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Intersection Algebras and Pointed Rational Cones

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INTERSECTION ALGEBRAS AND POINTED RATIONAL CONES

by

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Under the Direction of Florian Enescu

ABSTRACT

In this dissertation we study the algebraic properties of the intersection algebra of two ideals I and J in a Noetherian ring R. A major part of the dissertation is devoted to the finite generation of these algebras and developing methods of obtaining their generators when the algebra is finitely generated.

We prove that the intersection algebra is a finitely generated R-algebra when R is a Unique Factorization Domain and the two ideals are principal, and use fans of cones to find the algebra generators. This is done in Chapter 2, which concludes with introducing a new class of algebras called fan algebras.

Chapter 3 deals with the intersection algebra of principal monomial ideals in a polynomial ring, where the theory of semigroup rings and toric ideals can be used. A detailed investigation of the intersection algebra of the polynomial ring in one variable is obtained. The intersection algebra in this case is connected to semigroup rings associated to systems of linear diophantine equations with integer coefficients, introduced by Stanley.

In Chapter 4, we present a method for obtaining the generators of the intersection algebra for arbitrary monomial ideals in the polynomial ring.

INDEX WORDS: Commutative algebra, Semigroup rings, Fan algebras

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ON THE INTERSECTION ALGEBRA OF PRINCIPAL IDEALS

by

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DEDICATION

This dissertation is dedicated to my husband and the rest of my family, without whose patience and encouragement it would not have been possible.

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CHAPTER 1

INTRODUCTION

1.1 Preliminaries

In this thesis, we study the intersection of powers of two ideals in a commutative Noetherian ring. This is achieved by looking at the structure called the intersection algebra, a recent concept, which is associated to the two ideals.

The purpose of this dissertation is to study the finite generation of this algebra, and to show that it holds in a few significant cases, namely principal ideals in a UFD and monomial ideals in a polynomial ring.

In the general case, not much is known about the intersection algebra, and there are many questions that can be asked. Various aspects of the intersection algebra have been studied by J. B. Fields in [1, 2]. There, he proved several interesting things, including the finite generation of the intersection algebra of two monomial ideals in the power series ring over a field. He also studied the relationship between the finite generation of the intersection algebra and the polynomial behavior of a certain function involving lengths of Tors. It is interesting to note that this algebra is not always finitely generated, as shown by Fields.

The finite generation of the intersection algebra has also appeared in the work of Ciupercă, Enescu, and Spiroff in [3] in the context of asymptotic growth powers of ideals.

We will start with the definition of the intersection algebra. Throughout this dissertation, R will be a commutative Noetherian ring.

Definition 1.1.1. Let R be a ring with two ideals I and J. Then the *intersection algebra of I and J* is $\mathcal{B} = \bigoplus_{r,s\in\mathbb{N}} I^r \cap J^s$. If we introduce two indexing variables u and v, then $\mathcal{B}_R(I,J) = \sum_{r,s\in\mathbb{N}} I^r \cap J^s u^r v^s \subseteq R[u,v]$. When R, I and J are clear from context, we will simply denote this as $\mathcal{B}$. We will often think of $\mathcal{B}$ as a subring of $R[u,v]$, where there is a natural $\mathbb{N}^2$-grading on monomials $b \in \mathcal{B}$ given by $\deg(b) = (r,s) \in \mathbb{N}^2$. If this algebra is finitely generated over R, we say that I and J have *finite intersection algebra.*

Example 1.1.2. If $R = \mathbb{R}[x,y]$, $I = (x^2y)$ and $J = (xy^3)$, then an example of an element in $\mathcal{B}$ is $2x + 3x^5y^9u^2v^3 + x^{10}y^{15}u^4v$, since $2 \in I^0 \cap J^0 = \mathbb{R}[x,y]$, $x^5y^9u^2v^3 \in I^2 \cap J^3u^2v^3 = (x^4y^2)\cap(x^3y^9)u^2v^3 = (x^4y^9)u^2v^3$, and $x^{10}y^{15}u^4v \in I^4 \cap Ju^4v = (x^8y^4)\cap(xy^3)u^4v = (x^8y^4)u^4v$.

We remark that the intersection algebra has connections to the double Rees algebra $R[Iu, Jv]$, although in practice they can be very different. This relationship is significant due to the importance of the Rees algebra, but the two objects behave differently. The source for the different behavior lies in the obvious fact that the intersection $I^r \cap J^s$ is harder to predict than I^rJ^s as r and s vary. These differences in behavior are of great interest and should be further explored.

1.2 Semigroups

This thesis relies heavily on semigroup theory. A number of definitions and results are given below that we will reference later on. The following results come from [4] and [5].

Definition 1.2.1. A *semigroup* is a set together with a closed associative binary operation. A semigroup generalizes a monoid in that it need not contain an identity element. We call a semigroup an *affine semigroup* if it is finitely generated and isomorphic to a subsemigroup of $\mathbb{Z}^d$ for some d. An affine semigroup is called *pointed* if it contains the identity, which is the only invertible element of the semigroup.

We will be dealing with a special class of semigroups called polyhedral cones.

Definition 1.2.2. A *polyhedral cone* C in $\mathbb{R}^d$ is the intersection of finitely many closed linear half-spaces in $\mathbb{R}^d$, each of whose bounding hyperplanes contains the origin. A hyperplane H containing the origin is called a *supporting hyperplane* if $H \cap C \neq 0$ and C is contained in one of the closed half-spaces determined by H. If H is a supporting hyperplane of C, then $H \cap C$ is called a *face* of C. Every polyhedral cone C is finitely generated, i.e. there exist $\boldsymbol{c_1}, \dots, \boldsymbol{c_r} \in \mathbb{R}^d$ with

$$C = \{\lambda_1 \boldsymbol{c_1} + \cdots + \lambda_n \boldsymbol{c_n} | \lambda_1, \dots, \lambda_r \in \mathbb{R}_{\geq 0}\}.$$

We call the cone C *rational* if $\boldsymbol{c_1}, \dots, \boldsymbol{c_r}$ can be chosen to have rational coordinates, and C is *pointed* if $C \cap (-C) = \{\mathbf{0}\}$.

A special kind of collection of cones are called fans.

Definition 1.2.3. A *fan* is a collection Σ of cones $\{C_i\}_{i \in I}$, where I is a finite set, the faces of each $C_i \in \Sigma$ are also in Σ, and the intersection of every pair of two cones in Σ is a common face of both of them. Note that the empty set is considered a face of any cone.

One major property of these cones that we will use is that they are finitely generated.

Theorem 1.2.4. (Proposition 7.15 in [4]) *Any pointed affine semigroup Q has a unique finite minimal generating set $\mathcal{H}_Q$.*

Theorem 1.2.5. (Theorem 7.16 in [4]) (Gordan's Lemma) *If C is a rational cone in $\mathbb{R}^d$, then $C \cap A$ is an affine semigroup for any subgroup A of $\mathbb{Z}^d$.*

Definition 1.2.6. Let C be a rational pointed cone in $\mathbb{R}^d$, and let $Q = C \cap \mathbb{Z}^d$. Then the unique finite minimal generating set $\mathcal{H}_Q$ is called the *Hilbert Basis of the cone C*.

The next few definitions and theorems will be used to prove Fields' result in the next section of this introduction.

Definition 1.2.7. Let $Y \subset N^n$, and let $\leq$ be the usual partial order on $\mathbb{N}^n$. Define the set $\text{Minimals}_{\leq}(Y)$ to be the minimal set of elements of Y with respect to the order $\leq$.

Theorem 1.2.8. (Theorem 5.1 in [5]) (Dickson's Lemma) *Let N be a nonempty subset of $\mathbb{N}^n$. The set $M = Minimals_{\leq}(N)$ has finitely many elements.*

Proof. We use induction on n. For $n = 1$, the result follows easily from the fact that $\leq$ is a well order on $\mathbb{N}$. Assume that the statement is true for $n - 1$. Choose an element $(a_1, \ldots, a_n) \in M$. For each $1 \leq i \leq n$ and each $0 \leq j \leq a_i$ define

$$M_{ij} = \{(x_1, \ldots, x_n) \in M | x_i = j\}$$

and

$$B_{ij} = \{(x_1, \ldots, x_{n-1}) \in \mathbb{N}^{n-1} | (x_1, \ldots, x_{i-1}, j, x_i, \ldots, x_{n-1}) \in M_{ij}\}.$$

Observe that $\text{Minimals}_{\leq}(M) = M$ and for this reason $\text{Minimals}_{\leq}(M_{ij}) = M_{ij}$ and $\text{Minimals}_{\leq}(B_{ij}) = B_{ij}$. By the induction hypothesis, B_{ij} must be finite. Hence, M_{ij} is finite as well. Since there are finitely many sets M_{ij}, the set $\bigcup M_{ij}$ is again finite and nonempty. Hence it is enough to show that $M \subseteq \bigcup M_{ij}$. Take $(x_1, \ldots, x_n)$ to be an element in M. There exists $i \in \{1, \ldots, n\}$ such that $x_i \leq a_i$ (if this were not the case, $(a_1, \ldots, a_n) < (x_1, \ldots, x_n)$ and this is impossible, since $(x_1, \ldots, x_n)$ is a minimal element of N). Hence $(x_1, \ldots, x_n) \in M_{ix_i}$. □

Proposition 1.2.9. (Corollary 5.3 in [5]) *Let $\{x_i | i \in I\}$, $I \subseteq \mathbb{N}$ be a nonempty subset of $\mathbb{N}^n$ such that $x_j < x_i$ whenever $i < j$. Then I has finitely many elements.*

Proof. Clearly, the set

$$A = \{y | y \leq x_1\}$$

is finite and nonempty. Observe that $\{x_i | i \in I\} \subseteq A$. Thus I is finite, since all elements in $\{x_i | i \in I\}$ are different. □

Corollary 1.2.10. (Corollary 5.4 in [5]) *Let N be a nonempty subset of $\mathbb{N}^n$ and let $M = Minimals_{\leq}(N)$. Then for every $x \in N$ there exists $m \in M$ such that $m \leq x$.*

Proof. Let $x \in N$. If $x \in M$, the proof is trivial. Assume that $x \in N \setminus M$. By the definition of minimal element, there exists an element $x_1 \in N$ such that $x_1 < x$. If $x_1 \in M$, then we are done. Otherwise, there exists $x_2 \in N$ such that $x_2 < x_1 < x$. By Proposition 1.2.9, this process stops, and it does so when there is an i such that $x_i \in M$. Since by transitivity $x_i < x$, we obtain the desired result. □

Theorem 1.2.11. (Introduction to Chapter 7 in [5]) *Let A be an $m \times n$ matrix with integer entries. Define*

$$S = \{x \in \mathbb{N}^m | Ax = 0\},$$

and let $G(S)$ be the group generated by S. (Observe that $0 \in S$, and that if $x, y \in S$, then so is $x + y$. Thus S is a submonoid of $\mathbb{N}^k$.) Then

1. *$G(S) \cap \mathbb{N}^n = S$.*

2. *S is an affine semigroup.*

Proof. If $x, y \in S$, then $A(x - y) = Ax - Ay = 0$. So if $x - y \geq 0$, then $x - y \in S$, so $G(S) \cap \mathbb{N}^m = S$.

To show 2, note that by Theorem 1.2.8, the set $\text{Minimals}_{\leq}(S \setminus \{0\})$ is finite. Let

$$M = \text{Minimals}_{\leq}(S \setminus \{0\}) = \{m_1, \ldots, m_t\}.$$

Take x to be an element of $S \setminus \{0\}$. If $x \notin M$, then by Corollary 1.2.10 there exists $m_{i_1} \in N$ such that $m_{i_1} < x$. Define $x_1 = x - m_{i_1}$, which belongs to $G(M) \cap \mathbb{N}^n = S$. Once more, we check whether $x_1 \in M$. If not, there must be another element $s_{i_2} \in M$ such that $s_{i_2} < x_1$. Set $x_2 = x_1 - s_{i_2}$. By Corollary 1.2.9, this process must stop in a finite number of steps. In other words, there exists $k \in \mathbb{N}$ such that $x_k = s_{i_k} \in M$ and this leads to $x = \sum_{i=1}^{k} s_{i_j}$. Therefore S is an affine semigroup. □

1.3 Bruce Fields' Work

Bruce Fields proved the following result in his thesis [1], which is reproduced below for the convenience of the reader. First, a necessary lemma.

Lemma 1.3.1. if Q_1 and Q_2 are finitely generated subsemigroups of $\mathbb{N}^n$, then so is $Q = Q_1 \cap Q_2$.

Proof. Let $\alpha_{\mathbf{i},\mathbf{1}}, \dots, \alpha_{\mathbf{i},\mathbf{e_i}} \in \mathbb{N}^n$ generate Q_i. Define a new semigroup $Q' \subset \mathbb{N}^{n+e_1+e_2}$ as the set of all $(a_1, \dots, a_n, b_{1,1}, \dots, b_{1,e_1}, \dots, b_{2,e_2})$ satisfying the $2m$ equations

$$(a_1, \dots, a_n) = b_{1,1}\alpha_{1,1} + \cdots + b_{1,e_1}\alpha_{1,e_1}$$

$$(a_1, \dots, a_n) = b_{2,1}\alpha_{2,1} + \cdots + b_{2,e_2}\alpha_{2,e_2}$$

Since $Q \subset \mathbb{N}^n$ is the set of solutions to a finite set of $\mathbb{Z}$-linear equations, then Q is a finitely generated semigroup by 1.2.11. So in this case, Q' is finitely generated, and it is clear that $Q = Q_1 \cap Q_2$ is the image of Q' under the map that projects $\mathbb{N}^{n+e_1+e_2}$ onto the first n coordinates. So Q is finitely generated.

□

Theorem 1.3.2. *Let R be a Noetherian ring, and I and J monomial ideals in $A = R[x_1, \dots, x_n]$. Then I and J have finite intersection algebra.*

Proof. Let B be a sub-R-algebra of A generated by monomials in $x_1, \dots, x_n$. Then B is also a sub-k-module of A, and is generated over k by those same monomials, together with $x_1, \dots, x_n$. The set of exponents $\alpha \in \mathbb{N}^n$ of the monomials $\mathbf{x}^\alpha$ that generate B as a module over k form a subsemigroup of $\mathbb{N}^n$. Call that semigroup Q. Then we claim that Q is a finitely generated semigroup if and only if B is a finitely generated R-algebra.

If Q is finitely generated, say by $\{\alpha_{\mathbf{1}}, \dots, \alpha_{\mathbf{d}}\}$, then for any $\alpha \in Q$, $\alpha = \sum n_i \alpha_{\mathbf{i}}$, with $n_i \in \mathbb{N}$. But this means that, for $r \in R$, $r\mathbf{x}^\alpha = r\mathbf{x}^{\sum n_i \alpha_{\mathbf{i}}} = r\mathbf{x}^{n_1\alpha_{\mathbf{1}}} \cdots \mathbf{x}^{n_d\alpha_{\mathbf{d}}}$, so B is finitely generated as an R-algebra by the $\mathbf{x}^{\alpha_{\mathbf{i}}}$. The converse is similar.

Now consider our ideals, and let $\mathcal{B}$ be the intersection algebra of I and J. Consider

$$\mathcal{B}_1 = R[Iu, v] \subset R[u, v]$$

$$\mathcal{B}_2 = R[u, Jv] \subset R[u, v]$$

Let $I = (\mathbf{x}^{\alpha_1}, \ldots, \mathbf{x}^{\alpha_m})$. Then then $\mathcal{B}_1$ is generated over R by $\{v, \mathbf{x}^{\alpha_1}u, \ldots, \mathbf{x}^{\alpha_m}u\}$, and similarly $\mathcal{B}_2$ is finitely generated. The intersection of $\mathcal{B}_1$ and $\mathcal{B}_2$ is $\mathcal{B}$. Let Q be the subsemigroup of $\mathbb{N}^{n+2}$ consisting of all exponent vectors $(a_1, \ldots, a_n, b_1, b_2)$ which occur as the exponents of the monomials $x_1^{a_1} \cdots x_n^{a_n} u^{b_1} v^{b_2}$ in $\mathcal{B}$. In the same way, define semigroups Q_1 and Q_2 that correspond to $\mathcal{B}_1$ and $\mathcal{B}_2$. Then Q_1 and Q_2 are finitely generated, since $\mathcal{B}_1$ and $\mathcal{B}_2$ are, and by the above lemma, $Q = Q_1 \cap Q_2$ is finitely generated as well.

□

1.4 Semigroup Rings

When R is a polynomial ring, the intersection algebra of two monomial ideals is a semigroup ring, as we will show in Chapters 3 and 4.

Definition 1.4.1. Let k be a field. The *semigroup ring* $k[Q]$ of a semigroup Q is the k-algebra with k-basis $\{t^a | a \in Q\}$ and multiplication defined by $t^a \cdot t^b = t^{a+b}$.

Definition 1.4.2. For $c \in \mathbb{N}^n$, we set $\boldsymbol{x}^c = x_1^{c_1} \cdots x_n^{c_n}$. Let f be a monomial in R. The exponent vector of $f = \boldsymbol{x}^\alpha$ is denoted by $\log(f) = \alpha \in \mathbb{N}^n$. If F is a collection of monomials in R, $\log(F)$ denotes the set of exponent vectors of the monomials in F.

Definition 1.4.3. Let $R = k[\boldsymbol{x}] = k[x_1, \ldots, x_n]$ be the polynomial ring over a field k in n variables. Let $F = \{f_1, \ldots, f_q\}$ be a finite set of distinct monomials in R such that $f_i \neq 1$ for all i. The *monomial subring spanned by* F is the k-subalgebra

$$k[F] = k[f_1, \ldots, f_q] \subset R.$$

Note that when $F = \{f_1, \ldots, f_q\}$ is a collection of monomials in R, $k[F]$ is equal to the semigroup ring $k[Q]$, where $Q = \mathbb{N}\log(f_1) + \cdots + \mathbb{N}\log(f_q)$ is the subsemigroup of $\mathbb{N}^q$ generated by $\log(F)$. It is easy to see that multiplying monomials in the semigroup ring amounts to adding exponent vectors in the semigroup.

When R is a polynomial ring over a field k, we can consider $\mathcal{B}$ both as an R-algebra and as a k-algebra, and it is important to keep in mind which structure one is considering when proving results. While there are important distinctions between the two, finite generation as an algebra over R is equivalent to finite generation as an algebra over k.

Theorem 1.4.4. *Let R be a ring that is finitely generated as an algebra over a field k. Then $\mathcal{B}$ is finitely generated as an algebra over R if and only if it is finitely generated as an algebra over k.*

Proof. Let $\mathcal{B}$ be finitely generated over k. Then since $k \subset R$, $\mathcal{B}$ is automatically finitely generated over R. Now let $\mathcal{B}$ be finitely generated over R, say by elements $b_1, \ldots, b_n \in \mathcal{B}$. Then for any $b \in \mathcal{B}$, $b = \sum_{i=1}^{q} r_i b_i^{\alpha_i}$ with $r_i \in R$. But R is finitely generated over k, say by elements $k_1, \ldots, k_m$, so $r_i = \sum_{j=1}^{p} a_{ij} k_j^{\beta_{ij}}$, with $a_{ij} \in k$. So $b = \sum_i^q (\sum_j^p a_{ij} k_j^{\beta_{ij}}) b_i^{\alpha_i}$, and $\mathcal{B}$ is finitely generated as an algebra over k by $\{b_1, \ldots, b_n, k_1, \ldots, k_m\}$. □

Since $\mathcal{B}$ is a semigroup ring in certain cases, we can use some facts about semigroup rings and toric ideals to produce a presentation of $\mathcal{B}$ as a quotient of a polynomial ring. The necessary results are listed below.

Definition 1.4.5. Let $S = k[x_1, \ldots, x_m]$ and A be an abelian group together with a list of generators $a_1, \ldots, a_n$, and write Q for the subsemigroup of A generated by $a_1, \ldots, a_n$. Let L denote the kernel of the group homomorphism from $\mathbb{Z}^n$ to A that sends e_i to a_i for $i = 1, \ldots, n$. Then L is a lattice in $\mathbb{N}^n$, and the lattice ideal $I_L \subset S$ associated to L is the ideal

$$I_L = \langle \mathbf{x}^{\mathbf{u}} - \mathbf{x}^{\mathbf{v}} \mid \mathbf{u}, \mathbf{v} \in \mathbb{N}^m \text{ with } \mathbf{u} - \mathbf{v} \in L \rangle.$$

Theorem 1.4.6. (Theorem 7.3 in [4]) *The semigroup ring $k[Q]$ is isomorphic to the quotient S/I_L.*

If Q generates A, and A is the cokernel of an integer matrix $\mathbf{L} = (l_{ij}$ with n rows, I_L can be calculated more easily. The lattice L is generated by the columns of $\mathbf{L}$. Form the ideal $I_{\mathbf{L}}$ in S that is generated by

$$\prod_{i \text{ with } l_{ij}>0} x_i^{l_{ij}} - \prod_{i \text{ with } l_{ij}<0} x_i^{-l_{ij}},$$

where j runs over all column indices of the matrix $\mathbf{L}$.

Lemma 1.4.7. (Lemma 7.6 in [4]) The lattice ideal I_L is computed from $I_{\mathbf{L}}$ by taking the saturation with respect to the product of all the variables:

$$I_L = (I_{\mathbf{L}} : \langle x_1 \cdots x_m \rangle^{\infty}),$$

which by definition is the ideal $\{y \in S \mid (x_1 \cdots x_m)^p y \in I_{\mathbf{L}} \text{ for some } p > 0\}$.

Certain properties of semigroups carry over into semigroup rings, namely normality. The following background definition and theorem come from Chapter 6 of ([15]).

Given a semigroup C, there is a 'smallest' group G containing C, characterized by the fact that every homomorphism from C to a group factors in a unique way through G. We write $\mathbb{Z}C$ for G, and denote $\mathbb{Q}C = \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}C$ and $\mathbb{R}C = \mathbb{R} \otimes_{\mathbb{Z}} \mathbb{Z}C$

Definition 1.4.8. An affine semigroup C is called *normal* if it satisfies the following condition: if $mz \in C$ for some $z \in \mathbb{Z}C$ and $m \in \mathbb{N}$, $m > 0$, then $z \in C$.

All normal semigroups give rise to normal semigroup rings.

Theorem 1.4.9. (Theorem 6.1.4 in [15]) *Let C be an affine semigroup, and k be a field. Then the following are equivalent:*

1. *C is a normal semigroup;*
2. *$k[C]$ is normal.*

By a theorem of Hochster, normal semigroup rings are all Cohen-Macaulay.

Theorem 1.4.10. (Theorem 6.3.5 in [15])(Hochster) *Let C be a normal semigroup, and k be a field. Then $k[C]$ is a Cohen-Macaulay ring.*

Semigroup rings are also graded rings, and our semigroup ring is a special kind of graded ring called a *local ring.

Definition 1.4.11. A graded ideal $\mathfrak{m}$ of a graded ring R is called **maximal* if every graded ideal that properly contains $\mathfrak{m}$ equals R. The ring R is called **local* if it has a unique *maximal ideal $\mathfrak{m}$. We define the **dimension* of R as the height of $\mathfrak{m}$ and denote it by $*\dim R$. If $n = *\dim R$, and $x_1, \ldots, x_n$ are homogeneous elements such that $(x_1, \ldots, x_n)$ is $\mathfrak{m}$-primary, then $x_1, \ldots, x_n$ is called a *homogeneous system of parameters.*

1.5 The Dimension of $\mathcal{B}$

Recall that $\mathcal{B}_R(I, J) = \bigoplus_{r,s} (I^r \cap J^s)\, u^r v^s$.

Theorem 1.5.1. *Let R be a Noetherian domain of dimension n with ideals I and J, both nonzero. Then* $\dim \mathcal{B}_R(I, J) = n + 2$.

Proof. Let Q be a prime ideal in $\mathcal{B}$ and let $P = Q \cap R$ be its restriction to R. Then the dimension inequality [10] says that

$$\text{ht}Q + \text{tr.deg}_{\kappa(P)}\kappa(Q) \leq \text{ht}P + \text{tr.deg}_R\mathcal{B},$$

where $\kappa(P)$ and $\kappa(Q)$ denote the field of fractions of R/P and R/Q, respectively. Then since $\mathcal{B}$ is a domain, both $\frac{u}{1}, \frac{v}{1} \neq 0$ in the fraction field of $\mathcal{B}$, so $\{u, v\}$ form a transcendence basis for $\mathcal{B}$ over R. Thus $\text{tr.deg}_R\mathcal{B} = 2$, and since $\dim R = n$, $\text{ht}P \leq n$. So $\text{ht}Q \leq n+2$, and thus $\dim \mathcal{B} \leq n + 2$.

Define the following ideal

$$\mathcal{B}_+ = \{b \in \mathcal{B} \subset R[u, v] | b \text{ has no constant term}\},$$

and consider the localization $\mathcal{B}_{\mathcal{B}_+}$. Note that $u, v \in \mathcal{B}_{\mathcal{B}_+}$, since $u = \frac{Iu}{I}$ and $I \notin \mathcal{B}_+$, and $(u, v)\mathcal{B}_{\mathcal{B}_+} = \mathcal{B}_+\mathcal{B}_{\mathcal{B}_+} = \mathfrak{m}$, the maximal ideal in $\mathcal{B}_{\mathcal{B}_+}$. Since $\mathcal{B}$ is a domain, $\mathcal{B}_{\mathcal{B}_+}$ is too. We claim $\dim \mathcal{B}_{\mathcal{B}_+} \geq 2$.

Assume that $\dim \mathcal{B}_{\mathcal{B}_+} = 1$. So since $0 \neq u \in (u,v)\mathcal{B}_{\mathcal{B}_+}$, $\mathrm{ht}(u) = 1$. So $\mathrm{ht}P = 1$ for every $P \in \mathrm{Min}(\mathcal{B}_{\mathcal{B}_+}/(u)\mathcal{B}_{\mathcal{B}_+})$, and in fact every prime ideal in $\mathcal{B}_{\mathcal{B}_+}$ has height 1, since $\mathcal{B}_{\mathcal{B}_+}$ is local of dimension 1.

So we have a chain

$$0 \subset (u) \subset P = \mathfrak{m},$$

and therefore $\sqrt{(u)} = \mathfrak{m}$. So there exists an $n \in \mathbb{N}$ such that $\mathfrak{m}^n \subset (u)$. But $v \in \mathfrak{m}$, so $v^n \in (u)\mathcal{B}_{\mathcal{B}_+}$, which implies that there exists a $b \in \mathcal{B}$,$z \notin \mathcal{B}_+$ such that $v^n = u\frac{b}{z}$. But then $zv^n = ub$, and u does not divide v^n, so u must divide z. But this is false, because z has a nonzero constant term $z' \in R$, and u can not divide z'.

So $\dim \mathcal{B}_{\mathcal{B}_+} \geq n+2$ and therefore ht $\mathcal{B}_+ \geq n+2$.

Since $\mathcal{B}/\mathcal{B}_+ \cong R$, any chain of primes in R can be extended by 2 primes to a chain in $\mathcal{B}$, and $\dim \mathcal{B} \geq n+2$. □

1.6 Outline of the Dissertation

In the next chapter, we prove the first major result of this dissertation.

Theorem 1.6.1. *If R is a UFD and I and J are principal ideals, then $\mathcal{B}$ is finitely generated as an algebra over R.*

The proof of this theorem comes from the main idea underlying the work presented here: that many algebras can be associated with fans of cones, and the generators of these algebras can be produced from the Hilbert bases of the underlying cones. We also give a description of these Hilbert bases.

Next we outline a generalization of these algebras coming from cones, which we call fan algebras. We prove that they are finitely generated, and provide a description of their generating sets.

In Chapter 3, we extend the idea of the main theorem of Chapter 2 to the special case of polynomial rings.

Theorem 1.6.2. *If R is a polynomial ring in n variables over k, and I and J are ideals generated by monomials (i.e. monic products of variables) in R, then $\mathcal{B}$ is a semigroup ring.*

Since $\mathcal{B}$ is a semigroup ring, we can provide a list of generators of $\mathcal{B}$ as a k-algebra.

Theorem 1.6.3. *Let $I = (x_1^{a_1} \cdots x_n^{a_n})$ and $J = (x_1^{b_1} \cdots x_n^{b_n})$ be principal ideals in $R = k[x_1, \dots, x_n]$, and let $\Sigma_{\mathbf{a},\mathbf{b}}$ be the fan associated to $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b_1, \dots, b_n)$. Let*

$$Q_i = C_i \cap \mathbb{Z}^2 \text{ for every } C_i \in \Sigma_{\mathbf{a},\mathbf{b}}$$

and $\mathcal{H}_{Q_i}$ be its Hilbert basis of cardinality n_i for all $i = 0, \dots, n$. Further, let Q be the subsemigroup in $\mathbb{N}^2$ generated by

$$\{(a_1 r_{ij}, \dots, a_i r_{ij}, b_{i+1} s_{ij}, \dots, b_n s_{ij}, r_{ij}, s_{ij}) | i = 0, \dots, n, j = 1, \dots, n_i\} \cup \log(x_1, \dots, x_n),$$

where $(r_{ij}, s_{ij}) \in \mathcal{H}_{Q_i}$ for every $i = 0, \dots n, j = 1, \dots, n_i$. Then $\mathcal{B} = k[Q]$.

We also use the algorithm presented in the proof to give a function in Macaulay2 that computes a generating set for the intersection algebra of two principal monomial ideals.

The fact that $\mathcal{B}$ is a semigroup ring in this case allows us to use properties of toric ideals to obtain a representation of $\mathcal{B}$ as a quotient of a polynomial ring.

Lastly, we examine the specific case of two principal monomial ideals in $k[x]$. First, we describe how to obtain a Hilbert basis for a pointed rational cone in the plane, which is then applied to a specific case of two monomial ideals.

Next, we compute a regular sequence for $\mathcal{B}$:

Theorem 1.6.4. *A regular sequence on $\mathcal{B}((x^a), (x^b))$ is $\{x, x^{a+b}u^b v^a, x^a u + x^b v\}$.*

To conclude this chapter, we approach the intersection algebra of (x^a) and (x^b) via a system of linear diophantine equations. This allows us to construct a generating set for the canonical ideal of their intersection algebra.

In the final chapter, we extend our algorithm to produce a generating set for the intersection algebra of two non-principal monomial algebras in a polynomial ring.

Corollary 1.6.5. *Let* $R = k[\mathbf{x}]$, $\mathbf{x} = (x_1, \ldots, x_n)$, *and let*

$$I = (\mathbf{x}^{a_1}, \mathbf{x}^{a_2}, \ldots, \mathbf{x}^{a_m}), J = (\mathbf{x}^{b_1}, \mathbf{x}^{b_2}, \ldots, \mathbf{x}^{b_p}),$$

where $a_i = (a_{i1}, \ldots, a_{in}), b_i = (b_{i1}, \ldots, b_{in}) \in \mathbb{N}^n$ *for all* i. *Then there exists a finite fan of cones* C_i *that fill all of* $\mathbb{N}^{m+p+2}$ *such that*

$$\mathcal{B} = \mathcal{B}(I, J) = k[Q], \text{ where } Q = \bigcup_i (C_i \cap \mathbb{N}^{m+p+2}),$$

and $\mathcal{B}$ *is finitely generated by the set*

$$\{\mathbf{x}^{\mathbf{q}_j} | \mathbf{q}_j \in \mathcal{H}_{Q_i} \text{ for all } i, j\}.$$

The difficulty in this case lies in the complexity of the associated fan: it is no longer simply a collection cones in the plane. We provide a method to compute the extremal rays of a cone given its defining inequalities.

CHAPTER 2

THE INTERSECTION ALGEBRA OF TWO PRINCIPAL IDEALS IN A UNIQUE FACTORIZATION DOMAIN

2.1 The Intersection Algebra of Two Principal Ideals in a UFD

First, we state the main result of this chapter.

Theorem 2.1.1. *If R is a UFD and I and J are principal ideals, then $\mathcal{B}$ is finitely generated as an algebra over R.*

The following gives more detail on the structure of the Hilbert bases that will form the foundation of the proof, and shows that they are finite and unique in the case of a semigroup coming from a pointed rational cone.

Theorem 2.1.2. (Theorem 16.4 in [6]) *Every rational cone $C \subset \mathbb{R}^n$ generated by integral vectors $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_t\}$, $\mathbf{a}_i \in \mathbb{N}^n$ admits a Hilbert basis, and that basis is contained within the finite set $B = \{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$ of all integral vectors contained in the polytope*

$$Z := \{\mathbf{x} \in \mathbb{R}^n | \mathbf{x} = \sum_{i=1}^{t} \lambda_i a_i, 0 \leq \lambda_i \leq 1, i = 1, \ldots, t\}.$$

If C is pointed, the Hilbert basis is unique.

Proof. Let $\mathbf{p}$ be any integral point in C. Then, we have

$$\mathbf{p} = \sum_{i=1}^{t} \lambda_i a_i, \lambda_i \geq 0, i = 1, \ldots, t,$$

for some λ_i. This can be rewritten as

$$\mathbf{p} = \sum_{i=1}^{t} \lfloor \lambda_i \rfloor a_i + \sum_{i=1}^{t} (\lambda_i - \lfloor \lambda_i \rfloor) a_i, \text{ whence}$$

$$\mathbf{p} - \sum_{i=1}^{t} \lfloor \lambda_i \rfloor a_i = \sum_{i=1}^{t} (\lambda_i - \lfloor \lambda_i \rfloor) a_i.$$

All terms on the left hand side are integers, so it is an integer vector. The right hand side lies in Z, since $0 \leq \lambda_i - \lfloor \lambda_i \rfloor < 1$. So the right hand side is an integer vector in Z, and must be one of the $\mathbf{b}_1, \ldots, \mathbf{b}_k$. Since $\mathbf{a}_1, \ldots, \mathbf{a}_t$ are contained in $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$, p decomposes as a nonnegative integer combination of $\mathbf{b}_1, \ldots, \mathbf{b}_k$, so any minimal generating set is amongst $\{\mathbf{b}_1, \ldots, \mathbf{b}_k\}$.

Now suppose C is pointed. Define

$$H := \{\mathbf{x} \in B \setminus \{\mathbf{0}\} | \mathbf{x} \text{ is not the sum of two other vectors in } B\}.$$

We claim H is the Hilbert basis for C.

Every vector in H must clearly be inside any Hilbert basis for C, since otherwise there would be no way to generate that vector from those remaining, so $H \subset \mathcal{H}(C)$. To see the converse, note that $\mathcal{H}(C) \subset B$, so it is enough to show that every vector in $B \setminus H$ can be represented as a nonnegative integer combination of vectors in H. Suppose not: that there exists $\mathbf{b} \in B \setminus H$ that violates this property, and choose such a vector $\mathbf{b}$ minimizing $\mathbf{c}^\intercal \mathbf{b}$, where $\mathbf{c}$ is a vector such that $\mathbf{c}^\intercal x > 0$ for all nonzero $x \in C$, and $\mathbf{c}^\intercal$ denotes the usual vector transpose. It is known that the existence of $\mathbf{c}$ is guaranteed because C is pointed. Because $\mathbf{b} \notin H$, $\mathbf{b} = \mathbf{b}_i + \mathbf{b}_j$ for some nonzero vectors $\mathbf{b}_i, \mathbf{b}_j \in B$. So $\mathbf{c}^\intercal \mathbf{b} = \mathbf{c}^\intercal \mathbf{b}_i + \mathbf{c}^\intercal \mathbf{b}_j$, and all terms are positive. Thus $\mathbf{c}^\intercal \mathbf{b}_i < \mathbf{c}^\intercal \mathbf{b}$ and $\mathbf{c}^\intercal \mathbf{b}_j < \mathbf{c}^\intercal \mathbf{b}$. But we assumed that $\mathbf{c}^\intercal \mathbf{b}$ is minimal under the condition that $\mathbf{b} \notin H$, so both $\mathbf{b}_i, \mathbf{b}_j \in H$, which is a contradiction. So $H = \mathcal{H}(C)$.

□

So the Hilbert basis of a pointed rational cone is the collection of vectors inside B that are not sums of other vectors in B. In $\mathbb{N}^2$, given a cone C defined by two integer vectors $\mathbf{c}_1, \mathbf{c}_2$, Z is the parallelogram defined by the convex hull of $\mathbf{0}, \mathbf{c}_1, \mathbf{c}_2$ and $\mathbf{c}_1 + \mathbf{c}_2$, so B are all the integral vectors inside Z. So it suffices to collect those integer vectors inside the parallelogram that cannot be written as sums of any of the others.

The following example will be useful in Chapter 3, where we will use it to compute some facts about a particular intersection algebra.

Example 2.1.3. The Hilbert basis of the cone in $\mathbb{N}^2$ defined by the vectors $(1, a)$ and $(1, 0)$ is $\{(1, 0), (1, 1), (1, 2), \dots, (1, a-1), (1, a)\}$, and the Hilbert basis of the cone defined by $(0, 1)$ and $(1, a)$ is $\{(0, 1), (1, a)\}$.

Proof. Call the first cone in the example $C_0 = \{\lambda_1(1, a) + \lambda_2(1, 0) | \lambda_1, \lambda_2 \geq 0\}$, and using the terminology above, define the parallelogram Z_0 to be the convex hull of $(0, 0), (1, a), (2, a)$, and $(1, 0)$. The only integer vectors inside Z_0 are the ones defining the boundary along with $(1, 1), (1, 2), \dots, (1, a-1)$, so $\mathcal{H}(C_0)$ must be among these vectors. Again, $(0, 0)$ and $(2, a)$ can be discarded, and obviously none of the rest can be sums of the others, since the first coordinate of all of them is 1. So $\mathcal{H}(C_0) = \{(1, 0), (1, 1), \dots, (1, a-1), (1, a)\}$.

The second cone, call it C_1, is the narrow wedge of the first quadrant defined by $C_1 = \{\lambda_1(0, 1) + \lambda_2(1, a) | \lambda_1, \lambda_2 \geq 0\}$, and the parallelogram Z_1 that contains the Hilbert basis is the convex hull of the points $(0, 0), (0, 1), (1, a+1)$, and $(1, a)$. These are clearly the only integer vectors inside that parallelogram, and since $\mathbf{0}$ is never inside a Hilbert basis, and $(1, a+1)$ is a sum of the remaining two vectors, $\mathcal{H}(C_1) = \{(0, 1), (1, a)\}$.

□

Remark 2.1.4. This theorem allows one to produce a rough upper bound for the Hilbert basis of a cone, namely the number of integral vectors inside the parallelogram Z.

Now we will provide a list of generators for $\mathcal{B}$. These last definitions will provide the structure of the fans and cones that we will use to build our generating set.

For any two strings of numbers

$$\mathbf{a} = \{a_1, \dots, a_n\}, \mathbf{b} = \{b_1, \dots, b_n\} \text{ with } a_i, b_i \in \mathbb{N},$$

we can associate to them a fan of pointed, rational cones in $\mathbb{N}^2$.

Definition 2.1.5. We will call two such strings of numbers *fan ordered* if

$$\frac{a_i}{b_i} \geq \frac{a_{i+1}}{b_{i+1}} \text{ for all } i = 1, \ldots, n.$$

By convention, if $b_i = 0$, we will say that $\frac{a_i}{b_i} = \infty$. Assume $\mathbf{a}$ and $\mathbf{b}$ are fan ordered. Additionally, let $a_{n+1} = b_0 = 0$ and $a_0 = b_{n+1} = 1$. Then for all $i = 0, \ldots, n$, let

$$C_i = \{\lambda_1(b_i, a_i) + \lambda_2(b_{i+1}, a_{i+1}) | \lambda_i \in \mathbb{R}_{\geq 0}\}.$$

Let $\Sigma_{\mathbf{a},\mathbf{b}}$ be the fan formed by these cones and their faces, and call it the *fan of* $\mathbf{a}$ *and* $\mathbf{b}$ *in* $\mathbb{N}^2$. Hence

$$\Sigma_{\mathbf{a},\mathbf{b}} = \{C_i | i = 0, \ldots, n\}.$$

Then, since each C_i is a pointed rational cone, $Q_i = C_i \cap \mathbb{Z}^2$ has a Hilbert Basis, say

$$\mathcal{H}_{Q_i} = \{(r_{i1}, s_{i1}), \ldots, (r_{in_i}, s_{in_i})\}.$$

Note that any $\Sigma_{\mathbf{a},\mathbf{b}}$ partitions all of the first quadrant of $\mathbb{R}^2$ into cones, so the collection $\{Q_i | i = 0, \ldots, n\}$ partitions all of $\mathbb{N}^2$ as well, so for any $(r, s) \in \mathbb{N}^2$, $(r, s) \in Q_i$ for some $i = 0, \ldots, n$.

In this chapter, we are studying the intersection algebra when I and J are principal, so the order of the exponents in their exponent vectors does not matter. In general, for any two strings of numbers $\mathbf{a}$ and $\mathbf{b}$, there is essentially a unique way to rearrange the ratios in a non-increasing fashion. So a unique fan can be associated to any two vectors. For the purposes of this section, we will assume without loss of generality that the exponent vectors are fan ordered.

Here is the major result of this section, proving both the finite generation of $\mathcal{B}(I, J)$ and providing a list of algebra generators.

Theorem 2.1.6. *Let R be a UFD with principal ideals $I = (p_1^{a_1} \cdots p_n^{a_n})$ and $J = (p_1^{b_1} \cdots p_n^{b_n})$, where $p_i, i = 1, \ldots, n$ are irreducible elements, and let $\Sigma_{\mathbf{a},\mathbf{b}}$ be the fan associated to $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$. Then $\mathcal{B}$ is generated over R by the set*

$$\{p_1^{a_1 r_{ij}} \cdots p_i^{a_i r_{ij}} p_{i+1}^{b_{i+1} s_{ij}} \cdots p_n^{b_n s_{ij}} u^{r_{ij}} v^{s_{ij}} | i = 0, \ldots, n, j = 1, \ldots, n_i\},$$

where (r_{ij}, s_{ij}) run over the Hilbert basis for each $Q_i = C_i \cap \mathbb{Z}^2$ for every $C_i \in \Sigma_{\mathbf{a},\mathbf{b}}$.

Proof. Since $\mathcal{B}$ has a natural $\mathbb{N}^2$ grading, it is enough to consider only homogeneous monomials $b \in \mathcal{B}$ with $\deg(b) = (r, s)$. Then $(r, s) \in Q_i = C_i \cap \mathbb{Z}^2$ for some $C_i \in \Sigma_{\mathbf{a},\mathbf{b}}$. In other words, $r, s \in \mathbb{N}^2$ and

$$\frac{a_i}{b_i} \geq \frac{s}{r} \geq \frac{a_{i+1}}{b_{i+1}}.$$

So $a_i r \geq b_i s$, and by the ordering on the a_i and the b_i, $a_j r \geq b_j s$ for all $j < i$. Also, $a_{i+1} r \leq b_{i+1} s$, and again by the ordering, $a_j r \leq b_j s$ for all $j > i$. So

$$\begin{aligned} b \in I^r \cap J^s u^r v^s &= (p_1^{a_1} \cdots p_n^{a_n})^r \cap (p_1^{b_1} \cdots p_n^{b_n})^s u^r v^s \\ &= (p_1^{a_1 r} \cdots p_i^{a_i r} \cdot p_{i+1}^{b_{i+1} s} \cdots p_n^{b_n s}) u^r v^s. \end{aligned}$$

So $b = f \cdot p_1^{a_1 r} \cdots p_i^{a_i r} \cdot p_{i+1}^{b_{i+1} s} \cdot p_n^{b_n s} u^r v^s$ for some monomial $f \in R$.

Since $(r, s) \in Q_i$, the pair has a decomposition into a sum of Hilbert basis elements. So we have $(r, s) = \sum_{j=1}^{n_i} m_j (r_{ij}, s_{ij})$ with $m_j \in \mathbb{N}$, and $r = \sum_{j=1}^{n_i} m_j r_{ij}$, $s = \sum_{j}^{n_i} m_j s_{ij}$. Therefore

$$\begin{aligned} b =& f (p_1^{a_1 r} \cdots p_i^{a_i r} p_{i+1}^{b_{i+1} s} \cdots p_n^{b_n s} u^r v^s) \\ =& f \prod_{j=1}^{n_i} p_1^{m_j (a_1 r_{ij})} \cdots p_i^{m_j (a_i r_{ij})} p_{i+1}^{m_j (b_{i+1} s_{ij})} \cdots p_n^{m_j (b_n s_{ij})} u^{m_j (r_{ij})} v^{m_j (s_{ij})} \\ =& f \prod_{j=1}^{n_i} (p_1^{a_1 r_{ij}} \cdots p_i^{a_i r_{ij}} p_{i+1}^{b_{i+1} s_{ij}} \cdots p_n^{b_n s_{ij}} u^{r_{ij}} v^{s_{ij}})^{m_j}. \end{aligned}$$

So b is generated over R by the given finite set as claimed. □

Remark 2.1.7. This theorem extends and refines the main result in [7]

Theorem 2.1.8. *This generating set is minimal, in that no generator is a product of the others.*

Proof. First an easier case: Say $I = (p^a)$ and $J = (p^b)$, where p is an irreducible in R, and say that one generator $p^{\max(ar,bs)}u^r v^s$ is a product of the others, in other words, using the same notation as in the above proof,

$$p^{\max(ar,bs)}u^r v^s = \prod_i \left(p^{\max(ar'_i,bs'_i)}u^{r'_i}v^{s'_i}\right)^{c_i} \tag{2.1}$$

where (r'_i, s'_i) are elements of $\mathcal{H}_{Q_0} \cup \mathcal{H}_{Q_1}$ and $c_i \in \mathbb{N}$.

We collect all the (r'_i, s'_i) from Q_0 into one pair (r_0, s_0), and those from Q_1 into another, (r_1, s_1). However, since both Q_0 and Q_1 contain any points on the face separating them, we must make this partition well defined. Define the two sets

$$\Lambda_0 = \{i|(r'_i, s'_i) \in \mathcal{H}_{Q_0}\} \text{ and } \Lambda_1 = \{i|(r'_i, s'_i) \in \mathcal{H}_{Q_1} \setminus \mathcal{H}_{Q_0}\},$$

and define

$$(r_0, s_0) = \sum_{i \in \Lambda_0} c_i(r'_i, s'_i) \text{ and } (r_1, s_1) = \sum_{i \in \Lambda_1} c_i(r'_i, s'_i).$$

In other words, (r_0, s_0) is the sum of all Hilbert basis elements present in the decomposition of (r, s) (including coefficients) that come from the cone Q_0, and (r_1, s_1) is the sum of all Hilbert basis elements in the decomposition of (r, s) (including coefficients) that come from Q_1, not including the face between Q_1 and Q_0. Note that if the decomposition of (r, s) doesn't contain an element from the cone Q_k, set $(r_k, s_k) = (0, 0)$. Then by 2.1,

$$(r, s) = (r_0, s_0) + (r_1, s_1) \tag{2.2}$$

and

$$\max(ar, bs) = \max(ar_0, bs_0) + \max(ar_1, bs_1). \tag{2.3}$$

Since $(r_0, s_0) \in Q_0$, $\max(ar_0, bs_0) = bs_0$. Similarly, $\max(ar_1, bs_1) = ar_1$.

Assume $(r, s) \in Q_0$. Then $\max(ar, bs) = bs$, and by 2.3 and 2.2,

$$bs = bs_0 + ar_1 = bs_0 + bs_1.$$

Therefore, $ar_1 = bs_1$, so $a/b = s_1/r_1$, i.e. (r_1, s_1) lies on the face between Q_0 and Q_1, which contradicts the definition of (r_1, s_1).

So $(r, s) \in Q_1$ and $\max(ar, bs) = ar$. By 2.3 and 2.2,

$$ar = bs_0 + ar_1 = ar_0 + ar_1.$$

So $bs_0 = ar_0$, and thus (r_0, s_0) lies on the face between Q_0 and Q_1. But this means $(r_0, s_0) \in Q_1$, and also (r_1, s_1) and $(r, s) \in Q_1$. But (r, s) is a Hilbert basis element of Q_1, and $(r, s) = (r_0, s_0) + (r_1, s_1)$, which is a contradiction.

The proof of the general case is similar. Let $p_1, \ldots, p_n$ be irreducibles in R, and $I = (p_1^{a_1} \ldots p_n^{a_n}), J = (p_1^{b_1} \ldots p_n^{b_n})$. Say that one generator is a product of the others, i.e.

$$\prod_{i=1}^{m} (p_1^{\max(a_1 r_i', b_1 s_i')} \cdots p_n^{\max(a_n r_i', b_n s_i')})^{c_i} = p_1^{\max(a_1 r, b_1 s)} \cdots p_n^{\max(a_n r, b_n s)}, \tag{2.4}$$

where (r_i', s_i'), $i = 1, \ldots, m$ and (r, s) are Hilbert basis elements of one of the cones, and $c_i \in \mathbb{N}$.

We claim we can assume that $b_i \neq 0$ for all $i = 1, \ldots, n$. To see this, note that if there exists an h such that $b_h = 0$, then $b_1, \ldots, b_{h-1} = 0$ by the fan-ordering. Also, if $b_1 = 0$, then $a_1 \neq 0$ (otherwise, p_1 would simply not be in the decompositions of the generators of I and

J). So

$$a_1 r = \sum_{i=1}^{m} c_i a_1 r_i' \text{ implies } r = \sum_{i=1}^{m} c_i r_i'.$$

So by cancelling in 2.4,

$$\prod_{i=2}^{m} (p_2^{\max(a_2 r_i', b_1 s_i')} \cdots p_n^{\max(a_n r_i', b_n s_i')})^{c_i} = p_2^{\max(a_2 r, b_2 s)} \cdots p_n^{\max(a_n r, b_n s)}.$$

If $b_2 = 0$, we continue in the same way with cancelling the p_2 terms in 2.4, until the first nonzero b, say b_h. Then we have 2.4 with only terms $p_h, \ldots, p_n$, and $b_h, \ldots, b_n$ are all nonzero.

Now assume all $b_i \neq 0$.

We again partition all the (r_i', s_i'), and sum them into $n+1$ pairs $(r_i, s_i), i = 0, \ldots, n$. To make this partition well defined, define the $n+1$ sets

$$\Lambda_0 = \{i | (r_i', s_i') \in \mathcal{H}_{Q_0}\} \text{ and } \Lambda_k = \{i | (r_i', s_i') \in \mathcal{H}_{Q_k} \setminus \mathcal{H}_{Q_{k-1}}\} \text{ for all } k = 1, \ldots, n\}.$$

Then define

$$(r_k, s_k) = \sum_{i \in \Lambda_k} c_i (r_i', s_i').$$

By convention, if $(b_i, a_i) = k(b_{i+1}, a_{i+1})$ for some $k \in \mathbb{Q}_+$ and some i from 0 to n, we say that $(r_i, s_i) = (0, 0)$.

Then by 2.4,

$$(r, s) = \sum_{i=0}^{n} (r_i, s_i) \tag{2.5}$$

and

$$\max(ar, bs) = \sum_{i=0}^{n} \max(ar_i, bs_i). \tag{2.6}$$

Assume that $(r, s) \in Q_j$, so by the cone structure

$$p_1^{\max(a_1 r, b_1 s)} \cdots p_n^{\max(a_n r, b_n s)} = p_1^{a_1 r} \cdots p_j^{a_j r} p_{j+1}^{b_{j+1} s} \cdots p_n^{b_n s}.$$

Therefore

$$
\begin{aligned}
a_1 r &= \sum_{i=0}^{n} \max(a_1 r_i, b_1 s_i) \\
\vdots &= \vdots \\
a_j r &= \sum_{i=0}^{n} \max(a_j r_i, b_j s_i) \\
b_{j+1} s &= \sum_{i=0}^{n} \max(a_{j+1} r_i, b_{j+1} s_i) \\
\vdots &= \vdots \\
b_n s &= \sum_{i=0}^{n} \max(a_n r_i, b_n s_i).
\end{aligned}
$$

Also, since every nonzero (r_k, s_k) is in $Q_k \setminus Q_{k-1}$, for all $k = 0, \ldots n$,

$$
\begin{aligned}
a_i r_k &\geq b_i s_k && \text{for all } i < k \\
a_k r_k &> b_k s_k \\
a_i r_k &\leq b_i s_k && \text{for all } i > k.
\end{aligned}
$$

Therefore, the above sums become

$$
\begin{array}{ccccc}
a_1 r & = & b_1 \sum_{i=0}^{0} s_i & + & a_1 \sum_{i=1}^{n} r_i \\
a_2 r & = & b_2 \sum_{i=0}^{1} s_i & + & a_2 \sum_{i=2}^{n} r_i \\
\vdots & = & \vdots & & \vdots \\
a_j r & = & b_j \sum_{i=0}^{j-1} s_i & + & a_j \sum_{i=j}^{n} r_i \\
b_{j+1} s & = & b_{j+1} \sum_{i=0}^{j} s_i & + & a_{j+1} \sum_{i=j+1}^{n} r_i \\
\vdots & = & \vdots & & \vdots \\
b_n s & = & b_n \sum_{i=0}^{n-1} s_i & + & a_n \sum_{i=n}^{n} r_i.
\end{array}
$$

By (2.5), we can again clear terms on both sides to obtain

$$\begin{array}{rcl} a_1 \sum_{i=0}^{0} r_i & = & b_1 \sum_{i=0}^{0} s_i \\ a_2 \sum_{i=0}^{1} r_i & = & b_2 \sum_{i=0}^{1} s_i \\ \vdots & = & \vdots \\ a_j \sum_{i=0}^{j-1} r_i & = & b_j \sum_{i=0}^{j-1} s_i \\ b_{j+1} \sum_{i=j+1}^{n} s_i & = & a_{j+1} \sum_{i=j+1}^{n} r_i \\ \vdots & = & \vdots \\ b_n \sum_{i=n}^{n} s_i & = & a_n \sum_{i=n}^{n} r_i. \end{array}$$

By the last equation in the above collection, $b_n s_n = a_n r_n$. So (r_n, s_n) lies on the line with slope a_n/b_n. But this is the face in between Q_n and Q_{n-1}, contradicting the definition of (r_n, s_n). So there are no generators coming from Q_n, so $(r_n, s_n) = (0, 0)$.

From the $n-1$th equation in the collection, $b_{n-1}s_{n-1} + b_{n-1}s_n = a_{n-1}r_{n-1} + a_{n-1}r_n$. But since $(r_n, s_n) = (0, 0)$, $a_{n-1}/b_{n-1} = s_{n-1}/r_{n-1}$. So (r_{n-1}, s_{n-1}) lies on the face in between Q_{n-1} and Q_{n-2}, contracting the definition of (r_{n-1}, s_{n-1}). So again, $(r_{n-1}, s_{n-1}) = 0$. Continuing in this way, we see that $(r_k, s_k) = (0, 0)$ for all $j + 1 \leq k \leq n$.

By the first equation in the list, $a_1 r_0 = b_1 s_0$, so (r_0, s_0) is on the line between Q_0 and Q_1.

The second equation says $a_2 r_0 + a_2 r_1 = b_2 s_0 + b_2 s_1$. But since $(r_0, s_0) \in Q_0$ and $(r_1, s_1) \in Q_1$, $a_2 r_0 \leq b_2 s_0$ and $a_2 r_1 \leq b_2 s_1$. Therefore $a_2 r_0 = b_2 s_0$ and $a_2 r_1 = b_2 s_1$.

This together with the fact that $a_1 r_0 = b_1 s_0$ gives

$$\frac{a_2}{b_2} = \frac{s_1}{r_1} = \frac{s_0}{r_0} = \frac{a_1}{b_1}.$$

Therefore, by convention, $(r_1, s_1) = (0, 0)$.

In a similar way, the third equation $a_3(r_0 + r_2) = b_3(s_0 + s_2)$, together with $a_3 r_0 \leq b_3 s_0$ and $a_3 r_2 \leq b_3 s_2$ implies that $\frac{a_3}{b_3} = \frac{a_2}{b_2}$, and so by convention $(r_2, s_2) = (0, 0)$. Continuing in this fashion shows that $(r_k, s_k) = (0, 0)$ for all $1 \leq k \leq j$.

Therefore, $(r, s) = (r_0, s_0)$, contradicting the choice of (r, s) as a Hilbert basis element. So (r, s) is not a product of other Hilbert basis elements, and its corresponding algebra generator is not a product of other generators. □

2.1.1 Relationship to Work of Samuel and Nagata

Remark 2.1.9. For any two ideals I and J in R with $J \subset \sqrt{I}$, where I is not nilpotent and $\cap_k I^k = (0)$, define $v_I(J, m)$ to be the largest integer n such that $J^m \subseteq I^n$ and $w_J(I, n)$ to be the smallest m such that $J^m \subseteq I^n$. The two sequences $\{v_I(J, m)/m\}_m$ and $\{w_J(I, n), n\}_n$ have limits $l_I(J)$ and $L_J(I)$, respectively. See [8–10] for related work.

Given two principal ideals I and J in a UFD R whose radicals are equal (i.e. the factorizations of their generators use the same irreducible elements), our procedure to determine generators also shows that the vectors (b_1, a_1) and (b_n, a_n) are related to the pairs of points (r, s) where $I^r \subseteq J^s$ (respectively $J^s \subseteq I^r$): notice that C_0 is the cone between the y-axis and the line through the origin with slope a_0/b_0, and for all $(r, s) \in C_0 \cap \mathbb{N}^2$, $I^r \subseteq J^s$. Therefore $l_J(I) = a_0/b_0$. Similarly, C_n, the cone between the x-axis and the line through the origin with slope a_n/b_n, contains all $(r, s) \in \mathbb{N}^2$ where $J^s \subseteq I^r$, so $l_J(I) = a_n/b_n$. Then, since $l_I(J)L_J(I) = 1$, this gives that $L_J(I) = a_1/b_1$ and $L_I(J) = b_n/a_n$ as well. This agrees with the observations of Samuel and Nagata as mentioned in [3].

2.2 Converting from Fans to Algebras

We have shown that every intersection algebra of two principal ideals in a UFD corresponds to a fan. In fact, the converse is true, too.

Theorem 2.2.1. *Every fan of pointed rational cones that fill the first quadrant of* $\mathbb{N}^2$ *corresponds to an intersection algebra.*

Proof. Let F be a fan in $\mathbb{N}^2$ with faces given by the vectors $C_0 = \langle 1, 0\rangle$, $C_{n+1} = \langle 0, 1\rangle$ and $C_i = \langle a_i, b_i\rangle$ where $a_i, b_i \in \mathbb{N}$ and $\frac{a_i}{b_i} \geq \frac{a_{i+1}}{b_{i+1}}$ for all i. Let $I = (x_1^{a_1} \cdots x_n^{a_n})$ and $J = (x_1^{b_1} \cdots x_n^{b_n})$

be ideals in $k[x_1, \ldots, x_n]$. It is clear by the construction of the intersection algebra that $\mathcal{B} = \bigoplus_{r,s} I^r \cap J^s u^r v^s$ has F as its corresponding fan. $\square$

Definition 2.2.2. Let F be a fan in $\mathbb{N}^2$ with faces given by the vectors $C_0 = \langle 1, 0 \rangle$, $C_{n+1} = \langle 0, 1 \rangle$ and $C_i = \langle a_i, b_i \rangle$ where $a_i, b_i \in \mathbb{N}$ and $\frac{a_i}{b_i} > \frac{a_{i+1}}{b_{i+1}}$ and $(a_i, b_i) = 1$ for all i. Let I and J be constructed as in the above proof. Then $\mathcal{B}$ is called the *minimal intersection algebra for* F, denoted $\mathcal{B}(F)$.

We can consider $\mathcal{B}$ as an algebra over any ring which contains the ideals I and J. When the ring used is unclear, we denote that $\mathcal{B}$ is an R-algebra by $\mathcal{B}_R$.

2.3 Fan Algebras

This process of first obtaining semigroup generators and then extending them in a natural way generalizes nicely. Before we formally define this generalization, we need to define a special kind of function.

Definition 2.3.1. Given a fan of cones $\Sigma_{\mathbf{a},\mathbf{b}}$, a function $f : \mathbb{N}^2 \to \mathbb{N}$ is called *fan-linear* if it is nonnegative and linear on each subgroup $Q_i = C_i \cap \mathbb{Z}^2$ for each $C_i \in \Sigma_{\mathbf{a},\mathbf{b}}$, and subadditive on all of $\mathbb{N}^2$, i.e.

$$f(r, s) + f(r', s') \geq f(r + r', s + s') \text{ for all } (r, s), (r', s') \in \mathbb{N}^2.$$

In other words, $f(r, s)$ is a piecewise linear function where

$$f(r, s) = g_i(r, s) \text{ when } (r, s) \in C_i \cap \mathbb{N}^2 \text{ for each } i = 0, \ldots n, \text{ and each } g_i \text{ is linear on } C_i \cap \mathbb{N}^2.$$

Note that each piece of f agrees on the faces of the cones, that is $g_i = g_j$ for every $(r, s) \in C_i \cap C_j \cap \mathbb{N}^2$.

Example 2.3.2. Let $\mathbf{a} = \{1\} = \mathbf{b}$, so $\Sigma_{\mathbf{a},\mathbf{b}}$ is the fan defined by

$$C_0 = \{\lambda_1(0,1) + \lambda_2(1,1) | \lambda_i \in \mathbb{R}_{\geq 0}\}$$
$$C_1 = \{\lambda_1(1,1) + \lambda_2(1,0) | \lambda_i \in \mathbb{R}_{\geq 0}\},$$

and set $Q_i = C_i \cap \mathbb{Z}^2$. Also let

$$f = \begin{cases} g_0(r,s) = r + 2s & \text{if } (r,s) \in Q_0 \\ g_1(r,s) = 2r + s & \text{if } (r,s) \in Q_1 \end{cases}.$$

Then f is a fan-linear function. It is clearly nonnegative and linear on both Q_0 and Q_1. The function is also subadditive on all of $\mathbb{N}^2$: Let $(r,s) \in Q_0$ and $(r',s') \in Q_1$, and say that $(r+r', s+s') \in Q_0$. Then

$$f(r,s) + f(r',s') = g_0(r,s) + g_1(r',s') = r + 2s + 2r' + s$$
$$f(r+r', s+s') = g_0(r+r', s+s') = r + r' + 2(s+s').$$

Comparing the two, we see that

$$f(r,s) + f(r',s') \geq f(r+r', s+s') \text{ whenever } r + 2s + 2r' + s \geq r + r' + 2(s+s'),$$

or equivalently when $r' \geq s'$. But that is true, since $(r',s') \in Q_1$. The proof for $(r+r', s+s') \in Q_1$ is similar. The two pieces of f also agree on the boundary between Q_0 and Q_1, since the intersection of Q_0 and Q_1 is the ray in $\mathbb{N}^2$ where $r = s$, and

$$g_0(r,r) = 3r = g_1(r,r).$$

So f is a fan-linear function.

If, instead of intersecting ideals together, we apply fan-linear functions to them, the resulting algebras are still finitely generated.

Theorem 2.3.3. *Let $I_1 \ldots, I_n$ be ideals in a domain R and $\Sigma_{\mathbf{a},\mathbf{b}}$ be a fan of cones in $\mathbb{N}^2$. Let $f_1, \ldots, f_n$ be fan-linear functions on $\Sigma_{\mathbf{a},\mathbf{b}}$. Then the algebra*

$$\mathcal{B} = \bigoplus_{r,s} I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} u^r v^s$$

is finitely generated.

Proof. First notice that the subadditivity of the functions f_i guarantees that $\mathcal{B}$ is a subalgebra of $R[u, v]$ with the natural grading. Since $\mathcal{B}$ has a natural $\mathbb{N}^2$-grading, it is enough to consider only homogeneous monomials $b \in \mathcal{B}$ with $\deg(b) = (r, s)$. Then $(r, s) \in Q_i = C_i \cap \mathbb{Z}^2$ for some $C_i \in \Sigma_{\mathbf{a},\mathbf{b}}$. Since Q_i is a pointed rational cone, it has a Hilbert basis

$$H_{Q_i} = \{(r_{i1}, s_{i1}), \ldots, (r_{in_i}, s_{in_i})\}.$$

So we can write

$$(r, s) = \sum_{j=1}^{n_i} m_j (r_{ij}, s_{ij}).$$

Then, since each f_k is nonnegative and linear on Q_i, we have

$$f_k(r, s) = \sum_{j=1}^{n_i} m_j f_k(r_{ij}, s_{ij})$$

for each $k = 1, \ldots, n$. Since R is Noetherian, for each i, there exists a finite set $\Lambda_{i,j,k} \subset R$ such that

$$I_k^{f_k(r_{ij}, s_{ij})} = (x | x \in \Lambda_{i,j,k}).$$

So

$$\begin{aligned} b \in \mathcal{B}_{r,s} &= I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} u^r v^s \\ &= I_1^{\sum_{j=1}^{n_i} m_j f_1(r_{ij},s_{ij})} \cdots I_n^{\sum_{j=1}^{n_i} m_j f_n(r_{ij},s_{ij})} u^{\sum_{j=1}^{n_i} m_j r_{ij}} v^{\sum_{j=1}^{n_i} m_j s_{ij}} \\ &= I_1^{m_1 f_1(r_{i1},s_{i1})} \cdots I_1^{m_{n_i} f_1(r_{in_i},s_{in_i})} \cdots I_n^{m_1 f_n(r_{i1},s_{i1})} \cdots I_n^{m_{n_i} f_n(r_{in_i},s_{in_i})} \\ &\qquad u^{m_1 r_{i1}} \cdots u^{m_{n_i} r_{in_i}} v^{m_1 s_{i1}} \cdots v^{m_{n_i} s_{in_i}} \\ &= \left(I_1^{f_1(r_{i1},s_{i1})} \cdots I_n^{f_n(r_{i1},s_{i1})} u^{r_{i1}} v^{s_{i1}} \right)^{m_1} \cdots \left(I_1^{f_1(r_{in_i},s_{in_i})} \cdots I_n^{f_n(r_{in_i},s_{in_i})} u^{r_{in_i}} v^{s_{in_i}} \right)^{m_{n_i}}. \end{aligned}$$

So $\mathcal{B}$ is generated as an algebra over R by the set

$$\{x_1 \cdots x_n u^{r_{ij}} v^{s_{ij}} | (r_{ij}, s_{ij}) \in \mathcal{H}_{Q_i}, x_k \in \Lambda_{i,j,k}\}.$$

□

This result justifies the following definition.

Definition 2.3.4. Given ideals $I_1, \ldots, I_n$ in a domain R, $\Sigma_{\mathbf{a},\mathbf{b}}$ a fan of cones in $\mathbb{N}^2$, and $f_1, \ldots, f_n$ are fan-linear functions, we define

$$\mathcal{B}(\Sigma_{\mathbf{a},\mathbf{b}}, f) = \bigoplus_{r,s} I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} u^r v^s$$

to be the *fan algebra of f on $\Sigma_{\mathbf{a},\mathbf{b}}$*, where $f = (f_1, \ldots, f_n)$.

Remark 2.3.5. The intersection algebra of two principal ideals $I = (p_1^{a_1} \cdots p_n^{a_n})$ and $J = (p_1^{b_1} \cdots p_n^{b_n})$ in a UFD is a special case of a fan algebra. Let $I_i = (p_i)$ and $f_i = \max(ra_i, sb_i)$ for each $i = 1, \ldots, n$, and define the fan $\Sigma_{\mathbf{a},\mathbf{b}}$ to be the fan associated to $\mathbf{a} = (a_1, \ldots, a_n)$ and $\mathbf{b} = (b_1, \ldots, b_n)$. Then

$$\mathcal{B}(I, J) = \bigoplus_{r,s} (p_1)^{\max(ra_1, sb_1)} \cdots (p_n)^{\max(ra_n, sb_n)} u^r v^s.$$

This is a fan algebra since the max function is fan-linear: it is subadditive on all of $\mathbb{N}^2$, and linear and nonnegative on each cone, since the faces of each cone in $\Sigma_{\mathbf{a},\mathbf{b}}$ are defined by lines through the origin with slopes a_i/b_i for each $i = 0, \ldots, n$. So, as in the proof of Theorem 2.1.6, for any pair $(r, s) \in Q_i = C_i \cap \mathbb{Z}^2$ for every $C_i \in \Sigma_{\mathbf{a},\mathbf{b}}$, we have that

$$\frac{a_i}{b_i} \geq \frac{s}{r} \geq \frac{a_{i+1}}{b_{i+1}}.$$

So $a_i r \geq b_i s$, and by the ordering on the a_i and the b_i, $a_j r \geq b_j s$ for all $j < i$. Also, $a_{i+1} r \leq b_{i+1} s$, and again by the ordering, $a_j r \leq b_j s$ for all $j > i$. Since $f_k = \max(ra_k, sb_k)$ for all $k = 1, \ldots, n$, we have that

$$f_k = ra_k \text{ for all } k \leq i \text{ and } f_k = sb_k \text{ for all } k > i.$$

So each f_k is linear on each cone, and the above theorem applies.

2.3.1 Higher-Dimensional Fan Algebras

Fan algebras arise naturally in higher dimensions from two generalizations of the UFD case: first from intersecting more than two ideals, and second from intersecting non-principal ideals. What is lost in higher dimensions is the ease of calculation of the extremal rays of the cones of the fan, and also the natural ordering of the cones $C_0, \ldots, C_n$ that we had previously.

First, we define a more general fan algebra, without requiring that the fan be fan-ordered (since no analogue exists in higher dimensions). However, as before, we still require that the fan of cones must fill all of the positive orthant of the space it inhabits.

Definition 2.3.6. Let Σ be a fan of cones in $\mathbb{N}^m$, let R be a domain with ideals $I_1, \ldots, I_n$, and $f_1(\mathbf{r}), \ldots, f_n(\mathbf{r})$ be fan-linear functions on Σ from $\mathbb{N}^m \to \mathbb{N}$. Additionally, let $\mathbf{u}$ denote indeterminates $u_1, \ldots, u_m$. Then we define

$$\mathcal{B}(\Sigma, f) = \bigoplus_{\mathbf{r} \geq 0} I_1^{f_1(\mathbf{r})} \cdots I_n^{f_n(\mathbf{r})} \mathbf{u}^{\mathbf{r}}$$

to be the *fan algebra of* f *on* Σ, where $f = (f_1, \ldots, f_n)$.

Theorem 2.3.7. *The fan algebra* $\mathcal{B}(\Sigma, f)$ *is finitely generated.*

Proof. The subadditivity of the functions f_i guarantees that $\mathcal{B}$ is a subalgebra of $R[u_1, \ldots, u_m]$ with the natural grading. Since $\mathcal{B}$ has a natural $\mathbb{N}^m$-grading, it is enough to consider only homogeneous monomials $b \in \mathcal{B}$ with $\deg(b) = (r_1, \ldots, r_m) = \mathbf{r}$. Then $\mathbf{r} \in Q_i = C_i \cap \mathbb{Z}^m$ for some $C_i \in \Sigma$. Since Q_i is a pointed rational cone, it has a Hilbert basis

$$H_{Q_i} = \{\mathbf{r}_{i1}, \ldots, \mathbf{r}_{in_i}\} \text{ with } \mathbf{r}_{ij} \in \mathbb{Z}^m.$$

So we can write

$$\mathbf{r} = \sum_{j=1}^{n_i} m_j \mathbf{r_{ij}}.$$

Then, since each f_k is nonnegative and linear on Q_i, we have

$$f_k(\mathbf{r}) = \sum_{j=1}^{n_i} m_j f_k(\mathbf{r}_{ij}) \text{ for each } k = 1, \ldots, n.$$

Since R is Noetherian, for each i, there exists a finite set $\Lambda_{i,j,k} \subset R$ such that

$$I_k^{f_k(\mathbf{r}_{ij})} = (x | x \in \Lambda_{i,j,k}).$$

So

$$\begin{aligned}
b \in \mathcal{B}_{\mathbf{r}} &= I_1^{f_1(\mathbf{r})} \cdots I_n^{f_n(\mathbf{r})} \mathbf{u}^{\mathbf{r}} \\
&= I_1^{\sum_{j=1}^{n_i} m_j f_1(\mathbf{r})} \cdots I_n^{\sum_{j=1}^{n_i} m_j f_n(\mathbf{r})} \mathbf{u}^{\sum_{j=1}^{n_i} m_j \mathbf{r}_{ij}} \\
&= I_1^{m_1 f_1(\mathbf{r}_{i1})} \cdots I_1^{m_{n_i} f_1(\mathbf{r}_{in_i})} \cdots I_n^{m_1 f_n(\mathbf{r}_{i1})} \cdots I_n^{m_{n_i} f_n(\mathbf{r}_{in_i})} \mathbf{u}^{m_1 \mathbf{r}_{i1}} \cdots \mathbf{u}^{m_{n_i} \mathbf{r}_{in_i}} \\
&= \left(I_1^{f_1(\mathbf{r}_{i1})} \cdots I_n^{f_n(\mathbf{r}_{i1})} \mathbf{u}^{\mathbf{r}_{i1}} \right)^{m_1} \cdots \left(I_1^{f_1(\mathbf{r}_{in_i})} \cdots I_n^{f_n(\mathbf{r}_{in_i})} \mathbf{u}^{\mathbf{r}_{in_i}} \right)^{m_{n_i}} .
\end{aligned}$$

So $\mathcal{B}$ is generated as an algebra over R by the set

$$\{x_1 \cdots x_n \mathbf{u}^{\mathbf{r}_{ij}} | \mathbf{r}_{ij} \in \mathcal{H}_{Q_i}, x_k \in \Lambda_{i,j,k}\}.$$

□

2.3.2 Fan Algebras with Functional Exponents

Another generalization of the fan algebra comes from allowing the exponents of the dummy variables to be fan-linear functions.

Definition 2.3.8. Let Σ be a fan of cones that fills the positive orthant of $\mathbb{N}^p$, let R be a domain with ideals $I_1, \ldots, I_n$, with fan-linear functions

$$f_1(r_1, \ldots, r_p), \ldots, f_n(r_1, \ldots, r_p)$$

and linear functions

$$g_1(r_1, \ldots, r_p), \ldots, g_m(r_1, \ldots, r_p)$$

on Σ from $\mathbb{R}^p \to \mathbb{R}$. Then we define

$$\mathcal{B}(\Sigma, f, g) = \bigoplus_{r_1, \ldots, r_p \geq 0} I_1^{f_1(r_1, \ldots, r_p)} \cdots I_n^{f_n(r_1, \ldots, r_p)} u_1^{g_1(r_1, \ldots, r_p)} \cdots u_m^{g_m(r_1, \ldots, r_p)}$$

to be the *fan algebra of* f *and* g *on* Σ, where $f = (f_1, \ldots, f_n), g = (g_1, \ldots, g_m)$.

Theorem 2.3.9. *The algebra* $\mathcal{B} = \mathcal{B}(\Sigma, f, g)$ *is finitely generated.*

Proof. The subadditivity of the functions f_i, g_i guarantees that $\mathcal{B}$ is a subalgebra of $R[u_1, \ldots, u_m]$ with the natural grading. Since $\mathcal{B}$ has a natural $\mathbb{N}^p$-grading, it is enough to consider only homogeneous monomials $b \in \mathcal{B}$ with $\deg(b) = (r_1, \ldots, r_p) = \mathbf{r}$. Then $\mathbf{r} \in Q_i = C_i \cap \mathbb{Z}^p$ for some $C_i \in \Sigma$. Since Q_i is a pointed rational cone, it has a Hilbert basis

$$H_{Q_i} = \{\mathbf{r}_{i1}, \ldots, \mathbf{r}_{in_i}\} \text{ with } \mathbf{r}_{ij} \in \mathbb{Z}^p.$$

So we can write

$$\mathbf{r} = \sum_{j=1}^{n_i} m_j \mathbf{r_{ij}}.$$

Then, since each f_k is nonnegative and linear on Q_i, we have

$$f_k(\mathbf{r}) = \sum_{j=1}^{n_i} m_j f_k(\mathbf{r}_{ij}) \text{ for each } k = 1, \ldots, n,$$

and

$$g_k(\mathbf{r}) = \sum_{j=1}^{n_i} m_j f_k(\mathbf{r}_{ij}) \text{ for each } k = 1, \ldots, m.$$

Since R is Noetherian, for each i, there exists a finite set $\Lambda_{i,j,k} \subset R$ such that

$$I_k^{f_k(\mathbf{r}_{ij})} = (x | x \in \Lambda_{i,j,k}).$$

So

$$\begin{aligned} b \in \mathcal{B}_{\mathbf{r}} &= I_1^{f_1(\mathbf{r})} \cdots I_n^{f_n(\mathbf{r})} u_1^{g_1(\mathbf{r})} \cdots u_m^{g_m(\mathbf{r})} \\ &= I_1^{\sum_{j=1}^{n_i} m_j f_1(\mathbf{r})} \cdots I_n^{\sum_{j=1}^{n_i} m_j f_n(\mathbf{r})} u_1^{\sum_{j=1}^{n_i} m_j g_1(\mathbf{r}_{ij})} u_m^{\sum_{j=1}^{n_i} m_j g_m(\mathbf{r}_{ij})} \\ &= \left(I_1^{f_1(\mathbf{r}_{i1})} \cdots I_n^{f_n(\mathbf{r}_{i1})} u_1^{g_1(\mathbf{r}_{i1})} \cdots u_m^{g_m(\mathbf{r}_{i1})} \right)^{m_1} \cdots \\ &\quad \left(I_1^{f_1(\mathbf{r}_{in_i})} \cdots I_n^{f_n(\mathbf{r}_{in_i})} u_1^{g_1(\mathbf{r}_{in_i})} \cdots u_m^{g_m(\mathbf{r}_{in_i})} \right)^{m_{n_i}}. \end{aligned}$$

So $\mathcal{B}$ is generated as an algebra over R by the set

$$\{x_1 \cdots x_n u_1^{g_1(\mathbf{r}_{ij})} \cdots u_n^{g_n(\mathbf{r}_{ij})} | \mathbf{r}_{ij} \in \mathcal{H}_{Q_i}, x_k \in \Lambda_{i,j,k}\}.$$

□

CHAPTER 3

THE POLYNOMIAL RING CASE

3.1 The General Theorem

In this section, we will show that in the special case where R is a polynomial ring in finitely many variables over a field, then the intersection algebra of two principal monomial ideals is a semigroup ring whose generators can be algorithmically computed.

Let k be a field and Q a semigroup. Recall that $k[Q]$ is a semigroup ring, namely the k-algebra with k-basis $\{t^a | a \in Q\}$ and multiplication defined by $t^a \cdot t^b = t^{a+b}$. Also recall our notation: when x is a homogeneous element in a semigroup ring, $\log(x)$ denotes its exponent vector, and if X is a collection of homogeneous elements, $\log(X)$ refers to the set of exponent vectors of all the monomials in X.

Theorem 3.1.1. *If R is a polynomial ring in n variables over k, and I and J are ideals generated by monomials (i.e. monic products of variables) in R, then $\mathcal{B}$ is a semigroup ring.*

Proof. Since I and J are monomial ideals, $I^r \cap J^s$ is as well for all r and s. So each (r, s) component of $\mathcal{B}$ is generated by monomials, therefore $\mathcal{B}$ is a subring of $k[x_1, \ldots, x_n, u, v]$ generated over k by a list of monomials $\{b_i | i \in \Lambda\}$. Let Q be the semigroup generated by $\{\log(b_i) | i \in \Lambda\}$. Then $\mathcal{B} = k[Q]$, and $\mathcal{B}$ is a semigroup ring over k. □

Since a polynomial ring $R = k[x_1, \ldots, x_n]$ is a UFD, we will use the result of the previous chapter to show that intersection algebras of principal ideals in polynomial rings are finitely generated over R. Then, since R is generated over k by the variables $x_1, \ldots, x_n$, adding those variables to our list of generators will provide a generating set for $\mathcal{B}$ over k.

Theorem 3.1.2. *Let* $I = (x_1^{a_1} \cdots x_n^{a_n})$ *and* $J = (x_1^{b_1} \cdots x_n^{b_n})$ *be principal ideals in* $R = k[x_1, \ldots, x_n]$, *and let* $\Sigma_{\mathbf{a},\mathbf{b}}$ *be the fan associated to* $\mathbf{a} = (a_1, \ldots, a_n)$ *and* $\mathbf{b} = (b_1, \ldots, b_n)$. *Let*

$$Q_i = C_i \cap \mathbb{Z}^2 \text{ for every } C_i \in \Sigma_{\mathbf{a},\mathbf{b}}$$

and $\mathcal{H}_{Q_i}$ *be its Hilbert basis of cardinality* n_i *for all* $i = 0, \ldots, n$. *Further, let* Q *be the subsemigroup in* $\mathbb{N}^2$ *generated by*

$$\{(a_1 r_{ij}, \ldots, a_i r_{ij}, b_{i+1} s_{ij}, \ldots, b_n s_{ij}, r_{ij}, s_{ij}) | i = 0, \ldots, n, j = 1, \ldots, n_i\} \cup \log(x_1, \ldots, x_n),$$

where $(r_{ij}, s_{ij}) \in \mathcal{H}_{Q_i}$ *for every* $i = 0, \ldots n, j = 1, \ldots, n_i$. *Then* $\mathcal{B} = k[Q]$.

Proof. Since R is a UFD, by Theorem 2.1.6, $\mathcal{B}$ is generated over R by

$$\{x_1^{a_1 r_{ij}} \cdots x_i^{a_i r_{ij}} x_{i+1}^{b_{i+1} s_{ij}} \cdots x_n^{b_n s_{ij}} u^{r_{ij}} v^{s_{ij}} | i = 0, \ldots, n, j = 1, \ldots, n_i\}.$$

Then, since R is generated as an algebra over k by $x_1, \ldots, x_n$, it follows that $\mathcal{B} \subset k[x_1, \ldots, x_n, u, v]$ is generated as an algebra over k by the set

$$P = \{x_1, \ldots, x_n, x_1^{a_1 r_{ij}} \cdots x_i^{a_i r_{ij}} x_{i+1}^{b_{i+1} s_{ij}} \cdots x_n^{b_n s_{ij}} u^{r_{ij}} v^{s_{ij}} | i = 0, \ldots, n, j = 1, \ldots, n_i\}.$$

This is a set of monomials in $k[x_1, \ldots, x_n, u, v]$. Now note that therefore

$$\begin{aligned}\log(P) =& \{(a_1 r_{ij}, \ldots, a_i r_{ij}, b_{i+1} s_{ij}, \ldots, b_n s_{ij}, r_{ij}, s_{ij}) | i = 0, \ldots, n, j = 1, \ldots, n_i\} \\ & \cup \log(x_1, \ldots, x_n).\end{aligned}$$

In conclusion, $\log(P) = Q$ and hence $\mathcal{B} = k[Q]$.

□

Example 3.1.3. Let $I = (x^5 y^2)$ and $J = (x^2 y^3)$. Then $a_1 = 5, a_2 = 2$ and $b_1 = 2, b_2 = 3$, and $5/2 \geq 2/3$. Then we have the following cones:

$$C_0 = \{\lambda_1(0,1) + \lambda_2(2,5) | \lambda_i \in \mathbb{R}_{\geq 0}\}$$
$$C_1 = \{\lambda_1(2,5) + \lambda_2(3,2) | \lambda_i \in \mathbb{R}_{\geq 0}\}$$
$$C_2 = \{\lambda_1(3,2) + \lambda_2(1,0) | \lambda_i \in \mathbb{R}_{\geq 0}\}$$

C_0 is the wedge of the first quadrant between the y-axis and the vector $(2,5)$, C_1 is the wedge between $(2,5)$ and $(3,2)$, and C_3 is the wedge between $(3,2)$ and $(1,0)$. It is easy to see that this fan fills the entire first quadrant. Intersecting these cones with $\mathbb{Z}^2$ is equivalent to only considering the integer lattice points in these cones.

The Hilbert Basis of $Q_0 = C_0 \cap \mathbb{Z}^2$ is $\{(0,1),(1,3),(2,5)\}$, and their corresponding monomials in $\mathcal{B}$ are given by the generators of $\mathcal{B}_{r,s}$ for each (r,s):

$(0,1) : (I^0 \cap J^1)u = (x^2y^3)v$ – generator is x^2y^3v

$(1,3) : (I^1 \cap J^3)uv^3 = ((x^5y^2) \cap (x^6y^9))uv^3 = (x^6y^9)uv^3$ – generator is $x^6y^9uv^3$

$(2,5) : (I^2 \cap J^5)u^2v^5 = ((x^{10}y^4) \cap (x^{10}y^{15}))u^2v^5 = (x^{10}y^{15})u^2v^5$ – generator is $x^{10}y^{15}u^2v^5$.

Notice that all the generator monomials are of the form $x^{b_1s}y^{b_2s}u^rv^s$, with $b_1 = 2, b_2 = 3$, and (r,s) is a Hilbert Basis element, as shown earlier.

The Hilbert Basis of Q_1 is $\{(1,1),(1,2),(3,2),(2,5)\}$. In the same way as above, their monomials are $x^5y^3uv, x^5y^6uv^2, x^{15}y^6u^3v^2, x^{10}y^{15}u^2v^5$, all of which have the form $x^{a_1r}y^{b_2s}u^rv^s$ with $a_1 = 5, b_2 = 3$ and (r,s) a basis element.

Lastly, the Hilbert Basis of Q_2 is $\{(1,0),(2,1),(3,2)\}$, which gives rise to generators $x^5y^2u, x^{10}y^4u^2, x^{15}y^6u^3v^2$, all of which look like $x^{a_1r}y^{a_2r}u^rv^s$ with $a_1 = 5, a_2 = 2$.

Notice there are a few redundant generators in this list: those arise from lattice points that lie on the boundaries of the cones. So $\mathcal{B}$ is generated over R by

$$\{x^5y^2u, x^{10}y^4u^2, x^{15}y^6u^3v^2, x^5y^3uv, x^5y^6uv^2, x^2y^3v, x^6y^9uv^3, x^{10}y^{15}u^2v^5\}.$$

Then, since R is generated over k by x and y, $\mathcal{B}$ is generated over k by

$$\{x, y, x^5y^2u, x^{10}y^4u^2, x^{15}y^6u^3v^2, x^5y^3uv, x^5y^6uv^2, x^2y^3v, x^6y^9uv^3, x^{10}y^{15}u^2v^5\}.$$

Using this technique, we have written a program in Macaulay2 that will provide the list of generators of $\mathcal{B}$ for any I and J. First it fan orders the exponent vectors, then finds the Hilbert Basis for each cone that arises from those vectors. Finally, it computes the corresponding monomial for each basis element. The code is below:

```
loadPackage "Polyhedra"
--function to get a list of exponent vectors from an ideal I
expList=(I) ->(
     flatten exponents first flatten entries gens I
)


algGens=(I,J)->(
     B:=(expList(J))_(positions(expList(J),i->i!=0));
     A:=(expList(I))_(positions(expList(J),i->i!=0));
     L:=sort apply(A,B,(i,j)->i/j);
     C:=flatten {0,apply(L,i->numerator i),1};
     D:=flatten {1, apply(L,i->denominator i),0};
     M:=matrix{C,D};
     G:=unique flatten apply (#C-1, i-> hilbertBasis
      (posHull submatrix(M,{i,i+1})));
     S:=ring I[u,v];
     flatten apply(#G,i->((first flatten entries gens
      intersect(I^(G#i_(1,0)),J^(G#i_(0,0)))))*u^(G#i_(1,0))*v^(G#i_(0,0)))
)
```

3.2 Presentation Ideals of $\mathcal{B}$

Next we compute the presentation of $\mathcal{B}$. In some cases this can be done by hand, but more complicated examples require the use of semigroup ring theory. The presentation of this first example can be shown directly, and connects this intersection algebra with determinantal rings. We credit Yongwei Yao with the proof.

Theorem 3.2.1. *Let $R = k[x]$ and $a \in \mathbb{N}$, $a > 0$. Then*

$$\mathcal{B}((x^a),(x)) \cong \frac{k[x_1, \ldots, k_{a+3}]}{I_2(M)},$$

where $I_2(M)$ is the ideal generated by the 2×2 minors of the matrix

$$M = \begin{pmatrix} x_1 & x_3 & x_4 & \cdots & x_{a+2} \\ x_2 & x_4 & x_5 & \cdots & x_{a+3} \end{pmatrix}.$$

Proof. By Example 2.1.3 in the previous chapter, our algorithm shows that

$$\mathcal{B} = \mathcal{B}((x^a),(x)) \cong k[x, xv, x^a u, x^a uv, x^a uv^2, \ldots, x^a uv^a].$$

Construct the map

$$\varphi : k[x_1, \ldots, x_{a+3}] \to k[x, xv, x^a u, x^a uv, x^a uv^2, \ldots, x^a uv^a]$$

by sending x_1 to x, x_2 to xv, and so on. We claim that $\ker \varphi = I_2(M)$.

It is easy to see that $I_2(M) \subseteq \ker \varphi$. For the other inclusion, we will proceed by induction on a. Assume that $\ker \varphi \subsetneq I_2(M)$, and let $f(\mathbf{x}) \in \ker \varphi \setminus I_2(M)$. Choose a representative $g(\mathbf{x}) \in f(\mathbf{x}) + I_2(M)$ such that the total degree of $g(\mathbf{x})$ with respect to x_{a+3} is minimal and positive. Then $g(\mathbf{x})$ can be written as $g_1(x_1, \ldots, x_{a+2}) + g_2(\mathbf{x})$ for some g_1 and g_2 where x_{a+3} divides $g_2(\mathbf{x})$.

Since $x_{a+3} | g_2(\mathbf{x})$, we can use the relations in $I_2(M)$ to rewrite $g_2(\mathbf{x})$ in terms of only the variables x_2, x_{a+2}, and x_{a+3}, and $g_2(x_2, x_{a+2}, x_{a+3}) \neq 0$.

Then apply φ to g, so

$$\varphi(g) = \varphi(g_1 + g_2) = g_1(\mathbf{y}) + g_2(xv, x^a uv^{a-1}, x^a uv^a) = 0.$$

Let z be a monomial in $\varphi(g_1)$, and w be a monomial in $\varphi(g_2)$. Then

$$\begin{aligned} z &= x^{\alpha_1}(xv)^{\alpha_2}(x^a u)^{\alpha_3} \cdots (x^a uv^{a-1})^{\alpha_{a+2}} \\ w &= (xv)^{\beta_1}(x^a uv^{a-1})^{\beta_2}(x^a uv^a)^{\beta_3}, \end{aligned} \tag{3.1}$$

and note that $\beta_3 > 0$. We will show that $z \neq w$.

Assume not. Then, by matching degrees in (3.1), we have the following equations:

$$\alpha_1 + \alpha_2 + a(\alpha_3 + \cdots + \alpha_{a+2}) = \beta_1 + a\beta_2 + a\beta_3 \tag{3.2}$$

$$\alpha_3 + \alpha_4 + \ldots + \alpha_{a+2} = \beta_2 + \beta_3 \tag{3.3}$$

$$\alpha_2 + \alpha_4 + 2\alpha_5 + \ldots + (a-1)\alpha_{a+2} = \beta_1 + (a-1)\beta_2 + a\beta_3. \tag{3.4}$$

Combining (3.2) and (3.3), we obtain

$$\alpha_1 + \alpha_2 = \beta_1. \tag{3.5}$$

Then, by (3.4) and (3.5), and because $\beta_3 > 0$:

$$\begin{aligned} \beta_1 + (a-1)\beta_2 + a\beta_3 &= \beta_1 + (a-1)(\beta_2 + \beta_3) + \beta_3 \\ &= \beta_1 + \beta_3 + (a-1)(\alpha_3 + \ldots + \alpha_{a+2} \\ &\geq \beta_1 + \beta_3 + \alpha_4 + \cdots + (a-1)\alpha_{a+2} \\ &= \alpha_1 + \alpha_2 + \beta_3 + \alpha_4 + \cdots + (a-1)\alpha_{a+2} \\ &> \alpha_2 + \alpha_4 + \cdots + (a-1)\alpha_{a+2} \end{aligned}$$

which contradicts (3.4). So no monomials in g_1 and g_2 will cancel after taking their images under φ.

In addition, no two monomials within g_2 will cancel after applying φ. Choose two distinct monomials in g_2 whose images are equal. Then we have

$$x_2^{\beta_1} x_{a+2}^{\beta_2} x_{a+3}^{\beta_3} \neq x_2^{\beta_1'} x_{a+2}^{\beta_2'} x_{a+3}^{\beta_3'}$$

and

$$(xv)^{\beta_1}(x^a uv^{a-1})^{\beta_2}(x^a uv^a)^{\beta_3} = (xv)^{\beta_1'}(x^a uv^{a-1})\beta_2'(x^a uv^a)^{\beta_3'}.$$

But this is impossible, since the monomials' exponent vectors $(1,0,1),(a,1,a-1),(a,1,a)$ are all linearly independent.

So since $\varphi(g) = \varphi(g_1 + g_2) = \varphi(g_1) + \varphi(g_2) = 0$, we see that $\varphi(g_1) = \varphi(g_2) = 0$, and also $g_2 = 0$. Thus $f = g_1$, which does not involve the variable x_{a+3}. So by the induction hypothesis, $g_1 \in I_2(M)$.

$\square$

In this case, we can apply the language of determinantal rings to obtain another description of $\mathcal{B}$. Let

$$X = \begin{pmatrix} x_{11} & x_{12} & x_{13} & \cdots & x_{1(a+1)} \\ x_{21} & x_{22} & x_{23} & \cdots & x_{2(a+1)} \end{pmatrix}$$

and let $T = k[X]/I_2(X)$. Then

$$\mathcal{B} \cong \frac{T}{(x_{22} - x_{13}, x_{23} - x_{14}, \dots, x_{2a} - x_{1(a+1)})}.$$

Using the results from [4] reproduced in Section 1.4, we have a more algorithmic way of producing a presentation of $\mathcal{B}$.

Example 3.2.2. The following example shows the process of constructing a presentation for the intersection algebra of $\mathcal{B} = \mathcal{B}((x^n),(x^{n+1}))$. We will prove towards the end of this section in Example 3.3.8 that the R-algebra generating set of $\mathcal{B}$ is

$$\{x^{n+1}v, x^n u, x^{n+1}uv, x^{2n}u^2v, x^{3n}u^3v^2, \ldots, x^{(n+1)n}u^{n+1}v^n\}.$$

We form our matrix by using the exponent vectors of the generators as column vectors, together with $(1,0,0)^\intercal$, which is the exponent vector of x. This matrix is

$$M = \begin{pmatrix} n+1 & n & n+1 & 2n & 3n & \cdots & n^2 & n(n+1) & 1 \\ 0 & 1 & 1 & 2 & 3 & \cdots & n & n+1 & 0 \\ 1 & 0 & 1 & 1 & 2 & \cdots & n-1 & n & 0 \end{pmatrix}.$$

Next, we compute the matrix of the nullspace vectors of M. This can easily be shown to be the $(n+4) \times (n+1)$ matrix

$$\mathbf{L} = \begin{pmatrix} -1 & n & n-1 & \cdots & 2 & 1 \\ -1 & 0 & -1 & \cdots & -(n-2) & -(n-1) \\ 1 & -n(n+1) & -(n-1)(n+1) & \cdots & -2(n+1) & -(n+1) \\ 0 & 0 & 0 & \cdots & 0 & n \\ 0 & 0 & 0 & \cdots & n & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & n & \cdots & 0 & 0 \\ 0 & n & 0 & \cdots & 0 & 0 \\ n & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

.

Next we form the ideal $I_{\mathbf{L}}$ by forming binomials from each column of $\mathbf{L}$. Therefore

$$\begin{aligned} I_{\mathbf{L}} =& (x_{n+4}^n x_3 - x_1x_2, x_1^n x_{n+3}^n - x_3^{n(n+1)}, x_1^{n-1}x_{n+2}^n - x_2x_3^{(n-1)(n+1)}, \\ & x_1^{n-2}x_{n+1}^n - x_2^2x_3^{(n-2)(n+1)}, \cdots, x_1^2x_5^2 - x_2^{n-2}x_3^{2(n+1)}, x_1x_4^n - x_2^{(n-1)}x_3^{(n+1)}). \end{aligned}$$

Finally, we take the saturation of this ideal with respect to the product of all the variables to obtain the lattice ideal. Our computations in Macaulay 2 [11] show that the following statement is very likely to be true.

Conjecture 3.2.3.

$$I_L = I_{\mathbf{L}} : (x_1 \cdots x_{n+3})^\infty = I_{\mathbf{L}} + I_2(M) \text{ where}$$

$$M = \begin{pmatrix} x_2^n & x_{n+4}^n & x_4^n & x_5^n & \cdots & x_{n+1}^n & x_{n+2}^n \\ x_4^n & x_3^n & x_5^n & x_6^n & \cdots & x_{n+2}^n & x_{n+3}^n \end{pmatrix}.$$

3.3 Two Principal Ideals in One Variable

3.3.1 Constructing a Hilbert Basis

The case of the intersection algebra when $I = (x^a)$ and $J = (x^b)$ and a and b are relatively prime is surprisingly much more complicated. In two dimensions, finding the Hilbert Basis of a cone is equivalent to finding a particular continued fraction decomposition. The following result is due to Van Der Corput, and it expresses the same idea in terms of finding a minimal basis of the integer points satisfying a system of two Diophantine equations.

This proof relies on two other results, also due to van der Corput. The following exposition follows his original papers [12, 13], included for the convenience of the reader.

In these theorems, $x = (x^{(1)}, x^{(2)}, \ldots, x^{(m)})$ is a lattice point in the m-dimensional space, meaning each $x^{(i)} \in \mathbb{Z}$, and S is a system of l equations and r inequalities

$$f_\lambda(x) = 0,\ \lambda = 1, \ldots, l; \quad g_l(x) \geq 0,\ l = 1, \ldots r,$$

where l and r are both ≥ 0, $f_\lambda(x)$ are all linear forms, and $g_l(x)$ denote integer linear forms in $x_1, x_2, \ldots, x_m$. Let S_0 be the system of equations

$$f_\lambda(x) = 0,\ \lambda = 1, \ldots, l; \quad g_l(x) = 0,\ l = 1, \ldots, r.$$

Definition 3.3.1. The integer solutions $x_1, \ldots, x_s$ form a *basis of* S if each integer solution x of S can be written as

$$x = p_1 x_1 + p_2 x_2 + \cdots + p_s x_s$$

with integer non-negative coefficients p_i.

A basis of S with the property that there is no basis for S with fewer elements is called a *minimal basis of* S. A minimal basis of S is the zero element if and only if coordinate origin is the only integer solution of S.

Theorem 3.3.2. *Let $r = 0$, so that S is a system of l linear homogeneous equations*

$$S : f_\lambda(x) = 0 \text{ for } \lambda = 1, 2 \ldots, l.$$

Assume that S has at least one nontrivial solution, so that the integer solutions x of S form a free abelian group N. Assume that $rkN = n \geq 1$, and say that N is generated by the lattice points $a_1, \ldots, a_n$. Define the $n+1$ points $x_1, \ldots, x_{n+1}$ to be

$$x_i = q_{i,1} a_1 + q_{i,2} a_2 + \cdots + q_{i,n} a_n,\ i = 1, \ldots, n+1. \tag{3.6}$$

Then the following are true:

1. *The size of the minimal basis of S is $n + 1$.*

2. *The points $x_1, \ldots, x_{n+1}$ form a minimum basis of S if and only if the $n+1$ determinants*

$$D_i = \begin{vmatrix} q_{i+1,1} & \cdots & q_{i+1,n} \\ \vdots & \cdots & \vdots \\ q_{n+1,1} & \cdots & q_{n+1,n} \\ q_{1,1} & \cdots & q_{1,n} \\ \vdots & \cdots & \vdots \\ q_{i-1,1} & \cdots & q_{i-1,n} \end{vmatrix}, \quad i = 1, \ldots, n+1$$

are all relatively prime, and are either all positive or all negative.

Proof. Note that this theorem implies that the $n+1$ points $-(a_1+a_2+\cdots+a_n), a_1, a_2, \ldots, a_n$ form a minimal basis of S, since for these points, the determinants $D_i = 1$ for all i.

In this proof, we will omit commas between subscripts in the matrix entries when the meaning is clear.

1. Assume the determinants D_i are all relatively prime, and assume that they are either all positive or all be negative.

   Let N' be the free abelian group generated by $x_1, \ldots, x_{n+1}$. Since each $x_i \in N$, $N' \subseteq N$. We will show $N' = N$.

   Let $v_1 = 0$ and v_i, $i = 2, \ldots, n+1$ be equal to cofactor associated to the $q_{i,1}$ minor of the determinant

$$D_1 = \begin{vmatrix} q_{2,1} & \cdots & q_{2,n} \\ \vdots & \cdots & \vdots \\ q_{n+1,1} & \cdots & q_{n+1,n} \end{vmatrix}.$$

   Then since the cofactor expansion for D_1 along the first column is

$$D_1 = \sum_{i=2}^{n+1} v_i q_{i1},$$

   and also that $\sum_{i=2}^{n+1} v_i q_{ij} = 0$ for $j \neq 1$, we have that

$$\begin{aligned} \sum_{i=2}^{n+1} v_i x_i &= \sum_{i=2}^{n+1} v_i \sum_{j=1}^{n} q_{ij} a_j \text{ by (3.6)} \\ &= \sum_{j=1}^{n} a_j \sum_{i=2}^{n+1} v_i q_{ij} = a_1 D_1. \end{aligned}$$

   .

So $a_1 D_1 \in N'$. Similarly, N' also contains the points $a_1 D_2, \ldots, a_1 D_{n+1}$, and since $D_1, \ldots D_{n+1}$ are all relatively prime, $a_1 \in N'$. The same procedure with a_i instead of a_1 shows that the points $a_1, \ldots, a_n$ all belong to N', so $N = N'$.

So each point $x \in N$ may be written in the form

$$x = v_1 x_1 + v_2 x_2 + \cdots + v_{n+1} x_{n+1}, v_i \in \mathbb{Z}.$$

Then, since

$$\begin{aligned}\sum_{i=1}^{n+1} D_i x_i &= \sum_{i=1}^{n+1} D_i \sum_{j=1}^{n} q_{ij} a_j \text{ by (3.6)} \\ &= \sum_{j=1}^{n} a_j \sum_{j=1}^{n+1} D_i q_{ij} = 0,\end{aligned}$$

we also have that for every integer a, $a \sum_{i=1}^{n+1} D_i x_i = 0$. So

$$x = (v_1 + aD_1)x_1 + (v_2 + aD_2)x_2 + \cdots + (v_{n+1} + aD_{n+1})x_{n+1}. \tag{3.7}$$

Since the determinants $D_1, \ldots, D_{n+1}$ are either all positive or all negative, one can determine an integer a such that the $n+1$ coefficients $(v_i + aD_i)\; i = 1, \ldots, n+1$ are all positive, and from (3.7) it is clear that the numbers $x_1, \ldots, x_{n+1}$ form a basis of S.

To show that they form a minimal basis of S, we will prove that n integer solutions $X_1, \ldots, X_n$ of S can never form a basis for S. Assume not. Then, for each point x of N, we can find $V_1, \ldots, V_n \in \mathbb{N}$ with

$$x = V_1 X_1 + \cdots + V_n X_n.$$

Since the rank of N is n, these equations are linearly independent, therefore, the coefficients $V_1, \ldots, V_n$ for a given x are uniquely determined. Let $x = -X_1$. Then the only solution is

$$V_1 = -1, V_2 = V_3 = \cdots = V_n = 0,$$

which is a contradiction.

Hence we have shown that the points $x_1, \ldots, x_{n+1}$ form a minimal basis.

2. Let the numbers $x_1, \ldots, x_{n+1}$ form a basis of S, and assume the $n+1$ determinants $D_1, \ldots, D_{n+1}$ are all divisible by the same prime number p. Recall the definition of D_i:

$$D_i = \begin{vmatrix} q_{i+1,1} & \cdots & q_{i+1,n} \\ \vdots & \cdots & \vdots \\ q_{n+1,1} & \cdots & q_{n+1,n} \\ q_{1,1} & \cdots & q_{1,n} \\ \vdots & \cdots & \vdots \\ q_{i-1,1} & \cdots & q_{i-1,n} \end{vmatrix}, \quad i = 1, \ldots, n+1,$$

and note that the cofactor expansion of D_i across the first row is

$$D_i = \sum_{j=1}^{n} q_{i+1,j} C_{1,j},$$

where $C_{1,j}$ denotes the cofactor associated to the minor of $q_{i+1,j}$.

So since p divides every D_i, there exist n integers $c_1, \ldots, c_n$, not all of which are divisible by p, with the property that p is a divisor of the $n+1$ numbers

$$\sum_{j=1}^{n} c_j q_{ij}, \; i = 1, \ldots, n+1.$$

Assume without loss of generality that c_j is not divisible by p. Since the integers $x_1, \ldots, x_{n+1}$ form a basis of S, for each point $a_1, \ldots a_n$ there exist $n+1$ integers

$w_1, \ldots, w_{n+1}$ with

$$\begin{aligned} a_\omega &= w_1 x_1 + \cdots + w_{n+1} x_{n+1} \\ &= \sum_{i=1}^{n+1} w_i \sum_{j=1}^{n} q_{ij} a_j \text{ by (3.6)} \\ &= \sum_{j=1}^{n} a_j \sum_{i=1}^{n+1} \omega_i q_{ij}. \end{aligned}$$

Therefore if $j \neq \omega$, $\sum_{i=1}^{n+1} w_i q_{i\omega} = 0$, and if $j = \omega$, $\sum_{i=1}^{n+1} w_i q_{i\omega} = 1$. It follows that

$$c_\omega = \sum_{j=1}^{n} c_j \sum_{i=1}^{n+1} w_i q_{ij} = \sum_{i=1}^{n+1} w_i \sum_{j=1}^{n} c_j q_{ij},$$

which is a contradiction, as the right side is divisible by p, but not the left. Therefore, the $n+1$ determinants $D_1, \ldots, D_{n+1}$ have a greatest common divisor of 1, and in particular do not all vanish simultaneously.

We conclude by demonstrating that the determinants $D_1, \ldots, D_{n+1}$ are either all positive or all negative. Assume not. Then we can select a lattice point $(v_1, \ldots, v_n)$ such that, for every a, the set

$$\{v_i + aD_i | i = 1, \ldots, n+1\}$$

contains at least one negative number. The point

$$x = v_1 x_1 + \cdots + v_{n+1} x_{n+1}$$

is then an integer solution of S, and for any collection of coefficients $(p_1, \ldots, p_{n+1})$ with

$$x = p_1 x_1 + \cdots + p_{n+1} x_{n+1},$$

we have that $p_i = v_i + aD_i$ for every $i = 1, \ldots, n+1$ by (3.7). Thus any collection of coefficients contains at least one negative number. This contradicts the assumption that the numbers $x_1, \ldots, x_{n+1}$ form a basis of S, thus the determinants $D_1, \ldots, D_{n+1}$

are either all positive or all negative.

□

Definition 3.3.3. Let $r \geq 1$, and denote by $K(S)$ the set of lattice points $u = (u_1, \ldots, u_r) \neq \mathbf{0}$ with $u_i \geq 0$, where there is a lattice point x

$$f_\lambda(x) = 0,\ \lambda = 1, \ldots, l; \quad g_l(x) = u_l,\ l = 1, \ldots, r$$

such that for every lattice point ξ with

$$f_\lambda(\xi) = 0,\ \lambda = 1, \ldots, l; \quad 0 \leq g_l(\xi) \leq u_l,\ l = 1, \ldots, r$$

either

$$g_l(\xi) = 0 \text{ for all } l = 1, \ldots, r \text{ or } g_l(\xi) = u_l \text{ for all } l = 1, \ldots, r.$$

Theorem 3.3.4. *Let $r \geq 1$, and let the integer solutions of the system*

$$S_0 : f_\lambda(x) = 0,\ \lambda = 1, \ldots, l; \quad g_l(x) = 0,\ l = 1, \ldots, r$$

form a rank n free abelian group.

Let $B(S_0)$ be a basis of S_0, $B(S)$ a basis of S, and let $M(S), M(S_0)$ denote minimal bases of S and S_0, respectively. Each point $u = (u_1, \ldots u_r) \in K(S)$ is the image of a point x via the g_l's, i.e.

$$f_\lambda(x) = 0,\ \lambda = 1, \ldots, l; \quad g_l(x) = u_l,\ l = 1, \ldots, r. \tag{3.8}$$

Denote this collection of lattice points x satisfying (3.8) for some $u \in K(S)$ by $\mathfrak{B}(S)$.

Also let $\mathfrak{M}(S) \subset \mathfrak{B}(S)$ be a collection of lattice points x such that each $u \in K(S)$ is the image of a unique x via (3.8).

Then the following are true:

1. *The set $B(S_0) + \mathfrak{B}(S)$ is a basis of S.*

2. *The points of the basis $B(S)$ of S that satisfy the system S_0 form a basis of S_0.*

3. *Any point $u \in K(S)$ is the image of an $x \in B(S)$ via the g_l's (note that since every $u \in K(S)$ is nonzero, this corresponding point x does not satisfy the system S_0).*

4. *$M(S_0) + \mathfrak{M}(S)$ is a minimal basis of S.*

5. *Each minimum basis $M(S)$ of S contains exactly 0 (respectively $n+1$) points that satisfy the system S_0, depending on whether $n = 0$ or $n \geq 1$, and these points form a minimum basis of S_0.*

6. *If $M(S)$ is any minimal basis of S, then any point x of $M(S)$ which does not satisfy the system S_0 can be unambiguously associated to a point u of $K(S)$ via (3.8).*

Proof. Note that Theorems 3.3.2 and 3.3.4 show that every system S of linear homogeneous equations and inequalities, where the inequalities are integer linear forms, has a finite basis.

Let $k = |K(S)|$.

1. Let $x_1, \ldots, x_k$ be points of $\mathfrak{B}(S)$ with the property that each point $u = (u_1, \ldots, u_t)$ of $K(S)$ can be associated with a point x_i for some $i = 1, \ldots, k$ via

$$g_l(x_i) = u_l,\ l = 1, \ldots, r.$$

Let$B(S_0) = \{X_1, \ldots, X_s\}$, and assume that $B(S_0) + \mathfrak{B}(S)$ is not a basis for S. Then we can select an integer solution x of S that is not of the form

$$x = \sum_{j=1}^{s} P_j X_j + \sum_{i=1}^{k} p_i x_i \tag{3.9}$$

with non-negative integer coefficients, and also has the property that

$$\sum_{l=1}^{r} g_l(x)$$

is as small as possible. Consequently, x does not satisfy S_0, for every integer solution x of S_0 has the form

$$x = \sum_{j=1}^{s} P_j X_j$$

with non-negative integer coefficients, as $B(S_0)$ is a basis of S_0. The point with the r coordinates $g_l(x)$ for all $l = 1, \ldots, r$ does not belong to the set $K(S)$; otherwise, for a suitably chosen i, $1 \leq i \leq k$,

$$g_l(x) = g_l(x_i),\ l = 1, \ldots, r,$$

and then $x - x_i$ would be a point of S_0, so

$$x - x_i = \sum_{j=1}^{s} P_j X_j, \text{ and therefore } x = x_i + \sum_{j=1}^{s} P_j X_j$$

with suitably chosen non-negative integer coefficients, contradicting the assumption that x cannot be written as a combination of basis elements.

Since the point $(g_1(x), \ldots, g_r(x))$ does not belong to the set $K(S)$, it follows from the definition of this set that one lattice point ξ exists with

$$f_\lambda(\xi) = 0, \lambda = 1, \ldots, l; \quad 0 \leq g_l(\xi) \leq g_l(x),\ l = 1, \ldots, r$$

and

$$0 < \sum_{l=1}^{r} g_l(\xi) < \sum_{l=1}^{r} g_l(x).$$

Then

$$\sum_{l=1}^{r} g_l(\xi) < \sum_{l=1}^{r} g_l(x) \text{ and } \sum_{l=1}^{r} g_l(x - \xi) < \sum_{l=1}^{r} g_l(x).$$

The points x and $x - \xi$ are integer solutions of S, and since x was chosen to be the integer solution of S not satisfying (3.9) such that $\sum_{l=1}^{r} g_l(x)$ is minimal, we have that both ξ and x can be written as sums of basis elements as

$$\xi = \sum_{j=1}^{s} Q_j X_j + \sum_{i=1}^{k} q_i x_i \text{ and } x - \xi = \sum_{j=1}^{s} W_j X_j + \sum_{i=1}^{k} w_i x_i$$

with non-negative integer coefficients. Hence it follows that

$$x = \xi + x - \xi = \sum_{j=1}^{s} (Q_j + w_j) X_j + \sum_{i=1}^{k} (q_i + w_i) x_i$$

so x can be written in the form of (3.9) with

$$P_j = Q_j + w_j \geq 0,\ j = 1, \ldots, s; \quad p_i = q_i + w_i \geq 0,\ i = 1, \ldots, k$$

which is a contradiction.

2. Let $X_1, \ldots, X_s$ denote the points of the basis $B(S)$ satisfying the system S_0, and let $x_1, \ldots x_t$ denote the points of $B(S)$ which do not satisfy S_0, so any integer solution x of S, also, a fortiori, any integer solution x of S_0 can be written as

$$x = \sum_{j=1}^{s} P_j X_j + \sum_{i=1}^{t} p_i x_i \tag{3.10}$$

with non-negative integer coefficients. Since x_i is not a solution of the system S_0 then

$$\sum_{l=1}^{r} g_l(x_i) > 0,\ i = 1, \ldots, t.$$

It follows that all coefficients p_i in (3.10) vanish, otherwise

$$\begin{aligned} 0 = \sum_{l=1}^{r} g_l(x) &= \sum_{j=1}^{s} P_j \sum_{l=1}^{r} g_l(X_j) + \sum_{i=1}^{t} p_i \sum_{l=1}^{r} g_l(x_i) \\ &= \sum_{i=1}^{t} p_i \sum_{l=1}^{r} g_l(x_i) > 0. \end{aligned}$$

Therefore by (3.10), any integer solution x of S_0 has the form

$$x = \sum_{j=1}^{s} P_j X_j$$

with integer non-negative coefficients, so the points $X_1, \ldots, X_s$ form a basis of S_0.

3. Let $X_1, \ldots, X_s$ be the points of the basis $B(S)$ that satisfy the system S_0, and $x_1, \ldots, x_t$ be the points of $B(S)$ that do not satisfy S_0. Also, assume by way of contradiction that there is a $u = (u_1, \ldots, u_r) \in K(S)$ such that u is not the image of any of the x_i under the g_l's. In other words, that there exists no h, $1 \leq h \leq t$ such that

$$g_l(x_h) = u_l,\ l = 1, \ldots, r. \tag{3.11}$$

Now there exists an integer solution x of S with

$$g_l(x) = u_l,\ l = 1, \ldots, r. \tag{3.12}$$

As $B(S)$ is a basis of S, by (3.10) we have

$$x = \sum_{j=1}^{s} P_j X_j + \sum_{i=1}^{t} p_i x_i$$

for some $P_j, p_i \in \mathbb{N}$, and since $u \in K(S)$, u is nonzero. So x is nonzero, and there exists at least one i, $i = 1, \ldots, t$, such that $p_i \geq 1$.

Since each X_j is a solution of S_0,

$$g_l(x) = \sum_{j=1}^{s} P_j X_j + \sum_{i=1}^{t} p_i x_i = \sum_{i=1}^{t} p_i x_i,\ l = 1, \ldots, r.$$

So $u_l = g_l(x) \geq g_l(x_i) \geq 0$ for all $l = 1, \ldots, r$, and since $u \in K(S)$, either $g_l(x_i) = 0$ or $g_l(x_i) = u_l$. But this is a contradiction, since x_i is not a solution of S_0, and u is not an image of one of the x_i's.

4. That $M(S_0) + \mathfrak{M}(S)$ is a basis of S follows from the first claim. First assume n, the free rank of the group of solutions of S_0, is 0. Then by assertions 2 and 3, every basis $B(S)$ of S consists of at least $k = |K(S)|$ points, none of which satisfy the system S_0.

   Then assume $n \geq 1$. Then again by assertions 2 and 3, along with Theorem 3.3.2, $B(S)$ contains at least $n + 1 + k$ points, at least $n + 1$ of which satisfy S_0, and the remaining k do not.

   Since the set $M(S_0) + \mathfrak{M}(S)$ contains either k or $n + k + 1$ elements, depending on whether $n = 0$ or $n > 1$, this set must be a minimal basis of S_0.

(5)–(6) After the fourth assertion, the minimal basis $M(S)$ consists of k (respectively $n+k+1$) points, depending on whether $n = 0$ or $n > 1$. From the second assertion, it follows that $M(S)$ contains at least 0 (respectively $n+1$) integer solutions of S_0, and the third assertion shows that $M(S)$ has at least k points that do not satisfy the system S_0. Therefore $M(S)$ contains exactly 0 (respectively $n + 1$) solutions of S_0, which form a basis of S_0 by the second assertion, so that by Theorem 3.3.2, those solutions are a minimal basis of S_0. Also $M(S)$ contains exactly k points that do not satisfy the system S_0, and by the third assertion, one can unambiguously assign to each of these points a point u of $K(S)$ by (3.8).

□

Theorem 3.3.5. *If* $\mathbf{0}$ *is the only integer solution of* S_0*, then the minimal basis* $M(S)$ *of* S *is the set of the integral solutions* $x = (x^{(1)}, \ldots, x^{(m)}) \neq \mathbf{0}$ *of* S*, with the property that no lattice point* $\xi = (\xi^{(1)}, \ldots, \xi^{(m)}) \neq \mathbf{0}$ *and* $\neq x$ *exists satisfying the following relations*

$$f_\lambda(\xi) = \mathbf{0} \quad \lambda = 1, \ldots, l; \quad 0 \leq g_l(\xi) \leq g_l(x) \quad l = 1, \ldots, r. \tag{3.13}$$

Proof. If $r = 0$, then S and S_0 are same systems, so that $\mathbf{0}$ is the only integer solution of S, and the minimal basis $M(S)$ of S is empty. Then no lattice point $\xi = (\xi^{(1)}, \ldots, \xi^{(m)}) \neq \mathbf{0}$ of the system S exists, and also there is no lattice point $\xi \neq \mathbf{0}$ satisfying (3.13). So the proof is clear in the special case with $r = 0$.

So we may assume $r \geq 1$. Recall that n is the rank of the free abelian group formed by the integer solutions of S_0. Since the origin is the only integer solution of S_0, $n = 0$. By the fifth assertion of Theorem (3.3.4), $M(S)$ does not contain any points that satisfy S_0. Therefore, the minimal basis $M(S)$ also does not contain $\mathbf{0}$.

If x is any lattice point with

$$f_\lambda(x) = 0,\ \lambda = 1, \ldots, l,$$

and where

$$g_l(x) = u_l,\ l = 1, \ldots, r,$$

then not only is $u = (u_1, \ldots, u_r)$ uniquely determined by x, but also x is uniquely determined by u; because if there were a second lattice point X with

$$f_\lambda(X) = 0,\ \lambda = 1, \ldots, l \text{ and } g_l(X) = u_l,\ l = 1, \ldots, r,$$

then $X - x$ would be a nonzero integer solution of S_0.

By the sixth assertion of Theorem 3.3.4, each $x \in M(S)$ can be uniquely associated to a $u \in K(S)$. So the integer solutions x of S belongs to the minimal basis $M(S)$ of S if and only if, for each lattice point ξ with

$$f_\lambda(\xi) = 0,\ \lambda = 1, \ldots, l; \quad 0 \leq g_l(\xi) \leq u_l,\ l = 1, \ldots, r,$$

either the relations

$$f_\lambda(\xi) = 0, \quad \lambda = 1, \ldots, l; \quad g_l(\xi) = 0, \quad l = 1, \ldots, r \tag{3.14}$$

or the relations

$$f_\lambda(\xi) = 0, \quad \lambda = 1, \ldots, l; \quad g_l(\xi) = u_l, \quad l = 1, \ldots, r \tag{3.15}$$

are satisfied. The relations (3.14) are satisfied if and only if $\xi = \mathbf{0}$, and the relations (3.15) are only satisfied if and only $\xi = x$. Therefore, from the sixth assertion of Theorem 3.3.4, it is clear that an integer solution $x \neq \mathbf{0}$ of S belongs to the minimal basis $M(S)$ of S if and only if each lattice point ξ satisfying (3.13) is either zero or x. □

Theorem 3.3.6. *If* $\mathbf{0}$ *is the only integer solution of* S_0*, and* S *undergoes an integral unimodular transformation*

$$x^{(\mu)} = \sum_{\tau=1}^{m} c_{\mu\tau}\xi^{(\tau)}, \ \mu = 1, \ldots, m \tag{3.16}$$

into a system

$$\Sigma : \varphi_\lambda(\xi) = 0, \ \lambda = 1, \ldots, l; \quad \chi_p(\xi) \geq 0, \ l = 1, \ldots, r$$

then the minimal basis $M(S)$ *of* S *is turned into the minimal basis* $M(\Sigma)$ *of* Σ *via this same transformation.*

Proof. If the minimal basis $M(\Sigma)$ of Σ is empty, then the origin is the only integer solution of Σ, and consequently of S, so that then the minimal basis $M(S)$ is empty. So assume that $M(\Sigma)$ is not empty. Let $\xi_1, \ldots, \xi_s$ denote the points of $M(\Sigma)$, and define the points $x_1, \ldots, x_s$ via the equations

$$x_j^{(\mu)} = \sum_{\tau=1}^{m} c_{\mu\tau}\xi_j^{(\tau)}. \tag{3.17}$$

Then we claim the points $x_1, \ldots, x_s$ form a basis for S.

To see this, let x be any integer solution of S. By 3.16, the lattice point ξ satisfies the system Σ and can be written in the form

$$\xi = \sum_{j=1}^{s} p_j \xi_j \tag{3.18}$$

with integer coefficients $p_j \geq 0$. It follows from (3.16), (3.17) and (3.18) that

$$x = \sum_{j=1}^{s} p_j x_j,$$

so that the points $x_1, \ldots, x_s$ are in fact a basis of S. They even provide a minimal basis of S: otherwise, there exists a minimal basis of S with s' elements, where $s' < s$. Exchanging S and Σ above, one would find a basis of Σ with s' elements, and this is impossible, as the minimum basis of S would have to contain exactly s, but also more than s', elements. So the points $x_1, \ldots, x_s$ form a minimal basis of S.

□

We are now ready to prove the main result of van der Corput, which will produce our Hilbert basis elements for $\mathcal{B}((x^a), (x^b))$, when a and b are relatively prime.

Theorem 3.3.7. *Let $a, b, c, d \in \mathbb{Z}$ with $ad - bc \neq 0$ where a and b are relatively prime, and c and d are relatively prime, and let u and v be two integers with $av - bu = 1$ Set $q = ad - bc$ and $p = ud - vc$, and decompose p/q into a continued fraction as follows:*

$$\frac{p}{q} = g_1 - \frac{1|}{|g_2} - \cdots - \frac{1|}{|g_{k-1}} \quad \text{if } q > 0$$
$$\frac{p}{q} = g_1 + \frac{1|}{|g_2} - \cdots - \frac{1|}{|g_{k-1}} \quad \text{if } q < 0$$

where $g_1, \ldots, g_{k-1} \in \mathbb{Z}$ and $g_2, g_3, \ldots, g_{k-1}$ are all ≥ 2, so the continued fraction expansion for p and q is unique.

Let $P_x/Q_x, (x = 1, 2, \ldots, k-1)$ be the convergents of p/q, so that

$$P_1 = g_1, Q_1 = 1, P_2 = g_1 g_2 \mp 1, Q_2 = g_2, \cdots$$

$$P_{k-1} = p, Q_{k-1} = q \text{ for all } q > 0, P_{k-1} = -p, Q_{k-1} = -q \text{ for all } q < 0$$

and where

$$Q_0 = 0 \text{ and } P_0 = 1 \text{ when } q > 0, P_0 = -1 \text{ when } q < 0.$$

Then the minimal basis of lattice points satisfying the system of equations S:

$$ax + by \geq 0, cx + dy \geq 0$$

is the set of k points

$$\{(-bP_x + vQ_x, aP_x - uQ_x)|x = 0, \ldots, k-1, p_x \in \mathbb{N}.\} \tag{3.19}$$

Moreover, each integer solution (x, y) of S is of the form

$$x = \sum_{x=0}^{k-1} p_x(-bP_x + vQ_x), \qquad y = \sum_{x=0}^{k-1} p_x(aP_x - uQ_x)$$

may be written with non-negative integer coefficients p_x. These coefficients p_x need not be uniquely determined for a given (x, y). However, we will show that every integer solution (x, y) of S has a unique expression

$$x = A(-bP_x + vQ_x) + B(-bP_{x+1} + vQ_{x+1}) \tag{3.20}$$

$$y = A(aP_x - uQ_x) + B(aP_{x+1} - uQ_{x+1}) \tag{3.21}$$

where $0 \leq x \leq k-2$, and $A, B \in \mathbb{N}$.

Proof. By the unimodular integer transformation

$$x = v\xi - b\eta, \qquad y = -u\xi + a\eta$$

i.e. by the transformation

$$\xi = ax + by, \qquad \eta = ux + vy$$

S transforms into the system of equations Σ:

$$\xi \geq 0, \quad -p\xi + q\eta \geq 0,$$

and the k points mentioned in (3.19) become just (Q_x, P_x), $x = 0, ...k-1$.

To show that the points mentioned in (3.19) belong to the minimal basis $M(S)$ of S, it suffices by Theorem 3.3.6 to show that the points (Q_x, P_x) belong to the minimal basis $M(\Sigma)$ of Σ.

If $q > 0$, then the points (Q_x, P_x), $x = 0, ..., k-1$ belong to the minimal basis $M(\Sigma)$ of Σ.

The point $(0, 1)$ satisfies the system Σ because it is nonzero, $q > 0$, and additionally $(0, 1)$ satisfies (3.13): specifically, at each integer lattice point (ξ, η) with

$$0 \leq \xi \leq 0, \qquad 0 \leq -p\xi + q\eta \leq q,$$

either $\xi = 0, \eta = 0$ or $\xi = 0, \eta = 1$. So by Theorem 3.3.5, the point $(Q_0, P_0) = (0, 1)$ belongs to the minimum basis $M(\Sigma)$.

Any other point (Q_x, P_x) with $1 \leq x \leq k-1$ satisfies the system Σ because $Q_x > 0$ and

$$-pQ_x + qP_x = qQ_x \left(\frac{P_x}{Q_x} - \frac{p}{q} \right) > 0.$$

Also the point (Q_x, P_x) is nonzero. To show that (Q_x, P_x), $1 \leq x \leq k-1$ belongs to the minimal basis $M(\Sigma)$, by Theorem 3.3.5 it suffices to show that any integer solution (ξ, η) from Σ with

$$0 \leq \xi \leq Q_x, \qquad 0 \leq -p\xi + q\eta \leq -pQ_x + qP_x \tag{3.22}$$

equals either $\mathbf{0}$ or (Q_x, P_x). There are two different cases:

1. Let

$$\xi P_x - \eta Q_x \geq 0 : \tag{3.23}$$

If $\xi = 0$, it follows from (3.22) that $\eta \geq 0$, but by (3.23) $\eta \leq 0$, so $\eta = 0$. Therefore (ξ, η) is $\mathbf{0}$.

If $\eta > 0$, then because of (3.22) and (3.23)

$$\frac{p}{q} \leq \frac{\eta}{\xi} \leq \frac{P_x}{Q_x}.$$

It is a property of convergents of continued fractions that any convergent is closer to the continued fraction than any other fraction whose denominator is less than that of the convergent. In other words, since P_x/Q_x is a convergent of p/q, either $(\xi, \eta) = (Q_x, P_x)$ or $\xi > Q_x$; $\xi > Q_x$ is impossible by (3.22), so $(\xi, \eta) = (Q_x, P_x)$.

2. If

$$\xi P_x - \eta Q_x < 0 \tag{3.24}$$

then

$$\xi(\eta - P_x) - \eta(\xi - Q_x) > 0.$$

By (3.22)

$$0 \leq Q_x - \xi \leq Q_x, \quad 0 \leq -p(Q_x - \xi) + q(P_x - \eta) \leq -pQ_x + qP_x.$$

In this case, the relations (3.22) and (3.23) are satisfied with $Q_x - \xi$ instead of ξ and with $P_x - \eta$ instead of η, so that by Theorem 3.23, the point $(Q_x - \xi, P_x - \eta)$ is either the origin or (Q_x, P_x), so that either $(\xi, \eta) = (0, 0)$ or $(\xi, \eta) = (Q_x, P_x)$.

First assume $q > 0$. We claim that each integer solution (ξ, η) of Σ can be written uniquely as

$$\xi = AQ_x + BQ_{x+1}, \qquad \eta = AP_x + BP_{x+1} \tag{3.25}$$

where $0 \leq x \leq k - 2$, and $A, B \in \mathbb{N}$.

Proof of claim: If $\xi = 0$, then, since (ξ, η) is a solution of Σ, $\eta \geq 0$, so since $Q_0 = 0$ and $P_0 = 1$ when $q > 0$,

$$\xi = 0 = \eta Q_0 \text{ and } \eta = \eta P_0$$

so then (3.25) holds with $x = 0$, $A = \eta$, $B = 0$ and $\xi = 0$. Then, since $Q_1, \ldots, Q_{k-1}$ are positive, the expression defined in (3.25) must be unique.

Now let $\xi > 0$, so since (η, ξ) is a solution to Σ, $\eta/\xi \geq p/q$. Since P_x/Q_x are the convergents of p/q, we have

$$\frac{p}{q} = \frac{P_{k-1}}{Q_{k-1}} < \frac{P_{k-2}}{Q_{k-2}} < \cdots < \frac{P_1}{Q_1} < \frac{P_0}{Q_0},$$

so that with a suitably selected x, $0 \leq x \leq k-2$,

$$\frac{P_{x+1}}{Q_{x+1}} \leq \frac{\eta}{\xi} < \frac{P_x}{Q_x}.$$

Since P_x/Q_x and P_{x+1}/Q_{x+1} are two successive convergents, (3.25) will be satisfied with suitably selected non-negative integer coefficients A and B.

To prove the uniqueness of the expression given in (3.25), assume that in addition

$$\xi = CQ_\tau + DQ_{\tau+1}, \qquad \eta = CP_\tau + DP_{\tau+1}, \tag{3.26}$$

where $0 \leq \tau \leq k-2$, and C and D are non-negative integers.

If η/ξ is equal to one of the convergents P/Q of p/q, then it follows that

from (3.25): $B = 0, Q_x = Q, P_x = P$ or $A = 0, Q_{x+1} = Q, P_{x+1} = P_x$

from (3.26): $D = 0, Q_\tau = Q, P_\tau = P$ or $C = 0, Q_{\tau+1} = Q, P_{\tau+1} = P,$

so that by both (3.25) and (3.26),

$$\xi = KQ \text{ and } \eta = KP$$

and the writing of (η, ξ) is unique.

If η/ξ is not equal to one of the convergents of p/q, then it follows from (3.25) and (3.26) that

$$\frac{P_{x+1}}{Q_{x+1}} < \frac{\eta}{\xi} < \frac{P_x}{Q_x} \text{ and } \frac{P_{\tau+1}}{Q_{\tau+1}} < \frac{\eta}{\xi} < \frac{P_\tau}{Q_\tau}.$$

So $\tau = x$, and (3.26) becomes

$$\eta = CQ_x + DQ_{x+1}, \qquad \eta = CP_x + DP_{x+1}$$

and because $Q_xP_{x+1} - Q_{x+1}P_x \neq 0$, it follows from (3.25) that $A = C$ and $B = D$, so the expression in (3.25) is unique.

So if $q > 0$, the previous claim allows us to express (ξ, η) uniquely in the form (3.25). Applying the transformation

$$x = v\xi - b\eta, \qquad y = -u\xi + a\eta$$

results in a unique expression of (x, y) in the form (3.20) as well.

To conclude, let $q < 0$, and apply the unimodular integral transformation $x = x'$, $y = -y'$ to obtain the system

$$S' : ax' - by' \geq 0, \qquad cx' - dy' \geq 0.$$

The conditions of this theorem now fulfilled when S, b, d, u, q, g_1, and P_x are replaced by $S', -b, -d, -u, -q, -g_1$, and $-P_x$, and now $-q > 0$. By what we have already done (with $-q$ instead of q), the points

$$(-bP_x + vQ_x, -aP_x + uQ_x)x = 0, \ldots, k-1$$

form the minimal basis of S', so that by Theorem 3.3.6, the points referred to in (3.19) form a minimum basis of S. As in the $q < 0$ case, applying the transformation

$$x = v\xi - b\eta, \qquad\qquad y = -u\xi + a\eta$$

again with $-q$ instead of q, shows that each integer solution $(x, -y)$ of S' has the desired unique expression.

$$\begin{aligned} x =& A(-bP_x + vQ_x) + B(-bP_{x+1} + vQ_{x+1}) \\ -y =& A(-aP_x + uQ_x) + B(-aP_{x+1} + UQ_{x+1}) \end{aligned}$$

where $0 \leq x \leq k-2$, and A and B denote integer coefficients. Therefore the proof is completed for $q < 0$.

□

Now we can apply this result to our problem. Let $R = k[x]$ and $I = (x^a)$, $J = (x^b)$, with a and b relatively prime. We need to find a Hilbert basis for each of the two cones Q_0, defined by the equations $-ar + bs \geq 0$ and $r \geq 0$, and Q_1, defined by $ar - bs \geq 0$ and $s \geq 0$. Since these cones are pointed and rational, the minimal bases referred to above is the same as the unique Hilbert basis.

The cone Q_0: In the language of the theorem, where S is the system

$$ax + by \geq 0, cx + dy \geq 0,$$

we have that

$$x = r, y = s, a = -a, b = b, c = 1, d = 0.$$

So let u and v be two integers with $-av - bu = 1$. Then by the theorem, $q = -b$ and $p = -v$. Since $q < 0$, and the Hilbert basis elements for Q_0 are determined by the continued fraction decomposition

$$\frac{p}{q} = \frac{-v}{-b} = \frac{v}{b} = g_1 + \frac{1|}{|g_2} - \cdots - \frac{1|}{|g_{k-1}}$$

as outlined in the theorem.

The cone Q_1: For this cone, we have

$$x = r, y = s, a = a, b = -b, c = 0, d = 1.$$

Let u and v be two integers with $av + bu = 1$. By the theorem, $q = a$ and $p = u$. Since $q > 0$, and the Hilbert basis elements for Q_1 are determined by the terms in the continued fraction decomposition

$$\frac{p}{q} = \frac{u}{a} = g_1 - \frac{1|}{|g_2} - \cdots - \frac{1|}{|g_{k-1}}.$$

Below is one example of a case where the Hilbert basis elements are easy to compute.

Example 3.3.8. The intersection algebra $\mathcal{B}$ of $I = (x^n), J = (x^{n+1})$:

Q_0 consists of the lattice points satisfying $-nr + (n+1)s \geq 0$ and $r \geq 0$. So $a = -n, b = n + 1, c = 1, d = 0$. Then $u = -1, v = 1$ work to satisfy $-nv - (n+1)u = 1$, so $q = ad - bc = -n - 1$ and $p = ud - vc = -1$. Then $p/q = 0 + 1/(n+1)$, with $q < 0$, and we have $g_1 = 0$ and $g_2 = n + 1$. So the set of 3 points in $\{-bP_x + vQ_x, aP_x - uQ_x)|x = 0, \ldots, 2\}$ are

$$\begin{aligned}
&P_0 = -1, \quad Q_0 = 0 && \rightarrow (-(n+1)(-1), (-n)(-1)) = (n+1, n) \\
&P_1 = 0, \quad Q_1 = 1 && \rightarrow (-(n+1) \cdot 0 + 1 \cdot 1, -n \cdot 0 - (-1) \cdot 1) = (1, 1) \\
&P_2 = -1, \quad Q_2 = -n - 1 && \rightarrow (-(n+1)(-1) + 1(-n-1), (-n(-1) - (-1)(-n-1)) = (0, 1).
\end{aligned}$$

Q_1 is the set of lattice points satisfying $nr - (n+1)s \geq 0$ and $s \geq 0$. So $a = n, b = -(n+1), c = 0, d = 1$, and we can choose $u = 1$ and $v = -1$ to satisfy $nv - (-n-1)u = 1$. Then

$$\frac{p}{q} = \frac{1}{n} = 1 - \frac{n-1}{n} = 1 - \frac{1|}{|2} - \cdots - \frac{1|}{|2},$$

where there are $n - 1$ copies of 2 in the fraction decomposition. Therefore the convergents and their corresponding set of points are

$$\begin{array}{lll} P_0 = 1, & Q_0 = 0 & \to ((n+1)\cdot 1 - 1\cdot 0, n\cdot 1 - 1\cdot 0) = (n+1, n) \\ P_1 = 1, & Q_1 = 1 & \to ((n+1)\cdot 1 - 1\cdot 1, n\cdot 1 - 1\cdot 1) = (n, n-1) \\ P_2 = 1, & Q_2 = 2 & \to ((n+1)\cdot 1 - 1\cdot 2, n\cdot 1 - 1\cdot 2) = (n-1, n-2) \\ P_3 = 1, & Q_3 = 3 & \to ((n+1)\cdot 1 - 1\cdot 3, n\cdot 1 - 1\cdot 3) = (n-2, n-3) \\ \vdots & \vdots & \quad\vdots \\ P_n = 1, & Q_n = n & \to ((n+1)\cdot 1 - 1\cdot n, n\cdot 1 - 1\cdot n) = (1, 0) \end{array}$$

So the Hilbert basis corresponding to the fan associated to $\mathcal{B}$ consists of the points

$$\{(0,1), (1,1), (n+1,n), (n,n-1), (n-1,n-2), \ldots, (2,1), (1,0)\}.$$

3.3.2 Approach via Linear Diophantine Equations with Integer Coefficients

We give another approach to constructing $\mathcal{B}$. First, recall some facts from semigroup rings associated to linear diophantine equations with integer coefficients from Chapter I, Section 3 of [14].

Definition 3.3.9. Let Φ be an $r \times n$ $\mathbb{Z}$-matrix, $r \leq n$, and rank $\Phi = r$. Define

$$E_\Phi := \{\beta \in \mathbb{N}^n | \Phi\beta = 0\}.$$

Then E_Φ is clearly a submonoid of $\mathbb{N}^n$.
Let $R_\Phi := kE_\Phi$, the monoid algebra of E_Φ over k. We identify $\beta \in E_\Phi$ with $x^\beta = x_1^{\beta_1} x_2^{\beta_2} \cdots x_n^{\beta_n}$, so that $R_\Phi \subseteq k[x_1, \ldots, x_n]$ as a subalgebra graded by monomials.

To translate our problem into this language, we must describe all monomials in $\mathcal{B}$ as having exponents that are solutions to a system of equations. Since any monomial must have the form $x^m u^r v^s$, with $m \geq \max(ar, bs)$, there exists an $h, k \in \mathbb{N}$ such that the log of any monomial must satisfy

$$m = ar + h = bs + k,$$

so

$$\Phi = \Phi_{a,b} = \begin{pmatrix} a & 0 & 1 & 0 & -1 \\ 0 & b & 0 & 1 & -1 \end{pmatrix}.$$

Then $E_\Phi = E_{\Phi_{a,b}} = \{\beta \in \mathbb{N}^n | \Phi\beta = 0\}$.

Then E_Φ is a subsemigroup in $\mathbb{N}^5$, and to recover our semigroup Q where $\mathcal{B} = k[Q]$, we project E_Φ onto $\mathbb{N}^3$ by sending h and k to zero. In fact, Q and E_Φ are isomorphic, and so $\mathcal{B}$ and R_Φ are as well.

Reformulating $\mathcal{B}$ in terms of this matrix allows us to easily prove some more properties of this algebra. This requires a few more results from [14].

Definition 3.3.10. $\beta \in E_\Phi$ is *fundamental* if $\beta = \gamma + \delta$, $\gamma, \delta \in E_\Phi$ implies $\gamma = \beta$ or $\delta = \beta$.

$$\text{FUND}_\Phi := \text{ set of fundamental elements of } \Phi.$$

It is clear that FUND_Φ generates E_Φ and that any set which generates E_Φ contains FUND_Φ. In particular, $|\text{FUND}_\Phi| < \infty$ and

$$R_\Phi = k[x^\delta | \delta \in \text{FUND}_\Phi].$$

Definition 3.3.11. $\beta \in E_\Phi$ is *completely fundamental* if whenever $n > 0$ and $n\beta = \gamma + \delta$ for $\gamma, \delta \in E_\Phi$, then $\gamma = n_1\beta$ for some $0 \leq n_1 \leq n$.

$$\text{CF}_\Phi := \text{ set of completely fundamental elements of } E_\Phi.$$

Since any generating set for E_Φ contains FUND_Φ, and we already know a generating set for $Q = E_\Phi$, FUND_Φ must be among the points that we get from our generators. We claim that the two sets are in fact equal.

First, recall a few facts about the construction of $\mathcal{B}((x^a), (x^b))$. Its generating set is built from two cones

$$C_0 = \{\lambda_1(0,1) + \lambda_2(b,a) | \lambda_i \in \mathbb{R}_+\}$$

$$C_1 = \{\lambda_1(b,a) + \lambda_2(1,0) | \lambda_i \in \mathbb{R}_+\}$$

and, when intersected with $\mathbb{Z}$, they form two semigroups, Q_0 and Q_1, each with a corresponding Hilbert basis $\mathcal{H}_{Q_0}$ and $\mathcal{H}_{Q_1}$. Each basis element has a corresponding algebra generator depending on which cone it comes from, either

$$\{x^m u^r v^s | (r,s) \in \mathcal{H}_{Q_0}, m = bs\} \text{ or } \{(r,s,m) | (r,s) \in \mathcal{H}_{Q_1}, m = ar\}, \tag{3.27}$$

where Q_0 is the cone where $bs \geq ar$ and Q_1 is the cone where $ar \geq bs$. These monomials, together with x, generate $\mathcal{B}$ over k. Note that for the rest of this section, we will assume a and b are relatively prime. The general case follows.

Theorem 3.3.12. *Let $a, b \in \mathbb{N}$, and $\Phi = \Phi_{a,b}$. Then the set of fundamental elements of E_Φ is*

$$\begin{aligned} FUND_\Phi = & \{(r,s,bs-ar,0,bs) | (r,s) \in \mathcal{H}_{Q_0}\} \\ & \cup \{(r,s,0,ar-bs,ar) | (r,s) \in \mathcal{H}_{Q_1}\} \cup \{(0,0,1,1,1)\}. \end{aligned}$$

Proof. First, we translate each algebra generator into elements of E_Φ. Recall that $m = ar + h = bs + k$ for every $\beta = (r,s,h,k,m) \in E_\Phi$. So the three sets in equation 3.27 become

$$\{(r,s,bs-ar,0,bs) | (r,s) \in \mathcal{H}_{Q_0}\} \text{ and } \{(r,s,0,ar-bs,ar) | (r,s) \in \mathcal{H}_{Q_1}\},$$

and x corresponds to the vector $(0,0,1,1,1)$.

Let $\beta = (r,s,bs-ar,0,bs) \in E_\Phi$ with $(r,s) \in \mathcal{H}_{Q_0}$, and say that $\beta = \gamma + \delta$. Then

$$\gamma = (r', s', h', 0, m') \text{ and } \delta = (r'', s'', h'', 0, m''),$$

where

$$ar' + h' = bs' = m' \text{ and } ar'' + h'' = bs'' = m'',$$

which implies that $bs' \geq ar'$ and $bs'' \geq ar''$, so both (r', s') and (r'', s'') are in Q_0. But $r = r' + r''$ and $s = s' + s''$, and (r, s) is a Hilbert basis element. So either $(r, s) = (r', s')$ or $(r, s) = (r'', s'')$, and thus either $\beta = \delta$ or $\beta = \gamma$, and so the fundamental elements are the same as the ones that arise from the Hilbert basis algorithm.

□

Now, $\mathrm{CF}_\Phi \subseteq \mathrm{FUND}_\Phi$, so to determine the completely fundamental elements, we need only discard those fundamental elements which do not fit the definition.

Theorem 3.3.13. *Let $a, b \in \mathbb{N}$ and $\Phi = \Phi_{a,b}$. Then the completely fundamental elements of E_Φ are*

$$CF_\Phi = \{(1, 0, 0, a, a), (0, 1, b, 0, b), (b, a, 0, 0, ab), (0, 0, 1, 1, 1)\}.$$

Proof. First, it is clear from the positions of the zeros in each of the above points that they are completely fundamental. Let $\beta \in \mathrm{FUND}_\Phi$ that is not one of the above four, and say without loss of generality that

$$\beta = (r, s, bs - ar, 0, bs).$$

Then since

$$(r, s) \in Q_0 = C_0 \cap \mathbb{Z}, \text{ where } C_0 = \{\lambda_1(0, 1) + \lambda_2(b, a) | \lambda_i \in \mathbb{R}^+\},$$

$(r, s) = \lambda_1(0, 1) + \lambda_2(b, a)$ for some $\lambda_1, \lambda_2 \in \mathbb{R}^+$. So $r = \lambda_2 b$ and $s = \lambda_1 + \lambda_2 a$, and since $(r, s) \in \mathbb{N}^2$, $\lambda_1, \lambda_2 \in \mathbb{Q}$. Clearing denominators then shows that there exists an $n > 0$ such that

$$n(r, s) = \lambda'_1(0, 1) + \lambda'_2(b, a).$$

To conclude, note that

$$\begin{aligned}\lambda_1'(0,1,b,0,b)+\lambda_2'(b,a,0,0,ab) =&(\lambda_2'b,\lambda_1'+\lambda_2'a,\lambda_1',0,\lambda_1'b+\lambda_2'ab)\\ =&(nr,ns,nbs-nar,0,nbs)\\ =&n(r,s,bs-ar,0,bs),\end{aligned}$$

since $nbs - nar = b(\lambda_1' + \lambda_2'a) - a(\lambda_2'b) = \lambda_1'$. So β is not in CF_Φ, and similarly with any element of the form $(r, s, 0, ar - bs, ar)$ that is not in the list in the theorem. So those four are the only completely fundamental elements. □

If we define C_Φ to be the polyhedral cone of $\mathbb{R}^+$-solutions β to $\Phi\beta = 0$, it is known that CF_Φ are the integer points nearest 0 on each extreme ray of C_Φ. Notice that this is also the case with CF_Φ for our Φ.

The completely fundamental elements can be used to obtain information about the Hilbert series of the intersection algebra. The necessary results from [14] are excerpted below.

Theorem 3.3.14. (Corollary 3.8 in [14]) *The Hilbert series of R_Φ is $F(R_\Phi, \lambda) = \sum_{\beta\in E_\Phi} \lambda^\beta$. When it is written in lowest terms, the denominator is $\Pi_{\beta\in CF_\Phi}(1 - \lambda^\beta)$.*

Corollary 3.3.15. *Let $a, b \in \mathbb{N}$ and $\Phi = \Phi_{a,b}$. The Hilbert series of $\mathcal{B}((x^a) \cap (x^b))$ is*

$$\sum_{\beta\in E_\Phi} \lambda^\beta,$$

where $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$. When it is written in lowest terms, the denominator is

$$(1-\lambda_1\lambda_4^a\lambda_5^a)(1-\lambda_2\lambda_3^b\lambda_5^b)(1-\lambda_1^b\lambda_2^a\lambda_5^{ab})(1-\lambda_3\lambda_4\lambda_5)$$

3.3.3 A Regular Sequence in $\mathcal{B}((x^a),(x^b))$

Another fact from [14] allows us to find a regular sequence in $\mathcal{B}((x^a),(x^b))$.

Theorem 3.3.16. (Theorem 3.7 in [14]) *Let* $\delta_1, \ldots, \delta_t \in E_\Phi$*, and* $S = k[x^{\delta_1}, \ldots, x^{\delta_t}] \subseteq R_\Phi]$*. Then* R_Φ *is a finitely generated* S*-module (equivalently: integral over* S*) if and only if for every* $\beta \in CF_\Phi$ *there are* $1 \leq i \leq t$ *and* $j > 0$ *such that* $\delta_i = j\beta$*.*

Corollary 3.3.17. *Let* $a, b \in N$ *be relatively prime, and* $\Phi = \Phi_{a,b}$*. Then* $\mathcal{B}$ *is integral over* $k[x, x^a u, x^b v, x^{a+b} u^b v^a]$*.*

Proof. Since $\mathrm{CF}_\Phi = (0,0,1,1,1), (1,0,0,a,a), (0,1,b,0,b), (b,a,0,0,a+b)$, choosing the δ_i to be the fundamental elements completes the proof. $\square$

Lemma 3.3.18. Let a, b and $\mathcal{B}$ be as above. Then

$$\dim \frac{\mathcal{B}}{(x, x^{a+b}u^b v^a, x^a u + x^b v)} = 0$$

Proof. Define the following map

$$D = \frac{k[x,y,z,w]}{(x^{ab}y - z^b w^a)} \xrightarrow{\varphi} k[x, x^{ab}u^b v^a, x^a u, x^b v] = C$$

by sending x to x, y to $x^{ab}u^b v^a$, z to $x^a u$, and w to $x^b v$. Then since $\varphi(x^{ab}y - z^b w^a) = 0$, φ is surjective. Then

$$\frac{D}{(x, y, z+w)} = \frac{k[z,w]}{(z^b w^a, z+w)} = \frac{k[x]}{(w^{a+b})},$$

which is a dimension 0 ring.

Then we have the following maps:

$$\frac{C}{(x, x^{a+b}u^b v^a, x^a u + x^b v)} \xrightarrow{i} \frac{C}{(x, x^{a+b}u^b v^a, x^a u + x^b v)\mathcal{B} \cap C} \xrightarrow{j} \frac{B}{(x, x^{a+b}u^b v^a, x^a u + x^b v)}.$$

By the above corollary, $\mathcal{B}$ is integral over C, and the map i is surjective. So $\dim \frac{B}{(x, x^{a+b}u^b v^a, x^a u + x^b v)} = 0$. $\square$

Theorem 3.3.19. *Given the assumptions of the previous lemma, a regular sequence for* $\mathcal{B}((x^a), (x^b))$ *is* $x, x^{a+b}u^b v^a, x^a u + x^b v$*.*

Proof. Since $\mathcal{B}$ is a domain, x is obviously a nonzerodivisor in $\mathcal{B}$. Next, $x^{a+b}u^bv^a$ is a nonzerodivisor in $\mathcal{B}/x\mathcal{B}$, since if

$$rx^{ab}u^bv^a = 0 \in \mathcal{B}/(x)\mathcal{B} \text{ for some } r \in \mathcal{B}$$
$$\Rightarrow rx^{ab}u^bv^a = sx \text{ for some } s \in \mathcal{B}$$
$$\Rightarrow rx^{ab-1}u^bv^a = s.$$

We claim this implies $r \in (x)\mathcal{B}$.

Since $r, s \in \mathcal{B}$, $s = x^m u^r v^s$ and $r = x^{m'}u^{r'}v^{s'}$, where $m \geq \max(ar, bs)$ and $m' \geq \max(ar', bs')$. So

$$x^{m'+ab-1}u^{r'+b}v^{s'+a} = x^m u^r v^s$$

and so $r' + b = r$ and $s' + a = s$, and thus

$$\max(ar, bs) = \max(a(r'+b), b(s'+a)) = \max(ar'+ab, bs'+ab) = \max(ar', bs') + ab.$$

So $m' = m - ab + 1 \geq \max(ar', bs') + 1$, and therefore

$$r = x(x^{m'-1}u^{r'}v^{s'}) \text{ with } m' \geq \max(ar', bs')$$

so $r \in x\mathcal{B}$.

It remains to show that $x^a u + x^b v$ is a nonzerodivisor in $\mathcal{B}/(x, x^{ab}u^bv^a)\mathcal{B}$. Assume not. Then since $\mathcal{B}$ is Cohen-Macaulay, $\mathcal{B}/(x, x^{ab}u^bv^a)\mathcal{B}$ is too. So by the unmixedness property of Cohen-Macaulay rings, there exists a prime ideal $P \in \text{Ass}(P)$ such that $\text{ht}P = 2$, and $(x, x^{a+b}u^bv^a, x^au + x^bv) \subseteq P$. Then

$$\frac{\mathcal{B}}{(x, x^{a+b}u^bv^a, x^au + x^bv)} \twoheadrightarrow \frac{\mathcal{B}}{P},$$

so by the above lemma, $\dim \frac{\mathcal{B}}{P} = 0$. But $\dim \mathcal{B} = 3$, and $\text{ht}P = 2$, which is a contradiction.

So $x^a u + x^b v$ is a nonzerodivisor in $\mathcal{B}/(x, x^{ab}u^b v^a)\mathcal{B}$, and $x, x^{a+b}u^b v^a, x^a u + x^b v$ is a regular sequence on $\mathcal{B}$. □

Definition 3.3.20. We say $\beta \in E_\Phi$ is *positive*, denoted $\beta > 0$, if each coordinate of β is positive. A positive $\beta \in E_\Phi$ is called minimal if for any $\gamma > 0$, $\gamma \in E_\Phi$, then $\gamma - \beta \geq 0$

We are moving towards describing the canonical ideal of $\mathcal{B}((x^a), (x^b))$. The following result provides this description.

Theorem 3.3.21. (Corollary 13.1 in [14]) *Denote the canonical ideal of R_Φ by $\Omega(R_\Phi)$. Then*

$$\Omega(R_\Phi) = k\{x^\beta | \beta \in E_\Phi, \beta > 0\}.$$

Since any element in the canonical ideal is a linear combination of positive monomials, it is easy to see that the set of minimal positive elements provides a generating set for this ideal.

We will construct this set of minimal positive elements from E_Φ for our Φ, and obtain a generating set for the canonical module of $\Omega(\mathcal{B})$.

Theorem 3.3.22. *Let A be the set*

$$\{\gamma + (0,0,1,1,1) | \gamma \in \mathit{FUND}_\Phi \text{ different from } (0,0,1,1,1), (1,0,0,a,a), (0,1,b,0,b)\}.$$

Then the set of minimal positive $\beta \in E_\Phi$ is equal to A.

Proof. Recall that any element of FUND_Φ (other than $(0,0,1,1,1)$) looks like either

$$\{(r, s, bs - ar, 0, bs) | (r, s) \in \mathcal{H}_{Q_0}\} \text{ or } \{(r, s, 0, ar - bs, ar) | (r, s) \in \mathcal{H}_{Q_1}\}.$$

So A is the collection

$$\{(r, s, bs - ar + 1, 1, bs + 1) | (r, s) \in \mathcal{H}_{Q_0}\} \cup \{(r, s, 1, ar - bs + 1, ar + 1) | (r, s) \in \mathcal{H}_{Q_1}\}.$$

First, we show that any minimal positive element of E_Φ is in A. Let (r,s,h,k,m) be a minimal positive element of E_Φ, and assume without loss of generality that $ar \geq bs$. Then $ar + h = m = bs + k$, and we can decompose (r,s) into a sum of Hilbert basis elements $\sum_h l_h(r_h, s_h)$. Then

$$(r,s,h,k,n) = \left(\sum_h l_h r_h, \sum_h l_h s_h, m - a\sum_h l_h r_h, m - b\sum_h l_h s_h, m\right)$$

so we claim there exists an h' such that

$$(r,s,h,k,m) \geq (r_{h'}, s_{h'}, 1, ar_{h'} - bs_{h'} + 1, ar_{h'} + 1).$$

This inequality is obviously satisfied for the first three components. Since $h, k \geq 1$, and $ar \geq bs$, $m \geq ar + 1$, so $m \geq ar_{h'} + 1$.

Lastly, since $k = m - bs$ and $m \geq ar$,

$$k \geq ar - bs = \sum_h l_{h'}(ar_{h'} - bs_{h'}),$$

so (r,s,h,k,m) is greater than an element of A. But (r,s,h,k,m) is minimal, so it must be equal to an element of A.

Next, we show that any element from A is minimal. First, note that any two elements of A are incomparable: let $\beta \in A$ of the form

$$(r, s, bs - ar + 1, 1, bs + 1) \text{ for some } (r,s) \in \mathcal{H}_{Q_0},$$

and let

$$\beta' = (r', s', bs' - ar' + 1, 1, bs' + 1) \text{ for another } (r', s') \in \mathcal{H}_{Q_0}, \text{ with } (r', s') > (r, s).$$

Note that, since (r, s) and (r', s') are both in Q_0,

$$\frac{b}{a} \geq \frac{r}{s} \text{ and } \frac{b}{a} \geq \frac{r'}{s'}.$$

We claim $bs' - ar' + 1 < bs - ar + 1$, since this is equivalent to

$$\begin{aligned} b(s' - s) &< a(r' - r) \\ \Leftrightarrow \frac{b}{a} &< \frac{r' - r}{s' - s}. \end{aligned}$$

Assume this last inequality does not hold. Then

$$\frac{r' - r}{s' - s} \in Q_1,$$

thus

$$(r, s) + (r' - r, s' - s) = (r', s').$$

But (r', s') is in $\mathcal{H}_{Q_1}$, and by definition cannot be a sum of two elements in Q_1. So $bs' - ar' + 1 < bs - ar + 1$ while $(r', s') > (r, s)$, and therefore β and γ are incomparable.

Lastly, let $\beta \in A$ not minimal in E_Φ. Therefore there is an element $\gamma < \beta$. And, by the first half of this proof, γ is larger than some element $\beta' \in A$. So $\beta > \gamma > \beta'$, which contradicts that all elements of A are incomparable. Therefore any element of A is minimal. $\square$

Corollary 3.3.23. *Let $a, b \in \mathbb{N}$, $\mathcal{B} = \mathcal{B}((x^a), (x^b))$, and $\Phi = \Phi_{a,b}$. Then the canonical module of $\mathcal{B}$ is*

$$\Omega(\mathcal{B}) \cong k\{x^\beta | \beta \in E_\Phi, \beta > 0\}.$$

Corollary 3.3.24. *Let a, b and $\mathcal{B}$ be as above. Then the number of minimal elements, and thus the number of generators of the ideal $\Omega(\mathcal{B})$ is $|Q_0| + |Q_1| - 3$.*

Proof. There is one fundamental element for each Hilbert basis element in Q_0 and Q_1, plus one for the monomial x. However, one is double-counted, since $(b, a) \in \mathcal{H}_{Q_0} \cap \mathcal{H}_{Q_1}$. So $|\text{FUND}_\Phi| = |\mathcal{H}_{Q_0}| + |\mathcal{H}_{Q_1}|$. Then, since $|\text{CF}_\Phi| = |\text{FUND}_\Phi| - 3$ by Theorem 3.3.22, the corollary is proved. □

Stanley's methods allow us to easily determine whether $\mathcal{B}$ is Gorenstein.

Theorem 3.3.25. (Corollary 13.2 in [14]) *R_Φ is Gorenstein if and only if there exists a unique minimal $\beta > 0$ in E_Φ (i.e. if $\gamma > 0$, $\gamma \in E_\Phi$, then $\gamma - \beta \geq 0$).*

Corollary 3.3.26. *$\mathcal{B}$ is Gorenstein if and only if there exists a unique minimal $\beta > 0$ in E_Φ.*

There are very few Gorenstein intersection algebras of principal monomial ideals. In fact, if we stay within the assumption that a and b are relatively prime, there is only one such algebra.

Corollary 3.3.27. *Let $\mathcal{B} = \mathcal{B}(I, J)$ be the intersection algebra of two principal monomial ideals I and J in $R = k[x]$ that is Gorenstein. Then $I = J = (x)$.*

Proof. Let $I = (x^a)$ and $J = (x^b)$, where $a, b \in \mathbb{N}$ are relatively prime. First, note that if either a or b are zero, then $\mathcal{B} = \bigoplus I^r u^r v^s$ or $\mathcal{B} = \bigoplus J^s u^r v^s$ respectively, neither of which are Gorenstein. So assume, by way of contradiction, that both $a > 1$ and $b > 1$.

The number of minimal elements of $\mathcal{B}$ is equal to $|\mathcal{H}_{Q_0}| + |\mathcal{H}_{Q_1}| - 3$, and since $\mathcal{B}$ is Gorenstein, $|\mathcal{H}_{Q_0}| + |\mathcal{H}_{Q_1}| = 4$. Recall that both Hilbert bases must contain at least two elements, namely the generators of their cones. So $(0, 1), (b, a) \in \mathcal{H}_{Q_0}$ and $(1, 0), (b, a) \in \mathcal{H}_{Q_1}$, and since there are only four in total (including repetitions), these must be the only Hilbert basis elements.

Since $(a, b) > (1, 1)$, the point $(1, 1)$ must lie in one of the cones, say Q_0. Since $(1, 1)$ cannot be a Hilbert basis element by assumption, there exists $n_1, n_2 \in \mathbb{N}$ such that

$$(1, 1) = n_1(0, 1) + n_2(b, a)$$

which implies that $n_2 b = 1$ and $n_1 + n_2 a = 1$. But $a, b, n_1, n_2 \in \mathbb{N}$, so $n_2 = b = 1$, and therefore $a = 1$ as well, contradicting our assumption that both a and b are different from 1. $\square$

One can approach the general case by replacing a and b by their quotients when divided by the greatest common divisor of a and b, and see that any Gorenstein intersection algebra of principal monomial ideals in one variable is of the form $\mathcal{B}((x^a),(x^a))$ for some $a \in \mathbb{N}$, $a \geq 1$.

Lastly, examining these minimal positive elements led us to an upper bound on the number of Hilbert basis elements for a fan of two cones in the plane.

Theorem 3.3.28. *Let R, a, b and Φ be as before, and assume without loss of generality that $a > b$. Set $a = bq + l$, with $1 \leq l \leq b - 1$. Then the number of minimal elements of R_Φ is bounded above by $a - l + 1$.*

Proof. First, note that the case $a = b$ is covered above, and the number of minimal elements is determined.

Let (r, s, h, k, m) be an element of E_Φ and recall that $ar + h = m = bs + k$. Define $\mu = ar - bs = k - h$, with $\mu > 0$. So we may write our element as $(r, s, k, k + \mu, ar + k)$.

Notice that the condition $\mu > 0$ is equivalent to $ar - bs > 0$, which is the same as considering all the points $(r, s) \in Q_1$. Then $(1, 1)$ is a minimal positive element in this cone: for this pair, $\mu = a - b > 0$, and there is no other point with smaller positive coordinates. The smallest element in E_Φ corresponding to $(1, 1)$ must have $k > 0$, so the smallest such element is $(1, 1, 1, \mu + 1, a + 1)$. So, if $\mu > a - b$,

$$\begin{aligned}(r, s, k, \mu + k, ar + k) &\geq (r, s, 1, \mu + 1, ar + 1)\\ &\geq (1, 1, 1, 1 + \mu, a + 1)\\ &\geq (1, 1, 1, 1 + a - b, a + 1).\end{aligned}$$

If $\mu < a - b$, we claim that for each such μ, there exists a unique smallest r_μ, s_μ such that $\mu = ar_\mu - bs_\mu$: since $ar - bs = \mu$ determines a line in the (r, s) plane, let (r_μ, s_μ) be the point with smallest integer coordinates on that line. Obviously, $(r,s) \geq (r_\mu, s_\mu)$, and the smallest element in E_Φ corresponding to (r_μ, s_μ) is $(r_\mu, s_\mu, 1, 1+\mu, ar_\mu + 1)$. Then for $\mu \leq a-b$,

$$\begin{aligned}(r, s, k, k+\mu, ar+k) &\geq (r, s, 1, \mu+1, ar+1) \\ &\geq (r_\mu, s_\mu, 1, 1+\mu, ar_\mu + 1).\end{aligned}$$

If $\mu = 0$, then $ar = bs$, and, since a and b are relatively prime (because $l > 0$),

$$\begin{aligned}(r, s, 1, \mu+1, ar+1) &= (r, s, 1, 1, ar+1) \\ &\geq (b, a, 1, 1, ab+1).\end{aligned}$$

So, for $\mu > 0$, i.e. Q_1, the number of minimal elements no more than $a-b+1$: $a-b-1$ elements for each $0 < \mu \leq a-b$, one for $\mu > a-b$, and one for $\mu = 0$.

Now consider the points of E_Φ that correspond to pairs $(r,s) \in Q_0$, i.e. where $bs - ar = \eta > 0$. Then $\eta = k - h$, so we can write the full element of E_Φ as $(r, s, h+\eta, h, bs+h)$.

Since $a = bq + l$, we claim that the smallest (r,s) where $bs - ar > 0$ (i.e. the smallest pair $(r,s) \in Q_1$), is $(1, q+1)$. This pair is certainly in Q_1, since $\eta = b(q+1) - a = bq + b - a = b - l > 0$. Also, $s > q$, because

$$s = \frac{ar+\eta}{b} > q \Leftrightarrow r(bq+l) + \eta > bq \Leftrightarrow bqr + lr + \eta > bq,$$

which is true since $r > 0$ and $\eta > 0$. Since $s \in \mathbb{N}$, $s \geq q+1$. So, if $\eta \geq b - l$,

$$\begin{aligned}(r, s, h+\eta, h, bs+h) &\geq (r, s, 1+\eta, 1, bs+1) \\ &\geq (1, q+1, 1+\eta, 1, b(q+1)+1) \\ &\geq (1, q+1, 1+b-l, 1, b(q+1)+1).\end{aligned}$$

If $0 < \eta < b - l$, we will minimize for every η as we did for every μ before. The same logic applies, and for every (r,s) with $\eta < b - l$, there is a unique smallest $(r_\eta, s_\eta) \leq (r,s)$.

Then

$$(r, s, h+\eta, h, bs+h) \geq (r, s, 1+\eta, 1, bs+1)$$
$$\geq (r_\eta, s_\eta, 1, 1+\eta, bs_\eta+1).$$

So for $\eta > 0$, we have at most $1+b-l-1 = b-l$ minimal elements, giving a total of $a-b+1+b-l = a-l+1$ at most minimal elements for E_Φ. □

Notice that this bound is much sharper than the one provided by Remark 3.3.28.

3.4 Properties of Fan Algebras over Polynomial Rings

As mentioned in the introduction, fan algebras have two important properties, namely that they are both normal and Cohen-Macaulay.

One notable property of fan algebras coming from ideals in a polynomial ring is that they are all normal.

Theorem 3.4.1. *Let R be the n-dimensional polynomial ring over a field, and $I_1 = (x_1), \dots, I_n = (x_n)$ be principal monomial ideals of R. Also, let $\Sigma_{\mathbf{a},\mathbf{b}}$ be a fan of cones in $\mathbb{N}^2$, with fan-linear functions $f = f_1, \dots, f_n$. Then $\mathcal{B}(\Sigma_{\mathbf{a},\mathbf{b}}, f)$ is normal.*

Proof. Since $\mathcal{B}$ is a fan-algebra of principal monomial ideals over a polynomial ring, $\mathcal{B}$ is a semigroup ring. So $\mathcal{B} = k[Q]$ for the semigroup Q which consists of all the exponent vectors of all the elements of $\mathcal{B}$. Let $z = (z_1, \dots, z_n, r, s)$, $m \in \mathbb{N}$ and $mz = \in Q$. We claim $z \in \mathcal{B}$.

Since Q is the semigroup that defines $\mathcal{B}$ as a semigroup ring,

$$Q = \{(a_1, \dots, a_n, r, s) | a_j \geq f_j(r, s) \text{ for all } j = 1, \dots n, (r, s) \in \mathbb{N}^2\}.$$

And since $mz \in Q$, $(mr, ms) \in Q_i$ for some i. Then we have the following:

$$
\begin{aligned}
mz_1 &\geq f_1(mr, ms) \\
&\vdots \\
mz_n &\geq f_n(mr, ms).
\end{aligned}
$$

But each of the functions f_k are fan-linear, so on Q_i, $f_k(mr, ms) = mf_k(r, s)$. So we have

$$mz_k \geq mf_k(r, s) \text{ for all } k = 1, \ldots, n$$

and so

$$z_k \geq f_k(r, s) \text{ for all } k = 1, \ldots, n,$$

hence $z = (z_1, \ldots, z_n, r, s) \in Q$.

Therefore Q is normal, hence $\mathcal{B} = k[Q]$ is normal. □

As mentioned in the introduction, all normal semigroup rings are Cohen-Macaulay.

Corollary 3.4.2. *Let $\mathcal{B}(\Sigma_{\mathbf{a},\mathbf{b}}, f)$ be as defined in Theorem 3.4.1. Then $\mathcal{B}$ is Cohen-Macaulay.*

CHAPTER 4

SEMIGROUP RINGS COMING FROM CONES

We can use an extension of these ideas to find generating sets of more general kinds of semigroup rings, which will allow us to study non-principal ideals as well.

Theorem 4.0.3. *Let* $C = \bigcup C_i$ *be a fan of pointed rational cones that fills the entire first orthant of* $\mathbb{R}^n$*, and let* $Q_i = C_i \cap \mathbb{Z}^n$ *and* $Q = \bigcup Q_i$*. Let* $\mathcal{H}_{Q_i}$ *be the Hilbert basis of* Q_i*. Then* $k[Q]$ *is finitely generated by the set*

$$\{\mathbf{x}^{\mathbf{q}_j} | \mathbf{q}_j \in \mathcal{H}_{Q_i} \text{ for all } i, j\}.$$

Proof. It is enough to show that each homogeneous element of $k[Q]$ is finitely generated. Let $c\mathbf{x}^{\mathbf{q}}$ be a homogeneous element of $k[Q]$, with $c \in k$ and $\mathbf{q} \in Q$. Then Since $\mathbf{q} \in Q$, $\mathbf{q} \in Q_i$ for some i, $\mathbf{q}$ can be expressed as a sum of Hilbert basis elements

$$\mathbf{q} = \sum_j a_j \mathbf{q}_j,\ a_j \in \mathbb{N}, \mathbf{q}_j \in \mathcal{H}_{Q_i}.$$

Therefore

$$c\mathbf{x}^{\mathbf{q}} = c \prod_j (\mathbf{x}^{\mathbf{q}_j})^{a_j}$$

and

$$k[Q] = k[\mathbf{x}^{\mathbf{q}_j} | \mathbf{q}_j \in \mathcal{H}_{Q_i} \text{ for all } i, j].$$

□

One application of this idea is intersections of monomial ideals.

Corollary 4.0.4. *A particular case of the above is the intersection algebra of non-principal* *monomial ideals. Let* $R = k[\mathbf{x}]$, $\mathbf{x} = (x_1, \ldots, x_n)$, *and let*

$$I = (\mathbf{x}^{a_1}, \mathbf{x}^{a_2}, \ldots, \mathbf{x}^{a_m}), J = (\mathbf{x}^{b_1}, \mathbf{x}^{b_2}, \ldots, \mathbf{x}^{b_p})$$

where $a_i = (a_{i1}, \ldots, a_{in}), b_i = (b_{i1}, \ldots, b_{in}) \in \mathbb{N}^n$ *for all* i. *Then there exists a finite fan of cones* C_i *that fill all of* $\mathbb{N}^{m+p+2}$ *such that*

$$\mathcal{B} = \mathcal{B}(I, J) = k[Q], \textit{ where } Q = \bigcup_i (C_i \cap \mathbb{N}^{m+p+2}),$$

and $\mathcal{B}$ *is finitely generated by the set*

$$\{\mathbf{x}^{\mathbf{q}_j} | \mathbf{q}_j \in \mathcal{H}_{Q_i} \textit{ for all } i, j\}$$

Proof. Let $z \in \mathcal{B}_{(r,s)}$. Then

$$z \in I^r \cap J^s u^r v^s = (\mathbf{x}^{a_1}, \mathbf{x}^{a_2}, \ldots, \mathbf{x}^{a_m})^r \cap (\mathbf{x}^{b_1}, \mathbf{x}^{b_2}, \ldots, \mathbf{x}^{b_p})^s u^r v^s$$

Since

$$\begin{aligned} I^r =& \langle (\mathbf{x}^{a_1})^{i_1} (\mathbf{x}^{a_2})^{i_2} \cdots (\mathbf{x}^{a_m})^{i_m} | (i_1, \ldots, i_m) \in \mathbb{N}^m \text{ with } \sum_{k=1}^m i_k = r \rangle \\ =& \langle \mathbf{x}^{\sum_k i_k a_k} | (i_1, \ldots, i_m) \in \mathbb{N}^m \text{ with } \sum_{k=1}^m i_k = r \rangle \end{aligned}$$

and

$$\begin{aligned} J^s =& \langle (\mathbf{x}^{b_1})^{j_1} (\mathbf{x}^{b_2})^{j_2} \cdots (\mathbf{x}^{b_p})^{j_p} | (j_1, \ldots, j_p) \in \mathbb{N}^p \text{ with } \sum_{k=1}^p j_k = s \rangle \\ =& \langle \mathbf{x}^{\sum_k j_k b_k} | (j_1, \ldots, j_p) \in \mathbb{N}^p \text{ with } \sum_{k=1}^p j_k = s \rangle \end{aligned}$$

we have

$$
\begin{aligned}
I^r \cap J^s = &\langle \mathbf{x}^{\mathbf{q}} | \mathbf{q} = \max\left(\sum_k i_k a_k, \sum_k j_k b_k\right), \\
&\qquad (i_1, \ldots, i_m) \in \mathbb{N}^m \text{ with } \sum_k i_k = r, \, (j_1, \ldots, j_p) \in \mathbb{N}^p \text{ with } \sum_k j_k = s \rangle,
\end{aligned}
$$

where $\mathbf{q} = (q_1, \ldots, q_n) = \max\left(\sum_k i_k a_k, \sum_k j_k b_k\right)$ is defined componentwise, i.e.

$$
q_i = \max\left(\sum_{k=1}^{m} i_k a_{ki}, \sum_{k=1}^{p} j_k b_{ki}\right) \text{ for all } i = 1, \ldots, n.
$$

We will show that these vectors $(\mathbf{q}, r, s)$ come from cones as described. Define the fan

$$
\Sigma = \{C_{\mathbf{c}} | \mathbf{c} \in [2]^n\}
$$

where $C_{\mathbf{c}}$, $\mathbf{c} = (c_1, \ldots, c_n)$ denotes the cone

$$
\left\{ (q_1, \ldots, q_n) \in \mathbb{Z}^n \middle| q_i = \max\left(\sum_{k=1}^{m} i_k a_{ki}, \sum_{k=1}^{p} j_k b_{ki}\right) = \left\{ \begin{array}{ll} \sum_{k=1}^{m} i_k a_{ki} & \text{if } c_i = 1 \\ \sum_{k=1}^{p} j_k b_{ki} & \text{if } c_i = 2 \end{array} \right\} \right.
$$

In other words, if $c_i = 1$, the exponent on x_i in that cone comes from the sum in the exponent I^r, and if $c_i = 2$, the exponent on x_i comes from J^s.

Then each cone $C_{\mathbf{c}}$ consists of all points satisfying a linear homogeneous system of n inequalities

$$
\begin{aligned}
\sum_{k=1}^{m} i_k a_{ki} &\geq \sum_{k=1}^{p} j_k b_{ki} \quad \text{for all } i \text{ where } c_i = 1 \\
\sum_{k=1}^{p} j_k b_{ki} &\geq \sum_{k=1}^{m} i_k a_{ki} \quad \text{for all } i \text{ where } c_i = 2.
\end{aligned} \tag{4.1}
$$

These inequalities, together with

$$
\begin{aligned}
i_k \geq 0 \text{ for every } \quad & k = 1, \ldots, m \\
j_k \geq 0 \text{ for every } \quad & k = 1, \ldots, p
\end{aligned}
$$

and

$$\sum_{k=1}^{m} i_k = r \geq 0, \sum_{k=1}^{m} i_k = r \leq 0$$
$$\sum_{k=1}^{p} j_k = s \geq 0, \sum_{k=1}^{p} j_k = s \leq 0$$

form a system in $m+p+2$ variables that completely describes the exponent vectors in $C_{\mathbf{c}}$.

Since any $(i_1, \ldots, i_m, j_1, \ldots, j_p, r, s)$ will satisfy at least one of the above systems of inequalities (since we have written all possible combinations of inequalities), that point will therefore land in at least one cone. So it is clear that $\bigcup_{\mathbf{c}} C_{\mathbf{c}}$ fills all of $\mathbb{N}^{m+p+2}$. Also, these cones can only intersect on their faces: if there is a point $(q_1, \ldots, q_n) \in C_{\mathbf{c}} \cap C_{\mathbf{c}'}$, where $\mathbf{c} = (c_1, \ldots, c_n)$ and $\mathbf{c}' = (c'_1, \ldots, c'_n)$, then

$$q_i = \max\left(\sum_{k=1}^{m} i_k a_{ki}, \sum_{k=1}^{p} j_k b_{ki}\right) = \begin{cases} \sum_{k=1}^{m} i_k a_{ki} & \text{if } c_i = 1 \\ \sum_{k=1}^{p} j_k b_{ki} & \text{if } c_i = 2 \end{cases}$$

and

$$q_i = \max\left(\sum_{k=1}^{m} i_k a_{ki}, \sum_{k=1}^{p} j_k b_{ki}\right) = \begin{cases} \sum_{k=1}^{m} i_k a_{ki} & \text{if } c'_i = 1 \\ \sum_{k=1}^{p} j_k b_{ki} & \text{if } c'_i = 2 \end{cases}.$$

Therefore for any indices i where $c_i \neq c'_i$, one of the following equations hold:

$$q_i = \sum_{k=1}^{m} i_k a_{ki} = \sum_{k=1}^{p} j_k b_{ki} \text{ if } c_i = 1, c'_i = 2$$
$$q_i = \sum_{k=1}^{p} j_k b_{ki} = \sum_{k=1}^{m} i_k a_{ki} \text{ if } c_i = 2, c'_i = 1.$$

therefore $(q_1, \ldots, q_n)$ lies on the face that separates $C_{\mathbf{c}}$ and $C_{\mathbf{c}'}$.

so by the previous theorem,

$$\mathcal{B} = k[Q] = k[\mathbf{x}^{\mathbf{q}_j} | \mathbf{q}_j \in \mathcal{H}_{Q_i} \text{ for all } i, j].$$

□

In order to construct these generating sets, we must develop some methods of computing the boundary rays of a cone defined by a homogeneous system of inequalities. In $\mathbb{N}^2$, these boundaries are obvious, but in higher dimensions, more work must be done. The following comes from [16].

Consider the system of inequalities

$$\begin{aligned} a_{11}x_1 + b_{12}x_2 + \cdots + a_{1n}x_n &\geq 0 \\ \vdots \qquad\qquad &\geq \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq 0 \end{aligned} \tag{4.2}$$

and the corresponding system of equations

$$\begin{aligned} a_{11}x_1 + b_{12}x_2 + \cdots + a_{1n}x_n &= 0 \\ \vdots \qquad\qquad &= \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= 0 \end{aligned} \tag{4.3}$$

We denote the solution domain of the system 4.2 by $\mathcal{C}$, and the solutions to 4.3 by $\mathcal{L}$.

Each of the equations in 4.2 defines some half-space. Therefore the region determined by the given system can be represented as the intersection of m half-spaces, each of which contains the origin. So $\mathcal{C}$ is a pointed polyhedral cone with its vertex at the origin.

Since any pointed polyhedral cone is finitely generated, $\mathcal{C}$ can also be represented as a

$$\mathcal{C} = \{t_1B_1 + t_2B_2 + \ldots + t_qB_q | t_i \in R_+\},$$

where $B_1, B_2, \ldots B_q$ are a set of points, one selected from each edge of the cone $\mathcal{C}$. We can find such points by proceeding as follows: each point B_i belongs to $\mathcal{C}$, that is, satisfies the system 4.2, and also belongs to the intersection of $n-1$ distinct hyperplanes – that is satisfies $n-1$ independent equations from the system 4.3.

If the only point satisfying both conditions is the origin, then $\mathcal{C}$ reduces to the origin.

Example 4.0.5. Let $R = k[x, y]$ and $I = (xy^4, x^3y^2, x^5y)$, $J = (x^2y^3)$. Then I^r is generated by all monomials

$$(xy^4)^i(x^3y^2)^j(x^5y)^k \text{ where } i + j + k = r,$$

or, equivalently, all monomials

$$x^{i+3j+5k}y^{4i+2j+k} \text{ where } i + j + k = r.$$

Therefore, any degree (r, s) piece of $\mathcal{B} = \mathcal{B}(I, J)$ must satisfy one of the following four case for some $i, j, k \in \mathbb{N}$ with $i + j + k = r$, each of which determines a polyhedral cone

$$\begin{aligned} C_{(1,1)} : i + 3j + 5k &\geq 2s \\ 4i + 2j + k &\geq 3s \end{aligned}$$

$$\begin{aligned} C_{(1,2)} : i + 3j + 5k &\geq 2s \\ 4i + 2j + k &\leq 3s \end{aligned}$$

$$\begin{aligned} C_{(2,1)} : i + 3j + 5k &\leq 2s \\ 4i + 2j + k &\geq 3s \end{aligned}$$

$$\begin{aligned} C_{(2,2)} : i + 3j + 5k &\leq 2s \\ 4i + 2j + k &\leq 3s \end{aligned}$$

As before, the Hilbert bases for these cones determine the R-algebra generators. The first step is to find the extremal rays of each cone.

The full system of inequalities for first cone, $C_{(1,1)}$, is as follows:

$$\begin{aligned} i + 3j + 5k - 2s &\geq 0 \\ 4i + 2j + k - 3s &\geq 0 \\ i + j + k - r &\geq 0 \\ -i - j - k + r &\geq 0 \\ i &\geq 0 \\ j &\geq 0 \\ k &\geq 0 \\ r &\geq 0 \\ s &\geq 0. \end{aligned}$$

Note that this system can be simplified somewhat by directly substituting $i + j + k$ for r, and some of these equations may be redundant, but we will keep the full system for the sake of clarity. Since this is a system of inequalities in 5 variables, we must solve every possible system (i, j, k, r, s) of 4 equations from the following list:

$$i + 3j + 5k - 2s = 0 \tag{1}$$

$$4i + 2j + k - 3s = 0 \tag{2}$$

$$i + j + k - r = 0 \tag{3}$$

$$i = 0 \tag{4}$$

$$j = 0 \tag{5}$$

$$k = 0 \tag{6}$$

$$r = 0 \tag{7}$$

$$s = 0. \tag{8}$$

- Equations (1,2,3,4): $i = 0$, so $r = j + k$, $2j + k = 3s$, $3j + 5k = 2s$

  So

$$s = 2/3j + 1/3k = 3/2j + 5/2k \Rightarrow 4j + 2k = 9j + 15k \Rightarrow -5j = 13k$$

  which gives a solution of $(0, 13, -5, 8, 7)$, which is discarded because it doesn't satisfy $k \geq 0$.

- (1,2,3,5): $j = 0$, so $r = i + k$, $i + 5k = 2s$, $4i + k = 3s$

  So

$$s = 1/2(i + 5k) = 1/3(4i + k) \Rightarrow 3i + 15k = 8i + 2k \Rightarrow 13k = 5i$$

  which gives a solution of $(13, 5, 0, 18, 19)$, which satisfies all inequalities.

- (1,2,3,6): $k = 0$, so $r = i + j$, $i + 3j = 2s$, $4i + 2j = 3s$

  So

$$s = 1/2(i + 3j) = 1/3(4i + 2j) \Rightarrow 3i + 9j = 9i + 4j \Rightarrow 5j = 5i \Rightarrow j = i$$

  which gives a solution of $(1, 1, 0, 2, 2)$, which satisfies all inequalities.

- (1,2,3,7): $i + 3j + 5k = 2s$, $4i + 2j + k = 3s$, $i + j + k = 0$.

  So

$$-2i - j = i + 1/3j \Rightarrow -6i = 3j = 3i + j \Rightarrow -4j = 9i$$

  which gives a solution of $(-4, 9, -5, 0, -2)$, which fails $i \geq 0$.

- (1,2,3,8): $i + 3j + 5k = 0$, $4i + 2j + k = 0$, $i + j + k = r$

  So

$$1/5(i + 3j) = 4i + 2j \Rightarrow i + 3j = 20i + 10j \Rightarrow -19i = 7j$$

  which gives a solution of $(-7, 19, -10, 2, 0)$, which fails $i \geq 0$.

  For the remainder, only rays that satisfy all inequalities are shown:

- (1,3,5,6): $(2, 0, 0, 2, 1)$

- (2,3,4,5): $(0,0,3,3,1)$
- (2,3,4,6): $(0,3,0,3,2)$
- (3,4,5,8): $(0,0,1,1,0)$
- (3,4,6,8): $(0,1,0,1,0)$
- (3,5,6,8): $(1,0,0,1,0)$

So $C_{(1,1)}$ is defined by the extremal rays

$$\{(13,0,5,18,19),(1,1,0,2,2),(2,0,0,2,1),(0,0,3,3,1),(0,3,0,3,2),$$
$$(0,0,1,1,0),(0,1,0,1,0),(1,0,0,1,0)\}.$$

The polyhedra package for Macaulay2 [11] can be used to compute the Hilbert basis, $\mathcal{H}_{(1,1)}$, which is

$$\{(0,1,0,1,0),(0,0,1,1,0),(1,0,0,1,0),(2,0,0,2,1),(1,1,0,2,2),$$
$$(1,1,0,2,1),(2,0,1,3,3),(3,0,1,4,4),(0,2,0,2,1),(1,0,1,2,1),$$
$$(2,0,1,3,2),(0,1,1,2,1),(0,3,0,3,2),(1,0,2,3,2),(0,0,3,3,1),$$
$$(13,0,5,18,19)\}$$

and each Hilbert basis element (i,j,k,r,s) gives rise to an R-algebra generator

$$(xy^4)^i(x^3y^2)^j(x^5y)^k u^r v^s = x^{i+3j+5x}y^{4i+2j+k}u^r v^s.$$

Therefore, the R-algebra generators coming from semigroup elements in $C_{(1,1)}$ are

$$\{xy^4u,\ x^3y^2u,\ x^5yu,\ x^2y^8u^2v,\ x^4y^6u^2v^2,$$
$$x^4y^6u^2v,\ x^7y^9u^3v^3,\ x^8y^{13}u^4v^4,\ x^6y^4u^2v,\ x^6y^5u^2v,$$
$$x^7y^9u^3v^2,\ x^8y^3u^2v,\ x^9y^6u^3v^2\ x^{11}y^6u^3v^2,\ x^{10}y^3u^3v,$$
$$x^{38}y^{55}u^{18}v^{19}\}.$$

The computations for the R-algebra generators for the next three cones are similar, so we will neglect them.

Via the same process, $C_{(1,2)}$ is defined by the extremal rays

$$\{(13,0,5,18,19),(1,1,0,2,2),(0,0,2,2,5),(0,2,0,2,3),(0,0,3,3,1),(0,3,0,3,2)\}$$

and $\mathcal{H}_{(1,2)}$ is

$$\begin{aligned}\{&(0,1,0,1,1),(0,0,1,1,1),(0,1,1,2,1),(1,1,0,2,2),(0,0,2,2,1),\\ &(1,0,1,2,2),(0,3,0,3,2),(0,0,3,3,1),(1,0,2,3,2),(2,0,1,3,3),\\ &(1,0,1,2,3),(0,0,1,1,2),(0,2,0,2,3),(4,0,2,6,7),(0,1,1,2,4),\\ &(7,0,3,10,11),(0,0,2,2,5),(10,0,4,14,15),(13,0,5,18,19)\}.\end{aligned}$$

$C_{(2,1)}$ is defined by the extremal rays

$$\{(13,0,5,18,19),(1,1,0,2,2),(2,0,0,2,1),(3,0,0,3,4)\},$$

$\mathcal{H}_{(2,1)}$ is

$$\begin{aligned}\{&(1,1,0,2,2),(3,0,1,4,4),(1,0,0,1,1),(2,0,0,2,1),(11,0,4,15,16),\\ &(9,0,3,12,13),(7,0,2,9,10),(5,0,1,6,7),(3,0,0,3,4),(13,0,5,18,19)\}.\end{aligned}$$

Finally, $C_{(2,2)}$ is defined by the extremal rays

$$\{(13,0,5,18,19),(1,1,0,2,2),(0,0,2,2,5),(0,2,0,2,3),(3,0,0,3,4),(0,0,0,0,1)\}$$

and $\mathcal{H}_{(1,1)}$ is

$$\{13,0,5,18,19),(1,1,0,2,2),(0,0,2,2,5),(0,2,0,2,3),(3,0,0,3,4),$$
$$(0,0,0,0,1),(1,0,0,1,2),(2,0,0,2,3),(0,1,0,1,2),(1,0,1,2,3),$$
$$(2,0,1,3,4),(3,0,1,4,5),(4,0,1,5,6),(5,0,1,6,7),(0,2,0,2,3),$$
$$(0,0,1,1,3),(4,0,2,6,7),(5,0,2,7,8),(6,0,2,8,9),(7,0,2,9,10),$$
$$(0,1,1,2,4),(7,0,3,10,11),(8,0,3,11,12),(0,0,2,2,5),(10,0,4,14,15),$$
$$(11,0,4,15,16)\}.$$

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