begin{bmatrix} \pm & + \\ - & 0 \end{bmatrix}$ allows two distinct nonzero real eigenvalues. Indeed, $B_1 = \begin{bmatrix} \pm 3 & 1 \\ -2 & 0 \end{bmatrix}$ has two distinct nonzero real eigenvalues 1 and 2 (or -1 and -2). By perturbing $\begin{bmatrix} \pm 3 & 1 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ slightly, we can obtain a matrix $B \in Q(A)$ whose eigenvalues are sufficiently close to 0,1, and 2 (or 0, -1, and -2). Thus, there are three disjoint discs on the complex plane with centers on the real axis such that each disc contains precisely one eigenvalue of B. It follows that B has three real eigenvalues. However, by emphasizing the negative 2-cycle $a_{12}a_{21}$, we can obtain a matrix in $Q(A)$ with two nonreal eigenvalues. Thus, by Theorem 2.6, $A \notin \mathcal{DE}$. Similarly, if $a_{13} = -$, then by considering the principal submatrix obtained by deleting the second row and column, we can prove that $A \notin \mathcal{DE}$. We may now assume that $a_{13} = 0$ and $a_{21} = 0$. Because A is assumed to have a negative 2-cycle, we must have $a_{32} = -$. There are two possibilities:

$$\begin{bmatrix} + & + & 0 \\ 0 & 0 & + \\ + & - & 0 \end{bmatrix} \text{ and } \begin{bmatrix} - & + & 0 \\ 0 & 0 & + \\ + & - & 0 \end{bmatrix}.$$

The first one is in $\mathcal{DE}$ by Example 2.21. Since the matrix $\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{1}{4} & \frac{-1}{4} & 0 \end{bmatrix}$ has eigenvalues $0, \frac{-1}{2}$ and $\frac{-1}{2}$, we get that the second sign pattern is not in $\mathcal{DE}$. The proof is now complete. $\square$

For $n \geq 4$, the sign patterns in $\mathcal{DE}$ are not well understood. In particular, it is an open problem to characterize the upper Hesssenberg sign patterns that are in $\mathcal{DE}$.

3. 4×4 Sign Patterns Requiring Four Distinct Real Eigenvalues

Lemma 3.1. Let $A \in Q_n$. If A has a negative 2-cycle or a k-cycle with $k \geq 3$, then A does not require all real eigenvalues, that is, there exists a real matrix $B \in Q(A)$ such that B has at least one pair conjugate nonreal eigenvalues.

Proof: Let γ be a negative 2-cycle or a k-cycle with $k \geq 3$. By emphasizing the cycle γ (namely, by choosing a matrix $B \in Q(A)$ such that the entries of B in the positions indicated by γ have absolute value 1, while all other entries of B have absolute values equal to 0 or equal to a sufficiently small $\varepsilon > 0$, refer to [5]), from Lemma 1.4 we get a matrix $B \in Q(A)$ having at least one pair conjugate nonreal eigenvalues. $\square$

Lemma 3.1 immediately yields the following two results.

Lemma 3.2. Let $A = (a_{ij}) \in Q_4$ require four distinct real eigenvalues. Then the following conditions hold.

(1) For $1 \leq i, j \leq 4$, $a_{ij}a_{ji} \geq 0$.

(2) A has no k-cycles for $k \geq 3$.

Theorem 3.3 Let $A \in Q_4$ be irreducible and require four distinct real eigenvalues. Then A is a symmetric tree sign pattern, and up to equivalence, A is one of the following forms

$$A_1 = \begin{bmatrix} * & 1 & 0 & 0 \\ 1 & * & 1 & 0 \\ 0 & 1 & * & 1 \\ 0 & 0 & 1 & * \end{bmatrix}, \quad A_2 = \begin{bmatrix} * & 1 & 1 & 1 \\ 1 & * & 0 & 0 \\ 1 & 0 & * & 0 \\ 1 & 0 & 0 & * \end{bmatrix},$$

where $*$ may be 1, -1 or 0.

Lemma 3.4 ([5],[15]) If A is an $n \times n$ symmetric irreducible tridiagonal sign pattern, then A requires n distinct real eigenvalues.

Lemma 3.5. Let $A \in Q_4$ be a symmetric star sign pattern having the form

$$A = \begin{bmatrix} a_1 & 1 & 1 & 1 \\ 1 & a_2 & 0 & 0 \\ 1 & 0 & a_3 & 0 \\ 1 & 0 & 0 & a_4 \end{bmatrix},$$

where a_i may be 1, -1 or 0 for $i = 1, 2, 3, 4$. Then A requires four distinct real eigenvalues if and only if a_2, a_3 and a_4 are not the same.

Proof: Let

$$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & b_{14} \\ b_{21} & b_{22} & 0 & 0 \\ b_{31} & 0 & b_{33} & 0 \\ b_{41} & 0 & 0 & b_{44} \end{bmatrix} \in Q(A).$$

Take a nonsingular diagonal matrix

$$D = \operatorname{diag}\left[\ 1,\ \sqrt{\frac{b_{21}}{b_{12}}},\ \sqrt{\frac{b_{21}b_{31}}{b_{12}b_{13}}},\ \sqrt{\frac{b_{21}b_{31}b_{41}}{b_{12}b_{13}b_{14}}}\ \right].$$

Then

$$B_1 = D^{-1}BD = \begin{bmatrix} b_{11} & b_2 & b_3 & b_4 \\ b_2 & b_{22} & 0 & 0 \\ b_3 & 0 & b_{33} & 0 \\ b_4 & 0 & 0 & b_{44} \end{bmatrix} \in Q(A),$$

where $b_i = \sqrt{b_{1i}b_{i1}}$ for $i = 2, 3, 4$. It implies that B can be similar to the symmetric matrix B_1 in $Q(A)$. Thus, we only need to consider all real symmetric matrices in $Q(A)$.

For any real symmetric matrix $B \in Q(A)$, all eigenvalues of B are real, and B is diagonalizable. Thus, for each eigenvalue of B, the geometric multiplicity is equal to the algebraic multiplicity.

We consider the following five cases.

Case 1: $a_2 = a_3 = a_4 = 0$.

For any real symmetric matrix $B \in Q(A)$, it is clear that $\text{rank}(B) = 2$, and B has the zero eigenvalue with the algebraic multiplicity 2. Thus, A does not require four distinct real eigenvalues.

Case 2: a_2, a_3 and a_4 have exactly one nonzero element.

Up to equivalence, we may assume that $a_2 \neq 0$ and $a_3 = a_4 = 0$. Let $B \in Q(A)$ be a real symmetric matrix, and λ be an eigenvalue of B. It is easy to see that $\text{rank}(\lambda I - B) = 3$, and so λ is algebraically simple. Then B has four distinct real eigenvalues, and A requires four distinct real eigenvalues.

Case 3: a_2, a_3 and a_4 have exactly two nonzero elements.

By the similar method to Case 2, we may prove that in this case, A requires four distinct real eigenvalues.

Case 4: $a_2 = a_3 = a_4 \neq 0$.

Take a real symmetric matrix $B = (b_{ij}) \in Q(A)$ such that $b_{22} = b_{33} = b_{44} = b$. It is clear that $\text{rank}(bI - B) = 2$. Then $\lambda = b$ is an eigenvalue of B, and the geometric multiplicity (thus, the algebraic multiplicity) of $\lambda = b$ is 2. It implies that A does not require four distinct real eigenvalues.

Case 5: a_2, a_3 and a_4 are all nonzero, and have different signs. (The signs are not all the same.)

Let $B \in Q(A)$ be any real symmetric matrix. Let λ be an eigenvalue of B. It is easy to see that $\text{rank}(\lambda I - B) = 3$, and so λ is algebraically simple. Then B has four distinct real eigenvalues, and A requires four distinct real eigenvalues.

Combining the above five cases, the lemma follows. $\square$

From Theorem 3.3 and Lemmas 3.4 and 3.5, we now have the following.

Theorem 3.6. Let $A \in Q_4$ be irreducible. Then A requires four distinct real eigenvalues if and only if up to equivalence,

(1) A is a symmetric tridiagonal sign pattern having the form

$$A = \begin{bmatrix} * & + & 0 & 0 \\ + & * & + & 0 \\ 0 & + & * & + \\ 0 & 0 & + & * \end{bmatrix},$$

where $*$ may be 1, -1 or 0; or

(2) A is a symmetric star sign pattern having the form

$$A = \begin{bmatrix} a_1 & + & + & + \\ + & a_2 & 0 & 0 \\ + & 0 & a_3 & 0 \\ + & 0 & 0 & a_4 \end{bmatrix},$$

where a_i may be $+$, $-$ or 0 for $i = 1, 2, 3, 4$, and a_2, a_3 and a_4 are not the same.

4. 4×4 Sign Patterns Requiring Four Distinct Non-Real Eigenvalues

Lemma 4.1. Let $A \in Q_n$. If A has a positive even cycle or an odd cycle, then A does not require all nonreal eigenvalues, that is, there exists a real matrix $B \in Q(A)$ such that B has at least one real eigenvalues.

Proof: Let γ be a positive even cycle or an odd cycle. By emphasizing the cycle γ, from Lemma 1.4 we get a matrix $B \in Q(A)$ having at least one real eigenvalue. $\square$

Lemma 4.1 immediately yield the following two results.

Lemma 4.2. Suppose that $A = (a_{ij}) \in Q_4$ requires four distinct nonreal eigenvalues. Then the following conditions hold.

(1) $MR(A) = mr(A) = 4$, that is, A is sign nonsingular.

(2) All diagonal entries of A are zero.

(3) A has no positive 2-cycles and positive 4-cycles.

(4) A has no 3-cycles.

Theorem 4.3. Let $A \in Q_4$ be irreducible and require four distinct nonreal eigenvalues. Then, up to equivalence,

$$A = \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & 0 \end{bmatrix}, \tag{4.1}$$

or A has the following form

$$A = \begin{bmatrix} 0 & + & 0 & a_1 \\ -a_2 & 0 & + & 0 \\ 0 & -a_3 & 0 & + \\ - & 0 & -a_4 & 0 \end{bmatrix}, \tag{4.2}$$

where a_i may be $+$ or 0 for $i = 1, 2, 3, 4$.

Lemma 4.4. ([5, 15]) If A is an $n \times n$ skew-symmetric irreducible tri-diagonal sign pattern, then A requires n distinct pure imaginary (possibly including zero) eigenvalues.

Lemma 4.5. Let $A \in Q_4$ be irreducible and have the form (4.2). Then A requires four distinct nonreal eigenvalues if and only if

$$a_1 = a_2 = a_3 = a_4 = 0.$$

Proof: Sufficiency. Let $a_1 = a_2 = a_3 = a_4 = 0$. By Lemma 5.1, it is clear that the sufficiency holds.

Necessity. Let A require four distinct nonreal eigenvalues. Contradicting $a_1 = a_2 = a_3 = a_4 = 0$, we assume that at least one of a_1, a_2, a_3 and a_4 is nonzero.

Case 1: a_1, a_2, a_3 and a_4 have exactly one nonzero.

Up to equivalence, we may assume that $a_1 \neq 0$ and $a_2 = a_3 = a_4 = 0$. Take

$$B = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \in Q(A).$$

We have that $\sigma(B) = \{-i, -i, i, i\}$, it is a contradiction.

Case 2: a_1, a_2, a_3 and a_4 have exactly two nonzeros.

Up to equivalence, A has two forms, one is that $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 = a_4 = 0$, and the other is that $a_1 \neq 0$, $a_3 \neq 0$ and $a_2 = a_4 = 0$.

For the first form, take

$$B = \begin{bmatrix} 0 & 1 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix} \in Q(A).$$

We have that $\sigma(B) = \{-i, -i, i, i\}$, it is a contradiction.

For the second form, take

$$B = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & -4 & 0 & 1 \\ -2 & 0 & 0 & 0 \end{bmatrix} \in Q(A).$$

We have that $\sigma(B) = \{-i\sqrt{6}, -i\sqrt{6}, i\sqrt{6}, i\sqrt{6}\}$, it is a contradiction.

Case 3: a_1, a_2, a_3 and a_4 have exactly three nonzero.

Up to equivalence, we may assume that $a_1 \neq 0$, $a_2 \neq 0$, $a_3 \neq 0$ and $a_4 = 0$.

Take

$$B = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -2 & 0 & 2 & 0 \\ 0 & -4 & 0 & \frac{17}{4} \\ -2 & 0 & 0 & 0 \end{bmatrix} \in Q(A).$$

We have that $\sigma(B) = \{-i\sqrt{7}, -i\sqrt{7}, i\sqrt{7}, i\sqrt{7}\}$, it is a contradiction.

Case 4: All a_1, a_2, a_3 and a_4 are nonzero.

Take

$$B = \begin{bmatrix} 0 & 2 & 0 & 2 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 0 & 2 \\ -2 & 0 & -2 & 0 \end{bmatrix} \in Q(A).$$

We have that $\sigma(B) = \{-i\sqrt{6}, -i\sqrt{6}, i\sqrt{6}, i\sqrt{6}\}$, it is a contradiction.

Combining the above four cases, we see that the necessity follows. $\square$

From Theorem 4.3 and Lemmas 4.4 and 4.5, we now have the following.

Theorem 4.6. Let $A \in Q_4$ be irreducible. Then A requires four distinct nonreal eigenvalues if and only if up to equivalence,

$$A = \begin{bmatrix} 0 & + & 0 & 0 \\ - & 0 & + & 0 \\ 0 & - & 0 & + \\ 0 & 0 & - & 0 \end{bmatrix}, \quad \text{or} \quad A = \begin{bmatrix} 0 & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \\ - & 0 & 0 & 0 \end{bmatrix}.$$

5. 4×4 Sign Patterns Requiring Precisely Two Distinct Real Eigenvalues

Lemma 5.1. Suppose that $A \in Q_4$ is irreducible and A requires a pair of conjugate nonreal eigenvalues and two distinct real eigenvalues. Then

(1) $MR(A) \geq mr(A) \geq 3$.

(2) A has at most two nonzero diagonal entries.

(3) A is not symmetric.

(4) A has no negative 4-cycles.

(5) If γ is a composite 4-cycle in A consisting of two 2-cycles, then both 2-cycles have different signs.

Proof: (1)–(3) are clear. We only prove (4) and (5).

For (4), let γ be a negative 4-cycle in A. By emphasizing the cycle γ, we get a matrix $B \in Q(A)$ with two pairs conjugate nonreal eigenvalues, which is a contradiction. Thus, (4) follows.

For (5), let $\Gamma = \gamma_1\gamma_2$ be a composite 4-cycle in A, where both γ_1 and γ_2 are 2-cycles. If both γ_1 and γ_2 are negative (respectively, positive), then by emphasizing the cycle Γ, we get a matrix $B \in Q(A)$ with two pairs conjugate nonreal eigenvalues (respectively, four real eigenvalues), which is a contradiction. Thus, γ_1 and γ_2 have different signs, so (5) follows. □

Lemma 5.2. Let $A = (a_{ij}) \in Q_4$ be irreducible with $MR(A) = 3$. Then A requires a pair conjugate nonreal eigenvalues and two distinct real eigenvalues if and only if up to equivalence,

(1) A has no (simple or composite) 4-cycles;

(2) A has at least one 3-cycle, and all 3-cycles in A are negative;

(3) A has no positive 2-cycles;

(4) A has at most one nonzero diagonal entry, and if $a_{ii} \neq 0$, then $a_{ii} = -$ and $a_{ij}a_{ji} = 0$ for any $i \neq j$.

Proof: Let $B \in Q(A)$, and the characteristic polynomial of B be

$$P_B(x) = \det(xI - B) = x^4 + ax^3 + bx^2 + cx + d, \tag{5.1}$$

where a, b, c, d are real constants. Because $MR(A) = 3$, it is clear that $d = 0$. Thus,

$$Res(P_B(x), P_B'(x)) = (a^2b^2 - 4b^3 - 4a^3c + 18abc - 27c^2)c^2. \tag{5.2}$$

Sufficiency. Let (1)–(5) hold. Then $a \geq 0$, $b \geq 0$, $c > 0$, and $ab \leq c$. Thus by (5.2), $\text{Res}(P_B(x), P_B'(x)) \leq (c^2 - 4b^3 - 4a^3c + 18c^2 - 27c^2)c^2 = (-4b^3 - 4a^3c - 8c^2)c^2 < 0$. By Lemma 1.3, A requires all distinct eigenvalues. On the other hand, let γ be a 3-cycle. By emphasizing the cycle γ, we get a matrix $B \in Q(A)$ with a pair conjugate nonreal eigenvalues and two distinct real eigenvalues. From Lemma 1.2, the sufficiency follows.

Necessity. Let A require a pair conjugate nonreal eigenvalues and two distinct real eigenvalues. Since $MR(A) = 3$, it is clear that (1) holds, and there is a (simple or composite) 3-cycle in A.

If there is no simple 3-cycle in A, then A is a star sign pattern with at most two nonzero diagonal entries. Without loss of generality, we assume that $a_{11} \neq 0$ and $a_{12}a_{21} \neq 0$. Take a matrix $B_1 = (b_{ij}) \in Q(A)$ such that $|b_{12}| = |b_{21}| = 1$, $|b_{11}|$ is sufficiently large, and the absolute values of other nonzero entries are sufficiently small, thus, $\text{Res}(P_{B_1}(x), P_{B_1}'(x)) > 0$ from (5.2). On the other hand, by Lemma 5.1(3), A is not symmetric. Thus, there is a negative 2-cycle. By emphasizing this negative 2-cycle, we can get a matrix $B_2 \in Q(A)$ with $\text{Res}(P_{B_2}(x), P'_{B_2}(x)) < 0$ from (5.2). Note the fact that $\text{Res}(P_B(x), P'_B(x))$ is a continuous function of entries of B. There is $B \in Q(A)$ such that $\text{Res}(P_B(x), P'_B(x)) = 0$. By Lemma 1.3, it is a contradiction. Therefore there is a simple 3-cycle in A.

We now have that all the 3-cycles have the same sign. If not, it is not difficult to verify that there is $B \in Q(A)$ such that $Res(P_B(x), P'_B(x)) = 0$ from (5.2). By Lemma 1.3, it is a contradiction. Up to equivalence, we may assume that each 3-cycle is negative. Thus, (2) holds.

For (3), let γ_1 be a positive 2-cycle in A. Since $MR(A) = 3$, each real matrix in $Q(A)$ has a zero eigenvalue. By emphasizing γ_1, we may get a matrix $B \in Q(A)$ such that B has three real eigenvalues, it is a contradiction. Thus, (3) follows.

For (4), let A have two nonzero diagonal entries. By emphasizing two nonzero diagonal entries, we may get a matrix $B \in Q(A)$ such that B has two nonzero real eigenvalues and one zero eigenvalue, it is a contradiction. Thus, A has at most one nonzero diagonal entry.

We now let A have exactly one nonzero diagonal entry, without loss of generality, we assume that $a_{11} \neq 0$ and $a_{ii} = 0$ for $i = 2, 3, 4$. If $a_{1i}a_{i1} \neq 0$ for some $2 \leq i \leq 4$, then we take a matrix $B_1 = (b_{ij}) \in Q(A)$ such that $|b_{1i}| = |b_{i1}| = 1$, $|b_{11}|$ is sufficiently large, and the absolute values of other nonzero entries are sufficiently small. It implies that there is a matrix $B_1 \in Q(A)$ such that $\text{Res}(P_{B_1}(x), P'_{B_1}(x)) > 0$. On the other hand, by emphasizing a 3-cycle, we get a matrix $B \in Q(A)$ such that $\text{Res}(P_B(x), P'_B(x)) < 0$. Note the fact that $\text{Res}(P_B(x), P'_B(x))$ is a continuous function of entries of B. Then there is $B \in Q(A)$ such that $\mathrm{Res}(P_B(x), P'_B(x)) = 0$. By Lemma 1.3, it is a contradiction. Thus, $a_{1i}a_{i1} = 0$ for all $2 \leq i \leq 4$, so (4) follows.

The lemma now follows. $\square$

Lemma 5.3. Let $A = (a_{ij}) \in Q_4$ be irreducible, $MR(A) = 4$, and all diagonal entries of A be zero. If A has a 4-cycle, then A requires a pair conjugate nonreal eigenvalues and two distinct real eigenvalues if and only if

(1) Each 4-cycle in A is positive;

(2) If there is a 3-cycle in A, then each 2-cycle is negative (if there is); and

(3) If there is a composite 4-cycle in A consisting of two 2-cycles, then both 2-cycles have different signs.

Proof: Sufficiency. Let $B \in Q(A)$, and the characteristic polynomial of B be

$$P_B(x) = \det(xI - B) = x^4 + ax^3 + bx^2 + cx + d,$$

where a, b, c, d are real constants. Since that all diagonal entries of A are zero, it is clear that $a = 0$. Thus,

$$\mathrm{Res}(P_B(x), P'_B(x)) = -4b^3c^2 - 27c^4 + 16b^4d + 144bc^2d - 128b^2d^2 + 256d^3$$

$$= 16d(b^2 - 4d)^2 - 27c^4 + c^2(144bd - 4b^3).$$

From (1) and (3), we have $d = \det(B) < 0$. If there is no 3-cycle in A, then $c = 0$, and so

$$\mathrm{Res}(P_B(x), P'_B(x)) = 16d(b^2 - 4d)^2 < 0.$$

By Lemma 1.3, A requires all distinct eigenvalues. If there is a 3-cycle in A, then from (2), each 2-cycle is negative (if there is), and so $b \geq 0$. Thus, $\mathrm{Res}(P_B(x), P'_B(x)) < 0$, and A requires all distinct eigenvalues.

On the other hand, let Γ be a positive 4-cycle. By emphasizing the cycle Γ, we get a matrix $B \in Q(A)$ with a pair conjugate nonreal eigenvalues and two real eigenvalues. From Lemma 1.2, the sufficiency follows.

Necessity. Let A require a pair conjugate nonreal eigenvalues and two distinct real eigenvalues. Then (1) and (3) hold from Lemma 5.1. In order to prove (2), we assume that γ_1 is a 3-cycle and γ_2 is a positive 2-cycle in A. Let $B \in Q(A)$, and the characteristic polynomial of B be

$$P_B(x) = \det(xI - B) = x^4 + ax^3 + bx^2 + cx + d,$$

where a, b, c, d are real constants. Clearly, $a = 0$ and $d = \det(B) < 0$. Thus,

$$\mathrm{Res}(P_B(x), P'_B(x)) = -4b^3c^2 - 27c^4 + 16b^4d + 144bc^2d - 128b^2d^2 + 256d^3$$

$$= 16d(b^2 - 4d)^2 - 27c^4 + c^2(144bd - 4b^3).$$

It is not difficult to verify that by emphasizing both γ_1 and γ_2, respectively, a simple 4-cycle, we can get two matrices B_1 and B_2 in $Q(A)$ such that $\mathrm{Res}(P_{B_1}(x), P'_{B_1}(x)) > 0$, respectively, $\mathrm{Res}(P_{B_1}(x), P'_{B_1}(x)) < 0$. Note the fact that $\mathrm{Res}(P_{B_1}(x), P'_{B_1}(x))$ is a continuous function of entries of B. There is $B \in Q(A)$ such that $\mathrm{Res}(P_B(x), P'_B(x)) = 0$. By Lemma 1.3, it is a contradiction. Thus, (2) holds. $\square$

Lemma 5.4. Let $A = (a_{ij}) \in Q_4$ be irreducible, $MR(A) = 4$, and all diagonal entries of A be zero. If A has no 4-cycle, then A requires a pair conjugate nonreal eigenvalues and two distinct real eigenvalues if and only if

(1) A has a composite 4-cycle, and each composite 4-cycle in A consists of one positive 2-cycle and one negative 2-cycle; and

(2) A has no 3-cycles.

Proof: Sufficiency. Let $B \in Q(A)$, and the characteristic polynomial of B be

$$P_B(x) = \det(xI - B) = x^4 + ax^3 + bx^2 + cx + d,$$

where a, b, c, d are real constants. Because all the diagonal entries of A are zero, it is clear that $a = 0$. From (1) and (2), we have that $d = \det(B) < 0$ and $c = 0$. Thus,

$$\text{Res}(P_B(x), P_B'(x)) = 16b^4d - 128b^2d^2 + 256d^3 = 16d(b^2 - 4d)^2 < 0.$$

By Lemma 1.3, A requires all distinct eigenvalues.

On the other hand, let $\gamma = \gamma_1\gamma_2$ be a composite 4-cycle, where γ_1 is a positive 2-cycle and γ_2 is a negative 2-cycle. By emphasizing the cycle γ, we get a matrix $B \in Q(A)$ with a pair conjugate nonreal eigenvalues and two distinct real eigenvalues. From Lemma 1.2, the sufficiency follows.

Necessity. Since $MR(A) = 4$, from the hypotheses and Lemma 5.1(5), we have that A has a composite 4-cycle, and each composite 4-cycle consists of two 2-cycles that have different signs. Thus (1) follows. In order to prove (2), we assume that Γ is a 3-cycle in A.

Let $B \in Q(A)$, and the characteristic polynomial of B be

$$P_B(x) = \det(xI - B) = x^4 + ax^3 + bx^2 + cx + d,$$

where a, b, c, d are real constants. Clearly, $a = 0$ and $d = \det(B) < 0$ from the hypotheses and Lemma 5.1(5). Thus,

$$\text{Res}(P_B(x), P_B'(x)) = -4b^3c^2 - 27c^4 + 16b^4d + 144bc^2d - 128b^2d^2 + 256d^3$$

$$= 16d(b^2 - 4d)^2 - 27c^4 + c^2(144bd - 4b^3).$$

It is not difficult to verify that by emphasizing both Γ and one positive 2-cycle, respectively, both Γ and one negative 2-cycle, we can get two matrices B_1 and B_2 in $Q(A)$ such that $\mathrm{Res}(P_{B_1}(x), P'_{B_1}(x)) > 0$, and, $\mathrm{Res}(P_{B_2}(x), P'_{B_2}(x)) < 0$, respectively. Note the fact that $\mathrm{Res}(P_B(x), P'_B(x))$ is a continuous function of the entries of B. There is $B \in Q(A)$ such that $\mathrm{Res}(P_B(x), P'_B(x)) = 0$. By Lemma 1.3, it is a contradiction. Thus, (2) holds.

$\square$

We as of yet do not have a characterization of the 4×4 irreducible sign patterns A that require precisely two distinct real eigenvalues when $\mathrm{MR}(A) = 4$ and when A has at least one nonzero diagonal entry. We do know, however,that in this case, $\det A = -$.

6. Some Questions and Open Problems

The problem of characterizing the sign pattern matrices that require distinct eigenvalues is in general a very difficult problem. In this thesis we have given the solution for order 2 and order 3 matrices, and most of the solution for order 4. Future research will concentrate on completing the order 4 case, and also work on sign patterns of order greater than 4.

Question 1. Suppose that $A \in Q_4$ is irreducible, $MR(A) = 4$, and A has some nonzero diagonal entries. What are the necessary and sufficient conditions for A to require a pair of conjugate nonreal eigenvalues and two distinct real eigenvalues? We have shown that in this case, $\det A = -$.

Question 2. Suppose that $A \in \mathcal{DE} \cap Q_n$. What is the maximum number of nonzero entries in A? (Equivalently, what is the minimum number of zero entries in A?)

We note that for $n \geq 2$, the minimum number of nonzero entries of an $n \times n$ sign pattern that requires n distinct eigenvalues can be easily seen to be $n - 1$, which is achieved by an $(n - 1)$-cycle sign pattern.

Question 3. Find some new sufficient and/or necessary conditions for an irreducible $n \times n$ sign pattern to be in $\mathcal{DE}$.

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