Summer 8-10-2021

Size Ramsey Numbers Involving Double Stars and Brooms

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by

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Under the Direction of Guantao Chen, PhD

A Thesis Submitted in Partial Fulfillment of the Requirements for the Degree of

Master of Science

in the College of Arts and Sciences

Georgia State University

2021

ABSTRACT

The topics of this thesis lie in graph Ramsey theory. Given two graphs G and H, by the Ramsey theorem, there exist infinitely many graphs F such that if we partition the edges of F into two sets, say *Red* and *Blue*, then either the graph induced by the red edges contains G or the graph induced by the blue edges contains H. The minimum order of F is called the *Ramsey number* and the minimum of the size of F is called the *size Ramsey number*. They are denoted by $\mathrm{r}(G, H)$ and $\hat{\mathrm{r}}(G, H)$, respectively. We will investigate size Ramsey numbers involving double stars and brooms.

INDEX WORDS: Ramsey theory, Size Ramsey number, Double star, Broom.

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August 2021

DEDICATION

This thesis is dedicated to my parents and friends.

ACKNOWLEDGEMENTS

I started my M.S. program in August 2020 and will complete my study in July 2021. In this year, I have learned a lot of knowledge and received the guidance and help from teachers and classmates. Here I would like to express my most sincere thanks to them.

I would like to thank my supervisors, Prof. Guantao Chen as well as Prof. Yaping Mao from Qinghai Normal University firstly. Without their help, I would not have the opportunity to study in GSU. In addition, this thesis was completed under the guidance of these two professors. They gave good suggestions on the topic selection, conception, writing and final revision of the thesis.

I would like to thank Ning Song from Shandong University of Technology, who gave me great guidance in writing Chapter 2. I would also like to thank Meng Ji from Tianjin Normal University, who gave me great guidance in writing Chapter 3.

Finally, I would like to express my special thanks to my parents for their meticulous care.

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CHAPTER 1

INTRODUCTION

In this chapter, we first introduce the some basic concepts and terminologies, then introduce the research background and related research progress of size Ramsey number, and finally introduce the research results of this thesis.

1.1 Basic concepts and terminologies

All the graphs in this paper are undirected, finite and simple. For more concepts and symbols in graph theory, please refer to textbooks of graph theory $[1,2,3,4]$.

For a graph G, the vertex set, edge set, order and size of G are denoted by $V(G)$, $E(G)$, $v(G)$ and $e(G)$, respectively. The *neighborhood*, $N_G(v)$ or $N(v)$ for short, of a vertex v of G is the set of vertices adjacent to it. The *degree*, $\deg_G(v)$ or $\deg(v)$ for short, of a vertex v of G is the number of edges incident to it. The *minimum degree* of G, denoted by $\delta(G)$, is the smallest of the degrees of vertices in G and the *maximum degree*, denoted by $\Delta(G)$, of G is the largest of the degrees of the vertices in G. For any subset X of $V(G)$, let $G[X]$ denote the subgraph induced by X; similarly, for any subset F of $E(G)$, let $G[F]$ denote the subgraph induced by F. We use $G - X$ to denote the subgraph of G obtained by removing all the vertices of X together with the edges incident with them from G; similarly, we use $G \setminus F$ to denote the subgraph of G obtained by removing all the edges of F from G. If $X = \{v\}$ and $F = \{e\}$, we simply write $G - v$ and $G \setminus e$ for $G - \{v\}$ and $G \setminus \{e\}$, respectively.

A graph G is *connected* if every two distinct vertices of $V(G)$ are the ends of at least one path in G. A connected graph G with order $v(G)$ and size $v(G) - 1$ is called a tree, denoted by T. An *acyclic* graph is one that contains no cycles, acyclic graphs are also called forests. The *pendent* vertices are vertices of degree 1. A *double star* $D(m, n)$ is a tree containing exactly two non-pendant vertices x and y with $\deg(x) = n + 1$ and $\deg(y) = m + 1$, where x and y are called the *n-center* and *m-center* (or just a *center* for short), respectively. A *broom* $B(m, n)$ is a tree obtained from a path P_{m+1} and a star $K_{1,n}$ by identifying an end-vertex of P_{m+1} with the center of $K_{1,n}$. $D(m, n)$ and $B(m, n)$ are shown in the Figure 1.1.

A *red-blue coloring* (R, B) of the edges in a graph G is a partition of $E(G)$ into R and B, where we say an edge e is colored by *red* if $e \in R$, and colored by *blue* if $e \in B$. For

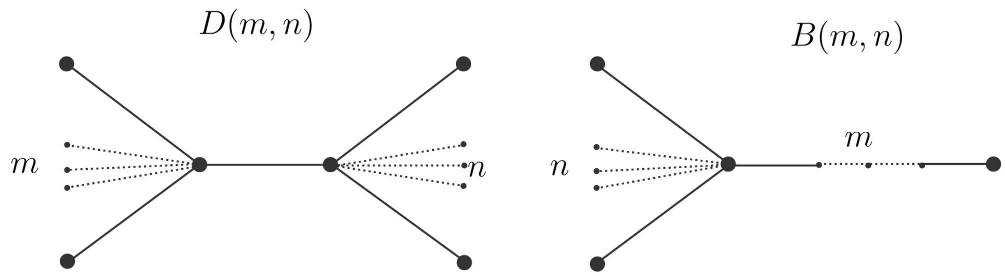

Figure 1.1. $D(m,n)$ and $B(m,n)$

Table 1.1. The difference between Ramsey number and size Ramsey number

Conception:	Ramsey number	size Ramsey number
Symbol:	$\mathrm{r}(G,H)$	$\hat{\mathrm{r}}(G,H)$
Definition:	$\mathrm{r}(G,H)=\min\{v(F) : F \to (G,H)\}$	$\hat{\mathrm{r}}(G,H)=\min\{e(F) : F \to (G,H)\}$

convenience, we also denote by R and B the induced subgraph by R and B, respectively, and call them the *red subgraph* and *blue subgraph*. Then for a vertex v, the meanings of $\deg_R(v)$, $\deg_B(v)$, $N_R(v)$ and $N_B(v)$ are clear.

Given graphs F, G and H, we write $F \to (G,H)$ if in any red-blue edge-coloring of F, F contains a red copy of G or a blue copy of H. Conversely, if there is a red-blue edge-coloring of F such that F contains neither a red copy of G nor a blue copy of H, then we write $F \nrightarrow (G,H)$. For graphs G and H, the smallest order of a graph F with $F \to (G,H)$ is called the *Ramsey number* for G and H, denoted by $\mathrm{r}(G,H)$; and the smallest size of a graph F with $F \to (G,H)$ is called the *size Ramsey number* for G and H, denoted by $\hat{\mathrm{r}}(G,H)$. The difference between Ramsey number and size Ramsey number can be seen in the Table 1.1.

1.2 Research background and related research progress

In 1978, Erdős, Faudree, Rousseau and Schelp introduced the notion of size Ramsey number and obtained the exact value of $\hat{\mathrm{r}}(K_m,K_n)$ and $\hat{\mathrm{r}}(K_{1,m},K_{1,n})$.

Theorem 1.2.1. [5] For positive integers m and n,

$$\hat{\mathrm{r}}(K_m,K_n)=\binom{\mathrm{r}(K_m,K_n)}{2}.$$

Theorem 1.2.2. [5] For positive integers m and n,

$$\hat{r}(K_{1,m}, K_{1,n}) = n + m - 1.$$

Later Burr, Erdős, Faudree, Rousseau and Schelp generalized the result of $\hat{r}(K_{1,m}, K_{1,n})$.

Theorem 1.2.3. [6] For positive integers k, l, m and n,

$$\hat{r}(mK_{1,k}, nK_{1,l}) = (m + n - 1)(k + l - 1).$$

Given graphs F, $G_1, \cdots, G_t$, we write $F \to (G_1, \cdots, G_t)$ if in any $t \geq 2$ edge-coloring of F, F contains a monochromatic subgraph G_i of i-color$(1 \leq i \leq t)$. The smallest size of a graph F with $F \to (G_1, \cdots, G_t)$ denoted by $\hat{r}(G_1, \cdots, G_t)$. A more general result was proved by Zhang, who considered $t \geq 2$ kinds of color edge-coloring.

Theorem 1.2.4. [7] For positive integers m_i and n_i for $(1 \leq i \leq t)$ with $t \geq 2$,

$$\hat{r}(m_1K_{1,n_1}, m_2K_{1,n_2}, \cdots, m_tK_{1,n_t}) = \Big(\sum_{i=1}^{t}(m_i - 1) + 1\Big)\Big(\sum_{i=1}^{t}(n_i - 1) + 1\Big).$$

Note that $K_{1,2} \cong P_3$, Faudree and Sheehan obtained the exact value of $\hat{r}(K_{1,2}, K_n)$.

Theorem 1.2.5. [8] For positive integers $n \geq 2$,

$$\hat{r}(K_{1,2}, K_n) = 2(n - 1)^2.$$

Note that $S_{k,n}$ is a star-like graph, which is the graph obtained from a star $K_{1,n}$ by subdividing one of the edge k times. $S_{k,n}$ is also called a broom. Bielak obtained the exact value of $\hat{r}(S_{1,n}, S_{1,n})$.

Theorem 1.2.6. [9] For positive integers $n \geq 3$,

$$\hat{r}(S_{1,n}, S_{1,n}) = 4n - 2.$$

Faudree, Rousseau and Sheehan obtained the exact value of $\hat{r}(K_{1,n}, K_{2,m})$ and $\hat{r}(K_{1,n}, K_{2,2})$.

Theorem 1.2.7. [10] For positive integers $m \geq 9$, if n is sufficiently large, then

$$\hat{\mathrm{r}}(K_{1,n}, K_{2,m}) = 4n + 2m - 4.$$

For positive integers $n \geq 3$,

$$\hat{\mathrm{r}}(K_{1,n}, K_{2,2}) = 4n.$$

If $G = H$, we write $\hat{\mathrm{r}}(G, H)$ as $\hat{\mathrm{r}}(G)$. Beck obtained the upper bound of $\hat{\mathrm{r}}(P_n)$.

Theorem 1.2.8. [11]

$$\hat{\mathrm{r}}(P_n) \leq 900n.$$

The upper bound $900n$ was subsequently improved in [12, 13, 14, 15]. In 2017, Dudek and Pralat [16] improved the upper bound $\hat{\mathrm{r}}(P_n) \leq 74n$.

Beck obtained the upper bound of $\hat{\mathrm{r}}(T_n)$.

Theorem 1.2.9. [11] For a tree T with n edges, if n is sufficiently large, then

$$\hat{\mathrm{r}}(T_n) \leq \Delta(T)n(\log n)^{12}.$$

Friedman and Pippenger obtained another expression of the upper bound of $\hat{\mathrm{r}}(T_n)$.

Theorem 1.2.10. [17] For a tree T of order n,

$$\hat{\mathrm{r}}(T_n) \leq c(\Delta(T))^4 n,$$

where c is an absolute constant.

Ke improved the upper bound given by Friedman and Pippenger.

Theorem 1.2.11. [18] For a tree T of order n,

$$\hat{\mathrm{r}}(T_n) \leq c(\Delta(T))^2 n,$$

where c is an absolute constant.

Let T be a tree and have bipartition $V(T) = V_1 \cup V_2$. For $i = 1, 2$, let $t_i = |V_i|$ and $\Delta_i = \max\{\deg(v) : v \in V_i\}$. Further, let $\beta(T) = t_1\Delta_1 + t_2\Delta_2$. Beck obtained the lower and upper bound of $\hat{\mathrm{r}}(T_n)$.

Theorem 1.2.12. [19] For any tree T,

$$\frac{\beta(T)}{4} \leq \hat{\mathrm{r}}(T_n) \leq c\beta(T)(\log|T|)^{12},$$

where c is an absolute constant.

Haxell and Kohayakawa improved the upper bound given by Beck.

Theorem 1.2.13. [20] For any tree T,

$$\hat{\mathrm{r}}(T_n) \leq c\beta(T)(\log \Delta(T)),$$

where c is an absolute constant.

Let G_n be a graph with order n and $\Delta(G) = 3$. Rödl and Szemerédi obtained the lower bound of $\hat{\mathrm{r}}(G_n)$.

Theorem 1.2.14. [21] There exist positive integers c and α, and an infinite sequence of graphs G_n, where G_n is of order n and $\Delta(G_n) = 3$ such that

$$\hat{\mathrm{r}}(G_n) \geq cn(\log_2 n)^{\alpha}.$$

Erdős, Faudree, Rousseau and Schelp obtained the lower and upper bound of complete bipartite graph $\hat{\mathrm{r}}(K_{m,n})$.

Theorem 1.2.15. [5] For $m \geq 2$ and sufficiently large n.

$$e^{-1}m2^{m-1}n \leq \hat{\mathrm{r}}(K_{m,n}) \leq \frac{28}{9}m^2 2^{m-1}n.$$

Also, Erdős and Rousseau [22] obtained the lower and upper bound of balanced complete bipartite graph $\hat{\mathrm{r}}(K_{n,n})$ as follows.

$$\frac{1}{60}n^2 2^n \leq \hat{\mathrm{r}}(K_{n,n}) \leq \frac{3}{2}n^3 2^n.$$

For the size Ramsey number of some small graphs such as K_2, P_3, P_4, K_3, $K_{1,3}$, C_4, $K_{1,3}+e$, K_4-e, K_4 and others, see the survey by Faudree [23].

1.3 Research results

In this thesis, we focus the size Ramsey numbers involving double stars and brooms together.

In Chapter 2, we consider the size Ramsey numbers involving double stars and get the following results.

- $n+2m+1 \leq \hat{r}(P_3, D(m,n)) \leq n+2m+4$ for $n \geq m \geq 2$,
- $\hat{r}(P_3, D(n,n)) = 3n+4$ for $n \geq 5$,
- $\hat{r}(P_3, D(1,n)) = n+5$ for $n \geq 4$,
- $s(n+1)+m \leq \hat{r}(sP_2, D(m,n)) \leq s(n+m+1)$ for $n \geq m \geq 2$ and $s \geq 2$,
- $\hat{r}(2P_2, D(2,2)) = 10$,
- $\hat{r}(2P_2, D(1,n)) = 2n+4$ for $n \geq 3$, and
- $\hat{r}(sP_2, D(1,n)) = s(n+2)$ for $s \geq 2$ and $n \geq \lceil (s^2+3s-2)/2 \rceil$.

In Chapter 3, we consider the size Ramsey numbers involving brooms and get the following results.

- $n+\frac{3}{2}m \leq \hat{r}(P_3, B(m,n)) \leq n+2m$ for $m \geq 3, n \geq 1$,
- $\hat{r}(P_3, B(3,n)) = n+6$ for $n \geq 5$,
- $\hat{r}(P_3, B(4,n)) = n+8$ for $n \geq 7$,
- $\hat{r}(2P_2, B(m,n)) \leq 2n+2m-2$ for $m \geq 4, n \geq 1$,
- $\hat{r}(2P_2, B(m,n)) \geq 2n+m+2$ for $m \geq 3, n \geq m+2$,
- $\hat{r}(2P_2, B(3,n)) = 2n+5$ for $n \geq 5$, and
- $\hat{r}(2P_2, B(4,n)) = 2n+6$ for $n \geq 6$.

CHAPTER 2

SIZE RAMSEY NUMBERS INVOLVING DOUBLE STARS

Note that a double star $D(m, n)$ is a tree containing exactly two non-pendant vertices x and y with $\deg(x) = n + 1$ and $\deg(y) = m + 1$. Next, we give some essential parameters of $D(m, n)$. The order of $D(m, n)$, $v(D(m, n)) = n + m + 2$. The size of $D(m, n)$, $e(D(m, n)) = n + m + 1$. The maximum degree of $D(m, n)$, $\Delta(D(m, n)) = \max\{n + 1, m + 1\}$.

2.1 Size Ramsey numbers of P_3 versus double stars

In this section, we give some results on the size Ramsey numbers for 3-path versus double stars. At first, we have an upper and lower bound for $\hat{r}(P_3, D(m, n))$ as the following theorem. Before giving the following theorems and proofs, we first give a very important fact.

Fact 2.1.1. *To avoid red copy of P_3 in graph F, the red edges in F form a matching.*

Theorem 2.1.2. *For $n \geq m \geq 2$,*

$$n + 2m + 1 \leq \hat{r}(P_3, D(m, n)) \leq n + 2m + 4.$$

Proof. Let uvw be a path of order 3, and G be a graph obtained from uvw and three stars $K_{1,m+1}$, $K_{1,n}$ and $K_{1,m+1}$ by identifying the center of $K_{1,m+1}$ with u, the center of $K_{1,n}$ with v and the center of $K_{1,m+1}$ with w, which shown in the Figure 2.1.

Giving a red-blue edge-coloring of G, let R and B denote the red and blue subgraph, respectively. Suppose that R does not contain a P_3, by Fact 2.1.1, the red edges in F form a matching. Without loss of generality, assume $uv \in B$. Since the stars at u and w can only contain at most one red edge, we get a blue $D(m, n)$ contained at u and v, and hence $\hat{r}(P_3, D(m, n)) \leq e(G) = n + 2m + 4$.

To show $\hat{r}(P_3, D(m, n)) \geq n + 2m + 1$, we let F be a graph with at most $n + 2m$ edges. We can assume $e(F) = n + 2m$. It suffices to show that there exists a red-blue edge-coloring of F such that F contains neither a red copy of P_3 nor a blue copy of $D(m, n)$. If F contains no subgraphs isomorphic to $D(m, n)$, then we choose an edge e in F and color it red, and

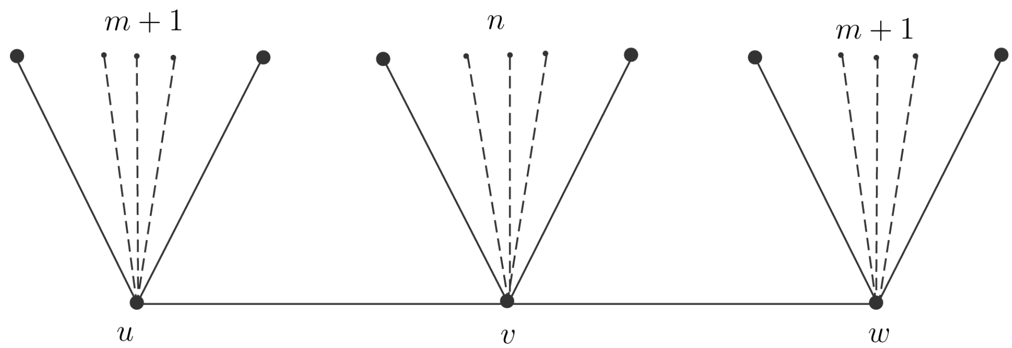

Figure 2.1. Upper Bound of P_3 versus $D(m, n)$

the edges in $F \setminus e$ are colored blue. One can see that F contains neither a red copy of P_3 nor a blue copy of $D(m, n)$ under such a red-blue edge-coloring. From now on, we assume that there exists a subgraph H of F such that $H \cong D(m, n)$. Let x, y be the n-center and m-center of H in F, respectively.

Claim 1. *If we color xy red and the other edges blue, then there is neither a red copy of P_3 nor a blue copy of $D(m, n)$.*

Proof. Clearly, there is no red copy of P_3 in F. Assume, to the contrary, that F contains a blue copy of $D(m, n)$, say H'. Let x', y' be the n-center and m-center of H' in F, respectively. Since $x'y'$ is colored by blue, it follows that $xy \neq x'y'$ and $H \neq H'$. If $x' \neq x$, then $e(H \cap H') \leq m + 1 \leq n + 1$, and hence $e(H \cup H') \geq 2(n + m + 1) - e(H \cap H') \geq n + 2m + 1 > n + 2m = e(F)$, a contradiction. If $x' = x$, then $e(H \cap H') \leq n + 1$, and hence $e(H \cup H') \geq 2(n + m + 1) - e(H \cap H') \geq n + 2m + 1 > n + 2m = e(F)$, a contradiction. □

From Claim 1, F contains neither a red copy of P_3 nor a blue copy of $D(m, n)$ under such edge-coloring, as desired. □

From Theorem 2.1.2, $3n + 1 \leq \hat{r}(P_3, D(n, n)) \leq 3n + 4$ if we take $m = n$. Furthermore, we will show $\hat{r}(P_3, D(n, n)) = 3n + 4$ for $n \geq 5$. For a maximum matching M of a graph F, we denote

$$V(M) = \{v \in V(F) \mid \deg(v) \geq n + 1 \text{ and } v \text{ covered by } M\}.$$

Then we have the following lemma.

Lemma 2.1.3. *Let F be a graph with $e(F) \leq 3n + 3$ and $\Delta(F) \geq n + 1$, where $n \geq 5$. Let M be a maximum matching of F such that $|V(M)|$ is maximized. For any two vertices u, v of degree at least $n + 1$, if $uv \in E(F)$, then M covers both u and v.*

Proof. If M covers neither u nor v, then $M \cup \{uv\}$ is also a matching of F, which contradicts the maximality of M. Suppose that M covers one of u and v. Without loss of generality, assume u is covered by M but v is not. Then there exists $w \in N(u) - v$ such that $uw \in M$. If $\deg(w) \leq n$, then a new matching $(M \setminus uw) \cup \{uv\}$ covers more vertices of degree at least $n+1$, a contradiction. From now on, we assume $\deg(w) \geq n+1$. Since $\deg(v) \geq n+1$ and $n \geq 5$, it follows that $|N(v) - \{u, w\}| \geq 4$. Choose four vertices x_1, x_2, x_3 and x_4 from $N(v) - \{u, w\}$. For each i ($1 \leq i \leq 4$), since $x_i v \notin M$ and M is a maximum matching, it follows that there exists $x_i' \in N(x_i)$ such that $x_i x_i' \in M$.

Let L be the set of edges incident to u or v or w. Noting that v is not covered by M. Then $L \cap M = \{uw\}$, and hence $x_i x_i' \notin L$ for each i ($i = 1, 2, 3, 4$). Let $L' = L \cup \{x_i x_i' \mid i = 1, 2, 3, 4\}$. Then $|L'| = |L| + 4 \geq 3n + 4 > 3n + 3 \geq e(F)$, a contradiction. □

Theorem 2.1.4. *For $n \geq 5$,*

$$\hat{\mathrm{r}}(P_3, D(n, n)) = 3n + 4.$$

Proof. By Theorem 2.1.2, we have $\hat{\mathrm{r}}(P_3, D(n, n)) \leq 3n+4$. To show $\hat{\mathrm{r}}(P_3, D(n, n)) \geq 3n+4$, we let F be a graph with at most $3n+3$ edges. Similarly to the proof of Theorem 2.1.2, we can assume that F contains a copy of $D(n, n)$. We will show that there is a red-blue coloring of the edges of F such that F contains neither a red copy of P_3 nor a blue copy of $D(n, n)$.

Let s denote the number of vertices of degree at least $n+2$ in F.

Claim 2. $s \geq 2$.

Proof. Let M be a maximum matching of F such that $|V(M)|$ is maximized. Color all the edges in M red, and the other edges are colored blue. Then F contains no red copy of P_3 clearly. Suppose F has a blue copy of $D(n, n)$, denoted by H, with centers u and v. Furthermore, $\deg_B(u) \geq n+1$ and $\deg_B(v) \geq n+1$. From Lemma 2.1.3, we have $\deg_R(u) \geq 1$ and $\deg_R(v) \geq 1$, and hence $\deg(u) \geq n+2$ and $\deg(v) \geq n+2$. So $s \geq 2$. □

Let t denote the number of vertices of degree $n+1$. Suppose that F is not connected. Clearly, F contains a subgraph $P_3, D(n, n)$ if and only if there exists a connected component C of F such that F contains a subgraph $P_3, D(n, n)$, respectively. We now assume F is also connected. By the choice of s and t, we have

$$2e(F) \geq (n+1)t + (n+2)s + (v(F) - s - t) = nt + (n+1)s + v(F),$$

Since $e(F) \leq 3n+3$, it follows that $nt + ns + s \leq 6n + 6 - v(F)$. Since F contains a copy of $D(n, n)$, it follows that $v(F) \geq v(D(n, n)) = 2n + 2$. From Claim 2, we have $s \geq 2$, and hence $ns + nt \leq 6n + 6 - (2n + 2) - 2 = 4n + 2$, and so $s + t \leq 4 + \frac{2}{n}$. Noting $n \geq 5$, we have $s + t \leq 4$. If $s + t = 2$, then the conclusion holds clearly, and thus we assume $s + t \geq 3$. Furthermore, we have

$$s + t = 3 \text{ or } 4, \quad \text{and } s \geq 2,$$

and hence $(t, s) = (1, 2),\ (2, 2),\ (0, 3),\ (1, 3)$ or $(0, 4)$.

Denote by W the set of vertices of degree at least $n + 1$. Then

$$\begin{aligned} e(F) &\geq e(W,\ V(F) - W) + e(W) = \left(\sum_{v \in W} \deg(v)\right) - e(W) \\ &\geq (n + 1)t + (n + 2)s - e(W). \end{aligned}$$

Since $e(F) \leq 3n + 3$, it follows that

$$(n + 1)t + (n + 2)s \leq 3n + 3 + e(W). \tag{2.1}$$

Claim 3. $(t, s) = (1, 2)$ *or* $(t, s) = (0, 3)$.

Proof. Assume, to the contrary, that $(t, s) = (2, 2),\ (0, 4)$ or $(1, 3)$, then $|W| = 4$ and $e(W) \leq 6$, thus by (2.1), we have $n \leq 3$, which contradicts to the fact $n \geq 5$. □

From Claim 3, $(t, s) = (1, 2)$ or $(t, s) = (0, 3)$. Then $|W| = 3$. Let $W = \{w_1, w_2, w_3\}$ with $\deg(w_1) \leq \deg(w_2) \leq \deg(w_3)$.

Case 1. $(t, s) = (1, 2)$.

In this case, $\deg(w_1) = n+1$, $\deg(w_2) \geq n+2$ and $\deg(w_3) \geq n+2$. Since $(t, s) = (1, 2)$, it follows that $e(W) \geq 2$ by (2.1).

Choose two edges, say e_1, e_2, from $F[W]$ and they have a common vertex in W. If w_2 is the common vertex, then $F[\{e_1, e_2\}]$ is 3-path $w_1w_2w_3$. Choose $w_1' \in N(w_1) - \{w_2, w_3\}$ and color w_2w_3, w_1w_1' red, and the other edges in F are colored blue. Since $\deg(w_1) = n + 1$, one can check that there is neither a red copy of P_3 nor a blue copy of $D(n, n)$ under this red-blue edge-coloring. The same is true if w_3 is the common vertex of e_1, e_2. If w_1 is the common vertex, then $F[\{e_1, e_2\}]$ is 3-path $w_2w_1w_3$. If $w_2w_3 \in E(F)$, then w_2 is the common vertex and we have done on this case. So we assume $w_2w_3 \notin E(F)$. Choose $w_1'' \in N(w_1)$. We now color w_1w_1'' red, and the other edges in F are colored blue. Noting $\deg(w_1) = n + 1$ and $w_2w_3 \notin E(F)$, one can check there is neither a red copy of P_3 nor a blue copy of $D(n, n)$ under this red-blue edge-coloring.

Case 2. $(t,s)=(0,3)$.

For each i $(i=1,2,3)$, we have $\deg(w_i)\geq n+2$.

Claim 4. *For each i $(i=1,2,3)$, $\deg(w_i)=n+2$.*

Proof. Assume, to the contrary, that one of w_1,w_2,w_3 is of degree at least $n+3$. Then $3n+3\geq e(F)\geq 3n+7-3=3n+4$, a contradiction. □

Since $(t,s)=(0,3)$, it follows that $e(W)=3$ by (2.1), and hence $F[W]$ is a 3-cycle. Since $n\geq 5$, we can choose $w_1'\in N(w_1)-\{w_2,w_3\}$, $w_2'\in N(w_2)-\{w_1,w_3,w_1'\}$ and $w_3'\in N(w_3)-\{w_1,w_2,w_1',w_2'\}$. We color the edges in $\{w_iw_i'\,|\,1\leq i\leq 3\}$ red, and the other edges in F are colored blue. Suppose there is a blue copy of $D(n,n)$, denoted by H. Then the centers of H belong to W, and assume w_1,w_2 are the two centers of H. Note that $|N_B(w_1)-\{w_2,w_3\}|=|N_B(w_2)-\{w_1,w_3\}|=n-1$ and w_3 is a common neighbor of w_1 and w_2, and so H is not a blue copy of $D(n,n)$. □

In the end of this section, we give the exact value of $\hat{r}(P_3,D(1,n))$ as the following theorem.

Theorem 2.1.5. *For $n\geq 4$,*

$$\hat{r}(P_3,D(1,n))=n+5.$$

Proof. We first give the upper bound $\hat{r}(P_3,D(1,n))\leq n+5$. Let G be a graph obtained from a star $K_{1,n}$ with center u and a K_4^- by identifying u and a vertex of degree 3 in K_4^-. Note that $e(F)=n+5$; see Figure 2.2.

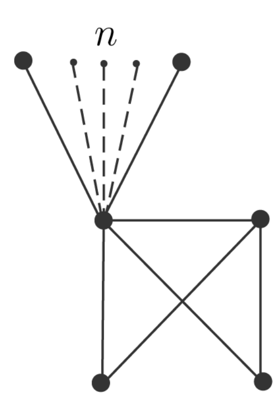

Figure 2.2. Upper Bound of P_3 versus $D(1,n)$

Giving a red-blue edge-coloring of G, let R and B denote the red and blue subgraph, respectively. Suppose that R does not contain a P_3, by Fact 2.1.1, the red edges in F form a matching. Since $\Delta(G) = n+3$, let $v \in V(G)$ such that $\deg(v) = n+3$. If the edges incident to vertex v are all blue, note that $e(G-v) = 2$, and in $G - v$ only one edge can be colored red, so there must be a $D(1,n)$ in B. Similarly, if an edge incident to vertex v is colored red, it is easy to check that there is a $D(1,n)$ in B, and hence $\hat{r}(P_3, D(1,n)) \leq e(G) = n+5$.

Next, we prove the lower bound $\hat{r}(P_3, D(1,n)) \geq n+5$. Let F be a graph of size at most $n+4$. We can assume $e(F) = n+4$. It suffices to show that there exists a red-blue edge-coloring of F such that F contains neither a red copy of P_3 nor a blue copy of $D(1,n)$. Similarly to the proof of Theorem 2.1.2, we can assume that F contains a copy of $D(1,n)$. Since P_3 and $D(1,n)$ is connected, we can assume F is connected.

Claim 5. *There is exactly one vertex of degree at least $n+1$ in F.*

Proof. Assume, to the contrary, that there are at least two vertices of degree at least $n+1$. Then $e(F) \geq 2n+1$, and hence $n \leq 3$, which contradicts to the fact $n \geq 4$. □

From Claim 5, there is the unique vertex u such that $\deg(u) \geq n+1$. If $\deg(u) = n+1$, then take one edge incident to u, color it red and color the other edges in F blue, which coloring is our desired coloring. Since $e(F) = n+4$, it follows that $\deg(u) \leq n+4$. If $\deg(u) = n+4$, then there is no copy of $D(1,n)$ in F, as desired. If $\deg(u) = n+3$, then there is exactly one edge not incident to u. We color the edge not incident to u red, and the other edges in F are colored blue. Then F contains neither a red copy of P_3 nor a blue copy of $D(1,n)$ under this coloring.

From now on, we assume $\deg(u) = n+2$. Denote by S the set of edges incident to u, and let $S' = E(F) \setminus S$. Then $|S| = n+2$ and $|S'| = 2$. Let $S' = \{e_1, e_2\}$. If e_1 and e_2 are not adjacent, then we color the edges in S blue, and e_1, e_2 are colored red. Then F contains neither a red copy of P_3 nor a blue copy of $D(1,n)$ under this coloring. Assume that e_1 and e_2 have a common vertex w. Let $e_1 = ww_1$ and $e_2 = ww_2$.

Since F is connected, $\{w, w_1, w_2\} \cap N[u] \neq \emptyset$. Since $S \cap S' = \emptyset$, we have $u \notin \{w, w_1, w_2\}$. Suppose that one of w, w_1, w_2 belongs to $N(u)$, say $w_1 \in N(u)$, but $w, w_2 \notin N(u)$. We color ww_1 red and the other edges in F are colored blue. This coloring is our desired coloring. Suppose that at least two of w, w_1, w_2 belong to $N(u)$. If $w_1, w_2 \in N(u)$ and $w \notin N(u)$, then we color uw_1 and ww_2 red, and the other edges in F are colored blue. Such a coloring is our desired coloring. If $w, w_1 \in N(u)$ and $w_2 \notin N(u)$, then we take an edge $e \in S$, color e and ww_2 red, and the other edges in F are colored blue. Note that the case of $w, w_2 \in N(u)$

and $w_1 \notin N(u)$ can be similarly proved. If $\{w, w_1, w_2\} \subseteq N(u)$, then we color uw_1 and ww_2 red and the other edges in F are colored blue, and hence the blue subgraph is a copy of $D(1, n-1)$, as desired. $\square$

2.2 Size Ramsey numbers of matchings versus double stars

In this section, we focus on size Ramsey numbers for matchings versus double stars. At first, we give an upper and lower bound for $\hat{r}(sP_2, D(m, n))$ as the following theorem. Before giving the following theorems and proofs, we first give a very important fact.

Fact 2.2.1. *To avoid red copy of $2P_2$ in graph F, the red edges in F can only be a star or a triangle.*

Theorem 2.2.2. *For $n \geq m \geq 2$ and $s \geq 2$,*

$$s(n+1) + m \leq \hat{r}(sP_2, D(m, n)) \leq s(n+m+1).$$

Proof. We first prove the upper bound. Let $G = sD(m, n)$. For any red-blue coloring of $E(G)$ without any red copy of sP_2, there is at least one blue copy of $D(m, n)$ clearly. Hence, $\hat{r}(sP_2, D(m, n)) \leq e(sD(m, n))) = s(n+m+1)$.

Next we give the lower bound $\hat{r}(sP_2, D(m, n)) \geq s(n+1) + m$. Let F be a graph of size at most $s(n+1) + m - 1$ and containing a copy of $D(m, n)$. Next by an induction on s, we will show that there is a red-blue coloring of $E(F)$ such that there is neither a red copy of sP_2 nor a blue copy of $D(m, n)$.

Assume $s = 2$. Then $e(F) \leq 2n + m + 1$. Since F contains a copy of $D(m, n)$, then $\Delta(F) \geq n + 1$, Take a vertex u of maximum degree, and color the edges incident to u red and the other edges in F blue. Then there is no red copy of $2P_2$ clearly. And the number of blue edges is at most $n + m$, thus there is no blue copy of $D(m, n)$ either. Suppose the conclusion holds for $s - 1$. Again, take a vertex u of maximum degree. Denote $F' = F - u$. Then $e(F') \leq (s-1)(n+1) + m - 1$. If F' has no copy of $D(m, n)$, then it is sufficient to color the edges incident to u red and color $E(F')$ blue. Assume F' contains a copy of $D(m, n)$. Then by inductive hypothesis, there is a red-blue coloring of $E(F')$ such that there is neither a red copy of $(s-1)P_2$ nor a blue copy of $D(m, n)$. Now keep this coloring, and further color the edges red incident to u, then we get the red-blue edge-coloring which we want. $\square$

Next we give some exact value of size Ramsey numbers for matchings versus double stars. At first, we show $\hat{r}(2P_2, D(2,2)) = 10$.

Theorem 2.2.3.

$$\hat{r}(2P_2, D(2,2)) = 10.$$

Proof. By Theorem 2.2.2, we have $\hat{r}(2P_2, D(2,2)) \leq 10$. Next we show $\hat{r}(2P_2, D(2,2)) \geq 10$. Let F be a graph of size 9 and containing a copy of $D(2,2)$. Then $\Delta(F) \geq 3$. If $\deg(w) \geq 5$, then $e(F-w) \leq 4$, and it is sufficient to color all edges incident to w red and color $E(F-w)$ blue. Then in this coloring, there is neither a red copy of $2P_2$ nor a blue copy of $D(2,2)$. Hence we can assume $\Delta(F) = \deg(w) = 4$ or 3, and then $e(F-w) = e(D(2,2))$ or $e(D(2,2)) + 1$. If $F - w$ contains no copy of $D(2,2)$, then it is sufficient to color all edges incident to w red and color $E(F-w)$ blue. Thus assume $F - w$ contains a copy of $D(2,2)$, denoted by H, with two centers u and v. Since $H \subseteq F - w$, then $w \neq u, v$.

Assume $\Delta(F) = \deg(w) = 4$. Then $e(F-w) = 5 = e(D(2,2))$. Thus $F - w = H$. It follows that the vertices of degree at least three in F exactly are u, v and w. Then it is sufficient to color $E(\{u,v,w\})$ red and the other edges in F blue.

Next assume $\Delta(F) = \deg(w) = 3$. Then $e(F \setminus w) = 6$ and thus there is the unique edge $e \in E(F-w) \setminus E(H)$. Since $\Delta(F) = 3$, then e is incident to none of u, v and w, and $u, v \notin N(w)$. Denote $N(w) = \{w_1, w_2, w_3\}$, $N(u) = \{v, u_1, u_2\}$ and $N(v) = \{u, v_1, v_2\}$. If $e = u_1u_2$ or v_1v_2, say $e = u_1u_2$, then color u_1u_2, u_1u and u_2u red. And noting $v \notin N(w)$, such coloring is what we want. If $e = u_iv_j$ for $i, j \in \{1,2\}$, say $e = u_1v_1$, then

- when $v_1 \notin N(w)$, it is sufficient to color the edges incident to u red and color the other edges in F blue;
- when $v_1 \in N(w)$, it is sufficient to color the edges incident to v red and color the other edges in F blue.

Thus assume e is incident to at most one vertex in $V(H)$.

If e is not incident to any vertex in $V(H)$, then w, u and v are the unique three vertices of degree at least 3 in F. Noting $u, v \notin N(w)$, it is sufficient to color uv red and the other edges in F blue. Thus assume e is incident to exact one vertex in $V(H)$. Considering e is not incident to u or v, we assume that e is incident to u_1 and not to any other vertex in $V(H)$. Then there are at most four vertices of degree 3 in F, which are w, u, v and u_1. It is sufficient to color the edges incident to u red and the other edges in F blue. Noting $u, v \notin N(w)$, such coloring is what we want. $\square$

Next consider $\hat{r}(sP_2, D(1,n))$. By an induction, we will show $\hat{r}(sP_2, D(1,n)) = s(n+2)$ for sufficiently large n. And the following theorem will be the inductive basis.

Theorem 2.2.4. *For $n \geq 3$,*

$$\hat{r}(2P_2, D(1,n)) = 2n + 4.$$

Proof. By Theorem 2.2.2, we have $\hat{r}(2P_2, D(1,n)) \leq 2n+4$. Next we show $\hat{r}(2P_2, D(1,n)) \geq 2n+4$. Let F be a graph of size $2n+3$ and containing a copy of $D(1,n)$. Take $w \in V(F)$ such that $\deg(w) = \Delta(F)$. Denote F' by the induced subgraph of the set of edges not incident to w. Since F contains a copy of $D(1,n)$, then $\Delta(F) \geq n+1$. If $\Delta(F) \geq n+2$, then $e(F') \leq n+1$ and F' has no copy of $D(1,n)$. It is sufficient to color the edges incident to w red and color the other edges in F blue. Thus assume that $\Delta(F) = n+1$. Then $e(F') = n+2 = e(D(1,n))$. If $F' \ncong D(1,n)$, then it is sufficient to color the edges incident to w red and color the other edges in F blue, again. Thus assume $F' \cong D(1,n)$. Let u and v be the n-center and 1-center, respectively. If $N(w) \cap V(F') = \emptyset$, then it is sufficient to color one edge of F' red and the other edges in F blue. Thus assume $N(w) \cap V(F') \neq \emptyset$. If $v \in N(w) \cap V(F')$, then it is sufficient to color the edges incident to v red and color the other edges in F blue. Thus assume $v \notin N(w) \cap V(F')$. If $u \in N(w) \cap V(F')$, then $\deg(u) \geq n+2$, which contradicts $\Delta(F') = n+1$. Thus assume $u \notin N(w) \cap V(F')$. If there exists $x \in N(w) \cap V(F')$ such that $x \in N_{F'}(u)$, then It is sufficient to color ux and wx red and color the other edges in F blue. Thus assume no neighbor of u belong to $N(w) \cap V(F')$.

Denote by v_1 the other neighbor of v than u in F'. Then $N(w) \cap V(F') = \{v_1\}$. It is sufficient to color vv_1 and wv_1 red and other edges in F blue. And such edge-coloring is what we want. □

Before to get $\hat{r}(sP_2, D(1,n)) = s(n+2)$, we prove the the following lemma first.

Lemma 2.2.5. *Assume F is a graph of size $s(n+2)-1$ where $s \geq 2$ and $n \geq \lceil (s^2+3s-2)/2 \rceil$. Denote by W the set of vertices of degree at least $n+1$ in F. If $|W| \geq s$, then $|W| = s$.*

Proof. Let $t = |W|$. If $t \geq n+3$, then $s(n+2) - 1 = e(F) \geq (n+1)t/2 \geq (n+1)(n+3)/2$, i.e.,

$$n^2 + (4-2s)n + (5-4s) \leq 0.$$

Thus

$$n \leq (s-2) + \sqrt{s^2-1} \leq 2s-2 = \frac{1}{2}[(s^2+3s-2) - (s^2-s+2)] < \frac{s^2+3s-2}{2},$$

which contradicts $n \geq \lceil (s^2+3s-2)/2 \rceil$. Thus $t \leq n+2$. And further,

$$e(F) \geq [2e(W) + e(W,\ V(F)-W)] - e(W) = \left(\sum_{x \in W} \deg(x)\right) - e(W) \geq t(n+1) - \frac{t(t-1)}{2}.$$

Noting $e(F) = s(n+2)-1$, we have

$$t(t-1) \geq 2(t-s)n + 2(t-2s+1). \tag{2.2}$$

Because $t \leq n+2$, we have

$$t(t-1) \geq 2(t-s)(t-2) + 2(t-2s+1).$$

That is

$$t[t-(2s+1)] + 2 \leq 0.$$

Then $t \leq 2s+1$.

Suppose $t \geq s+1$. Then $t \in [s+1,\ 2s+1]$. By (2.2), we have

$$n \leq \frac{t(t-1) - 2(t-2s+1)}{2(t-s)} = \frac{1}{2}\left[(t-s) + \frac{s^2+s-2}{t-s} + 2s - 3\right]. \tag{2.3}$$

Denote

$$f(t) := \frac{1}{2}\left[(t-s) + \frac{s^2+s-2}{t-s} + 2s - 3\right].$$

Since $t \in [s+1,\ 2s+1]$ and $s \geq 2$, we have

$$\begin{aligned}\max_{t \in [s+1,\ 2s+1]} f(t) &= \max\{f(s+1),\ f(2s+1)\}\\ &= \max\left\{\frac{s^2+3s-4}{2},\ 2s-1-\frac{2}{s+1}\right\} = \frac{s^2+3s-4}{2}.\end{aligned}$$

Then by (2.3), we have

$$n \leq f(t) \leq \frac{s^2+3s-4}{2}$$

But that contradicts $n \geq \lceil (s^2+3s-2)/2 \rceil > (s^2+3s-4)/2$. □

Theorem 2.2.6. *For $s \geq 2$ and $n \geq \lceil (s^2+3s-2)/2 \rceil$,*

$$\hat{\mathrm{r}}(sP_2, D(1,n)) = s(n+2).$$

Proof. We will use induction on s. When $s = 2$, by Theorem 2.2.4, we have $\hat{r}(2P_2, D(1, n)) = 2n+4$. Suppose $\hat{r}((s-1)P_2, D(1, n)) = (s-1)(n+2)$. Next we prove $\hat{r}(sP_2, D(1, n)) = s(n+2)$.

By Theorem 2.2.2, we have $\hat{r}(sP_2, D(1, n)) \leq s(n + 2)$. Next we show $\hat{r}(sP_2, D(1, n)) \geq s(n + 2)$. Let F be a graph of size $s(n + 2) - 1$ and containing a copy of $D(1, n)$. Then $\Delta(F) \geq n+1$. Denote by W the set of vertices of degree at least $n+1$ in F. If $|W| \leq s-1$, then it is sufficient to color the edges incident to w red for each $w \in W$ and color the other edges in F blue. Next assume $|W| \geq s$. By Lemma 2.2.5, we have $|W| = s$ and let $W = \{w_1, w_2, \ldots, w_s\}$.

Assume $\Delta(F) = n + 1$. If $w_1w_2 \in E(F)$ then it is sufficient to color w_1w_2 and the edges incident to w_i red for each $i \in [3, s]$ and color the other edges in F blue. Clearly, in such edge-coloring, there is neither a red copy of sP_2 nor a blue copy of $D(1, n)$. Thus assume $w_1w_2 \notin E(F)$, and symmetrically, $w_iw_j \notin E(F)$ for any $i, j \in [1, s]$. If there exists $u \in N(w_1) \cap N(w_2)$, then it is sufficient to color w_1u, uw_2 and the edges incident to w_i red for each $i \in [3, s]$ and color the other edges in F blue. Thus assume $N[w_1] \cap N[w_2] = \emptyset$, and symmetrically, $N[w_i] \cap N[w_j] = \emptyset$ for any $i, j \in [1, s]$. Thus $e(F - W) = s - 1$. Then color the edges incident to w_i blue for each $i \in [1, s]$ and color $F - W$ red. Since $e(F - W) = s - 1$, there is no red copy of sP_2 . Since $N[w_i] \cap N[w_j] = \emptyset$ for any $i, j \in [1, s]$, there is no blue copy of $D(1, n)$.

Assume $\Delta(F) \geq n + 2$ and $\deg(w_1) = \Delta(F)$. Denote $F' = F - \{w_1\}$. Then $e(F') \leq (s - 1)(n + 2) - 1$. By induction hypergraphs, there is a red-blue edge-coloring such that in F', there is neither a red copy of $(s - 1)P_2$ nor a blue copy of $D(1, n)$. And further color the edges incident to w_1 red. Then in the new red-blue edge-coloring, there is no red copy of sP_2 , and no blue copy of $D(1, n)$. □

CHAPTER 3

SIZE RAMSEY NUMBERS INVOLVING BROOMS

Note that a broom $B(m,n)$ is a tree obtained from a path P_{m+1} and a star $K_{1,n}$ by identifying an end-vertex of P_{m+1} with the center of $K_{1,n}$. Next, we give some essential parameters of $B(m,n)$. The order of $B(m,n)$, $v(B(m,n)) = n+m+1$. The size of $D(m,n)$, $e(B(m,n)) = n+m$. The maximum degree of $B(m,n)$, $\Delta(B(m,n)) = n+1$.

3.1 Size Ramsey numbers of P_3 versus brooms

In this section, we give some results on the size Ramsey numbers for 3-path versus brooms. At first, we have an upper and lower bound for $\hat{r}(P_3, D(m,n))$ as the following theorem.

Theorem 3.1.1. *For $m \geq 3$ and $n \geq 1$,*

$$n + \frac{3}{2}m \leq \hat{r}(P_3, B(m,n)) \leq n + 2m.$$

Proof. To show $\hat{r}(P_3, B(m,n)) \leq n+2m$, let G be a graph obtained from a $(m+1)$-path $P_{m+1} = v_1 \cdots v_{m+1}$ and a star $K_{1,n+1}$ with center u by identifying v_1 and u (that is $u = v_1$), then incident edges $v_i v_{i+2}$ for all $i \in \{1, \cdots, m-1\}$, which show in Figure 3.1.

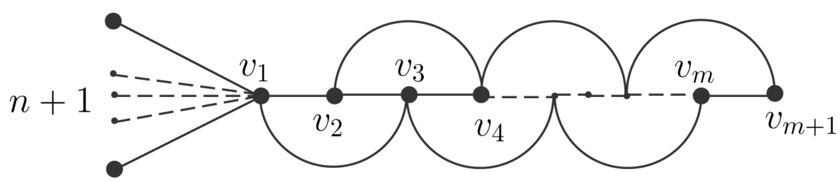

Figure 3.1. Upper Bound of P_3 versus $B(m,n)$

Giving a red-blue edge-coloring of G, let R and B denote the red and blue subgraph, respectively. Suppose that R does not contain a P_3, by Fact 2.1.1, the red edges in F form a matching. In star $K_{1,n+1}$, at most one edge is colored red, so there are at least n

blue edges in $K_{1,n+1}$. In path P_{m+1} with edges v_iv_{i+2} for all $i \in \{1, \cdots, m-1\}$, if any matching is deleted, there must be a path P_{m+1}. Thus, there is a $B(m,n)$ in B, and hence $\hat{r}(P_3, B(m,n)) \le e(G) = n + 2m$.

Next we show $\hat{r}(P_3, B(m,n)) \ge n + \frac{3}{2}m$. Let F be a graph with at most $n + \frac{3}{2}m - 1$ edges. We can assume that $e(F) = n + \frac{3}{2}m - 1$ and F containing a copy of $B(m,n)$. Since $B(m,n)$ contains a $(m+2)$-path copy, then we can color at least $\frac{m}{2}$ matchings in $B(m,n)$ red and color the other edges in F blue. Note that blue edges in F at most $n+m-1$, F contains no blue copy of $B(m,n)$, as desired. □

Next we give the exact value of $\hat{r}(P_3, B(3,n))$ and as $\hat{r}(P_3, B(3,n))$ the following theorems.

Theorem 3.1.2. *For $n \ge 5$,*

$$\hat{r}(P_3, B(3,n)) = n + 6.$$

Proof. By Theorem 3.1.1, we have $\hat{r}(P_3, B(3,n)) \le n+6$. Next we show $\hat{r}(P_3, B(3,n)) \ge n+6$. Let F be a graph with at most $n+5$. We can assume that $e(F) = n+5$. Let s be the number of vertices of degree $\ge n+1$.

Claim 6. $s = 1$.

Proof. Assume, to the contrary, that $s \ge 2$. Then there exist two vertices, say u_1, u_2, such that $\deg(u_i) \ge n+1$ for $i = 1, 2$, and hence there are at least $2(n+1) - 1 \ge n+6 > e(F)$, since $n \ge 5$, a contradiction. □

If $\Delta(F) \le n$, we choose an edge e in F and color it red, and the other edges in $F - e$ are colored blue. Then there is neither a red copy of P_3 nor a blue copy of $B(3,n)$. If $\Delta(F) = n+1$, we choose an edge e incident to v in F and color it red, and the other edges in F are colored blue. Then there is neither a red copy of P_3 nor a blue copy of $B(3,n)$.

Suppose $\Delta(F) = n+3+i$ $(i = 0, 1, 2)$. Then there exists a vertex v such that $\deg(v) = \Delta(F) = n+i$, and $e(F-v) = 2-i$. If $i = 1, 2$, then we color the edges incident to v red. If $i = 0$, then there are two edges in $F - v$ and color one of then red. Then we color the other edges in F blue. One can easily check that there is neither a red copy of P_3 nor a blue copy of $B(3,n)$, as desired.

Suppose $\Delta(F) = n+2$. Then there exists a vertex v such that $\deg(v) = \Delta(F) = n+2$, and $e(F-v) = 3$. If $F - v \ncong K_3$ and $F - v \ncong K_{1,3}$, then there is a $2P_2$ in $F - v$, and we color $2P_2$ red, and the other edges of F are colored blue. Then there is neither a red copy of P_3 nor a blue copy of $B(3,n)$.

Suppose $F - v \cong K_3$ or $F - v \cong K_{1,3}$. Let $N(v) = \{v_i \,|\, 1 \leq i \leq n+2\}$. The center vertex of $K_{1,3}$ is denoted as u, and the other three degree vertices in $K_{1,3}$ are denoted as u_1, u_2 and u_3. Since F is connected, $|N(v) \cap V(F-v)| \neq \emptyset$.

If $|N(v) \cap V(F-v)| = 1$, let $\{x\} = N(v) \cap V(F-v)$, then we color the edge vx red, the other edges in F blue. One can easily check that there is neither a red copy of P_3 nor a blue copy of $B(3,n)$.

Case 3.

$$F - v \cong K_3$$

Subcase 3.1. *If $|N(v) \cap V(F-v)| = 2$, then we color the edge vv_i ($v_i \in N(v)$ and $\deg(v_i) = 1$) red, the other edges in F blue. Then $v(F - v_i) = n+3 < v(B(3,n))$, so there is neither a red copy of P_3 nor a blue copy of $B(3,n)$.*

Subcase 3.2. *If $|N(v) \cap V(F-v)| = 3$, then $v(F) = n+3 < v(B(3,n))$, so $B(3,n) \nsubseteq F$.*

Case 4.

$$F - v \cong K_{1,3}$$

Subcase 4.1. *If $N(v) \cap V(F-v) = \{u, u_1\}$, then we color the edges uu_2 and vv_i ($v_i \in N(v)$ and $\deg(v_i) = 1$) red, the other edges in F blue. Then $v(F - u_2 - v_i) = n+3 < v(B(3,n))$, so there is neither a red copy of P_3 nor a blue copy of $B(3,n)$.*

Subcase 4.2. *If $N(v) \cap V(F-v) = \{u_1, u_2\}$, then we color the edges uu_3 and vv_i ($v_i \in N(v)$ and $\deg(v_i) = 1$) red, the other edges in F blue. Then $v(F - u_3 - v_i) = n+3 < v(B(3,n))$, so there is neither a red copy of P_3 nor a blue copy of $B(3,n)$.*

Subcase 4.3. *If $|N(v) \cap V(F-v)| = 3$, then we color the edge vv_i ($v_i \in N(v)$ and $\deg(v_i) = 1$) red, the other edges in F blue. Then $v(F - v_i) = n+3 < v(B(3,n))$, so there is neither a red copy of P_3 nor a blue copy of $B(3,n)$.*

Subcase 4.4. *If $|N(v) \cap V(F-v)| = 4$, then $v(F) = n+3 < v(B(3,n))$, so $B(3,n) \nsubseteq F$, as desired.*

$\square$

Theorem 3.1.3. *For $n \geq 7$,*

$$\hat{r}(P_3, B(4,n)) = n+8.$$

Proof. By Theorem 3.1.1, we have $\hat{r}(P_3, B(4,n)) \leq n{+}8$. Next we show $\hat{r}(P_3, B(4,n)) \geq n{+}8$. Let F be a graph with at most $n+7$. We can assume that $e(F) = n+7$. Let s be the number of vertices of degree $\geq n+1$.

Claim 7. $s = 1$.

Proof. Assume, to the contrary, that $s \geq 2$. Then there exist two vertices, say u_1, u_2, such that $\deg(u_i) \geq n+1$ for $i = 1, 2$, and hence there are at least $2(n+1) - 1 \geq n+8 > e(F)$, since $n \geq 7$, a contradiction. □

Suppose there exists a vertex v such that $\deg(v) = \Delta(F)$

If $\Delta(F) \leq n$, we choose an edge e in F and color it red, and the other edges in $F - e$ are colored blue. Then there is neither a red copy of P_3 nor a blue copy of $B(4,n)$. If $\Delta(F) = n+1$, we choose an edge e incident to v in F and color it red, and the edges in $F - e$ are colored blue. Then there is neither a red copy of P_3 nor a blue copy of $B(4,n)$.

Suppose $n+3 \leq \Delta(F) \leq n+7$. Then the number of edges that not incident to v is at most 4. Notice that $B(4,n) \subseteq F$ and has 3 edges that not incident to v. We can color red $2P_2$ in $F - v$, and the number of edges left in $F - v$ is at most 2. One can easily check that there is neither a red copy of P_3 nor a blue copy of $B(4,n)$, as desired.

Suppose $\Delta(F) = n+2$. Then there exists a vertex v such that $\deg(v) = \Delta(F) = n+2$, and $e(F - v) = 5$, denote $H = F - v$. Clearly, $P_4 \subseteq H$ and H has at least 2 matching. If the number of maximun matching of H is 3, then we can color red $3P_2$ in H and there is neither a red copy of P_3 nor a blue copy of $B(4,n)$.

In fact, there is only one graph with 5 edges that contains P_4 as subgraph and the number of maximum matching is 2, and delete any matching in H, there is always $P_4 \subseteq H$, which show in Figure 3.2.

If $|\{u_1, u_2, u_3, u_4\} \cap N(v)| = 1$, let $\{u_i\} = \{u_1, u_2, u_3, u_4\} \cap N(v)$, then we color the edge vu_i red, the other edges in F blue. One can easily check that there is neither a red copy of P_3 nor a blue copy of $B(4,n)$.

If $2 \leq |\{u_1, u_2, u_3, u_4\} \cap N(v)| \leq 4$, we color the edge vv_i ($v_i \in N(v)$ and $\deg(v_i) = 1$) red, the other edges in F blue. Then $v(F - v_i) \leq n+4 < v(B(4,n))$, so there is neither a red copy of P_3 nor a blue copy of $B(4,n)$. □

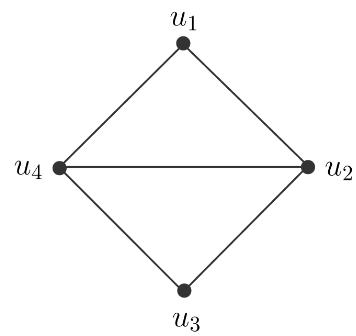

Figure 3.2. The Graph H

3.2 Size Ramsey numbers of matchings versus brooms

In this section, we focus on size Ramsey numbers for matchings versus brooms. At first, we give an upper and lower bound for $\hat{r}(2P_2, B(m,n))$ as the following theorem.

Theorem 3.2.1. *For $m \geq 4, n \geq 1$,*

$$\hat{r}(2P_2, B(m,n)) \leq 2n + 2m - 2.$$

For $m \geq 3, n \geq m+2$,

$$\hat{r}(2P_2, B(m,n)) \geq 2n + m + 2.$$

Proof. To show $\hat{r}(2P_2, B(m,n)) \leq 2n + 2m - 2$, let G be a graph obtained from a $2m$-cycle $C_{2m} = v_1v_2 \ldots v_{2m}v_1$ and two stars $K_{1,n-1}, K_{1,n-1}$ with centers u_1, u_2, respectively, by identifying u_1, v_1 and u_2, v_m (that is $u_1 = v_1$ and $u_2 = v_m$), which show in Figure 3.3.

Giving a red-blue edge-coloring of G, let R and B denote the red and blue subgraph, respectively. Suppose that R does not contain a $2P_2$, by Fact 2.2.1, the red edges in F form a star or a triangle. Since G is triangle-free, the red edges in F form a star. If the center of the red star is on the pendent vertices of G, then it is obvious that $B(m,n)$ is in B. Similarly, if the center of the red star is on the cycle of G, it is easy to check that there is a $B(m,n)$ in B, and hence $\hat{r}(2P_2, B(m,n)) \leq e(G) = 2n + 2m - 2$.

Next we show $\hat{r}(2P_2, B(m,n)) \geq 2n+m+2$. Let F be a graph with at most $2n+m+1$. We can assume that $e(F) = 2n + m + 1$. If $\Delta(F) \leq n$, we choose an edge e in F and color it red, and the other edges in $F - e$ are colored blue. Then there is neither a red copy of

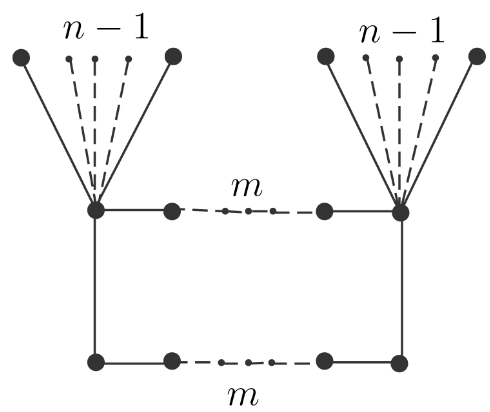

Figure 3.3. Upper Bound of $2P_2$ versus $B(m,n)$

$2P_2$ nor a blue copy of $B(m,n)$. If $\Delta(F) \geq n+2$, then there exists a vertex v such that $\deg(v) = \Delta(F) \geq n+2$. We color the edges incident to v red, and the other edges are colored blue. One can easily check that there is neither a red copy of $2P_2$ nor a blue copy of $B(m,n)$.

Suppose $\Delta(F) = n+1$. Let s be the number of vertices of degree $n+1$.

Claim 8. $s \leq 2$.

Proof. Assume, to the contrary, that $s \geq 3$. Then there exist three vertices, say u_1, u_2, u_3, such that $\deg(u_i) = \Delta(F) = n+1$ for $i = 1, 2, 3$, and hence there are at least $3(n+1) - 3 \geq 2n + m + 2 > e(F)$, since $n \geq m+2$, a contradiction. □

From Claim 8, we have $s \leq 2$. If $s = 1$, then there exists a vertex $v \in V(F)$ such that $\deg(v) = \Delta(F) = n+1$. We color the edges incident to v red, and then color the other edges blue. One can easily check that there is neither a red copy of $2P_2$ nor a blue copy of $B(m,n)$, as desired.

Suppose $s = 2$. Then there exist two vertices $u, v \in V(F)$ such that $\deg(u) = \deg(v) = \Delta(F) = n+1$. If $uv \in E(F)$, then we color uv red, and then color the other edges in F blue. Since $\Delta(F - uv) \leq n$, $F - uv$ contains no blue copy of $B(m,n)$ as its subgraph, as desired. Suppose that $uv \notin E(F)$ and $N(u) \cap N(v) \neq \emptyset$. Choose $w \in N(u) \cap N(v)$. We color the edges uw, vw red, and then color the other edges in F blue. It is clear that $\Delta(F - uw - vw) \leq n$, and hence $F - uw - vw$ contains no blue copy of $B(m,n)$ as its subgraph, as desired. Suppose that $uv \notin E(F)$ and $N(u) \cap N(v) = \emptyset$. If $F - u \ncong B(m,n)$, then we color the edges incident to u red, and then color the other edges in F blue. It is our desired edge-coloring. Similarly, we can get the desired edge-coloring if $F - v \ncong B(m,n)$. Thus, $F - u \cong B(m,n)$ and $F - v \cong B(m,n)$. Note that $e(F - u - v) = m - 1$, then these $m - 1$ edges must form a path, which show in Figure 3.4.

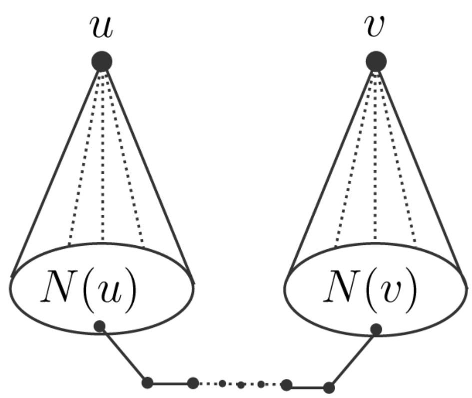

Figure 3.4. The Graph F

We color one edge from this path red, and then color the other edges in F blue. One can easily check that there is neither a red copy of $2P_2$ nor a blue copy of $B(m,n)$, as desired. □

The upper bound of $\hat{r}(2P_2, B(3,n))$ is some different, we need to construct a new graph to obtain the upper bound.

Corollary 1. *For $n \geq 5$,*

$$\hat{r}(2P_2, B(3,n)) = 2n + 5.$$

Proof. To show $\hat{r}(2P_2, B(3,n)) \leq 2n + 5$, let G be a graph obtained from a 6-cycle $v_1v_2, \cdots, v_6v_1$ and two stars $K_{1,n}, K_{1,n-1}$ with centers u_1, u_2, respectively, by identifying u_1, v_1 and u_2, v_3 (that is $u_1 = v_1$ and $u_2 = v_3$), which show in Figure 3.5.

Giving a red-blue edge-coloring of G, let R and B denote the red and blue subgraph, respectively. Suppose that R does not contain a $2P_2$, by Fact 2.2.1, the red edges in F form a star or a triangle. Since G is triangle-free, the red edges in F form a star. If the center of the red star is on the pendent vertices of G, then it is obvious that $B(3,n)$ is in B. If u_1 is the center of red star, since $G - u_1 \cong B(3,n)$, there is a $B(3,n)$ in B. If u_2 is the center of red star, there is also a $B(3,n)$ in B. Similarly, if the other vertices on the cycle of G are centers of the red stars, it is easy to check that there is a $B(3,n)$ in B, and hence $\hat{r}(2P_2, B(3,n)) \leq e(G) = 2n + 5$.

From Theorem 3.2.1, let $m = 3$ then we get $\hat{r}(2P_2, B(3,n)) \geq 2n + 5$ for $n \geq 5$. □

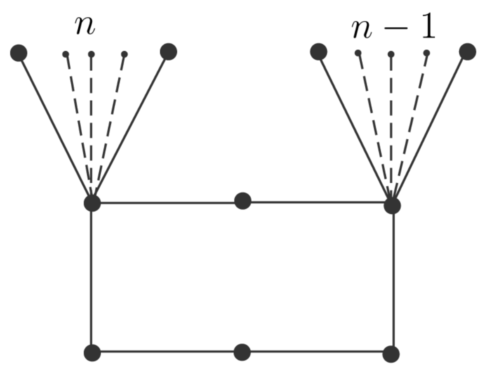

Figure 3.5. Upper Bound of $2P_2$ versus $B(3, n)$

Let $m = 4$, the exact value of $\hat{r}(2P_2, B(4, n))$ can get from Theorem 3.2.1 directly.

Corollary 2. *For $n \geq 6$,*

$$\hat{r}(2P_2, B(4, n)) = 2n + 6.$$

CHAPTER 4

SUMMARY AND DISCUSSION

4.1 Summary

This thesis mainly studies the size Ramsey numbers of P_3, matchings versus double stars and brooms, and get some exact values and upper and lower bounds.

In order to prove the upper bound, we need to construct a graph G such that G contains a red copy of P_3 (or matchings) or a blue copy of double star (or broom) under any red-blue edge-coloring of G.

It is much more difficult to prove the lower bound. Our aim is to give an edge-coloring for all graphs with fixed size, so that there is neither a red copy of P_3 (or matchings) nor a blue copy of double star (or broom).

4.2 Future work

In the following research, the following problems can still be considered.

- Improve the lower bound (or get the exact value) of $\hat{\mathrm{r}}(sP_2, D(m,n))$,
- Get the exact value of $\hat{\mathrm{r}}(P_3, D(m,n))$,
- Get the exact value of of $\hat{\mathrm{r}}(2P_2, B(m,n))$, and
- Get the exact value of $\hat{\mathrm{r}}(P_3, B(m,n))$.

Conjecture 4.2.1. *For $m \geq 5$ and $n \geq 2m-1$,*

$$\hat{\mathrm{r}}(P_3, B(m,n)) = n + 2m.$$

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