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Smoothed Empirical Likelihood for the Difference of Two Quantiles with Paired Sample

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SMOOTHED EMPIRICAL LIKELIHOOD FOR THE DIFFERENCE OF TWO QUANTILES WITH PAIRED SAMPLE

by

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Under the Direction of Yichuan Zhao, PhD

ABSTRACT

In this thesis, we propose a smoothed empirical likelihood method for the difference of quantiles with paired samples. The empirical likelihood for the difference of two quantiles with independent samples has been studied by some researchers. However, for many variables, we cannot ignore the correlation between the data. In this study, we construct two estimating equations for the difference of two quantiles and introduce a nuisance parameter in our proposed smoothed empirical likelihood. The limiting distribution of the smoothed empirical likelihood is the χ^2 distribution. Simulation studies demonstrate that our method is valid for the difference of quantiles. We also apply the proposed method to a real data set to illustrate the interval estimate of the quantile difference of GDP between different years.

INDEX WORDS: Quantile difference, Paired sample, Smoothed empirical likelihood

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DEDICATION

To all dedicated Educators.

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I would like to thank my advisor Professor Yichuan Zhao for his guidance in my thesis. In December 2019, I told Dr. Zhao that I wanted to do research. After knowing my background, he decided that I was suitable for research. During the winter vacation, Dr. Zhao sent me a paper to read. I read this paper carefully during the winter vacation, but it was difficult for me. I was going to give up at that time. About February, Dr. Zhao asked me how my research was going on. I told him my difficulty in understanding the paper. Then he sent me some other papers. After reading these papers, I gradually understood the empirical likelihood method. From then on, I would go to Dr. Zhao's office almost every week to ask him questions, and he patiently explained them to me. During this period, I successfully reproduced the code of the paper. After the middle of March, the school started online classes because of COVID-19, and I attended Dr. Zhao's weekly online seminar. At the end of the spring semester of 2020, I completed the code for my thesis, and I did a presentation in Dr. Zhao's seminar. Because the proof of the method I used was challenging, Dr. Zhao shared another way to estimate the difference of quantiles of paired data. I have been working on this method all the summer and had weekly meetings with Dr. Zhao. I appreciate he spared time to discuss my research during the summer. By August, I completed my thesis. Dr. Zhao has paid a lot of effort in my research. It was Dr. Zhao who led me to do research in the area of statistics for the first time. After I finished my thesis, Dr. Zhao instructed me to do some other research, and I learned a lot from Dr. Zhao. I also would like to thank Lauren Drinkard, Brain Daniel Pidgeon, Iyanuoluwa Ayodele and Ali Jinnah for their proofreading.

Although I will go to another university to study for a Doctor degree in the future, I hope I can still have the opportunity to get the guidance of Dr. Zhao in my research.

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LIST OF ABBREVIATIONS

- CI - Confidence Interval
- EL - Empirical Likelihood
- AEL - Adjusted Empirical Likelihood

CHAPTER 1

INTRODUCTION

The quantile is an interesting topic in statistics and probability. The p-th quantile is defined as $F^{-1}(p)$ =inf$\{x \in R : F(x) \geq p\}$, where $0 < p < 1$. The quantile is broadly applied in risk measurement, econometrics and many other subjects. To estimate the quantiles, Harrell and Davis (1982) proposed a distribution-free quantile estimator. Hutson (2002) developed the estimator of the tail extrapolation quantile function. Parrish (1990) compared ten nonparametric quantile estimators in terms of the bias and mean squared error. In the aspect of confidence interval of quantiles, Chen and Hall (1993) developed the confidence interval of quantiles by smoothed empirical likelihood. Zhou and Jing (2003a) introduced an adjusted empirical likelihood method to construct the confidence interval of quantiles. For the difference of quantiles, Zhou and Jing (2003b) proposed a smoothed empirical likelihood method for the difference of one-sample quantiles. In order to improve the performance and the efficiency of empirical likelihood on one-sample difference of quantiles, Yang and Zhao (2018) developed a smoothed jackknife empirical likelihood method. In the two-sample scenario, Qin and Zhao (1997) constructed the confidence interval for the difference of two independent population quantiles. Yu et al. (2014) compared the upper quantile difference of two independent groups by the classical empirical likelihood and 'plug-in' empirical likelihood methods. Yang and Zhao (2016) proposed a smoothed jackknife empirical likelihood method for the quantiles difference with two independent samples. For the difference of two dependent samples quantiles, Wilcox (1992, 2006) compared the quantiles of dependent groups based on Harrell–Davis estimator (Harrell and Davis (1982)). Lombard (2005) compared the marginals of a bivariate distribution by the quantile comparison function. Wilcox and Erceg-Hurn (2012) obtained the quantiles for two marginal distributions by the Harrell–Davis estimator, and then compared the quantiles of two dependent groups using the bootstrap technique. To the best of our knowledge, empirical likelihood for the quantile difference of two correlated samples has not been studied. In this thesis, we proposed a smoothed empirical likelihood method for the difference of quantiles with paired samples and obtain the interval estimate.

Since Owen (1988, 1990, 1991, 2001) introduced the empirical likelihood (EL) method and obtained the interval estimate, it has been applied to many statistical problems. For instance, Qin and Lawless (1994) considered the empirical likelihood in general estimating equations. Qin (1994) constructed the interval estimate for the difference of two-sample means by semi-empirical likelihood. Chen, Peng, and Zhao (2009) constructed confidence intervals for copulas based on smoothed empirical likelihood. Many versions of empirical likelihood were developed by researchers, for example, adjusted empirical likelihood (Chen, Variyath and Abraham (2008)), jackknife empirical likelihood (Jing, Yuan and Zhou (2009)), mean empirical likelihood (Liang, Dai and He (2019)), Bayesian empirical likelihood (Lazar (2003)), Bayesian jackknife empirical likelihood (Cheng and Zhao (2019)) and different combinations of these above mentioned methods.

Unlike one sample and two independent samples, it is challenging to develop normal approximation confidence interval for the paired-sample difference of quantiles and to estimate the difference of quantiles for the paired data. Lopez et al. (2009) developed empirical likelihood for non-smooth criterion functions. Rather than using the discrete functions, we propose a smoothed nonparametric estimating functions. In our simulation study, we observed that the non-smooth empirical likelihood has a severe over-coverage problem. After applying a smoothing kernel on the estimating equations, we were able to solve the over-coverage problem significantly. We will compare our method with the method (Method M) proposed by Wilcox and Erceg-Hurn (2012) in terms of coverage probability and average length of 95% confidence interval in the simulation study.

The rest of the thesis is formed as follows. In Chapter 2, we develop the inference procedure of the smoothed empirical likelihood for the difference of paired-sample quantiles. A simulation study is conducted to compare our method with existing methods in Chapter

3. In Chapter 4, we apply our method to a real data set from the Penn World Tables (PWT) database. A discussion is provided in Chapter 5, and we present the proofs in the Appendix.

CHAPTER 2

INFERENCE PROCEDURE

Consider a two-dimensional random variable $X = (X_1, X_2)$ with the distribution function $F(x_1, x_2)$. Let $(X_{1i}, X_{2i}),\ i = 1, ..., n$ be independent samples from the distribution function $F(x_1, x_2)$. Denote marginal distribution functions $F(x_1), F(x_2)$ of X_1 and X_2, respectively. Define the difference of quantiles $\xi = F_1^{-1}(p) - F_2^{-1}(p)$, where $F_1^{-1}(p)$ and $F_2^{-1}(p)$ denote the quantile functions of $F_1(p)$ and $F_2(p)$, respectively.

Denote $\hat{F}_j(t) = 1/n \sum_{i=1}^n I(X_{ji} \le t),\ j = 1, 2$. Define $\hat{F}_j^{-1}(p)$ as an empirical estimate of $F_j^{-1}(p),\ j = 1, 2$. $\hat{\xi} = \hat{F}_1^{-1}(p) - \hat{F}_2^{-1}(p)$ is an estimate of ξ. Let $t_1 = F_1^{-1}(p),\ t_2 = F_2^{-1}(p)$. It is clear that $F_1(t_1) - p = 0$ and $F_2(t_2) - p = 0$. As an estimation of these two equations, we obtain $1/n \sum_{i=1}^n (I(X_{1i} \le t_1) - p) = 0$, and $1/n \sum_{i=1}^n (I(X_{2i} \le t_2) - p) = 0$. Then, we propose two smoothed equations as follows,

$$\begin{aligned} W_i(\xi, t_1) &= \begin{pmatrix} W_{1i}(\xi, t_1) \\ W_{2i}(\xi, t_1) \end{pmatrix} \\ &= \begin{pmatrix} K(\frac{t_1 - X_{1i}}{h_1}) - p \\ K(\frac{t_1 - \xi - X_{2i}}{h_2}) - p \end{pmatrix} \\ &= 0,\ i = 1, ..., n, \end{aligned} \tag{2.1}$$

where $K(t) = \int_{-\infty}^t k(u)du$, $k(\cdot)$ is a kernel density function, and h_1 and h_2 are the bandwidths.

Now, the empirical likelihood ratio for (ξ, t_1) based on $W_i(\xi, t_1)$ is defined as

$$r(\xi, t_1) = \sup_{p_1, ..., p_n} \left\{ \prod_{i=1}^n (np_i) : \sum_{i=1}^n p_i = 1, \sum_{i=1}^n W_i p_i = 0, p_i \ge 0, i = 1, ..., n \right\}. \tag{2.2}$$

Let $\lambda^T = (\lambda_1, \lambda_2)$. Using the Lagrange multiplier method, the empirical log-likelihood ratio can be derived as

$$l(\xi, t_1, \lambda^T) = -2\log r(\xi, t_1) \\ = 2\sum_{i=1}^{n} \log(1 + \lambda^T W_i(\xi, t_1)), \tag{2.3}$$

where λ satisfies

$$\frac{1}{n}\sum_{i=1}^{n} \frac{W_i(\xi, t_1)}{1 + \lambda^T W_i(\xi, t_1)} = 0. \tag{2.4}$$

We profile the nuisance parameter t_1 from $\log r(\xi, t_1)$, and obtain

$$-2\log r(\xi) = \min_{t_1}[-2\log r(\xi, t_1)]. \tag{2.5}$$

Denote the solutions of the two equations (2.4) and (2.5) as $t_1 = \tilde{t}_1$ and $\lambda^T = \tilde{\lambda}^T$. It is easy to see that

$$-2\log r(\xi) = 2\sum_{i=1}^{n} \log[1 + \tilde{\lambda}^T W_i(\xi, \tilde{t}_1)]. \tag{2.6}$$

For the detailed computation of the coverage probability and the confidence interval, we propose the corresponding Algorithm 1, Algorithm 2 and Algorithm 3 in the Appendix. In order to establish theoretical results, similar to Zhou and Jing (2013b), we assume the following conditions.

C.1 The $(r-1)$-th derivatives of $F_1(x)$ and $F_2(x)$ exist in the corresponding neighborhood of $t_{1p} = F_1^{-1}(p)$ and $t_{2p} = F_2^{-1}(p)$, and are continuous at t_{1p} and t_{2p}, respectively for some integer $r \geq 2$. Specifically, we denote the first derivatives of $F_1(x)$ and $F_2(x)$ as $f_1(x) = F_1'(x)$ and $f_2(x) = F_2'(x)$, and assume $f_1(t_{1p})f_2(t_{2p}) > 0$.

C.2 The kernel function $k(\cdot)$ and its second derivative $k''(\cdot)$ are bounded. We assume

$$\int u^j k(u)du = \begin{cases} 1, & j = 0, \\ 0, & 1 \leq j \leq r-1, \\ C_0, & j = r, \end{cases}$$

where C_0 is a constant.

C.3 $h_2 = O(h_1), nh_1^{4r} \to 0, nh_2^{4r} \to 0, n^{4s-1}h_1^4 \to \infty, n^{4s-1}h_2^4 \to \infty$, as $n \to \infty$, where

$1/3 < s < 1/2$.

Condition C.1 requires the distribution functions F_1 and F_2 are smooth in the corresponding neighborhood of t_{1p} and t_{2p}. Condition C.2 is a common requirement for kernel methods. Condition C.3 ensures that the convergence of the bandwidths to zero is neither too fast nor too slow. Then, the Wilk's theorem is established as follows.

Theorem 2.1. Under regularity conditions, as $n \to \infty$, we have

$$-2\log r(\xi) \xrightarrow{\mathcal{D}} \chi_1^2. \tag{2.7}$$

Therefore, the EL confidence interval with $(1-\alpha)$ confidence level for ξ is constructed as

$$I_{EL}(\xi) = \big\{\xi : -2\log r(\xi) \le \chi_1^2(\alpha)\big\}, \tag{2.8}$$

where $\chi_1^2(\alpha)$ is the upper α-quantile of χ_1^2.

Chen, Variyath, and Abraham (2008) introduced the adjusted empirical likelihood (AEL) to increase the coverage probability. The AEL method adds one artificial data point such that it guarantees the inclusion of zero in the convex hull. We adapt the AEL in the estimation of quantile difference for the paired samples. Let $a_n = \max(1, \log(n)/2)$. Denote

$$W_{n+1}(\xi, t_1) = -a_n/n \sum_{i=1}^{n} W_i(\xi, t_1). \tag{2.9}$$

The adjusted EL ratio is

$$\begin{aligned} l^{adj}(\xi, t_1, \lambda^T) &= -2\log r^{adj}(\xi, t_1) \\ &= 2\sum_{i=1}^{n+1} \log(1 + \lambda^T W_i(\xi, t_1)), \end{aligned} \tag{2.10}$$

where λ satisfies

$$\frac{1}{n+1}\sum_{i=1}^{n+1} \frac{W_i(\xi, t_1)}{1 + \lambda^T W_i(\xi, t_1)} = 0. \tag{2.11}$$

We profile the nuisance parameter t_1 from $\log r^{adj}(\xi, t_1)$, and obtain

$$-2\log r^{adj}(\xi) = \min_{t_1}[-2\log r^{adj}(\xi, t_1)]. \tag{2.12}$$

Denote $t_1 = \tilde{t}_1$ and $\lambda^T = \tilde{\lambda}^T$ are the solutions of the two equations (2.11) and (2.12). We can obtain that

$$-2\log r^{adj}(\xi) = 2\sum_{i=1}^{n} \log[1 + \tilde{\lambda}^T W_i(\xi, \tilde{t}_1)]. \tag{2.13}$$

Then, we establish the following Wilk's theorem for AEL.

Theorem 2.2. Under regularity conditions, as $n \to \infty$, we have

$$-2\log r^{adj}(\xi) \xrightarrow{\mathcal{D}} \chi_1^2. \tag{2.14}$$

Therefore, the AEL $100(1-\alpha)\%$ confidence interval for ξ is constructed as

$$I_{AEL}(\xi) = \left\{\xi : -2\log r^{adj}(\xi) \leq \chi_1^2(\alpha)\right\}. \tag{2.15}$$

CHAPTER 3

NUMERICAL STUDIES

In this chapter, Monte Carlo simulations are used to examine the performance of our method. The coverage probabilities and the average lengths of confidence intervals for ξ are studied under different distributions of $F(X_1, X_2)$. For k, the standard normal kernel function, which satisfies condition C.2 with $r = 2$, is used and the cross-validation method is applied to select the optimal bandwidth. We consider nine scenarios to generate paired data with a specific correlation (ρ) as follows.

$S_1 : X_1 \sim Normal(1, 1)$ and $X_2 \sim Normal(2, 1)$, $\rho = 0.4$,

$S_2 : X_1 \sim Normal(1, 1)$ and $X_2 \sim Normal(2, 1)$, $\rho = 0.8$,

$S_3 : X_1 \sim Uniform(0, 1)$ and $X_2 \sim Uniform(0, 1)$, $\rho = 0.8$,

$S_4 : X_1 \sim Uniform(0, 1)$ and $X_2 \sim Uniform(0, 1)$, $\rho = 0.1$,

$S_5 : X_1 \sim Weibull(0.5, 2)$ and $X_2 \sim Weibull(0.5, 2)$, $\rho = 0.75$,

$S_6 : X_1 \sim Exponential(1)$ and $X_2 \sim Exponential(1)$, $\rho = 0.8$,

$S_7 : X_1 \sim Gamma(0.6, 1.6)$ and $X_2 \sim Gamma(0.6, 1.6)$, $\rho = 0.7$,

$S_8 : X_1 \sim Normal(0, 1)$ and $X_2 \sim Exponential(1)$, $\rho = 0.75$,

$S_9 : X_1 \sim Normal(0, 1)$ and $X_2 \sim Uniform(0, 1)$, $\rho = 0.8$.

In different scenarios, we set the probability parameter p at 0.25, 0.5, 0.75, and the sample sizes are chosen as 30, 40, 50. We conduct the simulation studies 2000 times at the nominal level 95%. The coverage probabilities under different scenarios are shown in Table 3.1 and the corresponding average lengths of confidence intervals are displayed in Table 3.2.

In all scenarios, we can see that the coverage probabilities of AEL are larger than EL and the corresponding average lengths are longer. As the sample size increases, the average length of 95% confidence interval decreases for EL and AEL. For Method M (Wilcox and Erceg-Hurn (2012)), the coverage probabilities have over-coverage for most scenarios, and the average lengths of 95% confidence interval for Method M are longer than those for EL and AEL in most cases.

Table (3.1) Coverage probability of 95% confidence interval for the difference of quantiles

		S_1			S_2			S_3		
p	n	EL	AEL	M	EL	AEL	M	EL	AEL	M
0.25	30	0.917	0.931	0.951	0.892	0.915	0.961	0.947	0.958	0.960
	40	0.922	0.941	0.961	0.906	0.918	0.971	0.946	0.964	0.959
	50	0.910	0.922	0.954	0.906	0.914	0.960	0.957	0.964	0.970
0.5	30	0.949	0.959	0.950	0.939	0.956	0.966	0.954	0.965	0.964
	40	0.953	0.962	0.955	0.948	0.958	0.963	0.959	0.968	0.958
	50	0.943	0.950	0.954	0.946	0.957	0.961	0.952	0.960	0.970
0.75	30	0.911	0.927	0.945	0.895	0.910	0.957	0.934	0.949	0.964
	40	0.924	0.937	0.954	0.921	0.929	0.964	0.940	0.950	0.961
	50	0.918	0.930	0.949	0.912	0.923	0.970	0.931	0.934	0.963
		S_4			S_5			S_6		
p	n	EL	AEL	M	EL	AEL	M	EL	AEL	M
0.25	30	0.945	0.959	0.943	0.922	0.936	0.967	0.956	0.964	0.964
	40	0.949	0.957	0.946	0.929	0.945	0.953	0.955	0.965	0.959
	50	0.955	0.964	0.962	0.934	0.950	0.954	0.946	0.955	0.957
0.5	30	0.948	0.962	0.954	0.947	0.957	0.961	0.936	0.949	0.961
	40	0.952	0.959	0.951	0.942	0.955	0.965	0.949	0.958	0.965
	50	0.955	0.961	0.957	0.937	0.944	0.966	0.950	0.960	0.966
0.75	30	0.945	0.955	0.951	0.967	0.963	0.955	0.949	0.956	0.952
	40	0.937	0.947	0.957	0.967	0.969	0.964	0.948	0.954	0.963
	50	0.937	0.945	0.947	0.971	0.973	0.964	0.949	0.949	0.965
		S_7			S_8			S_9		
p	n	EL	AEL	M	EL	AEL	M	EL	AEL	M
0.25	30	0.948	0.960	0.963	0.900	0.907	0.947	0.864	0.890	0.898
	40	0.936	0.951	0.954	0.898	0.891	0.948	0.855	0.872	0.893
	50	0.937	0.946	0.953	0.901	0.903	0.946	0.865	0.883	0.884
0.5	30	0.948	0.963	0.960	0.940	0.949	0.944	0.943	0.958	0.839
	40	0.953	0.965	0.967	0.949	0.955	0.950	0.947	0.957	0.819
	50	0.960	0.973	0.961	0.951	0.960	0.944	0.944	0.955	0.785
0.75	30	0.947	0.952	0.952	0.914	0.936	0.941	0.875	0.899	0.928
	40	0.947	0.950	0.958	0.902	0.925	0.948	0.861	0.882	0.934
	50	0.947	0.950	0.962	0.907	0.916	0.958	0.865	0.884	0.940

Table (3.2) Average length of 95% confidence interval for the difference of quantiles

			S_1			S_2			S_3	
p	n	EL	AEL	M	EL	AEL	M	EL	AEL	M
0.25	30	0.987	1.054	1.071	0.626	0.669	0.722	0.192	0.206	0.216
	40	0.857	0.903	0.938	0.547	0.576	0.645	0.170	0.180	0.195
	50	0.767	0.801	0.850	0.493	0.515	0.589	0.154	0.161	0.181
0.5	30	0.897	0.959	0.973	0.562	0.601	0.657	0.226	0.241	0.245
	40	0.777	0.819	0.849	0.489	0.516	0.584	0.202	0.212	0.221
	50	0.696	0.727	0.767	0.441	0.460	0.533	0.185	0.193	0.205
0.75	30	0.988	1.056	1.065	0.626	0.668	0.719	0.193	0.206	0.218
	40	0.865	0.911	0.942	0.551	0.581	0.646	0.171	0.180	0.196
	50	0.768	0.803	0.851	0.494	0.516	0.592	0.155	0.162	0.181
			S_4			S_5			S_6	
p	n	EL	AEL	M	EL	AEL	M	EL	AEL	M
0.25	30	0.337	0.361	0.360	0.117	0.127	0.080	0.282	0.303	0.324
	40	0.295	0.311	0.323	0.094	0.100	0.064	0.241	0.255	0.288
	50	0.267	0.279	0.293	0.079	0.083	0.055	0.213	0.223	0.260
0.5	30	0.397	0.423	0.418	0.240	0.263	0.281	0.488	0.527	0.554
	40	0.352	0.371	0.370	0.191	0.205	0.229	0.423	0.448	0.485
	50	0.319	0.333	0.339	0.163	0.172	0.200	0.378	0.396	0.443
0.75	30	0.336	0.360	0.366	0.980	1.055	0.972	1.034	1.110	0.970
	40	0.294	0.311	0.321	0.832	0.890	0.799	0.900	0.955	0.857
	50	0.266	0.278	0.293	0.729	0.769	0.695	0.801	0.841	0.779
			S_7			S_8			S_9	
p	n	EL	AEL	M	EL	AEL	M	EL	AEL	M
0.25	30	0.120	0.129	0.123	0.646	0.665	0.777	0.644	0.716	0.730
	40	0.100	0.106	0.105	0.594	0.657	0.686	0.578	0.624	0.643
	50	0.086	0.090	0.094	0.548	0.593	0.614	0.524	0.557	0.576
0.5	30	0.240	0.260	0.272	0.605	0.643	0.661	0.581	0.631	0.644
	40	0.203	0.217	0.234	0.530	0.558	0.583	0.513	0.542	0.567
	50	0.180	0.189	0.211	0.477	0.498	0.529	0.464	0.481	0.514
0.75	30	0.622	0.669	0.563	0.670	0.718	0.708	0.665	0.719	0.735
	40	0.536	0.570	0.490	0.582	0.615	0.626	0.591	0.624	0.640
	50	0.473	0.498	0.441	0.520	0.545	0.573	0.536	0.553	0.582

CHAPTER 4

REAL DATA ANALYSIS

In this chapter, one real data set from the Penn World Tables (PWT) database is used in our analysis. The database includes many kinds of data, for example, Real GDP, employment, population levels, etc. We apply the EL method to the expenditure-side real GDP of selected fifty countries in the years 1970 and 1990. The magnitude is measured in 10 billions U.S. dollars. The correlation of the expenditure-side real GDP between the year 1970 and the year 1990 is 0.912, and the sample size is 50. Table 4.1 shows the 95% confidence intervals of the data.

We compare EL, AEL and Method M (Wilcox and Erceg-Hurn (2012)) in this section. The lower bound, the upper bound, and the interval length for the difference of quantiles at 95% confidence level are shown in Table 4.1. We notice that the interval length of the quantile difference of AEL is longer than that of EL, which is consistent with our findings from the numerical studies. The interval length of Method M is longer than EL, and shorter than AEL for most quantiles. This indicates that EL performs better than Method M. As the probability p increases, the magnitudes of the lower bound, the upper bound and the interval length of the quantile difference increase for EL, AEL and Method M. This is due to the increase in magnitude of GDP as p increases. Because the gap is calculated by subtracting the GDP of the year 1990 from the GDP of the year 1970, the negative difference means the GDP of the year 1990 is larger than the GDP of the year 1970. At each p, zero is not included in the confidence interval, implying the quantiles of the expenditure-side real GDP in the year 1970 and the year 1990 are significantly different.

Table (4.1) Estimation of the difference of quantiles at 95% confidence level

	EL			AEL			M		
p	Lower	Upper	Length	Lower	Upper	Length	Lower	Upper	Length
0.1	-3.736	-0.078	3.658	-3.836	-0.012	3.824	-5.658	-1.809	3.849
0.15	-5.312	-1.620	3.692	-5.409	-1.551	3.858	-6.551	-2.796	3.755
0.2	-6.530	-2.842	3.688	-6.631	-2.771	3.860	-7.282	-3.586	3.696
0.25	-7.796	-3.864	3.932	-7.918	-3.792	4.126	-8.190	-4.020	4.170
0.3	-9.128	-4.760	4.368	-9.267	-4.688	4.579	-9.171	-4.413	4.758
0.35	-10.490	-5.600	4.890	-10.648	-5.522	5.126	-10.307	-4.980	5.327
0.4	-11.887	-6.397	5.490	-12.065	-6.310	5.755	-11.521	-5.697	5.824
0.45	-13.353	-7.065	6.288	-13.555	-6.961	6.594	-12.753	-6.246	6.507
0.5	-15.010	-7.797	7.213	-15.245	-7.682	7.563	-14.415	-7.084	7.331
0.55	-17.125	-8.702	8.423	-17.427	-8.573	8.854	-16.409	-7.820	8.589
0.6	-20.104	-9.819	10.285	-20.558	-9.671	10.887	-18.520	-8.326	10.194
0.65	-24.555	-11.170	13.385	-25.258	-11.001	14.257	-21.669	-9.6400	12.029
0.7	-28.326	-12.300	16.026	-29.838	-12.069	17.769	-27.164	-10.527	16.637
0.75	-35.597	-11.122	24.475	-36.083	-10.888	25.195	-33.453	-10.972	22.481
0.8	-38.197	-11.055	27.142	-38.578	-10.946	27.632	-39.751	-12.998	26.753

CHAPTER 5

CONCLUSIONS

Due to the correlation between two samples, the estimation of the difference of quantiles becomes more challenging. The difference of quantiles for paired samples plays an important role in statistics. In this thesis, we propose the smoothed empirical likelihood method for the difference of quantiles of paired samples and establish the Wilk's theorem. The simulation studies show that the EL and AEL methods have good performance in terms of the coverage probability and the interval length. The efficiency of the proposed methods are illustrated using a real data set to construct confidence intervals. Although the method performs well, it has room for improvement. The introduction of the nuisance parameter makes the calculation complicated. Yang and Zhao (2018) applied the smoothed jackknife empirical likelihood in the estimation of the one-sample difference of quantiles. Their idea might be valid to estimate the difference of quantiles with the paired sample. In the future, the novel empirical likelihood may be developed for this problem.

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APPENDICES

Proofs of Theorems

In the following proofs, C represents a constant and has different values at different occasions. Denote $\delta = h_1^r + n^{-s}$, where $1/3 < s < 1/2$.

Lemma 6.1. Under the conditions C.1-C.3, for t_1 satisfying $|t_1 - t_{1p}| \le \delta$, we have

$$\mathrm{E}[W_{1i}(\xi, t_1)] = F_1(t_1) - p + O(h_1^r), \tag{6.1}$$

$$\mathrm{E}[W_{2i}(\xi, t_1)] = F_2(t_1 - \xi) - p + O(h_2^r), \tag{6.2}$$

$$\mathrm{Var}[W_{1i}(\xi, t_1)] = F_1(t_1)(1 - F_1(t_1)) + O(h_1^r), \tag{6.3}$$

$$\mathrm{Var}[W_{2i}(\xi, t_1)] = F_2(t_1 - \xi)[1 - F_2(t_1 - \xi)] + O(h_2^r), \tag{6.4}$$

$$\mathrm{E}[W_{1i}(\xi, t_1)W_{2i}(\xi, t_1)] = F(t_1, t_1 - \xi) - p\big[F_1(t_1) + F_2(t_1 - \xi)\big] + p^2 + O(h_1^2). \tag{6.5}$$

Proof. Under the conditions C.1 to C.3, by Taylor expansions, we have

$$\begin{aligned}
\mathrm{E}[W_{1i}(\xi, t_1)] &= \int_{-\infty}^{+\infty} K\left(\frac{t_1 - z}{h_1}\right) f_1(z)dz - p \\
&= h_1 \int_{-\infty}^{+\infty} K(v) f_1(t_1 - h_1 v)dv - p \\
&= -\int_{-\infty}^{+\infty} K(v) dF_1(t_1 - h_1 v) - p \\
&= \int_{-\infty}^{+\infty} k(v) F_1(t_1 - h_1 v)dv - p \\
&= \int_{-\infty}^{+\infty} k(v)\big[F_1(t_1) - F_1'(t_1)h_1 v\big]dv + O(h_1^2) - p \\
&= F_1(t_1) - p + O(h_1^r).
\end{aligned} \tag{6.6}$$

Thus, equation (6.1) is proved. Equation (6.2) can be proved in the same way.

To prove equation (6.3), we have

$$\begin{aligned}
\mathrm{E}[W_{1i}^2(\xi,t_1)] &= \int_{-\infty}^{+\infty} K^2\left(\frac{t_1-z}{h_1}\right) f_1(z)dz - 2p\int_{-\infty}^{+\infty} K\left(\frac{t_1-z}{h_1}\right) f_1(z)dz + p^2 \\
&= h_1\int_{-\infty}^{+\infty} K^2(v) f_1(t_1-h_1v)dv - 2pF_1(t_1) + O(h_1^r) + p^2 \\
&= -\int_{-\infty}^{+\infty} K^2(v) dF_1(t_1-h_1v) - 2pF_1(t_1) + O(h_1^r) + p^2 \\
&= \int_{-\infty}^{+\infty} 2K(v)k(v)F_1(t_1-h_1v)dv - 2pF_1(t_1) + O(h_1^r) + p^2 \\
&= \int_{-\infty}^{+\infty} 2K(v)k(v)\big[F_1(t_1) - F_1'(x)h_1v\big]dv - 2pF_1(t_1) + O(h_1^r) + p^2 \\
&= F_1(t_1) - 2pF_1(t_1) + p^2 + O(h_1^r).
\end{aligned}$$

Using equation (6.1), equation (6.3) is proved as follows,

$$\begin{aligned}
\mathrm{Var}[W_{1i}(\xi,t_1)] &= \mathrm{E}[W_{1i}^2(\xi,t_1)] - [\mathrm{E}W_{1i}(\xi,t_1)]^2 \\
&= F_1(t_1)(1-F_1(t_1)) + O(h_1^r).
\end{aligned}$$

Equation (6.4) can be proved in the same way. The proof of equation (6.5) is as follows.

$$\begin{aligned}
\mathrm{E}[W_{1i}(\xi,t_1)W_{2i}(\xi,t_1)] &= \iint_{-\infty}^{+\infty} K\left(\frac{t_1-z_1}{h_1}\right)K\left(\frac{t_1-\xi-z_2}{h_2}\right) f(z_1,z_2)dz_1dz_2 \\
&\quad - p\iint_{-\infty}^{+\infty}\left[K\left(\frac{t_1-z_1}{h_1}\right) + K\left(\frac{t_1-\xi-z_2}{h_2}\right)\right] f(z_1,z_2)dz_1dz_2 + p^2 \\
&= F(t_1,t_1-\xi) - p\big[F_1(t_1) + F_2(t_1-\xi)\big] + p^2 + O(h_1^2).
\end{aligned}$$

In the proof, equations (6.7), (6.8) and (6.9) are used.

$$
\begin{aligned}
&\iint_{-\infty}^{+\infty} K\left(\frac{t_1 - z_1}{h_1}\right) K\left(\frac{t_1 - \xi - z_2}{h_2}\right) f(z_1, z_2) dz_1 dz_2 \\
&= h_1 h_2 \iint_{-\infty}^{+\infty} K(v_1) K(v_2) f(t_1 - h_1 v_1, t_1 - \xi - h_2 v_2) dv_1 dv_2 \\
&= h_1 h_2 \int_{-\infty}^{+\infty} K(v_2) dv_2 \int_{-\infty}^{+\infty} K(v_1) f(t_1 - h_1 v_1, t_1 - \xi - h_2 v_2) dv_1 \\
&= -h_2 \int_{-\infty}^{+\infty} K(v_2) dv_2 \int_{-\infty}^{+\infty} K(v_1) dF'_{v_2}(t_1 - h_1 v_1, t_1 - \xi - h_2 v_2) \\
&= h_2 \int_{-\infty}^{+\infty} K(v_2) dv_2 \int_{-\infty}^{+\infty} K'(v_1) F'_{v_2}(t_1 - h_1 v_1, t_1 - \xi - h_2 v_2) dv_1 \\
&= h_2 \int_{-\infty}^{+\infty} K'(v_1) dv_1 \int_{-\infty}^{+\infty} K(v_2) F'_{v_2}(t_1 - h_1 v_1, t_1 - \xi - h_2 v_2) dv_2 \\
&= - \int_{-\infty}^{+\infty} K'(v_1) dv_1 \int_{-\infty}^{+\infty} K(v_2) dF(t_1 - h_1 v_1, t_1 - \xi - h_2 v_2) \\
&= \int_{-\infty}^{+\infty} K'(v_1) dv_1 \int_{-\infty}^{+\infty} K'(v_2) F(t_1 - h_1 v_1, t_1 - \xi - h_2 v_2) dv_2 \\
&= \iint_{-\infty}^{+\infty} K'(v_1) K'(v_2) \big[F(t_1, t_1 - \xi) - F'_1(t_1, t_1 - \xi) h_1 v_1 - F'_2(t_1, t_1 - \xi) h_2 v_2\big] dv_1 dv_2 \\
&+ O(h_1^2) + O(h_2^2) + O(h_1 h_2) \\
&= F(t_1, t_1 - \xi) + O(h_1^2) + O(h_2^2) + O(h_1 h_2).
\end{aligned}
\tag{6.7}
$$

$$
\begin{aligned}
\iint_{-\infty}^{+\infty} K\left(\frac{t_1 - z_1}{h_1}\right) f(z_1, z_2) dz_1 dz_2 &= \int_{-\infty}^{+\infty} K\left(\frac{t_1 - z_1}{h_1}\right) f_1(z_1) dz_1 \\
&= - \int_{-\infty}^{+\infty} K(v) dF_1(t_1 - h_1 v) \\
&= \int_{-\infty}^{+\infty} k(v) F_1(t_1 - h_1 v) dv \\
&= \int_{-\infty}^{+\infty} k(v) \big[F_1(t_1) - F'_1(t_1) h_1 v\big] dv + O(h_1^2) \\
&= F_1(t_1) + O(h_1^2).
\end{aligned}
\tag{6.8}
$$

$$
\begin{aligned}
\iint_{-\infty}^{+\infty} K\left(\frac{t_1-\xi-z_2}{h_1}\right) f(z_1,z_2)dz_1dz_2 &= \int_{-\infty} K\left(\frac{t_1-\xi-z_2}{h_2}\right) f_2(z_2)dz_2 \\
&= -\int_{-\infty}^{+\infty} K(v)dF_1(t_1-\xi-h_2v) \\
&= \int_{-\infty}^{+\infty} k(v)F_2(t_1-\xi-h_2v)dv \\
&= \int_{-\infty}^{+\infty} k(v)\big[F_2(t_1-\xi)-F_2'(t_1-\xi)h_2v\big]dv + O(h_2^2) \\
&= F_2(t_1-\xi)+O(h_2^2).
\end{aligned}
\tag{6.9}
$$

Lemma 6.2. Under the conditions C.1-C.3, uniformly for $t_1 \in \{t_1 : |t_1 - t_{1p}| \le \delta\}$, we have

$$
\begin{aligned}
\overline{W}_1(\xi,t_1) &= \frac{1}{n}\sum_{i=1}^{n} W_{1i}(\xi,t_1) \\
&= O_p(\delta),
\end{aligned}
\tag{6.10}
$$

$$
\begin{aligned}
\overline{W}_2(\xi,t_1) &= \frac{1}{n}\sum_{i=1}^{n} W_{2i}(\xi,t_1) \\
&= O_p(\delta),
\end{aligned}
\tag{6.11}
$$

$$
\begin{aligned}
S_1^2(\xi,t_1) &= \frac{1}{n}\sum_{i=1}^{n} W_{1i}^2(\xi,t_1) \\
&= p(1-p)+O_p(\delta+h_1),
\end{aligned}
\tag{6.12}
$$

$$
\begin{aligned}
S_2^2(\xi,t_1) &= \frac{1}{n}\sum_{i=1}^{n} W_{2i}^2(\xi,t_1) \\
&= p(1-p)+O_p(\delta+h_2),
\end{aligned}
\tag{6.13}
$$

$$
\begin{aligned}
S_{12}(\xi,t_1) &= \frac{1}{n}\sum_{i=1}^{n} W_{1i}(\xi,t_1)W_{2i}(\xi,t_1) \\
&= \gamma - p^2 + O_p(\delta+h_1),
\end{aligned}
\tag{6.14}
$$

where $\gamma = F(t_{1p}, t_{1p}-\xi)$.

Proof. Since

$$W_{1i}(\xi,t_1)-W_{1i}(\xi,t_{1p})=\frac{t_1-t_{1p}}{h_1}k\Big(\frac{t_1-X_{1i}}{h_1}\Big)+\frac{(t_1-t_{1p})^2}{2h_1^2}k'\Big(\frac{t_i-X_{1i}}{h_1}\Big),$$

where t_i is between t_{1p} and t_1, we have

$$\overline{W}_1(\xi,t_1)=\overline{W}_1(\xi,t_{1p})+\frac{t_1-t_p}{nh_1}\sum_{i=1}^{n}k\Big(\frac{t_1-X_{1i}}{h_1}\Big)+\frac{(t_1-t_{1p})^2}{2nh_1^2}\sum_{i=1}^{n}k'\Big(\frac{t_i-X_{1i}}{h_1}\Big). \tag{6.15}$$

Combining Lemma 6.1 and the CLT, it follows that

$$\begin{aligned}\overline{W}_1(\xi,t_{1p})&=\mathrm{E}[W_{1i}(\xi,t_{1p})]+[\overline{W}_1(\xi,t_{1p})-\mathrm{E}[W_{1i}(\xi,t_{1p})]]\\&=O(h_1^r)+O(n^{-\frac{1}{2}})\\&=O_p(\delta).\end{aligned} \tag{6.16}$$

As the similar way of proving equation (6.40) later in the thesis, we can get

$$\frac{1}{nh_1}\sum_{i=1}^{n}\Big[k\Big(\frac{t_{1p}-X_{1i}}{h_1}\Big)-\mathrm{E}k\Big(\frac{t_{1p}-X_{1i}}{h_1}\Big)\Big]\longrightarrow 0 \quad a.s. \tag{6.17}$$

By Taylor expansions, we obtain

$$\begin{aligned}\mathrm{E}k\Big(\frac{t_{1p}-X_{1i}}{h_1}\Big)&=\int_{-\infty}^{+\infty}k\Big(\frac{t_{1p}-z}{h_1}\Big)f_1(z)dz\\&=h_1\int_{-\infty}^{+\infty}k(v)f_1(t_{1p}-h_1v)dv\\&=h_1\int_{-\infty}^{+\infty}k(v)\big[f_1(t_{1p})+O(h_1)\big]dv\\&=h_1f_1(t_{1p})+O(h_1^2).\end{aligned} \tag{6.18}$$

Combining equations (6.17) and (6.18), we get

$$\frac{t_1-t_p}{nh_1}\sum_{i=1}^{n}k\Big(\frac{t_1-X_{1i}}{h_1}\Big)=O(\delta). \tag{6.19}$$

Regarding to condition C.2, we have

$$k'\Big(\frac{t_i - X_{1i}}{h_1}\Big) < C. \tag{6.20}$$

Under condition C.3, we have

$$\begin{aligned}\frac{\delta}{h_1^2} &= h_1^{r-2} + \frac{n^{-s}}{h_1^2} \\ &= h_1^{r-2} + \frac{n^{s-1/2}}{(h_1^4 n^{4s-1})^{1/2}} \\ &< C.\end{aligned} \tag{6.21}$$

Combining equations (6.20) and (6.21), we have

$$\frac{(t_1 - t_{1p})^2}{2nh_1^2}\sum_{i=1}^{n} k'\Big(\frac{t_i - X_{1i}}{h_1}\Big) = O(\delta). \tag{6.22}$$

Equation (6.10) is proved following equations (6.16), (6.19) and (6.22). The rest equations in the lemma can be proved similarly.

Lemma 6.3. Under the conditions C.1-C.3, for $t_1 \in \{t_1 : |t_1 - t_{1p}| \le \delta\}$, as $n \to \infty$, we have

$$\begin{aligned}\overline{W}_1(\xi, t_1) &= O\big(\delta + h_1^r + n^{-1/2}(\log n^{1/2})\big) \\ &= O(\delta) \quad a.s.,\end{aligned} \tag{6.23}$$

$$\begin{aligned}\overline{W}_2(\xi, t_1) &= O\big(\delta + h_2^r + n^{-1/2}(\log n^{1/2})\big) \\ &= O(\delta) \quad a.s.,\end{aligned} \tag{6.24}$$

$$\begin{aligned}S_1^2(\xi, t_1) &= p(1-p) + O\big(\delta + h_1 + n^{-1/2}(\log n^{1/2})\big) \\ &= p(1-p) + O(h_1) \quad a.s.,\end{aligned} \tag{6.25}$$

$$\begin{aligned}S_2^2(\xi, t_1) &= p(1-p) + O\big(\delta + h_2 + n^{-1/2}(\log n^{1/2}\big) \\ &= p(1-p) + O(h_2) \quad a.s.,\end{aligned} \tag{6.26}$$

$$
\begin{aligned}
S_{12}(\xi, t_1) &= \gamma - p^2 + O\big(\delta + h_1 + n^{-1/2}(\log n^{1/2})\big) \\
&= \gamma - p^2 + O(h_1) \quad a.s.
\end{aligned} \tag{6.27}
$$

Proof. Let $Y_1, ..., Y_n$ be i.i.d. random variables satisfying $P(|Y_i - EY_i| \le m) = 1$, $i = 1, ..., n$, where m is any fixed number and $0 < m < \infty$. Following the Bernstein inequality (Serfling (1980), p.95), for $t > 0$,

$$
P(|\overline{Y} - \mu| \ge t) \le 2 \exp\big(- \frac{nt^2}{2Var(Y_1) + 2/3mt} \big).
$$

Let $Y_i = W_{1i}(\xi, t_1), m = 1, t = dn^{-1/2}(\log n)^{1/2}$ for $d > 0$. Noting equation (6.3), we obtain

$$
\begin{aligned}
\sum_{n=1}^{\infty} P\big(|\overline{W}_1(\xi, t_1) - \mathrm{E}\overline{W}_1(\xi, t_1)| \ge dn^{-1/2}(\log n)^{1/2}\big) &\le 2 \sum_{n=1}^{\infty} \exp\big(- \frac{d^2 \log n}{2C + 2/3dn^{-1/2}(\log n)^{1/2}} \big) \\
&\le 2 \sum_{n=1}^{\infty} \exp(-2 \log n) \\
&= 2 \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,
\end{aligned}
$$

for d sufficiently large. According to the Borel–Cantelli Lemma, it follows that

$$
|\overline{W}_1(\xi, t_1) - \mathrm{E}\overline{W}_1(\xi, t_1)| < dn^{-\frac{1}{2}}(\log n)^{\frac{1}{2}} \ a.s. \tag{6.28}
$$

By Taylor expansions, we get

$$
F_1(t_1) - F_1(t_{1p}) = (t_1 - t_{1p}) f_1(t_i) = O(\delta), \tag{6.29}
$$

where t_i is between t_1 and t_{1p}. Combining equations (6.1), (6.28) and (6.29), we prove equation (6.23). Other equations can be proved in the similar way.

Lemma 6.4. Given t_1, denote the solution of equation (2.4) as $\lambda^T(t_1) = \big(\lambda_1(t_1), \lambda_2(t_1)\big)$. Under the conditions C.1-C.3, for $t_1 \in \{t_1 : |t_1 - t_{1p}| \le \delta\}$, we have

$$
\lambda_1(t_1) = O_p(\delta), \quad \lambda_2(t_1) = O_p(\delta), \tag{6.30}
$$

$$\lambda_1(t_{1p} \pm \delta) = O\big(\delta + h_1^r + n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}\big) \quad a.s., \tag{6.31}$$

$$\lambda_2(t_{1p} \pm \delta) = O\big(\delta + h_2^r + n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}}\big) \quad a.s. \tag{6.32}$$

Proof. Let $\rho = \sqrt{\lambda_1^2(t_1) + \lambda_2^2(t_1)}$, $Z_n = \max\limits_{1\le i\le n} ||W_i(\xi, t_1)||$ and $S = \begin{pmatrix} S_1^2(\xi, t_1), S_{12}(\xi, t_1) \\ S_{12}(\xi, t_1), S_2^2(\xi, t_1) \end{pmatrix}$. Define $\theta = \lambda(t_1)/\rho$. Denote $g(\lambda) = 1/n \sum_{i=1}^n \frac{W_i(\xi,t_1)}{1+\lambda^T W_i(\xi,t_1)}$. From equation (2.4), we have

$$\begin{aligned}
\big|\theta^T g(\rho\theta)\big| &= \frac{1}{n}\Big|\theta^T \sum_{i=1}^n \frac{W_i(\xi, t_1)}{1 + \rho\theta^T W_i(\xi, t_1)}\Big| \\
&= \frac{1}{n}\Big|\theta^T \sum_{i=1}^n \frac{W_i(\xi, t_1)[1 + \rho\theta^T W_i(\xi, t_1)] - \rho W_i(\xi, t_1)\theta^T W_i(\xi, t_1)}{1 + \rho\theta^T W_i(\xi, t_1)}\Big| \\
&= \frac{1}{n}\Big|\theta^T \sum_{i=1}^n W_i(\xi, t_1) - \rho\theta^T \sum_{i=1}^n \frac{W_i(\xi, t_1)\theta^T W_i(\xi, t_1)}{1 + \rho\theta^T W_i(\xi, t_1)}\Big| \\
&= \frac{1}{n}\Big|\theta^T \sum_{i=1}^n W_i(\xi, t_1) - \rho\theta^T \sum_{i=1}^n \frac{W_i(\xi, t_1)W_i^T(\xi, t_1)}{1 + \rho\theta^T W_i(\xi, t_1)}\theta\Big| \\
&\ge \frac{\rho}{n}\theta^T \sum_{i=1}^n \frac{W_i(\xi, t_1)W_i^T(\xi, t_1)}{1 + \rho\theta^T W_i(\xi, t_1)}\theta - \frac{1}{n}\theta^T \sum_{i=1}^n W_i(\xi, t_1) \\
&\ge \frac{\rho\theta^T S\theta}{1 + \rho Z_n} - \frac{1}{n}\Big[\big|\sum_{i=1}^n W_{1i}(\xi, t_1)\big| + \big|\sum_{i=1}^n W_{2i}(\xi, t_1)\big|\Big].
\end{aligned} \tag{6.33}$$

From equation (2.4), we have $|\theta^T g(\rho\theta)| \le ||g(\rho\theta)|| = 0$. With equation (6.33), it follows that

$$\frac{\rho\theta^T S\theta}{1 + \rho Z_n} \le |\overline{W}_1(\xi, t_1)| + |\overline{W}_2(\xi, t_1)|. \tag{6.34}$$

By Lemma 6.2, we have

$$\begin{aligned}
S &= \begin{pmatrix} p(1-p), \gamma - p^2 \\ \gamma - p^2, p(1-p) \end{pmatrix} + O_p(\delta + h_1) \\
&=: S_0 + O_p(\delta + h_1).
\end{aligned} \tag{6.35}$$

Let σ_p be the minimal eigenvalue of S_0. Then

$$\theta^T S_0 \theta \geq \sigma_p. \tag{6.36}$$

With Lemma 6.2, combining equations (6.34), (6.35), and (6.36), we have

$$\frac{\rho}{1+\rho Z_n} = O_p(\delta).$$

Noting $Z_n < C$, it follows that $\rho = O_p(\delta)$, which proves equation (6.26).

From Lemma 6.3, we have

$$S = S_0 + O\big(\delta + h_1 + n^{-\frac{1}{2}}(\log n^{\frac{1}{2}})\big) \quad a.s., \tag{6.37}$$

$$\overline{W}_i(\xi, t_{1p} \pm \delta) = O\big(\delta + h_1^r + n^{-\frac{1}{2}}(\log n^{\frac{1}{2}})\big) \quad a.s.,\ i = 1, 2. \tag{6.38}$$

Combining equations (6.34), (6.36), (6.37) and (6.38), we obtain $\rho = O(\delta) \quad a.s.$ This proves equations (6.31) and (6.32).

Lemma 6.5. Assume conditions C.1-C.3 hold. Then with probability one, as $n \to \infty$, $l(\xi, t_1, \lambda^T)$ attains its minimum value at $t_1 = \tilde{t}_1$ and $\lambda^T = \tilde{\lambda}^T$, where $\tilde{t}_1, \tilde{\lambda}^T$ is a solution to equations (2.4) and (2.5).

Proof. Let $t_1' = t_{1p} + \delta$. By Lemma 6.3, we have

$$\begin{aligned} S(t_1') &= \begin{pmatrix} S_1^2(\xi, t_1'), S_{12}(\xi, t_1') \\ S_{12}(\xi, t_1'), S_2^2(\xi, t_1') \end{pmatrix} \\ &= S_0 + O_p(\delta + h_1 + n^{-\frac{1}{2}}(\log n)^{\frac{1}{2}})\ a.s. \end{aligned}$$

With equation (2.4), similar to the proof in Owen (1990), we get

$$\lambda^T(t_1') = S^{-1}(t_1')(\overline{W}_1(\xi, t_1'), \overline{W}_2(\xi, t_1'))^T + O(\delta^2)\ a.s. \tag{6.39}$$

By Taylor expansion, we have

$$
\begin{aligned}
-2\log r(\xi,t_1') &= 2\sum_{i=1}^{n}\log(1+\lambda^T(t_1')W_i(\xi,t_1'))\\
&= 2\sum_{i=1}^{n}\lambda^T(t_1')W_i(\xi,t_1') - \sum_{i=1}^{n}[\lambda^T(t_1')W_i(\xi,t_1')]^2 + O(n\delta^3)\ a.s.\\
&= 2n\lambda^T(t_1')\big(\overline{W}_1(\xi,t_1'),\overline{W}_2(\xi,t_1')\big) - n\lambda^T(t_1')S(t_1')\lambda(t_1') + O(n\delta^3)\ a.s.\\
&= n\big(\overline{W}_1(\xi,t_1'),\overline{W}_2(\xi,t_1')\big)S^{-1}(t_1')\big(\overline{W}_1(\xi,t_1'),\overline{W}_2(\xi,t_1')\big)^T + O(n\delta^3)\\
&= n\left(\overline{W}_1(\xi,t_{1p}) + \frac{1}{nh_1}\sum_{i=1}^{n}K'\left(\frac{t_{1i}^*-X_{1i}}{h_1}\right)\delta, \overline{W}_2(\xi,t_{1p}) + \frac{1}{nh_2}\sum_{i=1}^{n}K'\left(\frac{t_{1i}^{**}-X_{2i}}{h_2}\right)\delta\right)\\
&\quad S^{-1}(t_1')\left(\overline{W}_1(\xi,t_{1p}) + \frac{1}{nh_1}\sum_{i=1}^{n}K'\left(\frac{t_{1i}^*-X_{1i}}{h_1}\right)\delta, \overline{W}_2(\xi,t_{1p}) + \frac{1}{nh_2}\sum_{i=1}^{n}K'\left(\frac{t_{1i}^{**}-X_{2i}}{h_2}\right)\delta\right)^T\\
&\quad + O(n\delta^3)\ a.s.\\
&\geq n\sigma\Gamma^T\Gamma + O(n\delta^3)\ a.s.,
\end{aligned}
$$

where t_{1i}^* is between t_{1p} and $t_{1p}+\delta$, t_{1i}^{**} is between $t_{1p}-\xi$ and $t_{1p}-\xi+\delta$,

$$
\Gamma^T = \Big(\overline{W}_1(\xi,t_{1p}) + \frac{1}{nh_1}\sum_{i=1}^{n}K'\big(\frac{t_{1i}^*-X_{1i}}{h_1}\big)\delta, \overline{W}_2(\xi,t_{1p}) + \frac{1}{nh_2}\sum_{i=1}^{n}K'\big(\frac{t_{1i}^{**}-X_{2i}}{h_2}\big)\delta\Big),
$$

and σ is the minimal eigenvalue of $S^{-1}(t_1')$. Note that t_{1i}^*'s are independent and X_{1i}'s are independent. Following Bernstein inequality, we have

$$
\begin{aligned}
\sum_{i=1}^{\infty}P\left(\left|\frac{1}{nh_1}\sum_{i=1}^{n}\left[K'\left(\frac{t_{1i}^*-X_{1i}}{h_1}\right) - EK'\left(\frac{t_{1i}^*-X_{1i}}{h_1}\right)\right]\right| \geq dh^2\right) &\leq 2\sum_{i=1}^{\infty}\exp\frac{-nd^2h^4}{2C+2/3dh^2}\\
&\leq \sum_{i=1}^{\infty}\exp(-2\log n)\\
&= \sum_{i=1}^{\infty}\frac{1}{n^2} < \infty,
\end{aligned}
$$

for d sufficiently large. By the Borel–Cantelli Lemma, we have

$$\frac{1}{nh_1}\sum_{i=1}^{n}\left[K'\left(\frac{t_{1i}^{*}-X_{1i}}{h_1}\right)-EK'\left(\frac{t_{1i}^{*}-X_{1i}}{h_1}\right)\right]\to 0 \quad a.s. \tag{6.40}$$

On the other hand,

$$\begin{aligned}\frac{1}{h_1}EK'\left(\frac{t_{1i}^{*}-X_{1i}}{h_1}\right)&=\frac{1}{h_1}\int_{-\infty}^{+\infty}K'\left(\frac{t_{1i}^{*}-x}{h_1}\right)f_1(x)dx\\&=\frac{1}{h_1}\int_{-\infty}^{+\infty}K'\left(\frac{t_{1i}^{*}-t_{1p}}{h_1}+\frac{t_{1p}-x}{h_1}\right)f_1(x)dx\\&=\frac{1}{h_1}\int_{-\infty}^{+\infty}K'\left(\frac{t_{1i}^{*}-t_{1p}}{h_1}+x\right)f_1(t_{1p}-h_1x)dx\\&=\frac{1}{h_1}\int_{-\infty}^{+\infty}K'\left(\frac{t_{1i}^{*}-t_{1p}}{h_1}+x\right)f_1(t_{1p})dx+O(h_1)\\&=f_1(t_{1p})+O(h_1).\end{aligned}$$

Hence,

$$\begin{aligned}\frac{1}{nh_1}\sum_{i=1}^{n}K'\left(\frac{t_{1i}^{*}-X_{1i}}{h_1}\right)&=\frac{1}{nh_1}\sum_{i=1}^{n}k\left(\frac{t_{1i}^{*}-X_{1i}}{h_1}\right)\\&\to f_1(t_p)\ a.s.\end{aligned} \tag{6.41}$$

Similarly,

$$\begin{aligned}\frac{1}{nh_2}\sum_{i=1}^{n}K'\left(\frac{t_{1i}^{**}-X_{2i}}{h_2}\right)&=\frac{1}{nh_2}\sum_{i=1}^{n}k\left(\frac{t_{1i}^{**}-X_{2i}}{h_2}\right)\\&\to f_2(t_{1p}-\xi)\ a.s.\end{aligned} \tag{6.42}$$

From Lemma 6.3, $\overline{W}_j(\xi,t_p)=O(\delta)$ $a.s.$, $j=1,2$. Therefore by equations (6.41) and (6.42), it follows that $\Gamma^T\Gamma=f_1^2(t_{1p})\delta^2+f_2^2(t_{1p}-\xi)\delta^2+O(\delta^2)$. Thus, $-2\log r(\xi,t_1')\geq(C-\epsilon_n)n\delta^2$, where $\epsilon_n\to 0$ $a.s.$ At t_{1p}, by the law of the iterated logarithm, we have

$$\begin{aligned}-2\log r(\xi,t_{1p})&=n\big(\overline{W}_1(\xi,t_{1p}),\overline{W}_2(\xi,t_{1p})\big)S^{-1}(t_{1p})\big(\overline{W}_1(\xi,t_{1p}),\overline{W}_2(\xi,t_{1p})\big)^T+O(n\delta^3)\\&=o(n\delta^2).\end{aligned}$$

Hence, $-2\log r(\xi,t_{1p}+\delta)>-2\log r(\xi,t_{1p})$ $a.s.$ Similarly, $-2\log r(\xi,t_{1p}-\delta)>-2\log r(\xi,t_{1p})$ $a.s.$

Because $-2\log r(\xi, t_1)$ is differentiable in the neighborhood of $t_1 \in [t_{1p}-\delta, t_{1p}+\delta]$, there exists $\tilde{t}_1 \in [t_{1p}-\delta, t_{1p}+\delta]$ such that $-2\log r(\xi, t_1)$ attains its minimum and $\tilde{t}_1, \tilde{\lambda}^T(\tilde{t}_1)$ satisfy equations (2.4) and (2.5).

Lemma 6.6. Under the conditions C.1-C.3, for $\tilde{t}_1, \tilde{\lambda}_1 = \lambda_1(\tilde{t}_1)$ and $\tilde{\lambda}_1 = \lambda_2(\tilde{t}_1)$ satisfying equations (2.4) and (2.5), we have

$$\sqrt{n}(\tilde{t}_1 - t_{1p}) \xrightarrow{\mathcal{D}} N\Big(0, \frac{(p-p^2)^2 - (\gamma - p^2)^2}{c_0}\Big), \tag{6.43}$$

$$\tilde{\lambda}_1 = -\frac{f_2(t_{1p}-\xi)}{f_1(t_{1p})}\lambda_2(\tilde{t}_1) + o_p(n^{-\frac{1}{2}}), \tag{6.44}$$

$$\sqrt{n}\tilde{\lambda}_2 \xrightarrow{\mathcal{D}} N\Big(0, \frac{f_1^2(t_{1p})}{c_0}\Big), \tag{6.45}$$

where $c_0 = (p-p^2)\big[f_1^2(t_{1p}) + f_2^2(t_{1p}-\xi)\big] - 2(\gamma - p^2)f_1(t_1)f_2(t_1-\xi)$.

Proof. Define

$$Q_1(t_1, \lambda_1, \lambda_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{W_{1i}(\xi, t_1)}{1 + \lambda_1 W_{1i}(\xi, t_1) + \lambda_2 W_{2i}(\xi, t_1)},$$

$$Q_2(t_1, \lambda_1, \lambda_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{W_{2i}(\xi, t_1)}{1 + \lambda_1 W_{1i}(\xi, t_1) + \lambda_2 W_{2i}(\xi, t_1)},$$

$$Q_3(t_1, \lambda_1, \lambda_2) = \frac{1}{n}\sum_{i=1}^{n} \frac{\frac{\lambda_1}{h_1}K'\big(\frac{t_1 - X_{1i}}{h_1}\big) + \frac{\lambda_2}{h_2}K'\big(\frac{t_1-\xi-X_{2i}}{h_1}\big)}{1 + \lambda_1 W_{1i}(\xi, t_1) + \lambda_2 W_{2i}(\xi, t_1)}.$$

By Lemma 6.2, we obtain

$$
\begin{aligned}
\frac{\partial Q_1(t_{1p},0,0)}{\partial t_1} &= \frac{1}{nh_1}\sum_{i=1}^{n} K'\left(\frac{t_{1p}-X_{1i}}{h_1}\right) \\
&\to f_1(t_{1p}) \ a.s., \\
\frac{\partial Q_1(t_{1p},0,0)}{\partial \lambda_1} &= -\frac{1}{n}\sum_{i=1}^{n} W_{1i}^2(\xi,t_{1p}) \\
&\to -p(1-p) \ a.s., \\
\frac{\partial Q_1(t_{1p},0,0)}{\partial \lambda_2} &= -\frac{1}{n}\sum_{i=1}^{n} W_{1i}(\xi,t_{1p})W_{2i}(\xi,t_{1p}) \\
&\to -(\gamma-p^2) \ a.s., \\
\frac{\partial Q_2(t_p,0,0)}{\partial t_1} &= \frac{1}{nh_2}\sum_{i=1}^{n} K'\left(\frac{t_{1p}-\xi-X_{2i}}{h_2}\right) \\
&\to f_2(t_{1p}-\xi) \ a.s., \\
\frac{\partial Q_2(t_{1p},0,0)}{\partial \lambda_1} &= -\frac{1}{n}\sum_{i=1}^{n} W_{1i}(\xi,t_{1p})W_{2i}(\xi,t_{1p}) \\
&\to -(\gamma-p^2) \ a.s., \\
\frac{\partial Q_2(t_{1p},0,0)}{\partial \lambda_2} &= -\frac{1}{n}\sum_{i=1}^{n} W_{2i}^2(\xi,t_{1p}) \\
&\to -p(1-p) \ a.s., \\
\frac{\partial Q_3(t_{1p},0,0)}{\partial t_1} &= 0, \\
\frac{\partial Q_3(t_{1p},0,0)}{\partial \lambda_1} &= \frac{1}{nh_1}\sum_{i=1}^{n} K'\left(\frac{t_{1p}-X_{1i}}{h_1}\right) \\
&\to f_1(t_{1p}) \ a.s., \\
\frac{\partial Q_3(t_{1p},0,0)}{\partial \lambda_2} &= \frac{1}{nh_2}\sum_{i=1}^{n} K'\left(\frac{t_{1p}-\xi-X_{2i}}{h_2}\right) \\
&\to f_2(t_{1p}-\xi) \ a.s.
\end{aligned}
$$

Denote

$$\hat{S}_n = \begin{pmatrix} \frac{\partial Q_1(t_{1p},0,0)}{\partial t_1} & \frac{\partial Q_1(t_{1p},0,0)}{\partial \lambda_1} & \frac{\partial Q_1(t_{1p},0,0)}{\partial \lambda_2} \\ \frac{\partial Q_2(t_{1p},0,0)}{\partial t_1} & \frac{\partial Q_2(t_{1p},0,0)}{\partial \lambda_1} & \frac{\partial Q_2(t_{1p},0,0)}{\partial \lambda_2} \\ \frac{\partial Q_3(t_{1p},0,0)}{\partial t_1} & \frac{\partial Q_3(t_{1p},0,0)}{\partial \lambda_1} & \frac{\partial Q_3(t_{1p},0,0)}{\partial \lambda_2} \end{pmatrix} \to \begin{pmatrix} f_1(t_{1p}) & -p(1-p) & -(\gamma - p^2) \\ f_2(t_{1p}-\xi) & -(\gamma-p^2) & -p(1-p) \\ 0 & f_1(t_{1p}) & f_2(t_{1p}-\xi) \end{pmatrix}$$
$$=: S_n. \tag{6.46}$$

Expanding $Q_i(t_1, \lambda_1, \lambda_2)$ at $(\tilde{t}_1, 0, 0), i = 1, 2, 3$, we have

$$Q_i(\tilde{t}_1, \tilde{\lambda}_1, \tilde{\lambda}_2) = Q_i(t_{1p}, 0, 0) + (\tilde{t}_1 - t_{1p})\frac{\partial Q_i(t_{1p}, 0, 0)}{\partial t_1} + \tilde{\lambda}_1 \frac{\partial Q_i(t_{1p}, 0, 0)}{\partial \lambda_1} + \tilde{\lambda}_2 \frac{\partial Q_i(t_{1p}, 0, 0)}{\partial \lambda_2} + O_p(\delta^2). \tag{6.47}$$

From equations (2.4) and (2.5), we have $Q_i(\tilde{t}_1, \tilde{\lambda}_1, \tilde{\lambda}_2) = 0, i = 1, 2, 3$. Combining equations (6.46) and (6.47), we obtain

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} Q_1(t_{1p}, 0, 0) \\ Q_2(t_{1p}, 0, 0) \\ Q_3(t_{1p}, 0, 0) \end{pmatrix} + S_n \begin{pmatrix} \tilde{t}_1 - t_{1p} \\ \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \end{pmatrix} + O_p(\delta^2).$$

Noting that

$$\delta^2 = h_1^{2r} + 2h_1^r n^{-s} + n^{-2s} = o_p(n^{-\frac{1}{2}}), \tag{6.48}$$

we have

$$\begin{pmatrix} \tilde{t}_1 - t_{1p} \\ \tilde{\lambda}_1 \\ \tilde{\lambda}_2 \end{pmatrix} = S_n^{-1} \begin{pmatrix} Q_1(t_{1p}, 0, 0) \\ Q_2(t_{1p}, 0, 0) \\ Q_3(t_{1p}, 0, 0) \end{pmatrix} + o_p(n^{-\frac{1}{2}}),$$

where

$$S_n^{-1} = \begin{pmatrix} \frac{(p-p^2)f_1(t_{1p})-(\gamma-p^2)f_2(t_{1p}-\xi)}{c_0} & \frac{(p-p^2)f_2(t_{1p}-\xi)-(\gamma-p^2)f_1(t_{1p})}{c_0} & \frac{(p-\gamma)(p+\gamma-2p^2)}{c_0} \\ -\frac{f_2^2(t_{1p}-\xi)}{c_0} & \frac{f_1(t_{1p})f_2(t_{1p}-\xi)}{c_0} & \frac{(p-p^2)f_1(t_{1p})-(\gamma-p^2)f_2(t_{1p}-\xi)}{c_0} \\ \frac{f_1(t_{1p})f_2(t_{1p}-\xi)}{c_0} & -\frac{f_1^2(t_{1p})}{c_0} & \frac{(p-p^2)f_2(t_{1p}-\xi)-(\gamma-p^2)f_1(t_{1p})}{c_0} \end{pmatrix}.$$

Noting $Q_3(t_{1p}, 0, 0) = 0$, we obtain

$$\begin{aligned}
\tilde{t}_1 - t_{1p} = & \frac{(p - p^2)f_1(t_{1p}) - (\gamma - p^2)f_2(t_{1p} - \xi)}{c_0} Q_1(t_{1p}, 0, 0) \\
+ & \frac{(p - p^2)f_2(t_{1p} - \xi) - (\gamma - p^2)f_1(t_{1p})}{c_0} Q_2(t_{1p}, 0, 0)] + o_p(n^{-\frac{1}{2}}), \\
\tilde{\lambda}_1 = & -\frac{f_2^2(t_{1p} - \xi)}{c_0} Q_1(t_{1p}, 0, 0) + \frac{f_1(t_{1p})f_2(t_{1p} - \xi)}{c_0} Q_2(t_{1p}, 0, 0) + o_p(n^{-\frac{1}{2}}), \\
\tilde{\lambda}_2 = & \frac{f_1(t_{1p})f_2(t_{1p} - \xi)}{c_0} Q_1(t_{1p}, 0, 0) - \frac{f_1^2(t_{1p})}{c_0} Q_2(t_{1p}, 0, 0) + o_p(n^{-\frac{1}{2}}).
\end{aligned}$$

Combining the fact that

$$\sqrt{n} \begin{pmatrix} Q_1(t_{1p}, 0, 0) \\ Q_2(t_{1p}, 0, 0) \end{pmatrix} \xrightarrow{\mathcal{D}} N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} p(1-p) & \gamma - p^2 \\ \gamma - p^2 & p(1-p) \end{pmatrix} \right),$$

we complete the proof.

Proof of Theorem 2.1. From equations (6.39) and (6.48), we have

$$\overline{W}_1(\tilde{t}_1) = \tilde{\lambda}_1 S_1^2(\xi, \tilde{t}_1) + \tilde{\lambda}_2 S_{12}(\xi, \tilde{t}_1) + o_p(n^{-\frac{1}{2}}),$$

$$\overline{W}_2(\tilde{t}_1) = \tilde{\lambda}_1 S_{12}(\xi, \tilde{t}_1) + \tilde{\lambda}_2 S_2^2(\xi, \tilde{t}_1) + o_p(n^{-\frac{1}{2}}).$$

Therefore, by Taylor expansions, we have

$$\begin{aligned}
-2\log r(\xi, \tilde{t}_1) &= 2\sum_{i=1}^{n} \log(1 + \tilde{\lambda}^T W_i(\xi, \tilde{t}_1)) \\
&= 2\sum_{i=1}^{n} \tilde{\lambda}^T W_i(\xi, \tilde{t}_1) - \sum_{i=1}^{n} [\tilde{\lambda}^T W_i(\xi, \tilde{t}_1)]^2 + o_p(1) \\
&= n\tilde{\lambda}_1^2 S_1^2(\xi, \tilde{t}_1) + n\tilde{\lambda}_2^2 S_2^2(\xi, \tilde{t}_1)) + 2n\tilde{\lambda}_1^2 \tilde{\lambda}_2^2 S_{12}(\xi, \tilde{t}_1) + o_p(1) \\
&= n\lambda_2^2(\tilde{t}_1) \left[\frac{f_2^2(t_{1p} - \xi)}{f_1^2(t_{1p})} S_1^2(\xi, \tilde{t}_1) - 2\frac{f_2(t_{1p} - \xi)}{f_1(t_p)} S_{12}(\xi, \tilde{t}_1) + S_2^2(\xi, \tilde{t}_1) \right] + o_p(1).
\end{aligned}$$

From Lemma 6.3, with some simple calculations, we get

$$\begin{aligned}\frac{f_2^2(t_{1p}-\xi)}{f_1^2(t_{1p})}S_1^2(\xi,\tilde{t}_1)-2\frac{f_2(t_{1p}-\xi)}{f_1(t_{1p})}S_{12}(\xi,\tilde{t}_1)+S_2^2(\xi,\tilde{t}_1) &\to \frac{f_2^2(t_{1p}-\xi)}{f_1^2(t_{1p})}(p-p^2)\\ &\quad -2\frac{f_2(t_{1p}-\xi)}{f_1(t_{1p})}(\gamma-p^2)+(p-p^2) \quad a.s.\\ &= \frac{c_0}{f_1^2(t_{1p})} \quad a.s.\end{aligned}$$

From equation (6.45), the proof is completed.

Proof of Theorem 2.2. By the arguments in Chen, Variyath and Abraham (2008), Theorem 2.2 can be proved.

Algorithms

Denote the true difference of quantiles as ξ_0 and the upper α-quantile of χ_1^2 as $\chi_1^2(\alpha)$.

Algorithm 1: Coverage probability calculation

---

1. Generate the paired data.
2. Use Algorithm 3 to calculate $l(\xi_0, \tilde{t}_1, \tilde{\lambda}^T)$.
3. Compare $l(\xi_0, \tilde{t}_1, \tilde{\lambda}^T)$ and $\chi_1^2(\alpha)$.

Algorithm 2: Confidence interval calculation

---

1. Generate the paired data.
2. Propose an estimator $\hat{\xi}$ of ξ.
3. Use Algorithm 3 to calculate $l(\hat{\xi}, \tilde{t}_1, \tilde{\lambda}^T)$.
4. Repeat steps 2 and 3 until $l(\hat{\xi}, \tilde{t}_1, \tilde{\lambda}^T) < \chi_1^2(\alpha)$.
5. Solve the equation $l(\hat{\xi}, \tilde{t}_1, \tilde{\lambda}^T) - \chi_1^2(\alpha) = 0$ using the bisection method.
6. Keep the two solutions ξ_l and ξ_u as the lower and upper bounds of ξ.

Algorithm 3: Compute $l(\xi, \tilde{t}_1, \tilde{\lambda}^T)$

---

1. Given ξ, solve the equations (2.4) and (2.5) and $\tilde{t}_1, \tilde{\lambda}^T$ are obtained.
2. Plug ξ, $\tilde{t}_1$ and $\tilde{\lambda}^T$ into the equation (2.3), and $l(\xi, \tilde{t}_1, \tilde{\lambda}^T)$ is calculated.
