Minimum Degree Conditions for Tilings in Graphs and Hypergraphs

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We consider tiling problems for graphs and hypergraphs. For two graphs $G$ and $F$, an $F$-tiling of $F$ is a subgraph of $G$ consisting of only vertex disjoint copies of $F$. By using the absorbing method we give a short proof that in a balanced tripartite graph $G$, if every vertex is adjacent to $(2/3 + \gamma)$ of the vertices in each of the other vertex partitions, the $G$ has a $K_3$ tiling. Previously Magyar and Martin [14] proved the same result (without $\gamma$) by using the Regularity Lemma.

In a 3-uniform hypergraph $H$, let $\delta_2(H)$ denote the minimum number of edges that contain $\{u,v\}$ for all pairs $\{u,v\}$ of vertices. We show that if $\delta_2(H) \geq \left(1 - \frac{2}{k(k-2)}\right)n$ there exists a $K^3_k$-tiling of $H$ that misses at most $k^2$ vertices of $H$. On the other hand, we show that there exist hypergraphs $H$ such that $\delta_2(H) = \left(1 - \frac{1}{k}\right)n - 2$ and $H$ does not have a perfect $K^3_k$-tiling. These extend the results of Pikhurko [17] on $K^3_3$-tilings.

INDEX WORDS: Graph tiling, Graph packing, Absorbing method, Hypergraph Codegree
MINIMUM DEGREE CONDITIONS FOR TILINGS IN GRAPHS AND HYPERGRAPHS

by

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This thesis is dedicated to Luzy, who stands beside me on all of my adventures.
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Chapter 1

INTRODUCTION

For two graphs $G$ and $F$, an $F$-tiling (or $F$-packing) of $G$ is a subgraph of $G$ consisting of vertex disjoint copies of $F$. When $F$ is a single (hyper)edge we call an $F$-tiling a matching. If the $F$-tiling covers all of the vertices of $G$ we say that the tiling is perfect or refer to the tiling as an $F$-factor. For a perfect tiling to exist the order of $F$ must divide the order of $G$.

The purpose of this paper is to determine bounds on the minimum degree necessary to ensure a perfect or near perfect $F$-tiling. An early result by Dirac [6] proves that any graph on $n$ vertices with minimum degree at least $n/2$ is Hamiltonian. This result allows us to obtain a perfect matching in $G$ by deleting every other edge from the Hamiltonian cycle. For $F = K_h$, the complete graph on $h$ vertices, Hajnal and Szemerédi [8] provide the following result: If $G$ is a graph with $hk$ vertices and minimum degree at least $(h - 1)k$, then $G$ contains $k$ vertex disjoint copies of $K_h$. Later, using Szemerédi’s Regularity Lemma [22], Alon and Yuster [2, 3] were able to provide minimum degree conditions that guarantee an $F$-factor for arbitrary $F$. Kühn and Osthus [12] were able to find the best possible minimum degree conditions for finding an $F$-factor.

Tiling in multipartite graphs has a shorter history. A graph $G$ is called $r$-partite if the vertex set $V(G)$ can be partitioned in $r$ sets $V_1, \ldots, V_r$ such any that two vertices $u, v \in V_i$ are not adjacent. The Marriage Theorem by König and Hall (see e.g. [4]) implies that a bipartite graph ($r = 2$) $G$ with partition sets of size $n$ contains a 1-factor if $\delta(G) \geq n/2$. In an $r$-partite graph $G$ with $r \geq 2$, let $\bar{\delta}(G)$ be the minimum degree from a vertex in one partition set to each other partition set (so $\bar{\delta}(G) = \delta(G)$ when $r = 2$). An $r$-partite graph is balanced if all partition sets have the same order.

Fischer [7] conjectured the following $r$-partite version of the Hajnal-Szemerédi Theorem and
proved it asymptotically for $r = 3, 4$: if $G$ is an $r$-partite graph with $n$ vertices in each partition set and $\bar{\delta}(G) \geq \frac{r-1}{r}n$, then $G$ contains a $K_r$-factor. Magyar and Martin [14] used the following theorem to show that Fischer’s conjecture is slightly wrong for $r = 3$ (off by only 1): For $G$ a balanced tripartite graph on $3N$ vertices with $\bar{\delta}(G) \geq (2/3)N + 1$ then $G$ contains a perfect $K_3$-tiling. As written, this is a weaker form of the actual theorem, as they prove that $G$ can be perfectly tiled with triangles when $\bar{\delta}(G) \geq (2/3)N$ as long as it is not the graph $\Gamma_3(N/3)$. The case when $G$ is $\Gamma_3(N/3)$ is what disproves Fischer’s conjecture and necessitates the extra edge to complete the tiling. Notice in Figure 1 that there can be no $K_3$-tiling of $\Gamma_3$. To form $\Gamma_3(N/3)$, replace each vertex with a cluster of $N/3$ vertices and each edge with the complete bipartite graph $K_{N/3, N/3}$. Since $\Gamma_3$ cannot be perfectly tiled by triangles, neither can the blown up version $\Gamma_3(N/3)$ unless you add a single edge. Martin and Szemerédi [15] showed that Fischer’s conjecture is true for $r = 4$. Note that in general, a tiling result for multipartite graphs does not follow from a corresponding result for arbitrary graphs. On the other hand, given a graph $G$ of order $nr$, we can easily obtain (by taking a random partition) an $r$-partite balanced spanning subgraph $G'$ such that $\bar{\delta}(G') \geq \delta(G)/r - o(n)$. Therefore a tiling result for multipartite graphs immediately gives a slightly weaker tiling result for arbitrary graphs.

The next chapter will focus on a tripartite graph and will provide a lower bound on $\bar{\delta}(G)$, for balanced $G$, in order to obtain a perfect $K_3$-tiling, often referring to $K_3$ as a triangle. Here we use the absorbing lemma, though previously Magyar and Martin [14], by using Szemerédi’s Regularity Lemma, were able to avoid $\gamma$. The advantage in using the absorbing method is that we will achieve a much smaller order graph than is necessary with the Regularity Lemma.
Theorem 1.1. For any $\gamma > 0$, there exists $n_0$ such that for all $n > n_0$ the following holds: Let $G$ be a balanced tripartite graph on $n = 3N$ vertices with $\bar{\delta}(G) \geq (2/3 + \gamma)N$, then $G$ contains a $K_3$-factor.

The last chapter focuses on tiling problems in hypergraphs. We say that a hypergraph $H$ is $k$-uniform, also called a $k$-graph, if every edge in $E(H)$ contains exactly $k$ vertices. We denote the complete $k$-graph on $n$ vertices by $K^k_n$. For a set $T$ of size $l < k$ in $H$, we define $\text{deg}(T)$ to be the number of edges in $H$ that contain $T$ and $\delta_l(H)$ be the minimum $l$-degree of $H$. For $l = k - 1$, we say that $\delta_{k-1}(H)$ is the minimum vertex codegree of $H$. All hypergraphs in this chapter will be 3-graphs.

Definition 1.2. Let $t^k_l(n, F)$, for all integers $k > l \geq 1$ and $n \in k\mathbb{Z}$, denote the minimum $t$ such that every $k$-uniform hypergraph $H$ on $n$ vertices satisfying $\delta_l(H) \geq t$ contains a perfect $F$-tiling.

In their survey on the subject, Rödl and Ruciński [18] point out this result from Kühn and Osthus [10]:

$$t^3_2(n, C^{(3,1)}_4) \sim n/4,$$

where the graph $C^{(3,1)}_4$ is the $(3,1)$-cycle graph on 4 vertices.

When $k = 2$ this is exactly the graph case and has been discussed above. For $k \geq 3, l = k - 1$ Kühn and Osthus [11], as well as Rödl et al. [19–21], investigated the number $t^k_{k-1}(n, F)$. Notably, Rödl, Ruciński and Szemerédi [20] determined $t^k_{k-1}(n, F)$ for arbitrary $k \geq 3$ and sufficiently large $n$, showing $t^k_{k-1}(n, F) = n/2 - k + c_{k,n}$ where $c_{k,n} \in \{3/2, 3, 5/2, 3\}$ based on the parities of $k$ and $n$. Continuing this work, Pikhurko [17] provided the bounds

$$\frac{3}{4}n - 2 \leq t^k_l(n, K^3_4) \leq \frac{2 + \sqrt{10}}{6}n + O(\sqrt{n \log N}),$$

where the upper bound was also proved, independently by Keevash and Zhao (unpublished).

For the upper bound on $t$ for $K^3_k$-tilings we extend an argument from Fischer [7] by introducing a weight function to handle the added complexity of the hypergraph.

Theorem 1.3. Let $H$ be a 3-graph of order $n$ with $\delta_2(H) \geq \left(1 - \frac{2}{k(k-2)}\right)n$ and $k|n$. Then there exists a tiling of vertex disjoint copies of $K^3_k$ in $H$ covering all but at most $k^2$ vertices.
Lo and Markström [13] have a proof that extends this proof to all $K_{t_k}$-tilings, obtaining the same bound.

To show the lower bound on $t$ we we extend a construction from Pikhurko [17] to show that $\mathcal{H}$ may not contain a $K_{k}^3$-factor.

**Proposition 1.4.** Let $\mathcal{H}$ be 3-graph on $n = 2kq + r$ for integers $k, q \geq 0$ and $r \in \{0, k\}$, we have

$$\delta_2(\mathcal{H}) \geq 2(k - 1)q + r - 2 \geq \left(1 - \frac{1}{k}\right)n - 2.$$

Lo and Markström [13] also extended this construction to all $K_{k}^t$ and achieved an improved bound.
Chapter 2

PROOF OF THEOREM 1.1

Let $\gamma > 0$ and $n_0(\gamma)$ be the minimum positive integer satisfying the following two conditions:

(i) $2\gamma^2 n_0^2 + \frac{5}{3} \gamma n_0^2 + 1 \geq 3\gamma n_0 + n_0$

(ii) $6\gamma^2 n_0^2 + 2 \geq 7\gamma n_0 + \frac{2}{3} n_0$

Also let $G = (V_1, V_2, V_3, E)$ be a balanced tripartite graph of order $n = 3N$ with $\bar{\delta} \geq (2/3 + \gamma)N$. We prove Theorem 1.1 in three steps. First we show that for an arbitrary $T = \{v_1, v_2, v_3\}, v_i \in V_i$, there are many absorbing 6-sets. Next we show that $G$ will have a near perfect tiling that misses only six vertices. Last, we will show that the final six vertices can be absorbed into the tiling.

2.1 Absorbing Sets

We use Proposition 2.1 to establish an absorbing structure in $G$ and prove that the edge density provides enough absorbing 6-sets for an arbitrary $T$ to be added to a partial tiling. The proof follows from Lemma 10 (Absorbing Lemma) by Hán et. al. [9].

**Proposition 2.1.** For $G$, as in the theorem, there exists a tiling $M$ in $G$ of size $|M| \leq \frac{1}{2} \gamma^2 N$ such that for every set $W \subset V \setminus V(M)$ of size at most $\frac{1}{2} \gamma^6 N$ there exists a tiling covering exactly the vertices in $V(M) \cup W$.

*Proof.* In $G$ we say that a set $A = A_1 \cup A_2 \cup A_3, A_i \in (V_i \choose 2)$, is an absorbing 6-set for $T$ if $A$ spans a tiling of size 2 and $A \cup T$ spans a tiling of size 3. Lemma 2.2 determines how many such $A$ exist for arbitrary $T$. 
Lemma 2.2. For every $T$ in $G$, there are at least $\frac{2}{9} \gamma^2 N^6$ absorbing 6-sets for $T$.

Proof. Fix a set $T$. We wish to build the structure in Figure 2.1, so we begin by finding a triangle containing $v_1$ but not $v_2$ or $v_3$. By the degree condition, $v_1$ has at least $(2/3 + \gamma)N - 1$ vertices in $V_2$ that are not $v_2$. Let $u_2 \neq v_2$ be a neighbor of $v_1$ and consider $N_{V_3}(v_1) \cap N_{V_3}(u_2)$. The shared neighborhood of $v_1$ and $u_2$ that avoids $v_3$ must be at least

$$(2/3 + \gamma)N + (2/3 + \gamma)N - N - 1 = (1/3 + 2\gamma)N - 1$$

vertices $u_3 \neq v_3$. Thus, we have in total

$$((2/3 + \gamma)N - 1)((1/3 + 2\gamma)N - 1) \geq \frac{2}{9} N^2$$

triangles that contain $v_1$ and not $v_2$ or $v_3$, as $N \to \infty$.

Fix one such triangle $\{v_1, u_2, u_3\}$ and let $U_1 = \{u_2, u_3\}$. Now suppose we are able to choose a set $U_2$ such that it is disjoint to $U_1 \cup T$ and both $U_2 \cup \{u_2\}$ and $U_2 \cup \{v_2\}$ are triangles in $G$. Suppose further that we are able to choose a set $U_3$ such that it is disjoint to $U_1 \cup U_2 \cup T$ and both $U_3 \cup \{u_3\}$ and $U_3 \cup \{v_3\}$ are triangles in $G$. Then we call such a choice for $U_2$ and $U_3$ good, motivated by $U_1 \cup U_2 \cup U_3$ being an absorbing 6-set for $T$, which describes the structure shown in Figure 2.1.

Focus on the number of good sets for $U_2$. The shared neighborhood of $u_2$ and $v_2$ in $V_1$ is at least $(1/3 + 2\gamma)N - 1$ vertices avoiding $v_1$. Fix a vertex $x_1 \neq v_1$ and count how many of its neighbors in $V_3$ are also adjacent to both $v_2$ and $u_2$, while avoiding $v_3$. The vertices $x_1, v_2$ and $u_2$
will have at least \((1/3 + 2\gamma)N + (2/3 + \gamma)N - N - 2 = 3\gamma N - 2\) common neighbors in \(V_3\) that avoid \(v_3\) and \(u_3\). We have in all at least
\[
((1/3 + 2\gamma)N - 1)(3\gamma N - 2) \geq \gamma N^2
\]
good choices for \(U_2\). The same analysis hold for the number of choices for \(U_3\).

Using equations (2.1) and (2.2), we see that the total number of absorbing 6-sets for \(T\) is
\[
\frac{2}{9} N^2 \times (\gamma N^2)^2 = \frac{2}{9} \gamma^2 N^6.
\]

\[\square\]

To continue the proof of Proposition 2.1, we let \(\mathcal{L}(T)\) denote the family of all the 6-sets that can absorb the \(T\) fixed in Lemma 2.2. We know that \(|\mathcal{L}(T)| \geq \frac{2}{9} \gamma^2 N^6\), again from Lemma 2.2. Choose a family \(\mathcal{F}\) of 6-sets by selecting each of the \(\binom{N}{2}^3\) possible 6-sets independently with probability
\[
p = \frac{\gamma^3}{N^5}.
\]
Then we can use the following result by Chernoff (see [1]) to determine how big \(\mathcal{F}\) is likely to be.

**Proposition 2.3.** If \(X_i, 1 \leq i \leq n\), be mutually independent random variables with
\[
Pr[X_i = +1] = Pr[X_i = -1] = \frac{1}{2}
\]
and set
\[
S_n = X_1 + \cdots + X_n.
\]
Let \(a > 0\). Then
\[
Pr[S_n > a] < e^{-a^2/2n}.
\]
Therefore, with probability \(1 - o(1)\), as \(N \to \infty\) the family \(\mathcal{F}\) fulfills the following properties:
\[
|\mathcal{F}| \leq 2E(|\mathcal{F}|) \leq \frac{\gamma^3}{N^5} \binom{N}{2}^3 \leq \frac{1}{4} \gamma^3 N
\]
(2.3)
\[ |\mathcal{L}(T) \cap \mathcal{F}| \geq \frac{1}{2} \mathbb{E}(|\mathcal{L}(T) \cap \mathcal{F}|) \geq \frac{1}{2} \left( \frac{\gamma^3}{N^5} \right) \times \frac{2}{9} \gamma^2 N^6 \geq \frac{1}{9} \gamma^5 N \]  \hspace{1cm} (2.4)

Moreover we can bound the expected number of intersecting 6-sets by choosing a 6-set, a vertex in the 6-set, a second vertex in same partition and a pair of vertices from each of the other two partitions:

\[ \left( \frac{N}{2} \right)^3 \times 6(N-1) \left( \frac{N}{2} \right)^2. \]

Then, the probability of choosing both sets is

\[ p^2 \left( \frac{N}{2} \right)^3 \times 6(N-1) \left( \frac{N}{2} \right)^2 \leq \frac{1}{4} \gamma^6 N \]  \hspace{1cm} (2.5)

Now, in order to upper bound the number of intersecting sets we use Markov’s bound (also in [1]).

**Proposition 2.4.** Suppose that \( Y \) is an arbitrary nonnegative random variable, \( \alpha > 0 \). Then

\[ \Pr[Y > \alpha \mathbb{E}[Y]] < 1/\alpha. \]

Therefore, with probability at least 1/2

\( \mathcal{F} \) contains at most \( \frac{1}{2} \gamma^6 N \) intersecting pairs.

Therefore, with positive probability the family \( \mathcal{F} \) has the properties stated in (2.3), (2.4) and (2.5). Since some of the 6-sets will not absorb any \( T \) and some will intersect each other, we delete all of these undesired 6-sets in the family \( \mathcal{F} \) to get a subfamily \( \mathcal{F}' \) consisting of pairwise disjoint absorbing 6-sets which satisfies

\[ |\mathcal{L}(T) \cap \mathcal{F}'| \geq \frac{1}{9} \gamma^5 N - \frac{1}{2} \gamma^6 N \geq \frac{1}{2} \gamma^6 N. \]

Finally, the thinned out family \( \mathcal{F}' \) consists of pairwise disjoint absorbing 6-sets and \( G[V(\mathcal{F}')] \) contains a perfect tiling \( M \) of size at most \( \frac{1}{2} \gamma^3 N \). Also, for any subset \( W \subset V \setminus V(M) \) of size \( \frac{1}{2} \gamma^6 N \)
we can partition $W$ into sets of size 3 and successively absorb them using a different absorbing 6-set each time. This gives us a tiling that covers exactly the vertices in $V(F') \cup W$. 

\[ \square \]

2.2 Complete Tiling

To complete the proof of the theorem, we find in $G$ an absorbing family $M$ guaranteed by Proposition 2.1. We let $G' = G - V(M)$ and observe that

$$
\bar{\delta}(G') \geq (2/3 + \gamma)N - \frac{3}{2} \gamma^3 N \geq \frac{2}{3} N \geq \frac{2}{3} N'
$$

where $N'$ is the number of vertices in each partition set of $G'$. Notice further that $G'$ is still balanced and we can apply Proposition 3.2 in Fischer [7] to find an incomplete tiling in $G'$.

**Proposition 2.5.** If $G$ is a tripartite graph with vertex partitions $V_1, V_2$ and $V_3$ of size $N$, such that each vertex in any partition has at least $\frac{2}{3} N$ neighbors in each of the other partitions, then $G$ contains $N - 2$ disjoint triangles.

This proposition gives us an almost perfect tiling of $G'$, leaving only a set $W$ containing 6 vertices uncovered. By Proposition 2.1 we can divide $W$ into sets of 3 and use $M$ to absorb each triple and complete the perfect tiling on $G$. 
Chapter 3

PROOFS ON 3-GRAPHS

In this chapter we provide a minimum degree condition that guarantees an almost perfect tiling of a 3-graph $H$ that misses at most $k^2$ vertices. Next we will provide a construction that shows that if the minimum degree condition is too small, we cannot guarantee a perfect tiling of $H$.

3.1 Proof of Theorem 1.3

This proof is adapted from the proof of Lemma 6.1 by Pikhurko [17] which adapts the proof of Theorem 2.1 by Fischer [7].

Proof. Let $H$ be a 3-graph on $n$ vertices with $\delta_2(H) \geq \left(1 - \frac{2}{k(k-2)}\right)n$ and $k|n$. Begin with a partition $P$ of the vertex set $V(H)$ into sets of size $k$, $V_1, \ldots, V_{n-k}$. Let $G_i$ be the largest complete graph in $V_i$. If $V_i$ is an independent set, we define $|G_i| = 2$. Denote by $w : \{2, \ldots, k\} \to \mathbb{R}$ the function defined by $w(2) = 0$ and $w(j + 1) - w(j) = 1 - \frac{1}{k^j}$ for $2 \leq j \leq k - 1$. We say that $w(P)$, the weighting of $P$, is $\sum_{1 \leq j \leq n/k} w(|G_j|)$. Assume that $P$ is chosen such that $w(P)$ is maximal.

We will now show that for each weight class $2 \leq i \leq k - 1$ there are at most $k - 1$ sets $V_j$ in $P$ with $|G_j| = i$. Suppose, for a contradiction, that $|G_1| = \cdots = |G_k| = i < k$. Since $|G_j| < k$ for $1 \leq j \leq k$ we can find at least one $v_j \in V_j \setminus G_j$. Now, for $1 \leq j \leq k$ and vertex $v \not\in V_j$, we say the pair $(v, j)$ is a connection if and only if $\{v\} \cup G_j$ spans a complete hypergraph. If there are any connections $(v, j)$ with $v \in V_1 \cup \cdots \cup V_k$ then switching $v$ with any vertex $v_j$ will result in a new partition $P'$. Note that since

$$1 - \frac{1}{k^i} \geq 1 - \frac{1}{k^{i-1}}$$
we have
\[ w(i + 1) - w(i) \geq w(i) - w(i - 1) \]
which is
\[ w(i + 1) + w(i - 1) \geq 2w(i) \]
and we immediately provide a contradiction to \( w(P) \) being maximal. Thus, we can assume there are no connections with \( v \in V_1 \cup \cdots \cup V_k \) and \( 1 \leq j \leq k \).

Using the condition on \( \delta_2(H) \), for \( 1 \leq j \leq k \) we can determine a lower bound on the number of connections there are by double counting the number of adjacencies among the \( G_j \)'s. An arbitrary pair of vertices in \( G_j \) is adjacent to at least \( \delta_2(H) \) vertices. If we let \( c \) be the number of connections to \( G_j \) then
\[ \left( \begin{array}{c} i \\ 2 \end{array} \right) \delta_2(H) \leq \left( \begin{array}{c} i \\ 2 \end{array} \right) c + \left( \left( \begin{array}{c} i \\ 2 \end{array} \right) - 1 \right) (n - c) \]
and
\[ c \geq \left( \begin{array}{c} i \\ 2 \end{array} \right) \delta_2(H) - \left( \left( \begin{array}{c} i \\ 2 \end{array} \right) - 1 \right) n \geq \frac{(k - i)n}{k} \]
where the last inequality is true since \( i < k \).

Now there are at least \( (k - i)n \) connections \((v, j)\) with \( v \notin V_1 \cup \cdots \cup V_k \) and \( 1 \leq j \leq k \). Since \( n > k \) we can choose \( V'_j \) such that there are more than \( k(k - i) \) connections \((v', j)\) for \( v' \in V'_j \) and \( 1 \leq j \leq k \). Consider the bipartite graph \( B \) with parts \( \{G_1, \ldots, G_k\} \) and \( V'_j \) whose edge set consists of those pairs that make a connection. Since \( B \) has at least \( k(k - i) \) edges, the König-Egerváry Theorem (see [4] Theorem 8.32) shows that \( B \) contains a matching of size at least \( k - i + 1 \). Now by moving \( v'_j \) to \( V_j \) for \( 1 \leq j \leq k - i + 1 \) and \( \{v_1, \ldots, v_{k-i+1}\} \) to \( V'_j \), see Figure 3.1, \( w(P) \) increases by
\[
(k - i + 1)(w(i + 1) - w(i)) - (w(|G'_j|) - w(\max\{2, |G'_j| - k + 1 + i\})) \\
\geq (k - i + 1) \left( 1 - \frac{1}{k^i} \right) - \left( k + 1 - i - \frac{k - i + 1}{k} \right) \\
= \frac{(k^i - 1)(k - i + 1)}{k^{i+1}} > 0
\]
a contradiction.
3.2 Proof of Proposition 1.4

We now provide a construction that proves that the codegree of $H$ must be larger than $(1 - 1/k)n - 2$ if we are to be guaranteed a perfect tiling.

Proof. For $n = 2kq + r$, if $r = k$ let $a_0 = 2q + 1$. Otherwise we let $a_0$ be either $2q + 1$ or $2q - 1$, with both choices giving the same bound. Partition $V(H) = A_0 \cup A_1 \cup \cdots \cup A_{k-1}$ into parts of sizes $a_0 + a_1 + \cdots + a_{k-1} = n$, where $a_1, \ldots, a_{k-1}$ are nearly equal, that is $|a_i - a_j| \leq 1$ for $1 \leq i < j \leq k - 1$. Let $H$ be the 3-graph on $n$ vertices whose edge set consists of all triple excluding any that satisfy one of the following (mutually exclusive) properties:

(i) have exactly three vertices in $A_0$

(ii) have one vertex in $A_0$ and two vertices in $A_i$ for some $1 \leq i \leq k - 1$

(iii) intersect each of $A_1, A_2$ and $A_3$.

Figure 3.2 shows examples of edges that are excluded from $H$. To see why there can be no $K^3_k$-tiling, consider any $K^3_k$-subgraph $K$ of $H$. By Property (i), $K$ cannot intersect $A_0$ in more than two vertices. Suppose that $K$ intersects $A_0$ in exactly one vertex and avoids at least one partition. Then by the pigeon hole principle there is a partition $A_i$ for $1 \leq i \leq k - 1$ that contains at least
two vertices of $K$. Property (ii) forbids the edge spanning the vertex in $A_0$ along with any pair in $A_i$. So if $K$ is to intersect $A_0$ in exactly one vertex, $K$ must also intersect every other partition in exactly one vertex. By property (iii), the edge with a vertex in $A_1, A_2$ and $A_3$ is forbidden, so $K$ cannot intersect $A_0$ in one vertex in this manner either.

Therefore every $K^3_k$-subgraph of $H$ has an even number of vertices in $A_0$. This makes a perfect tiling impossible, since $|A_0| = 2q \pm 1$, which is odd.

A case by case analysis gives the desired bound.

**Case 1** Two vertices in $A_0$ are in an edge with every vertex in $A_i$ for $1 \leq i \leq k - 1$, so the codegree is $\frac{k-1}{k}n$;

**Case 2** One vertex in $A_0$ and one vertex in $A_i$ for $1 \leq i \leq k - 1$ are in an edge with every other vertex in $A_0$ and every vertex in $A_j$ for $j \neq i$ and $1 \leq j \leq k - 1$, so the codegree is $\frac{k-1}{k}n - 1$;

**Case 3** Two vertices in $A_i$ for $1 \leq i \leq k - 1$ are in an edge with every other vertex in $A_i$ and every vertex in $A_j$ for $j \neq i$ and $1 \leq j \leq k - 1$, so the codegree is $\frac{k-1}{k}n - 2$;

**Case 4** One vertex in $A_i$ and one vertex in $A_j$ for $i, j \in [3]$ and $i \neq j$ are in an edge with every vertex in $A_0$, every other vertex in $A_i$ and $A_j$ and every vertex in $A_\ell$ for $4 \leq \ell \leq k - 1$, so the codegree is $\frac{k-1}{k}n - 2$;
**Case 5** One vertex in $A_i$ for $i \in [3]$ and one vertex in $A_j$ for $4 \leq j \leq k - 1$ are in an edge with every other vertex of $\mathcal{H}$, so the codegree is $n - 2$.

**Case 6** Two vertices in $A_i$ for $4 \leq i \leq k - 1$ are in an edge with every other vertex of $\mathcal{H}$, so the codegree is $n - 2$.

We take the minimum of these codegrees, which is $\frac{k-1}{k} n - 2$. \qed
REFERENCES


