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"Move the Decimal Point and Divide": An Exploration of Students' Introduction to Division with Decimals

Sharon Hooper

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The Dissertation Advisory Committee and the student’s Department Chairperson, as representatives of the faculty, certify that this dissertation has met all standards of excellence and scholarship as determined by the faculty. The Dean of the College of Education concurs.

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PRESENTATIONS AND PUBLICATIONS:


“MOVE THE DECIMAL POINT AND DIVIDE”: AN EXPLORATION OF
STUDENTS’ INTRODUCTION TO DIVISION WITH DECIMALS

by

SHARON BARTLETT HOOPER

Under the Direction of Lynn C. Hart, Ph.D.

ABSTRACT
This study explores the pedagogical approaches used by fifth grade teachers to introduce division with decimals and the resultant understandings of students in their classrooms. The study is important because of the need for students to gain conceptually-based understandings in mathematics and the limited research on instruction and related learning of the very difficult and complex concept of division with decimals. In particular, there is limited research on strategies teachers use to develop students’ conceptual understanding of division with decimals. Therefore, the research questions are as follows.

- What strategies do teachers use to introduce division with decimals?
When first learning to divide decimal numbers, how do fifth-grade students explain the strategies they use?

The study is grounded in social constructivist learning theory and uses a collective case study methodology. Following the study design, three fifth-grade teachers from three schools were interviewed before and after an introductory lesson to division with decimals. They also were observed teaching the study lesson. Following the lesson, one to three students from each class (six in all) were interviewed on their understandings of division with decimals using their classwork from the lesson as a point of entry. The design includes three sources of data: transcriptions from semi-structured interviews of teachers and students, field notes from classroom observations, and artifacts from lessons. Results suggest that instruction of division with decimals varies such that the differences can be captured along a continuum of traditional to reform practices. The placement of the decimal point in the quotient is the focus of the discussion regardless of where the instruction lies on the continuum. Interestingly, as instruction moves towards the traditional end of the continuum, student engagement was a result of interaction with the teacher, whereas closer to the reform end of the spectrum students were engaged with the mathematics.

INDEX WORDS: Decimals, Division, Elementary Mathematics, Elementary Instruction
“MOVE THE DECIMAL POINT AND DIVIDE”: AN EXPLORATION OF
STUDENTS’ INTRODUCTION TO DIVISION WITH DECIMALS

by

SHARON BARTLETT HOOPER

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in
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DEDICATION

This work is dedicated to my mom and dad for their unwavering love and support, and to my husband for his enduring love and patience.
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DIVISION WITH DECIMALS: A REVIEW OF THE LITERATURE

Guiding Question

Very little is known about what fifth grade teachers do to help students learn division with decimals. While there have been many studies on how children understand the concept of division (e.g., Fischbein, 1987; Fischbein, Deri, Nello, & Marino, 1985; Greer, 1987; Parmar, 2003), and how students come to understand decimals (e.g., Moss, 2003; Steinle & Stacey, 2004), in over 20 years only three studies were identified regarding classroom teaching methods for division with decimals (Bell, Swan, & Taylor, 1981; Bonotto, 2005; Okazaki & Koyama, 2005). Overall, research shows that students’ understanding of division affects their understanding of division with decimals (Graeber & Tosh, 1990) and using problems that create disequilibrium\(^1\) within students causes students to rethink the meaning of division with decimal numbers (Okazaki & Koyama, 2005); however, there is a void in the research literature regarding classroom instruction of division with decimals. This review makes it apparent that there is a need for more research on what teachers do to help students understand division with decimals.

Facility with decimal operations is considered a prerequisite skill for many avenues of study in mathematics (National Mathematics Advisory Panel, 2008). It is often the case, however, that students have difficulty with the specific operation of division with decimals. The extant research shows numerous misconceptions students have about division with decimals (Bell, Fischbein, & Greer, 1984, Graeber & Tirosh, 1990; Greer, 1987), illustrating the gap between what students need to understand in order to be considered competent in the concept and what they actually do understand.

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\(^1\) The problem causes conflict with student’s current understanding of a concept, in this case division with decimals.
Several researchers suggest that a student’s ability to think and reason can be hindered by teaching only traditional algorithms (Kamii, 1994; Wearne & Hiebert, 1988; Yang, 2005), however there is limited research available regarding alternative approaches specific to teaching division with decimals. The Common Core State Standards for Mathematics (CCSSM), published in 2010, suggests that students should build on their understanding of place value and the operations of multiplication and division to acquire a conceptual understanding of division with decimals. There is clearly a need to determine how and if teachers use these requisite concepts to build understanding of division with decimal numbers when introducing the concept and if it makes a difference in student understanding. Therefore, the guiding question for this review of the literature is, what does research tell us about effective strategies to introduce the concept of division with decimals?

Introduction

Many mathematics teachers in the United States use strategies that have been used for generations to teach division with decimals. This is most probably the result of a dearth of research informing the process and a lack of new pedagogical approaches for teaching the topic. This review of the literature will identify and synthesize studies on division with decimals. This review is important because (a) it highlights the need for additional research to be conducted on this topic, (b) it is timely in relation to the adoption the CCSSM in most states in the United States, (c) it demonstrates how minimal the discussion of division with decimals is in the CCSSM, and (d) it aims to shed light on the question of how division with decimals should be introduced to students.

This review will begin with an overview of division with decimals within the CCSSM and its peripheral documents. Teacher guidance is provided in the CCSSM and its ancillary
documents. The CCSSM includes the Standards for Mathematical Practice (SMP), which delineates ways students should interact with the mathematics concepts in fifth grade. Also, in 2012, the Common Core Standards Writing Team drafted the *Progressions for the Common Core State Standards in Mathematics [Draft], Grades K-5, Number and Operations in Base Ten*² (progressions documents), which illustrate current thinking regarding the pedagogical approach to division with decimals. This is followed by an overview of relevant research on the understanding of division with decimals. These studies were sorted into four areas: (a) research that emphasizes conceptual understanding of division with decimals, (b) research that uses number sense strategies to deepen students’ understanding of division problems with decimals, (c) research that highlights the need for students to explore division with decimals using related-problem experiences, and (d) research that identifies common student misconceptions about dividing decimals and how this informs the instruction of division with decimals.

For the purposes of this study, conceptual understanding is defined using Hiebert and Lefevre (1986) meaning, “Knowledge that is rich in relationships” (p. 3). To develop conceptual understanding, students need to build a complex web of connections to other topics in mathematics and to their lived experiences. As division with decimals is introduced, conceptual understanding may be developed by asking students to draw upon their understanding of operations, decimals, and place value, to name a few. Conceptual understanding may be contrasted to procedural understanding which is, “composed of the formal language, or symbol representation system, of mathematics…[and] the algorithms, or rules for completing mathematical tasks” (Hiebert & Lefevre, 1986, p. 6). While both types of understanding are important, research has shown that the development of conceptual understanding leads a student

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² All of the progressions documents can be found at http://ime.math.arizona.edu/progressions/
to construct procedural understanding. This constructed knowledge is more readily remembered and applied in new situations (Hiebert et al., 1997).

An extensive electronic search on division with decimals was conducted to complete this review of the literature. The initial search was conducted in the following databases: Academic Search Complete (EBSCO), ERIC (EBSCO), and JSTOR, and focused on peer-reviewed articles in scholarly journals using the following search terms: division, decimals, and elementary. The Academic Search Complete (EBSCO) search gave 57 results, the ERIC (EBSCO) search provided 137 results, and the JSTOR initially provided over 2,200 results. The limits added to JSTOR included “not fractions” and the dates 1995 – 2015. This resulted in 137 articles. The results from all three databases were further filtered through a review of each title and abstract.

Studies included in this review of the literature dealt with student understanding of division and decimal notation as well as teaching and understanding division with decimals. The inclusion of decimals and division added additional foundational studies on students’ understanding of the operation of division and decimal numbers, including misconceptions on these two topics. Studies dealing with fractions were excluded. Through the databases a total of 28 articles were found. Further studies were identified through the reference lists of the initial studies. The search was considered exhausted when reviews of article references presented no new relevant studies. The identified articles were reviewed in more detail, by accessing the article and carefully reviewing its contents. This allowed the number of applicable articles to be culled to the 10 articles under review.
Review

Common Core State Standards.

The CCSSM (2010) was created by the National Governors Association Center for Best Practices and the Council of Chief State School Officers in an effort to create coherent standards for mathematics instruction in the United States. Their goal was to offer standards that are focused and reflective of standards from high performing countries (e.g. Singapore, Hong Kong, and Korea), where fewer mathematics topics are addressed in more depth. Currently, 43 states have adopted the CCSSM, making it an influential document in the United States (“Standards in Your State,” 2015).

Prerequisite standards for division with decimals in the CCSSM in fifth grade include the concepts of place value, division, and decimals. Students in fifth grade are expected to acquire place value understanding to the thousandths place\(^3\) including fluency in multiplying and dividing numbers by powers of ten\(^4\). For example, students should recognize that multiplying 0.2 by 100 has the effect of moving the decimal two places to the right, equaling 20. Alternately, 0.2 divided by 100 has the effect of moving the decimal two places to the left, making 0.002. These patterns of decimal point placement and the movement of digits when multiplying or dividing by a power of ten are useful when dividing decimals and provide a foundational understanding for division with decimals.

\(^3\) 5.NBT.3 Read, write, and compare decimals to thousandths. (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010)

\(^4\) 5.NBT.2 Explain patterns in the number of zeros of the product when multiplying a number by powers of 10, and explain patterns in the placement of the decimal point when a decimal is multiplied or divided by a power of 10. Use whole-number exponents to denote powers of 10. (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010)
The CCSSM states that fifth grade students will continue their experience with division, dividing up to four digits by two-digit divisors. However, while students are expected to multiply fluently using the standard algorithm, fluency with the standard algorithm for division is not expected until the end of sixth grade. Students are introduced to division with a decimal in both the dividend and divisor in fifth grade. What follows is the CCSSM standard for division with decimals.

5.NBT.7 Add, subtract multiply, and divide decimals to hundredths using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used. (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010)

While the language of the CCSSM objectives indicates that student understanding should include models or drawings, the Progressions for the Common Core State Standards in Mathematics [Draft], Grades K-5, Number and Operations in Base Ten (2015) focuses on an understanding of place value and the relationship between multiplication and division. In the Grades K-5, Number and Operations in Base Ten section of the progressions documents, the authors of the CCSSM clarify this through standard 5.NF.7.b. Using students’ understanding of division with unit fractions and inverse operations, students look for patterns when dividing by 0.1 and 0.01. Then, students are asked to think of the problem 8 ÷ 0.1 as how many tenths are in 8. If there are 10

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5.NBT.6 Find whole-number quotients of whole numbers with up to four-digit dividends and two-digit divisors, using strategies based on place value, the properties of operations, and/or the relationship between multiplication and division. Illustrate and explain the calculation by using equations, rectangular arrays, and/or area models. (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010)

5.NF.7.b Apply and extend previous understandings of division to divide unit fractions by whole numbers and whole number by unit fractions. . . . (b) Interpret division of a whole number by a unit fraction, and compute such quotients. For example, create a story context for 4 ÷ (1/5), and use a visual fraction model to show the quotient. Use the relationship between multiplication and division to explain that 4 ÷ (1/5) = 20 because 20 × (1/5) = 4. (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010)
tenths in one, there must be $8 \times 10 = 80$ tenths in 8. After several similar problems, students may note that $8 \div 0.1$ yields the same results as $80 \div 1$, showing that the place value in both dividend and divisor can be shifted one place to the left when dividing by a tenth. This reasoning continues when dividing by 0.2. For example, $8 \div 0.2$ is the same as $80 \div 2 = 40$. Alternatively, students could reason that 0.2 is the same as $2 \times 0.1$ and first divide 8 by 2 leaving 4, and then $4 \div 0.1 = 40 \div 1 = 40$.

Students can summarize the results of their reasoning as specific numerical patterns then as one general overall pattern such as “when the decimal point in the divisor is moved to make a whole number, the decimal point in the dividend should be moved the same number of places.” (Common Core Standards Writing Team, 2015, p. 18) The focus is on understanding through reasoning and building on a basic understanding to solve more complex problems.

The CCSSM (2010) includes content standards as well as the standards for mathematical practice (SMP). The SMP are consistent at each grade level, indicating how students will interact with mathematics as a student develops mathematical maturity. According to the SMP, students should be asked to use “overarching habits of mind of a productive mathematical thinker” (McCallum, 2011). This requires a teacher to create a low-risk climate in which students reason, argue, and critique the reasoning of others in order to develop a student’s productive disposition. Also, students need to be given problems that engage them in mathematical modeling, the strategic use of appropriate tools, and the opportunity to look for repeated reasoning and structure in order to develop a student’s procedural fluency.
**Emphasis on conceptual understanding.**

Four studies were found that looked at methods helping students understand division with decimals. These methods include (a) using a context that is familiar to students (Bonotto, 2005), (b) encouraging student-generated algorithms (Ambrose, Baek, & Carpenter, 2003), and (c) challenging students’ misconceptions (Bell, Swan, & Taylor, 1981; Okazaki and Koyama, 2005). These methods emphasize conceptual understanding.

Bonotto (2005) used authentic artifacts from students’ daily life to develop students’ understanding of decimals. Using qualitative methods, Bonotto observed two upper elementary classrooms of students as they solved problems with decimals. During these observations, problems were presented within a context that was familiar to students, including grocery receipts in one case and centimeter rulers in the other. Bonotto “show[s] how suitable cultural artifacts, with their incorporated mathematicals, can play a fundamental role in bringing students' out-of-school reasoning experiences into play by creating a new tension between school mathematics and the students' everyday-life knowledge” (p. 315). Bonotto found that by using real-life artifacts, students were able to see mathematics outside of school and were better able to understand the application and importance of mathematics. This gave students an opportunity to solve meaningful rather than contrived problems, to practice mathematical operations (skills), and to develop new mathematical understanding.

A second method was prompting students to use their own invented strategies for division with decimals to determine if students developed a deeper conceptual understanding of the operation. Ambrose, Baek, and Carpenter (2003) spent a year observing in 6 classrooms (grades 3-5), as well as conducting clinical interviews with students from one classroom at each grade level. Students were encouraged to approach division in a way that made sense to them
and then share their strategies with the class. Students’ invented strategies for division included repeated subtraction, partitioning with friendly numbers (representing equal groups using numbers or pictures), abstract modeling (where every group may not be represented), decomposing the dividend, and building up to the dividend using addition (including doubling or quadrupling). Because multiplication and division with decimals were taught concurrently, many of the strategies mirrored students’ invented strategies for multiplication. Students demonstrated an intuitive understanding of the associative and distributive properties in their solution strategies. Ambrose et al. disputed the claims that building division understanding on addition and subtraction leads to misconceptions when working with rational numbers. They argued that “this kind of sense-making activity would mitigate against the kind of rule-based reasoning that multiplication always makes bigger and division always makes smaller found by Bell, Fischbein, and their associates” (p. 322). Ambrose et al. found the advantages of invented algorithms include (a) students breaking the problem into manageable tasks reduced processing demands, so they were less likely to make careless errors; and (b) underlying concepts were usually made apparent. The researchers also found students’ facility with invented algorithms can lead to a deeper understanding of other mathematics skills, such as converting a fraction to a decimal. For example, using an understanding of decomposition students found halves of a number. The researchers observed this student-invented strategy for changing $3/8$ into a decimal.

First, students recognized that $3/8 = 1/4 + 1/8$. Students knew that $1/4 = .25$ and that $1/8$ is half of $1/4$. So they found half of $.25$, which is half of $.2$ or $.1$, and half of $.05$, which is $.025$. Putting the pieces back together, students get $3/8 = .25 + .1 + .025$, which is $.375$. (p. 333)
The researchers contend that early flexibility with basic operations allows students to think flexibly later when dealing with converting fractions to decimals. This is because many of the invented algorithms could be easily used to convert fractions to decimals.

Cognitive conflict has been used in several studies in order to develop an understanding of division and to dispel the misconceptions students have regarding division with decimals. During the third phase of their study, Bell, Swan, and Taylor (1981) incorporated this approach using calculator games. The researchers studied a class of 27 students who were 15 years old; the results include the 18 students present for all assessments. Bell et al. taught a class using eight activities developed after pre-tests determined where students’ misunderstandings and misconceptions occurred. The researchers asked students to play decimal games that included division with decimals. One activity, a game called Target, required students to use a given number and the division operation key on a calculator to hit the target of 100 (e.g. $34 \div 2.9 = 117.24137931$). Through this game, students were able to compare their estimated quotient with the exact result as displayed by the calculator. Bell et al. found the percentage of students who understood the order of the dividend and divisor was not reversible increased 38% (from 39% to 67%); however this gain was not maintained as indicated by the delayed post-test result of 39% of students having this understanding. By making students aware of misconceptions, conflicting understandings can be discussed and conflicts can be resolved (Mangan, 1989). This process is not only interactive, it is also reflective through the use of metacognition.

The more recent study by Okazaki and Koyama (2005), investigated whether cognitive conflict is an effective strategy. In their study, students were given a decimal story problem and they had to determine which operation should be used to solve the problem. The researchers believed that by working through disequilibrium, the students would make the most growth in
their understanding of division with decimals. Using a “diagnostic teaching approach” (p. 229) they planned six lessons in a fifth grade class of 38 students (20 boys and 18 girls) in a Japanese primary school. The lessons were planned and taught in a cyclical manner, as each subsequent lesson was planned after the researchers debriefed with the teacher regarding the previous lesson.

The researchers had access to students’ notebooks in which students regularly recorded what they understood and what they were still unsure about at the end of each lesson. After each lesson, Okazki and Koyama used these artifacts and planned for the next day. The authors describe disequilibrium during the third lesson when students were asked to solve the following problem: “The price of 0.8 liters of juice is 116 yen. How much does 1 liter cost?” (p. 227). Because the divisor is less than 1, the partitive division model with which the students were very comfortable is difficult to apply in this situation. Students were conflicted as to whether the correct operation to solve this problem should be multiplication, division, or both. Through discussion and several models (operational number lines, double number lines, and proportion) students came to recognize that the problem could be solved by dividing by 0.8 or by multiplying by 1.25. Students were able to use their understanding of multiplying by 1.25 and their understanding of inverse operations and reciprocals to make sense of dividing by 0.8. The researchers believed that the methods used to work through this problem helped students move from a concrete operational stage (referring to Piaget’s theory of development) to a formal operational stage. Also, they determined that the use of cognitive conflict was successful in developing students’ understanding of division with decimals.
Understanding through number sense.

Howden (1989) suggests, “Number sense can be described as good intuition about number and their relationships” (p. 11). Yang (2005) elaborates by adding, number sense is an understanding of numbers allowing one to flexibly compute, recognize the reasonableness of answers, and explain strategies clearly, leading to efficient strategies. Estimation complements number sense; when students use estimation, they reflect on the accuracy of their answers (National Council of Teachers of Mathematics, 2000).

Developing number sense through estimation, the use of benchmark numbers, and judging the reasonableness of an answer can all lead to deeper understandings of decimal operations. This section describes three studies related to the development of number sense as a strategy to enhance decimal division understanding (Bonotto, 2005; Graeber & Tirosh, 1990; Yang, 2005).

The insights middle grades students have when multiplying and dividing decimals was studied by Graeber and Tirosh (1990). The researchers interviewed 90 fourth and fifth graders (60 from the United States and 30 from Israel) regarding common multiplication and division misconceptions such as multiplication results in a larger number and division results in a smaller number. They found that these misconceptions persist in both countries and made some suggestions to improve student understanding of division with decimals. These included (a) beginning estimation in the early grades, allowing students to develop an understanding of division with both whole numbers and decimals; (b) ensuring students have a strong understanding of decimal values; and (3) promoting student proficiency in the measurement model for division with whole numbers before moving to decimals.
Bonotto (2005) explored how the use of real-world cultural artifacts could be used to help students make sense of formal mathematics computation, in this case multiplication of decimals. His study, as mentioned in the previous section, included a fourth grade class in two schools in Italy, one with 23 students, and the other with 21 students. The cultural artifacts were used from students’ lived experiences as a link to the skills learned in school. Bonotto found the context of the artifacts were effective in providing meaning to the mathematics computation. The researcher also explored number sense by incorporating estimation into the problem solving process. Before students were expected to find the exact answer, students were allowed to discuss discrepancies between expected answers and actual answers. Bonotto also believed that estimating provided the students more freedom to discuss the reasonableness of answers.

Yang (2005) focused on number sense in the study of sixth grade students in Taiwan. In this section, elaboration on Yang’s interviews with 21 sixth grade students is provided. One school was affiliated with a local university; the other three schools served children from economically diverse families with varying educational backgrounds. The students were randomly chosen from three achievement groups, low-, middle-, and high-level students, based on the student’s previous year’s academic performance. Yang described students groups as follows: “high—the top 10%; middle—those in the 40–60% range; and low—the bottom 10%” (p. 320). Yang’s study is interesting because while Taiwanese students are proficient in mathematics, number sense has not taken a prominent role in mathematics instruction. The researcher voiced surprise that no initiatives in curriculum or pedagogy have placed number sense in a prominent role in Taiwan. To investigate if Taiwanese students could use number sense strategies to solve problems, Yang created an interview instrument with questions that evaluated a student’s use of number sense. Questions included problems to evaluate a student’s
understanding of decimals and operations with decimals, including division. For example, “Without calculating an exact answer, circle the best estimate for: $72 \div 0.025$. The selection choices were (1) A lot less than 72; (2) A little less than 72; (3) A little more than 72; and * (4) A lot more than 72” (p.322). Each interview was videotaped, transcribed, and coded by two researchers. Overall, Yang found very few students used number sense techniques to solve the problems presented in the interview instrument. While 26% of the “high-level” students used number sense-based responses to determine correct answers, only 7% of the students in the “low-level” group gave number sense based-responses. Yang concluded that “the low (7%) and middle (12%) level students were least likely to use number sense-based explanations” (p. 331). Across groups, students were more likely to explain their response using a rule-based explanation or students could not explain their thinking. What the researcher did not explicitly state, but is evident in “Table 2. Students’ responses to interview questions (percentages)” (Yang, 2005, p.331) is that all students who used a number sense-based response, regardless of the student’s level, gave correct answers.

**Related-problem experiences.**

The importance of providing related-problem experiences when introducing concepts such as division with decimals was illustrated in four studies. Bell, Swan, and Taylor (1981) suggested that students be allowed to find similarities between problems where the numerical values vary between whole numbers and decimals, while the context remains consistent. In addition, Greer (1994) described the need for students to explore problems in multiple contexts and to use numbers that include rational numbers. Brousseau, Brousseau, and Warfield (2008), created a series of closely-related problems to help students construct an understanding of division of rational numbers. And, Williams and Copley (1994) studied how problems where the
digits are the same but the placement of the decimal varies allows students to discuss and generalize patterns that are observed.

Bell, Swan, and Taylor (1981) interviewed pairs of students with problems that included division with decimals to investigate the “conceptual systems with which pupils approached the set of tasks” (p. 403). During the first phase of their research, 40 students were interviewed ranging in age from 12-16 years old. The student participants were able to choose the correct operation (multiplication) when given numbers greater than one. However, when one of the values in the problem was less than one, students typically changed the operation to division. Bell et al. found that students would assume the operation needed to change if the numbers used were smaller (e.g. less than 1). Therefore, they concluded that students need to understand that the operation does not vary as the numbers change. The researchers asserted, "The invariance of multiplication and division over the number involved is a powerful idea that potentially can be harnessed to overcome the limitations of intuition" (p. 77).

Building on previous research, Greer (1987) studied students’ nonconservation of the operations of multiplication and division. To further investigate how the type of number used affects nonconservation, Greer gave 65 students aged 12-13 in a school in Northern Ireland a written test. From the initial pool, interviews of 15 students who showed “evidence of nonconservation” (p. 40) were conducted. Nonconservation means the students were given a series of similar problems, requiring the same operation, but the students were often unable to recognize that only the numbers had changed and the operation should remain the same. A problem requiring students to find the cost of 2.6 gallons of petrol at £1.79 per gallon was solved correctly by 63% of the students, while only 38% of the students solved the problem correctly when there was 0.8 gallons of petrol at £1.85 per gallon. Greer concluded that the type of number
used in a problem has a significant effect on the difficulty of identifying the correct operation. Also, he concluded it is important to help students understand that the operation does not change because of the types or size of numbers used.

Brousseau, Brousseau, and Warfield (2008), created and tested “a carefully structured, tightly woven and interdependent sequence of ‘situations’” (p. 79), to study how related problems help to develop a student’s understanding of the inverse operations of multiplication and division with rational numbers. The situations were used with two classes of 10-11 year-old students in Talence, France in over 65 lessons. In one lesson, students created situations in which division was required and discussed whether the operation would change if the situations included decimals or fractions. Following lessons used a pantograph to model multiplication and division with decimals by determining the scale factor used to enlarge or reduce figures. Through the series of lessons, students were able to develop an understanding of division with decimals, not as discrete skills to be mastered, but as related problems that helped students construct an understanding of division, division of rational numbers, and the relationship between multiplication and division.

Williams and Copley (1994) observed how a sixth grade class used calculators to observe and discuss patterns that occurred when dividing decimals. This study was a part of a larger research project conducted in conjunction with a school district in the Houston, Texas area. Students were given problems such as “Given: 2726 ÷ 58 = 47; Predict: 272.6 ÷ 58 = _____; 2.726 ÷ 58 =_____; 27260 ÷ 58 =_____; 0.2726 ÷ 58 = _____” (p. 73). Students were asked to examine the results and discuss patterns identified, and then make predictions about the cause of the patterns. Students generated rules about the size of the quotient in relation to the dividend and divisor as well as the movement of the digits and the decimal point. After working on this
problem, students began questioning what would happen if the digits stayed the same, but the placement of the decimal in the divisor changed or the placement of the decimal in both the dividend and divisor changed. By observing the patterns in the dividend, divisor, and quotient of related problems, students “discovered” how the dividend and divisor were multiplied by the same power of ten.

**Common student misconceptions.**

There were no studies focused on misconceptions about decimals and division with decimals after 1994. Three studies of interest to this review are Bell, Swan, and Taylor (1981), Fischbein, Deri, Nello, and Marino (1985), and Graeber and Tirosh (1990), each of which is reviewed briefly.

Bell, Swan, and Taylor (1981), interviewed 12 to 16 year-olds (N = 40) to explore students’ conceptual understanding and misconceptions regarding operations with decimals. As described in an earlier section, the interviews involving pairs of students occurred during the first phase of the research. The participants were given several problems and asked to choose the correct expression to solve each. Students were interviewed in pairs and encouraged to discuss their answer choices. If they had difficulties, a hint would be given that focused on employing heuristics such as using a diagram, trying smaller or easier numbers, retelling the problem in their own words, or using a calculator. Even if the students chose the correct expression, some of the hints were given to determine if they could use different strategies. One example requiring multiplication of decimals was, “If petrol was £1.17 per gallon, what would be the cost of filling a tank containing 8.6 gallons?” (Bell, Swan, and Taylor, 1981, p. 405). For this example the students immediately chose the correct answer, 1.17 × 8.6. However, if the values were changed, for example, “If petrol was £1.20 per gallon, what would be the cost of filling a gallon can?”
(Bell, et al., 1981, p. 405), they thought the operation was division. The students determined the cost was “different because you’ve got a lesser amount. It’s under £1.20, so obviously it’s $1.20 ÷ 0.22$ or something like that. . . . Neither they nor any other pupil interviewed, considered it incompatible that the needed operation should change as the numbers changed” (Bell, et al., 1981, p. 405). Bell, et al. found that 89% of the students, believed that when multiplying the product would be larger and conversely when dividing the quotient would be smaller. As a result, if students intuitively believed the result would be smaller, they would choose division as the operation.

Inspired by Bell et al. (1981) research, Fischbein, Deri, Nello, and Marino (1985) studied how a students’ intuitive model influences their choice of operation for a given problem. A student’s intuitive model of an operation may not, for example, account for decimal numbers, making the correct operation seem inappropriate. The researchers used a 43-item instrument containing 14 division problem items in which students were prompted to identify the operation necessary to solve a problem. A total of 628 students participated from grade 5, 7, and 9. Although the students were traditionally introduced to division with decimals in fifth grade, the researchers found that 65% of the student participants switched the divisor and dividend, making the dividend the larger number. Also, when students were required to divide 900 by 0.75, approximately one-third did not respond. The researchers concluded that this item conflicted with the students’ implicit understanding that the divisor must be a whole number.

Fischbein et al. (1985) theorized that two primitive models of division may implicitly influence computation with decimals. In the first model, *partitive division*, items are shared equally among a given number of groups. With this interpretation, the dividend must be larger than the divisor, the divisor must be a whole number, and the quotient must be smaller than the
dividend. The second model, *quotative division*, also known as measurement division, is essentially repeated subtraction when working with whole numbers. In this interpretation, the dividend must be larger than the divisor so that it can be determined how many of a quantity is contained within another quantity. The researchers theorized that students have difficulties when dividing by decimals because of these implicit beliefs, which they framed as a *primitive intuitive model*. Fischbein et al. found that when an operation is required for two numbers, the operation choice is influenced by the primitive intuitive model. Fischbein et al. asserted, "Most of our tacit, intuitive models are imperfect mediators, leading often to incorrect or incomplete interpretations" (p. 126). Therefore, the researchers concluded that the models used with young children for division (i.e., partitioning, measurement) effect the ease with which students are able to extrapolate these models to decimals. A sample problem that had a decimal divisor was given as an example: “I spent 900 lire for 0.75 hg of cocoa. What is the price of 1 hg?” (Fischbein et al., 1985, p.9). A divisor of 0.75 had no intuitive meaning for the students, therefore nearly one-fourth of the students left the item blank and just over one-half of the students got an incorrect answer; nearly one-fourth of the students with an incorrect answer chose to multiply. The challenge or question is, how can instruction help students move beyond these basic representations, or how can we teach division in a way that does not develop a primitive model.

A common misconception was observed by Graeber and Tirosh (1990) during interviews of 30 U.S. American fourth and fifth grade students from the Washington, DC area. As described in a previous section, the researchers were studying the influence of students’ beliefs about the operations of multiplication and division with whole numbers when they began multiplying and dividing decimals. Only one-sixth of the students were able to write the correct expression for a problem where the divisor was larger than the dividend (i.e., \(5 \div 15\)); only one student was able
to give the expression without hesitation. The researchers concluded that the models students develop for division when working with whole numbers, partitive and quotative models, limit the values that make sense to students. When a decimal number is presented that does not fit with a student's model of division, she becomes confused as to the correct operation.

**Conclusion**

This review of the literature exposes some key factors regarding student understandings of division with decimals. There is a need for teachers to establish a learning environment where connections are frequently made between mathematics topics as well as between mathematics topics and real-world experiences (Bell, Swan, & Taylor, 1981; Greer, 1994; Bonotto, 2005). This type of learning environment is also encouraged in the CCSSM (2010), in particular the productive disposition as described in the SMP. The need to make connections is elaborated by Bonotto (2005),

> Many teachers introduce decimal numbers by extending the place-value convention. They tend to spend little time allowing children to understand the meaning of decimal numeration or reflect on decimal number properties and relationships; efforts to connect decimal numbers and decimal measures are lacking. As a consequence, children learn to carry out the required computations, but have difficulty in mastering the relationship between symbols and their referents, and between fractional and decimal representations. (p. 320)

A key aspect of understanding is flexibility with number. Students should have opportunities to compute fluently using mental mathematic strategies, estimation, and written computation. Overall, a goal of number sense should be evident, because for students without number sense
“mathematics is a conglomeration of disconnected facts and formulas that they must memorize and practice” (Yang, 2005, p. 319).

Students should have an opportunity to make sense of decimals and division with decimals through a variety of contextual experiences. Students may be struggling to reconcile expected answers and mathematical notation, but this will dispel primitive intuitive models that may have developed when operations were exclusively with whole numbers. All of this provides a framework around which an understanding of how students make sense of division with decimals can be built.

References


“Move the Decimal Point and Divide”: An Exploration of Students’ Introduction to
Division with Decimals

Facility with decimal operations is considered a prerequisite skill for many avenues of study in mathematics (National Mathematics Advisory Panel, 2008). The extant research shows numerous misconceptions students have about division with decimals (Graeber & Tirosh, 1990; Greer, 1992), illustrating the gap between what students need to understand, in order to be considered competent in the concept, and what they actually do understand. Also, several researchers suggest that a student’s ability to think and reason can be hindered by teaching only traditional algorithms (Kamii, 1994, Wearne & Hiebert, 1988; Yang, 2005), however there is limited research available regarding alternative approaches specific to teaching division with decimals. When considering the Common Core State Standards for Mathematics (CCSSM), published in 2010, it is expected that students build on their understanding of place value and the operations of multiplication and division to acquire a conceptual understanding of division with decimals. However, within the CCSSM no specific direction is given on how to teach division with decimals. There is clearly a need to determine how and if teachers use these requisite concepts to build understanding of division with decimal numbers when introducing the concept and if it makes a difference in student understanding. Therefore, the research questions are as follows:

1. What strategies do teachers use to introduce division with decimals?

2. When first learning to divide decimal numbers, how do fifth-grade students explain the strategies they use?
Theoretical framework

Based on an interpretive framework, understandings about the world are socially constructed. Constructivists espouse that knowledge is actively constructed rather than found or discovered. “Human beings…invent concepts, models, and schemes to make sense of experience, and we continually test and modify these constructions in the light of new experience” (Schwandt, 2000, p. 197). However, “we do not construct our interpretations in isolation but against a backdrop of shared understandings, practices, language, and so forth” (Schwandt, 2000, p. 197). Constructivist epistemologies bear similarities to the theories of social constructivism focused on student learning. When considering the interaction of language and learning, Vygotsky (1978) explains:

*The most significant moment in the course of intellectual development, which gives birth to the purely human forms of practical and abstract intelligence, occurs when speech and practical activity, two previously completely independent lines of development, converge.*

(p. 24, italics in original)

Therefore, beliefs about student learning and epistemological stance regarding qualitative inquiry are closely aligned, which provides a more consistent lens with which to view this research.

The aims for this study are interpretive rather than critical. The focus is on the perceptions of the participants and making sense of the events as they unfold rather than critique what is observed. According to Denzin and Lincoln (2005), “the constructivist paradigm assumes a relativist ontology (there are multiple realities), a subjectivist epistemology (knower and respondent cocreate [sic] understandings), and a naturalistic (in the natural world) set of methodological procedures” (p. 24). Given the tenets of social constructivist theory, this research design focuses on multiple perspectives, uses semi-structured interviews, and incorporates
observations in a natural setting. This allows the researcher (a) to interpret the multiple realities (of the teachers and students) when division of decimals is introduced, (b) to co-create an understanding of how a teacher introduces division with decimals, and (c) to observe the lesson in a naturalistic setting (i.e., the classroom).

**Conceptual Framework**

Theoretical framing is considered in this study as well as current mathematical standards such as the CCSSM (2010). The CCSSM is a set of standards that “define what students should understand and be able to do in their study of mathematics” (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). The standards were based on mathematics learning trajectories for children (see Sarama & Clements, 2009), standards developed by states such as Massachusetts, nations such as Singapore, and organizations in the United States such as the National Council of Teachers of Mathematics (see Curriculum Focal Points for Prekindergarten through Grade 8 Mathematics: A Quest for Coherence, 2006). The CCSSM includes standards for mathematical practice (SMP); these standards are consistent at each grade level, with the expectation that students’ interaction with mathematics develops along with their mathematical maturity. According to the SMP, students should be asked to use “overarching habits of mind of a productive mathematical thinker” (McCallum, 2011). The SMP require a teacher to create a low-risk climate in which students reason, argue, and critique the reasoning of others in order to develop a student’s *productive disposition*. Also, students need to be given problems that engage them in mathematical modeling, the strategic use of appropriate tools, and the opportunity to look for repeated reasoning and structure in order to develop a student’s *procedural fluency*. 
Methodology

An extensive search yielded no research documenting strategies teachers currently use to introduce the concept of division with decimals, thus suggesting that this topic is largely unexamined. Yet, it remains important for the knowledge base in mathematics education. Therefore, this study explores what teachers do to help their students learn division with decimals. The study design includes interviewing three teachers from three schools before and after an introductory lesson on division with decimals, observing the teachers during the study lesson, and interviewing students on their understandings after the lesson. A collective case study will be used as the research strategy.

A collective case study uses multiple cases to gain an understanding of a phenomenon. In this study, the phenomenon was students’ first classroom experience with division with decimals. This study took place in three fifth grade classrooms, and the interviews also took place at each participant’s school. There were clear boundaries for this phenomenon because the study took place in the classroom, during the time that mathematics instruction occurs, and when the topic division with decimals was introduced. Multiple sources of data were collected via observations, interviews (of both teachers and students), and documents (work samples). Triangulation, “a process of using multiple perceptions to clarify meaning, verifying the repeatability of an observation or interpretation,” (Stake, 2005, p.454) provided a comprehensive picture of the phenomenon under study.

A collective case study allows a researcher to look for patterns and themes within each case as well as across multiple cases. For this study a case is one mathematics classroom during the introduction of division with decimals. Generalizations were drawn from the individual cases and the collective case. The collected data was used to provide a detailed description of each
lesson in which division with decimals was introduced. Three schools were chosen to allow sufficient data to look for patterns and themes across the cases.

Throughout the presentation of this study, readers should be able to visualize the experience as it unfolds. It is understood that each participant during the study viewed the events through the filters of their lived experiences, and the researcher’s record of these events is influenced by her lived experiences. However, the goal is to present the events of the classroom and make them as transparent as possible. To do this, the experience will be viewed from several angles: (a) through teacher interviews before and after the initial day of the unit of study, (b) through observation in the classroom, (c) through documents created during the lesson, and (d) through student interviews after the lessons. A multifaceted view of the experience is given using all the data from one site as an individual case.

**Sampling Strategy**

The study took place in three fifth grade classrooms. According to CCSSM (2010), fifth grade is when students are first introduced to division with decimals. Because the inclusion of decimals is where implicit understandings of division break down (Fischbein, Deri, Nello, & Marino, 1985), observing classroom discussion of this topic was critical.

Purposeful sampling was used in order to “build in variety and acknowledge opportunities for intensive study” (Stake, 2005). Three schools were identified that met the following criteria: (a) they made Adequate Yearly Progress (AYP) for the previous two years, (b) they were in reasonable proximity for ease in data collection, and (c) they used different core mathematics texts. To identify teacher participants, the mathematics coach, or if there was no coach in the building the principal, recommended a fifth grade teacher whose students consistently performed well in mathematics (i.e. the participant’s students’ mathematics scores
on standardized tests needed to be equally good or better than other teachers in the building.)

While there are limitations with this process for identifying “effective” teaching, it is not the purpose of this study to develop a definition of effective teachers but merely to identify teachers who are typically successful in moving students forward using standardized measures. Once the teachers were chosen, each teacher selected three students who were neither struggling nor excelling in mathematics. It was hoped that these students would be helpful in determining if the classroom instruction was effective for students who typically do well, but often have to work to develop an understanding. Also each teacher was asked to consider students who would be comfortable articulating their mathematical thinking, so that shyness did not limit the ability to interview the student.

The three schools in this study are Title I schools, due to a high percent of students being from low-income families. According to the United States Department of Education, at least 40% of the student population must come from low-income families in order for the school to be eligible for Title 1 funds (U. S. Department of Education, 2014, June 4). The teachers in this study, Christy, Ashley, and Steve (pseudonyms) are Caucasian. The students interviewed following the study lesson are also Caucasian. Christy’s school is located in a small city in the southeastern United States, about 70 miles from the state capital. Steve’s school also is in the southeastern United States, about 45 miles from the state capital. Ashley’s school is located in a small city in the northeastern United States, about 45 miles from the state capital (see demographics in Figure 1).
Figure 1. Demographics of the three schools

Data Collection

Interviews.

Three teachers were interviewed prior to and following the study lesson using a semi-structured format. Selected students were interviewed following the study lesson. All interviews were audio-taped and transcribed. The teachers were interviewed in their classrooms either during their planning period or after school. Each teacher interview lasted between 15 and 30 minutes. The pre-lesson interviews were held 1-3 days before the study lesson. The post-lesson interviews were held within a week following the study lesson. The student interviews were held in a location in the school thought to be free from distractions. The student interviews were held immediately following the study lesson or the following day (see Appendix for interview questions). During the initial interviews the teachers were asked about their background and the format of a typical lesson. After the lesson was observed each teacher responded to questions regarding the study lesson and future plans regarding division with decimals. The times for all interviews were chosen by the classroom teacher; the student interview times were influenced by the student’s schedule.

Observed lessons and field notes.

The teachers determined when they would be introducing division with decimals based on their district’s curriculum map. The three lessons were observed in mid-November, early
December of 2013, and early March of 2014. Prior to the study lesson, typically on the day of the pre-lesson interview, field notes were taken on the classroom in which the study lesson would be held. The arrangement of the student desks, layout of the classroom, and posters or charts on the walls were recorded during this initial visit. In each case, the first lesson in which the division of decimals was taught lasted only one class period, which varied from 50 minutes to 90 minutes. When the lesson was observed, notes were taken in an unobtrusive manner from a desk in the back of the classroom unless a request was made by the teacher to observe a student conversation or students’ work. The goal of the note-taking was to record as much detail as possible. Areas of focus included the structure of the lesson and what the teacher and students were doing and saying. By the end of the lesson 9-17 pages of field notes were taken in each classroom.

**Artifacts.**

Documents were collected to clarify aspects of each lesson. After Christy’s lesson, the information from the cards placed in the bins at each table was recorded. This provided clarification regarding what problems the students were expected to solve at each table. After Ashley’s lesson, student work was collected as evidence of the strategies students in the class used. Also, these samples showed the work around which class discussions were held. Finally, during Steve’s lesson photographs were taken of problems that were worked on the board. These problems informed discussions surrounding the solution to the problems. They were either written by the teacher with student input or written on the board by selected students.

**Analysis**

The study design included three forms of data: transcriptions from semi-structured interviews, field notes from classroom observations, and lesson artifacts. The analysis of the data began with the transcription of all interviews, pre- and post-lesson teacher interviews and post-
lesson student interviews. Additionally, field notes taken from the classroom and the study lesson were combined in a single transcript, detailing the events of the lesson as thoroughly as possible.

The data within one case, including the lesson transcript and the interview transcripts, were analyzed using initial coding (Saldaña, 2013). This allowed ideas from the pre- or post-lesson interview, the student interviews, and the lesson transcript to be integrated into a case summary. When writing each summary, artifacts from the lesson (problems given to students and student work samples) were used to clarify events in the classroom and to illustrate student comments or conversations.

The summaries, written for each case, were analyzed using initial coding for the cross-case analysis. As expected, this generated numerous categories and subcategories that represent observed phenomenon found in the data (Strauss & Corbin, 1998). Following the initial coding, 23 codes were used. Most commonly used were “placement of decimal” and “student conceptual understanding.” From these categories, constant comparative methods (Lincoln & Guba, 1985) were used to generate more refined categories within the codes. Some codes were either merged with similar codes or eliminated if they were used infrequently or did not collapse into a category. For example, the codes “textbook” and “problem-solving approach” each was only used once across all three case summaries and neither became part of a category, therefore they were eliminated. The code “time” was only used four times, but it was collapsed into “teacher’s role in the classroom” because the pieces of data regarding time referred to the instructional decisions the teacher made to maximize time. After looking at the data associated with the most frequently used codes, and collapsing related categories, three themes emerged: (a) the influence of time on instructional decisions, (b) the influence of instructional approaches on classroom
routines and student discourse, and (c) the influence of instructional approaches on conversations regarding the placement of the decimal point in the quotient.

**Researcher’s role.**

For more than 30 years, the researcher has been teaching mathematics at the elementary, middle and high school levels. Additionally, she served as a mathematics specialist and coach. She conducted workshops and training for in-service mathematics teachers and has spent many hours observing teachers in their classrooms. The researcher drew upon these experiences while observing teacher participants. As a teacher with extensive experience, this researcher implemented curriculum at various levels and developed her own specialized mathematical content knowledge (Ball, Hill, & Bass, 2005). But with this experience, she also brought established beliefs about how mathematics should be taught, what engages students, and what type of mathematics background a teacher should have. As a researcher, her role was to watch the lesson unfold and record what happened, paying attention to the methods used by the teacher to develop the concept of division with decimals and to be as objective as possible. Considering her background, there is no doubt her experiences influenced what she noticed.

**Trustworthiness.**

Four factors are significant when considering the trustworthiness of a study (Bowen, 2005). The first is credibility. Through triangulation of the data (teacher interviews, observations, work samples, and student interviews) the findings are credible because evidence was drawn from several sources. The second, transferability, means that the study presents the findings in a detailed manner allowing other researchers to easily incorporate the findings in their own study. “Thick description” (Stake, 2005, p. 457) is provided as an observer and sections of interview transcripts and student work are included as evidence of patterns or emerging themes. The last
two, dependability and confirmability, both refer to the stability of the conclusions drawn from the data. The conclusions should be stable over time and when viewed by other researchers. Referring back to the research questions throughout this study to guide (a) observations during the lessons, (b) the analysis of the field notes and student work, and (c) the questions asked during the interviews helped to keep the conclusions dependable and confirmable.

**Results**

Each of the three cases are presented in total, including information about the teacher’s (a) teaching experience, (b) math experience as a student, (c) beliefs about mathematics instruction, (d) professional learning, (e) typical lesson, (f) summary of study lesson, (g) reflections on lesson and plans for moving forward, and (h) students’ understanding of division with decimals. After each case is given, a cross case analysis is presented with conclusions and implications. Pseudonyms are used throughout this study.

**Within Case Results – Christy**

Christy began teaching more than 30 years ago. As an educator, she worked in a variety of settings. She started her career teaching kindergarten. From there she worked in special education settings for both elementary and then secondary students. After teaching adult education, she returned to special education in a middle school before returning to elementary education. For the last 7 years, Christy has been teaching mathematics in a departmentalized, general education, fifth-grade classroom. When asked how her special education expertise has translated to instruction in a general education classroom she said, “It has helped me know that the kids just need so much hands on things and that’s so important for them to learn and to internalize all the things that we’re trying to teach them” (Post-lesson interview).
As the daughter of a mathematics teacher, Christy shared how she “really liked math” (Pre-lesson interview) as a young student. She did not question mathematics procedures, “we just were told, ‘ours is not to question why’” (Pre-lesson interview). In middle school, Christy was placed in a class for “really good math students” (Pre-lesson interview) where she first encountered some struggle. “I remember word problems; I remember my mother and I really struggling with the word problems…having a hard time, in algebra. Not knowing how the word problems worked” (Pre-lesson interview). As a young adult, Christy recalled,

I didn’t even know how to do percentages until I started working at a drug store and had to give 10% off and 20% off. And so it just, you know, made sense, you know, when somebody showed me you just move the decimal point and that’s 10% and you can double it and be 20%. So, you know, somebody had to teach me that, when I was in college. (Pre-lesson interview)

Although Christy had no recollection of being taught how to use percentages, she remembered being able to find a percent off with ease after being told the steps to use.

Christy credited her beliefs about mathematics education to a mathematics consultant, Nancy, hired by her district about 7 years ago when the state was transitioning to new mathematics standards. The consultant spent two years at Christy’s school modeling lessons, providing professional learning, and observing lessons. Christy commented on how students were encouraged to explore and to think about mathematics when Nancy modeled a lesson. Nancy “would come in and like have the kids come in and write down the problem…and then she would…ask questions about it and extend the problem, and get them to think, you know, different ways, about what was going on” (Pre-lesson interview). Christy recalled watching Nancy orchestrate a lesson. “I just really liked, you know, how she encouraged the kids to work
together, the way she worked…the room, getting kids to think for themselves” (Pre-lesson interview). It was through Nancy’s coaching that Christy gained the confidence she needed to try some of the strategies that Nancy modeled. After spending time under Nancy’s tutelage, Christy began using problems to begin her own lessons. “[Nancy] would teach us so that we could go and, you know, teach the kids that way, and so, you know, it was successful” (Pre-lesson interview).

Christy stays current by working with district mathematics coaches. Each elementary school in her district has one mathematics coach. Each coach is an expert in the mathematics standards for one grade level (K-5), and provides coaching for the teachers at their school. The fifth grade coach, located at a different school, offers “collaborative planning maybe about three to four times a year” (Pre-lesson interview) for all fifth grade teachers in the district. These planning sessions provide a venue for teachers to discuss grade-level tasks and to share new tasks. David, the coach at Christy’s school provides support in planning lessons and, when Christy wants to learn more about topics, he models lessons in her classroom.

Christy’s school district has not adopted a mathematics textbook recently. Instead, teachers are expected to use the curriculum maps and units developed by the state. However, Christy shared two mathematics resources that her district used in the past, which are important in her teaching. Several years ago, her district used Saxon Math (currently published by Houghton Mifflin Harcourt). “I always tried to be hands on and ah, using a lot of manipulatives…and I really liked the early Saxon Math where they had so many good things for the kids to do, to discover why we do stuff” (Pre-lesson interview). The other textbook Christy uses is Investigations in Number, Data, and Space (the most recent publication is through Pearson). “I didn’t like all of Investigations, but I do like, you know, doing percents and fractions
and decimals” (Pre-lesson interview). The *Investigations* series offers a unit called *What’s that Portion?* an integration of decimals, fractions, and percents. Students learn how the different numbers are related and use this relationship to understand relative size. Strategies and materials found in both resources continue to influence her teaching today.

While Christy discussed the use of hands-on materials to explore mathematics topics, she also spoke about the time that those strategies require and the need she feels for students to move quickly beyond the concrete stage. “[I] wish we had more time for hands on ‘cause I feel like…we’re in a rush to get everything taught. Sometimes I feel like we’re just scooping…and we should be digging a little deeper in a lot of things” (Pre-lesson interview). Also, she commented on how long it takes students to develop a conceptual understanding of the topics being taught. “I think I want them to have it all in one day, but you know they’ve gotta, you know, it’s a gradual thing” (Post-lesson interview). Sometimes she has to make choices regarding what she asks the students to do during class. One of these choices is between the 5-a-Day\(^7\) and Number Talks\(^8\). “[I] just don’t have time, I really don’t. You think I have an hour and a half? ‘Cause I just really think this is so important to get the skills done…we’re [only] able to do mathematics talks this week ‘cause we didn’t do a 5-a-Day” (Pre-lesson interview). Finally, Christy shared a recent lesson when she was able to give students the time they needed to explore tangrams; watching the students work and their expressed enthusiasm was what she considered one of the rewards of teaching.

Ah, what are the rewards? You know, seeing kids get it and understand it and get excited about math. Um, on Tuesday we did tangrams….Just to see how kids put those puzzles

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\(^7\) 5-a-Day are cyclical review worksheets that Christy created to ensure students review mathematics topics frequently throughout the year.

\(^8\) Number Talks is a mathematics routine used to develop fluency with computation. Students use mental math, patterns, and place value, among other strategies, to solve problems.
together, see how frustrated they get with them, they said, ‘I can’t do this!’ ‘It’s too hard!’ ‘These pieces don’t fit,’ you know, ‘can I use more pieces?’ And, and then when they do it they’re just so excited about it, so you know, with a math problem they’re really getting excited about. (Pre-lesson interview)

It was a lesson where students could have the time to use hands-on materials uninterrupted until they were able to work through their problem that held the most reward for Christy.

**Christy’s teaching.**

The events and quotations in this section came from the lesson observation notes, unless noted otherwise. Christy believes students should learn mathematics through “exploration, practice, and refinement of skills” (Pre-lesson interview). These three key aspects are reflected in her typical lesson plan. Christy starts her lesson with a word problem. She chooses problems worthy of discussion, problems that “make kids think about real life experiences . . . [and are] more than one step” (Pre-lesson interview). Students work on this problem individually first, which Christy calls, “struggle time.” Then students discuss their work with a partner and come to a consensus regarding the correct answer. Finally, Christy initiates a class discussion about the problem, typically referring to work students wrote on the white board.

I want the kids to come in and if we have a problem on the board, ah, they’re to work them [sic] out by themselves first and then they’re talking as a group and, ah, I do a lot of letting them compare answers, and finding out why their error – answers were different, and then I encourage the students to argue their point, ah, why they think they’re right, why they think the other person is wrong. (Pre-lesson interview)

After discussing the opening problem, Christy provides brief directions and some questions to ponder while working. After the “mini lesson,” students engage in an extended task or project,
which she considers the “practice” part of her lesson. Throughout the lesson, Christy sees her role as “a facilitator, and to ask them [students] you know, to think about things and um, to make sure they’re doing what they’re supposed to do, staying on task…I’m a task master” (Pre-lesson interview). The last 15 to 20 minutes of the mathematics class is spent completing “5-a-Day,” cyclic review sheets. She created these because they were a big part of curriculum materials from *Saxon Math* which she used earlier in her career. Often, Christy closes the lesson by asking students to put the solution to a problem on the white board for discussion. This is when Christy tries to solidify the lesson for the day. “You know, the kids put up the problems and stuff and we also have our closing and try to, you know, bring together what we learned” (Pre-lesson interview).

**The division with decimals lesson – problem-solving phase.**

During the observed lesson, a tune from *Fraggle Rock* was playing when the students enter the room. Once students were settled they opened their mathematics journals and worked on the problem posted on the board. “Mr. Ellis runs 1.6 miles around the block. He sees a telephone pole every 0.2 miles. How many telephone poles does he see?”

Christy observed students as they worked and moved from student to student commenting on student work and asking questions. She stopped near one student and suggested, “You may need to draw a picture for this problem; mathematicians see pictures in their heads.” She continued to observe other students’ work and provide feedback, “Think about how you would solve it. Can you draw a picture that proves that? I want you to draw [a picture] that proves it. I want to see a picture that proves your answer.” When Christy was standing near a group of students, one student explained his thinking about what operation to use. “I think this is going to be division because you said next week we will be working on division.” Without
responding, Christy gave students permission to discuss the problem, “Once you have something on your paper you can talk about your answers.”

Throughout the lesson Christy continued to move around the room commenting and asking questions in rapid-fire succession. For clarification she asked a student who had written 3.2 on her paper, “Can you tell me what two tenths of a telephone pole looks like? Mr. Ellis is running and sees two tenths of a telephone pole? What does three and two tenths mean? Can he see two tenths of a telephone pole?” When another student said she does not know what a telephone pole is, Christy brought her out of the classroom to show her one on the street.

As students worked, Christy selected students to share their solution on the board. When she stopped at one student’s desk she asked, “You think it’s going to be eight telephone poles? Go write it on the board.” The student drew 1.6 using base ten blocks. She then circled every two-tenths rod and got a total of eight circled groups. She explained, “I did one whole and six tenths, then I circled two of each because of two tenths, and every circle is a telephone pole.” Christy quickly reiterated what the student said, but erroneously identified each circle as 2 telephone poles instead of the distance of 2 tenths, the interval of each telephone pole. Neither the student nor Christy corrected the error.

A second student used a bar model to solve the problem (see Figure 2). First, he wrote what he called an answer statement, “He sees _____ telephone poles.” Then he drew a bar model to illustrate the problem. While everything he wrote made sense, his answer was 3.2. Christy probed, “How did you get 3.2?” The student replied, “I multiplied 0.2 by 1.6 tenths.” Christy responded, “Is your decimal in the right place?” seemingly more concerned about the placement of the decimal than about not recognizing the problem required division or finding a missing factor. Once the student recognized that the product would be 0.32, Christy asked, “Have you
ever seen 32 hundredths of a telephone pole?” focusing on the incorrect answer to the question, rather than the operation required to find the answer.

![Bar model for the problem 1.6 ÷ 0.2=](image)

Figure 2. Bar model for the problem 1.6 ÷ 0.2=.

The last student who shared used the standard algorithm, He said, “0.2 divided by 1.6” reversing the division problem, but dividing correctly. “[I] moved the decimal one place for both, you have to make the decimal a whole number.”

**The division with decimals lesson – student exploration phase.**

Christy used the last student’s strategy as an introduction to the next part of the lesson. She asked, “How did you do that?” He repeated, “I moved the decimal in both the divisor and dividend.”

Then she continued, “Today we’re going to talk about this and find out why that is a strategy you can use.” The students took out a sheet of paper; they seemed to know the routine and what to expect next. Christy asked the students to think about “what happens to the decimal, how does the decimal point move?” Then she wrote on the board, “What happens to the decimal when I divide?”

Christy set up this part of the lesson to address many parts of the CCSS standard, 5.NBT.7[^1]. She had students using concrete models (base-ten blocks, play money, and clear tenths squares). As they worked, they were asked to talk about what they observed and later explain the reasoning they used. There was a bin on each of six tables; each bin held one of three different strategies.

[^1]: Add, subtract, multiply, and divide decimals to hundredths, using concrete models or drawings and strategies based on place value, properties of operations, and/or the relationship between addition and subtraction; relate the strategy to a written method and explain the reasoning used (Common Core State Standards Writing Team, 2012).
tasks. Christy planned to have the student groups rotate periodically so that they would have an opportunity to complete each of the tasks. Christy briefly explained the contents of each bin. One task required students to use teacher-made clear squares with a different number of tenths shaded on each. In a previous lesson, students were asked to use the materials in the bin to show a product of two decimals. For example, 56 hundredths is found by overlapping two clear squares with 8 tenths shaded on one square and 7 tenths shaded on the other. The shaded tenths are placed perpendicularly over one other, so the overlapping color denotes the product (see Figure 3). For this lesson, students were asked to think about how to use the manipulatives to find the quotient in problems such as 0.56 ÷ 0.7. In another bin there was play money. Christy asked the students to think about “what happens when you divide into a dollar; how many dimes do you get when you divide ten cents into a dollar?” Finally, she showed them a bin of base-ten blocks. For those, she asked the students to divide by a whole number using the base ten blocks. For example, “3.68 hundredths divided into 3 groups.” She finished by reminding students about expectations. “I’m looking for teams working together and thinking about what happens to the decimal.” She then reiterated the directions, “With the money group, how many quarters are in a dollar? So how many quarters would be in four dollars? With the base ten blocks, when you divide by a whole number, what happened to the decimal? See if it stays the same.”
exception of the money bin, the students were instructed to think about the placement of the
decimal in each of the problems. When explaining each bin, Christy asked students to use
“traditional algorithm” thinking (e.g. how many times does ___ go into ___?). The problems
posed in each bin are given in Figure 4.

The students began working and Christy visited the group with the clear tenths squares.
The students told her, “56 hundredths divided by 7 tenths is 8 tenths,” then they modeled this by
putting a square with seven-tenths shaded and a square with the eight-tenths shaded on top of
each other so that the shaded areas overlapped to create 56 hundredths. Christy focused on the
size of the quotient when dividing. “You have to say where the decimal is, do you get a bigger
number than you divided by?”

The students replied, “Lesser number.”

Christy asked, “So how do you make 64 hundredths, eight-tenths times? What makes 64
hundredths?

The students replied, “eight-tenths”

As Christy moved between the tables, she checked in on students, asking them to explain
or clarify their thinking. Typically, questions ended with an examination of the location of the
decimal point. For example, at the money table Christy asked, “When you divide hundredths by
hundredths what happens to the decimal?” At the table with clear unit squares, Christy asked,
Bin 1: The base 10 bin contained the following problems.

3.68 ÷ 2 =
6.25 ÷ 5 =
6 ÷ 5 =
0.12 ÷ 8 =
12 ÷ 0.3 =

Bin 2: The “How Much Money?” bin contained the following questions.

How many quarters are in $5.50? Dimes? Pennies? Nickels?
How many quarters are in $7.50? Dimes? Pennies? Nickels?
How many quarters are in $25.50? Dimes? Pennies? Nickels?

Bin 3: The clear unit squares bin contained the following overarching question, “How does the decimal place change when you divide a hundredth by a tenth?” Also, it contained the following problems.

0.56 ÷ 0.7 =
0.64 ÷ 0.8 =
0.27 ÷ 0.3 =
0.32 ÷ 0.4 =
1.36 ÷ 0.4 =
5.35 ÷ 0.5 =

Figure 4. Problem sets contained in table bins.

“If I divide a hundredth by a tenth I get tenths?” After the students rotated to a new group, Christy visited the students at the money table. They were finding out how many quarters are in $5.50. She guided them by asking, “How many quarters are in one dollar?”

The students responded, “There’re four.”

“How many quarters in 5 dollars?” and then, “How many in 50 cents?” Finally, “So how many quarters are in 5.50? Is that a whole number or is that a decimal?” After the students responded, she continued, “What happens when you divide by dimes?” She counted out the answer, “30, 40, 50, so 55 dimes. Is there a decimal in the answer?” Again, the focus was on the location of the decimal point in the answer.

At the table with base-ten blocks, students were working on sharing 3.68 into two groups. The students traded one whole for ten tenths in order to share the amount equally. Christy asked,
“So what’s our answer?” The students responded that the answer is 1.84. Christy continued, “Where did the decimal move? What happened to the decimal? You get a smaller number but what happens to the decimal? So did the decimal move?” The students recognized that the decimal did not move. As Christy left the group she added, “See if it happens on the next one.” The next group Christy visited was also working on sharing 3.68 into two equal groups (see Figure 5). Christy asked, “What does 3 and 68 hundredths look like?” Once the students showed her 3.68 with the base-ten blocks she asked, “How many groups is it divided into?” The students replied, “Two.” “So could you divide it into two groups? Do you have two equal groups?” The students recognized that they had to make a trade, “We have to swap this for tenths.” Christy agreed, “Right, so swap that for tenths. Share the tenths you have and the ones. So what’s my answer?” The students began to add some blocks to the total, but Christy corrected, “We can’t add anything.” Once the students equally shared the blocks, Christy asked, “So what’s our answer? What’s our whole number? And what’s the…” The students replied, “One and eighty-four hundredths”
After a short conversation Christy asked, “What do we notice about the decimal in our dividend? What do we notice about the decimal in the quotient? Did it move, stay the same?” The students responded that the decimal did not move. “So when you divide a whole number by a decimal what happens to the decimal?”

Christy continued asking questions to the students at each group as she visited tables. “When you multiply tenths by tenths you get hundredths, so what happens when you divide hundredths by tenths? When you divide a hundredth by a hundredth, what happens to your decimal?” As she was looking at student work, she asked some students to put their solutions on the board. One student wrote, “When you divide a decimal by a whole number, the decimal stays the same” Christy stopped and asked all of the students to write that statement down. Another student wrote on the board the solution to the problem from the clear squares bin, $0.56 \div 0.7 = 0.8$. He used the standard algorithm and kept the decimal point throughout his work. Then he added, “When you divide hundredths by a tenth you get a tenth.” After two rotations, Christy asked students to clean up their tables, leaving approximately 20 minutes to close the lesson.

*The division with decimals lesson – lesson close.*

To close the lesson Christy discussed the opening problem with the telephone poles. “What about our opening problem? When we divided a tenth by a tenth we got a whole. Now let’s go back to Edgar telling us about moving the decimal [$1.6 \div 0.2 =$]. Edgar, what did you do to move the decimal?”

Edgar replied, “You have to make it a whole number.”

Christy questioned, “Do I have to change it to a whole number? How do I do that?”

Edgar answered, “Move it to the right.”

Christy wondered, “But what gave him permission to do that?”
A student responded, “He could do that because you can’t have a part of a telephone pole”

Christy continued, “Okay. What gives us permission to do this?”

Christy moved to the money example, 5.50 ÷ 0.25=. “How many quarters was that?”

Students replied, “22.”

Christy said, “So our decimal is gone.”

One student said, “Oh the zero!”

Christy clarified, “You can take away the zero, you have to move the decimal two here and two here [points to the digits to the right of the decimal in the divisor and dividend]…What strategy are we using? What are we multiplying by? Or how are you doing it? What gives us permission?”

One student suggested, “It’s like multiplying, we move it two times, multiplying it by how many places you have.”

Christy wondered,

Multiplying it by how many places you have…25 hundredths times what? What do I need to multiply by to move that decimal over two places? We practiced this so many times. Ten to the second power means you’ll have two zeros behind it and we’ll have to multiply by 100.

Students cheered, “Yes!”

Christy continued, “What did I multiply by here?”

Another student replied, “A ten.”

“What would I multiply 0.125 by to get a whole number?” Christy asked.

One student said, “Times 1000.”
Christy summarized, “I wanted you to find out today why we could do that. We’ll talk more about that tomorrow.” At this point, the students began their work on a “5-a-Day” worksheet, where they had five mixed review problems they completed daily.

**Christy’s reflections on the lesson.**

The first topic that Christy raised the day after the lesson, during the post-lesson interview, was time. She said she thought the students were beginning to identify patterns regarding location of the decimal in the quotient, but there was not enough time to fully develop this idea.

I think some of them were beginning to see and I really wanted to see, you know, what the pattern is and how it was changing….You know from why, how we multiply by or multiply by ten or by a hundred so I don’t think they were quite ready to get that . . . you know maybe I just went too fast, trying to get all those things in.

When asked what she would do differently the next time she teaches this lesson she said she should have given the students more time to explore. “I just feel like I rushed them [the students] just trying to get them to find out what was going on first instead of letting them discover it so much.” Again indicating the students needed more time to identify and describe patterns they noticed when dividing related decimal problems.

While Christy often shared her belief that students need time with manipulatives, during the post-lesson interview she described the lesson she taught the day following the study lesson. She repeated bin #2 “How Much Money?” but did not let the students use the play money. While Christy commented on the value of letting students explore using manipulatives, she explained she found classroom management more difficult with the manipulatives.
Those coins were a little problem, but you know most of them, I mean, all of them have been doing money since, you know, first grade. They pretty much, you know, know how many quarters are in a dollar and things like that. So we did talk about money, you know, one dollar, what would two dollars be, what would three dollars be [e.g. 4 quarters in one dollar, 8 quarters in two dollars, 12 quarters in three dollars, etc.].

She made the focus of the lesson observing how the decimal point moved, which she said was the next step in understanding division with decimals.

**The students’ understanding of division with decimals.**

Two students, Doug and Sue, shared their understanding of division with decimals during the interviews that followed the study lesson. Both students relied on memorization of rules to find the correct placement of the decimal in the quotient. They had varying degrees of success in the application of these rules. Neither of the students was able to explain how following these rules resulted in accurate quotients.

Doug began by explaining how he used missing factor problems to solve division problems. “If you are times-ing, or if you’re dividing a hundredths by a tenths you get, you, your answer will be a tenth… I times my answer to my dividend and it gives me my divisor” (Doug interview). It was unclear whether Doug misspoke, reversing dividend and divisor, or if he was still unclear about the inverse relationship.

During the lesson, Doug copied down several pieces of information. He read his notes out loud. “It says, ‘one hundredth divided by another hundredth equals one whole’” (Doug interview). When asked why that was true, he began reviewing what he remembered about the relationship between dividend, divisor, and quotient with different types of decimal numbers.
Doug …if you have a hundredth by a tenth you get a tenth.

Interviewer And why is that?

Doug Because you get a, a big number divided by a littler number gives you the little number.

Interviewer Can you give me an example of that?

Doug Like, let’s say zero and thirty-two divided by… zero and four tenths… equals what? And what you would do is you would get your hundredth with your tenth and you would, you would work it out and you would get a tenth as your answer.

Interviewer And how would you work it out?

Doug Um you would, you would …this is how I remember it. A hundredth divided by a tenth will always give you a tenth, and, and like if you, if you know your multiplication facts really well then if you get your answer, two tenths and time one, times them with each other and if you don’t get the, whole number, the hundredth number then, you probably have it wrong unless you have a remainder. (Doug interview)

Doug was able to reiterate the rules he had written down during the lesson, and used the inverse relationship of multiplication and division, however he was not able to explain how he knew these relationships to be true. Additionally, he used rote memorization when dividing one hundredth by one tenth, “This is how I remember it. A hundredth divided by a tenth will always give you a tenth” Doug described this as “a big number divided by a littler number gives you the little number” (Doug interview).
When discussing the placement of the decimal point in the quotient, Doug showed how to get the correct answer but was unable to explain how he knew the procedure worked (see Figure 6).

![Figure 6](image)

*Figure 6. Arrows indicate the movement of the decimal point before dividing 5.6 by 3.8.*

Doug: See if we had 3 and 8 tenths and 5 and 6 tenths…

Interviewer: Um, um

Doug: And you have an 8 and it’s asking you to move one place, so you just move it, and then you don’t have a decimal and then you move this in there (moving the decimal in the dividend)

Interviewer: So, how do you know to do that?

Doug: Well, it’s like, because it tells you, because you have, like if you read the problem, and there’s like a um, there’s like, ‘cause you really can’t divide the decimal, so you put your decimal on the end…

Interviewer: Uh, huh

Doug: And then you like work it out and it just really helps me because, [our teacher] walks us through step by step and so… (Doug interview)

Doug was able to follow the steps of moving the decimals in the divisor and dividend, but was unable to explain why that is a viable method for finding the quotient.

Like Doug, Sue was able to describe the rules she learned during the lesson.

Interviewer: Okay, the first thing I want you to tell me is, what do you know about division with decimals?
Sue: That, when you divide that, if it’s like a whole number

Interviewer: Um, hum

Sue: And divide it with a number that has a decimal

Interviewer: Um, hum

Sue: That, that you just gotta put the decimal, in the same place with the, the, the one you were dividing with

Interviewer: Um, hum

Sue: And if you’re dividing with a decimal, a decimal, you just gotta make the decimal a whole number and just divide and put your decimal on the other side, on the end

Interviewer: Um, hum

Sue: And, and that’s how you can get your answer. (Sue interview)

Here, Sue explained that when you divide a decimal by a whole number the decimal stays “in the same place” when placed in the quotient, and when a decimal is divided by a decimal, the divisor needs to be made “a whole number” and the decimal in the dividend must move to “the end.” For example, Sue used these memorized rules to divide a decimal number by a whole number with the problem 3.68 ÷ 2 (see Figure 7). First, she read the problem incorrectly (2 ÷ 3.68), then she erroneously moved the decimal in the dividend, but she placed the decimal correctly in the quotient.

Figure 7. Arrows indicate how Sue moved the decimal point before solving 3.68 ÷ 2.
Sue: Umm, I did this, “Two divided by three and, three and 68 hundredths…”

and I just moved my decimal two places.

Interviewer: Um, hmm…

Sue: And I got one and 84 hundredths.

Interviewer: But you moved your decimal point over.

Sue: Oh, it’s over here, right?

Interviewer: No, it’s in the right place, how did that work?

Sue: Oh because when you multiply by, I mean divide by a whole number, the decimal doesn’t move.

Interviewer: Ah, so why did you draw these arrows underneath?

Sue: Because you want to make it a whole number not, just leave it like that.

Interviewer: But your decimal point goes in the same place?

Sue: Um, hmm. (Sue interview)

Sue’s explanations showed that she has memorized some rules, but forgot the correct use of the rules learned. When asked why the decimal did not move when dividing by a whole number, she replied, “Mmm, I think it works because when you, cause, when you divide it by with a whole number, you can’t, like, I can’t say it…I got it in my mind I just can’t” (Sue interview). She recognized that she still had some difficulty recognizing where the decimal belonged in the quotient.

Within Case Results – Ashley

Ashley remembered mathematics as procedures to memorize when she was a child. She recalled the point in her life when she realized she had memorized procedures but had little conceptual understanding of mathematics.
When I was growing up and learning, I really did learn a lot of tricks and I was really good at getting down those procedures. I was successful. But then when it came time to like really use the math, when I was wanting to be a teacher and I was tutoring um, some friends of mine and I was like, oh, geez I really have no idea why I’m doing this. (Pre-lesson interview)

Like many teachers in the classroom today, when Ashley was a student, she had no experience of a student-centered classroom. Early in her teaching career she pursued opportunities to learn more about teaching mathematics. She participated in courses offered through her district and the state department of education. Currently she is participating in the Poincaré Institute through Tufts University.

To gain a better understanding of her state standards Ashley mentioned two particularly influential courses. The first was a semester-long course that encompassed many mathematical topics. Ashley learned strategies new to her that she could develop with her students. She recalled learning about partial products, “I was like, ‘Oh!’ that works…and it made me think about how I can maybe not have to use my algorithm to come up with an answer” (Post-lesson interview). The second was offered through her district where Ashley added to her knowledge of alternative strategies for fifth grade mathematics. This course focused on teachers’ “conceptual understanding, computational fluency, and problem solving” (U.S. Department of Education OESE/Mathematics and Science Partnerships, 2010, p. 26) and was designed to “solicit the participants’ solutions and then discuss the pedagogical considerations that these answers entail” (U.S. Department of Education OESE/Mathematics and Science Partnerships, 2010, p. 56). She found these courses caused her to critically examine her instruction. The courses “really focused
on the conceptual ideas and why you were doing things, and so I think that kinda changed my ideas” (Post-lesson interview).

Her district adopted the mathematics textbook, *Investigations in Number, Data, and Space* a reform-oriented textbook series published by Pearson. Ashley credited the adoption of the textbook with further influencing her pedagogy in mathematics. “Even just teaching the *Investigations* program, a lot of that was …making sense of new ideas that I hadn’t thought about before…so it was like, eye-opening to see new methods” (Pre-lesson interview). She recalled how much of the instruction was based on asking students to share their work and to explain how they thought about a problem. Also, students were asked to make sense of ideas presented, all of which were new to Ashley. Until recently Ashley used the standard algorithm exclusively to solve multiplication problems. Now she also uses mental mathematics strategies. When multiplying by 5 she multiplies by 10 first and cuts the product in half. Or she uses the distributive property and breaks one factor into tens and ones. She developed enough strategies to be able to choose the best strategy for a given problem.

Finally, Ashley was taking a series of three courses through the Poincaré Institute at Tufts University. During this program she realized that she needed to focus on facilitating and cultivating her students’ ideas. “So now I really try to let the kids come up with their ideas…because a lot of time you get the ‘Uh, Oh! Okay,’…you can tell that they are seeing something new or something they didn’t see” (Post-lesson interview). Ashley found the institute particularly informed her understanding of operations with decimals. For example, earlier in her career she indicated that she would say, “oh yeah, I just take them [the decimals] out and put them back in or. . .however many times you slide the divisor, you slide the dividend, too” (Post-lesson interview). Now she spends time exploring decimal numbers and operating with the
decimals, “really showing how we’re multiplying those pieces” (Pre-lesson interview) rather than just ignoring them and placing them after performing the operation.

Ashley discussed how her teaching practice has changed since she began teaching. She sees herself as a facilitator now, but earlier in her career she was more teacher-centered.

I would say . . . when I first was teaching I really did a lot more telling than facilitating, and so now I really try to let the kids come up with their ideas, I really let, like to have them share their work, and they like to share their work. (Post-lesson interview)

Ashley wants students to experiment, build on previous knowledge, and make connections. In order to achieve this, Ashley builds on students’ skills, rather than requiring mastery of just one skill or procedure. “I like the kids to try to come up with ideas and think about how that would work and then maybe if they saw a strategy, try to like apply it to what they are doing” (Pre-lesson interview). She prefers students come to an understanding of topics in mathematics on their own, through discourse and experience. “‘Cause if you just tell somebody something I don’t think they get it as much, some people are successful that way but not all kids” (Pre-lesson interview).

In her teaching Ashley uses questioning, facilitating a lesson rather than leading a lesson. “I try to ask questions and I don’t want to give answers, so I’m trying to ask questions and see if they can come up with ideas” (Pre-lesson interview). At times Ashley struggles with this. Rather than tell a student how to do something, or that their strategy is correct, she wants students “to try to think and try it out” (Pre-lesson interview). She wants students to engage in discourse, think about and make sense of other students’ ideas, and ask themselves if it would be helpful to add others’ ideas or strategies to those they already use. To encourage discussion, Ashley posted anchor charts in front of the area in which the class meets to discuss student work. One chart
reminds students of questions they can ask to help others clarify how they found their answer, “How did you get that answer? Do you have an example? Could you please clarify that for me? Why?” (Classroom Observation Notes). Below it was a smaller poster that contained prompts to help students remember how to add to a conversation, “I’d like to add on to your answer…, I disagree with your answer because…” (Classroom Observation Notes). Both of these support student conversations and mathematics discourse.

Using these strategies often caused an internal struggle for Ashley due to time constraints.

Well there are things I’d like to spend a lot more time on but maybe you don’t just, you don’t get to because there’s a, there is a timeline and you do have to make sure you expose kids to the standards, so in that respect I think it makes me have, makes me, maybe not spend as much time as I’d like to for everybody…and you’re always kind of fighting, not wanting to just tell them how to do it. (Post-lesson interview)

She recognized that she does not have a lot of time for mathematics instruction, but she used her schedule flexibly to ensure she has enough time for a task at hand. “The math block is so tiny so sometimes I do end up splitting like one lesson into two” or “[I] take writer’s workshop time to kind of wrap that up [which is] what we did today, ‘cause we ran out of time” (Pre-lesson interview).

**Ashley’s teaching.**

During the pre-lesson interview, Ashley described the format of a typical lesson in her class and her plans for the study lesson. (The events and quotations in this section came from the pre-lesson interview, unless otherwise noted.) Her lessons follow a three-part structure: engage, explore, and explain (Bybee, 1977). First, she introduces the topic with a question. She uses the
question to engage and to gauge student knowledge and understanding of the topic. The class discusses strategies they have that may help them with the task. The next part of the lesson is work time for the students. They either work on a problem or practice a skill. The third part of the lesson lasts about 20 minutes. During this time students share their work and the class discusses the strategies used. “At the end of my math class… [I] usually pick one or two problems at the end of the practice work that we can talk about, and I don’t always get there but when I do, we have good conversations.”

Ashley also described her use of questioning during the lesson as a way to facilitate rather than lead the lesson. “I try to ask questions and I don’t want to give answers, so I’m trying to ask questions and see if they can come up with ideas…” She said at times she struggles with not giving answers. Rather than tell a student how to do something, or that their strategy is correct, she wants students “to try to think and try it out.” She wants students to engage in discourse, think about and make sense of other students’ ideas, asking themselves if it would be helpful to add others’ ideas or strategies to those they already use.

When reflecting on previous years, Ashley shared that her students could manipulate algorithms but did not have an understanding of the operation of division with decimals.

In past years I’ve really had kids who know strategies like, steps and procedures, like algorithms, but they don’t know why they’re doing them. . . . I’m hoping that they’ll see some, some things like just that the fact, that the answer’s gonna be larger than the dividend, ‘cause I’ve been making it my point, since a lot of kids when we started division [said,] ‘Well the bigger number goes in, in the inside,’ and I, I’ve been making a point to say it doesn’t just go in the inside, that’s not true.
To avoid the common misunderstanding that the quotient is always smaller than the dividend, as is the typical case with whole number division problems (Bell, Swan and Taylor, 1981), Ashley wanted to make students aware of this misconception. Because of this, she hoped that the students would not be surprised by the size of the quotient when dividing by a decimal smaller than one.

When discussing where students might have difficulty, Ashley considered the value of productive struggle. “Maybe if they got an answer that was more than…12 they’re gonna think, wait a minute what happened? So I think they’ll struggle in that way which is a good thing.” After the lesson she returned this idea of productive struggle.

I think at the end they were all kind of questioning, well, what do you do with a decimal point? And that, I thought was really powerful because now they’re gonna be interested in thinking about how that really happens for the next step. (Post-lesson interview)

Also, Ashley knew that students are reflective about the reasonableness of their answers, “I’ve noticed over the past couple of months that some, most of the time when these kids get an answer, they really do think, ‘Wait a minute does this make sense?’ So I will be curious to see what happens when they do that.”

The division with decimals lesson – engage.

Ashley began the students’ first experience with decimal division by telling the students to “Chat with some people next to you. What comes to mind when we say estimate? What do you guys think? What comes to mind?” (Unless noted otherwise, all quotations from Ashley’s lesson came from the Lesson Observation Notes). Examples of student responses were:

Next biggest or lowest number,

Might use rounding,
Making numbers easier,

Making a reasonable guess,

Compatible numbers,

To be sure you have enough,

Zeros.

Ashley commented, ‘Why that’s an interesting one.’

Student replied, ‘Because when I round I think of zeros.’

Ashley gave brief directions regarding the problem the students would be given.

We have estimated when we add and subtract numbers. Using both reasonable and compatible numbers, try to find a reasonable estimate [for the problem they will solve during the lesson], think about the strategy you’re going to use so you can present it. Try to draw a model, we’re going to share later. Work with your partner. Show a clear, concise solution. If you finish early, you can try to find an exact answer, and then try some challenge problems on the back.

After stating the expectations, Ashley gave students a worksheet with the following directions.

Work together: Answer the problem below. Discuss your strategies with your partner and be sure to clearly show your solution. A 12.5 meter piece of construction material needs to be cut into pieces 0.65 meter long. About how many pieces can be made? (Student Sheet)

Ashley chose this task because the problem was similar to those used when introducing operations with whole numbers. “I’m not sure what they’re gonna do when they see the decimals, so, but I am interested to see” (Pre-lesson interview).
The division with decimals lesson – explore.

Students were put in small groups of two or three. She used groups that were heterogeneous, but the students’ abilities were close enough that the students could work together successfully. Ashley asked the students to estimate first because she wanted the students to consider the relative size of the quotient before trying to find the exact answer. Ashley explained,

Where we’ve been dividing before…their quotients have been smaller, and so in this case, because they are dividing by something less than one, their answer’s gonna be bigger and I wanted to try to see what their thoughts were of that. (Pre-lesson interview)

She also wanted the students to be able to concentrate on the concept of division as equal parts rather than multi-step procedures that might be required to find the exact answer. "I wanted to try to see what their thoughts were…think about just the act of dividing and what we’re really doing as opposed to the numbers and any procedural stuff” (Pre-lesson interview).

The lesson was grounded on students’ understanding of division of whole numbers. Ashley wanted students “to understand that the dividend is what is being broken up” (Pre-lesson interview) and to recognize if an answer was reasonable. Ashley started the work time by checking in with each group and responding to identified needs. For example, to one group she commented, “Did you get a chance to read the problem, too? I want you to work together.” She stopped at another group that was using repeated subtraction as a strategy to find the exact answer to the problem. “Not the most efficient, but it works,” one group member commented to Ashley. Then to his partner the student suggested, “How about if we switch off, I’ll do one, you can do one,” indicating the repeated subtraction. They subtracted 65 hundredths 19 times to get the answer.
One pair that Ashley stopped to observe was using rounding to find the estimated answer. A student in the group wanted to round 12.5 to 12.1 because, “the 5 rounds to 10.” Ashley asked the other student, who was able to round 12.5 correctly, to explain using base-ten grid paper, why 12.5 rounded to 13. Once the partner explained how 12.5 rounds to 13, the student said, “Ahh, yeah.”

Another pair discussed how their understanding of the area model for multiplication and division could represent this problem. The partners explained to Ashley that the total length of the wood equals “0.65 times whatever number we get. So, whatever number [is] on the side we get at the end.” Thus explaining the relationship between the answer to the division problem and one of the dimensions of an area model representation of the problem.

Partners at one table ignored the decimals in the problem 12.50 divided by 0.65 by “lining up the decimals.” They divided 1250 by 65 (see Figure 8). One student explained that by lining up the decimals, they were making sure there were two places after the decimal for both the dividend and the divisor and then divided as if using two whole numbers. They got the answer 19 r15 which meant they could make 19 pieces of construction material.

Figure 8. Students lined up the numbers (bottom left) and divided without a decimal point.
A different pair used partial products to divide and kept the decimals throughout their work (see Figure 9). They showed the first partial product of \(0.65 \times 10 = 6.5\) and subtracted 6.5 from 12.5 to get 6. They said they could not subtract 6.5 again so they did \(9 \times 0.65\) and got 5.85. They subtracted from 6 and got 0.15.

![Figure 9. Students’ work keeping the decimal point throughout the calculations.](image)

**The division with decimals lesson – explain.**

After 20 minutes of work time, the teacher collected the student work and invited the students to sit on the carpet at the front of the room to discuss their work. “Thank you for working pretty well together, yes?” Ashley then asked if the students felt confused about the problem, two hands went up. To those students she suggested, “Look at other people’s work and try to make sense of other people’s work. You need to be a teacher, and if you see something you don’t understand please ask.” Ashley spent a minute to skim through the work again and then commented to the class, “Lots of groups went right from the estimating to the actual answers. I didn’t think you’d be able to find the exact answers, so I think it’s great that many of you tried to do that.”
To determine what was understood about division with decimals, Ashley had the students meet at the front of the room near the document camera. As planned, she showed several students’ work. When their work was displayed, the students explained how they found their answer. Ashley wanted her students to “share out their strategies and see what the other kids have to say about [them]…about what we observed…and figure out where we are gonna go from there depending on what they know” (Pre-lesson interview).

The first group that shared their estimating strategy rounded both the dividend and the divisor to the ones place. The division problem they used to estimate was $13 \div 1 = 13$. This strategy was used by several groups and there were no questions or comments regarding the strategy. The next pair of students chose to round the divisor to 0.5 instead of a whole number (see Figure 10).

![Figure 10](image)

*Figure 10. Students rounded 0.65 to 0.5 in order to estimate the quotient.*

Once they found that $10 \times 0.5 = 5$, they recognized that $14 \times 0.5 = 7$, because 5 is half of 10 and 7 is half of 14. They said, “Our estimate was 25. We, 0.65 we rounded to 0.5. Then we divided
12.5 by 0.5.” The step where the pair found $14 \times 0.5 = 7$ confused several students. To explain it, one of the students said,

Student 1  10 divided by 2 is like multiplying by 0.5; 0.5 is half of a whole number, so…

Class  Ooh!

Teacher  Why are you saying, ‘Ooh?’

Student 2  Because 7 is half of 14; 5 is half of 10.

This pair of students had made sense of multiplying by one-half, recognizing it as the same as dividing by two and were able to explain to other students how they incorporated that understanding to solve the problem.

Ashley asked a pair of students, who used partial quotients to solve their problem, to share their work. When considering the placement of the decimal, they had the correct answer to the problem, 19 pieces, but they had difficulty making sense of the remainder. “We did 12.5 and we subtracted 0.65 and we put a 1 next to it because it represents one piece. We kept repeating that and counting up until we got our answer.” When asked what answer they got they responded, “We meant to put down 19 remainder 0.15, we were left with nothing so we made a subtraction error.”

Teacher asked, “How do you know where you put the decimal back in?”

He replied, “If you were doing repeated subtraction, you could just keep the decimal in your answer.”

Teacher clarified, “Are you thinking you could keep the decimal in the same spot? Is that what you’re thinking?”

Student replied, “Write the number where the decimal is and then keep it in the line.”
So when this group finished dividing 12.5 by 0.65, they knew they had 0.15 remaining (if the divisor is 0.65 then 0.15 is the correct remainder; if the divisor is 65 then a remainder of 15 is correct).

Another group, who used a similar strategy, recorded their answer differently. Instead of a remainder of 0.15, they assumed the decimal should be placed between the answer and the remainder, (i.e. 19.15). Because their remainder was in the form of a decimal, they did not realize that it was still a remainder. Ashley commented on this group’s work during the second lesson interview (see Figure 11).

Oh, yeah because they did…four times 65 hundredths. And got two, ah, two and 60 hundredths; so that…seems like an, an effective strategy, I’m curious as to how they got this. Fifteen hundredths, though, why did they put point 15, you know? (Post-lesson interview)

The correct answer is 19 remainder 0.15/0.65 or 19.23 (rounded to the hundredths place) if they had continued to divide.
The last group to share was Michael (one of the students interviewed as part of this study) and his partner. Ashley had discussed their work with the two boys during the “explore” part of the lesson. During the closing they thought about the decimal point and its location in the answer. Ashley placed their work under the document camera and asked if they wanted to talk about what they did (see Figure 8).

Partner 1  Twelve point five and 0.65, so the numbers have the same number of places [to the right of the decimal point] so we took away the decimal points.

Teacher  Comments?

Student  It’s a good strategy because instead of decimal points, you can divide whole numbers and put the decimal point in the answer.

Teacher  I didn’t understand why you could…

Partner 1  When you take away the decimal point it is the same thing.

Student  Is that true?

Partner 1  Yeah, because I do it all the time.

Teacher  How do you know where you put the decimal back in?

Partner 1  If you look at the question and then the answers, well…

Teacher  It looks like it might be easier, but how do you know where to put the decimal back in?

Student  Why are you doing….oh never mind

Student  I think we are working…

Student 2  If you were doing repeated subtraction and you could just keep the decimal in your answer.
Teacher  Are you thinking you could keep the decimal in the same spot? Is that what you’re thinking?

Student 2 Write the number where the decimal is and then keep it in the line.

Teacher  I think this idea of decimals and ignoring them and putting them back in… Let’s stop here and think about how we can put decimals back in, using our number sense, thinking about what the answers might be.

Student  The answer is not 0.19.

Teacher  Yeah the answer is not 0.19, but we need to think about where the decimal is, but this will be a great conversation to come. We are going to come back and discuss how [Partner 1] thought about ignoring the decimal.

The class ended with students debating whether the answer was 19 or 0.19. Because they took the decimal out, students were not convinced where it should be placed in the quotient. Ashley confirmed that 0.19 was not the correct answer, but left the correct answer open, leaving the students wondering about locating or placing the decimal in the quotient.

Ashley’s reflections on the lesson.

Moving forward, Ashley planned to provide further opportunities for students to explore multiplication and division with decimals later by analyzing patterns that occur during these operations.

When we talk more about multiplying decimals, we’re gonna look at the patterns of how the decimal moves and I think that it will be more appropriate to do that afterwards… um, once they see those patterns and kind of internalized that, but I don’t want, I want to stay away from just saying, move your decimal point… (Pre-lesson interview).
She planned to use these patterns to discuss how multiplying by a power of ten can be used to shift place values, allowing students to use strategies they have been using all along, such as partial quotients, which in her class is called the “super 7” (because when setting up the problem, the division symbol is extended down the right-hand side of the work, making the symbol look more like a large number 7).

A good strategy to divide decimals would be to divide your divisor and your dividend by a power of ten so that your dividend doesn’t have a decimal. And then um, you can use those same strategies they’ve been using, the ‘super 7’ or the um, division algorithm.

(Post-lesson interview)

Ashley planned to use exploration and discourse to help students identify patterns with decimals and then use that information to allow students to access decimal division using strategies students find familiar.

**The students’ understanding of division with decimals.**

The interviews of two students, Monica and Michael, will be used to show student understanding of division with decimals. They were interviewed the day following the lesson. Both their estimations and attempts to find the exact answer are discussed.

As mentioned previously, students were asked during the lesson to estimate first when solving the decimal division problem. To do this, some students rounded both the dividend and the divisor to the ones place. When thinking about estimating the answer, the students who were interviewed all demonstrated their understanding of the relationship of the dividend and divisor in a division problem. Monica was asked if she was surprised by the answer.

Monica We learned that the answer is actually like 19, with a remainder of 15, I think, and 12 wasn’t really close to that answer.
Interviewer  Hmm, why do you think that is?

Monica  Well, um, when we were doing division, we learned that if you move, that
if you change the dividend, or the divisor, then it’s going to affect your
answer a lot more, and we rounded up, it up by a lot. (Monica interview)

When Michael was discussing the reasonableness of his answer, he discussed the
relationship between the values in a division problem.

Michael  If you move the numbers around it’s, if you moved them, if you changed
the divisor, up that’s, you have to put it into more groups,

Interviewer  Um, hmm

Michael  So and then you’re going to get a smaller number. And if you round it
down, you’re gonna get a bigger number….So it’s more important that
you don’t change the divisor, and it’s kinda better to change the dividend.

(Michael interview)

Both students were able to articulate how changing one aspect of a division problem would
affect the outcome of the problem.

When Monica was interviewed, her work reflected many of the strategies used when she
divided whole numbers. She used bar diagrams and repeated subtraction as strategies to solve the
problem (see Figure 12).

Monica  Well, we ah, to show our work, we drew a bar diagram of our picture, of
our equation and then we, we started to get, we started to, try and find the
real answer, by, by repeated subtraction.

Interviewer  Hmm, so why did you stop?

Monica  Well we ran out of time,
Interviewer: Ran out of time, okay, so tell me what you’d do if you finished that.

Monica: Well, then I would, then I would subtract 65 hundredths again, from the partial quotient [remaining dividend] of 11 and 95 hundredths and then I would put a little two to remind myself that I subtracted, that that’s the second one I’ve subtracted.

Interviewer: Um, hmm

Monica: And then at the end I’d add up all the little numbers on the side and that would be my answer.

Interviewer: Okay, so um, how many times do you think you’re going to have to subtract 65 hundredths?

Monica: Well, I think it’s going to be somewhere around 12, because that was our estimated answer. (Monica interview)

Monica also explained how she chose between using the standard algorithm and partial quotients. “If it’s, an easier number, and you already see multiples then it’s easier to do the standard algorithm, but maybe if it’s harder numbers then you can do, the super 7 [partial quotients]” (Monica interview).
During Michael’s interview his confusion with eliminating the decimal became more apparent.

Michael So I divided the decimal. I did, first of all I knew we were working with 12.5 and 0.65 and I’m glad I had that partner to remind me that, ‘cause I would have got [sic] this wrong if I didn’t have my partner with me, I realized that the 12.5 and the 0.65, if you line them up, there’s a place holder so that’s why I did, I took away the decimals and that’s why I did the 1,250 instead of 125

Interviewer Ahh

Michael I would have made a mistake, and just put 125.

Interviewer Uh-huh

Michael That’s actually what I originally put too, so, and then he’s like [his partner], ‘Oh, did you know you could do that?’ (Michael interview)

Based on his partner’s experience, Michael removed the decimals while dividing. After he divided, he could not correctly answer the question regarding the number of pieces that could be made.

Interviewer So, I know your teacher was talking about this too. Can you, how can you just take away the decimal point?

Michael Um, well it’s easier to work with, I heard a lot of people commenting that it’s, it’s kind of cool and easier way to work with, and you can go down and fill in the spots with the decimal points and you can find out what that, how that is a decimal. ‘Cause I got 19 remainder 15, now my real answer would be point nineteen pieces.
Interviewer  Ahh, point nineteen, what does a point nineteen look like?
Michael   A point to the left, and then it has one tenth and nine thousandths, umm
Interviewer With the construction material, how would that look, point one nine?
Michael   Umm, oh, the meter, there would be a 19, a point nineteen meter piece of
construction material, yeah.
Interviewer What’s the question for the story problem?
Michael   About how many pieces can be made?
Interviewer Okay, what was your answer?
Michael   19 pieces
Interviewer Hmm. So if it’s point nineteen, what does it mean?
Michael   Umm, 19 hundredths, of construction material long, meter long
Interviewer So instead of how many pieces it’s gonna be, how long it is?
Michael   (long pause) 65 meters long, and 12.5 meter piece, I think that’s the same
meter long and meter piece is the same, so I think you could use them
both, in a way, so it would be a 19 hundredths piece of construction
material. (Michael interview)

Michael spent some time thinking about this, but could not reconcile the number of pieces with
the length of 0.19 meter.

**Within Case Results – Steve**

Steve taught fifth grade mathematics to four classes per day in 75 minute blocks. Steve
has been teaching at his current school for 15 years and has been a teacher for a total of 19 years.
His early experience in teaching was using *Saxon Math*, currently published by Houghton
Mifflin Harcourt. The publisher’s web page describes *Saxon Math* as having “an incremental
structure that distributes content throughout the year. This integrated and connected approach provides deep, long-term mastery of the content and skills called for in the Common Core State Standards” (Houghton Mifflin Harcourt, 2015). For the last seven years, Steve used *Houghton-Mifflin Math*, a traditional elementary textbook. This is his first year using *enVision MATH*, published by Pearson. Although this textbook has some online features including videos, it also follows a traditional textbook format. In addition to using a new textbook, this is the first year Steve is fully implementing the CCSS with a new district curriculum map adapted from the Agile Mind website created at the Dana Center in Texas\(^{10}\).

Steve remembered mathematics teachers he had as a student during his pre-lesson interview. Those teachers with whom he did well and those with whom he did not do so well. He recalls having a “terrible trig teacher in college and then…a great trig teacher in college which [sic] made it make sense. I think if you can bring it to life, and see why it works, it makes sense, not just memorizing…for school…I’ve had both good and bad.” In detail, he shares his first struggle in mathematics while in school.

Well…Ah…I was always good at math. I don’t know why it just clicked. I never really did anything and I wasn’t a good student, but, Miss Nelson, I finally got introduced to algebra, it was, here’s the problem, here’s how you do it, go. It was algebra, we didn’t, you know, we didn’t build up to algebra, we weren’t introduced to that in elementary school at the time, it didn’t click, and so finally, I asked my dad, he was a math guy, he had to teach me math. So we started to spend quality time at the kitchen table….And so he started teaching me math. And he’d get frustrated ‘cause it was always, what the hell

\(^{10}\) http://ccsstoolbox.agilemind.com/pdf/DanaCenter_Grade5_SequencedUnits_2013.pdf
does she teach you? What does she do? She shows us how to work a problem, assigns 30
and we’re finished. And then they had a parent teacher meeting. Come to the town hall,
so he showed up and other parents that were frustrated with their kids failing algebra and
I’m sitting here with him, she did her spiel and she said is [sic] there any questions, and
he says, yeah, how would you teach me to do this? And he gave her a problem and I
thought oh gosh, I know this voice ‘cause when he asks you questions, he already knows
the answer, he’s just leading you to the next one. I said, [in a whisper] don’t answer it. So
she told him, she showed him how to work it, he says, no I didn’t ask you to show me, I
asked you to teach me. And she just stood there kind of dumbfounded, he goes, anybody
can show our kids how to do it, how are you going to teach them to get there? So, Miss
Lawson didn’t talk to me for the rest of the year and I, he taught me math. (Pre-lesson
interview)

In college Steve took one mathematics methods course about which he said, “I don’t
remember anything about it. I couldn’t even tell you who the teacher was” (Pre-lesson
interview). Steve believes his experience in the classroom is the biggest influence on his
instruction, “just getting in and seeing what works and what doesn’t work” (Pre-lesson
interview). When asked about resources he frequently uses, he first looked to his experience.

Um, primarily I pull from past knowledge. I pull from Coach\textsuperscript{11} books, text book, I look
online to see what kind of task I can find that best fits what we need at the time, and I
rarely use the same materials each year. I just kinda go with each class and how it works
out. (Pre-lesson interview)

\textsuperscript{11} Coach books, published by Triumph Learning, are test prep books that are created specifically for the
state standards. In previous years, these books have been purchased through local school funding for use with
students in math.
When asked about his beliefs regarding mathematics instruction Steve was unsure if the focus should be on drill and practice or if the focus should be on open-ended problems. “I always thought of myself somewhere in the middle…I don’t know if there’s a magic formula. I think just good common sense and building on what you know” (Pre-lesson interview).

Steve grappled with the best approach for division in fifth grade. “I’m still at that point of which technique was better for the kids…the algorithm or the partial quotient” (Post-lesson interview). Because partial quotients was fairly new to Steve, he expressed some concerns in his ability to encourage its use.

If you had somebody that had a set plan and knew it [partial quotients] backwards and forwards then you may have a different outcome. But from the way I know and the way I taught it, the problem could be with me there….When they don’t know their facts, they’re gonna miss the problems. So as far as improving, you know, maybe have a, you know, more time, just on division, then you have a much easier transfer. (Post-lesson interview)

Steve was also struggling with helping his students make the transition from partial quotients to the standard algorithm for division. The fourth grade teachers used partial quotients to teach the standards for division; the state standards required similar division strategies in fifth grade. It was not until the end of sixth grade that the state standards required students to be fluent with the standard algorithm for division.

I’m still torn up, as far as fifth grade goes. I know 4th grade, and I’m a firm believer 4th grade should use partial quotient. I do believe that, and they should talk about this is, what it is. This is what division is and drive that home. Somewhere in fifth grade you’ve gotta make a change, and I know it’s set up for sixth grade, it’s just where do you leave
one and go to the other, exclusively. I did both this year and I don’t know if that confused some kids or if it helped. I’m thinking it helped some make the transition, others were ready for the algorithm at the beginning of the year, so, it’s a discussion we’ll have.

(Post-lesson interview)
Steve was working to become better with partial quotients while determining the best way to help his students transition from partial quotients to the standard algorithm for division.

Steve reflected on his current role as a fifth-grade mathematics teacher. “I’m pretty confident, you know. I believe in what I’m doing, I think I can explain why I’m doing it. If I don’t understand something I’m not afraid to ask questions…Ah, I think there’s room for improvement” (Pre-lesson interview). When thinking about challenges, he discussed how he tries to help all students access mathematics.

The hardest thing I guess is finding that different way for a kid that doesn’t see it. You don’t know why, because it’s just, it should be laid out for you. This is the way that, you know, it just all makes sense. You can do this, you should be able to make the transition and when they can’t, it’s tough to find something to make it work. (Pre-lesson interview)

**Steve’s teaching.**

During the pre-lesson interview, Steve described his plans for the division with decimals lesson based on his previous years’ experience. (Unless noted otherwise, all quotations from this section came from pre-lesson interview.) He planned to start with a review of division with whole numbers, helping students remember the steps and strategies they used earlier in the year.

Well normally I scaffold, just the steps, and we’ll, I go into the steps of just the basic division problem. Because we have the problem of what style of division do they use, do they use the algorithm or do they use the partial quotients, I’ll work two sides of the
board, so I’ll have, algorithm on one side, on every problem, partial quotient on the other. We’ll start with just the basic division problem, no decimals, we’ll work it out, make sure we’ve got the rhythm of it.

Steve planned to remind students to keep the place values lined up, “keeping everything lined up, being neat, not being a ‘squisher.’”

Next he planned to give the students the same division problem with a decimal in the dividend. In order to help students remember to place the decimal correctly in the quotient, he planned to have students relate the action with a sound and a movement.

Then we’re going to do the exact same problem and I’m going to put a decimal in the dividend. And we’ll talk about how it changes it, our only rule, and the way I get them to remember it is we take the decimal and it will go straight up. So every kid, every kid will, I’ll say, what’ll we do with the decimal? And every kid will [Steve makes a “raspberry tart” noise and points up with his finger]….And they’ll point up. Then every time we look at it, it’s, okay, you’ve got the setup, what do we do with our decimal? And everybody makes their noise…just to remember; just to kinda get one of those other senses involved.

When students are given a division problem with a decimal in the dividend and the divisor, Steve planned to divide a decimal number by 25 hundredths. He anticipated the problems students may have with this step and planned to have students multiply both the divisor and dividend by the same power of ten, what he calls the “over, over up” method. “We’ll go back to our equivalent fractions and show okay, really you’re just doing this [multiplying the numerator and denominator by the same power of 10]. It’s just a different format.” He planned to refer back to a story he used when teaching equivalent fractions.
And that goes back to our fractions, our numerators and the denominators are introduced as like brothers and sisters. They never really get along, so whatever a parent does to one, you have to do to the other, to keep it balanced.

Steve chose to move towards division with decimals incrementally because it is the strategy he used in past years. “[It’s] what I’ve used in the past, it’s just, the memory of it and why we do it, linking it to past skills, it’s always worked well.”

The division with decimals lesson – part 1, warm up.

The events and quotations in this section came from the lesson observation notes, unless otherwise noted. Steve began his lesson by asking students to complete two problems in their practice workbook. He told the students the problems are similar to what they will see on the state assessment (see Figure 13). As the students were working, Steve reminded them to, “Focus on what the question is, how you're going to solve that, and how your work will prove the answer. Remember the correct answer is not as important as how we get there.” Students worked very quietly. There was little discussion, but when there was a conversation between two students, it was in a whisper.

11. Tony bought a 72-ounce box of dog biscuits. How many pounds of dog biscuits did he buy?
(Remember: 1 pound = 16 ounces)
A 4 pounds
B 4.5 pounds
C 90 pounds
D 4,320 pounds

12. Janell uses 56 beads for each necklace she makes. She bought a bag of 500 beads. How many necklaces can she make?

Figure 13. Problems given during the “warm-up” part of the lesson (Pearson Education Inc., 2012, p. 7-4).
Once the students finished the problems, Steve led a conversation regarding the solution to each problem. He began by asking the students what they know about the first problem (Figure 13, Problem 11). The students identified the total amount of dog biscuits as 72 ounces and that 1 pound equals 16 ounces. They also gave the question, “How many pounds of dog biscuits did he buy?” Steve reminded the students about the importance of underlining the question. “Underline the question because that is what I need to answer. You can do all the work correctly but if you don’t answer the question, you won’t get it right.” The students recognized that the problem could be solved by dividing 72 by 16, because they needed to divide the number of ounces in the box by the number of ounces in 1 pound. To divide, Steve prompted the students using the standard algorithm for division. “Will it [16] go into 7? Will it go into 72?” Steve worked with the class, multiplying 16 by 10 and then by 5 before choosing to multiply 16 by 4 (see Figure 14). Once they found the solution using the standard algorithm, they used the partial quotients algorithm to find the solution. Here Steve focused the conversation on how many groups of 16 are in 72. He annexed a zero in the dividend, making it 72.0. “Now add a zero, if I add a zero will I change the value of the dividend?” He then answered the problem’s question, “How many pounds were in a 72 ounce box, 4.5 pounds or 4 1/2 pounds?”

![Figure 14. Solutions to problem 11, standard algorithm (left) and partial quotients (right).](image-url)
Steve followed a similar procedure when discussing the second problem, starting with what information was known in the problem. The students offered, “66 beads per necklace, she has a total of 500 beads.” Steve clarified that the question is asking how many necklaces can be made. Admitting that he did not know his 66 facts, Steve asked the students to estimate. “What if we estimated with 60? 70?” A student offered, “It could be 7 because of 490.” Steve multiplied 7 x 66 on the board, getting 462. They concluded that seven necklaces could be made and 38 beads would be left over. The warm-up took approximately 20 minutes.

The division with decimals lesson – part 2, instruction.

A Khan Academy video¹² was ready to start on the interactive white board in the front of the classroom. The problem shown was 1.03075 ÷ 0.25. Steve told his students that they will be watching the mathematics video “for fun” and that the students did not need to take notes because they would be taking notes together. Before beginning the video, Steve reviewed multiplying a decimal by a power of ten. “Who remembers our decimal boy? What happens to the decimal point when we multiply by 10?” A student replied, “Oh I know this. 10 x 7 = 70. The decimal went one place to the right.” Another student added, “If you divide, it goes to the left.” The next example was 5.7 x 100 = 570. Steve reminded the students of his story about equivalent fractions. “Numerators and denominators really don't like each other; they are like brothers and sisters. If sister gets 20 dollars and a hug, then the brother expects 20 dollars and a hug, it should be fair and equal.” Steve shared the following example, 3/4 × 10/10 = 30/40 and asked, “Is 30/40 the same as 3/4?” A student replied, “Yes, sir” (see Figure 15).

¹² https://www.youtube.com/watch?v=Nqts8zW8RxM
Steve then focused on the problem in the video, 1.03075 ÷ 0.25. “There is a decimal in the divisor. It is okay to have a decimal in the dividend; it is difficult to divide with a decimal in the divisor. So, we get rid of it.” Steve reminded students that fractional form is just another way to write a division problem, 4/3 is a different notation for 4 ÷ 3. “So could I write this problem [1.03075 ÷ 0.25] as 1.03075/0.25?” Once the students agreed the problem could be written as a fraction, Steve asked, “What could I possible multiply .25 by to get rid of the decimal?”

A student responded, “It would be 100 because that would move the decimal 2 places.”

Steve asked, “If I move the divisor two places, what else do we need to do?”

The students agreed that he would also need to multiply the dividend by 100. Following this discussion, Steve showed the video. After the video, the students solved the same problem. There was no conversation as the students worked. When finished, one student put the solution on the board using the standard algorithm, what Steve called, “the normal way.” Another student showed how she used partial quotients to find the solution.

The student who solved using the standard algorithm explained her thinking. She started by explaining that she knew her “25s from quarters and dollars.” Next, the student who used partial products explained her thinking (see Figure 16). When she was finished, Steve asked her, “One question, how could we organize the numbers along the side?” He reminded her, “[The] decimal has to follow it all the way up when you are stacking, and all the way down when using partial quotients. I haven't done this strategy enough to be completely comfortable.” At this point
the support teacher, who had worked the problem using partial quotients with a student said, “We pretended the decimal wasn't there, so we used 4123.”

Figure 16. Student solved $1.03075 \div 0.25$ using a partial quotients strategy.

There were about 6 calculators around the classroom. Students with a calculator were asked to check the problem twice. First entering $1.03075 \div 0.25 =$ and then entering $103.075 \div 25 =$, checking to see that they got the same answer.

*The division with decimals lesson – part 3, student practice and lesson close.*

Steve shifted gears, asking students about the following problem, $1.29 \div 0.3$. “How many spaces would I need to move the decimal? What am I really doing?” (Lesson Observation Notes). Students agreed they should multiply both dividend and divisor by 10. The students told Steve to write the “new” problem as $12.9 \div 3$. One student had difficulty reading the problem, first she said $3$ divided by $12.9$ which was corrected by the support teacher. Then she said $12$ divided by $3$ which was again corrected. Finally the student read $12.9 \div 3$. She followed the correct response with, “Whew!” Steve solved the problem on the board with student input.
The next problem Steve gave them was $4.62 \div 0.06$. He provided a few hints by saying, “0.06 is not the same as 6.0, we’re not really getting rid [of the decimal] but changing it. Rewrite the problem and solve it.” (Lesson Observation Notes). Again, there was no student conversation as they worked. Steve moved through the classroom, checking work and giving fist bumps to some students. The class ended with students putting their work on the board. Steve commented on the work, “Got the decimal out of the divisor, did the same thing in the dividend” (Lesson Observation Notes). Then he continued to explain each step in the standard algorithm. When discussing the partial quotients solution to the problem he clarified, “When you are using the partial products you are looking at the whole number, how many times can 6 go into 400” To this a student responded, “Ah!” Steve closed the lesson by telling the students that they will continue with more practice the next day and that the “division is starting to look better.”

**Steve’s reflections on the lesson.**

After the lesson, Steve shared some of his reflections regarding the lesson. He said that he felt one strength of the lesson was the use of the Khan Academy video. “I think using a different source besides me always helps. I guess that it gets their senses, some of the kids that don’t pay attention, they’ll listen to somebody else, even if it’s for two minutes.” (All of the quotations and comments in this section are taken from the post-lesson interview.) Also, he said that building on processes that the students already knew was a positive. He said the students may think, “Hey, I do kinda already know how to do most of this, and it’s just one more small step.” He thought adding to a process with which the students were familiar was a confidence builder, that the lower achieving students could think, “You know what? I can do this. It’s not overwhelming. Although it looks confusing, it’s really not much more than I can already do.”
Steve’s concerns regarding the lesson focused on the pacing of instruction near the beginning of the school year. “Earlier this year [I] should have spent more time on division, so that their basic division skills were better than they are…this group seems to be a little bit low, on their basic skills. I think that’s what held them back; conceptually, I think a majority of the kids understood what we were doing.” Also, he wished he had more time to spend practicing division with decimals.

Another concern regarded division with a two-digit divisor. Steve saw students struggle when dividing with a two-digit divisor and was thinking about ways to make that process easier for the students.

The main struggle came when we did two-digit divisors, ’cause they don’t feel as comfortable with those facts, and it’s a little more work, and a lot of times they don’t want to do that work off to the side. They feel, some of the lower level kids were the ones that don’t necessarily believe in their math skills. They felt defeated before they started…they just don’t want to do it and they feel they have already missed it before they’ve started.

Steve looked at how the students scored on the assessment following the instruction on division with decimals. He saw that the students were able to place the decimal correctly, but often made computational errors when dividing. “As far as the placement of decimals, they did really well. Even if we look back at the test, placement of decimals was pretty good; division, not so good. So conceptually, I think they did, they did well.”

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13 The work off to the side involves multiplying the two digit divisor by the estimated quotient. Some students use a “multiplication bank” where they do all of the multiplication work and then use the products when dividing.
The students’ understanding of division with decimals.

Immediately following the lesson, Joey, one of the students in Steve’s class, talked about his understanding of division with decimals. The comments quoted in this section came from Joey’s post-lesson interview. When asked to talk about a problem such as $4.16 \div 0.4$, Joey described the steps he would use to solve the problem (see Figure 17).

Now as we talked about today, we said that it would be a lot easier not to keep that decimal so, he [Steve] said we would want to move it one over, put it right here [multiplied $0.4 \times 10 = 4$] and we also had to be fair with the other number so we have to move it right here [multiplied $4.16 \times 10 = 41.6$] and then he said to rewrite it. I just need a minute….So then I know that four will go in to four one time….Drop the one, cannot do that so put a zero, you have to, drop, I mean minus zero, drop the six, then four will go into 16 four, eight, twelve, sixteen, four times, sorry, four times, 16 minus that is zero, and you always need to “up the decimal” and that is how I would do it.

He was able to use all of the steps he had learned in class to solve the problem.

Figure 17. Joey’s work for the problem $4.16 \div 0.4$. 
Joey was asked what was difficult about a problem like the one in Figure 17. He did not find anything difficult, “to me, I don’t see anything hard about it.” But he did comment on where other students might find the problem challenging.

    Well the real, main easy thing is that I think it would be hard for other people that didn’t know their math and their multiplication facts. That, well it would be harder, but I’m [sic] got my facts pretty good down [sic] so that’s really easy for me ‘cause I can tell that, you know, 16 - 16 is 0.

    When asked to focus specifically on the decimal points, Joey found dividing with decimals easier when he ignored the decimal while doing the division.

    Well the hardest thing about the decimal points is really, if you really concentrate on it being there it makes it hard, but I figure if you concentrate on that not even being there, and then just pop it up when you need to pop it up, then it can very, be very simple….‘Cause you don’t have to worry about using anything ‘cause you really don’t have to use it, except when you have to bring it up on top to get your right answer.

    Because Joey commented on the ease of dividing decimals, he was asked a question to determine his understanding of the process of division with decimals.

    Interviewer Can you think of a, a story problem that could be used for this division problem?

    Joey Okay…Haley had four and tenth, I mean 4 and 16ths beads, and she wanted to put four tenths of those beads to make one full bracelet. So how many full bracelets can she make with this problem, with this problem?

    Interviewer Okay, and what would the answer be, how many full bracelets could she make?
Joey She would actually be able to make [whispers to himself, ‘Let’s see.’]

She’d actually be able to make 10, 10 full, I think, 10 full, 10 full.

While Joey was able to explain the steps when solving a division problem with little difficulty, he was unable to create a realistic context for the division problem.

Joey’s final thought for being successful when dividing decimals was, “if you really study hard, work on it, and you really pay attention to what the teacher is saying, it should be very simple.”

**Teachers’ Cross Case Results - Differences**

The instructional strategies used by each teacher to introduce division with decimals are discussed in this section in order to answer the first research question: What strategies do effective teachers use to introduce division with decimals?

In order to consider the strategies each teacher used when introducing division with decimals, tables were utilized to compare the common strategies used by each teacher. The strategies used by all three teachers were, engaging students in class discussions, encouraging students to share their strategies, and questioning students about their strategies (see Figure 18).

![Figure 18. Strategies used by all three teachers](image-url)
The identified strategies used by two of the teachers were, providing contextualized problems and guiding students when working on problems, both used by Christy and Ashley, and step-by-step instruction used by Christy and Steve (see Figure 19).

<table>
<thead>
<tr>
<th>Strategies</th>
<th>Christy</th>
<th>Ashley</th>
<th>Steve</th>
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<tbody>
<tr>
<td>Class discussion</td>
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<td>Context</td>
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<td>Small group “stations”</td>
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<td>Step-by-step instruction</td>
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<td>Stories, movement, humor</td>
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<td>Student discourse</td>
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<td>Students share their strategies</td>
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<td>Teacher questions students</td>
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<tr>
<td>Think, pair, share</td>
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<tr>
<td>Visualization</td>
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Figure 19. Strategies used by pairs of teachers

The strategies used to introduce division with decimals to fifth graders were quite varied. The differences can be captured along a continuum of traditional to reform practices, with Steve exhibiting the most traditional, teacher-centered practices, Ashley displaying practices that align most closely with the 2000 recommendations of the National Council of Teachers of Mathematics (NCTM) and the CCSSM (2010) standards for mathematical practice, and Christy landing in between. All three teachers established classroom norms and routines that reflected their beliefs as well as their conceptual understanding of the mathematics. However, instructional practices were quite divergent.

Christy.

Christy used the most strategies, 11 total to introduce division with decimals. Strategies unique to Christy were the use of manipulatives, patterns, and small group “stations.” For the study lesson, Christy used a three-part format. She began with a word problem that required
division with decimals. She gave her students an opportunity to work on the problem individually first, and then compare answers. Afterwards, the class discussed the solution.

During the next part of the lesson student groups began exploring sets of related division problems at one of six “stations,” using materials with which the students were familiar (in this case teacher-made clear tenths squares with a different number of tenths shaded in on each, play money, and base-ten blocks). By solving several related division problems, Christy expected the students to look for patterns in the placement of the decimal point in the quotient. She gave students time to work on the problem sets. They were able to rotate through two stations over the course of the work time. During this time, Christy visited groups of students, asking and answering questions. During the last part of her lesson, Christy led a closing discussion. This final discussion focused on procedures for (a) how to place the decimal in the quotient when dividing by a whole number, and (b) how to use multiplication when dividing by a decimal to show how to “move the decimal.” Christy summarized the placement of the decimal when dividing a decimal number by a whole number and then, through guided conversation, discussed how multiplying both dividend and divisor by a power of ten allows one to “move the decimal” and then divide.

In her pre-lesson interview Christy verbalized that she wanted to allow students to come to an understanding of division with decimals on their own, through exploration. But time was an issue for her. She said, “I think they [the students] were beginning to see, I think we didn’t have enough time for them to fully develop all of the stations, and you know come to the realization [of] what was going on” (Christy, Post-lesson interview). Only a couple of students were able to articulate the understanding that Christy wanted the students to leave with. She used the work of those students to tell the rest of the class what she had hoped they would discover through
exploration. As a result, most students spent little time with productive struggle; instead they were given a strategy, i.e., they could multiply using a power of ten to place the decimal before dividing.

**Ashley.**

Ashley wanted her students’ to use reasoning to solve their first problem of division with decimals. She presented a problem of finding how many pieces of construction material 0.65 m in length can be cut from a piece of building material 12.5 meters in length. She provided a concrete context that allowed students to visualize the operation even though it potentially involved using an operation to which they had not been introduced. By knowing the size of each group (the length of the cut pieces), students were able to visualize a piece of material being cut in multiple sections of that length, a measurement division situation (Fischbein, Deri, Nello, & Marino, 1985). This can also be thought of as repeated subtraction, which many students used to find the exact answer.

Ashley used a total of eight strategies when introducing division with decimals. Three strategies that were unique to her instruction were partner work, visualization, and student discourse. Her lesson focused almost entirely on student interaction, exploration, and discussion. She began by asking students to share their understanding of rounding, suggesting that they might be able to estimate a solution for the problem. She then asked students to work with a partner or in a small group to plan and work the problem together. Finally, she demonstrated how she valued discussion by allowing students to share their approach and thoughts regarding the solution to the problem. She began the discussion with the strategy that was most accessible to her students and built upon that to a solution that caused many students to pause and wonder about correctly placing the decimal.
Ashley encouraged her students to make sense of the process of dividing with decimals. She provided a problem for them to ponder, allowing accessibility for all students by requiring students to estimate, but also challenging students by encouraging them to find the exact answer. When she visited students working, she asked questions to further discussion. Most importantly, at the end of class she left the students to ponder the correct answer.

Steve.

For Steve, division with decimals is the pinnacle of the operation of division. In his lesson, he methodically took his students through a step-by-step introduction to the procedure. He began by reviewing division with whole numbers. If students struggled when dividing whole numbers, he or the support teacher helped those students individually, focusing on the process of division. Next, he introduced dividing with a decimal dividend and a whole number. Finally, he modeled division with a decimal divisor and dividend after showing a short video where the steps required to divide two decimals are described. In his interview Steve shared that he frequently uses stories, movement, and or humor to help students remember a process. In this case, he modeled making a “raspberry tart” noise and pointing up, indicating the decimal point location moves straight up into the quotient. This was followed by a problem he modeled dividing a decimal in the dividend and in the divisor. For this problem he called upon a story he used with equivalent fractions, reminding students that the numerator and denominator of a fraction are like a jealous brother and sister; whatever you do for one, you must do the same for the other. Therefore, to “get rid of the decimal” the students were taught to think of the division problem as a fraction and multiply both the numerator and denominator by the same power of ten. Finally, once the decimal point was in place, he showed the students they could merely follow the steps for the whole number division algorithm, ignoring the decimal, because it would
already be placed in the quotient. Steve indicated that he thought his incremental instructional strategy was beneficial to his students. He would continue to add additional steps until solving a division with decimals problem would require just one more step, placing the decimal point. Steve indicated that using this step-by-step strategy made difficult tasks manageable. He said he wanted students to think, “I do kinda already know how to do most of this, and it’s just one more small step” (Steve, Post-lesson interview).

Steve used the fewest number of strategies (modeling, step-by-step instruction, humorous stories, questioning, class discussion, and student shares) when introducing division with decimals. He followed a set routine. He began class by modeling the new step the students needed to solve to be able to do the next level of the problem (in this case division with decimals). Students then worked quietly for some time to solve the problem. Finally, Steve asked a student to put her solution on the board. While the student wrote her work on the board, Steve led the discussion, referring to the student’s work.

Steve worked very hard to prevent his students from struggling, teaching the steps to follow so that they would not have to experience confusion. Steve saved word problems and problem-solving situations until after students had successfully mastered a technique for dividing with decimals.

**Teachers’ Cross Case Results - Similarities**

When considering the strategies teachers used to introduce division with decimals, there are several commonalities. All three teachers used whole-class discussions, asked students to share their strategies, and used questioning to help students articulate understandings or procedures. In all cases the teachers questioned students in order to understand what they understood about division with decimals. Often, this included asking students to share their work
with the class and to explain their process in solving the problem. While the underlying strategy was the same, how the strategy was carried out in the classroom varied just as each teacher’s instructional style varies.

Christy and Ashley both introduced division with decimals by providing a story in a context that could be visualized or represented through illustration. Also, both teachers served as a guide during parts of the lesson. As a guide, they asked clarifying questions, suggested paths to explore, and used words of encouragement. Christy and Steve both used the strategy of step-by-step instruction or providing rules to follow. They told students what steps to follow when dividing decimals (e.g. multiply the dividend and divisor by a power of ten, place the decimal point in the quotient, and then divide). Both teachers indicated the importance of students being able to follow correctly steps for dividing decimals. Of particular importance was the movement and placement of the decimal.

Steve and Christy both preferred their fifth-grade students transitioned from partial quotient strategies to the standard algorithm for division. While students are not expected to be fluent in the standard algorithm for division until the end of sixth grade, neither teacher felt comfortable enough with the strategies used in fourth grade to help the students use those strategies with fifth grade topics. When Christy was asked about students dividing decimals using the division strategies learned in fourth grade, she said “[It] is really messing them up, or messing me up when I was trying to grade their papers [laughs]” (Christy, Post-lesson interview). Steve commented to his class when giving an example of division with decimals using partial quotients, “I haven't done this strategy enough to be completely comfortable. . .” (Steve, Lesson Transcript).
Returning to the CCSSM (2010), which indicates that students should build on place value understanding and the operations of multiplication and division to build a conceptual understanding of division with decimals, all three teachers used at least one aspect of this. Christy built on her students’ understanding of the inverse relationship between multiplication and division when they used the clear tenths squares. Ashley built on her students’ understanding of division of whole numbers to introduce division with decimals. Both Christy and Steve used what their students knew about multiplication of powers of ten in order to properly place the decimal point in the quotient.

**Students’ Cross Case Results – Differences**

Analysis of student interviews will be used to answer the second research question: How do children explain the strategies they use when first learning to divide decimal numbers? Student interviews for each teacher will be discussed.

**Christy’s students.**

Both of Christy’s students, Sue and Doug, referred to rules they had written down during the lesson, but neither could explain how or why following these rules gave the correct answer. For example, Sue moved the decimal in the dividend even though there was no decimal in the divisor. Her action is an example of a misunderstanding or an incorrect application of a rule she was shown during the lesson. Sue recognized she had not mastered the process because when asked why dividing decimals is hard, she said, “‘Cause, I sometimes don’t know where to put the decimal” (Sue interview). Doug also relied primarily on memorization or statements written in his notes. When asked to explain why he moved the decimal in a problem he either referred to his notes or repeated what he had been told during class. During his interview he referred to his notes,
Well I have a lot of notes one of them says, when you divide by a whole number the decimal stays the same, when you divide by [sic] a hundredth, by a tenth you get a tenth….So all you’re really doing is just switching them around.

Doug relied heavily on his memorization of facts and skills, and credited his understanding of mathematics with his ability to remember.

**Ashley’s students.**

Ashley’s students were able to articulate an understanding of the concept of division as well as how adjusting the size of the divisor (by rounding) impacts the quotient. None of the students interviewed from Ashley’s class seemed surprised by the size of the estimated quotient in relation to the dividend. In fact, Michael was able to articulate his understanding very clearly when explaining how the size of the divisor affects the size of the quotient.

Michael If you changed the divisor, up that’s, you have to put it into more groups,

Interviewer Um, hmm

Michael So and then you’re going to get a smaller number. And if you round it down, you’re gonna get a bigger number. So

Interviewer Oh, I see, so if you’re rounding the divisor you’re talking about

Michael So it’s more important that you don’t change the divisor, and it’s kinda better to change the dividend (Michael interview)

Michael was able to move beyond estimating to consider the resultant quotient because he had a strong understanding of how the quotient is affected when the divisor changes.

During the “explain” part of the lesson Ashley kept guiding the students’ conversation to what the quotient meant in terms of the context of the problem. Their confusion occurred when placing the decimal in the exact quotient. Although the context of the problem provided ways to
visualize the process described in the problem, the students focused on where the decimal point should be in the answer; even when a decimal answer did not make sense.

**Steve’s students.**

The interview with Joey, one of Steve’s students indicated that Joey was able to follow the steps Steve had modeled during the lesson. Joey explained that he is successful in mathematics because he is good at following steps laid out by his teacher. However, when asked to create a context for the division with decimals problem, \(4.16 \div 0.4\), Joey’s problem was unrealistic. He created a scenario where he had 4.16 beads. He wanted to find out how many necklaces he could make if each necklace required four-tenths of the beads. He completed the problem and indicated he could make 10 full necklaces, which is mathematically correct, but he was oblivious to the absurdity of the context of a necklace made with 1.4 beads. Following steps made it easy for Joey to solve the division problem, but he was unable to make sense of the mathematics.

**Students’ Cross Case Results – Similarities**

The strategies used by all of the students during the interviews reflected the strategies their teachers used. Also, the students in all three classes left the study lesson with a partially developed understanding of division with decimals. But unlike the other students, both Monica and Michael, from Ashley’s class, had a well-developed understanding of the operation of division. Both students were able to explain in detail how the quotient is affected by the value of the divisor.

Steve’s and Christy’s students were still developing their understanding of division with decimals. These students showed limited understanding of the operation of division and when
asked to clarify the reasoning used when dividing decimals, relied solely on the steps or rules given in class.

**Discussion**

Considering the strategies that are used by teachers and students to understand division with decimals, this discussion incorporates the researcher’s background knowledge, interpretations of what was seen during the study, mathematics standards in use, and the extant literature to discuss the strategies effective teachers are using to introduce division with decimals.

**Themes Found Across Cases**

This section discusses the observed themes in the data, specifically (a) how the commodity of time seemed to influence instructional decisions; (b) how instructional approaches seemed to influence classroom routines and student conversation; and (c) how all teachers discussed the placement of the decimal point in the quotient, but that the nature of the discussion varied based on the instructional approach each teacher chose.

All three teachers found time to be a valuable resource and they all discussed the best ways to maximize time. Steve considered time in terms of the curriculum map; he thought about how he could include more time with division of whole numbers to prepare students for division with decimals. Christy discussed exchanging one aspect of her lesson (5-a-day) for another (number talks). She also struggled with balancing the time students spent exploring concepts with the time students spent practicing procedures. Ashley discussed how concerned she was about time, but she was the only teacher who gave the students the time to process the introduction of division with decimals in relation to the operation of division with whole numbers.
The strategies teachers used to introduce students to division with decimals seemed to influence many other aspects of typical classroom routines. When the mathematics was presented as a step-by-step procedure, with a focus on building skills, the students rarely spoke to one another. The more student centered the class, the more students interacted, with the conversation focused on student understanding and thinking. Also, in the latter class there was much more variety in the approaches students used to solve problems and value was given to each strategy.

Student conversation seemed to be influenced by the structure the teacher chose to introduce division with decimals. In the teacher-centered classroom students rarely (if at all) discussed the problem they were solving or the strategies they were using. In the more student centered classroom students spent time discussing their work either in small groups or as a whole group. Strategies could be placed on a continuum, from those that were unrelated to mathematics (stories about brothers and sisters being treated fairly) to strategies that enhanced solving real mathematical problems and clarifying student understanding.

All three teachers discussed the placement of the decimal point. Steve and Christy wanted their students to understand how to move and place the decimal in the quotient by the end of the lesson. Only in Ashley’s class did the movement and placement of the decimal build from the discussion of several students’ work, and the question remained unanswered even at the end of the lesson. Ashley’s was the only class that began to construct an understanding of the concept of division with decimals. While Christy’s lesson contained the beginnings of conceptual development, time was limited (or students were given too much to do) and in the end, the students were simply told what they should understand about division with decimals. In Steve’s lesson, the students were told the procedures first, and no opportunities for productive struggle
were given. It is interesting to note that Ashley and Steve both worked to build on students’ previous knowledge. Ashley worked to build her students’ conceptual understanding of division to include decimals, while Steve worked on building his students’ procedural understanding of division to include decimals.

**Similarities and Differences in Introducing Division with Decimals**

With all teachers there are similarities and differences; this section describes similarities and the subtle differences that are evidenced in the data. The teachers in this study have similar backgrounds; they all received similar mathematics instruction as young students and as a pre-service teacher. They did not learn mathematics in ways teachers are expected to teach in the era of the Common Core State Standards. The standards for mathematical practice suggest that young learners need to engage in mathematics through practices such as reasoning, arguing, and modeling (National Governors Association Center for Best Practices and Council of Chief State School Officers, 2010). However, as learners of elementary mathematics, all of these teachers described experiences as young students where they were asked to follow procedures with little or no understanding.

While all three teachers in this study were identified as exemplary teachers by their principal or coach and the teachers’ lessons had many common aspects, there were significant differences in implementation and resultant impact on their students. Two of the three teachers cite professional learning during their career as impacting their instructional approach. Ashley cites several examples of how her mathematical thinking and reasoning changed due to courses she took in her district. Christy describes how a consultant hired by her district affected her teaching. On the other hand, Steve provides no examples of courses or influential professional learning.
Other similarities include the use of ongoing review as a strategy to revisit earlier topics in mathematics. Christy wrote her own review sheets, which she called “Daily 5,” and ended her class period working on these review problems. Steve and Ashley chose a handful of previous problems from the textbook and began their lesson with a discussion of those problems.

When introducing division with decimals, Christy and Ashley both used a three-part lesson format with an opening, work time, and closing, but experienced different success as demonstrated by student understanding; whereas Steve used a gradual release lesson structure in which students were given incremental pieces of new information to add on to previous knowledge.

Both Christy and Steve stressed the importance of keeping work neat and organized, carefully lining up the decimal and place values. Also, they both suggested that students need to be successful with simple, procedural division problems before experiencing more difficult problems. Christy planned to follow her introduction to division with decimals by priming her students “with some simple ah, division problems that, moving that decimal, and making sure that things are lined up, and, ah, everything’s clear, and concise notation…. [Then] we’re gonna do word problems that force us, you know, to use division” (Christy, Post-lesson interview). Steve planned his lesson to include review of basic division and the importance of organization. “We’ll start with just the basic division problem, no decimals, we’ll work it out, make sure we’ve got the rhythm of it. Talk about reviewing, as far as, ah, keeping everything lined up, being neat, not being a ‘squisher’” (Steve, Pre-lesson interview). At the end of the unit, Steve planned a problem-solving lesson, allowing students to use their division skills. In contrast, students in Ashley’s classroom began with a word problem dilemma and were encouraged to explore and make sense of the problem without direct instruction. And, while students in
Christy’s class were given some opportunities to state, explain, and defend their mathematical thinking, their thinking was limited because Christy began the lesson wanting students to learn a procedure on how to multiply the dividend and the divisor by a power of ten. Ashley allowed her students to build an understanding of division with decimals from their understanding of whole number division.

All three teachers worked in high-achieving schools and the percent of students who passed the fifth grade mathematics state assessment ranged from 89% to 98%. (These scores were based on state standards or a partial implementation of the CCSSM; the first assessment for the CCSSM was given in the spring of 2015, those results were unavailable.) However, the state assessment was not able to distinguish between students’ conceptual understanding and their procedural understanding. When CCSSM are fully assessed, including the standards for mathematical practice, the instrument may be better able to distinguish the level of student conceptual understanding. When introducing division with decimals it appears that the approach (student-centered, teacher-centered) did not make a difference in student understanding of division with decimals. Therefore, a substantial difference in student understanding could not be discerned from this data even though the teaching pedagogy of each teacher was considerably different.

Conclusions

This research provides a rare opportunity to “view” classroom instruction in three elementary schools in the eastern United States. The data provides insight into teachers’ thinking and students’ understanding of division with decimals. It clearly focuses on how division with decimals is introduced. This study is timely with the implementation of CCSSM over the past
few years. Finally, this study provides impetus for other studies with implications for how
division with decimals can be taught.

This study is important because over the last two decades there is little research from the
United States on the very difficult and complex concept of division with decimals. Therefore,
this study adds to the dearth of research on this topic. This is a first step in an attempt to uncover
approaches used by teachers to teach division with decimals in the classroom. This study will be
used as an impetus for other studies by reawakening awareness of this challenging topic. Before
research can be done regarding the effectiveness of strategies used to develop capacity, an
understanding of what is already happening in the classroom needs to be determined.

**Strategies Teachers Used**

The strategies teacher participants used varied from mathematical (e.g. look for patterns,
student strategies), to social (e.g. discourse, partner work), to instructional (teacher as a guide,
small group stations). While strategies were used by more than one teacher, only whole-class
discussion, students sharing their strategies, and teacher questioning were used by all three
teachers. Ashley was the only teacher to use partner work, student discourse, and visualization as
strategies when introducing division with decimals.

**Literature Related to Teacher Strategies**

The relevant literature on the division with decimals fell into four categories. These
included literature that supports the development of students’ (a) conceptual understanding, (b)
number sense, (c) related-problem experiences, and (d) understanding of common student
misconceptions. This section explores whether the strategies observed are supported in the
literature.
Ashley asked students to make sense of a problem that required division with decimals. While she told the students that they would be using division, she did not give the students any indications on how to divide decimals. While four studies found the development of conceptual understanding effective, Abrose, Baek, and Carpenter’s 2003 study in which students developed their own approach to division with decimals and then shared their strategies with the class closely aligns with the strategies Ashley used. Not only did the researchers find this approach effective in several ways, they also found that this strategy mitigated misunderstandings based on “rule-based reasoning” (p. 322).

To ensure all of her students would be successful with the problem, Ashley asked her students to estimate the quotient first and then work on finding the exact answer. This use of estimation as an effective strategy is found in two studies that focused on students’ number sense (Bonotto, 2005; Graeber & Tirosh, 1990). It was found that estimating first improved students’ understanding of division with decimals (Graeber & Tirosh, 1990) and provided an opportunity to discuss the reasonableness of answers (Bonotto, 2005).

The literature contains studies that highlight the effectiveness of asking students to explore division with decimals using related-problem experiences (Brousseau, Brousseau, & Warfield, 2008; Williams & Copley, 1994). This supports the use of related problem sets as Christy did with the tasks in the three bins and the strategy Ashley planned to use as the unit on division with decimals develops. These studies asked participants to make sense of the problems and the values used in the problems, focusing on the relationship between the dividend, divisor, and quotient or factors and product, whereas Christy asked students to look for patterns only in the quotients of the related problems she used. As the studies indicated, Ashley planned to ask
students to analyze patterns of how “the decimal point moves” when dividing with decimals, giving students an opportunity to internalize the patterns identified.

Studies conducted in the 1980s and 1990s that included division with decimals focused on common misconceptions students have when operating on numbers. While the focus was not on division with decimals, some prevalent misconceptions were highlighted in the literature. Most common was the student misconception that multiplication always gives a larger result, and division always gives a smaller result (Bell, Swan, & Taylor, 1981). Another common misconception with division was that the divisor must be smaller than the dividend and that a divisor must be a whole number (Fischbein, Deri, Nello, & Marino, 1985). This may be based on the model students use when thinking about division (partitive or quotative). If the values of the numbers do not fit in the model a student uses, the operation does not make sense to the student (Graeber & Tirosh, 1990). None of these misconceptions were evident during this study. However, observing only the initial lesson, limits accessibility to the different types of problems students are asked to solve throughout a unit of study.

Several strategies used by teachers were not in the body of literature for division with decimals. These were either social strategies or instructional strategies that would likely be found in the literature, just not in relation to division with decimals.

**Implications**

**For Research**

While this study is only a beginning, it provides insight and highlights the need for further research in the area of division with decimals. These needs include studies with more diversity, a larger number of participants, and varied schools. Also, a study of a unit on division with decimals is needed in order to explore much more than how the concept is introduced.
Finally, further exploration of teaching strategies used when students are learning division with decimals, is needed to clarify and categorize the instruction that is occurring currently in classrooms.

**For Classroom Practice**

The literature supports the development of conceptual understanding, related-number patterns, and estimation. However this was seen in only one of the three classrooms observed. This provides an opportunity for schools and districts to create units that include these key features. Development of units would also provide more continuity in how the topic of division with decimals is taught. Finally, teachers, coaches, and mathematics coordinators should consider what importance to place on locating the decimal point in the quotient before dividing. Should that be the focus of instruction?

**For Teacher Educators**

Pre-service teachers need to develop an understanding of effective strategies for teaching both division and division with decimals. They also need to develop their conceptual understanding of division with decimals, their ability to generalize patterns based on related-number problems, and their understanding of the importance of estimation for developing number sense and “operation sense.”

This study also has professional development implications for in-service teachers. Both Ashley and Christy credit in-service professional learning for their awareness and understanding of effective strategies. Their experience with long-term, imbedded professional learning for in-service teachers suggests this is an important consideration. Ashley’s content knowledge and use of effective strategies developed through in-service professional learning. During the post-lesson interview, she recalled a professional learning experience during her first year teaching.
I remember it was, you know, a semester long course and we would go for different days here and there and we were learning all kinds of technical things about numbers but then we were learning new methods; like that’s the first time I ever saw the partial products method and I was like, ‘Oh! That works!’ . . . it made me think about how I can maybe not have to use my algorithm to come up with an answer for 65 times 5, I could do that by breaking it apart and then I can do it in my head, and that’s a lot quicker.

During her pre-lesson interview Christy commented on the positive influence a consultant hired by the district had on her. “I kind of model the way I teach after her [the consultant]. I just really liked, you know, how she encouraged the kids to work together, the way she worked . . . the room, getting kids to think for themselves.” Christy commented on the effects the consultant had on her pedagogy, but not her content knowledge for teaching mathematics.

At times, Christy and Steve struggled with content knowledge for teaching mathematics. During the post-lesson interview Christy discussed how she wanted her students to move away from strategies they had learned in fourth grade. She feared that if they didn’t move to the standard algorithm for division (instead of strategies that used place-value understanding), the students would be unable to place the decimal correctly. During his lesson, Steve commented on division strategies used in fourth grade, “I haven't done this strategy enough to be completely comfortable [with it].” Ball, Sleep, Boerst, and Bass (2008) describe a knowledge of mathematics specific to elementary teachers as *mathematical knowledge for teaching*.

Skilled mathematics teaching requires more than simply learning how to enact particular pedagogical tasks. It also requires knowing and using mathematics in ways that are distinct from simply doing math oneself. . . . [and] require mathematical knowledge,
reasoning, and skill different from what it takes to do well in a math course as a student or to be good at other jobs that require mathematics.” (p. 461).

Teacher educators need to develop a teacher’s mathematical knowledge for teaching and offer imbedded, long-term, and relevant professional learning.

While we can place division with decimals under a microscope and focus in on how students understand and perform. The true understanding can only be determined over time, when students are faced with ever more difficult problems and concepts that require the use of division with decimals.

References


Teacher and Student Interview Questions

Teacher interview questions (pre-lesson):

- What are your beliefs about mathematics teaching and learning?
- What is the role of the teacher during mathematics instruction?
- How should students learn mathematics?
- What teacher resources, such as curricula, tools, and manipulatives, will you use to teach division with decimals?
- How do you plan to introduce division with decimals to your students? Why did you choose that method/strategy?
- What prior knowledge do you believe students should have to help understand division with decimals?
- Do you anticipate any students will have difficulties with understanding division with decimals? If so, what are your plans for helping all students understand this topic?

Teacher interview questions (post-lesson):

- What was particularly effective about the lesson?
- What could be improved or changed next time?
- What do you think students learned?
- What did students easily understand during the lesson?
- What did students struggle to learn during the lesson?
- What happened during the lesson that was expected?
- What happened during the lesson that was unexpected?
- What are your next steps regarding instruction of division with decimals?
• Is there anything else about the lesson introducing division with decimals that I need to know?

Students interview questions (post-lesson):

• What did you teacher do to help you learn about division with decimals? What was hard? What was easy?

• What do you know about division with decimals?

• Can you explain what you did? (Looking at student’s work from study lesson.)