Speed Variation Tolerance in Single Lane Traffic

Sean Keeler
Department of Mathematics and Statistics, GSU

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Introduction

As more drivers use the roadways in the U.S. and abroad, traffic problems only become more important to understand and, hopefully, resolve. Among the long list of traffic problems, traffic jams are quite near the top. The "shockwave traffic" theory has emerged recently as a promising way to model the propagation of these traffic jams by describing them as waves.[**1**] In this theory, the shockwaves are mostly described by differences in speeds between groups of cars. One of the most interesting aspects of this theory is the idea that traffic jams are not necessarily caused by tight roadway, but by the interaction between different speed groups, a fact that is backed up by experimental evidence.[**3**]

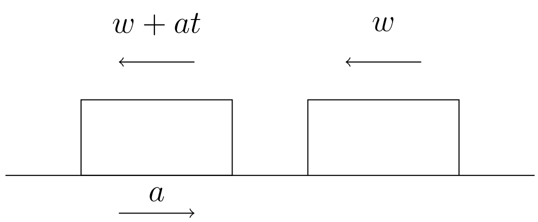
We must examine the interaction of two cars with different velocities. We do this by describing the motion of two cars having constant acceleration. We begin by examining the case where the cars have the same velocity.

Case 1: Same Accelerations

To start, we must define some parameters of the cars.

The perception-reaction time, r, is the time it takes for the second car to react to the motion of the first car.
The acceleration of the cars is denoted by a.
The initial and final velocities of the cars are w and u, respectively.
The initial distance between the back of the first car and front of the second car is denoted by d.
Naturally, w, u, d, and r must be greater than zero, a must be less than zero, and w must be greater than u.

We begin in the timeframe where the first car begun to slow down, but he second car has not.



In this first stage the time t is in between 0 and r. In other words, $0 \geq t > r$.
The relative velocity of the two cars is given by

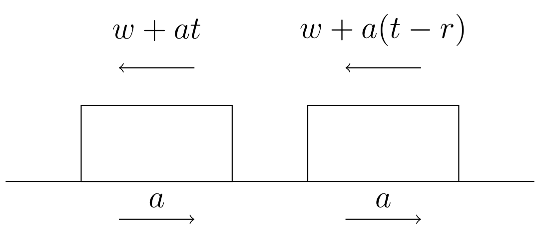
$$v_1(t) = (w + at) - w = at$$

Integrating from 0 to t and adding the initial following distance, we get the the relative distance,

$$s_1(t) = \int_0^t a\hat{t}\,\mathrm{d}\hat{t} + d = \frac{a}{2}t^2 + d$$

The next timeframe is where both the first and the second car are slowing down.

$$r \geq t > \frac{u - w}{a}$$



The relative velocity during this period is given by

$$v_2(t) = (w + at) - (w + a(t - r)) = ar$$

Integrating $v_2(t)$ from r to t and adding $s_1(r)$,

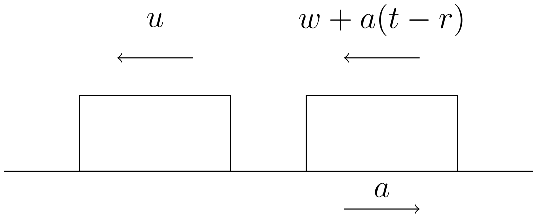
$$s_2(t) = \int_r^t ar\hat{t}\,\mathrm{d}\hat{t} + s_1(r) = d - \frac{ar^2}{2} + art$$

The third period occurs when the first car has slowed to velocity u and the second car is still slowing down.

$$\frac{u - w}{a} \geq t > \frac{u - w}{a} + r$$

The relative velocity during this period is given by

$$v_3(t) = u - (w + a(t - r)) = u - w - a(t - r)$$



Again, integrating this function from $\frac{u-w}{a}$ to t and adding $s_2(\frac{u-w}{a})$ gives us the distance between the cars.

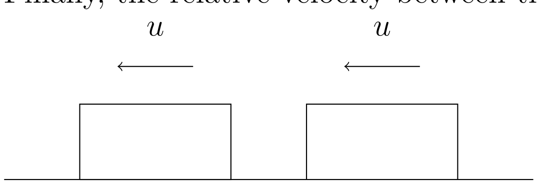
$$s_3(t) = \int_{\frac{u-w}{a}}^{t} u - w - a(\hat{t} - r)\,\mathrm{d}\hat{t} + s_2(\frac{u-w}{a})$$

$$= d - \frac{ar^2}{2} + ar(\frac{u-w}{a}) + (u-w)t - \frac{a}{2}t^2 + art - (\frac{(u-w)^2}{a} - \frac{a}{2}(\frac{u-w}{a})^2 + ar\frac{u-w}{a})$$

$$= d - \frac{ar^2}{2} + r(u-w) + (u-w)t - \frac{a}{2}t^2 + art - (\frac{(u-w)^2}{2a} + r(u-w))$$

$$= d - \frac{ar^2}{2} + (u-w)t - \frac{a}{2}t^2 + art - \frac{(u-w)^2}{2a}$$

Finally, the relative velocity between the two cars is 0.



$$v_f = u - u = 0$$

The final distance is given by $s_3(\frac{u-w}{a} + r)$.

$$s_f = s_3(\frac{u-w}{a} + r)$$

$$= d - \frac{ar^2}{2} + \frac{(u-w)^2}{a} + r(u-w) - \frac{a}{2}(\frac{u-w}{a} + r)^2 + r(u-w) + ar^2 - \frac{(u-w)^2}{2a}$$

$$= d + \frac{ar^2}{2} + \frac{(u-w)^2}{2a} - \frac{(u-w)^2}{2a} - r(u-w) - \frac{ar^2}{2} + 2r(u-w)$$

$$= d + r(u - w)$$
$$= d - r(w - u)$$

The quantity $w - u$ is the speed variation, which we will call Δv.

$$s_f = d - r\Delta v$$

In order for the cars to not crash, the final distance must be greater than zero.

$$s_f > 0$$
$$d - r\Delta v > 0$$
$$d > r\Delta v$$
$$\frac{d}{r} > \Delta v$$

This means that the ratio between the initial following distance and the perception-reaction time must be greater than the speed variation.

Before we discuss the implications of this result, we will perform a similar calculation for the case with different accelerations.

Case 2: Different Accelerations

In this case, parameters w, u, and r remain unchanged. We also define two new parameters, a_1 and a_2. These are the accelerations of the front and the back car, respectively. a_1 and a_2 must be negative.

Our timeframes are similar to the last calculation.

$$0 \geq t > r$$

Here, the relative velocity is given by

$$v_{1*} = (w + a_1 t) - w = a_1 t$$

Integrating from 0 to t and adding the initial following distance, we get the the relative distance,

$$s_{1*}(t) = \int_0^t a_1 \hat{t}\, \mathrm{d}\hat{t} + d = d + \frac{a_1}{2} t^2$$

The next timeframe has both cars slowing down.

$$r \geq t > \frac{u-w}{a_1}$$

In this period, the relative velocity is given by

$$v_{2*} = (w + a_1 t) - (w - a_2(t-r)) = a_1 t - a_2(t-r)$$

Once more, we integrate v_{2*} from r to t and add the previous distance at r.

$$s_{2*} = \int_r^t a_1\hat{t} - a_2(\hat{t}-r)\,\mathrm{d}\hat{t} + d + \frac{a_1 r^2}{2} = \frac{a_1}{2}t^2 - \frac{a_2}{2}t^2 + a_2 r t - (\frac{a_1 r^2}{2} - \frac{a_2 r^2}{2} + a_2 r^2) + d + \frac{a_1 r^2}{2}$$

$$= d - \frac{a_2 r^2}{2} + \frac{a_1}{2}t^2 - \frac{a_2}{2}t^2 + a_2 r t$$

The next period has the first car at the final velocity u while the second car is still slowing down.

$$\frac{u-w}{a_1} \geq t > \frac{u-w}{a_2} + r$$

Here, the velocity function is given by

$$v_{3*}(t) = u - (w + a_2(t-r))$$

Again, we integrate this velocity from $\frac{u-w}{a_1}$ to t and add the previous position function at $\frac{u-w}{a_1}$.

$$s_{3*}(t) = \int_{\frac{u-w}{a_1}}^t (u-w) - a_2\hat{t} + a_2 r\,\mathrm{d}\hat{t} + s_{2*}(\frac{u-w}{a_1})$$

$$= (u-w)t - \frac{a_2}{2}t^2 + a_2 r t - (\frac{(u-w)^2}{a_1} - \frac{a_2}{2} * \frac{(u-w)^2}{a_1^2} + a_2 r\frac{u-w}{a_1})$$

$$+ d - \frac{a_2 r^2}{2} + \frac{a_1}{2} * \frac{(u-w)^2}{a_1^2} - \frac{a_2}{2} * \frac{(u-w)^2}{a_1^2} + a_2 r\frac{u-w}{a_1}$$

$$= d - \frac{a_2 r^2}{2} - \frac{(u-w)^2}{2a_1} - \frac{a_2}{2}t^2 + a_2 r t + (u-w)t$$

The final velocity is zero, as both cars are at velocity u.
The final distance is simply $s_{3*}(\frac{u-w}{a_2} + r)$.

$$s_{f*} = s_{3*}(\frac{u-w}{a_2} + r)$$

$$s_{f*} = d - \frac{a_2 r^2}{2} - \frac{(u-w)^2}{2a_1} - \frac{a_2}{2}(\frac{u-w}{a_2} + r)^2 + a_2 r(\frac{u-w}{a_2} + r) + (u-w)(\frac{u-w}{a_2} + r)$$

$$s_{f*} = d + r(u-w) + \frac{(u-w)^2}{2a_2} - \frac{(u-w)^2}{2a_1}$$

$$s_{f*} = d - r\Delta v + \frac{(\Delta v)^2}{2a_2} - \frac{(\Delta v)^2}{2a_1}$$

If we set $a_1 = a_2$, then we recover s_f.

$$s_{f*} = d - r\Delta v + \frac{(\Delta v)^2}{2a} - \frac{(\Delta v)^2}{2a} = d - r\Delta v = s_f$$

Just like in the last case, we much find a condition based on the speed variation that prevents collisions.

$$s_{f*} > 0$$

$$d - r\Delta v + \frac{(\Delta v)^2}{2a_2} - \frac{(\Delta v)^2}{2a_1} > 0$$

Clearly, this relation is quadratic in Δv.// If $a_2 > a_1$, then

$$\frac{r - \sqrt{r^2 - 2d(\frac{1}{a_2} - \frac{1}{a_1})}}{\frac{1}{a_2} - \frac{1}{a_1}} < \Delta v$$

If $a_1 > a_2$, then

$$\frac{r + \sqrt{r^2 - 2d(\frac{1}{a_2} - \frac{1}{a_1})}}{\frac{1}{a_2} - \frac{1}{a_1}} < \Delta v$$

These relations allow for more precision in calculations. However, the simplicity of the equation not relying on accelerations makes it easy to apply in a variety of situations, with less reliance on experimental data. For this reason, we continue by examining a variety of scenarios using the distance\reaction-time relation.

Traffic Scenario 1

On average, the perception-reaction time is not going to change very much. For this reason, we initially assume the value of r to be 1 second. This is close to the reported average value of 1.01 seconds, and simplifies our calculations greatly.[**2**]

Next, we imagine a line of cars with a uniform speed. The car in the front of the line decreases its speed by 5 m/s. This means that the distance between adjacent cars must be at least 5 meters.

An interesting property in this scenario is that the specific speed of the cars does not matter, as long as it is initially uniform. This indicates that there is no limit on the speed, minimum or maximum, that these cars can travel in. A car traveling in a line with a speed near the speed of light would need the same distance to account of any given change in speed as a line of cars moving at a near stop.

Traffic Scenario 2

Again, we assume the value of r to be 1 second.

Perhaps a more realistic scenario is where the only information we have is the density of traffic. If we have the average density of traffic in a given area, then

$$\frac{1}{\rho} - L = d$$

where ρ is the density of traffic and L is the average length of cars.

Let's assume L to be 3 meters. Let's assume that there is 1 car every 10 meters, for a ρ value of .01 cars/meter.

$$\frac{1}{0.1} - 3 > \Delta v$$

$$7 > \Delta v$$

In this case, the change in speed can be up to 7 m/s.

Traffic Scenario 3

Here, we consider a decreasing perception-reaction time. This case comes into play in a case where someone becomes distracted. Let us use the distance between cars of 5 meters from the first scenario. As discussed in the first scenario, this distance can tolerate a change in speed of 5m/s.

If one driver in this line decides to, say, look at their phone, then their perception-reaction time will increase. Doubling this time cuts the tolerance in half; that is, at the current distance that the cars are apart from each other, the maximum change in speed allowed to prevent a crash is 2.5m/s.

If this change were to happen in the middle of a traffic situation with changing speeds, it is easy to see that becoming distracted can fairly directly lead to a crash.

Conclusion

Traffic issues continue to plague roadways around the world. Among these issues are those of traffic jams during times of high traffic flow. By analyzing the required distance needed between cars to allow for a certain speed variation, it becomes possible prevent some of these traffic accidents and decrease the travel time of everyone in the group. Further research in this area should provide further insights to prevent traffic jams.

Furthermore, the density of traffic, and not the speed of traffic, determines the tolerate speed variance. The utility of the equations found here allows for application to a large number of scenarios.

Lastly, increasing the amount of time it takes any given person to react to a speed change greatly decreases the speed variation tolerance. This indicates that work must be done to prevent distractions for drivers.

References

[1]Lu, X., & Skabardonis, A. (2006). Freeway Traffic Shockwave Analysis: Exploring the NGSIM Trajectory Data. Transportation Research Board, 86, 1-18.

[2]Olson, P. L., & Sivak, M. (1986). Perception-Response Time to Unexpected Roadway Hazards. Human Factors, 28(1), 91-96.

[3]Sugiyama, Y., Fukui, M., Kikuchi, M., Hasebe, K., Nakayama, A., Nishinari, K., Tadaki, S., & Yukawa, S. (2008). Traffic jams without bottlenecks?experimental evidence for the physical mechanism of the formation of a jam. New Journal of Physics New J. Phys., 10(3), 033001.