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GERŠGORIN DISCS AND GEOMETRIC MULTIPLICITY

by

RACHID MARSLI

Under the Direction of Prof. Frank J. Hall

ABSTRACT

If A is an $n \times n$ complex matrix and λ is an eigenvalue of A with geometric multiplicity k , then λ is in at least k of the Geršgorin discs D_i of A .

Let k, r, t be positive integers with $k \leq r \leq t$. Then there is a $t \times t$ complex matrix A and an eigenvalue λ of A such that λ has geometric multiplicity k and algebraic multiplicity t , and λ is in precisely r Geršgorin Discs of A . Some examples and related results are also provided.

INDEX WORDS: G-discs

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RACHID MARSLI

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

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Georgia State University

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DEDICATION

To my Father and his mother

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I would like to express my very great appreciation to Professor Frank J. Hall, who is my Professor and coauthor of the papers in which those results were published , and to Professor Vladimir Bondarenko for my formation and assistance during my study

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INTRODUCTION

One of the most important and useful results for the location of the eigenvalues of a matrix is the Geršgorin theorem, which goes back to 1931. As stated in [2], the main part of the theorem is the following:

Geršgorin Theorem:

Let A be an $n \times n$ complex matrix and let:

$$R'_i = \sum_{j=1, j \neq i}^n |a_{ij}|, \quad 1 \leq i \leq n$$

denote the deleted absolute row sums of A . Then all of the eigenvalues of A are located in the union of the n Geršgorin discs:

$$\bigcup_{i=1}^n D_i$$

where

$$D_i = \{z \in \mathbb{C} : |z - a_{ii}| \leq R'_i\}$$

The proof of the theorem involves a clever idea. Let λ be an eigenvalue of A , and suppose that

$$Ax = \lambda x, \quad x = [x_i] \neq 0.$$

Some entry of x has largest modulus, say $|x_p| \geq |x_i|$ for all $i = 1, 2, \dots, n$, and $x_p \neq 0$. Then

$$x_p(\lambda - a_{pp}) = \sum_{j=1, j \neq p}^n a_{pj} x_j$$

and hence

$$|x_p| |\lambda - a_{pp}| \leq |x_p| \sum_{j=1, j \neq p}^n |a_{pj}| = |x_p| R'_p$$

so that $|\lambda - a_{pp}| \leq R'_p$; that is, λ lies in the p^{th} Geršgorin disc.

If λ is a simple eigenvalue of A , that is the end of the story. However, if λ is associated with several linearly independent eigenvectors, how could that fact be used to extract more information from Geršgorin's theorem?

If the geometric multiplicity of λ , namely the dimension of the associated eigenspace (the null space of $(\lambda I - A)$, is 1, then we have no control over the position of a largest modulus entry of a corresponding eigenvector: every eigenvector is a nonzero scalar multiple of some given eigenvector, so that every eigenvector has its largest modulus entry of a corresponding eigenvector: every eigenvector is a nonzero scalar multiple of some given eigenvector, so that every eigenvector has its largest modulus entry in the same position. However, if the geometric multiplicity of λ is greater than 1, we have some flexibility in this regard. For example, let

$$A = \begin{pmatrix} 2 & 1 & 0 \\ -4 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then 0 is an eigenvalue of A with algebraic multiplicity 3 and geometric multiplicity 2. The vectors $[1, -2, 3]^T$ and $[0, 0, 1]^T$ are linearly independent eigenvectors associated with $\lambda = 0$, with both largest modulus entries occurring in the third position. However, $[1, -2, 0]^T$ and $[0, 0, 1]^T$ are also eigenvectors, and their largest modulus entries occur in different positions. Our proof of Geršgorin's theorem shows that the eigenvalue $\lambda = 0$ is in both the second and third Geršgorin discs; what we have observed here is no accident.

An $n \times n$ matrix A has n Geršgorin discs D_i , some of which may be duplicates, as in the trivial example of identity matrix. We claim that if λ is an eigenvalue of A with geometric multiplicity k , then λ is in at least k of the Geršgorin discs of A .

CHAPTER I GERSGORIN DISCS AND GEOMETRIC MULTIPLICITY

To prove the first part of our claim we need a key preliminary result: Each subspace S of C^n has a basis whose vectors have largest modulus entries in different positions. The argument uses a deflation process that has the same flavor as the proof in [1] of Schur's triangularization theorem.

Lemma 1.1

Let S be a k -dimensional subspace of C^n . There is a basis $\{v_1, v_2, \dots, v_k\}$ of S with the following property: for each $i = 1, 2, \dots, k$, there are distinct integers p_i , with $1 \leq p_i \leq n$ and $p_i \neq p_j$ for $i \neq j$, such that a largest modulus entry of each v_i is in position p_i .

Proof

We place the vectors of a basis $B = \{x_1, x_2, \dots, x_k\}$ of S as columns of an $n \times k$ full column rank matrix $X = [x_1 | \dots | x_k]$. Let $P_1 \in M_n$ be a permutation matrix (not necessarily unique) such that a largest modulus entry of x_1 (there could be more than one) is the first entry of $P_1 x_1 = y_1$.

Partition $P_1 X = [y_1 \ Y_2]$ and $y_1 = [y_{11} \ w^T]^T$. Let R_1 be an upper triangular matrix of the form

$$R_1 = \begin{pmatrix} 1 & z^* \\ 0 & I_{k-1} \end{pmatrix}$$

and choose the unique vector $z \in C^{n-1}$ such that:

$$(P_1)R_1 = [y_1 \ Y_2] \begin{pmatrix} 1 & z^* \\ 0 & I_{k-1} \end{pmatrix} = [y_1 \quad y_1 z^* + Y_2] = \begin{pmatrix} y_{11} & 0 \\ w & X^{(2)} \end{pmatrix}$$

has zero entries in the first row to the right of the $(1, 1)$ -entry. Now repeat this process on $X^{(2)}$, $X^{(3)}$, \dots to obtain $(P_{k-1} \cdots P_1)X(R_1 \cdots R_{k-1}) = Z$, A lower-triangular matrix whose diagonal entries are largest modulus entries in their respective columns. Moreover, $Z = P^* X^* R$, in which P is a product of $k - 1$ permutation matrices and R is a product of $k-1$ upper-triangular matrices

with 1s on the diagonal. Thus, P is a permutation matrix, R is upper-triangular and nonsingular, and Z has full column rank. Note that the column spaces of X and $X * R$ are the same. Thus, we see that the columns of $P^T * Z = X * R$ have the desired property.

We can now use Lemma 2.1 to prove our claim. In the following discussion, $A = [a_{i,j}]$ is always an $n \times n$ complex matrix.

Theorem 1.2

Let λ be an eigenvalue of A with geometric multiplicity k . Then λ is in at least k of the Geršgorin discs D_i of A .

Proof

Lemma 2.1 ensures that there is a basis $\{x_1, x_2, \dots, x_k\}$ of the eigenspace S of λ and distinct integers $p_1, \dots, p_k \in \{1, \dots, n\}$ such that each vector x_i has a largest modulus entry in position p_i . Our construction in the proof of the Geršgorin theorem shows that λ lies in Geršgorin discs D_{p_1}, \dots, D_{p_k} .

From Theorem 1.2, we see that an eigenvalue with geometric multiplicity at least $k \geq 1$ is contained in any union of $n - k + 1$ different Geršgorin discs of A . Now that's an improvement of Geršgorin's general theorem, which is the case $k = 1$ of our assertion!

Corollary 1.3

Let λ be an eigenvalue of A with geometric multiplicity at least $k \geq 1$. Then

$$\lambda \in \bigcup_{j=1}^{n-k+1} \left\{ z \in \mathbb{C} : \left| z - a_{i_j i_j} \right| \leq R'_{i_j} \right\}$$

for any choices of indices $1 \leq i_1 < \dots < i_{n-k+1} \leq n$. There are possibilities for such a union, and λ is contained in their intersection.

The rank-nullity theorem says that for a matrix B with n columns, the rank of B plus the dimension of the null space of B is equal to n . We apply this theorem and Theorem 1.2 to obtain the following result.

Corollary 1.4

Let λ be an eigenvalue of A . If $\text{rank}(A - \lambda I) \leq t$, then λ is in at least $n - t$ of the Geršgorin discs D_i of A .

Using the same two theorems, we can prove another interesting and useful result.

Corollary 1.5

If $|a_{i,i}| > R_i$ for q different values of i , then the geometric multiplicity of $\lambda = 0$ as an eigenvalue of A is at most $n - q$ and $\text{rank } A \geq q$.

Examples

The example (1) shows that the result in Theorem 2.2 is not valid for the algebraic multiplicity of an eigenvalue. Also, if λ is an eigenvalue of A with geometric multiplicity k , then λ may be in more than k of the Geršgorin discs D_i of A . For example, let

$$A = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$$

Then 2 is an eigenvalue of A with geometric multiplicity 1 (and algebraic multiplicity 2), but the eigenvalue 2 is contained in two Geršgorin discs of A . The same is true for the slightly more complicated example A_1 where

$$A_1 = \begin{pmatrix} 3 & -1/2 & 0 \\ 2 & 1 & 0 \\ -1 & -1 & 3 \end{pmatrix}$$

Which has an additional eigenvalue of 3.

Another example shows that an eigenvalue λ can be in t Geršgorin discs, for some t between the geometric multiplicity of λ and the algebraic multiplicity of λ . Let

$$A_2 = \begin{pmatrix} 3 & 0 & -1 \\ -2 & 4 & 6 \\ 1 & -1 & -1 \end{pmatrix}$$

Then 2 is an eigenvalue of A_2 with geometric multiplicity 1 and algebraic multiplicity 3; the eigenvalue 2 is contained in two Geršgorin discs of A_2 . The direct sum matrix

$$A_3 = \begin{pmatrix} A_2 & 0 \\ 0 & A_2 \end{pmatrix}$$

has 2 as an eigenvalue with geometric multiplicity 2 and algebraic multiplicity 6; the eigenvalue 2 is contained in four Geršgorin discs of A_2 .

At the end of this chapter, we close with a question, that is going to be the subject of a second paper and also, of the following chapter

Question:

Let k, r, t be positive integers with $k \leq r \leq t$. Is there a square complex matrix A and an eigenvalue λ of A such that λ has geometric multiplicity k and algebraic multiplicity t , and λ is in r Geršgorin discs of A ?

CHAPTER II CONTROL OF INCLUSION OF EIGENVALUES

To answer the posed question in the affirmative, we first introduce some notation and discuss the case that λ is equal to zero, and that the geometric multiplicity of λ is 1 (i.e., that there is just one Jordan block corresponding to zero), and we also take the algebraic multiplicity of λ to be n , the order of the matrix. We thus consider the $n \times n$ Jordan block

$$J_n(0) = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{pmatrix}$$

which has eigenvalue 0 with algebraic multiplicity n and geometric multiplicity 1. We shall transform $J_n(0)$ by similarity (thereby retaining the same two multiplicities) to a matrix which has the eigenvalue 0 in a number r of Geršgorin discs, for any given positive integer r , $1 \leq r \leq n$. To illustrate the idea of the proof, let us consider

$$J_4(0) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with not yet specified ξ_1 , ξ_2 and ξ_3 , define

$$S_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \xi_1 & 1 & 0 & 0 \\ 0 & \xi_2 & 1 & 0 \\ 0 & 0 & \xi_3 & 1 \end{pmatrix}$$

and transform $J_4(0)$ by similarity to $S_4^{-1}J_4(0)S_4$. Since

$$S_4^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -\xi_1 & 1 & 0 & 0 \\ \xi_2 \xi_1 & -\xi_2 & 1 & 0 \\ -\xi_3 \xi_2 \xi_1 & \xi_3 \xi_2 & -\xi_3 & 1 \end{pmatrix}$$

we obtain:

$$S_4^{-1} J_0(0) S_4 = \begin{pmatrix} \xi_1 & 1 & 0 & 0 \\ -\xi_1^2 & \xi_2 - \xi_1 & 1 & 0 \\ \xi_2 \xi_1^2 & \xi_2 \xi_1 - \xi_2^2 & \xi_3 - \xi_2 & 1 \\ -\xi_3 \xi_2 \xi_1^2 & \xi_3 \xi_2^2 - \xi_3 \xi_2 \xi_1 & \xi_3 \xi_2 - \xi_2^2 & -\xi_3 \end{pmatrix}.$$

The idea is to make choices for the ξ_i so that $\lambda = 0$ is out of the first three Geršgorin discs (G-discs for short), but necessarily in the fourth since its geometric multiplicity is 1. Then, depending on the required value of r , we re-adjust some of the values of ξ_i to zero. We see that if we choose $\xi_1 > 1$, then 0 is not in the first G-disc. Next, choose $\xi_2 > \xi_1$ and $\xi_2 > \xi_1 + \xi_1^2 + 1$, so that 0 is not in the second G-disc. Finally, choose $\xi_3 > \xi_2 + \xi_2 \xi_1^2 + |\xi_2 \xi_1 - \xi_2^2| + 1$, so that 0 is not in the third G-disc. This covers the case $r = 1$.

Suppose $r = 2$. In this case, keep ξ_1 and ξ_2 as before, but set $\xi_3 = 0$. Then, 0 is clearly not in the first and second G-discs, but is in the fourth one. But we also have

$$\xi_2 < \xi_2 \xi_1^2 + |\xi_2 \xi_1 - \xi_2^2| + 1$$

since $\xi_1 > 1$. So, 0 is in the third G-disc and we have covered the case $r = 2$. For the case $r = 3$, simply set $\xi_1 > 1$ and $\xi_2 = \xi_3 = 0$, so that 0 is in the last three G-discs. Of course, if $r = 4$, we set all the $\xi_i = 0$ and the eigenvalue 0 is in all four G-discs.

Example 2.1

Using (1) with $\xi_1 = 2$, $\xi_2 = 8$, $\xi_3 = 0$, we have that 0 is an eigenvalue of the matrix

$$\begin{pmatrix} 2 & 1 & 0 & 0 \\ -4 & 6 & 1 & 0 \\ 32 & -48 & -8 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

with the eigenvalue 0 having algebraic (geometric) multiplicity 4(1), respectively. Clearly, 0 is outside the first two, but inside the last two G-discs. We are now ready to prove our main result.

Theorem 2.2

Let k , r and n be positive integers such that $k \leq r \leq n$. Then for any complex number λ there is an $n \times n$ matrix A which has λ as an eigenvalue such that λ has geometric multiplicity k and algebraic multiplicity n , and λ is in precisely r Geršgorin discs of A .

Proof

We first consider the case where $k = 1$ and transform $J_n(0)$ by similarity to $S_n^{-1}J_0(0)S_n$, which has eigenvalue 0 with algebraic multiplicity n and geometric multiplicity 1. We use again

$$S_n = \begin{pmatrix} 1 & & & \\ \xi_1 & \ddots & & \\ & \ddots & 1 & \\ & & \xi_{n-1} & 1 \end{pmatrix}$$

with not yet specified entries ξ_i . It can be verified that the (i,j) entry of S_n^{-1} is

$$\begin{aligned} & 0 \text{ for } 1 \leq i < j \leq n, \\ & 1 \text{ for } i = j, \quad 1 \leq i \leq n, \\ & (-1)^{i-j} \xi_j \dots \xi_{i-1} \text{ for } 1 \leq j < i \leq n \end{aligned}$$

The matrix $A_0 = S_n^{-1}J_0(0)S_n$ has then its (i,j) entry a_{ij} such:

$$a_{ij} = \begin{cases} 0, & \text{if } j > i + 1 \\ 1, & \text{if } j = i + 1 \text{ and } i = 1, \dots, n - 1 \\ (-1)^{i-1} \xi_1 \left(\prod_{t=1}^{i-1} \xi_t \right), & \text{if } j = 1 \text{ and } 1 \leq i \leq n \\ (-1)^{i-j} (\xi_j - \xi_{j-1}) \left(\prod_{t=j}^{i-1} \xi_t \right), & \text{if } 1 < j \leq i \leq n \text{ and } a_{ij} \neq a_{nn} \\ -\xi_{n-1}, & \text{if } i = j = n \end{cases}$$

We now arrange that the eigenvalue 0 is not in the first $n - r$ G-discs of the matrix A_0 but in all the remaining r G-discs by choosing $\xi_{n-r+1} = \xi_{n-r+2} = \dots = \xi_{n-1} = 0$ and if $r < n$, the numbers $\xi_1, \xi_2, \dots, \xi_{n-r}$ recurrently as follows :

$$\xi_1 > 1$$

$$\xi_{t+1} > \xi_t + \sum_{j=1}^t |a_{t+1,j}| + 1, \quad \text{for } t \geq 1 \quad (2)$$

Indeed, the right-hand side of the last inequality in (2) contains only ξ_1, \dots, ξ_t ; if $r = 1$, the Geršgorin theorem ensures that the last G-disc contains the eigenvalue 0. In our case that $k = 1$, the matrix $A_0 + \lambda I_n$ satisfies the conditions of the theorem.

Finally, if the geometric multiplicity k is greater than one, it suffices to form a direct sum of the similarly constructed matrix $A_0 + \lambda I_{n-k+1}$ of order $n - k + 1$ and $k-1$ matrices of order one each with the entry λ to obtain the resulting matrix A .

We now turn our attention to generalizing the above construction to more than one distinct eigenvalue. In this case, our procedure will not allow the choice of more than one resulting eigenvalue to be arbitrary. For simplicity, we first consider two distinct eigenvalues.

Theorem 2.3

Let $k_1, k_2, r_1, r_2, n_1, n_2$, and n be positive integers such that $k_1 \leq r_1 \leq n_1, k_2 \leq r_2 \leq n_2$, and $n_1 + n_2 = n$. Then there is an $n \times n$ matrix A and eigenvalues λ_1, λ_2 such that λ_1 (λ_2) has geometric multiplicity k_1 (k_2) and algebraic multiplicity n_1 (n_2), and λ_1 (λ_2) is in precisely r_1 (r_2) G-discs of A , respectively. In fact, the eigenvalue λ_1 can be chosen to be any complex number.

Proof

As in Theorem 2.2, we can obtain an $n_1 \times n_1$ matrix A_{01} and an $n_2 \times n_2$ matrix A_{02} such that zero is an eigenvalue of A_{01} (A_{02}) with geometric multiplicity k_1 (k_2), algebraic multiplicity n_1 (n_2) and zero is in precisely r_1 (r_2) G-discs of A_{01} (A_{02}), respectively.

It is then clear that for suitable λ_1 and λ_2 , the requirements of the theorem are fulfilled in the matrix:

$$\begin{pmatrix} A_{01} + \lambda_1 I_1 & 0 \\ 0 & A_{02} + \lambda_2 I_2 \end{pmatrix}$$

One only has to ensure that the respective G-discs are separated.

Remark

It is immediate that Theorem 2.3 can be extended to more than two distinct eigenvalues in an analogous way.

Observation

An eigenvalue λ can, of course, be in q Geršgorin discs, where q is an arbitrary integer greater than the algebraic multiplicity of λ , but not exceeding the order of the matrix. For example, the following holds.

Let k , t and n be positive integers such that $k \leq t \leq n$. Then there is an $n \times n$ matrix A and an eigenvalue λ of A such that λ has geometric multiplicity k and algebraic multiplicity t , and λ is in all n Geršgorin discs of A . Indeed, we can simply use the eigenvalue $\lambda = 0$ and let

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

where A_1 is a $t \times t$ matrix which is a direct sum of k Jordan blocks of the type $J_i(0)$ and A_2 is a cyclic matrix of order $n - t$, that is,

$$A_2 = \begin{pmatrix} 0 & 1 & & \\ & 0 & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{pmatrix}$$

Note that the eigenvalues of A_2 are the $(n - t)^{\text{th}}$ roots of 1.

Remark 2.6

This work and the main results were originated by the author Rachid Marsli; the results in Chapter I will appear in the journal *The American Mathematical Monthly* [3], and the results in chapter II will appear in the journal *Linear Algebra and its Applications* [1].

There has been so much written on Geršgorin, his work, and follow-up results. We simply refer to Chapter 6 in the valuable book [2] and also to the important book [4].

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