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The Generators, Relations And Type Of The Backelin Semigroup

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ABSTRACT

We present an explicit minimal set of generators for the defining ideal of the family of Backelin semigroups, introduced first in [8], and find its Betti numbers. In particular, we compute the type of the semigroup as well, correcting a claim in the literature. Additionally, we show that the Betti numbers of the associated numerical semigroup ring coincide to those of its tangent cone.

INDEX WORDS: Commutative Algebra, Numerical Semigroups, Semigroups.

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DEDICATION

I would like to dedicate this document to my family for everything they've done.

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CHAPTER 1

INTRODUCTION

Numerical semigroups have been objects of interest since the late 19th century. Owing to their simplicity in definition, in addition to naturally arising in various problems across mathematics, they make it possible to state these problems in very simple terms, even if the solutions to said problems are themselves far from being trivial. To that end, let us start by defining numerical semigroups and related terminology in order to further motivate the study of these objects.

Definition 1.0.1 (Numerical semigroup)**.** *A numerical semigroup S is a non-empty subset of $\mathbb{N}$ that contains 0, closed under $+$ and satisfies $|S^c| < \infty$. If $n_1, n_2, \ldots, n_d \in \mathbb{N}$ satisfy $\gcd(n_1, \ldots, n_d) = 1$, then we say $S = \langle n_1, n_2, \ldots, n_d \rangle = \{\sum_{i=1}^{d} \lambda_i n_i | \lambda_i \in \mathbb{N}$ for all $i \in \overline{1, d}\}$ is the semigroup generated by $n_1, \ldots n_d$.*

As a convention, throughout this manuscript we will assume $0 \in \mathbb{N}$. In fact, one can show that any numerical semigroup S is a commutative monoid and is generated by some $n_1 \ldots n_d \in \mathbb{N}$. Initially, numerical semigroups were conceived as a way to study linear non-homogeneous equations with positive integer coefficients. This line of study, in fact gave rise to the Frobenius coin problem that seeks to find the largest monetary amount that can not be achieved using only coins of specified denominations. In technical terms, given a semigroup $S = \langle n_1, \ldots, n_d \rangle$, the coin problem seeks to find a formula for the largest integer not belonging to S. This integer is often called the Frobenius number of the semigroup S. The Frobenius problem remains unsolved to this day for general semigroups with four or more generators. In [4] the authors derive a rather surprising formula for the Frobenius number of a general two generated semigroup (a result originally due to Sylvester) and motivate the solution for the much more complicated case of a three generated semigroup.

While this classical approach presented a lot of interesting problems in semigroup theory in the late nineteenth century, numerical semigroups went overlooked for the better part of the early twentieth century. During the second half of the twentieth century, however, numerical semigroups started reappearing in the literature substantially, because of their new found applications to algebraic geometry and commutative algebra. It is known that the valuations of an analytically unramified one-dimensional local Noetherian domain R is a numerical semigroup if the completion $\hat{R}$ is a domain, and is reduced. In fact, many properties of these domains can be characterized via studying their associated value semigroup and conversely, one may use these numerical semigroups to construct one dimensional Noetherian local domains with desirable properties. In [2], the authors point out that an important class of examples of such rings are local rings of algebraic curves, which lends credence to the importance of the subject. This modern pursuit of numerical semigroups with a view towards algebraic geometry, also gave rise to the discovery of various important invariants like multiplicity, embedding dimension, pseudo-Frobenius number and type. We will define and study these invariants as we apply them to our problem at hand concerning the Backelin semigroup. On the other hand, over the course of the last fifty years various families of numerical semigroups have also been introduced to serve as counterparts of different objects in ring theory. Some examples, as pointed out in [12], are symmetric and pseudo-symmetric numerical semigroups, numerical semigroups with maximal embedding dimension and with the Arf property, saturated numerical semigroups and complete intersections. While the scope of numerical semigroups extends beyond the realm of algebraic geometry, into number theory, factorization on integral domains and linear integer programming – for the purposes of this short exposition, we refrain from discussing those applications.

In the following section, we will proceed to establish relevant definitions, terminology and some fundamental results from the literature on numerical semigroups to understand these objects a little better so that we may work with them. In section 1.3 we will introduce and study a smaller example problem due to Bresinsky [5] and present relevant work done in that area. In section 1.4 we will introduce the main problem concerning the Backelin semigroup, present relevant theorems from previous work done in this area, and motivate our proof.

CHAPTER 2

BACKGROUND

Let S be a numerical semigroup and let $S^{\times} = S \setminus \{0\}$. Denote by $S^{\times} + S^{\times}$ the elements of S that are the sum of two non-zero elements of S. In what follows, we will provide some simple results (without proof) that will be useful for further discussion. The reader is encouraged to refer to [12] for proofs and clarifications.

Theorem 2.0.1. *Let S be a numerical semigroup as above, and $S^{\times} = S \setminus \{0\}$. S admits a unique minimal system of generators given by $S^{\times} \setminus (S^{\times} + S^{\times})$. This minimal system is finite.*

With a unique minimal generating for any given numerical semigroup S guaranteed by Theorem 2.0.1, we can now concretely formulate the following definition.

Definition 2.0.2 (Multiplicity and Embedding dimension)**.** *Let S be a numerical semigroup and let $\{n_1 < n_2 < \cdots < n_h\}$ be its unique minimal generating set. We call n_1 the multiplicity of S, often denoted $m(S)$ and h the embedding dimension of S, denoted by* $\mathrm{embdim}(S)$.

A few things that are straightforward to note are that $m(S)$ is the least positive integer in S and that $\mathrm{embdim}(S) = 1$ if and only if $S = \mathbb{N}$. With this set up in place one can also formally define the two important invariants of S, namely the *Frobenius number* of S and the *genus* of S.

Definition 2.0.3 (Frobenius number and genus)**.** *If x in $\mathbb{N} \setminus S$ such that $x + n \in S$ for all $n \in \mathbb{N}$, then x is called the Frobenius number of S and is often denoted by $F(S)$. The set $\mathbb{N} \setminus S$ is known as the set of gaps of S, and its cardinality is called the genus of S.*

While the genus of S does not play an important role in this study, the Frobenius number is rather central to our exploration. Presented below is a result due to Sylvester on the Frobenius number of a general two generated semigroup. The reader is encouraged to refer Chapter 1 of [4] for a detailed proof.

Theorem 2.0.4. *Let $S = \langle a, b\rangle$, the Frobenius number of S is given by $F(\langle a, b\rangle) = ab - a - b$.*

One may relax the definition of the Frobenius number to allow for the so called pseudo-Frobenius numbers of S. An associated invariant is the so called type of the numerical semigroup. Both of these play an important role in our study, and are central to the connection between numerical semigroups and objects in commutative algebra.

Definition 2.0.5 (Pseudo-Frobenius numbers and Type)**.** *An integer $x \in \mathbb{N} \setminus S$ is called a pseudo-Frobenius number if $x + s \in S$ for all $s \in S^*$. The set of all pseudo-Frobenius numbers are often denoted by $PF(S)$. The cardinality of $PF(S)$ is known as the type of S and is often denoted by $t(S)$.*

An easy thing to notice is that $F(S) \in PF(S)$ and is, the maximal element of $PF(S)$. In fact, if we introduce an ordering $\leq_S$ over the integers given by $a \leq_s b \iff b - a \in S$, then it can be easily shown from the definition that $PF(S)$ is the same as the set of all maximal elements in the gaps of S with respect to $\leq_S$. The following example illustrates all the above definitions in action for a given semigroup S.

Example 1. *Let $S = \langle 5, 7, 11\rangle$. S is given by $\{0, 5, 7, 10, 11, 12, 14, 15, \dots\}$. The multiplicity of S is $m(S) = 5$, the Frobenius number of S is $F(S) = 13$. The gaps of S is the set $\{1, 2, 3, 4, 6, 8, 9, 13\}$ and which gives that the genus of S is 8. The pseudo-Frobenius numbers of S are given by $PF(S) = \{9, 13\}$, and thus the type of S is simply $t(S) = 2$.*

In fact, work in [8], extends Herzög's work in [9] to show that any three generated semigroup has type at most 2. This case of $\text{embdim}(S) = 3$ is also further explored in Chapter 9 of [12]. It is noteworthy that a lot of concrete computations in this concerning numerical semigroups can be made using `numericalsgps` package that comes integrated with the computer algebra software GAP.

In order to make further progress, we now want to establish the mechanism that lets us translate and reinterpret the above definitions into commutative algebraic terms. To that end, let K be a field and consider the following definition

Definition 2.0.6 (Numerical semigroup ring)**.** *To a given semigroup* $H = \langle a_1, \ldots, a_h \rangle$*, we can associate the so called numerical semigroup ring given by* $K[H] = K[t^a : a \in H]$*, which is simply a* K*-subalgebra of* $K[t]$*.*

In what follows we outline the necessary ideas and concepts from commutative algebra which will serve us well in our exploration. In section 3.1 we will reinterpret these ideas in the context of semigroup rings. A much more thorough treatment of the following concepts can be found in chapter 1 of [6] or chapter 15 of [7].

Definition 2.0.7 (Syzygy module)**.** *Let* $(R, \mathfrak{m})$ *be a commutative, Noetherian, local ring and let* M *be a finitely generated* R*-module, say with minimal generators* $z_1, \ldots, z_{\beta_0}$*. A syzygy of* M *is an element* $(a_1, \ldots, a_n) \in R^{\beta_0}$ *such that* $a_1 z_1 + \cdots + a_{\beta_0} z_{\beta_0} = 0$*. The set of all syzygies (relative to a given generating set) forms a submodule of* R^{β_0} *and is known as the (first) syzygy module of* M*. In other words, it is the kernel of the map* $\phi_0 : R^{\beta_0} \to M$ *where the canonical basis elements of* R^{β_0} *are mapped to the generators of* M*.*

It is also worth noting that for any two choices of minimal generating sets, their corresponding syzygy modules are isomorphic. Proceeding similarly, upon the construction of the first syzygy module, say $M_1 = \operatorname{Ker}(\phi_0)$ of M, one can choose a minimal generating set of M_1 to create the second syzygy module, of M, say $M_2 = \operatorname{Ker}(\phi_1)$ where $\phi_1 : R^{\beta_1} \to R^{\beta_0}$ defined similar to ϕ_1; and so on. This way one gets a minimal free resolution

$$\cdots \to R^{\beta_i} \to \cdots \to R^{\beta_1} \to R^{\beta_0} \to M \to 0$$

where the i^{th} syzygy module of M is simply $\operatorname{Ker}(\phi_{i-1})$.

While the above construction was made for the Noetherian local case, it is worth noting that the above definitions and discussions also hold for finitely generated graded modules over polynomial rings with a minimal graded set of generators.

Definition 2.0.8 (Betti Numbers). *With the above setup, we say β_i is the i^{th} Betti number of M.*

With the above set up in place, one can now define the projective dimension of M

Definition 2.0.9 (Projective dimension). *If for some $h \in \mathbb{N}$ we have $\beta_h \neq 0$ but $\beta_i = 0$ for all $i > h$, then, h is called the projective dimension of M, often denoted by $\operatorname{projdim}(M)$. If no such h exists, then $\operatorname{projdim}(M) = \infty$.*

Often times when working with Cohen-Macaulay (Noetherian) local rings R with finite projective dimension, one can concretely compute the projective dimension of an $R-$module M using the Auslander Buchsbaum formula presented below.

Theorem 2.0.10 (Auslander-Buchsbaum). *Let $(R, \mathfrak{m}, K)$ be a Noetherian local ring and M a finitely generated R module. If $\operatorname{projdim}(M) < \infty$ then we have*

$$\operatorname{projdim}(M) = \operatorname{depth}(R) - \operatorname{depth}(M).$$

While Theorem 2.0.10 is given for the Noetherian local case, a version of it is also true for a finitely generated graded module over a polynomial ring (over a field). That is to say, if we let $S = K[x_1, \ldots, S_h]$ and V a finitely generated graded module over S, then

$$\operatorname{projdim}(V) = h - \operatorname{depth}(V).$$

Without loss of generality, when we consider finitely generated graded modules over a polynomial ring, we will treat the above equation as the result given by Theorem 2.0.10. A special case of the above that is of importance to us is when $V = S/\mathfrak{p}$ where $\mathfrak{p}$ is some prime ideal of S. In this case, we get more flexibility because we know that V is Cohen-Macaulay and thus there is equality in depth and dimension. So we get

$$\operatorname{projdim}(V) = h - \dim(V) = h - (\dim(S) - \operatorname{ht}(\mathfrak{p})) = \operatorname{ht}(\mathfrak{p}).$$

The last non-trivial Betti number of M is known as the Cohen-Macaulay type of M and is denoted by $t(M)$ (a temporary abuse of notation, refer Section 3.1 for clarification), explicitly given by $\dim_K \operatorname{Ext}_M^h(K, M)$ where $h = \operatorname{depth}(M)$. However, by reducing the homological degrees, one is able to show that $t(M) = \dim_K \operatorname{Soc}(M/\mathbf{x}M)$, where $\mathbf{x}$ is a maximal regular sequence and $\operatorname{Soc}(\cdot)$ refers to the Socle, defined (in the Noetherian local case) as $\operatorname{Soc}(M) = (0 : \mathfrak{m})_M \cong \operatorname{Hom}_R(K, M)$; see [6].

Another object that is relevant to our study is the associated graded ring of R.

Definition 2.0.11. *Let I be an ideal of a ring R. The associated graded ring of R with respect to I is given by*

$$\operatorname{gr}_I R = \bigoplus_{i=0}^{\infty} \frac{I^i}{I^{i+1}}.$$

Letting $\frac{I^i}{I^{i+1}}$ be the homogeneous component of degree i we can see that $\operatorname{gr}_I R$ is a graded $K-$algebra under the standard grading. This object is often called the tangent cone of $K[H]$. Chapter 5 of [7] provides a very nice explanation on the reason for this terminology.

CHAPTER 3

PREVIOUS RESULTS

3.1 Set up

In this section we will reinterpret the commutative algebra tools established in the previous chapter in terms of semigroup rings. To this end, we start by constructing the semigroup ring associated to a given semigroup $H = \langle a_1, \ldots, a_h \rangle$.

Let $S = K[x_1, \ldots, x_h]$, and consider the natural homomorphism $\phi : S \to K[t]$ with $\phi(x_i) = t^{a_i}$. This can be thought of as a graded K-algebra homomorphism mapping onto $K[t^{a_1}, \ldots, t^{a_h}] = K[H]$ with the grading given by $\deg(x_i) = a_i$. The kernel of this map is the central object of study, and is generally known in the literature as the *presentaton ideal* or the *defining ideal* of the semigroup ring $K[H]$. We will denote this presentation ideal by I_H. The importance of I_H is clear as it describes the relations between the generators of H. Furthermore, it is clear that we have $S/I_H \cong K[H]$, and so obtaining the relations in I_H gives us a realistic way to understand $K[H]$. Furthermore, in [9], Herzög points out that I_H is generated by binomials and in fact,

$$I_H = \left\langle x^u - x^v : u, v \in \mathbb{N}^n \text{ with } \sum_{i=1}^{h} u_i a_i = \sum_{i=1}^{h} v_i a_i \right\rangle$$

where $x^u = \prod_{i=1}^{h} x_i^{u_i}$.

With this in place we first note that in our case, we have $M = K[H] \cong S/I_H$. So, we construct a minimal free resolution of $K[H]$ as

$$\ldots \xrightarrow{\phi_2} S^{\beta_1} \xrightarrow{\phi_1} S^1 \xrightarrow{\phi_0 = \phi} K[H] \to 0$$

It is clear that minimal generating set of I_H generates the first syzygy module of $K[H]$

and that the cardinality of the minimal generating set of I_H makes up the first Betti number of $K[H]$. Moreover, since $K[H]$ is a finitely generated module over the polynomial ring S, we can use Hilbert's syzygy theorem to produce non-trivial syzygies until the the projective dimension is reached.

In case of numerical semigroup rings we also enjoy more benefits, as we are able to concretely compute the projective dimension of $K[H]$ using the Auslander-Buchsbaum formula. Letting $M = S/I_H$ and $R = S$ in Theorem 2.0.10, it is straightforward to check that the hypothesis of Theorem 2.0.10 holds in this case. Since $K[H] \cong S/I_H$ is a one dimensional domain and S is just a polynomial ring over a field, both objects are Cohen-Macaulay and thus, we have equality in dimension and depth. This gives

$$\operatorname{projdim}(K[H]) = \dim(S) - \dim(S/I_H) = h - 1 = \operatorname{embdim}(H) - 1.$$

Therefore, the Betti sequence of $K[H]$ is simply $(\beta_0(K[H]), \beta_1(K[H]), \ldots, \beta_{h-1}(K[H]))$, where $\beta_0(K[H]) = 1$. It is also generally known that $\mu(I_H) = \beta_1(K[H])$ equals the minimal number of graded generators of I_H and this number is the same for *any* set of minimal graded generators for I_H.

Furthermore, in section 3 of [13], Stamate shows that the Cohen-Macaulay type of $K[H]$ coincides with the $t(H)$. Examples of families of semigroups with unbounded type and fixed embedding dimension have been always of interest and are scarce in the literature.

Finally, we also consider the associated graded ring of $K[H]$ with respect to the maximal ideal $\mathfrak{m} = \langle t^h : h \in H^* \rangle$. This is explicitly given as

$$\operatorname{gr}_{\mathfrak{m}}(K[H]) = \bigoplus_{i=0}^{\infty} \frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1}}.$$

We recall that, letting $\frac{\mathfrak{m}^i}{\mathfrak{m}^{i+1}}$ be the homogeneous component of degree i, we notice that

$\mathrm{gr}_{\mathfrak{m}}(K[H])$ is a graded K-algebra under the standard grading. Working with $\mathrm{gr}_{\mathfrak{m}}(K[H])$ lets us work around the non-standard grading on $K[H]$, and instead work with a standard grading, which often times, is much nicer.

An interesting fact about the tangent cone of $K[H]$ and its Betti numbers that is important to our exploration, is given by the following theorem

Theorem 3.1.1. *The Betti numbers of* $\mathrm{gr}_{\mathfrak{m}}(K[H])$ *bound by above the Betti numbers of* $K[H]$*. That is to say for any* i*, we have* $\hat{\beta}_i \geq \beta_i$ *where* $\hat{\beta}_i$ *is the* i^{th} *Betti number of* $\mathrm{gr}_{\mathfrak{m}}(K[H])$ *and* β_i *is the* i^{th} *Betti number of* $K[H]$*.*

Moreover, a special case of the above theorem requires its own terminology as defined below

Definition 3.1.2 (Homogeneous type)**.** *Let* $\hat{\beta}_i$ *and* β_i *be as in Theorem 3.1.1. When* $\hat{\beta}_i = \beta_i$ *for all* i*, then we say that* $K[H]$ *is of Homogeneous type.*

The reader may refer to [10] for further reading.

3.2 Previous results

In this section we will enumerate, explore and motivate some previous results in this area. When $\mathrm{embdim}(H) = h = 2$, I_H turns out to be principal. This can be seen as a consequence of Theorem 2.0.10. In particular, notice that we are in the Cohen-Macaulay local case with $M = S/I_H$ and I_H prime. So

$$1 = 2 - 1 = \mathrm{embdim}(K[H]) - 1 = \mathrm{projdim}(K[H]) = \mathrm{ht}(I_H).$$

Since we are working with a UFD and I_H is prime, we know that $\mathrm{ht}(I_H) = 1 \implies I_H$ is principal.

When $h = 3$, a result of Herzog in [9] shows that $\mu(I_H) \leq 3$ and in fact I_H can be described precisely. However, if $h \geq 4$, the situation changes drastically. Bresinsky was first to show that for $h = 4$ there is no upper bound for $\mu(I_H)$ in [5]. Specifically, Bresinsky's family of semigroups is given as follows: let $n \geq 2$ be an integer and let

$$H = \langle (2n-1)2n, (2n-1)(2n+1), 2n(2n+1), 2n(2n+1) + 2n - 1 \rangle$$

and $\mu(I_H) = 4n$. Subsequently, Arslan has also provided an example by letting

$$H = \langle n(n+1), n(n+1) + 1, (n+1)^2, (n+1)^2 + 1 \rangle,$$

for $n \geq 2$ and showing that $\mu(I_H) = 2n + 2$, see [1]. In what follows, we will look at Bresinsky's semigroup and use it as an example to motivate our problem at hand. To this end, let $n \geq 2$ and consider

$$H = \langle (2n-1)2n, (2n-1)(2n+1), 2n(2n+1), 2n(2n+1) + 2n - 1 \rangle.$$

Construct the canonical homomorphism as outlined in the previous chapter. In [5], Bresinsky introduces H only to show that $\mu(I_H)$ can be arbitrarily large. It is done so in the following fashion. We start by defining sets

$$\begin{aligned}
\mathcal{A}_1 &= \{f_i = x^{i+1}y^{2n-i} - z^{i-1}w^{2n-i} : 1 \leq i \leq 2n\} \\
\mathcal{A}_2 &= \{f = x^{\nu_1}z^{\nu_3} - y^{\mu_2}w^{\mu_4} : \nu_3, \mu_4 \leq 2n-1, f \in I_H\} \\
\mathcal{E} &= \{g_1 = z^{2n-1} - y^{2n}, g_2 = xw - yz\} \\
\mathcal{A} &= A_1 \cup A_2 \cup E \text{ and} \\
\mathcal{A}' &= A \cup \{x^{\nu_1}z^{\nu_3} - y^{\mu_2}w^{\mu_4} \in I_H\}
\end{aligned}$$

After defining these sets, Bresinsky took the following four-step approach

1. (*Lemma 2*) $(A') = I_H \implies (A) = I_H$

2. (*Lemma 3*) $(A') = I_H$

3. (*Lemma 4*) Elements of A_1 are necessarily in the minimal generating set.

4. (*Corollary 1*) $\mu(I_H) \geq 2n$

Since n is arbitrary, Corollary 1 ensures that $\mu(I_H)$ can be arbitrarily large. The heart of Bresinsky's argument lies in *Lemma 2*, and this technique proves to be vital in our problem as well. So, let us consider an example to illustrate what happens in *Lemma 2*.

Example 2. *Let $n = 2$ and so we have $H = \langle 12, 15, 20, 23\rangle$.*
Consider $f = x^6z^7 - y^8w^4 \in (A')$ and construct f' given by

$$f' = f + x^6z^4f_4 = f + x^6z^4(x^5 - z^3) = x^{11}z^4 - y^8w^4$$

Notice that we have decreased the exponent of z by 3. Similarly, we construct once more

$$f'' = f' + x^6z^4f_4 = x^{16}z^1 - y^8w^4$$

Finally, for one last time, we consider

$$f''' = f'' - y^8w^1f_1 = f'' - y^8w^1(x^2y^3 - w^3) = x^2(x^{14}z^1 - y^{11}w^1) \in A$$

Now, we simply work backwards to write

$$f = f''' + y^8wf_1 - 2x^6z^4f_4 \in A$$

The technique highlighted in the above example is what we in our work call the process of reduction. This is because at the end of the entire process, we have essentially written f as a combination of elements with lesser homogeneous degree than that of the binomial f – which allows us to shift our focus from f to the "reduced" components of f.

In a clever inductive argument, Bresinsky was able to also show that for any $h \geq 4$, $\mu(I_H)$ can be arbitrarily large as well. But of course, the set $\mathcal{A}_2$ is an infinite set in Bresinsky's example, and thus a closed form for the cardinality of the minimal generating set for I_H was unknown for a long period of time. It was only in 2014, a complete 44 years after Bresinsky's original work, that a generating set was completely written for Bresinsky's example. In [10] the authors revise Bresinksy's $\mathcal{A}_2$ set with

$$\mathcal{A}_3 = \{u_j = x^{2n+1-j}z^j - y^{2n-j}w^j : 0 \leq j \leq 2n-2\}$$

to write an explicit generating set for I_H. Notice that this replacement allows for the removal of g_1 from the set $\mathcal{E}$. So the final minimal generating set for I_H is given by $\mathcal{B} = A_1 \cup \{g_2\} \cup \mathcal{A}_3$ with $4n$ elements.

The last piece of the puzzle we need is a way to ensure the minimality of a generating set. To that end consider the following set up. Once again, let $S = K[x_1, \ldots, x_h]$ and I an ideal of S. Consider the standard grading on S where all variables have degree 1. For a nonzero element $f \in I$, the nonzero homogeneous part of f of lowest degree is called the initial form of f, and is denoted by f^*. Let $I^* = (f^* : f \in I)$. We say that $f_1, \ldots, f_k \in I$ form a standard basis for I if $I^* = (f_1^*, \ldots, f_k^*)$. The following theorem is useful in establishing the minimality of our generating set.

Theorem 3.2.1 (Herzog, see [10])**.** *Let $I \subseteq \mathfrak{n} = (x_1, \ldots, x_n)$ be an ideal in $S = K[x_1, \ldots, x_h]$. Let $\hat{S} = K[[x_1, \ldots, x_h]]$ and assume that x_1 is a nonzerodivisor on $\hat{S}/I\hat{S}$.*
Let $\pi : S \to K[x_2, \ldots, x_h]$ defined by $\pi(x_1) = 0, \pi(x_i) = x_i$ for all $i = 2, \ldots, h$ and denote $\overline{I} = \pi(I)$. Assume that $g_1, \ldots, g_r$ form a standard basis for $\overline{I}$ in $K[x_2, \ldots, x_h]$ and let $f_i \in I$ such that $\pi(f_i) = g_i$ and $deg(f_i^*) = deg(g_i^*)$ for all $i = 1, \ldots, r$. Let $\overline{S} = \pi(S) = K[x_2, \ldots, x_n]$ and $\overline{\mathfrak{n}} = \pi(\mathfrak{n}) = (x_2, \ldots, x_n)$.

1. *Then $f_1, \ldots, f_r$ form a standard basis for I.*

2. *x_1 is a nonzerodivisor on $gr_{\mathfrak{n}}(S/I)$.*

3. We have a graded K-algebra isomorphism

$$\frac{gr_{\mathfrak{n}}(S/I)}{x_1 \cdot gr_{\mathfrak{n}}(S/I)} \simeq gr_{\overline{\mathfrak{n}}}(\overline{S}/\overline{I}).$$

The above theorem plays a very important role in our study. This is because, with a clever representation of the minimal generating set and a good choice of π, the above theorem gives us a way to translate our binomial ideal I_H into a monomial ideal and reduces the entire problem into ensuring the minimal generation of a monomial ideal by the images, which is a much easier endeavor. In addition, since the Betti numbers are preserved under modding out by a nonzero-divisor, we know that all the Betti numbers are preserved through this map π. This reduces the computational complexity of the problem and boils the non-triviality down to simply just obtaining a proper generating set of I_H.

CHAPTER 4

THE MAIN RESULT

4.1 Introducing the problem and preliminary results

In what follows, we focus on the case where $h = 4$, and in particular we study Backelin's family of semigroups defined as done below:

Given $n \geq 2, r \geq 3n + 2$, let

$$H = H_{n,r} = \langle r(3n+2)+3, r(3n+2)+6, r(3n+2)+3n+4, r(3n+2)+3n+5 \rangle.$$

Fröberg, Gottlieb, and Häggkvist have communicated this example in their work [8] as the first family with fixed embedding dimension, but with unbounded type, attributing the claim to Backelin. In the paper, they verified the claim by showing that the type is at least $2n+2$, and also stated, however incorrectly, that the type equals $2n + 3$ (see Example on page 75 of their paper).

The goal of the chapter is multifold. We will produce an explicit minimal generating set for I_H and prove therefore that $\mu(I_H) = 3n+4$, which was stated in [13] based on numerical evidence obtained with Singular and GAP. Secondly, we verify that the type of this semigroup is $3n+2$ correcting the wrong claim from [8]. This discrepancy was noted by Stamate in his paper [13], based upon computer algebra computations, but we verify it here in full generality. Moreover, there has been interest in the literature in numerical semigroup rings for which the Betti numbers coincide with the Betti numbers of their tangent cone. We show that this is the case for the Backelin family of numerical semigroups, confirming numerical evidence from [13].

We start by letting n, r and $H = H_{n,r}$ as above. As done in chapter 2, we let $S =$

$K[x, y, z, w]$ and $\phi : S \to K[H]$, defined by

$$\phi(x) = t^{r(3n+2)+3},\ \phi(y) = t^{r(3n+2)+6},\ \phi(z) = t^{r(3n+2)+3n+4},\ \phi(w) = t^{r(3n+2)+3n+5}.$$

We put a grading on S by letting $\deg(x) = r(3n+2)+3, \deg(y) = r(3n+2)+6, \deg(z) = r(3n+2)+3n+4, \deg(w) = r(3n+2)+3n+5$ and consider the grading on $K[H]$ induced by the natural grading on $K[t]$. This turns ϕ into a graded homomorphism. As before, we let $I_H = \mathrm{Ker}(\phi)$ which is a graded ideal under this $\mathbb{N}$-grading. It is known that this equals the minimal number of graded generators of I and this number is the same for any set of minimal graded generators for I.

Definition 4.1.1. *Consider the following sets of elements in $K[x, y, z, w]$:*

$$S_1 = \{x^{n-k}z^{3k-1} - y^{n-k+1}w^{3k-2} : k = 1, \dots, n\},$$

$$S_2 = \{x^{r-k+3}y^{k-1} - z^{3(n-k)+2}w^{r-3(n-k)-1} : k = 1, \dots, n\},$$

$$S_3 = \{x^{r-(n+k)+3}y^{n+k} - z^{3(n-k)+1}w^{r-3(n-k)+1} : k = 1, \dots, n\},$$

and

$$E = \{xw^3 - yz^3, x^n w^2 - y^{n+1}z, x^{r-n+2}y^n z - w^{r+2}, x^{2n-1}zw - y^{2n+1}\}.$$

With these notations we can state the main results of this note.

Theorem 4.1.2. *The set $S_1 \cup S_2 \cup S_3 \cup E$ generates the defining ideal of $K[H]$.*

Moreover, the following result holds.

Corollary 4.1.3. *1. A minimal generating set of the presentation ideal of the semigroup ring $K[H]$ is given by $S_1 \cup S_2 \cup S_3 \cup E$.*

2. The type of $K[H]$ is $3n+2$ and the sequence of Betti numbers of $K[H]$ is $(1, 3n+4, 6n+5, 3n+2)$.

The proof of Theorem 4.1.2 will occupy the bulk of this article. Using it, one can show Corollary 4.1.3. So, we will show this first.

Proof of Corollary 4.1.3 while assuming Theorem 4.1.2. Our proof will use the notations in Theorem 3.2.1. Let $x_1 = x, x_2 = y, x_3 = z, x_4 = w$ and $\overline{I} = \pi(I)$, where π is defined by $\pi(x) = x, \pi(y) = y, \pi(z) = z, \pi(w) = 0$, as it is done in Theorem 3.2.1.

The minimal number of generators of I is given by the number of generators of a minimal graded set of generators. The Betti numbers of the graded ideal I is invariant under modding out by a nonzerodivisor on S/I, in this case w. So it is enough to find the Betti numbers of $\overline{I}$.

Theorem 4.1.2 shows that the defining ideal I of ϕ is (S_1, S_2, S_3, E). This shows that, if we let $w = 0$, the ideal $\overline{I}$ is generated by

$$x^{n-k}z^{3k-1}, x^{r-k+3}y^{k-1}, x^{r-(n+k)+3}y^{n+k}, k = 1, \ldots, n$$

and

$$yz^3, y^{n+1}z, x^{r-n+2}y^n z, y^{2n+1}.$$

Clearly this forms a minimal set of generators for $\overline{I}$ which is a monomial ideal. So, $\mu(\overline{I}) = 3n + 4$ which also shows that $\mu(I) = 3n + 4$. As the cardinality of $S_1 \cup S_2 \cup S_3 \cup E$ is also $3n + 4$ this proves that it is also a minimal set of generators for I.

For the second claim of the Corollary, we will first compute the type of $K[H] \simeq S/I$, which reduces to finding the type of $\overline{S}/\overline{I}$. This can be computed by finding the dimension of the socle of $K[x, y, z]/\overline{I}$, which is $\frac{\overline{I}:(x,y,z)}{\overline{I}}$ (see Definition 1.2.15 and Lemma 1.2.19 in [?]).

Hence, to compute the type, we will find the set $\mathcal{B}$ of monomials whose images form a basis for

$$\frac{\overline{I} : (x, y, z)}{\overline{I}}.$$

Since $\overline{I}$ is monomial, it can be easily seen by looking at the exponents, that the following

$3n+2$ monomials

$$x^{n-k}z^{3k-2} \text{ for } k = 2, \ldots, n$$

$$x^{r-(n+k)+3}y^{n+k-1}, x^{r-k+2}y^{k-1}z \text{ for } k = 1, \ldots, n$$

and

$$x^{r-n+1}y^n z, x^{n-2}y^n z^2, x^{r-2n+2}y^{2n}$$

have nonzero images in $\frac{\overline{I}:(x,y,z)}{\overline{I}}$.

Let us consider a monomial $x^a y^b z^c$ whose image belongs to the basis $\mathcal{B}$. We will show it belongs to the list above by examining each possible value for c. Clearly, $c \leq 3n-2$.

If $c = 0$, then $b \leq 2n$. If $b = 2n$, then the only possibility for a is $r - 2n + 2$. For $n \leq b < 2n$, then $b = n + k - 1$ for some k with $1 \leq k \leq n$. In this case, $a = r - (n+k) + 3$. Further examination shows that $b < n$ is not possible.

If $c = 1$, then $b \leq n$. If $b = n$, it follows easily that $a = n - 2$. If $b < n$, then $b = k - 1$ for some $k = 1, \ldots, n$ and it can be seen that $a = r - k + 2$.

The rest of the cases $c \geq 2$ can be examined similarly.

Finally, the zeroth Betti number is 1, the first Betti number is $3n+4$ and the third Betti number is $3n+2$. Since the alternating sum of Betti numbers is 0 it follows that the second Betti number is $6n+5$.

$\square$

The following Lemma is well-known and easy to check and so it will be used without proof.

Lemma 4.1.4. *Let K be a field and $f_1, \ldots, f_n$ be elements of $K[x_1, \ldots, x_n]$ that are homogenous under the standard grading. Let $I = (f_1, \ldots, f_n)$. Then $f_1, ..., f_n$ form a standard basis for I.*

We can derive now the following consequence of our work.

Corollary 4.1.5. *1. The set $S_1 \cup S_2 \cup S_3 \cup E$ forms a standard basis for the defining ideal I of $K[H]$.*

2. Let $\mathfrak{n} = (x, y, z, w)$ in $K[x, y, z, w]$ which maps onto the maximal graded ideal of $K[H] = K[x, y, z, w]/I$. Then $K[H]$ and $\operatorname{gr}_{\mathfrak{n}}(K[H])$ have the same Betti numbers.

Proof. This result is a straightforward consequence of Theorem 3.2.1. The reasoning is well known, mentioned explicitly in [10] in the proof of their Theorem 1.4.

As in Theorem 3.2.1, let $S = K[x, y, z, w]$, I be the defining ideal of $K[H]$, and $\overline{I}, \overline{\mathfrak{n}}, \overline{S}$ resulting from sending $x \to 0$.

The elements we obtain in $\overline{I}$ are all homogenous:

$$z^{3n-1} - yw^{3n-2}, y^{n-l+1}w^{3l-2}, z^{3(n-k)+2}w^{r-3(n-k)-1}, z^{3(n-k)+1}w^{r-3(n-k)+1}, yz^3, y^{n+1}z, w^{r+2}, y^{2n+1}$$

with $l = 1, \dots, n-1$ and $k = 1, \dots, n$. They form a standard basis for $\overline{I}$, according to Lemma 4.1.4. Now Theorem 3.2.1 gives the first part.

For the second part, on one hand, because x is a nonzerodivisor on $K[H]$, we have that the Betti numbers of $K[H]$ coincide to the Betti numbers of $\overline{S}/\overline{I}$.

On the other hand, because x is nonzerodivisor on $gr_{\mathfrak{n}}(S/I)$, then the Betti numbers of $gr_{\mathfrak{n}}(S/I)$ coincide to the Betti numbers of

$$\frac{gr_{\mathfrak{n}}(S/I)}{x \cdot gr_{\mathfrak{n}}(S/I)} \simeq gr_{\overline{\mathfrak{n}}}(\overline{S}/\overline{I}).$$

But $\overline{S}/\overline{I}$ and $gr_{\overline{\mathfrak{n}}}(\overline{S}/\overline{I})$ have the same Betti numbers, as $\overline{I}$ is a homogenous ideal.

$\square$

We will end this section by computing the pseudo-Frobenius numbers of the Backelin semigroup $H_{n,r}$.

Definition 4.1.6. *For a numerical semigroup H, the pseudo-Frobenius numbers are the elements of the set*

$$PF(H) = \{n \in \mathbb{Z} \setminus H : n + h \in H \text{ for all } h \in H \setminus \{0\}\}.$$

The cardinality of this set equals the type of H.

Mainting the notations used in introducing the Backelin semigroup, let us denote $a_1 = r(3n+2)+3, a_2 = r(3n+2)+6, a_3 = r(3n+2)+3n+4, a_4 = r(3n+2)+3n+5$.

Proposition 4.1.7. *Let $n \geq 2, r \geq 3n+2$. The pseudo-Frobenius numbers of $H_{n,r}$ are*

$$(n-k)a_1 + (3k-2)a_3 - a_4, \text{ for } k = 2, \dots, n$$

$$(r-(n+k)+3)a_1 + (n+k-1)a_2 - a_4, \text{ for } k = 1, \dots, n$$

$$(r-k+2)a_1 + (k-1)a_2 + a_3 - a_4, \text{ for } k = 1, \dots, n$$

$$(r-n+1)a_1 + na_2 + a_3 - a_4, (n-2)a_1 + na_2 + 2a_3 - a_4, (r-2n+2)a_1 + 2na_2 - a_4.$$

Proof. The homomorphism $\phi : S \to K[H]$, where $w \to 0$, induces an isomorphism $S/(w) \simeq K[H]/(t^{a_4})$. But $S/(w) = K[x,y,z]/\overline{I}$, where $\overline{I}$ is the ideal resulting from sending $w \to 0$. According to Lemma 8 in [3] we need compute the nonzero monomials of the socle of $K[H]/(t^{a_4}) \simeq K[x,y,z]/\overline{I}$ and this was in done in the proof of Corollary 4.1.3. $\square$

4.2 Proof of Theorem 4.1.2

The arguments behind the proof of Theorem 4.1.2 will cover the remainder of this paper and will be rather lengthy. We introduce some auxiliary notations needed in section.

As we mentioned in the Introduction, the defining ideal I is generated by binomials

$$x^{\nu_1}y^{\nu_2}z^{\nu_3}w^{\nu_4} - x^{\mu_1}y^{\mu_2}z^{\mu_3}w^{\mu_4},$$

with $\nu = (\nu_1, \ldots, \nu_4), \mu = (\mu_1, \ldots, \mu_4), \sum \nu_i a_i = \sum \mu_i a_i$. Let $d = \sum \nu_i a_i = \sum \mu_i a_i$ the total degree of the binomial under the natural grading induced by the semigroup H. Since the defining ideal is prime, it can be easily seen that each such binomial can be assumed to be a difference of non-overlapping monomials, that is, $\nu_i \cdot \mu_i = 0$, for each $i = 1, \ldots, 4$. In this section, we study all the different types of binomials in the presentation ideal.

Let J be the ideal generated by $S_1 \cup S_2 \cup S_3 \cup E$. Since $J \subseteq I$, we plan to show that $I \subseteq J$.

The proof of this statement will go by induction on $d = \sum \nu_i a_i = \sum \mu_i a_i$ in the following way:

to show that $I \subseteq J$, we will show that, for any $d \geq 1$, the binomial $x^{\nu_1} y^{\nu_2} z^{\nu_3} w^{\nu_4} - x^{\mu_1} y^{\mu_2} z^{\mu_3} w^{\mu_4}$ of degree d either belongs to J or it belongs to the ideal generated by binomials of I of degree strictly less than d.

The reader should note that, if a binomial belongs to the ideal generated by binomials in I of strictly lower degree, we will say that the binomial reduces to a lower degree.

Our analysis will consider all possible types of binomials in I, assumed non-overlapping.

First, we start with a lemma.

Lemma 4.2.1. *$xw^{3n+1} - z^{3n+2}$, $xyw^{6n-1} - z^{6n+1}$ belong to J.*

Proof. We know that S_1, respectively E, is in J and, so, $z^{3n-1} - yw^{3n-2}$ and, respectively $xw^2 - z^3$, is in J. Modulo J we have $z^{3n+2} = z^3 \cdot z^{3n-1} = z^3 \cdot yw^{3n-2} = xw^2 w^{3n-2} = xw^{3n+1}$. $\square$

Now, we move to analyzing each binomial based upon type.

4.2.1 Type: $x^{\nu_1} y^{\nu_2} - z^{\mu_3} w^{\mu_4}$

Consider $x^{\nu_1} y^{\nu_2} - z^{\mu_3} w^{\mu_4} \in I$. This binomial satisfies

$$\nu_1[r(3n+2)+3] + \nu_2[r(3n+2)+6] = \mu_3[r(3n+2)+3n+4] + \mu_4[r(3n+2)+3n+5] \quad (4.2.1.1)$$

Claim 4.2.2. *$\nu_1 + \nu_2 \geq r+2$ and $\mu_1 + \mu_2 \geq r+1$.*

Proof. In equation 4.2.1.1, denote $\nu_1 + \nu_2 = k$, $3\nu_2 = l$, $\mu_3 + \mu_4 = k'$ and $\mu_4 = l'$ and note that $0 \leq l \leq 3k$ and $0 \leq l' \leq k'$. So we get $k[r(3n+2)+3] + l = k'[r(3n+2)+3n+4] + l'$.

Case 1: If $k \leq k'$, we have $k' = k + k''$ with $k'' \geq 0$.
This gives $l = k(3n+1) + k''[r(3n+2)+3n+4] + l' > 3k$, contradiction!

Case 2: If $k > k'$, we have $k = k' + k''$, with $k'' > 0$. We get

$$k''[r(3n+2)+3] + l = (3n+1)k' + l' \leq (3n+2)k'$$

and so

$$k' \geq \frac{r(3n+2)+3}{3n+2}$$

which further gives $k' \geq r+1 \implies k \geq r+2$, proving our claim. □

Rearranging equation 4.2.1.1, we have

$$3\nu_1 + 6\nu_2 - 2\mu_3 - 3\mu_4 = p(3n+2) \tag{4.2.2.1}$$

where

$$(\mu_3 + \mu_4)(r+1) - (\nu_1 + \nu_2)r = p \tag{4.2.2.2}$$

It is easy to notice that $p > 0$. Now, if $\mu_3 + \mu_4 = r + m$ with $m \geq 1$, then $\nu_1 + \nu_2 \geq r + m + 1$. Let $\nu_1 + \nu_2 = r + m + m'$ with $m' \geq 1$. Going back to equation 4.2.2.2 we have

$$(r+m)(r+1) - (r+m+m')r = p > 0 \implies r + m - m'r = p > 0 \implies m + r(1-m') = p > 0.$$

Therefore, we have two cases to consider.

Case 1: $m' = 1 \implies m = p$. Let $\nu_1 + \nu_2 = r + p + 1 + k' - k' \implies \nu_1 = r + p + 1 - k'$ and $\nu_2 = k'$. Solving for μ_3, μ_4 we get $\mu_3 = 3(pn - k') + 2p - 3$ and $\mu_4 = r - 3(pn - k') + (3 - p)$. Since k', ν_1, ν_2, μ_3 and μ_4 all depend on p, let us start denoting them by k'_p, ν_{1p}, ν_{2p}, μ_{3p}

and μ_{4p} respectively.

Case 1.1 Now, when $p = 1$, we get

$$\nu_{11} = r + 2 - k_1', \quad \nu_{21} = k_1', \quad \mu_{31} = 3(n - k_1') - 1, \quad \mu_{41} = r - 3(n - k_1') + 2.$$

Notice that $\mu_{31} \geq 0$ implies that $k_1' \in \overline{0, n-1}$.

With the change of variables $k_1' \to k_1' - 1$ we obtain a set of binomials

$$S_2 = \{x^{r-k_1'+3}y^{k_1'-1} - z^{3(n-k_1')+2}w^{r-3(n-k_1')-1} : \qquad k_1' \in \overline{1, n}\}.$$

Case 1.2 Now, when $p = 2$, we get

$$\nu_{12} = r + 3 - k_2', \quad \nu_{22} = k_2', \quad \mu_{32} = 3(2n - k_2') + 1, \quad \mu_{42} = r - 3(2n - k_2') + 1$$

Notice that from $\mu_{32} \geq 0$ we obtain that $k_2' \in \overline{0, 2n}$.

This gives us the set of polynomials

$$\{x^{r-k_2'+3}y^{k_2'} - z^{3(2n-k_2')+1}w^{r-3(2n-k_2')+1} : \qquad k_2' \in \overline{0, 2n}\}$$

Some of the binomials in this set can be further reduced to lower degree: when $0 \leq k_2' = i \leq n - 1$ we see that $\nu_{12}|_{k_2'=i} > \nu_{11}|_{k_1'=i}$ and $\nu_{22}|_{k_2'=i} = \nu_{21}|_{k_1'=i} = i$.

So,

$$x^{\nu_{12}}y^{\nu_{22}} - z^{\mu_{32}}w^{\mu_{42}} = x^{\nu_{12}-\nu_{11}}y^{\nu_{22}-\nu_{21}}(x^{\nu_{11}}y^{\nu_{21}} - z^{\mu_{31}}w^{\mu_{41}}) + z^{\mu_{31}}w^{\mu_{42}}(x^{\nu_{12}-\nu_{11}}y^{\nu_{22}-\nu_{21}}w^{\mu_{41}-\mu_{42}} - z^{\mu_{32}-\mu_{31}}).$$

Note that $x^{\nu_{12}-\nu_{11}}y^{\nu_{22}-\nu_{21}}w^{\mu_{41}-\mu_{42}} - z^{\mu_{32}-\mu_{31}}$ equals $xw^{3n+1} - z^{3n+2}$ which is in J by

Lemma 4.2.1. Also $x^{\nu_{11}}y^{\nu_{21}} - z^{\mu_{31}}w^{\mu_{41}}$ is in J by the preceding case and so $x^{\nu_{12}}y^{\nu_{22}} - z^{\mu_{32}}w^{\mu_{42}}$ is in J for these special values of k_2'.

Also, when $k_2' = n$, we have $\nu_{12}|_{k_2'=n} = \nu_{11}|_{k_1'=n-1}$ and $\nu_{22}|_{k_2'=n} = n > \nu_{21}|_{k_1'=n-1} = n-1$. So this binomial can be reduced to lower degree in similar fashion as above using that $z^{3n-1} - yw^{3n-2}$ is in J. Therefore, we can eliminate all cases where $k_2' \in \overline{0,n}$ and under a renaming of variables $k_2' \to n + k_2'$ we have

$$S_3 = \{x^{r-(n+k_2')+3}y^{n+k_2'} - z^{3(n-k_2')+1}w^{r-3(n-k_2')+1} : \qquad k_2' \in \overline{1,n}\}$$

Case 1.3 We now aim to show that for $p \geq 3$, the set of binomials obtained belong to the ideal J or can be reduced to lower degree. We will employ the same method as before using binomials in J. In order to do this, we set up an induction on p. We plan show that all binomials obtained when $p \geq 3$, belong to J or can be reduced to lower degree.

Base Case: $p = 3$.

$$\nu_{13} = r - k_3' + 4, \quad \nu_{23} = k_3', \quad \nu_{33} = 3(3n - k_3') + 3, \quad \nu_4 = r - 3(3n - k_3').$$

Notice that $0 \leq k_3' \leq 3n + 1$.

This gives us the set of binomials

$$\{x^{r-k_3'+4}y^{k_3'} - z^{3(3n-k_3')+3}w^{r-3(3n-k_3')} : \qquad k_3' \in \overline{0, 3n+1}\}.$$

However, notice that

for $k_3' \in \overline{0, n-1}$, $\nu_{13}|_{k_3'=i} > \nu_{11}|_{k_1'=i+1}$ and $\nu_{23}|_{k_3'=i} = \nu_{21}|_{k_1'=i+1}$;

for $k_3' = n$, $\nu_{13}|_{k_3'=n} > \nu_{11}|_{k_1'=n}$ and $\nu_{23}|_{k_3'=n} > \nu_{21}|_{k_1'=n}$;

for $k_3' = n+1$, $\nu_{13}|_{k_3'=n+1} = \nu_{11}|_{k_1'=n}$ and $\nu_{23}|_{k_3'=n+1} > \nu_{21}|_{k_1'=n}$.

for $k'_3 \in \overline{n+2, 2n+1}$, $\nu_{13}|_{k'_3=i} = \nu_{12}|_{k'_2=i-n-1}$ and $\nu_{23}|_{k'_3=i} > \nu_{22}|_{k'_2=i-n-1}$.

for $k'_3 \in \overline{2n+2, 3n}$, $\mu_{33}|_{k'_3=i} > \mu_{31}|_{k'_2=i-2n}$ and $\mu_{43}|_{k'_3=i} > \mu_{41}|_{k'_2=i-2n}$.

for $k'_3 = 3n+1, \mu_{33}|_{k'_3=3n+1} = 0$ and $\mu_{43}|_{k'_3=3n+1} > r+2$.[1]

So all polynomials obtained in case $p = 3$ are reducible to lower degree with S_2, S_3 and E.

This concludes the base case.

Induction Step:

Fix $p \geq 4$. Assume that all polynomials obtained for $1, \ldots, p-1$ belong to J. We will show that this holds too for the binomials obtained for p.

Case p**:**

Notice that $\mu_{3p} = 3(pn - k'_p) + 2p - 3$

$\implies 0 \leq k'_p \leq pn + \left[\frac{2p-3}{3}\right]$, where $[x]$ denotes the greatest integer less than or equal to x.

For comparison, notice that in case $p-1$, $0 \leq k'_{p-1} \leq (p-1)n + \left[\frac{2(p-1)-3}{3}\right]$

$(p-1)n + \left[\frac{2(p-1)-3}{3}\right] = pn + \left[\frac{2p-3}{3} - \frac{2}{3}\right] - n = pn + \left[\frac{2p-3}{3}\right] - n$ or $pn + \left[\frac{2p-3}{3}\right] - n - 1$.

That is to say, $\max(k'_p) = n + \max(k'_{p-1})$ or $\max(k'_p) = n + 1 + \max(k'_{p-1})$

Notice that when $k'_p \in \overline{0, \max(k'_{p-1})}$, $\nu_{1p}|_{k'_p=i} > \nu_{1(p-1)}|_{k'_{p-1}=i}$ and $\nu_{2p}|_{k'_p=i} \geq \nu_{2(p-1)}|_{k'_{p-1}=i}$. So these polynomials can be reduced using the polynomials from case $p-1$. By the induction hypothesis, they can be reduced to lower degree using S_2, S_3 and E.

[1]This binomial is reducible to a lower degree with $x^{r-n+2}y^n z - w^{r+2} \in E$

Let us now look at $\mu_{3p}|_{k'_p=\max(k'_{p-1})+1}$.

When, $\max(k'_{p-1}) = (p-1)n + \left[\frac{2p-3}{3}\right] - 1$,

$$\mu_{3p} = 3(pn - (p-1)n - \left[\tfrac{2p-3}{3}\right]) + 2p - 3 = 3n - 3\left[\tfrac{2p-3}{3}\right] + 2p - 3$$

$$\mu_{3p} = 3n + 3\left(\tfrac{2p-3}{3}\right) - 3\left[\tfrac{2p-3}{3}\right] = 3n + 3\left\{\tfrac{2p-3}{3}\right\}$$

$$3n \leq \mu_{3p} < 3n + 3$$

Similarly when $\max(k'_{p-1}) = (p-1)n + \left[\frac{2p-3}{3}\right]$, we will have $3n - 3 \leq \mu_{3p} < 3n$. So, we have $3n-3 \leq \mu_{3p}|_{k'_p=\max(k'_{p-1})+1} \leq 3n+2$. This can now be studied in a case by case basis.

Let $\mu_{3p}|_{k'_p=\max(k'_{p-1})+1} = 3n - 3 > \mu_{31}|_{k'=1}$.
$\mu_{4p} = r + p - 3n + 3 \geq r + 4 - 3n + 3 = r - 3n + 7 > r - 3n + 5 = \mu_{41}|_{k'=1}$

Let $\mu_{3p}|_{k'_p=\max(k'_{p-1})+1} = 3n - 2 > \mu_{31}|_{k'=1}$.
$\mu_{4p} = r + p - 3n + 2 \geq r + 4 - 3n + 2 = r - 3n + 6 > r - 3n + 5 = \mu_{41}|_{k'=1}$

The other cases can be studied similarly. Therefore, when $k'_p = \max(k'_{p-1}) + 1$, the binomial we obtain in case p is reducible to lower degree using the polynomials in S_2. Since both μ_{3p} and μ_{31} decrease by 3, and at the same time both μ_{4p} and μ_{41} increase by 3, with each increment of k'; the inequalities are preserved. Therefore, when $k'_p \in \overline{\max(k'_{p-1}) + 1, \max(k'_p)}$ all the resulting polynomials are reducible to lower degree using S_2. Therefore, all polynomials in case p are reducible to lower degree using S_2, S_3 and E. This completes the induction ■

Case 2: $m' > 1$ and $m > r(m'-1)$

Let $\nu_1 = r+m+m'-k'$ and $\nu_2 = k'$. We have $\mu_3+\mu_4 = r+m$ and $\nu_1+\nu_2 = r+m+m'+k'-k'$.

Similar to previous case, solving for μ_3 and μ_4 using equation 3, we arrive at:

$\mu_3 = p(3n+2)-3m'-3k'$ and

$\mu_4 = r+m-p(3n+2)+3m'+3k'$.

Notice that the minimum value attained by ν_1 in $S_2 \cup S_3$ is $r-2n+3$ (in S_2 with $k'_1 = n$). This means, for $k' \leq m+m'+2n-3$, we would always have $\nu_1 > \nu_{11}$ and $\nu_2 = \nu_{21}$ or $\nu_1 > \nu_{21}$ and $\nu_2 \geq \nu_{22}$, meaning that we can use binomials from S_2 or S_3 to show that our binomial is in J. When $k' \geq m+m'+2n-2$ it is straightforward to show that all binomials in this case can be reduced using $z^{3n-1}-yw^{3n-2}$, $x^{r-2n+3}y^{2n}-zw^{r+1}$ and $w^{r+2}-x^{r-n+2}y^n z \in J$.

4.2.2 Type: $x^{\nu_1}z^{\nu_3}-y^{\mu_2}w^{\mu_4}$

Consider $x^{\nu_1}z^{\nu_3}-y^{\mu_2}w^{\mu_4} \in I$.

Just like our previous section, we can write out our relation and simplify to obtain

$$(\nu_1+\nu_3)[r(3n+2)+3]+(3n+1)\nu_3 = (\mu_4+\mu_2)[r(3n+2)+6]+(3n-1)\mu_4$$

For ease of writing, let $\nu_1+\nu_3 = k$, $(3n+1)\nu_3 = l$, $(\mu_2+\mu_4) = k'$, $(3n-1)\mu_4 = l'$, with $l \leq (3n+1)k$ and $l' \leq (3n-1)k'$.

We can therefore rewrite the above equation as the following

$$k[r(3n+2)+3]+l = k'[r(3n+2)+6]+l' \tag{4.2.2.3}$$

This brings us to three cases.

Case 1: $k = k'$. Write $\nu_3 - \mu_4 = p = \mu_2 - \nu_1$ for some p. Choose $\eta \in \mathbb{Z}$ such that $\nu_1 = p(n-1) - \eta$. We obtain

$$\mu_2 = pn - \eta,\ \nu_3 = 3\eta + 2p \text{ and } \mu_4 = 3\eta + p.$$

Since $\nu_1, \nu_3, \mu_2, \mu_4$ depend on p, let us now call them $\nu_{1p}, \nu_{3p}, \mu_{2p}$ and μ_{4p} respectively. Notice that from the constraints $\nu_{1p} \geq 0$ and $\mu_{4p} \geq 0$, we have $\frac{-p}{3} \leq \eta \leq p(n-1)$.

Let us first consider $0 \leq \eta \leq p(n-1)$.

Notice that when $p = 1$, $0 \leq \eta \leq n-1$ which yields $0 \leq \nu_{11} \leq n-1$ and $2 \leq \nu_{31} \leq 3n-1$.

Now, for $p > 1$, when $0 \leq \eta < (p-1)(n-1)$, it is possible to verify that $\nu_{1p} > n-1$ and $\nu_{3p} \geq 2$ for all $0 \leq \eta \leq (p-1)(n-1)$. Therefore, all these binomials are reducible to lower degree using $x^{n-1}z^2 - y^n w$ that we obtain when $p = 1$ and $\eta = 0$. When $(p-1)(n-1) \leq \eta \leq p(n-1)$, we observe that $0 \leq \nu_{1p} \leq n-1$ and $3(n-1)(p-1) + p \leq \nu_{3p} \leq 3(p+1)(n-1)$. Thus these binomials can be reduced by the ones obtained in the case $p = 1$.

This leaves the question of $\frac{-p}{3} \leq \eta < 0$. Let $\eta = -\eta'$ with $\eta' > 0$. In fact, $0 < \eta' \leq \frac{p}{3}$. (Notice that this implicitly means $p > 3$). Now let us consider the $\nu's$ and $\mu's$ with respect to η' and for purposes of comparison to other binomials, we rename them as $\tilde{\nu_1}, \tilde{\mu_2}, \tilde{\nu_3}, \tilde{\mu_4}$. One can check easily that $p < \tilde{\nu_3} \leq 2p$ and $p(n-1) < \tilde{\nu_1} \leq p(n-1) + \left\lfloor \frac{p}{3} \right\rfloor$. But this means, $\tilde{\nu_3} > 3$ and $\tilde{\nu_1} > 3(n-1)$. Therefore, all these binomials are reducible to lower degree using $x^{n-1}z^2 - y^n w$ that we obtain when $p = 1$ and $\eta = 0$. Therefore all binomials obtained with permitted negative η can be reduced, and are thus belonging to the ideal J. It is possible to reduce all polynomials obtained in the case $p > 1$ with the set of polynomials obtained in the case $p = 1$.

Case 2: $k < k'$. As in Claim 2.2, one can show that $\nu_1 + \nu_3 \geq r + 4$ and $\mu_2 + \mu_4 \geq r + 5$.

Writing the relations out and simplifying gives the following equation

$$(\nu_1 + \nu_3)r - (\mu_2 + \mu_4)r + (\nu_3 - \mu_4) = p'.$$

for some integer p'. Furthermore, since $k < k'$, we have

$$\nu_1 + \nu_3 < \mu_2 + \mu_4 \qquad \Longrightarrow 3\nu_1 + 3\nu_3 < 3\mu_2 + 3\mu_4$$

leading to

$$3\nu_1+2\nu_3 < 3\mu_3+3\mu_4 \qquad \Longrightarrow 3\nu_1+2\nu_3 < 6\mu_2+3\mu_4 \qquad \Longrightarrow 6\mu_2+3\mu_4-3\nu_1-2\nu_3 > 0 \qquad \Longrightarrow p' > 0.$$

Let $\nu_1 + \nu_3 = r + m$ and $\mu_2 + \mu_4 = r + m + m'$ where $m \geq 4, m' \geq 1$. So, we have

$$(r + m)r - (r + m + m')r + (\nu_3 - \mu_4) = p' > 0$$

leading to $\nu_3 - \mu_4 - rm' = p' > 0$ and so $\nu_3 > r \geq 3n + 2$.

It can be easily seen now that the binomials in this case be reduced to lower degree with $z^{3n-1} - yw^{3n-2}$.

Case 3: $k > k'$. As in Claim 2.2, one can show that $\nu_1 + \nu_3 \geq r + 2$ and $\mu_2 + \mu_4 \geq r + 3$. By using a very similar argument to that used in case 2, we notice that binomials attained through this case are also reducible using the binomials in S_1.

4.2.3 Type: $x^{\nu_1}w^{\nu_4} - y^{\mu_2}z^{\mu_3}$

We have two such binomials $xw^3 - yz^3$ and $x^n w^2 - zy^{n+1}$ in the list of generators of J. Now, we aim to show that these two are the only generators for this type needed to generate all other binomials of this type from I. Eliminating the trivial cases, we are led to binomials with the following restrictions that require a detailed analysis:

Restrictions on μ_2, μ_3 :

1. $\mu_2 = 0, \mu_3 = k \quad k \in \mathbb{N}$
2. $\mu_2 = j, \mu_3 = 0 \quad j \in \mathbb{N}$
3. $\mu_2 = 1, \mu_3 = 1$
4. $\mu_2 = 1, \mu_3 = 2$
5. $\mu_2 = i, \mu_3 = 1 \quad i < n + 1$

To these restrictions, we need to add the ones on ν_1 and ν_4, for which we get the following cases

1. $x^{\nu_1}w - y^{\mu_2}z^{\mu_3}$ where $\nu_1 < r + 2$.
2. $x^{\nu_1} - y^{\mu_2}z^{\mu_3}$ where $\nu_1 < r + 2$.
3. $w^{\nu_4} - y^{\mu_2}z^{\mu_3}$ where $\nu_4 < r + 2$.

Let us now tackle each case.

Case 1: $\nu_4 = 1$

$$x^{\nu_1}w - y^{\mu_2}z^{\mu_3} \in I \implies (\nu_1+1)r(3n+2)+3(\nu_1+1)+3n+2 = (\mu_2+\mu_3)r(3n+2)+6\mu_2+(3n+4)\mu_3$$

This further implies that

$$(\nu_1+1)r(3n+2)+(\mu_2+\mu_3)r(3n+2) = 6\mu_2+(3n+4)\mu_3-3(\nu_1+1)-(3n+2) = pr(3n+2)$$

This leads us to three subcases.

1. $p > 0$ We have

$$6\mu_2 + (3n+4)\mu_3 = (pr+1)(3n+2) + 3(\nu_1+1) > r(3n+2) \geq (3n+2)^2.$$

Looking at the conditions on μ_2, μ_4, most can be easily ruled out by straightforward substitution. We need to explain only the following two situations.

(a) $\mu_2 = 0, \mu_3 = k$: We need k such that $k(3n+4) > (3n+2)^2$. Note that $(3n-1)(3n+4) = 9n^2+9n-4 < (3n+2)^2$ implies $k > 3n-1$. But this means, this polynomial can be reduced using $z^{3n-1} - yw^{3n-2} \in J$.

(b) $\mu_2 = j, \mu_3 = 0$: We need j such that $6j > (3n+2)^2$ implies $j > \frac{(3n+2)^2}{6} > 3n+2$. But this means, this binomial can be reduced to lower degree using $y^{2n+1} - x^{2n-1}zw \in J$.

2. $p = 0$. Under this condition, the initial equation breaks down to $3\mu_2+(3n+1)\mu_3 = 3n+2$, which is impossible.

3. $p < 0$. Under this condition, substitute $p \to -p'$ where now $p' > 0$. Which then forces

$$\nu_1 + 1 \geq \frac{(p'r-1)(3n+2)}{3} = (n+\frac{2}{3})(p'r-1) \geq 2(r-1) > r+2$$

(which is true for all $r \geq 3n+2$). So, we get $\nu_1 \geq r+2$.

This means, this binomial will be reducible to lower degree using $x^{r+2} - z^{3n-1}w^{r-3n+2} \in J$.

So we have seen that in all three subcases, the binomials we encounter are either reducible using other binomials in our generating set, or our restrictions provide a contradiction. One can check that the same arguments work for the other two main cases on ν_1, ν_4 ($\nu_4 = 0$ and $\nu_1 = 0$), thereby completing the investigation of all binomials of this type.

4.2.4 Type $w^{\mu_4} - x^{\nu_1}y^{\nu_2}z^{\nu_3}$*:*

We will assume ν_1, ν_2, ν_3 are all positive otherwise this binomial has been treated in an earlier case.

As in Claim 2.2, one can show $\mu_4 \geq r + 2$. Since for any $Q = w^{\mu_4} - x^{\nu_1}y^{\nu_2}z^{\nu_3} \in I$ we have $\mu_4 \geq r + 2$, Q can be reduced using $w^{r+2} - x^{r-n+2}y^n z \in I$ and therefore $w^{r+2} - x^{r-n+2}y^n z$ is the only binomial we need of this type in J.

Using similar arguments, one may also deduce that $y^{2n+1} - x^{2n-1}zw$ is the only binomial we would need of the type $y^{\nu_2} - x^{\mu_1}z^{\mu_3}w^{\mu_4}$. It is also easily verified in a similar fashion that any binomial of the types $x^{\nu_1} - y^{\mu_2}z^{\mu_3}w^{\mu_4}$ and $z^{\nu_3} - x^{\mu_1}y^{\mu_2}w^{\mu_4}$ are reducible to lower degree using $x^{r+2} - z^{3n-1}w^{r-3n+2} \in S_2$ and $z^{3n-1} - yw^{3n-2} \in S_1$ respectively.

With this we have exhausted all types of binomials we could have in four variables and showed that J equals I, hence establishing a minimal generating set for I.

4.3 Different approach using indispensable binomials

We would like to discuss our results and the consequences presented in the proof in this section in relation to the classification of monomial curves embedded in a four dimensional affine space in [11].

We recall the following definition from [11].

Definition 4.3.1. *A bionomial $x_i^{c_i} - \prod_{j\neq i} x_j^{a_{ij}} \in I_H$ is called critical with respect to x_i if c_i is the least positive integer such that $c_i a_i \in \sum_{j\neq i} \mathbb{N}a_j$.*

The ideal generated by all critical binomials is called the critical ideal of H and is denoted by C_H.

In general, computing the critical binomials can be difficult. A consequence of our proof is that we can establish what the critical binomials are for the Backelin semigroup.

Proposition 4.3.2.

$$C_H = \langle \mathcal{C} \rangle = \langle z^{3n-1} - yw^{3n-2},\ y^{2n+1} - x^{2n-1}zw,\quad w^{r+2} - x^{r-n+2}y^n z,\ x^{r+2} - z^{3n-1}w^{r-3n+2} \rangle$$

Proof. It is clear that $\mathcal{C} \subseteq S_1 \cup S_2 \cup S_3 \cup E$. Proposition 3.2 from [11] ensures us that we just have to show for every $x_i^{c_i} - \prod_{j \neq i} x_j^{a_{ij}} \in \mathcal{C}$, c_i is the least positive integer such that $c_i a_i \in \sum_{j \neq i} \mathbb{N} a_j$. However, this is precisely what we do in subsection 2.4 with the help of Claim 2.2 for the case of $w^{r+2} - x^{r-n+2}y^n z$. All other critical binomials can be treated similarly. For purposes of completion, a similar proof as in claim 2.2 is provided below for the treatment of $x^{r+2} - z^{3n-1}w^{r-3n+2}$

Writing out the relations, we have

$$\nu_1[r(3n+2)+3] = \mu_2[r(3n+2)+6] + \mu_3[r(3n+2)+3n+4] + \mu_4[r(3n+2)+3n+5]$$

Let $\nu_1 = k$, $\mu_2 + \mu_3 + \mu_4 = k'$ and $(3n-2)\mu_3 + (3n-1)\mu_4 = l'$. Notice that $l' < (3n-1)k'$

So, we have $k[r(3n+2)+3] = k'[r(3n+2)+6] + l'$

If $k \leq k'$, let $k' = k + k''$ which gives

$$k[r(3n+2)+3] = k[r(3n+2)+6] + k''[r(3n+2)+6] + l' \geq k[r(3n+2)+6]$$

which is clearly a contradiction! as $k > 0$.

If $k > k'$, let $k = k' + k''$ this gives $k''[r(3n+2)+3] = 3k' + l' < (3n+2)k'^2$. So,

$$k' > \frac{k''[r(3n+2)+3]}{3n+2} \implies k' > r$$

So $k' \geq r+1$, which means $k \geq r+2$. So if we can show that there exists some $f \in I_H$ of this type with $\nu_1 = r+2$ then we will be done. Notice we have, $x^{r+2} - z^{3n-1}w^{r-3n+2} \in I_H$ of this type. So, $c_1 = r+2$. $\square$

Remark 4.3.3. $\mathcal{C}$ *is not uniquely determined. In fact, we have a* $\mathcal{C}'$ *such that*

$$C_H = \langle \mathcal{C}' \rangle = \langle z^{3n-1} - yw^{3n-2},\ y^{3n+2} - x^{3n-1}w^3,\ w^{r+2} - x^{r-n+2}y^n z,\ x^{r+2} - yw^r \rangle$$

Given an ideal $J \subseteq K[x,y,z,w]$, let M_J denote monomial ideal generated by all $\mathbf{x^u}$ for which there exists a non-zero $\mathbf{x^u} - \mathbf{x^v} \in J$. Now consider the following Theorems from [11] (re-written for our case) that will prove helpful in understanding our results from the context of critical and indispensable binomials.

Theorem 4.3.4 (Prop 1.5 in [11]). *The indispensable monomials of* J *are precisely the minimal generating set of* M_J

Theorem 4.3.5 (Prop 3.6 in [11]). *After permuting the variables (if necessary), there exists a minimal system of binomial generators* $\mathcal{C}$ *of* C_H *of the following form*

1. *CASE 1: If* $c_i a_i \neq c_j a_j$ *for any* $i \neq j$*, then* $\mathcal{C} = \{x_i^{c_i} - \boldsymbol{x}^{\boldsymbol{u}_i}, i = 1, \ldots, 4\}$

Theorem 4.3.6 (Theorem 3.10 in [11]). *The union of* $\mathcal{C}$*, the set* Γ *of all binomials in* I_H *of the form* $x_{i_1}^{u_{i_1}} x_{i_2}^{u_{i_2}} - x_{i_3}^{u_{i_3}} x_{i_4}^{u_{i_4}}$ *that satisfy* $0 < u_{i_j} < c_j$ *for* $j = 1, 2$ *and* $u_{i_3}, u_{i_4} > 0$ *with* $x_{i_3}^{u_{i_3}} x_{i_4}^{u_{i_4}}$ *indispensable is a minimal system of generators of* ${I_H}$[3]*.*

[2]notice that this automatically implies that $k'' \neq 0$ for non-trivial homogeneous degree.

[3]Borrowing notation from [11], since we are in CASE 1, by Theorem 4.3.5 we automatically have $\mathcal{R} = \varnothing$

Looking at Theorem 4.3.5, we see that the Backelin semigroup does indeed fall into CASE 1. This allows us to consult Theorem 4.3.6 to obtain a minimal generating set for I_H. Our work in proving Theorem 4.1.2 and Corollary 4.1.3 shows that in our case $\Gamma = (S_1 \cup S_2 \cup S_3 \cup E) \setminus \mathcal{C}$. In particular, Theorem 4.1.2 bounds u_{i_j} in Γ as required while Corollary 4.1.3 guarantees that $x_{i_3}^{u_{i_3}} x_{i_4}^{u_{i_4}}$ are indispensable. This proves once again that $S_1 \cup S_2 \cup S_3 \cup E$ is a minimal generating set of I_H. In addition to that, we obtain a lot more information about our generating set and consequently I_H. For instance, Proposition 3.9 from [11] informs us that I_H is generic. We also obtain from Theorem 3.11 in [11] that I_H does not have a unique minimal system of generators since C_H does not (see remark 4.3.3). Viewing the minimal generators from the context of indispensable binomials provides a new perspective and opens up various other avenues of exploration in close relation to classical semigroup theory.

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