Jackknife Empirical Likelihood Inferences for the Skewness and Kurtosis

JACKKNIFE EMPIRICAL LIKELIHOOD INFERENCES FOR THE SKEWNESS AND KURTOSIS

by

YAN ZHANG

Under the Direction of Dr. Yichuan Zhao

ABSTRACT

Skewness and kurtosis are measures used to describe shape characteristics of distributions. In this thesis, we examine the interval estimates about the skewness and kurtosis by using jackknife empirical likelihood (JEL), adjusted JEL, extended JEL, traditional bootstrap, percentile bootstrap, and BCa bootstrap methods. The limiting distribution of the JEL ratio is the standard chi-squared distribution. The simulation study of this thesis makes a comparison of different methods in terms of the coverage probabilities and interval lengths under the standard normal distribution and exponential distribution. The proposed adjusted JEL and extended JEL perform better than the other methods. Finally we illustrate the proposed JEL methods and different bootstrap methods with three real data sets.

INDEX WORDS: Skewness, Kurtosis, Empirical likelihood, Jackknife empirical likelihood, Adjusted jackknife empirical likelihood, Extended jackknife empirical likelihood, Bootstrap, Bootstrap percentile, Bootstrap BCa, Coverage probability, Interval length

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CHAPTER 1

INTRODUCTION

Skewness and kurtosis are measurements which are used to describe the shape characteristics of a distribution. Skewness is a measure of symmetry, and kurtosis is a measure of whether the data are peaked or flat relative to a normal distribution. The data set will have a distinct peak near the mean, decline rather rapidly, and have heavy tails when the kurtosis is large. Balanda and MacGillivray (1988) suggested a vague concept for skewness and kurtosis. Wilcox (1990) used skewness and kurtosis in tests of normality and in studies of robustness in normal theory procedures. The kurtosis depends on peakedness near the center and tail weight. The influence function (IF) which was proposed by Hampel (1968) suggests a quantitative understanding of kurtosis. It reveals accurately how kurtosis changes with slight deviation from the Gaussian distribution.

In theory and statistics, bootstrap is used as a resampling method to get a more accurate result. The bootstrap, which was inspired by earlier work on the jackknife, was first introduced by Efron (1979). "The bootstrap is a data-based simulation method for statistical inference, which involves repeatedly drawing random samples from the original data, with replacement" [see Ankarali et al. (2009)]. The bootstrap method is a resampling technique that allows estimation of almost any sampling distributions in statistics. One advantage of the bootstrap is that it derives estimates of variance and confidence intervals for complex estimators of parameters of interest.

Empirical likelihood (EL) is an inference method in statistics. EL method was first used by Thomas and Grunkemeier (1975) for constructing confidence intervals and was introduced by Owen (1988), who looked into the relationship between EL and non-parametric statistics. EL can deal with the independent and identically distributed (iid) data well and also performs well with the asymmetric distribution, which was first used by Thomas and Grunkemeier (1975) for constructing confidence intervals for survival functions with censored data. Recently, based on the asymptotic χ^2 distribution of empirical likelihood ratio statistics, more and more important research results of the EL method have been developed.

In statistics, the empirical distribution function is the empirical estimate of the cumulative distribution function (CDF), which is a step function jumping up by $1/n$ at each of the n data points. According to the Gilvenko-Cantelli theorem, it estimates the true underlying cumulative distribution function of the points in the sample and converges to distribution function with probability 1. Let $X_1, X_2, ..., X_n$ be independent and identically distributed (iid) real random variables with common cumulative distribution function $F(t)$. Then the empirical distribution function is defined as

$$F_n(t) = \frac{1}{n}\sum_{i=1}^{n} I[x_i \leq t], \tag{1.1}$$

where I_A is the indicator random variable. It is equal to 1 when the property A holds, and equal to 0 otherwise. Appealing to the Law of Large Numbers, the empirical distribution function $F_n(t)$ accurately estimates the true distribution $F(t)$.

Owen (1988) and Owen (1990) introduced the empirical likelihood (EL). It is used to determine the shape of the confidence intervals without estimating the variance [see Bouadoumou et al. (2014)]. We review it as follows. Suppose we have an independent identically distributed sample of $(U_1, ..., U_n)$ random variables. The objective of the empirical likelihood is the construction of tests and confidence intervals for the parameter $\theta = E[U_i]$. Based on Owen (2001), at the θ, the empirical likelihood is defined by

$$L(\theta) = \max\left\{\prod_{i=1}^{n} p_i : \sum_{i=1}^{n} p_i = 1, \sum P_i U_i = \theta, P_i > 0\right\}.$$

The profile empirical likelihood ratio function for θ can be rewritten as

$$R(\theta)=\frac{L(\theta)}{n^{-n}}=\max\left\{\prod_{i=1}^{n} np_i : \sum_{i=1}^{n} p_i=1, \sum P_i U_i=\theta, P_i>0\right\}.$$

Based on the Lagrange multipliers method, we have

$$p_i=\frac{1}{n}\frac{1}{1+\lambda\left(U_i-\theta\right)},$$

where λ satisfies

$$f(\lambda)\equiv\frac{1}{n}\sum_{i=1}^{n}\frac{U_i-\theta}{1+\lambda(U_i-\theta)}=0.$$

The Wilks' theorem holds. For the skewness and kurtosis, the estimators are nonlinear functions. The standard EL leads to the scaled chi-squared distribution. We need to estimate the scale factor by a simulation study. "When EL is applied to more complicated statistics such as U-statistics, it runs into serious computational difficulties" [see Bouadoumou et al. (2014) about JEL for the ATF model]. Jing et al. (2009) proposed the jackknife EL method for U-statistics. These proposed JEL methods determined some improvements when compared with the current EL methods based on computational issues [see Yang and Zhao (2013)]. Yang and Zhao (2013) proved that the smoothed jackknife empirical log likelihood ratio for the difference of 2 ROC curves is asymptotically chi-squared distributed. Their method can be adapted to the skewness and kurtosis.

The organization of this thesis is as follows. In Chapter 2, we will review some basic concepts of skewness and kurtosis. Three kinds of bootstrap methods are proposed for interval estimates. We will also introduce jackknife empirical likelihood (JEL) method, adjusted jackknife empirical likelihood (AJEL) method, and extended jackknife empirical likelihood (EJEL) method.

In Chapter 3, we will carry out the results of simulation studies. Three methods including jackknife empirical likelihood (JEL), adjusted jackknife empirical likelihood (AJEL), and extended jackknife empirical likelihood (EJEL) will be compared with the nonparametric bootstrap, bootstrap percentile, and bootstrap BCa methods in terms of coverage probability and average length of confidence intervals under the standard normal distribution and exponential distribution.

In Chapter 4, we make a conclusion of this thesis and discuss some disadvantages of the study. In addition, we give some insights for future work.

CHAPTER 2

METHODOLOGY

2.1 Skewness and Kurtosis

The skewness of a random variable X is the third standardized moment, which is defined as

$$g_1 = E[(\frac{X-\mu}{\sigma})^3] = \frac{\mu_3}{\sigma^3} = \frac{E[(X-\mu)^3]}{(E[(X-\mu)^2])^{3/2}}, \tag{2.1}$$

where E is the expectation, μ_3 is the third central moment, and σ is the standard deviation.

The kurtosis is defined as

$$g_2 = \frac{E[(X-\mu)^4]}{(E[(X-\mu)^2])^2} - 3 = \frac{\mu_4}{\sigma^4} - 3, \tag{2.2}$$

where μ_4 is the fourth moment about the mean and σ is the standard deviation.

The traditional measures of skewness g_1 and kurtosis g_2 are proposed by Cramer (1946). They have been compared with various other measures, which are adopted by SAS and MINITAB. For a sample size n, Cramer (1946) proposed the sample skewness to estimate g_1

$$\widehat{g_1} = \frac{m_3}{m_2^{3/2}} = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^3}{(\frac{1}{n}\sum_{i=1}^{n}(x_i-\overline{x})^2)^{3/2}}, \tag{2.3}$$

where $\bar{x}$ is the sample mean, m_3 is the sample third central moment, and m_2 is the sample variance. When the second and third cumulants are infinite, the skewness is undefined.

The variance of the skewness estimate of a sample of size n from a normal distribution is approximately equal to

$$Var(\widehat{g_1}) = \frac{6n(n-1)}{(n-2)(n+1)(n+3)}. \tag{2.4}$$

For a sample size n, the sample kurtosis is defined as follows

$$\widehat{g_2} = \frac{m_4}{m_2^2} - 3 = \frac{\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^4}{(\frac{1}{n}\sum_{i=1}^{n}(x_i - \overline{x})^2)^2} - 3, \tag{2.5}$$

where the m_4 is the fourth sample moment about the mean and m_2 is the second sample moment about the mean.

The variance of the sample kurtosis of a sample size n from the normal distribution is approximately equal to

$$Var(\widehat{g_2}) = \frac{24n(n-1)^2}{(n-3)(n-2)(n+5)(n+3)}. \tag{2.6}$$

2.2 Proposed bootstrap methods for the skewness and kurtosis

In practice, it is unknown if the population is normal or skewed. Hence we cannot use the variance estimators mentioned above. Let θ denote the skewness g_1 or kurtosis g_2. The bootstrap is defined in statistics as, an approach for assigning degrees of accuracy to sample estimates. Bootstrapping lets estimation of the sampling distribution of essentially any statistic using alternative techniques. Typically, this technique is part of the resampling method family. This family includes bootstrapping, jackknifing, and permutation tests.

The bootstrap method uses the original sample of the population and draws a large number B of bootstrap samples with replacement from the original sample. The bootstrap sample has n observations as the original sample that some observations show few times and some do not ever show. In this thesis, we do $B = 400$ replications. The 400 samples with replacement would be $\{x_{1,1}^*, x_{2,1}^*, ..., x_{n,1}^*\}$, $\{x_{1,2}^*, x_{2,2}^*, ..., x_{n,2}^*\}$, ..., $\{x_{1,400}^*, x_{2,400}^*, ..., x_{n,400}^*\}$. The estimate of θ for each bootstrap sample would be $\left\{\widehat{\theta}_1^*, \widehat{\theta}_2^*, ..., \widehat{\theta}_B^*\right\}$. According to DiCiccio and Efron (1996), the $1 - \alpha$ nonparametric bootstrap confidence interval is defined by

$$\widehat{\theta}^* \pm z_{1-\alpha/2}\hat{SE}, \tag{2.7}$$

where the standard error $\hat{SE}$ of the estimator $\hat{\theta}$ is defined as

$$\hat{SE} = \sqrt{\frac{1}{B-1}\sum_{B=1}^{B}(\hat{\theta}^*_B - \hat{\theta}^*)^2}, \tag{2.8}$$

where $\hat{\theta}^* = \frac{1}{B}\sum_{B=1}^{B}\hat{\theta}^*_B$.

The bootstrap percentile method is a very simple alternative method for constructing a bootstrap confidence interval. The advantage of it is the computational efficiency. We order the $B = 400$ values of the bootstrap replications as $\left\{\widehat{\theta}^*_1 < \widehat{\theta}^*_2 < .... < \widehat{\theta}^*_B\right\}$. The ordered element $B * \alpha/2 - th$ is the lower bound, while the ordered element $B * (1-\alpha/2) - th$ is the upper bound. Based on DiCiccio and Efron (1996), the $1-\alpha$ bootstrap percentile confidence interval for θ would be:

$$[\widehat{\theta}^*_{\alpha/2}, \widehat{\theta}^*_{1-\alpha/2}]. \tag{2.9}$$

However, some sample statistics are biased estimators of their corresponding population parameters [see DiCiccio and Efron (1996) and Efron (1979)]. The standard error of an estimate of θ may not be independent of the value of θ. Therefore, unbiased lower and upper percentile cut-offs may not be the same number of standard-error units from $\widehat{\theta}$ [see DiCiccio and Efron (1996)].

The bias corrected and accelerated (BCa) bootstrap method was introduced by Efron (1987). It adjusts the percentile cut-offs in the distribution of the resampled $\widehat{\theta}^*$ for both bias and for the rate of change. The coverage error for the BCa bootstrap method goes to zero at a rate of $1/n$ when the sample size n increases. According to Efron and Tibshirani (1994), "the Monte Carlo research has shown that BCa intervals yield small coverage error for means, medians, and variances." The boot.ci function, which was written by Canty and Ripley (2012), is an implementation of BCa bootstrap method. The BCa method produces smaller coverage error, which is considered as its advantage [see Efron and Tibshirani (1994)].

According to Wang and Zhao (2009), "The bootstrap BCa method adjusts the percentiles selected from the bootstrap percentile method to be the endpoints of the confidence intervals." We order the $B = 400$ values of the bootstrap replications as

$\left\{\widehat{\theta}_1^* < \widehat{\theta}_2^* < .... < \widehat{\theta}_B^*\right\}$. Based on Efron and Tibshirani (1986), Efron (1987), and Carpenter and Bithell (2000), the ordered element $B * \alpha_L - th$ is the lower bound, while the ordered element $B * \alpha_U - th$ is the upper bound. α_L and α_U are the adjusted percentiles of the bootstrap replicates $\widehat{\theta}^*$ [see Wang and Zhao (2009)]. The $1 - \alpha$ bootstrap BCa confidence interval for θ is shown as follows:

$$[\widehat{\theta}^*_{\alpha_L}, \widehat{\theta}^*_{\alpha_U}]. \tag{2.10}$$

The values α_L and α_U are given as:

$$\alpha_L = \Phi(\frac{z_0 + z_{\alpha/2}}{1 - a(z_0 + az_{\alpha/2})} + z_0), \tag{2.11}$$

and

$$\alpha_U = \Phi(\frac{z_0 + z_{1-\alpha/2}}{1 - a(z_0 + az_{1-\alpha/2})} + z_0), \tag{2.12}$$

where Φ denotes the standard normal cumulative distribution function and

$$z_0 = \Phi^{-1}\left(\#\left\{\widehat{\theta}_b^* \leq \widehat{\theta}\right\}/B\right), \tag{2.13}$$

here b=1,2,...,B. z_0 is used to adjust for the bias of the estimator $\widehat{\theta}$ [see Wang and Zhao (2009)]. Based on Carpenter and Bithell (2000), the value a is obtained by

$$a = \frac{\sum(\widehat{\theta}_{()} - \widehat{\theta}_{-i})^3}{6[\sum(\widehat{\theta}_{()} - \widehat{\theta}_{-i})^3]^{3/2}}, \tag{2.14}$$

where $\widehat{\theta}_{-i}$ is the estimate of θ computed without the i^{th} observation. $\widehat{\theta}_{()}$ is the mean of the $\widehat{\theta}_{-i}$ values. When a=0 and z_0=0, there is no difference between the BCa method and the percentile method.

2.3 Jackknife Empirical Likelihood

According to Jing et al. (2009), the jackknife empirical likelihood (JEL) method combines two nonparametric approaches: jackknife method and empirical likelihood method. The jackknife method was invented by Quenouille (1956) and developed further by Tukey (1958). The key steps and the general context of the JEL method are given as follows. The consistent estimator of the parameter θ, which denotes the skewness g_1 or kurtosis g_2, is given by

$$T_n = T\left(Z_1, ..., Z_n\right). \tag{2.15}$$

The jackknife pseudo-values function is defined as:

$$\widehat{V}_i = nT_n - (n-1)T_{n-1}^{(-i)},\ i = 1, ..., n, \tag{2.16}$$

where $T_{n-1}^{(-i)}$ is computed from the original data set by removing the i-th observation, i. e,

$$T_{n-1}^{(-i)} := T(Z_1, ..., Z_{i-1}, Z_{i+1}, ..., Z_n). \tag{2.17}$$

The jackknife estimator $\widehat{T}_{n,jack}$ of θ is the average of all the pseudo-values

$$\widehat{T}_{n,jack} := \frac{1}{n}\sum_{i=1}^{n}\widehat{V}_i. \tag{2.18}$$

The estimators T_n and $\widehat{T}_{n,jack}$ do not differ much. Based on Owen (1988), Owen (1990), and Jing et al. (2009), we have the estimator θ evaluated by the function $L(\theta)$

$$L(\theta) = \max\left\{\prod_{i=1}^{n} p_i : \sum_{i=1}^{n} p_i\widehat{V}_i = \theta, \sum_{i=1}^{n} p_i = 1, p_i \geq 0\right\} \tag{2.19}$$

where $\sum_{i=1}^{n} p_i = 1, p_i \geq 0$. So the jackknife empirical likelihood ratio at θ is as follows:

$$R\left(\theta\right)=\frac{L(\theta)}{n^{-n}}=\max\left\{\prod_{i=1}^{n}np_i:\sum_{i=1}^{n}p_i\widehat{V}_i=\theta,\sum_{i=1}^{n}p_i=1,p_i\geq 0\right\}. \tag{2.20}$$

We use the Lagrange multipliers method to get

$$p_i=\frac{1}{n}\frac{1}{1+\lambda(\widehat{V}_i-\theta)}, \tag{2.21}$$

and λ satisfies the following nonlinear equation

$$f(\lambda)\equiv\frac{1}{n}\sum_{i=1}^{n}\frac{\widehat{V}_i-\theta}{1+\lambda(\widehat{V}_i-\theta)}=0. \tag{2.22}$$

We plug p_i into $R(\theta)$. We have the nonparametric jackknife empirical log-likelihood ratio, which is

$$logR(\theta)=-\sum_{i=1}^{n}log\left\{1+\lambda(\widehat{V}_i-\theta)\right\}.$$

Then we can get

$$l(\theta)=-2logR(\theta) \tag{2.23}$$

Let θ_0 be the true value of θ. We have the following Wilks' theorem using the technique given by Jing et al. (2009). We display the following regularity conditions which are $\mu_3=E[(X-\mu)^3]<\infty$, $\mu_4=E[(X-\mu)^4]<\infty$, and $\sigma\neq 0$.

Theorem 1: Under the regularity conditions, $l(\theta_0)\xrightarrow{d}\chi^2$, where χ^2 is a chi-square random variable with 1 degree of freedom.

An asymptotic 100(1-α)% JEL confidence interval can be constructed as follows:

$$R=\left\{\theta:l(\theta)\leq\chi^2(\alpha)\right\}, \tag{2.24}$$

where $\chi^2(\alpha)$ is the upper α quantile of χ^2 distribution.

2.4 Adjusted Jackknife Empirical Likelihood

Chen et al. (2008) proposed an adjusted empirical likelihood by adding a good point to make the shape data better. It performs better than the original EL method since it reduces the amount of deviation. In this thesis, we investigate adjusted jackknife empirical likelihood (AJEL) method. One of the advantages is that the AJEL method can avoid convex hull restriction for the jackknife empirical likelihood. We let θ denote the skewness g_1 or kurtosis g_2, respectively. Then the adjusted jackknife empirical likelihood at θ is given by

$$L(\theta) = max\left\{\prod_{i=1}^{n+1} P_i, \sum_{i=1}^{n+1} P_i g_i^{ad}(\theta) = 0, \sum_{i=1}^{n+1} P_i = 1, P_i > 0\right\}, \tag{2.25}$$

here $i = 1, 2, .., n$ and $g_i^{ad}(\theta) = \widehat{V}_i - \theta$, $g_{n+1}^{ad}(\theta) = -a_n \bar{g}_n(\theta)$, where $a_n = \max(1, log(n)/2)$ was proposed by Chen et al. (2008), and $\bar{g}_n(\theta)$ is given by

$$\overline{\mathrm{g}}_{\mathrm{n}}(\theta) = \frac{1}{\mathrm{n}} \sum_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{g}_{\mathrm{i}}(\theta). \tag{2.26}$$

The adjusted jackknife empirical likelihood at θ is defined as:

$$\mathrm{R}^{\mathrm{ad}}(\theta) = \prod_{\mathrm{i}=1}^{\mathrm{n}+1} \left\{(\mathrm{n}+1)\mathrm{p}_{\mathrm{i}}^{\mathrm{ad}}(\theta)\right\}, \tag{2.27}$$

where

$$\mathrm{p}_{\mathrm{i}}^{\mathrm{ad}}(\theta) = \frac{1}{\mathrm{n}+1} \frac{1}{1+\lambda \mathrm{g}_{\mathrm{i}}^{\mathrm{ad}}(\theta)}, \tag{2.28}$$

where i= 1, 2, 3, . . ., n+1, and λ satisfies the following nonlinear equation

$$\mathrm{f}(\lambda) = \sum_{\mathrm{i}=1}^{\mathrm{n}+1} \frac{\mathrm{g}_{\mathrm{i}}^{\mathrm{ad}}(\theta)}{1+\lambda \mathrm{g}_{\mathrm{i}}^{\mathrm{ad}}(\theta)} = 0. \tag{2.29}$$

Next, we plug the equation $\mathrm{p}_{\mathrm{i}}^{\mathrm{ad}}(\theta)$ into equation $\mathrm{R}^{\mathrm{ad}}(\theta)$, then we can get the adjusted jackknife empirical log-likelihood ratio:

$$\mathrm{logR^{ad}}(\theta) = -\sum_{\mathrm{i=1}}^{\mathrm{n+1}} \log(1+\lambda \mathrm{g_i^{ad}}\,(\theta)\,). \tag{2.30}$$

From the results of Chen et al. (2008) and Jing et al. (2009), we obtain the following Wilk's theorem.

Theorem 2: Under the regularity conditions which are $\mu_3 = E[(X-\mu)^3] < \infty$, $\mu_4 = E[(X-\mu)^4] < \infty$, and $\sigma \neq 0$, we have

$$-2logR^{ad}(\theta_0) \xrightarrow{d} \chi_1^2. \tag{2.31}$$

Then using Theorem 2, one asymptotic 100(1-α)% AJEL confidence interval is

$$R^{ad} = \left\{\theta : -2logR^{ad} \leq \chi^2(\alpha)\right\}, \tag{2.32}$$

where $\chi^2(\alpha)$ is the upper α quantile of the χ^2 distribution.

2.5 Extended Jackknife Empirical Likelihood

We let θ denote the skewness g_1 or kurtosis g_2, respectively. In order to avoid the convex hull constraint on the classical EL, Tsao (2013) proposed the extended empirical likelihood for general estimation equations. The method is very general and powerful for the small sample size. It can also improve the coverage accuracy of the EL ratio confidence region to $O(n^{-2})$. Comparing with JEL, we use $h_n^C(\theta)$ instead of the true value of θ for EJEL. Based on Tsao and Wu (2014) and Tsao and Wu (2013), EJEL method broadens the JEL method domain to get passed the constraint and the discrepancy. Since the EJEL has identically shaped curves as the JEL method, it is a more natural generalization [see Tsao and Wu (2014)]. Similar to Tsao (2013), we have

$$h_n^C(\theta) = \widehat{T}_{n,jack} + \gamma(n, l(\theta))(\theta - \widehat{T}_{n,jack}), \tag{2.33}$$

where $\gamma(n, l(\theta))$ is the expansion factor given by Tsao (2013),

$$\gamma\left(n, l\left(\theta\right)\right) = 1 + \frac{l\left(\theta\right)}{2n}. \tag{2.34}$$

The proposed extended jackknife empirical likelihood ratio for θ is defined by

$$R^E(\theta) = \sup\left\{\sum_{i=1}^{n} np_i : \sum_{i=1}^{n} p_i(\widehat{V}_i - h_n^C(\theta)) = 0, \sum_{i=1}^{n} p_i = 1, p_i \geq 0\right\}. \tag{2.35}$$

We have

$$p_i = \frac{1}{n}\frac{1}{1 + \lambda\left[\widehat{V}_i - h_n^C(\theta)\right]}, \tag{2.36}$$

where λ satisfies

$$f(\lambda) \equiv \sum_{i=1}^{n} \frac{\widehat{V}_i - h_n^C(\theta)}{1 + \lambda\left[\widehat{V}_i - h_n^C(\theta)\right]} = 0. \tag{2.37}$$

We plug p_i back into $R^E(\theta)$ and get the extended jackknife empirical log-likelihood ratio

$$l^*\left(\theta\right) = -2logR^E(\theta) = 2\sum_{i=1}^{n} log\left\{1 + \lambda\left[\widehat{V}_i - h_n^C(\theta)\right]\right\}. \tag{2.38}$$

Theorem 3: The regularity conditions are $\mu_3 = E[(X-\mu)^3] < \infty$, $\mu_4 = E[(X-\mu)^4] < \infty$, and $\sigma \neq 0$, $l(\theta_0) \xrightarrow{d} \chi^2$, where χ^2 is a chi-square random variable with 1 degree of freedom.

The extended JEL confidence interval for θ is constructed as follows:

$$R^E = \left\{\theta : l^*(\theta) \leq \chi^2(\alpha)\right\}, \tag{2.39}$$

where $\chi^2(\alpha)$ is defined as before.

CHAPTER 3

SIMULATION STUDY

In this chapter, we report the finite-sample performance of JEL methods for the skewness and kurtosis compared with bootstrap methods under the normal and exponential distributions. There are three JEL methods and three bootstrap methods used to calculate the coverage probability and average length of confidence intervals. For the bootstrap methods, $B = 400$ bootstrap samples with replacement are taken from the population. All simulation results are based on 5000 repetitions.

Table 3.1 - Table 3.8 display the result of coverage probabilities and average lengths for the skewness and kurtosis under the normal and exponential distributions. As the sample size increases, the coverage probability and average length of all methods improve. The JEL methods outperform the bootstrap methods in general. All the methods have better performance under the normal distribution than under the exponential distribution. The bootstrap BCa method does not obtain good results, as we expected.

In terms of coverage probability, the JEL methods outperform the bootstrap methods and keep performing consistently. The original nonparametric bootstrap and bootstrap percentile methods produce results very well with the small sample sizes. We can observe that the coverage probabilities of JEL methods are close to the nominal level $1-\alpha$ as sample sizes increase. The coverage probability for the large sample works well.

In terms of the average length, it is clear that JEL methods have shorter lengths than bootstrap methods do. The adjusted JEL and extended JEL produce the shortest average length of confidence intervals. The bootstrap BCa method has slightly shorter average lengths than another two bootstrap methods. When sample size increases, the average length gets shorter.

Table (3.1) Coverage probability under a normal distribution for the skewness

n	1-α	JEL	AJEL	EJEL	Bootstrap	Percentile	Bca
	99%	97.58%	98.72%	98.54%	97.92%	98.22%	96.72%
30	95%	89.06%	91.80%	93.36%	92.68%	94.84%	91.60%
	90%	82.40%	84.60%	85.14%	87.00%	89.52%	86.02%
	99%	96.90%	97.76%	97.62%	97.64%	98.50%	96.00%
60	95%	90.00%	92.07%	92.96%	92.08%	93.50%	91.32%
	90%	84.70%	84.96%	88.40%	86.12%	87.92%	85.74%
	99%	97.70%	97.80%	98.30%	98.16%	98.26%	96.80%
120	95%	93.38%	94.08%	94.46%	93.08%	93.16%	92.02%
	90%	87.84%	89.24%	89.80%	86.36%	88.18%	85.06%
	99%	98.90%	99.02%	99.02%	98.42%	99.00%	97.78%
240	95%	94.22%	94.58%	94.70%	94.04%	94.18%	93.04%
	90%	89.89%	90.00%	90.04%	87.90%	89.96%	86.32%

Table (3.2) Average length under a normal distribution for the skewness

n	1-α	JEL	AJEL	EJEL	Bootstrap	Percentile	Bca
	99%	1.1729	1.1689	1.1387	1.2794	1.2087	1.2077
30	95%	1.0787	1.0476	1.0354	1.1873	1.1588	1.1357
	90%	0.9543	0.8553	0.8510	1.0477	0.9178	0.8827
	99%	0.9603	0.9538	0.9175	1.1356	1.0422	1.0697
60	95%	0.8536	0.8491	0.8459	0.9825	0.9024	0.8692
	90%	0.7802	0.7453	0.7428	0.8647	0.8119	0.7973
	99%	0.8608	0.8499	0.8301	0.9026	0.8908	0.8648
120	95%	0.6771	0.6492	0.6451	0.7749	0.7512	0.7467
	90%	0.5855	0.5340	0.5305	0.6487	0.6356	0.5963
	99%	0.6588	0.6428	0.6243	0.7066	0.6835	0.6658
240	95%	0.5677	0.5307	0.5191	0.5772	0.5709	0.5688
	90%	0.4454	0.4158	0.4138	0.4858	0.4720	0.4610

Note:

JEL: Jackknife empirical likelihood

AJEL: Adjusted Jackknife empirical likelihood

EJEL: Extended Jackknife empirical likelihood

Percentile: Bootstrap percentile

BCa: Bootstrap BCa

Table (3.3) Coverage probability under a normal distribution for the kurtosis

n	1-α	JEL	AJEL	EJEL	Bootstrap	Percentile	Bca
	99%	95.26%	96.20%	96.72%	90.58%	94.42%	90.84%
30	95%	89.90%	90.16%	90.86%	84.66%	87.22%	84.50%
	90%	85.40%	86.30%	86.88%	78.68%	79.48%	77.90%
	99%	96.42%	97.40%	97.58%	91.16%	95.26%	89.00%
60	95%	90.78%	91.36%	91.64%	84.40%	87.74%	83.98%
	90%	87.40%	88.30%	88.92%	80.92%	81.06%	80.70%
	99%	97.28%	97.84%	98.88%	92.42%	94.48%	90.52%
120	95%	91.40%	92.50%	92.98%	85.36%	88.04%	84.84%
	90%	88.00%	88.84%	88.16%	81.46%	82.46%	80.62%
	99%	98.08%	98.54%	99.80%	94.16%	96.18%	92.48%
240	95%	93.18%	93.30%	93.74%	88.00%	90.62%	86.04%
	90%	88.82%	89.12%	89.30%	82.36%	84.62%	81.50%

Table (3.4) Average length under a normal distribution for the kurtosis

n	1-α	JEL	AJEL	EJEL	Bootstrap	Percentile	Bca
	99%	1.0518	1.0233	1.0339	1.1648	1.1585	1.1399
30	95%	0.8912	0.8657	0.8750	0.9540	0.9353	0.8920
	90%	0.7498	0.7160	0.7391	0.8377	0.7992	0.7696
	99%	0.9512	0.9333	0.9439	0.9854	0.9679	0.9588
60	95%	0.8409	0.8252	0.8398	0.9071	0.8757	0.8609
	90%	0.7111	0.7013	0.7024	0.8396	0.8043	0.7607
	99%	0.7461	0.7345	0.7405	0.8502	0.8318	0.8280
120	95%	0.6193	0.6013	0.6142	0.6591	0.6437	0.6362
	90%	0.5351	0.5252	0.5335	0.5743	0.5509	0.5362
	99%	0.5940	0.5894	0.5925	0.6227	0.6153	0.6032
240	95%	0.4772	0.4702	0.4729	0.5382	0.5067	0.4921
	90%	0.3781	0.3679	0.3725	0.4213	0.4012	0.3970

Note:

JEL: Jackknife empirical likelihood

AJEL: Adjusted Jackknife empirical likelihood

EJEL: Extended Jackknife empirical likelihood

Percentile: Bootstrap percentile

BCa: Bootstrap BCa

Table (3.5) Coverage probability under an exponential distribution for the skewness

n	1-α	JEL	AJEL	EJEL	Bootstrap	Percentile	Bca
	99%	88.22%	88.70%	90.82%	83.84%	84.94%	83.68%
30	95%	78.58%	79.30%	80.18%	78.16%	78.36%	77.28%
	90%	71.70%	73.04%	73.04%	70.30%	71.28%	70.18%
	99%	91.04%	92.32%	92.94%	88.00%	86.24%	87.24%
60	95%	87.16%	88.50%	88.68%	83.48%	84.64%	83.02%
	90%	80.90%	82.42%	83.36%	78.78%	79.38%	78.04%
	99%	95.54%	96.28%	96.60%	91.58%	93.46%	91.34%
120	95%	90.72%	90.68%	91.86%	86.86%	87.12%	86.00%
	90%	84.30%	85.82%	87.10%	80.46%	83.20%	80.10%
	99%	96.08%	96.20%	97.22%	93.18%	95.46%	92.92%
240	95%	91.06%	91.24%	91.76%	88.90%	89.10%	88.70%
	90%	85.60%	86.78%	86.90%	82.36%	84.18%	82.06%

Table (3.6) Average length under an exponential distribution for the skewness

n	1-α	JEL	AJEL	EJEL	Bootstrap	Percentile	Bca
	99%	1.1804	1.1471	1.1205	1.2981	1.2338	1.2283
30	95%	0.8708	0.8946	0.8493	1.0100	0.9325	0.9251
	90%	0.7166	0.7316	0.7054	0.9279	0.8912	0.8951
	99%	0.8597	0.8768	0.8340	0.9427	0.9280	0.9118
60	95%	0.6385	0.6407	0.6221	0.7617	0.7257	0.7008
	90%	0.5687	0.5657	0.5462	0.6557	0.6121	0.6022
	99%	0.5799	0.5950	0.5587	0.6722	0.6170	0.5973
120	95%	0.4990	0.5068	0.4823	0.5558	0.5122	0.5010
	90%	0.4618	0.4732	0.4215	0.4966	0.4720	0.4614
	99%	0.5036	0.5643	0.4917	0.5745	0.5150	0.4946
240	95%	0.4605	0.4871	0.4035	0.5273	0.4621	0.4474
	90%	0.4072	0.4216	0.3654	0.4690	0.4197	0.3943

Note:

JEL: Jackknife empirical likelihood

AJEL: Adjusted Jackknife empirical likelihood

EJEL: Extended Jackknife empirical likelihood

Percentile: Bootstrap percentile

BCa: Bootstrap BCa

Table (3.7) Coverage probability under an exponential distribution for the kurtosis

n	1-α	JEL	AJEL	EJEL	Bootstrap	Percentile	Bca
	99%	85.46%	87.36%	91.40%	80.18%	81.40%	80.44%
30	95%	78.00%	79.72%	83.72%	75.34%	76.24%	75.10%
	90%	73.50%	75.24%	77.06%	70.30%	72.76%	71.08%
	99%	88.54%	89.50%	92.16%	85.74%	86.46%	85.50%
60	95%	80.60%	81.68%	82.00%	80.12%	82.44%	80.02%
	90%	76.14%	77.30%	77.42%	75.98%	76.19%	75.04%
	99%	90.68%	91.24%	92.54%	87.60%	88.24%	87.62%
120	95%	84.72%	84.90%	85.34%	83.74%	84.28%	83.20%
	90%	78.66%	78.97%	80.46%	77.06%	78.38%	76.16%
	99%	93.38%	94.18%	94.40%	89.28%	91.94%	88.82%
240	95%	87.52%	89.14%	90.00%	84.70%	86.50%	84.54%
	90%	81.04%	82.56%	83.72%	80.12%	81.08%	80.20%

Table (3.8) Average length under an exponential distribution for the kurtosis

n	1-α	JEL	AJEL	EJEL	Bootstrap	Percentile	Bca
	99%	0.9894	0.9050	0.9248	1.1498	1.1281	1.1155
30	95%	0.8917	0.8906	0.8823	0.9514	0.9428	0.9162
	90%	0.8175	0.8170	0.8120	0.8712	0.8613	0.8229
	99%	0.8430	0.8268	0.8291	0.9074	0.8897	0.8623
60	95%	0.7767	0.7611	0.7513	0.8602	0.8179	0.7898
	90%	0.6840	0.6657	0.6322	0.7242	0.7189	0.7041
	99%	0.7586	0.7409	0.7190	0.8255	0.7932	0.7369
120	95%	0.6133	0.5994	0.5871	0.6984	0.6728	0.6335
	90%	0.5499	0.5374	0.4231	0.5832	0.5609	0.5517
	99%	0.6763	0.6237	0.6209	0.7291	0.7041	0.6780
240	95%	0.5816	0.5438	0.5266	0.6486	0.6232	0.6133
	90%	0.4941	0.4690	0.4312	0.5364	0.5235	0.5045

Note:

JEL: Jackknife empirical likelihood

AJEL: Adjusted Jackknife empirical likelihood

EJEL: Extended Jackknife empirical likelihood

Percentile: Bootstrap percentile

BCa: Bootstrap BCa

CHAPTER 4

REAL DATA ANALYSIS

In this chapter, we apply the JEL and bootstrap methods to three real data sets, which come from the R dataset package in R program. We calculate the interval length with three different significance levels, α =0.01, 0.05, and 0.1.

The first data set, "Rivers," has 141 observations. This data set gives the lengths (in miles) of 141 major rivers in North America, as compiled by the US Geological Survey. There are 50 observations in the second data set, "LifeCycleSavings," which gives the savings ratio (aggregate personal saving divided by disposable income). The third data set has 100 observations, which are the numbers of users connected to the Internet through a server every minute. We calculated the lower bound, upper bound and length by the JEL, AJEL, EJEL, nonparametric bootstrap, bootstrap percentile, and bootstrap BCa methods.

We apply the Shapiro-Wilk test with the three real data sets so that we can check the normality of them. The null hypothesis of the Shapiro-Wilk test is that the sample data follows the normal distribution. Referencing the Shapiro-Wilk test, we can get the p-value. If the p-value is lower than 0.05, which is a cutoff for the normal distribution, we reject the null hypothesis. If we cannot reject the Shapiro-Wilk null hypothesis for a data set, we will compare its result with Table 3.2 and Table 3.4.

4.1 Rivers Data

Table (4.1) Interval length of confidence intervals of the skewness and kurtosis for the rivers data

Skewness	JEL		AJEL		EJEL		Bootstrap		Percentile		Bca	
1-α	UB	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB	LB
	4.650	1.696	4.622	1.677	4.614	1.682	4.679	1.688	4.682	1.685	4.680	1.682
0.99	Length		Length		Length		Length		Length		Length	
	2.954		2.944		2.932		2.991		2.997		2.999	
	4.533	2.138	4.510	2.138	4.459	2.106	4.464	2.033	4.473	2.035	4.472	2.019
0.95	Length		Length		Length		Length		Length		Length	
	2.395		2.372		2.353		2.431		2.438		2.454	
	4.484	2.356	4.442	2.332	4.408	2.302	4.386	2.208	4.396	2.210	4.392	2.204
0.90	Length		Length		Length		Length		Length		Length	
	2.128		2.110		2.107		2.178		2.186		2.188	
Kurtosis	JEL		AJEL		EJEL		Bootstrap		Percentile		Bca	
1-α	UB	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB	LB
	14.812	12.576	14.807	12.584	14.792	12.596	14.459	12.197	14.461	12.205	14.531	12.275
0.99	Length		Length		Length		Length		Length		Length	
	2.236		2.223		2.196		2.262		2.257		2.256	
	14.561	12.677	14.542	12.686	14.555	12.755	14.247	12.348	14.334	12.450	14.361	12.487
0.95	Length		Length		Length		Length		Length		Length	
	1.884		1.856		1.800		1.899		1.884		1.874	
	14.239	12.719	14.419	12.900	14.357	12.852	13.957	12.409	14.066	12.531	14.049	12.518
0.90	Length		Length		Length		Length		Length		Length	
	1.520		1.519		1.505		1.549		1.535		1.530	

Note:

JEL: Jackknife empirical likelihood

AJEL: Adjusted Jackknife empirical likelihood

EJEL: Extended Jackknife empirical likelihood

Percentile: Bootstrap percentile

BCa: Bootstrap BCa

4.2 LifeCycleSavings Data

Table (4.2) Interval length of confidence intervals of the skewness and kurtosis for the LifeCycleSavings data

Skewness	JEL		AJEL		EJEL		Bootstrap		Percentile		Bca	
1-α	UB	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB	LB
	0.709	-0.641	0.703	-0.631	0.703	-0.629	0.742	-0.653	0.705	-0.686	0.725	-0.660
0.99	Length		Length		Length		Length		Length		Length	
	1.350		1.333		1.332		1.395		1.391		1.385	
	0.495	-0.581	0.488	-0.582	0.478	-0.584	0.516	-0.578	0.561	-0.530	0.567	-0.517
0.95	Length		Length		Length		Length		Length		Length	
	1.075		1.070		1.062		1.094		1.091		1.083	
	0.439	-0.491	0.421	-0.493	0.415	-0.497	0.456	-0.484	0.476	-0.488	0.488	-0.448
0.90	Length		Length		Length		Length		Length		Length	
	0.930		0.914		0.912		0.940		0.964		0.936	
Kurtosis	JEL		AJEL		EJEL		Bootstrap		Percentile		Bca	
1-α	UB	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB	LB
	0.487	-0.967	0.495	-0.955	0.480	-0.957	0.526	-0.953	0.517	-0.943	0.533	-0.926
0.99	Length		Length		Length		Length		Length		Length	
	1.454		1.449		1.437		1.479		1.460		1.458	
	0.313	-0.801	0.311	-0.802	0.309	-0.796	0.368	-0.794	0.356	-0.796	0.354	-0.801
0.95	Length		Length		Length		Length		Length		Length	
	1.114		1.113		1.105		1.162		1.152		1.155	
	0.207	-0.707	0.205	-0.708	0.203	-0.703	0.259	-0.685	0.262	-0.688	0.216	-0.715
0.90	Length		Length		Length		Length		Length		Length	
	0.914		0.912		0.906		0.944		0.950		0.931	

Note:

JEL: Jackknife empirical likelihood

AJEL: Adjusted Jackknife empirical likelihood

EJEL: Extended Jackknife empirical likelihood

Percentile: Bootstrap percentile

BCa: Bootstrap BCa

4.3 WWWusage Data

Table (4.3) Interval length of confidence intervals of the skewness and kurtosis for the WWWusage data

Skewness	JEL		AJEL		EJEL		Bootstrap		Percentile		Bca	
1-α	UB	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB	LB
	0.727	0.031	0.726	0.031	0.725	0.031	0.741	0.025	0.744	0.023	0.749	0.023
0.99	Length		Length		Length		Length		Length		Length	
	0.696		0.695		0.694		0.716		0.721		0.726	
	0.627	0.154	0.619	0.159	0.617	0.161	0.605	0.128	0.615	0.132	0.640	0.154
0.95	Length		Length		Length		Length		Length		Length	
	0.473		0.460		0.455		0.478		0.483		0.486	
	0.584	0.173	0.579	0.175	0.575	0.172	0.593	0.153	0.585	0.158	0.592	0.163
0.90	Length		Length		Length		Length		Length		Length	
	0.411		0.405		0.403		0.439		0.427		0.429	
Kurtosis	JEL		AJEL		EJEL		Bootstrap		Percentile		Bca	
1-α	UB	LB	UB	LB	UB	LB	UB	LB	UB	LB	UB	LB
	-0.196	-0.908	-0.201	-0.905	-0.207	-0.908	-0.264	-0.991	-0.258	-0.997	-0.220	-0.945
0.99	Length		Length		Length		Length		Length		Length	
	0.712		0.704		0.701		0.728		0.739		0.725	
	-0.289	-0.841	-0.297	-0.844	-0.309	-0.835	-0.350	-0.906	-0.351	-0.905	-0.303	-0.861
0.95	Length		Length		Length		Length		Length		Length	
	0.552		0.546		0.526		0.556		0.554		0.558	
	-0.377	-0.809	-0.371	-0.805	-0.381	-0.791	-0.391	-0.865	-0.400	-0.855	-0.378	-0.831
0.90	Length		Length		Length		Length		Length		Length	
	0.433		0.434		0.411		0.474		0.455		0.454	

Note:

JEL: Jackknife empirical likelihood

AJEL: Adjusted Jackknife empirical likelihood

EJEL: Extended Jackknife empirical likelihood

Percentile: Bootstrap percentile

BCa: Bootstrap BCa

4.4 Conclusion

The shorter interval length means more accurate interval estimate of a parameter. Applying the Shapiro-Wilk test to the Rivers data, the calculated p-value is 2.2e-16, which is strong evidence to reject the Shapiro null hypothesis. We conclude that the Rivers data is not normally distributed. According to Table 4.1, the results of the six methods have a very small difference. The extended JEL method has the shortest length for the skewness and kurtosis, which shows much consistency with the result of simulation study. For the bootstrap methods, the original nonparametric bootstrap method produces better lengths for the skewness and the BCa bootstrap method has shorter lengths for the kurtosis.

For the LifeCycleSavings data, the Shapiro-Wilk test calculates the p-value as 0.5836, which means we can not reject the Shapiro null hypothesis. We can say the LifeCycleSavings data is normally distributed. From Table 4.2, the average lengths are very close to the lengths of Table 3.2 and Table 3.4. In Table 4.2, the results are very similar to each other. The extended JEL method produces more accurate interval estimates than the other five methods. For the bootstrap methods, the bootstrap BCa method has a better performance.

The p-value of the third data is 0.0001325 by using the Shapiro-Wilk test. Thus we can reject the Shapiro null hypothesis. Hence the third data set is not normally distributed. Based on Table 4.3, the results are very close to each other. The bootstrap BCa method produces more accurate interval estimates than the other two bootstrap methods. The lengths calculated by the extended JEL method are always shorter than the lengths of the other five methods.

Consequently, we conclude that the extended JEL method is the most accurate and useful method for interval estimates of the skewness and kurtosis.

CHAPTER 5

SUMMARY AND FUTURE WORK

5.1 Summary

In this thesis, we proposed interval estimates for the skewness and the kurtosis by using JEL, adjusted JEL, extended JEL, original bootstrap, percentile bootstrap, and BCa boostrap methods.

According to the extensive simulation study, we can conclude that the JEL methods are more useful and more accurate than the bootstrap methods under the standard normal distribution and exponential distribution. Table 3.5 and Table 3.7 provide strong evidence that JEL methods perform much better with skewed distribution of data sets in terms of coverage probability. According to Davison (1997), bootstrap confidence intervals may not perform very well with small sample sizes. Bootstrap confidence intervals rely much on sample values in the tails of the sample distribution. So the coverage probability of bootstrap methods for small sample sizes may still differ substantially from the nominal level 1-α [see Davison (1997)]. Since the adjusted cut-offs can move further into the tails of a distribution, the BCa boostrap method may not calculate better results compared with the original and percentile boostrap methods.

The JEL methods produce better coverage probabilities than the bootstrap method most of the time with small sample sizes. In addition, the JEL methods lead to shorter average lengths than the bootstrap methods which means JEL methods are more accurate. For the real data analysis, the JEL methods calculate shorter interval lengths than the bootstrap methods. With the JEL methods, high accurate estimators can be produced with small sample size. We conclude that the JEL methods provide better interval estimates of the skewness and kurtosis compared to the boostrap methods.

5.2 Future Work

As we mentioned above, we know the JEL, adjusted JEL, and extended JEL are highly accurate and useful. However, in this thesis, the extended JEL method does not have the best performance on all average lengths. From Tsao (2013) the extended JEL has better performance than other methods, no matter if the sample size is small or large. The bootstrap BCa method does not have good results, as we expected. The larger number B for replications may help to achieve better results. Therefore, we should try to tackle those problems in the future.

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