Regularity Criteria via the Frobenius Euler Characteristic

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by

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Under the Direction of Yongwei Yao, Ph.D.

A Dissertation Submitted in Partial Fulfillment of the Requirements for the Degree of

Doctor of Philosophy

in the College of Arts and Sciences

Georgia State University

2024

ABSTRACT

This dissertation explores the use of the Frobenius endomorphism to detect the regularity of a ring within the framework of commutative algebra. In commutative local rings of prime characteristic, invariants such as the Hilbert-Kunz multiplicity, F-signature, and Frobenius Betti numbers have been shown to detect regularity. Polstra and Smirnov [PS21] showed that the Frobenius Euler characteristic can be used to determine regularity for the class of strongly F-regular rings. Here, we extend their results by relaxing the condition of strong F-regularity. We show that the Frobenius Euler characteristic can detect regularity for rings with sufficient F-splitting and for rings that are Cohen-Macaulay.

INDEX WORDS: Commutative algebra, Noetherian, Cohen-Macaulay, Frobenius, F-finite, Frobenius Euler characteristic, Frobenius Betti numbers, F-splitting, regular

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August 2024

DEDICATION

This dissertation is dedicated to my parents, Michael and Karla, and to my wife, Taylor.

ACKNOWLEDGMENTS

First and foremost, I would like to express my deepest gratitude to my advisor, Dr. Yongwei Yao, for his unwavering support, guidance, patience, and encouragement throughout my entire graduate career at Georgia State University. I will be forever thankful for his support.

I am also immensely grateful to the members of my dissertation committee, Dr. Florian Enescu, Dr. Zongshan Li, and Dr. Mark Grinshpon, for their constructive feedback and suggestions, which have significantly improved the quality of this work.

I would like to acknowledge the financial support from Georgia State University, which made this research possible. I have had a wonderful experience working as a Graduate Lab, Teaching, and Research Assistant in the department. Thank you to all the instructors I have worked under, who have made it an incredibly enjoyable experience. I would like to particularly thank Dr. Mark Grinshpon, Dr. Kelvin Rozier, Dr. Michael Stewart, Dr. Mikhail Stroev, and Sutandra Sarkar for all of their mentorship and support.

On a personal note, I am deeply thankful to my family for their unconditional love, patience, and understanding. Their belief in my abilities has been a constant source of strength. I owe my heartfelt thanks to my wife, Taylor, for her endless support and encouragement during the challenging times of this journey.

This dissertation is dedicated to everyone who has played a part in my academic and personal growth. Thank you.

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CHAPTER 1
INTRODUCTION

Throughout this dissertation, all rings are assumed to be Noetherian, commutative rings with 1, unless otherwise noted. We use the notation $\subset$ to denote a strict inclusion, while $\subseteq$ has the possibility of equality. We use the notation $(R, \mathfrak{m})$ to indicate a local ring with unique maximal ideal $\mathfrak{m}$ and $(R, \mathfrak{m}, k)$ to further specify that $k := R/\mathfrak{m}$ is the residue field of R. We also use the convention that $\mathbb{N} = \{0, 1, 2, \dots\}$, while $\mathbb{N}_+ = \{1, 2, \dots\}$.

Let R be a ring of prime characteristic p. The Frobenius map, denoted by F, is a ring endomorphism that sends each element $r \in R$ to its pth power, i.e., $F(r) = r^p$. Various invariants associated with the Frobenius map have been developed to detect regularity. The works of Watanabe-Yoshida, Huneke-Yao, Huneke-Leuscke, and Aberbach-Li has shown that the Hilbert-Kunz multiplicity, F-signature, and Frobenius Betti numbers can all be used to detect regular rings.

If R is a regular local ring of prime characteristic and M is a finitely generated R-module, then $\chi_j^F(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$, where $\chi_j^F(-)$ denotes the jth Frobenius Euler characteristic and $\mathrm{e}_*(-)$ denotes either the Hilbert-Samuel or Hilbert Kunz multiplicity. We conjecture that in converse of this statement holds as well, under mild conditions:

Conjecture 4.0.2. *Let $(R, \mathfrak{m})$ be an unmixed local ring of prime characteristic p, and let M be a finitely generated R-module such that* $\operatorname{Assh}(M) = \operatorname{Assh}(R)$. *If $\chi_j^F(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$ for some $j \in \mathbb{N}$, then R is regular.*

Polstra and Smirnov demonstrated that the Frobenius Euler characteristic can be used

to determine regularity within strongly F-regular rings, and they conjectured that it could be proven under a weaker hypothesis. Specifically, they showed the following:

Theorem 3.5.4 ([PS21, Theorem 3.23])**.** *Let $(R, \mathfrak{m})$ be an F-finite, strongly F-regular ring of prime characteristic p. The following are equivalent:*

(1) *R is a regular local ring.*

(2) *$\chi_j({}^eR) = (-1)^j \operatorname{rank}({}^eR)$ for every $e \in \mathbb{N}_+$.*

(3) *$\chi_j({}^eR) = (-1)^j \operatorname{rank}({}^eR)$ for some $e \in \mathbb{N}_+$.*

(4) *$\chi_j^F(R) = (-1)^j$.*

The organization of this dissertation is as follows. In Chapter 2, we cover basic notions. In Section 2.1, we provide a review of the necessary background on homological algebra, Cohen-Macaulay rings and modules, and regular local rings. In Section 2.2, we review topics of rings in prime characteristic, tight closure, and the variants of F-regularity. In Chapter 3, provide an overview of many invariants, including the Hilbert-Samuel multiplicity, Hilbert-Kunz multiplicity, F-signature, Frobenius Betti numbers, and Frobenius Euler characteristic.

In Chapter 4, we present our main results. To start, we show that if Conjecture 4.0.2 holds for $M = R$, then it must hold for any finitely generated R-module M:

Theorem 4.1.6. *Let $(R, \mathfrak{m})$ be an unmixed local ring of prime characteristic p, and let $j \in \mathbb{N}$. The following implications are equivalent:*

(1) *R is regular if and only if $\chi_j^F(R) = (-1)^j$.*

(2) *R is regular if and only if $\chi_j^F(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$ for some finitely generated R-module M with $\operatorname{Assh}(M) = \operatorname{Assh}(R)$.*

(3) *R is regular if and only if $\chi^F_j(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$ for all finitely generated R-modules M.*

We then demonstrate how the Frobenius Euler characteristic can detect regularity in certain classes of rings. One such class includes rings with modules that possess a property we refer to as "sufficient F-splitting". This leads us to the following result:

Theorem 4.2.9. *Let $(R, \mathfrak{m})$ be an unmixed, F-finite, local ring of prime characteristic p, and let M be a finitely generated R-module such that M has sufficient F-splitting. If $\chi^F_j(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$ for some $j \in \mathbb{N}$, then R is regular.*

The other class of rings we cover are rings with Cohen-Macaulay modules. To do so, we go through the Frobenius Betti numbers via a theorem due to Aberbach and Li:

Theorem 3.4.5 ([AL08, Corollary 3.2])**.** *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p. The following are equivalent:*

(1) *R is regular.*

(2) *$\beta^F_j(R) = 0$ for all $j \in \mathbb{N}_+$.*

(3) *$\beta^F_j(R) = 0$ for some $j \in \mathbb{N}_+$.*

The above result enables us to demonstrate the following:

Theorem 4.3.1. *Let $(R, \mathfrak{m})$ be an unmixed Cohen-Macaulay local ring of prime characteristic p, let M be a finitely generated R-module such that $\operatorname{Assh}(M) = \operatorname{Assh}(R)$, and let $j \in \mathbb{N}$. If $\chi^F_j(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$, then R is regular.*

Both Theorem 4.2.9 and Theorem 4.3.1 recover the Theorem 3.5.4, as strongly F-regular do indeed have sufficient F-spltiting and are Cohen-Macaulay.

CHAPTER 2
PRELIMINARIES

In this chapter, we review and present the fundamental background from commutative algebra necessary for this dissertation.

2.1 Basic notions

In this section, we will outline some basic notions and conventions that will be used throughout this dissertation.

Let R be a ring. The *characteristic* of R, denoted $\text{char}(R)$, is the smallest positive integer n such that $n \cdot 1_R = 0$. If no such n exists, then the characteristic is zero. If an ideal I is generated by the elements $r_1, \ldots, r_n \in R$, then we write $I = (r_1, \ldots, r_n)$. Similarly, if an R-module M is generated by the elements $m_1, \ldots, m_n \in M$, then we write $M = (m_1, \ldots, m_n)$.

The set of all prime ideals of R is denoted by $\text{Spec}(R)$. The *height* of an ideal I, denoted $\text{ht}(I)$, is the supremum of lengths of chains of prime ideals contained in I. The *dimension* of R is given by $\dim(R) := \sup\{\text{ht}(\mathfrak{p}) \mid \mathfrak{p} \in \text{Spec}(R)\}$. More generally, given an R-module M, dimension of M, denoted $\dim(M)$, or $\dim_R(M)$ if we need to specify the ring, is $\dim(R/\text{Ann}(M)) = \dim(\text{Supp}(M))$.

For a ring R, the *radical* of an ideal $I \subset R$ is the set of all elements $r \in R$ such that $r^n \in I$ for some positive integer n. The *nilradical* of R is $\text{nil}(R) := \sqrt{(0)}$, and if $\text{nil}(R) = \{0\}$, we say that R is *reduced*. If $\sqrt{I} = \mathfrak{m}$, where $\mathfrak{m}$ is a maximal ideal of R, then we say that I is $\mathfrak{m}$-*primary*.

We denote R° as the set of elements in R that are not in any minimal prime of R. If R

is a ring of prime characteristic p and $e \in \mathbb{N}$, then for any ideal I of R, we denote $I^{[p^e]}$ as the ideal generated by $\{x^{p^e} \mid x \in I\}$.

Given a module M over a ring R, a chain of submodules of M of the form $M_0 \subset \cdots \subset M_n$ is said to have length n. The *length* of M, denoted $\ell_R(M)$, is the length of the largest chain of submodules, and if no such largest chain exists, then it is said to have infinite length.

A local ring $(R, \mathfrak{m})$ is said to be *equidimensional* (resp. *unmixed*) if $\dim(\widehat{R}/\mathfrak{p}) = \dim(\widehat{R})$ for all minimal (resp. associated) primes $\mathfrak{p}$. Given a module M over a ring R, the set of *associated prime ideals of* M, denoted $\operatorname{Ass}(M)$, is the set of all $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\mathfrak{p} = \operatorname{Ann}(m)$ for some $m \in M$. We denote the set of all prime ideals of R of highest dimension by $\operatorname{Assh}(R) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \dim(R/\mathfrak{p}) = \dim(R)\}$. More generally, for an R-module M, we denote $\operatorname{Assh}_R(M) := \{\mathfrak{p} \in \operatorname{Supp}_R(M) \mid \dim(R/\mathfrak{p}) = \dim(M)\}$.

A module M over a ring R is said to be *flat* if taking tensor product of M with any exact sequence over R remains exact. We say that M is *faithfully flat* if the tensor product with M preserves exactness of a sequence if and only if the original sequence is exact. A ring map $R \to S$ is (*faithfully*) *flat* if S is a (faithfully) flat R-module.

Let $(G, +)$ be a commutative monoid with identity element e and M be an R-module. We say that R is *G-graded* if there exists a family of subgroups $\{R_\sigma\}_{\sigma \in G}$ of R such that $R = \bigoplus_{\sigma \in G} R_\sigma$ and $R_\sigma R_\tau \subseteq R_{\sigma+\tau}$ for all $\sigma, \tau \in G$. If M is an R-module, then we say that M is G-graded if there exists a family of subgroups $\{M_\sigma\}_{\sigma \in G}$ of M such that $M = \bigoplus_{\sigma \in G} M_\sigma$ and $R_\sigma M_\tau \subseteq M_{\sigma+\tau}$ for all $\sigma, \tau \in G$. If I is an ideal of R, we say that I is G-graded if it is G-graded as an R-module. If R is $\mathbb{N}$-graded, the *associated graded ring of R with respect to*

I is $\mathrm{gr}_I(R) := \bigoplus_{i=0}^{\infty} I^i/I^{i+1}$, which is induced by the multiplication $I^i \times I^j \to I^{i+j}$.

2.1.1 *Homological algebra*

Homological algebra, a relatively young branch of mathematics, studies homology within algebraic contexts. At its core, it seeks to understand the properties and relationships of algebraic structures through exact sequences, chain complexes, and derived functors such as Tor and Ext.

Definition 2.1.1. A *chain complex*, or simply *complex*, is a sequence of module homomorphisms

$$C_\bullet : \cdots \to C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \to \cdots$$

such that the composition of any two consecutive maps is zero, i.e., $\partial_i \circ \partial_{i+1} = 0$ (or equivalently, $\operatorname{im} \partial_{i+1} \subseteq \ker \partial_i$) for all i. The maps ∂_i are called *differentials*. We write $(C_\bullet, \partial_\bullet)$ if we need to specify the differentials of the complex.

A *cochain complex* is the dual notion to a chain complex. It is sequence of module homomorphisms

$$C^\bullet : \cdots \leftarrow C^{i+1} \xleftarrow{\partial^i} C^i \xleftarrow{\partial^{i-1}} C^{i-1} \leftarrow \cdots$$

such that the composition of any two consecutive maps is zero, i.e., $\partial^i \circ \partial^{i-1} = 0$ (or equivalently, $\operatorname{im} \partial_{i-1} \subseteq \ker \partial_i$) for all i. We write $(C^\bullet, \partial^\bullet)$ if we need to specify the differentials of the complex.

Example 2.1.2.

(1) Every exact sequence is a complex.

(2) If $C_\bullet$ is a complex and $\mathcal{F}$ is a functor, then

$$\mathcal{F}(C_\bullet) : \cdots \to \mathcal{F}(C_i) \xrightarrow{\mathcal{F}(\partial_i)} F(C_{i-1}) \to \dots$$

is also a complex. However, if $C_\bullet$ is an exact sequence, then $\mathcal{F}(C_\bullet)$ need not be an exact sequence.

Definition 2.1.3. Let $(C_\bullet, \partial_\bullet)$ be a chain complex and $(C^\bullet, \partial^\bullet)$ be a cochain complex.

(1) The *ith homology group*, denoted $H_i(C_\bullet)$, is defined as $H_i(C_\bullet) = \ker \partial_i / \operatorname{im} \partial_{i+1}$.

(2) The *ith cohomology group*, denoted $H^i(C^\bullet)$, is defined as $H^i(C^\bullet) = \ker \partial^i / \operatorname{im} \partial^{i-1}$.

Remark 2.1.4. Given a short exact sequence of chain complexes

$$0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0,$$

there exists natural homomorphisms $\partial_i : H_i(C_\bullet) \to H_{i-1}(A_\bullet)$ for every $i \in \mathbb{Z}$ such that

$$\cdots \to H_{i+1}(C_\bullet) \xrightarrow{\partial_{i+1}} H_i(A_\bullet) \to H_i(B_\bullet) \to H_i(C_\bullet) \xrightarrow{\partial_i} H_i(A_\bullet) \to \cdots$$

is exact.

Definition 2.1.5. Let R be a ring and M be an R-module. Let F_i be a free R-module for every $i \in \mathbb{N}$. If

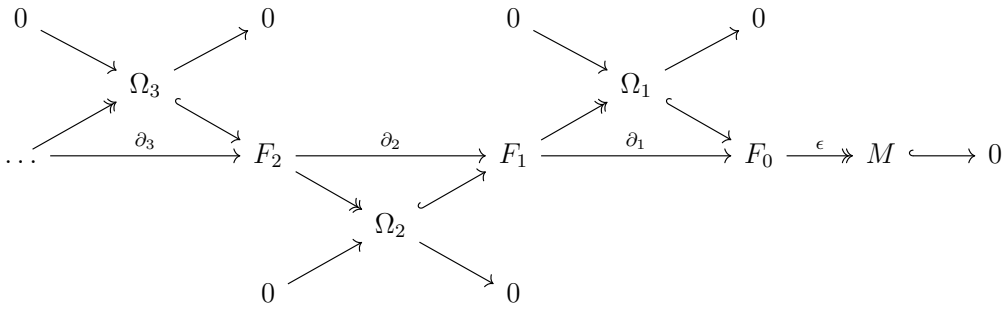
$$\cdots \xrightarrow{\partial_4} F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\epsilon} M \to 0$$

is exact, then we say that

$$F_\bullet : \cdots \xrightarrow{\partial_4} F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0$$

is a *free resolution of* M.

Free resolutions exist for every module, although they may not be unique even up to isomorphism. To construct one, we choose a set of generators for M and let F_0 be the free module with this set of generators as its basis. Then there is a natural surjective map $\epsilon\colon F_0 \to M$. Set $\Omega_1 = \ker(\epsilon)$. This gives us an exact sequence $0 \to \Omega_1 \hookrightarrow F_0 \xrightarrow{\epsilon} M \to 0$. We then choose another free module F_1 that can be surjected onto Ω_1 and define $\partial_1\colon F_1 \to F_0$ as the composition of the maps $F_1 \twoheadrightarrow \Omega_1$ and $\Omega_1 \hookrightarrow F_0$ and set $\Omega_2 = \ker(\partial_1)$. This gives us an exact sequence $0 \to \Omega_2 \hookrightarrow F_1 \twoheadrightarrow \Omega_1 \to 0$. Proceding in this manner gives us an exact sequence $0 \to \Omega_{j+1} \hookrightarrow F_j \twoheadrightarrow \Omega_j \to 0$ for every $j \in \mathbb{N}_+$. We can combine these short exact sequences in the following way:



This gives us a free resolution

$$F_\bullet : \cdots \xrightarrow{\partial_4} F_3 \xrightarrow{\partial_3} F_2 \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} 0.$$

Remark 2.1.6. By definition, a free resolution of a module M contains neither M nor the mapping to M in the exact sequence. If we want to include that mapping, we call it an *augmented free resolution*, or we will use parenthesis around the mapping to M to emphasize which module is being resolved, e.g.: $\cdots \to F_1 \to F_0 \to (M \to)\ 0$. If the resolution is

truncated at a syzygy, i.e., if it is of the form $0 \to \Omega_{j+1}(M) \to F_j \to \cdots \to F_0 \to 0$, then we refer to it as a *truncated free resolution* of M. However, it is common practice to refer to both augmented and truncated free resolutions simply as free resolutions, as the context usually makes it clear which type of resolution is being discussed.

We now briefly review the Tor and Ext functors, along with some basic properties.

Definition 2.1.7. Let R, M and $F_\bullet$ be as in Definition 2.1.5. Given another R-module N, we can form the complex

$$F_\bullet \otimes_R N : \cdots \xrightarrow{d_3} F_2 \otimes_R N \xrightarrow{d_2} F_1 \otimes_R N \xrightarrow{d_1} F_0 \otimes_R N \xrightarrow{d_0} 0,$$

where $d_i = \partial_i \otimes_R 1_N$ for each $i \in \mathbb{N}$. We define

$$\operatorname{Tor}_i^R(M, N) := H_i(F_\bullet \otimes_R N) = \ker d_i / \operatorname{im} d_{i+1},$$

We write $\operatorname{Tor}_i(M, N)$ instead of $\operatorname{Tor}_i^R(M, N)$ if the underlying ring is understood.

Remark 2.1.8. Let R be a ring and let M and N be R-modules. Then for all $i \in \mathbb{N}$, we have the following:

(a) $\operatorname{Tor}_i(M, N)$ is independent from of choice of free resolution for M.

(b) $\operatorname{Tor}_i(M, N) \cong \operatorname{Tor}_i(N, M)$.

(c) $\operatorname{Tor}_0(M, N) \cong M \otimes_R N$.

Proposition 2.1.9. *Let $(R, \mathfrak{m})$ be a local ring, M be a finitely generated R-module, L be an R-module of finite length, and $G_\bullet$ be a free resolution of L. Then $\ell_R(H_i(G_\bullet \otimes_R M)) < \infty$ for all $i \in \mathbb{N}$.*

Proof. Since M is finitely generated, there exists a free R-module F that surjects onto M. Tensoring this map with L gives us $L \otimes_R F \twoheadrightarrow L \otimes M$. Since F is free, $F \cong R^n$ for some $n \in \mathbb{N}_+$, so $L \otimes_R F \cong L^n$. Since L has finite length, L^n has finite length, so $\ell_R(H_0(G_\bullet \otimes M)) = \ell_R(L \otimes M) < \infty$.

Let $i \geq 1$. We show that $\mathrm{Supp}(\mathrm{Tor}_i(L, M)) = \{\mathfrak{m}\}$. Let $\mathfrak{p} \in \mathrm{Supp}(\mathrm{Tor}_i(L, M))$. Then $\mathrm{Tor}_i(L_\mathfrak{p}, M_\mathfrak{p}) \neq 0$, and in particular, $L_\mathfrak{p} \neq 0$ and $M_\mathfrak{p} \neq 0$. Since R is local, $0 \neq L_\mathfrak{p} \otimes_{R_\mathfrak{p}} M_\mathfrak{p} = (L \otimes_R M)_\mathfrak{p}$. Since $L \otimes_R M$ has finite length, $\mathfrak{p}$ is maximal. Hence, $H_i(G_\bullet \otimes_R M) = \mathrm{Tor}_i(L, M)$ has finite length. □

Definition 2.1.10. Let R, M and $F_\bullet$ be as in Definition 2.1.5. Given another R-module N, we can form the cochain complex

$$\mathrm{Hom}_R(F_\bullet, N) : \cdots \xleftarrow{d^3} \mathrm{Hom}_R(F_2, N) \xleftarrow{d^2} \mathrm{Hom}_R(F_1, N) \xleftarrow{d^1} \mathrm{Hom}_R(F_0, N) \xleftarrow{d^0} 0$$

We define

$$\mathrm{Ext}^i_R(M, N) := H^i(\mathrm{Hom}_R(F_\bullet, N)) = \ker d^{i+1} / \operatorname{im} d^i$$

Remark 2.1.11. Let R be a ring and let M and N be R-modules. Then for all $i \in \mathbb{N}$, we have the following:

(a) $\mathrm{Ext}^i_R(M, N)$ is independent from of choice of free resolution for M.

(b) $\mathrm{Ext}^0_R(M, N) \cong \mathrm{Hom}_R(M, N)$.

Proposition 2.1.12. *Let R be a ring, M be an R-module, and $0 \to A \to B \to C \to 0$ an exact sequence of R-modules. Then*

(1) *There is a long exact sequence*

$$\cdots \to \operatorname{Tor}_i^R(A,M) \to \operatorname{Tor}_i^R(B,M) \to \operatorname{Tor}_i^R(C,M) \to \operatorname{Tor}_{i-1}^R(A,M) \to \cdots$$
$$\to \operatorname{Tor}_1^R(C,M) \to A \otimes_R M \to B \otimes_R M \to C \otimes_R M \to 0.$$

(2) *There is a long exact sequence*

$$0 \to \operatorname{Hom}_R(C,M) \to \operatorname{Hom}_R(B,M) \to \operatorname{Hom}_R(A,M) \to \operatorname{Ext}_R^1(C,M) \to \cdots$$
$$\to \operatorname{Ext}_R^{i-1}(A,M) \to \operatorname{Ext}_R^i(C,M) \to \operatorname{Ext}_R^i(B,M) \to \operatorname{Ext}_R^i(A,M) \to \cdots .$$

While free resolutions exist for every module, these resolutions are not unique due to different choices of generating sets and mappings that can lead to different results for the same module. To find the most efficient free resolution, minimizing the relations at each step, we use the concept of a minimal free resolution, which relies on Nakayama's lemma. This concept is particularly meaningful in two main cases: the local case and the graded case. We present the local case:

Let $(R, \mathfrak{m}, k)$ be a local ring and M be a finitely generated R-module. The minimal number of generators $\mu(M)$ is well-defined because, by Nakayama's Lemma, any generating set of M can be reduced to a minimal generating set, and the number of elements in this minimal generating set is unique and equal to $\dim_k(M/\mathfrak{m}M)$. We can then construct our resolution as we did before, choosing minimal generating sets at each step to create a minimal free resolution.

Minimal free resolutions over local rings are uniquely determined by the module being resolved, up to isomorphism [BH93, §1.3], so the following are well-defined.

Definition 2.1.13. Let $(R, \mathfrak{m})$ be a local ring, M be a finitely generated R-module, and $(F_\bullet, \partial_\bullet)$ be a minimal free resolution of M. Set ∂_0 to be the map from F_0 to M. For each integer $i \geq 1$, the *ith syzygy of* M, denoted $\Omega_i(M)$, is defined to be $\ker(\partial_{i-1})$.

Definition 2.1.14. Let $(R, \mathfrak{m})$ be a local ring, M be a finitely generated R-module, and $F_\bullet$ be a minimal free resolution of M. For any $i \in \mathbb{N}$, the *ith Betti number of* M, denoted $\beta_i(M)$, is defined to be $\operatorname{rank}(F_i)$. Alternatively, Betti numbers can be expressed as

$$\beta_i(M) = \dim_k(\operatorname{Tor}_i^R(M, k)) = \dim_k(\operatorname{Ext}_R^i(M, k)).$$

Definition 2.1.15. Let $(R, \mathfrak{m})$ be a local ring, M be a finitely generated R-module, and $\beta_i(M)$ be the ith Betti number of M. The *ith Euler characteristic of* M is defined as

$$\chi_i(M) = \sum_{j=0}^{i} (-1)^j \beta_{i-j}(M).^1$$

Lemma 2.1.16 (Schanuel's lemma)**.** *Let*

$$0 \to N \to F \to M \to 0 \qquad \text{and} \qquad 0 \to N' \to F' \to M \to 0$$

be exact sequences of modules such that F and F' are free. Then $N \oplus F' \cong N' \oplus F$.

Proposition 2.1.17 ([GTT07, Proposition 6.6])**.** *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. If $F_\bullet$ is a minimal free resolution of M, then any free resolution of M is of the form $F_\bullet \oplus K_\bullet$, where $K_\bullet$ is a free resolution of* 0*.*

Corollary 2.1.18. *Let $(R, \mathfrak{m})$ be a local ring, M be a finitely generated R-module, and $i \in \mathbb{N}_+$. Then $\Omega_i(M)_\mathfrak{p} \cong \Omega_i(M_\mathfrak{p}) \oplus R_\mathfrak{p}^n$ for some $n \in \mathbb{N}$.*

[1] Some authors instead define χ_i using β_j instead of β_{i-j} in the sum.

Proof. Consider a minimal free resolution of M over R:

$$0 \to \Omega_i(M) \to R^{b_{i-1}} \to \cdots \to R^{b_1} \to R^{b_0} \to (M \to)\ 0.$$

We can localize this resolution at $\mathfrak{p}$ to get

$$G_\bullet : 0 \to \Omega_i(M)_\mathfrak{p} \to R_\mathfrak{p}^{b_{i-1}} \to \cdots \to R_\mathfrak{p}^{b_1} \to R_\mathfrak{p}^{b_0} \to (M_\mathfrak{p} \to)\ 0.$$

Compare this to a minimal free resolution of $M_\mathfrak{p}$ over $R_\mathfrak{p}$:

$$F_\bullet : 0 \to \Omega_i(M_\mathfrak{p}) \to R_\mathfrak{p}^{\beta_{i-1}} \to \cdots \to R_\mathfrak{p}^{\beta_1} \to R_\mathfrak{p}^{\beta_0} \to (M_\mathfrak{p} \to)\ 0.$$

By Proposition 2.1.17, $G_\bullet \cong F_\bullet \oplus K_\bullet$, where $K_\bullet$ is some free resolution of 0. It follows that $\Omega_i(M)_\mathfrak{p} \cong \Omega_i(M_\mathfrak{p}) \oplus L_i$, where $L_i = \ker(K_{i-1} \to K_{i-2})$ for each $i \geq 2$. Each L_i is free, since if we have any other free resolution of 0, we can show that $L_i \oplus R_\mathfrak{p}^r \cong R_\mathfrak{p}^s$ by Schanuel's lemma, for some $r, s \in \mathbb{N}$. Thus, $\Omega_i(M)_\mathfrak{p} \cong \Omega_i(M_\mathfrak{p}) \oplus L_i \cong \Omega_i(M_\mathfrak{p}) \oplus R_\mathfrak{p}^n$ for some $n \in \mathbb{N}$. □

Definition 2.1.19. Let $(R, \mathfrak{m})$ be a local ring, M be a finitely generated R-module, and $F_\bullet$ be a minimal free resolution of M. The *projective dimension* of M is defined to be

$$\mathrm{pd}_R(M) := \sup\{i \geq 0 \mid F_i \neq 0\}.$$

We often write $\mathrm{pd}(M)$ if the underlying ring is understood.

The term "projective dimension" is derived from the concept of resolving a module using projective modules. In the local case, these concepts are equivalent. We direct the reader to [Rot79] or [Wei94] for a detailed explanation and formal definitions.

2.1.2 *Cohen-Macaulay rings and modules*

Cohen-Macaulay modules play a significant role in commutative algebra due to their nice homological properties and their connection to regular sequences. In this section, we will explore the foundational aspects of Cohen-Macaulay modules, starting with their definition and key properties.

Recall that an element $r \in R$ is called a *nonzero divisor on* M if for every nonzero element $m \in M$, we have $rm \neq 0$.

Definition 2.1.20. Let R be a ring and M be a nonzero finitely generated R-module. A sequence of elements $x_1, x_2, \ldots, x_n \in R$ is called an *M-regular sequence* if

(1) $M/(x_1, \ldots, x_n)M \neq 0$, and

(2) for each $i = 1, 2, \ldots, n$, the element x_i is a nonzero divisor on $M/(x_1, \ldots, x_{i-1})M$.[2]

We call an R-regular sequence simply a *regular sequence*.

Example 2.1.21. Let R be a ring and k be a field.

(1) If $R = k[x_1, \ldots, x_n]$, then $x_1, \ldots, x_n$ is a regular sequence on R.

(2) If $R = k[x, y, z]$, then $x, y(1-x), z(1-x)$ is a regular sequence over R (but $y(1-x), z(1-x), x$ is not).

The following proposition is a well-known property of regular sequences:

Proposition 2.1.22 ([BH93, Exercise 1.1.10])**.** *Let R be a ring and M be a nonzero finitely generated R-module. If $x_1, x_2, \ldots, x_n$ is an M-regular sequence, then $x_1^{t_1}, x_2^{t_2}, \ldots, x_n^{t_n}$ is an M-regular sequence for any $t_1, t_2, \ldots, t_n \in \mathbb{N}$.*

[2]The case of $i = 1$ is interpreted as "x_1 is a nonzero divisor on M."

Depth is another important concept in the study of modules over rings. It measures the length of the longest regular sequence contained in an ideal. The following definition formalizes the notion of I-depth.

Definition 2.1.23. Let R be a ring, I be an ideal of R, and M be a finitely generated R-module. The *I-depth* of M, denoted $\operatorname{depth}_I(M)$, is defined as follows:

(1) If $IM = M$, then $\operatorname{depth}_I(M) = \infty$.

(2) If $IM \neq M$, then $\operatorname{depth}_I(M)$ is defined to be length of a maximal M-regular sequence in I.

If $(R, \mathfrak{m})$ is local, we denote $\operatorname{depth}_{\mathfrak{m}}(M)$ as $\operatorname{depth}(M)$.

An equivalent definition of I-depth, first proven by Rees, involves Ext functors and provides a more homological perspective on this invariant.

Theorem 2.1.24 (Rees, c.f. [BH93, Theorem 1.2.5])**.** *Let R be a ring, M be a finitely generated R-module, and I be an ideal such that $I \neq IM$. Then all maximal M-regular sequences in I have the same length n given by*

$$n = \min\{i \mid \operatorname{Ext}^i_R(R/I, M) \neq 0\}.$$

Proposition 2.1.25. *Let $(R, \mathfrak{m})$ be a local ring and $0 \to A \to B \to C \to 0$ be an exact sequence of finitely generated R-modules. Then*

(1) $\operatorname{depth} A \geq \min\{\operatorname{depth} B, \operatorname{depth} C + 1\}$, *and equality holds if* $\operatorname{depth} B \neq \operatorname{depth} C$.

(2) $\operatorname{depth} B \geq \min\{\operatorname{depth} A, \operatorname{depth} C\}$, *and equality holds if* $\operatorname{depth} A \neq \operatorname{depth} C - 1$.

(3) $\operatorname{depth} C \geq \min\{\operatorname{depth} A - 1, \operatorname{depth} B\}$, *and equality holds if* $\operatorname{depth} A \neq \operatorname{depth} B$.

Proof. The short exact sequence $0 \to A \to B \to C \to 0$ induces a long exact sequence:

$$\cdots \to \operatorname{Ext}^{i-1}(k, C) \to \operatorname{Ext}^i(k, A) \to \operatorname{Ext}^i(k, B) \to \operatorname{Ext}^i(k, C) \to \operatorname{Ext}^{i+1}(k, A) \to \cdots .$$

We prove the first statement; the other two statements are verified similarly. We break the argument into three cases.

Case 1: Suppose $i = \operatorname{depth} B < \operatorname{depth} C$. Then $\operatorname{Ext}^j(k, B) = \operatorname{Ext}^j(k, C) = 0$ for all $0 \leq j < i$, so $\operatorname{Ext}^j(k, A) = 0$ for all $0 \leq j < i$. Since $\operatorname{Ext}^i(k, C) = 0$ and $\operatorname{Ext}^i(k, B) \neq 0$, this forces $\operatorname{Ext}^i(k, A) \neq 0$. Therefore, $\operatorname{depth} A = \operatorname{depth} B$.

Case 2: Suppose $i = \operatorname{depth} C < \operatorname{depth} B$. Then $\operatorname{Ext}^j(k, B) = \operatorname{Ext}^j(k, C) = 0$ for all $0 \leq j < i$, so $\operatorname{Ext}^j(k, A) = 0$ for all $0 \leq j < i$. Since $\operatorname{Ext}^i(k, B) = 0$ and $\operatorname{Ext}^i(k, C) \neq 0$, This forces $\operatorname{Ext}^{i+1}(k, A) \neq 0$. Therefore, $\operatorname{depth} A = \operatorname{depth} C + 1$.

Case 3: Suppose $i = \operatorname{depth} B = \operatorname{depth} C$. Then $\operatorname{Ext}^j(k, B) = \operatorname{Ext}^j(k, C) = 0$ for all $0 \leq j < i$, so $\operatorname{Ext}^j(k, A) = 0$ for all $0 \leq j < i$. Therefore, $\operatorname{depth} A \geq \operatorname{depth} B = \operatorname{depth} C$. □

For local rings, depth provides a lower bound for the dimension of a module.

Proposition 2.1.26 ([BH93, Proposition 1.2.13])**.** *Let $(R, \mathfrak{m})$ be a local ring and M be a nonzero finitely generated R-module. Then* $\operatorname{depth}(M) \leq \dim(R/\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Ass}(M)$*. In particular,* $\operatorname{depth}(M) \leq \dim(M)$*.*

Cohen-Macaulay modules are modules in which equality holds in Proposition 2.1.26.

Definition 2.1.27. Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. We say that M is a *Cohen-Macaulay module* if $\operatorname{depth}(M) = \dim(M)$. We say that R is a *Cohen-Macaulay ring* if R is a Cohen-Macaulay module over itself.

Proposition 2.1.28. *Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring of dimension d, $n \in \mathbb{N}$, and M be a finitely generated R-module such that* $\operatorname{pd} M \geq n$. *Then*

$$\operatorname{depth} \Omega_{n+1}(M) = \begin{cases} \operatorname{depth} M + n + 1 & \text{if } n < d - \operatorname{depth} M. \\ d & \text{if } n \geq d - \operatorname{depth} M \end{cases}$$

Proof. Set $\Omega_i = \Omega_i(M)$ for all $i \geq 1$, and let $F_\bullet$ be a minimal free resolution of M. Note that for all $i \geq 0$, since F_i is a free R-module, we have that $\operatorname{depth}(F_i) = \operatorname{depth}(R) = d$.

Assume the claim is true for some $n \geq 0$. We have that $0 \to \Omega_{n+1} \to F_n \to \Omega_n \to 0$ is exact, so by Proposition 2.1.25, we have $\operatorname{depth} \Omega_{n+1} \geq \min\{d, \operatorname{depth} \Omega_n + 1\}$, with equality if $\operatorname{depth} \Omega_n \neq d$. We have two cases:

Case 1: Suppose $n < d - \operatorname{depth} M$. Then $n - 1 < d - \operatorname{depth} M$. By our assumption, we have that us that $\operatorname{depth} \Omega_n = \operatorname{depth} M + n < d$. Since $\operatorname{depth} \Omega_n \neq d$, we have that $\operatorname{depth} \Omega_{n+1} = \operatorname{depth} M + n + 1$.

Case 2: Suppose $n \geq d - \operatorname{depth} M$. If $n = d - \operatorname{depth} M$, then $n - 1 < d - \operatorname{depth} M$, and by our assumption, $\operatorname{depth} \Omega_n = \operatorname{depth} M + n = d$. On the other hand, if $n > d - \operatorname{depth} M$, then $n - 1 \geq d - \operatorname{depth} M$, so we again get $\operatorname{depth} \Omega_n = d$. Therefore, $\operatorname{depth} \Omega_{n+1} \geq d$, which forces $\operatorname{depth} \Omega_{n+1} = d$.

The case for $n = 0$ is proven similarly. So by induction on n, the proposition holds. □

Maximal Cohen-Macaulay modules are those whose dimension equals that of the ring itself. These modules represent the "largest" Cohen-Macaulay modules, in terms of dimension.

Definition 2.1.29. Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. We say that M a *maximal Cohen-Macaulay module* if it is a Cohen-Macaulay module such that $\operatorname{depth}(M) = \dim(R)$. By convention, we treat the 0 module as a maximal Cohen-Macaulay module.

Proposition 2.1.30. *Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring of dimension d, $n \in \mathbb{N}$, and M be a finitely generated R-module such that $\operatorname{pd}(M) \geq n$. If $n \geq d - \operatorname{depth}(M)$, then $\Omega_{n+1}(M)$ is a maximal Cohen-Macaulay R-module. In particular, if M is a maximal Cohen-Macaulay R-module, then $\Omega_{n+1}(M)$ is a maximal Cohen-Macaulay R-module*

Proof. It follows from Corollary 2.1.28 that if $n \geq d - \operatorname{depth} M$, then $\operatorname{depth} \Omega_{n+1}(M) = d$. Thus, $\Omega_{n+1}(M)$ is a maximal Cohen-Macaulay R-module. □

2.1.3 *Regular local rings*

Regular local rings form a fundamental class of rings in commutative algebra and algebraic geometry due to their well-behaved nature and desirable properties. We begin this section by introducing the concept of a system of parameters.

Definition 2.1.31. Given a local ring $(R, \mathfrak{m})$ of dimension d, a sequence of elements $x_1, \ldots, x_d$ that generate an $\mathfrak{m}$-primary ideal is called a *system of parameters*. An ideal I of R is called a *parameter ideal* if it can be generated by a system of parameters.

It is well-known that every local ring admits a system of parameters, so the minimal number of generators of $\mathfrak{m}$ is always at least the dimension of R. Regular local rings are those in which we have equality.

Definition 2.1.32. Let $(R, \mathfrak{m})$ be a local ring. We say that R is a *regular local ring* if $\mathfrak{m}$ can be generated by a system of parameters. Without the assumption of locality, we say that R is *regular* if every localization of R at a prime ideal is a regular local ring.

Example 2.1.33. Let $(R, \mathfrak{m})$ be a local ring and $n \in \mathbb{N}_+$.

(1) If $\dim(R) = 0$, then R is regular if and only if it is a field.

(2) If $\dim(R) = 1$, then R is regular if and only if discrete valuation ring.

(3) Let $S = k[x_1, \ldots, x_n]$ for some field k. Then $S_{\mathfrak{p}}$ is regular for any $\mathfrak{p} \in \operatorname{Spec}(S)$.

(4) The power series ring $k[\![x_1, \ldots, x_n]\!]$ over a field k is regular.

The following proposition implies that that every regular local ring is Cohen-Macaulay. In Chapter 4, we will present a criterion for regularity for Cohen-Macaulay rings of prime characteristic (Theorem 4.3.1).

Proposition 2.1.34 ([BH93, Proposition 2.2.5])**.** *Let $(R, \mathfrak{m}, k)$ be a local ring and $x_1 \ldots, x_n$ be minimal generators of $\mathfrak{m}$. Then the following are equivalent:*

(1) *R is regular.*

(2) *$x_1, \ldots, x_n$ is an R-regular sequence.*

(3) *The map $X_i \mapsto x_i \in \mathfrak{m}/\mathfrak{m}^2$ yields an isomorphism $k[X_1, \ldots, X_n] \cong \operatorname{gr}_{\mathfrak{m}}(R)$.*

One of the foundational results in homological algebra is the Auslander-Buchsbaum formula, which establishes a profound relationship between the depth and projective dimension of a finitely generated module over a local ring.

Proposition 2.1.35 (Auslander-Buchbaum formula)**.** *Let* $(R, \mathfrak{m})$ *be a local ring and* M *be a nonzero finitely generated* R*-module such that* $\operatorname{pd}(M) < \infty$*. Then*

$$\operatorname{depth}(R) = \operatorname{depth}(M) + \operatorname{pd}(M).$$

The following theorem can be viewed as an extension of the Auslander-Buchsbaum formula, offering a characterization of regular local rings in terms of the projective dimensions of their modules.

Theorem 2.1.36 (Auslander-Buchsbaum-Serre theorem)**.** *Let* $(R, \mathfrak{m}, k)$ *be a local ring. Then the following are equivalent:*

(1) R *is regular.*

(2) $\operatorname{pd}_R(M) < \infty$ *for every finitely generated* R*-module* M*.*

(3) $\operatorname{pd}_R(k) < \infty$*.*

Given a local ring $(R, \mathfrak{m})$, we say that a ring homomorphism $\phi : R \to R$ is a *contracting endomorphism* if $\phi^i(\mathfrak{m}) \subseteq \mathfrak{m}^2$ for some $i \in \mathbb{N}_+$. Denote ${}^{\phi^i}M$ as the R-module given $r \cdot m = \phi^i(r)m$ for any $r \in R$ and $m \in M$. This leads us to the following result by Avramov, Hochster, Iyengar, and Yao.

Theorem 2.1.37 ([AHIY12, Theorem 1.1])**.** *Let* R *be a local ring,* $\phi : R \to R$ *be a contracting endomorphism,* M *be a finitely generated* R*-module, and* $i \in \mathbb{N}_+$*. If* $\operatorname{pd}_R({}^{\phi^i}M) < \infty$*, then* R *is regular.*

2.2 Modules over rings of prime characteristic

In this section, we discuss properties of the Frobenius endomorphism and F-finite modules, introduce the concept of tight closure and variants of F-regularity, and introduce several invariants.

2.2.1 *Frobenius endomorphism*

Let $\phi : R \to S$ be a homomorphism of rings, and let M be an S-module. We can derive a left R-module structure on M by restriction of scalars via ϕ. Specifically, for any $r \in R$ and $m \in M$, the action is given by $r \cdot m = \phi(r)m$. We denote this derived R-module structure on M by ${}^{\phi}M$.

Definition 2.2.1. Let R be a ring of prime characteristic p. The *Frobenius endomorphism* is the map $F : R \to R$ defined by $r \mapsto r^p$ for any $r \in R$. For any $e \in \mathbb{N}$, the *iterated Frobenius endomorphism* is the map $F^e : R \to R$ defined by $r \mapsto r^{p^e}$ for any $r \in R$.

The Frobenius map is indeed a ring endomorphism: given a ring R of prime characteristic $p > 0$, for any $a, b \in R$, we have $(a + b)^p = a^p + b^p$, $(a \cdot b)^p = a^p \cdot b^p$, and $1^p = 1$.

For any R-module M and any $e \in \mathbb{N}$, we can derive a left R-module structure from the Frobenius endomorphism on the set M by $r \cdot m = r^{p^e} m$ for any $r \in R$ and $m \in M$. Specifically in the case of the Frobenius endomorphism, we denote this R-module by ${}^{e}M$ (some authors use the notation $F^e_* M$).

Without knowing beforehand whether an element m comes from the module M or ${}^{e}M$, the meaning of $r \cdot m$ may be ambiguous. For this reason, if it is necessary, we may use ${}^{e}m$ to

clarify that an element is from of ${}^eM$. We now observe the following, which we will be using throughout this dissertation without reference:

Proposition 2.2.2. *Let R be a ring of prime characteristic p. Let M and N be R-modules. For any $e \in \mathbb{N}$, we have that:*

(a) ${}^{e_0}({}^eM) = {}^{e_0+e}M$ *for all* $e_0 \in \mathbb{N}$.

(b) ${}^eR$ *is a ring isomorphic to* R, *with addition given by* ${}^ea + {}^eb = {}^e(a+b)$ *and multiplication given by* ${}^ea\,{}^eb = {}^e(ab)$ *for all* $a, b \in R$.

(c) M *is Artinian if and only if* ${}^eM$ *is Artinian.*

(d) ${}^eM$ *is an* ${}^eR$*-module via* $r \cdot {}^em = {}^e(rm)$ *for all* $r \in {}^eR$ *and* ${}^em \in {}^eM$.

(e) *If* $N \cong M$, *then* ${}^eM \cong {}^eN$.

(f) *If* $N \subseteq M$, *then* ${}^eN \subseteq {}^eM$.

(g) *If* $N \subseteq M$, *then* ${}^eM/\,{}^eN \cong {}^e(M/N)$ *as an* R*-module.*

(h) *If* $\mathfrak{p} \in \operatorname{Spec}(R)$, *then* ${}^e(M_{\mathfrak{p}}) \cong ({}^eM)_{\mathfrak{p}}$.

This twisted R-module structure provides insight about the regularity of a ring. For example, a Kunz famously proved that a ring is regular if and only if the Frobenius map is flat:

Theorem 2.2.3 ([Kun69, Theorem 2.1])**.** *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p. Then the following are equivalent:*

(1) R *is regular.*

(2) ${}^eR$ *is a flat* R*-module for all* $e \in \mathbb{N}$.

(3) ${}^eR$ *is a flat* R*-module for some* $e \in \mathbb{N}_+$.

Now present a consequence of Theorem 2.1.37, which was also proven separately as a special case in [AHIY12].

Theorem 2.2.4 ([AHIY12, Theorem 6.1])**.** *Let $(R, \mathfrak{m})$ be a local ring and M be a finitely generated R-module. If ${}^eM$ has finite projective dimension for some $e \in \mathbb{N}_+$, then R is regular.*

Definition 2.2.5. We say that a ring R is *F-finite* if ${}^eR$ is a finitely generated R-module for some, or equivalently all, $e \in \mathbb{N}_+$.

When R is reduced, Definition 2.2.5 is equivalent to the condition that R^{1/p^e}, the ring obtained by adjoining the p^eth roots of all elements of R, is module-finite over R for some, or equivalently all, $e \in \mathbb{N}_+$. This is clear by considering the commutative diagram

$$\begin{array}{ccc} R & \xrightarrow{\subseteq} & R^{1/p^e} \\ {\scriptstyle \mathrm{id}}\uparrow & & \cong\uparrow{\scriptstyle G} \\ R & \xrightarrow[F^e]{} & R \end{array}$$

where G maps $r \in R$ to $r^{1/p^e} \in R^{1/p^e}$.

Example 2.2.6. Recall that a field K of prime characteristic p is perfect if $K = K^p$.

(1) A perfect field with positive characteristic is F-finite.

(2) A complete local ring with a perfect residue field is F-finite.

Proposition 2.2.7. *Let $(R, \mathfrak{m})$ be an F-finite local ring of prime characteristic p. Then the following are equivalent:*

(1) *R is regular.*

(2) *${}^eR$ is a free R-module for all $e \in \mathbb{N}$.*

(3) ${}^eR$ *is a free* R*-module for some* $e \in \mathbb{N}_+$.

Proof. (2) $\implies$ (3) is trivial, and (3) $\implies$ (1) follows directly from Theorem 2.2.3.

(1) $\implies$ (2): Since R is regular and F-finite, ${}^eR$ is a finitely generated flat R-module for all $e \in \mathbb{N}$. Let $n = \mu(M)$. We have the exact sequence $0 \to K \to R^n \to {}^eR \to 0$, where $K = \ker(R^n \to {}^eR)$. Since ${}^eR$ is flat, $\operatorname{Tor}_1^R(k, {}^eR) = 0$, so we have a short exact sequence

$$0 \to k \otimes_R K \to k \otimes_R R^n \to k \otimes_R {}^eR \to 0.$$

By Nakayama's lemma, $\dim(k \otimes_R {}^eR) = n$, so $k \otimes_R R^n \cong k \otimes_R {}^eR$ as k-vector spaces. Thus, $k \otimes_R K = 0$, which implies that $K = 0$. Therefore, $R^n \cong {}^eR$, and ${}^eR$ is free. □

Many properties of F-finite rings are well-known and have been extensively studied due to their significance in the theory of rings and modules in prime characteristic. One of the most celebrated results on F-finite rings is a result of Kunz, who showed in [Kun76] that every F-finite ring is excellent.

The following proposition summarizes some basic properties of F-finite rings and modules over them:

Proposition 2.2.8. *Let* R *be an F-finite ring of prime characteristic* p.

(a) *If* M *is a finitely generated* R*-module, then* ${}^eM$ *is a finitely generated* R*-module for all integers* $e \in \mathbb{N}$.

(b) R/I *is F-finite for any ideal* I *of* R.

(c) $S^{-1}R$ *is F-finite for any multiplicative subset* S *of* R.

(d) $R[\![x_1, \ldots, x_n]\!]$ *is F-finite.*

(e) *If R is a local ring with residue field k, then $[k : k^p] < \infty$.*

In many situations, we can pass to an F-finite ring even if the original ring is not F-finite, as demonstrated in the following remark.

Remark 2.2.9. Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic p. Cohen's Structure Theorem implies that $\widehat{R}$ is a homomorphic image of a power series ring over k. Therefore, there exists a ring homomorphism from $k[\![x_1, \ldots, x_n]\!]$ onto $\widehat{R}$, where each x_i maps to the generators of $\mathfrak{m}\widehat{R}$. Set $S = \widehat{R} \otimes_{k[\![x_1,\ldots,x_n]\!]} \overline{k}[\![x_1, \ldots, x_n]\!]$, where $\overline{k}$ denotes the algebraic closure of k. The maps $R \to \widehat{R} \to S$ are faithfully flat and $\mathfrak{m}S$ is the maximal ideal of S, so $S/\mathfrak{m}S$ is perfect and S has an infinite residue field. Since S is a complete local ring with a perfect residue field, it is F-finite. Additionally, this extension preserves length and Krull dimension. Furthermore, R is Cohen-Macaulay if and only if S is Cohen-Macaulay, and R is regular if and only if S is regular.

Proposition 2.2.10. *Let R be a ring of prime characteristic p, M be an R-module, and I be and ideal of R. Then*

$$R/I \otimes_R {}^eM \cong {}^e(M/I^{[p^e]}M).$$

Proof. Consider the exact sequence $0 \to I \hookrightarrow R \twoheadrightarrow R/I \to 0$. Since tensoring is right exact, we obtain

$$I \otimes_R {}^eM \to {}^eM \twoheadrightarrow R/I \otimes_R {}^eM \to 0.$$

Since the image of $I \otimes_R {}^eM$ in ${}^eM$ is ${}^e(I^{[p^e]}M)$, we have that $R/I \otimes_R {}^eM \cong {}^e(M/I^{[p^e]}M)$. $\square$

Proposition 2.2.11. *Let $(R, \mathfrak{m})$ be an F-finite local ring of prime characteristic p. Then for any $e \in \mathbb{N}$, $\operatorname{depth}(M) = \operatorname{depth}({}^eM)$. In particular, if M is a maximal Cohen-Macaulay R-module, then ${}^eM$ is a maximal Cohen-Macaulay R-module.*

Proof. By Proposition 2.1.22, any maximal M-regular sequence is a ${}^eM$ regular sequence, and any maximal ${}^eM$-regular sequence is a M regular sequence. □

Let R be an ring of prime characteristic p. For every $\mathfrak{p} \in \operatorname{Spec}(R)$, we denote $\alpha(R_\mathfrak{p}) := \log_p[k(\mathfrak{p}) : k(\mathfrak{p})^p]$, where $k(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. That is, the field extension $k(\mathfrak{p})^p \subseteq k(\mathfrak{p})$ has degree $p^{\alpha(R_\mathfrak{p})}$. If $(R, \mathfrak{m}, k)$ is local, then we denote $\alpha(R) := \alpha(R_\mathfrak{m}) = \log_p[k : k^p]$.

Proposition 2.2.12. *Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic p such that $[k : k^p] < \infty$, M be an R-module of finite length, and $e \in \mathbb{N}$. Then $\ell_R({}^eM) = \ell_R(M)p^{e\alpha(R)}$.*

Proof. Set $\lambda = \ell_R(M)$. Then there exists a composition series

$$0 = M_0 \subset M_1 \subset \cdots \subset M_\lambda = M$$

such that $M_i/M_{i-1} \cong R/\mathfrak{m} = k$ for all $i = 1, 2, \ldots, \lambda$. Applying the Frobenius endomorphism to the above composition series gives us

$$0 = {}^eM_0 \subset {}^eM_1 \subset \cdots \subset {}^eM_\lambda = {}^eM,$$

where ${}^eM_i/{}^eM_{i-1} \cong {}^e(M_i/M_{i-1}) \cong {}^ek$ for all $i = 1, 2, \ldots, \lambda$. Therefore,

$$\begin{aligned}\ell_R({}^eM) = \sum_{i=1}^{\lambda} \ell_R({}^eM_i/{}^eM_{i-1}) &= \lambda \cdot \ell_R({}^ek) \\ &= \lambda \cdot \ell_R(k^{1/p^e}) \\ &= \lambda \cdot [k^{1/p^e} : k] \\ &= \lambda \cdot [k : k^{p^e}] = \ell_R(M)p^{e\alpha(R)}.\end{aligned}$$ □

Proposition 2.2.13 ([Kun76, Proposition 2.3])**.** *Let R be an F-finite ring of prime characteristic p. For all $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec}(R)$ with $\mathfrak{p} \subseteq \mathfrak{q}$, we have*

$$\alpha(R_\mathfrak{p}) = \alpha(R_\mathfrak{q}) + \dim(R_\mathfrak{q}/\mathfrak{p}R_\mathfrak{q}).$$

In particular, if $(R, \mathfrak{m})$ is local, then $\alpha(R_\mathfrak{p}) = \alpha(R) + \dim(R/\mathfrak{p})$.

Corollary 2.2.14. *Let $(R, \mathfrak{m})$ be an F-finite local ring of dimension d and prime characteristic p, let $\mathfrak{p} \in \operatorname{Min}(R)$, and M be a finitely generated R-module. Then for any integer $e \geq 0$,*

$$\ell_{R_\mathfrak{p}}({}^eM_\mathfrak{p}) = \ell_{R_\mathfrak{p}}(M_\mathfrak{p})(p^e)^{\alpha(R)+\dim(R/\mathfrak{p})}.$$

Proof. We have that $R_\mathfrak{p}$ is Artinian because R is Noetherian and $\mathfrak{p}$ is a minimal prime of R, and we have that $M_\mathfrak{p}$ is a finitely generated $R_\mathfrak{p}$-module. Therefore,

$$\begin{aligned}\ell_{R_\mathfrak{p}}({}^eM_\mathfrak{p}) &= (p^e)^{\alpha(R_\mathfrak{p})}\ell_{R_\mathfrak{p}}(M_\mathfrak{p}) && \text{(by Proposition 2.2.12)} \\ &= (p^e)^{\alpha(R)+\dim(R/\mathfrak{p})}\ell_{R_\mathfrak{p}}(M_\mathfrak{p}) && \text{(by Corollary 2.2.13)}\end{aligned}$$ □

Corollary 2.2.15. *Let $(R, \mathfrak{m})$ be an F-finite local domain of prime characteristic p. Then*

$\operatorname{rank}({}^eM) = \operatorname{rank}(M)(p^e)^{\alpha(R)+\dim(R)}$ *for any integer* $e \geq 0$.

Proof. Let Q be the field of fractions of R, so that $\operatorname{rank}(M) = \ell_Q(M \otimes_R Q)$. We have

$$\begin{aligned}
\operatorname{rank}({}^eM) &= \ell_Q({}^eM \otimes_R Q) \\
&= \ell_Q({}^e(M \otimes_R Q)) \\
&= \ell_Q(M \otimes_R Q)(p^e)^{\alpha(R)+\dim(R/0)} \\
&= \operatorname{rank}(M)(p^e)^{\alpha(R)+\dim(R)}.
\end{aligned}$$

□

We now introduce the concept of finite F-representation type (FFRT), a notion first introduced by Smith and Van den Bergh [SVdB97] in the context of the simplicity of rings of differential operators. This concept will be particularly relevant in Example 4.2.5.

Definition 2.2.16. Given an F-finite ring R and finitely generated R-module M, we say that M has *finite F-representation type* (*FFRT*) by finitely generated R-modules $M_1, \ldots, M_k$, if for all $e \geq 0$,

$${}^eM \cong \bigoplus_{i=1}^{k} M_i^{n_i}$$

for some $n_1, \ldots, n_k \in \mathbb{N}$.

Example 2.2.17. Let R be an F-finite ring.

(1) If R is Cohen-Macaulay, then ${}^eR$ is a Cohen-Macaulay R-module with maximal dimension, and hence, so is every indecomposable summand of ${}^eR$. Therefore, if R has only a finite number of indecomposable maximal Cohen-Macaulay modules, up to isomorphism, then R has FFRT.

(2) If R is a regular ring, then it is either a polynomial ring or a power series ring over a field, so ${}^eR$ is free over R. This implies that regular rings have FFRT.

2.2.2 *Frobenius functor*

Let $\phi : R \to S$ be a homomorphism of rings, and let N be an R-module and M be an S-module. Since M is an R-module via restriction of scalars through ϕ, $N \otimes_R M$ has an S-module structure given by $s \cdot (n \otimes m) = n \otimes (sm)$ for all $s \in S$, $n \in N$, and $m \in M$. We also note that if $r \in R$, then $(rn) \otimes m = n \otimes \phi(r)m$.

Definition 2.2.18. Let R be a ring of prime characteristic p and $e \in \mathbb{N}$. The *eth iterated Frobenius functor* $\mathcal{F}_R^e$ is a functor from the category of R-modules to itself given by extension of scalars along F^e, the eth iterated Frobenius map. That is, $\mathcal{F}_R^e(N) := N \otimes_R {}^eR$. We simply write $\mathcal{F}^e$ as opposed to $\mathcal{F}_R^e$ if the underlying ring is understood.

Remark 2.2.19. Given a free resolution $(L_\bullet, \partial_\bullet)$ of an R-module M, we can view $\mathcal{F}^e(L_\bullet)$ as raising the entries in the matrices representing the differentials $\partial_\bullet$ to the p^eth power. That is, if ∂_i is represented by the matrix (a_{ij}), then $\mathcal{F}^e(\partial_i)$ is represented by the matrix $(a_{ij}^{p^e})$.

Lemma 2.2.20. *Let R be a ring of prime characteristic p, M and N be R-modules, and $n, m \in \mathbb{N}$. Then*

$$\mathcal{F}^n(N) \otimes_R {}^mM \cong {}^m(\mathcal{F}^{n+m}(N) \otimes_R M).$$

Proof. More generally, consider ring homomorphisms $R \xrightarrow{\varphi} S \xrightarrow{\psi} T$. Let N be an R-module and M be an T-module. Then by restriction of scalars, ${}^\varphi S$ is a R-module and ${}^\psi M$ is S-module. So, $N \otimes_R {}^\varphi S$ is a S-module, and therefore, $(N \otimes_R {}^\varphi S) \otimes_S {}^\psi M$ is a T-module. We

have that, as T-modules,

$$(N \otimes_R {}^{\varphi}S) \otimes_S {}^{\psi}M \cong N \otimes_R ({}^{\psi\circ\varphi}M) \cong N \otimes_R ({}^{\psi\circ\varphi}T \otimes_T M) \cong (N \otimes_R {}^{\psi\circ\varphi}T) \otimes_T M.$$

Therefore, via restriction of scalars along ψ, we have, as T-modules,

$$(N \otimes_R {}^{\varphi}S) \otimes_S {}^{\psi}M \cong {}^{\psi}((N \otimes_R {}^{\varphi}S) \otimes_S {}^{\psi}M) \cong {}^{\psi}((N \otimes_R {}^{\psi\circ\varphi}T) \otimes_T M).$$

Setting $S = T = R$, $\varphi = F^n$, and $\psi = F^m$ gives us

$$\mathcal{F}_R^n(N) \otimes_R {}^{m}M \cong {}^{m}(\mathcal{F}_R^{n+m}(N) \otimes_R M). \qquad \square$$

Proposition 2.2.21. *Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic p such that $[k : k^p] < \infty$, $G_\bullet$ be a complex of finitely generated free R-modules, and M be a finitely generated R-module such that $G_\bullet \otimes_R M$ has homology of finite length. Then*

$$\ell_R(H_i(G_\bullet \otimes_R {}^{e}M)) = \ell_R(H_i(\mathcal{F}^e(G_\bullet) \otimes_R M))p^{e\alpha(R)}.$$

Proof. We have that

$$\begin{aligned}
\ell_R(H_i(G_\bullet \otimes_R {}^{e}M)) &= \ell_R(H_i({}^{e}(\mathcal{F}^e(N) \otimes_R M))) \\
&= \ell_R({}^{e}(H_i(\mathcal{F}^e(N) \otimes_R M))) \\
&= \ell_R(H_i(\mathcal{F}^e(N) \otimes_R M))p^{e\alpha(R)} \qquad \text{(by Proposition 2.2.12).}
\end{aligned}$$ $\square$

2.2.3 *Tight closure and F-regularity*

An important concept in the study of rings with prime characteristic p is tight closure. This notion was initially explored and developed by Hochster and Huneke in the 1980s. Closely

related to tight closure is the concept of F-regularity, which characterizes rings where the tight closure operation behaves particularly well. In this section, we will define tight closure and F-regularity and state some results.

Definition 2.2.22 ([HH88, Definition 1])**.** Let I be an ideal of R. We say that $r \in R$ is in the *tight closure* of I, denoted I^*, if there exists $c \in R^\circ$ such that $cr^{p^e} \in I^{[p^e]}$ for all $e \gg 0$. If $I = I^*$, then we say that I is *tightly closed*.

The notion of tight closure was first introduced by Hochster and Huneke in [HH88]. The tight closure is smaller than other closures, such as integral closure, and follows the usual characteristics of a closure operation. Hochster and Huneke showed in [HH90] that if R is a regular ring, then every ideal is tightly closed.

Definition 2.2.23 ([HH89; HH90])**.** Let R be a ring of characteristic p. If every ideal of R is tightly closed, then we say that R is *weakly F-regular*. If every localization of R is weakly F-regular, then R is *F-regular*. If R is F-finite, we say that R is *strongly F-regular* if for every $c \in R^\circ$, there exists p^e, with $e \geq 1$, such that the R-linear map $R \to R^{1/p^e}$ defined by $1 \mapsto c^{1/p^e}$ splits over R.

It is conjectured that all weakly F-regular, F-regular, and strongly F-regular rings coincide. Several definitions exist for strongly F-regular rings in the non-F-finite case. For an in-depth discussion of these various definitions, we refer the reader to [HY23, §2], which offers a comprehensive analysis and comparison of the different approaches to defining strong F-regularity.

We now summarize some results. We also direct the reader to a comprehensive diagram by Murayama in [Mur19, Figure A.1] relating these concepts, along with other F-singularities.

Theorem 2.2.24. *Let R be a ring of prime characteristic p and M be a finitely generated R-module.*

(1) [HH89, Corollary 3.1], [HH94, Theorem 4.2] *R is regular $\implies$ R is strongly F-regular $\implies$ R is F-regular $\implies$ R is weakly F-regular $\implies$ R is normal.*

(2) [HH94, Theorem 4.2] *If R is weakly F-regular and is homomorphic image of a Cohen-Macaulay ring, then R is Cohen-Macaulay.*

(3) [HH89, Corollary 2.4] *A direct summand of a weakly F-regular ring is weakly F-regular.*

(4) [LS99, Corollaries 4.3, 4.4] *If R is a weakly F-regular $\mathbb{N}$-graded ring, then R is F-regular. If R is also F-finite, then R is strongly F-regular.*

(5) [HH90, Corollary 4.15] *R is weakly F-regular if and only if $R_{\mathfrak{m}}$ is weakly F-regular for every maximal ideal $\mathfrak{m}$.*

(6) [HH94, Proposition 6.27] *If R is strongly F-regular, then R is Cohen-Macaulay.*

CHAPTER 3
INVARIANTS

3.1 Hilbert-Samuel multiplicity

Definition 3.1.1. Let $(R, \mathfrak{m})$ be a local ring, I be an $\mathfrak{m}$-primary ideal of R, and M be a finitely generated R-module. The *Hilbert-Samuel multiplicity* of M with respect to I is

$$\mathrm{e}(I, M) = \lim_{n\to\infty} \frac{d! \cdot \ell_R(M/I^n M)}{n^d},$$

where $d = \dim R$.

We use $\mathrm{e}_R(I, M)$ to clarify the underlying ring, if necessary, and we write $\mathrm{e}(I, R)$ as $\mathrm{e}(I)$ and $\mathrm{e}(\mathfrak{m}, M)$ as $\mathrm{e}(M)$. The Hilbert-Samuel multiplicity is always a nonnegative integer. Many properties of the Hilbert-Samuel multiplicity are well-known. For example, if $I \subseteq J$ are ideals with the same integral closure, then $\mathrm{e}(I) = \mathrm{e}(J)$; if R is equidimensional, then the converse holds as well [Ree61]. We present the following properties of the Hilbert-Samuel multiplicity, which are used in Corollary 3.6.2.

Proposition 3.1.2. *Let $(R, \mathfrak{m})$ be a local ring, I be an $\mathfrak{m}$-primary ideal, and M', M, and M'' be finitely generated R-modules.*

(1) *If $M \cong M'$, then $\mathrm{e}(I, M) = \mathrm{e}(I, M')$.*

(2) *$\mathrm{e}(I, M) = 0$ if and only if $\dim(M) < \dim(R)$.*[1]

(3) *If $0 \to M' \to M \to M'' \to 0$ is exact, then $\mathrm{e}(I, M) = \mathrm{e}(I, M') + \mathrm{e}(I, M'')$.*

Proof. (1) Clear.

[1] Some authors define $\mathrm{e}(-)$ using $d = \dim(M)$ instead of $d = \dim(R)$, and in this case, we have that $\mathrm{e}(I, M) > 0$, even when $\dim(M) < \dim(R)$.

(2) For sufficiently large n, the length $\ell_R(M/I^{n+1}M)$ is a polynomial in n (known as the Hilbert-Samuel polynomial) with degree equal to $\dim(M)$ [LW12, p. 313]. Therefore, $\mathrm{e}(I, M) = 0$ if and only if $\dim(M) < \dim(R)$.

(3) We can tensor the exact sequence with R/I^n to get the right exact sequence $M'/I^nM' \to M/I^nM \to M''/I^nM'' \to 0$. By the additivity of length,

$$\ell(M/I^nM) \leq \ell(M''/I^nM'') + \ell(M'/I^nM').$$

Therefore, $\mathrm{e}(I, M) \leq \mathrm{e}(I, M'')+\mathrm{e}(I, M')$. On the other hand, we can view M' as a submodule of M to get $M'' \cong M/M'$. This implies that $M''/I^nM'' \cong (M/I^nM)/(M'/M' \cap I^nM)$. Thus, we have an exact sequence

$$0 \to M'/(M' \cap I^nM) \to M/I^nM \to M''/I^nM'' \to 0.$$

By the Artin-Rees lemma, there exists $k \in \mathbb{N}_+$ such that $M' \cap I^nM = I^{n-k}(M' \cap I^kM)$ for all $n \geq k$. Thus,

$$\begin{aligned}\ell(M/I^nM) = \ell(M''/I^nM'') + \ell(M'/(M' \cap I^nM)) &= \ell(M''/I^nM'') + \ell(M'/I^{n-k}(M' \cap I^kM)) \\ &\geq \ell(M''/I^nM'') + \ell(M'/I^{n-k}M').\end{aligned}$$

Therefore, $\mathrm{e}(I, M) \geq \mathrm{e}(I, M'') + \mathrm{e}(I, M')$, and equality must hold. □

The following result was introduced in [Lec57, Theorem 2] and is known as Lech's lemma:

Lemma 3.1.3 (Lech's lemma)**.** *Let $(R, \mathfrak{m})$ be a local ring of dimension d, M be a finitely generated R-module such that* $\dim(M) = d$*, and $\underline{x} = x_1, \ldots x_d$ be a system of parameters of*

R. Then

$$\mathrm{e}((\underline{x}), M) = \lim_{\min\{n_1,\ldots,n_d\}\to\infty} \frac{\ell_R(M/(x_1^{n_1}, \ldots, x_d^{n_d})M)}{n_1 \ldots n_d}$$

The Hilbert-Samuel multiplicity behaves particularly well in the Cohen-Macaulay case, partially due to the following lemma:

Lemma 3.1.4. *Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring of characteristic d, M be a finitely generated R-module such that $\dim(M) = d$, and $\underline{x} = x_1, \ldots, x_d$ be a system of parameters of R. Then for any $n_1, \ldots, n_d \in \mathbb{N}$,*

$$\ell_R(M/(x_1^{n_1}, \ldots, x_d^{n_d})M) = n_1 \ldots n_d \ell_R(M/(\underline{x})M).$$

This leads to the following theorem:

Theorem 3.1.5 ([Mat89, Theorems 14.10 and 14.11])**.** *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of dimension d and M be a finitely generated R-module such that $\dim(M) = d$. Let $\underline{x} = x_1, \ldots, x_d$ be a system of parameters of R. Then*

$$\mathrm{e}((\underline{x}), M) = \ell_R(M/(\underline{x})M).$$

Corollary 3.1.6. *Let $(R, \mathfrak{m})$ be a local ring of dimension d and M be a finitely generated maximal Cohen-Macaulay R-module. Then $\mu(M) \leq \mathrm{e}(M)$.*

Proof. Without loss of generality, assume $|R/\mathfrak{m}| = \infty$. Then there exists a system of parameters $\underline{x} = x_1, \ldots, x_d \in \mathfrak{m}$ such that $\overline{(\underline{x})} = \mathfrak{m}$. Since $(\underline{x})$ and $\mathfrak{m}$ have the same integral closure, $\mathrm{e}(\mathfrak{m}, M) = \mathrm{e}((\underline{x}), M)$. Since M is Cohen-Macaulay, $\mathrm{e}((\underline{x}), M) = \ell(M/(\underline{x})M)$. Since

$(\underline{x}) \subseteq \mathfrak{m}$, we have

$$\mu(M) = \ell(M/\mathfrak{m}M) \leq \ell(M/(\underline{x})M) = \mathrm{e}(M). \qquad \square$$

The Hilbert-Samuel multiplicity can be used to detect regularity of a ring.

Theorem 3.1.7 ([Nag62, Theorem 40.6])**.** *Let $(R, \mathfrak{m})$ be an unmixed local ring. Then R is regular if and only if $\mathrm{e}(R) = 1$.*

3.2 Hilbert-Kunz multiplicity

Definition 3.2.1. Let $(R, \mathfrak{m}, k)$ be local ring of prime characteristic p and dimension d, I be an $\mathfrak{m}$-primary ideal of R, and M be a finitely generated R-module. The *Hilbert-Kunz multiplicity* of M with respect to I is

$$\mathrm{e}_{\mathrm{HK}}(I, M) := \lim_{e\to\infty} \frac{\ell_R(M/I^{[p^e]}M)}{(p^e)^d}.$$

In particular, if R is F-finite, we can apply Propositions 2.2.10 and 2.2.12 to get

$$\mathrm{e}_{\mathrm{HK}}(I, M) = \lim_{e\to\infty} \frac{\ell_R((R/I) \otimes_R {}^eM)}{(p^e)^{\alpha+d}}.$$

We often write $\mathrm{e}_{\mathrm{HK}}(I, R)$ as $\mathrm{e}_{\mathrm{HK}}(I)$ and $\mathrm{e}_{\mathrm{HK}}(\mathfrak{m}, M)$ as $\mathrm{e}_{\mathrm{HK}}(M)$. The origins Hilbert-Kunz multiplicity traces back to the following observation of Kunz:

Theorem 3.2.2 ([Kun69, Theorem 3.3])**.** *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p. Set $d = \dim(R)$. The following are equivalent:*

(1) *R is a regular local ring.*

(2) *$\ell_R(R/\mathfrak{m}^{[p]}) = p^d$.*

(3) *$\ell_R(R/\mathfrak{m}^{[p^e]}) = (p^e)^d$ for some $e \in \mathbb{N}_+$.*

(4) *$\ell_R(R/\mathfrak{m}^{[p^e]}) = (p^e)^d$ for all $e \in \mathbb{N}$.*

(3') $\ell_R(R/\mathfrak{m}^{[p^e]}) \leq (p^e)^d$ *for some* $e \in \mathbb{N}_+$.

(2') $\ell_R(R/\mathfrak{m}^{[p]}) \leq p^d$.

Remark 3.2.3. The Hilbert-Kunz multiplicity was shown to exist by Monsky in [Mon83, Theorem 1.8], and it is notoriously difficult to calculate. In his paper, Monsky initially suspected that the Hilbert-Kunz multiplicity was always rational. However, in [Mon08], he conjectured that

$$\mathrm{e_{HK}}\left(\frac{\mathbb{Z}/(2)[x,y,z,u,v]}{(uv+x^3+y^3+xyz)}\right) = \frac{4}{3} + \frac{5}{14\sqrt{7}}.$$

While this specific problem is still open, Brenner in [Bre13] found an example of a ring with a module that has an irrational Hilbert-Kunz multiplicity. It is now conjectured that some may even be transcendental.

Example 3.2.4 ([HM93])**.** Let

$$R = \mathbb{Z}_5[\![w,x,y,z]\!]/(w^4+x^4+y^4+z^4)$$

with maximal ideal $\mathfrak{m}$. Then $\ell(R/\mathfrak{m}^{[5^e]}) = \frac{168}{61}(5^e)^3 - \frac{107}{61}(3^e)$. Therefore, $\mathrm{e_{HK}}(R) = \frac{168}{61}$.

The Hilbert-Kunz multiplicity has similar properties to the Hilbert-Samuel multiplicity. While it is not always an integer, it is always at least 1, if $\dim(M) = \dim(R)$. Moreover, given ideals I and J such that $I \subseteq J$, if I and J have the same tight closure, then $\mathrm{e_{HK}}(I) = \mathrm{e_{HK}}(J)$; if R is complete and equidimensional, then the converse holds as well [HH90, Theorem 8.17]. The following proposition gives a relationship between the Hilbert-Samuel multiplicity and the Hilbert-Kunz multiplicity:

Proposition 3.2.5 ([Hun96, Lemma 6.1])**.** *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p and dimension d, let M be a finitely generated R-module, and let I be an $\mathfrak{m}$-primary ideal. Then*

$$\mathrm{e}(I, M)/d! \leq \mathrm{e}_{\mathrm{HK}}(I, M) \leq \mathrm{e}(I, M)$$

This, along with Corollary 3.1.6, gives us the following corollary:

Corollary 3.2.6. *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p and dimension d, and let M be a maximal Cohen-Macaulay R-module. Then*

$$\mu(M) \leq \mathrm{e}(M) \leq d!\, \mathrm{e}_{\mathrm{HK}}(M).$$

We now aim to list some properties of the Hilbert-Kunz multiplicity that will be used in Corollary 3.6.2. First, we need the following lemma:

Lemma 3.2.7 ([Hun13, Corollary 3.10])**.** *Let $(R, \mathfrak{m})$ be a local reduced ring of prime characteristic p and dimension d, let I be an $\mathfrak{m}$-primary ideal, and let M and N be finitely generated R-modules such that $M_{\mathfrak{p}} \cong N_{\mathfrak{p}}$ for all $\mathfrak{p} \in \mathrm{Assh}(R)$. Then*

$$|\ell_R(M/I^{[p^e]}M) - \ell_R(N/I^{[p^e]}N)| = O((p^e)^{d-1})$$

Proof. Let $S = R \setminus \bigcup_{p \in \mathrm{Assh}(R)} \mathfrak{p}$. Then $S^{-1}R$ is an Artinian ring, and by the Chinese Remainder Theorem, $S^{-1}R \cong \prod_{\mathfrak{p} \in \mathrm{Assh}(R)} R_{\mathfrak{p}}$. Therefore, we have

$$\prod_{\mathfrak{p} \in \mathrm{Assh}(R)} M_{\mathfrak{p}} = S^{-1}M \cong S^{-1}N = \prod_{\mathfrak{p} \in \mathrm{Assh}(R)} N_{\mathfrak{p}}.$$

Since $S^{-1} \operatorname{Hom}_R(M, N) \cong \operatorname{Hom}_R(S^{-1}M, S^{-1}N)$, there exists $\varphi \in \operatorname{Hom}_R(M, N)$ such that

$S^{-1}\varphi$ is an isomorphism. Set $C = \operatorname{coker}(\varphi)$. Since $S^{-1}C = 0$, we have that $\dim(C) < d$. Now, we have the exact sequence $M \xrightarrow{\varphi} N \to C \to 0$, which gives us

$$R/I^{[p^e]} \otimes_R M \to R/I^{[p^e]} \otimes_R N \to R/I^{[p^e]} \otimes_R C \to 0.$$

By the additivity of length and applying the isomorphism in Proposition 2.2.10, we have

$$\begin{aligned} \ell_R(N/I^{[p^e]}N) &\leq \ell_R(M/I^{[p^e]}M) + \ell_R(C/I^{[p^e]}C) \\ &= \ell_R(M/I^{[p^e]}M) + O((p^e)^{d-1}). \end{aligned}$$

□

Proposition 3.2.8. *Let $(R, \mathfrak{m})$ be a local ring, I be an $\mathfrak{m}$-primary ideal, and M', M, and M'' be finitely generated R-modules.*

(1) *If $M \cong M'$, then $\mathrm{e}_{\mathrm{HK}}(I, M) = \mathrm{e}_{\mathrm{HK}}(I, M')$.*

(2) *$\mathrm{e}_{\mathrm{HK}}(I, M) = 0$ if and only if $\dim(M) < \dim(R)$.*[2]

(3) *If $0 \to M' \to M \to M'' \to 0$ is exact, then $\mathrm{e}_{\mathrm{HK}}(I, M) = \mathrm{e}_{\mathrm{HK}}(I, M') + \mathrm{e}_{\mathrm{HK}}(I, M'')$.*

Proof. (1) Clear.

(2) We have that $\mathrm{e}(I, M) = 0$ if and only if $\dim(M) < \dim(R)$, by Proposition 3.1.2. Set $d = \dim(R)$. By Proposition 3.2.5, $\mathrm{e}(I, M)/d! \leq \mathrm{e}_{\mathrm{HK}}(I, M) \leq \mathrm{e}(I, M)$, so $\mathrm{e}_{\mathrm{HK}}(I, M) = 0$ if and only if $\dim(M) < \dim(R)$.

(3) Without loss of generality, we may assume that R is complete and k is perfect, and therefore, F-finite. If R is reduced, then $R_{\mathfrak{p}}$ is a field for all $\mathfrak{p} \in \operatorname{Assh}(R)$, and we have

[2]Some authors define $\mathrm{e}_{\mathrm{HK}}(-)$ using $d = \dim(M)$ instead of $d = \dim(R)$, and in this case, we have that $\mathrm{e}_{\mathrm{HK}}(I, M) > 0$, even when $\dim(M) < \dim(R)$.

$M'_{\mathfrak{p}} \oplus M''_{\mathfrak{p}} = M_{\mathfrak{p}}$. Applying, by Lemma 3.2.7, dividing by p^{ed}, and taking limits gives us

$$\lim_{e\to\infty} \left| \frac{\ell_R(M'/I^{[p^e]}M')}{p^{ed}} + \frac{\ell_R(M''/I^{[p^e]}M'')}{p^{ed}} - \frac{\ell_R(M/I^{[p^e]}M)}{p^{ed}} \right| = \lim_{e\to\infty} \frac{O(p^{e(d-1)})}{p^{ed}}.$$

Thus, $\mathrm{e}_{\mathrm{HK}}(I, M) = \mathrm{e}_{\mathrm{HK}}(I, M') + \mathrm{e}_{\mathrm{HK}}(I, M'')$.

If R is not reduced, then there exists $e_0 \in \mathbb{N}$ such that $\sqrt{0}^{[p^{e_0}]} = 0$. Therefore, for any finitely generated R-module N, we have $\sqrt{0} \cdot {}^{e_0}N = {}^{e_0}(\sqrt{0}^{[p^{e_0}]}N) = 0$. So, ${}^{e_0}N$ is an $R/\sqrt{0}$-module. This gives us that $0 \to {}^{e_0}M' \to {}^{e_0}M \to {}^{e_0}M'' \to 0$ is a short exact sequence of $R/\sqrt{0}$-modules. Since $R/\sqrt{0}$ is reduced, we can use the same argument above, as well as Proposition 2.2.12, to get

$$\lim_{e\to\infty} \left| \frac{\ell_R(M'/I^{[p^e]}M')}{p^{ed}} + \frac{\ell_R(M''/I^{[p^e]}M'')}{p^{ed}} - \frac{\ell_R(M/I^{[p^e]}M)}{p^{ed}} \right| = \lim_{e\to\infty} \frac{O(p^{e(d-1)})}{p^{ed}}.$$

Thus, $\mathrm{e}_{\mathrm{HK}}(I, M) = \mathrm{e}_{\mathrm{HK}}(I, M') + \mathrm{e}_{\mathrm{HK}}(I, M'')$. □

Like the Hilbert-Samuel multiplicity, the Hilbert-Kunz multiplicity can also be used to detect regularity.

Theorem 3.2.9 ([WY00, Theorem 1.5], c.f. [HY02, Theorem 3.1])**.** *Let $(R, \mathfrak{m})$ be an unmixed local ring of prime characteristic p. Then R is regular if and only if $\mathrm{e}_{\mathrm{HK}}(R) = 1$.*

In fact, this result provides an even stronger characterization than Theorem 3.1.7, since the Hilbert-Kunz multiplicity is always less than or equal to the Hilbert-Samuel multiplicity.

3.3 F-signature

For a local F-finite ring $(R, \mathfrak{m})$, R is regular if and only if ${}^eR$ is a free R-module. For a non-regular ring R, Smith and Van den Bergh [SVdB97, §3.3] investigated how the free part

of ${}^eR$ behaves as e grows, in the context of FFRT modules. Specifically, they were interested in understanding the proportion of free summands in these modules, which measures how "far away" the ring is from being regular. Huneke and Leuschke formalized this idea by introducing the F-signature [HL02, §2]. We first define the following:

Definition 3.3.1. Let $(R, \mathfrak{m})$ be an F-finite local ring and M be a finitely generated R-module. For each $e \in \mathbb{N}$, the *eth Frobenius splitting number of* M (or *free rank of* ${}^eM$), denoted $a_e(M)$, is the largest rank of a free summand of ${}^eM$.

With the setup in Definition 3.3.1, we can always write ${}^eM \cong R^{a_e(M)} \oplus M_e$, where M_e has no non-zero direct free summands. In fact, if we have two decompositions ${}^eM \cong R^{a_e(M)} \oplus M_e \cong R^{b_e(M)} \oplus N_e$ such that and both M_e and N_e have no free summands, then we must have that $a_e(M) = b_e(M)$. Indeed, without loss of generality, assume $a_e(M) \geq b_e(M)$. Since R is local and ${}^eM$ is finitely generated, we can cancel $b_e(M)$ copies of R in the direct sum to get $R^{a_e(M)-b_e(M)} \oplus M_e \cong N_e$ (see Remark 3.3.2). Since N_e has no free summands, $a_e(M) = b_e(M)$.

Remark 3.3.2. In general, direct summands cannot be canceled. That is, if $A \oplus C \cong B \oplus C$ for modules A, B, C of a ring R, it does not necessarily follow that $A \cong B$. For example, consider the ring $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 - 1)$. On one hand, we have $R^3 \cong R^2 \oplus R$. Consider the map $\varphi : R^3 \to R$ defined by $(a, b, c) \mapsto a\overline{x} + b\overline{y} + c\overline{z}$. This map is onto, so it splits, and we have $R^3 = \ker \varphi \oplus R$. It is known that $\ker \varphi$ is not free, so $R^2 \oplus R \cong \ker \varphi \oplus R$, but $R^2 \ncong \ker \varphi$. However, if R is local and A, B, C are finitely generated, then $A \oplus C \cong B \oplus C$ implies $A \cong B$. This follows from applying the Krull-Schmidt theorem to the $\mathfrak{m}$-adic

completion $\widehat{R}$, which allows us to conclude that $\widehat{A} \cong \widehat{B}$. This, in turn, implies that $A \cong B$ [Vas65, Proposition 6.1].

We are now ready to define the F-signature.

Definition 3.3.3. Let $(R, \mathfrak{m})$ be an F-finite local ring and M be a finitely generated R-module. Set $\alpha = \alpha(R)$ and $d = \dim(R)$. The *F-signature* of M is defined as

$$s(M) := \lim_{e \to \infty} \frac{a_e(M)}{(p^e)^{\alpha+d}}.$$

When it was formalized, it was unknown as to whether or not the F-signature existed in general, and it was not until Tucker [Tuc12] proved its existence for any F-finite, local ring of prime characteristic.

The F-signature is always a real number between 0 and 1, inclusively; at these extremes, the F-signature characterizes a ring in the following way:

Theorem 3.3.4. *Let $(R, \mathfrak{m})$ be an F-finite local ring of prime characteristic p.*

(1) [AL03, Theorem 0.2] *R is strongly F-regular if and only if $s(R) > 0$.*

(2) [HL02, Corollary 16] *R is regular if and only if $s(R) = 1$.*

In fact, for a ring R of dimension $d \geq 2$, R is regular if and only if $s(R) \geq 1 - \frac{1}{d!p^d}$ [Yao06, Theorem 3.1]. This implies that the F-signature cannot be arbitrarily close to 1 for rings of dimension $d \geq 2$, since if $s(R) \geq 1 - \frac{1}{d!p^d}$, then R is regular and $s(R) = 1$.

3.4 Frobenius Betti numbers

Recall that when R is an F-finite local ring and M is a finitely generated R-module, the Hilbert Kunz multiplicity of M can be expressed as $\mathrm{e_{HK}}(M) = \lim_{e\to\infty} \ell_R(k \otimes_R {}^eM)/(p^e)^{\alpha+d}$. Since $k \otimes_R {}^eM = \mathrm{Tor}_0^R(k, {}^eM)$, this leads to a natural extension of the Hilbert-Kunz multiplicity to higher Tor functors, called the Frobenius Betti numbers. Here, we present a more general definition that reduces to the above in the case of F-finite rings.

Definition 3.4.1. Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic p and dimension d, M be finitely generated R-module, L be an R-module of finite length, and $G_\bullet$ be a resolution of L by finitely generated free modules. The *ith Frobenius Betti number of M with respect to L* is

$$\beta_i^F(L, M) := \lim_{e\to\infty} \frac{\ell_R(H_i(\mathcal{F}^e(G_\bullet) \otimes_R M))}{p^{ed}}.$$

In particular, if R is F-finite, then we can apply Proposition 2.2.21 to get

$$\beta_i^F(L, M) = \lim_{e\to\infty} \frac{\ell_R(\mathrm{Tor}_i^R(L, {}^eM))}{(p^e)^{\alpha+d}},$$

where $\alpha = \alpha(R) < \infty$. We denote $\beta_i^F(M) = \beta_i^F(k, M)$.

Remark 3.4.2. Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p and dimension d, M be a finitely generated R-module, and L be an R-module of finite length. Let $R \to S$ be a faithfully flat ring homomorphism such that $\mathfrak{m}S$ is the unique maximal ideal of the local ring S. Then $\beta_i^F(L, M) = \beta_i^F(L \otimes_R S, M \otimes_R S)$. Indeed, let $G_\bullet$ be a free resolution of L. Since S is flat over R, we have that $\dim(S) = d$ and $G_\bullet \otimes_R S$ is a free resolution of $L \otimes_R S$.

Therefore,

$$\begin{aligned}\beta_i^F(L,M) := \lim_{e\to\infty} \frac{\ell_R(H_i(\mathcal{F}_R^e(G_\bullet)\otimes_R M))}{p^{ed}} &= \lim_{e\to\infty} \frac{\ell_S(H_i((\mathcal{F}_R^e(G_\bullet)\otimes_R M)\otimes_R S))}{p^{ed}} \\ &= \lim_{e\to\infty} \frac{\ell_S(H_i(\mathcal{F}_S^e(G_\bullet\otimes_R S)\otimes_S (M\otimes_R S)))}{p^{ed}} \\ &= \beta_i^F(L\otimes_R S, M\otimes_R S).\end{aligned}$$

Seibert's work in [Sei89] shows that Frobenius Betti numbers exist. We now present the relevant parts of the main theorem from Seibert [Sei89] that are necessary for our discussion. For the complete theorem, we refer the reader to the original source.

Theorem 3.4.3 ([Sei89, Main Theorem])**.** *Let $(R,\mathfrak{m})$ be an F-finite local ring of characteristic p, and g be a function from the family of all finitely generated R-modules to the integers. Assume that $g(N)\leq g(N')+g(N'')$ for all short exact sequences $0\to N'\to N\to N''\to 0$ of finitely generated R-modules. Then for any finitely generated R-module M, as $e\to\infty$, we have that $g({}^eM) = c(M)(p^e)^{\alpha(R)+\dim(M)}+O((p^e)^{\alpha(R)+\dim(M)-1})$ for some $c(M)\in\mathbb{R}$; moreover, $c(-)$ is an additive function on exact sequences.*

We now present the following properties of the Frobenius Betti numbers, which used for Corollary 3.6.2. The proofs for these results closely follow [Sei89, Proposition 1].

Proposition 3.4.4. *Let $(R,\mathfrak{m})$ be an F-finite local ring of prime characteristic p, let M', M, and M'' be finitely generated R-modules, let L be an R-module of finite length, and let $i\in\mathbb{N}$. Then*

(1) *If $M\cong M'$, then $\beta_i^F(L,M)=\beta_i^F(L,M')$.*

(2) *If $\dim(M)<\dim(R)$, then $\beta_i^F(L,M)=0$.*

(3) *If* $0 \to M' \to M \to M'' \to 0$ *is exact, then* $\beta_i^F(L, M') = \beta_i^F(L, M) + \beta_i^F(L, M'')$.

Proof. (1) is clear.

For (2) and (3), we may assume that R is complete and F-finite (in particular, we do not need to assume that k is perfect).

Let $G_\bullet$ be a free resolution of L. Define $g(P) = \ell_R(H_i(G_\bullet \otimes_R P))$ for all finitely generated R-modules P. For any short exact sequence $0 \to N' \to N \to N'' \to 0$ of finitely generated R-modules, we have the exact sequence $H_i(G_\bullet \otimes N') \to H_i(G_\bullet \otimes N) \to H_i(G_\bullet \otimes N'')$. Therefore, $g(N) \leq g(N) + g(N'')$. Since homology and length commute with direct sums, equality holds if the sequence splits, so Theorem 3.4.3 applies. Now, by Proposition 2.2.21, for any finitely generated R-module P,

$$\ell_R(H_i(\mathcal{F}_R^e(G_\bullet) \otimes P)) = \ell_R(H_i(G_\bullet \otimes {}^eP))p^{-e\alpha(R)} = g({}^eP)p^{-e\alpha(R)}.$$

Hence, in light of Theorem 3.4.3,

$$\beta_i^F(L, P) := \lim_{e\to\infty} \frac{\ell_R(H_i(\mathcal{F}_R^e(G_\bullet) \otimes P))}{p^{ed}} = \lim_{e\to\infty} \frac{g({}^eP)}{(p^e)^{\alpha(R)+d}} = \lim_{e\to\infty} c(P)(p^e)^{\dim P - d},$$

for some $c(P) \in \mathbb{R}$.

For (2), clearly if $\dim(M) < \dim(R)$, then $\beta_i^F(L, M) = 0$.

For (3), it is clear that $\beta_i^F(L, M) \leq \beta_i^F(L, M') + \beta_i^F(L, M'')$. For the other direction, since $c(-)$ is additive on exact sequences, we have that $c(M) = c(M') + c(M'')$. Without

loss of generality, assume $\dim(M) = \dim(M') \geq \dim(M'')$. Then

$$\begin{aligned}
\beta_i^F(L, M) &= \lim_{e\to\infty} c(M)(p^e)^{\dim M - d} \\
&= \lim_{e\to\infty} c(M')(p^e)^{\dim M - d} + \lim_{e\to\infty} c(M'')(p^e)^{\dim M - d} \\
&\geq \lim_{e\to\infty} c(M')(p^e)^{\dim M' - d} + \lim_{e\to\infty} c(M'')(p^e)^{\dim M'' - d} \\
&= \beta_i^F(L, M') + \beta_i^R(L, M'').
\end{aligned}$$

Hence, we have equality. □

Like the Hilbert-Kunz multiplicity, these higher in invariants also detect regularity, which was shown by Aberbach and Li. They established the following theorem, which serves as a cornerstone for Theorem 4.3.1.[3]

Theorem 3.4.5 ([AL08, Corollary 3.2])**.** *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p. The following are equivalent:*

(1) *R is regular.*

(2) *$\beta_i^F(R) = 0$ for all $i \in \mathbb{N}_+$.*

(3) *$\beta_i^F(R) = 0$ for some $i \in \mathbb{N}_+$.*

3.5 Frobenius Euler characteristic

In [DPY22], De Stefani, Polstra, and Yao extended the Frobenius Betti numbers to encompass a wide range of finitely generated modules over rings of prime characteristic, without assuming locality. One of the invariants they studied was the so-called Frobenius Euler

[3]We note that in their work, Aberbach and Li denote $\beta_i^F(R)$ as $\mathfrak{t}_i(R)$.

characteristic. While they extended this invariant to the global case, we define the local case:

Definition 3.5.1. Let $(R, \mathfrak{m}, k)$ be a local ring of prime characteristic p and dimension d, M be finitely generated R-module, L be an R-module of finite length, and $G_\bullet$ be a resolution of L by finitely generated free modules. The *jth Frobenius Euler characteristic* of M is

$$\chi_j^F(L, M) := \lim_{e\to\infty} \sum_{i=0}^{j} (-1)^i \frac{\ell_R(H_{j-i}(\mathcal{F}^e(G_\bullet) \otimes_R M))}{p^{ed}} = \sum_{i=0}^{j} (-1)^i \beta_{j-i}^F(L, M).$$

In particular, if R is F-finite, then we can apply Proposition 2.2.21 to get

$$\chi_j^F(L, M) = \lim_{e\to\infty} \sum_{i=0}^{j} (-1)^i \frac{\ell_R(\mathrm{Tor}_{j-i}^R(L, {}^eM))}{(p^e)^{\alpha+d}}$$

where $\alpha = \alpha(R) < \infty$. We denote $\chi_j^F(M) = \chi_j^F(k, M)$.

The Frobenius Euler characteristic also exists due to the results of Seibert in [Sei89]. We now present the following properties of the Frobenius Euler characteristic, which are used in Corollary 3.6.2.

Proposition 3.5.2. *Let $(R, \mathfrak{m})$ be an F-finite local ring of prime characteristic p, let M', M, and M'' be finitely generated R-modules, let L be an R-module of finite length, and let $j \in \mathbb{N}$. Then*

(1) *If $M \cong M'$, then $\chi_j^F(L, M) = \chi_j^F(L, M')$.*

(2) *If $\dim(M) < \dim(R)$, then $\chi_j^F(L, M) = 0$.*

(3) *If $0 \to M' \to M \to M'' \to 0$ is exact, then $\chi_j^F(L, M) = \chi_j^F(L, M') + \chi_j^F(L, M'')$.*

Proof. (1) Clear.

(2) Since $\dim(M) < \dim(R)$, we have that $\beta_i^F(L, M) = 0$ for all $0 \leq i \leq j$. Therefore $\chi_j^F(L, M) = \sum_{i=0}^{j}(-1)^i \beta_{j-i}^F(L, M) = 0$.

(3) We have that $\beta_i^F(L, M') - \beta_i^F(L, M) + \beta_i^F(L, M'') = 0$ for all i. Therefore,

$$\begin{aligned}
&\chi_j^F(L, M') - \chi_j^F(L, M) + \chi_j^F(L, M'') \\
&\qquad = \sum_{i=0}^{j}(-1)^i \beta_{j-i}^F(L, M') - \sum_{i=0}^{j}(-1)^i \beta_{j-i}^F(L, M) + \sum_{i=0}^{j}(-1)^i \beta_{j-i}^F(L, M'') \\
&\qquad = \sum_{i=0}^{j}(-1)^i \left(\beta_{j-i}^F(L, M') - \beta_{j-i}^F(L, M) + \beta_{j-i}^F(L, M'')\right) \\
&\qquad = 0.
\end{aligned}$$

□

Example 3.5.3 ([DHNB17, Example 3.2])**.** Let S be an F-finite regular local ring of characteristic p and set $R = S/(f)$ for some $f \in S$. We can write ${}^eR \cong R^{a_e(R)} \oplus M_e$, where $a_e(R)$ is the eth Frobenius splitting number of R and M_e has no free summands. Consider a minimal free resolution of ${}^eR$:

$$\cdots \to R^{\beta_2({}^eR)} \to R^{\beta_1({}^eR)} \to R^{\beta_0({}^eR)} \to ({}^eR \to)\ 0.$$

We have that $\beta_0({}^eR) = a_e(R) + \beta_0(M_e)$ and $\beta_i({}^eR) = \beta_i(M_e)$ for all $i > 0$. Since M_e is a maximal Cohen-Macaulay module with no free summands, $\beta_i(M_e) = \beta_0(M_e)$ for all $i > 0$

[Eis80, Proposition 5.3 and Theorem 6.1]. Therefore,

$$\begin{aligned}\chi_j({}^eR) &= \chi_j(R^{a_e(R)}) + \chi_j(M_e)\\ &= \sum_{i=0}^{j}(-1)^i\beta_{j-i}(R^{a_e(R)}) + \sum_{i=0}^{j}(-1)^i\beta_{j-i}(M_e)\\ &= (-1)^j a_e(R) + \sum_{i=0}^{j}(-1)^i\beta_0(M_e).\end{aligned}$$

Thus, $\chi_j({}^eR) = a_e(R) + \beta_0(M_e)$ if j is even, and $\chi_j({}^eR) = -a_e(R)$ if j is odd. Therefore, we have that

$$\chi_0^F(R) := \lim_{e\to\infty}\frac{\chi_0({}^eR)}{(p^e)^{\alpha+d}} = \lim_{e\to\infty}\frac{a_e(R)}{(p^e)^{\alpha+d}} + \lim_{e\to\infty}\frac{\beta_0(M_e)}{(p^e)^{\alpha+d}} = s(R) + \lim_{e\to\infty}\frac{\beta_0(M_e)}{(p^e)^{\alpha+d}}.$$

Note that this implies that $\lim_{e\to\infty}\frac{\beta_0(M_e)}{(p^e)^{\alpha+d}} = \chi_0^F(R) - s(R) = \mathrm{e_{HK}}(R) - s(R)$. Hence,

$$\chi_j^F(R) = \lim_{e\to\infty}\frac{\chi_j({}^eR)}{(p^e)^{\alpha+d}} = \begin{cases}\lim\limits_{e\to\infty}\dfrac{a_e(R)}{(p^e)^{\alpha+d}} + \lim\limits_{e\to\infty}\dfrac{\beta_0(M_e)}{(p^e)^{\alpha+d}} & \text{if } j \text{ is even}\\ \lim\limits_{e\to\infty}\dfrac{-a_e(R)}{(p^e)^{\alpha+d}} & \text{if } j \text{ is odd}\end{cases}$$
$$= \begin{cases}\mathrm{e_{HK}}(R) & \text{if } j \text{ is even}\\ -s(R) & \text{if } j \text{ is odd}\end{cases}$$

for all $j \geq 0$.

The Frobenius Euler characteristic has been shown to also detect regularity. For instance, Li demonstrated that $\chi_1^F(R) = -1$ if and only if R is regular [Li08, Theorem 2.2]. However, it remained an open question whether higher orders of this invariant could similarly detect regularity. Polsta and Smirnov addressed this question partially, proving results under the assumption that R is strongly F-regular. They conjectured that their results might hold under a weaker hypothesis. We present their results here:

Theorem 3.5.4 ([PS21, Theorem 3.23]). *Let $(R, \mathfrak{m})$ be an F-finite, strongly F-regular ring of prime characteristic p. The following are equivalent:*

(1) *R is a regular local ring.*

(2) *$\chi_j({}^eR) = (-1)^j \operatorname{rank}({}^eR)$ for every $e \in \mathbb{N}_+$.*

(3) *$\chi_j({}^eR) = (-1)^j \operatorname{rank}({}^eR)$ for some $e \in \mathbb{N}_+$.*

(4) *$\chi_j^F(R) = (-1)^j$.*

3.6 Associativity formula

Proposition 3.6.1 ("Associativity formula"). *Let R be a ring and f be a function from the family of all finitely generated R-modules to the integers such that for any finitely generated R-modules M', M, M'', the following hold:*

(a) *If $M \cong M'$, then $f(M) = f(M')$,*

(b) *If $\dim(M) < \dim(R)$, then $f(M) = 0$, and*

(c) *If $0 \to M \to M' \to M'' \to 0$ is exact, then $f(M) = f(M') + f(M'')$.*

Then for any finitely generated R-module M, we have

$$f(M) = \sum_{\mathfrak{p} \in \operatorname{Assh}(R)} \ell_{R_\mathfrak{p}}(M_\mathfrak{p}) f(R/\mathfrak{p}).$$

Proof. Consider a prime filtration of an R-module M as follows:

$$0 = M_n \subset M_{n-1} \subset \cdots \subset M_0 = M.$$

Then we have short exact sequences of the form $0 \to M_i \to M_{i-1} \to M_{i-1}/M_i \to 0$ for each

$i = 1, \ldots, n$. By the additivity of f,

$$\begin{aligned} f(M) = f(M) + f(M_0/M) &= f(N) + f(M/N) + f(M_0/M) \\ &= \cdots \\ &= f(M_n) + f(M_{n-1}/M_n) + \cdots + f(M/N) + f(M_0/M) \\ &= \sum_{i=1}^{n} f(M_{i-1}/M_i). \end{aligned}$$

Since the above filtration is a prime filtration, for each $i = 1, \ldots, n$, we have that $M_{i-1}/M_i \cong R/\mathfrak{p}_i$ for some $\mathfrak{p}_i \in \operatorname{Spec}(R)$. Define

$$\Lambda := \{\mathfrak{p} \in \{\mathfrak{p}_1, \ldots, \mathfrak{p}_n\} \mid \dim(R/\mathfrak{p}) = \dim(R)\} \subseteq \operatorname{Assh}(R),$$

and let $n_{\mathfrak{p}} := |\{i \in \{1, \ldots, n\} \mid \mathfrak{p}_i = \mathfrak{p}\}|$ for every $\mathfrak{p} \in \operatorname{Assh}(R)$. Since $f(R/\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\dim(R/\mathfrak{p}) < \dim(R)$,

$$f(M) = \sum_{i=1}^{n} f(M_{i-1}/M_i) = \sum_{i=1}^{n} f(R/\mathfrak{p}_i) = \sum_{\mathfrak{p} \in \operatorname{Assh}(R)} n_{\mathfrak{p}} f(R/\mathfrak{p}).$$

To finish the proof, we show that $n_{\mathfrak{p}} = \ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$ for every $\mathfrak{p} \in \operatorname{Assh}(R)$. Let $\mathfrak{p} \in \operatorname{Assh}(R)$ and localize the above filtration at $\mathfrak{p}$:

$$0 = (M_n)_{\mathfrak{p}} \subset (M_{n-1})_{\mathfrak{p}} \subset \cdots \subset (M_0)_{\mathfrak{p}} = M_{\mathfrak{p}}.$$

Then $(M_{i-1})_{\mathfrak{p}}/(M_i)_{\mathfrak{p}} \cong (M_{i-1}/M_i)_{\mathfrak{p}} \cong (R/\mathfrak{p}_i)_{\mathfrak{p}}$ for each $i = 1, 2, \ldots, n$. If $\mathfrak{p} \neq \mathfrak{p}_i$ for some i, then $(R/\mathfrak{p}_i)_{\mathfrak{p}} = 0$, since $\operatorname{Ann}(R/\mathfrak{p}_i) \not\subseteq \mathfrak{p}$. Otherwise, if $\mathfrak{p} = \mathfrak{p}_i$ for some i, then $(R/\mathfrak{p}_i)_{\mathfrak{p}} =$

$(R/\mathfrak{p})_{\mathfrak{p}}$. Therefore, we have

$$\ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \sum_{i=1}^{n} \ell_{R_{\mathfrak{p}}}((M_{i-1}/M_i)_{\mathfrak{p}}) = \sum_{i=1}^{n} \ell_{R_{\mathfrak{p}}}((R/\mathfrak{p}_i)_{\mathfrak{p}}) = \sum_{\mathfrak{p}_i=\mathfrak{p}} \ell_{R_{\mathfrak{p}}}(R/\mathfrak{p}) = \sum_{\mathfrak{p}_i=\mathfrak{p}} 1 = n_{\mathfrak{p}}.$$

Hence, $f(M) = \sum_{\mathfrak{p}\in \operatorname{Assh}(R)} \ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) f(R/\mathfrak{p})$. □

Corollary 3.6.2. *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p and M be a finitely generated R-module. Let f be any of the following invariants:*

- $\mathrm{e}(I, M)$ *for any $\mathfrak{m}$-primary ideal I;*
- $\mathrm{e}_{\mathrm{HK}}(I, M)$ *for any $\mathfrak{m}$-primary ideal I;*
- $\beta_i^F(L, M)$ *for any finite length R-module L and $i \geq 0$;*
- $\chi_j^F(L, M)$ *for any finite length R-module L and $j \geq 0$.*

Then

$$f(M) = \sum_{\mathfrak{p}\in \operatorname{Assh}(R)} \ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) f(R/\mathfrak{p}).$$

Corollary 3.6.3. *Let R, M, and f be as in Corollary* 3.6.2*. If R is F-finite, then*

$$f({}^eM) = (p^e)^{\alpha+d} f(M),$$

where $\alpha = \alpha(R)$ and $d = \dim(R)$.

Proof. We have

$$\begin{aligned} f({}^eM) &= \sum_{\mathfrak{p}\in \operatorname{Assh}(R)} \ell_{R_{\mathfrak{p}}}({}^eM_{\mathfrak{p}}) f(R/\mathfrak{p}) && \text{(by Proposition 3.6.2)} \\ &= (p^e)^{\alpha+d} \sum_{\mathfrak{p}\in \operatorname{Assh}(R)} \ell_{R_{\mathfrak{p}}}(M_{\mathfrak{p}}) f(R/\mathfrak{p}) && \text{(by Corollary 2.2.14)} \\ &= (p^e)^{\alpha+d} f(M). && \text{(by Proposition 3.6.2)} \end{aligned}$$

□

Corollary 3.6.4. *Let R, M, and f be as in Corollary* 3.6.2. *Further, assume that R is a domain. Then*

$$f(M) = f(R)\operatorname{rank}(M).$$

Proof. Let Q be the field of fractions of R, so that $\operatorname{rank}(M) = \ell_Q(M \otimes_R Q)$. We have

$$f(M) = \sum_{\mathfrak{p} \in \operatorname{Assh}(R)} \ell_{R_\mathfrak{p}}(M_\mathfrak{p}) f(R/\mathfrak{p}) = \ell_Q(M \otimes_R Q) f(R) = f(R)\operatorname{rank}(M). \qquad \square$$

CHAPTER 4
MAIN RESULTS

In this section, we will present out main results, which revolve around determining regularity of a ring by using the Frobenius Euler characteristic. In this work, we use a multiplicity invariant, and in most cases, the Hilbert-Samuel multiplicity and the Hilbert-Kunz multiplicity can be used interchangeably. Therefore, throughout this chapter, we will use $\mathrm{e}_*(-)$ represent either the Hilbert-Samuel multiplicity or the Hilbert-Kunz multiplicity. Moreover, throughout this chapter, we will set $d = \dim(R)$ and $\alpha = \alpha(R)$.

The following is a direct consequence of Kunz's work in [Kun69]. We present a proof for completeness.

Proposition 4.0.1. *Let $(R, \mathfrak{m})$ be a regular local ring of prime characteristic p, M be a finitely generated R-module, and $j \in \mathbb{N}$. Then $\chi_j^F(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$.*

Proof. Without loss of generality, we may assume that R is F-finite (see Remark 2.2.9 and Remark 3.4.2). Since R is regular, ${}^eR$ is free as an R-module by Corollary 2.2.7. Therefore ${}^eR \cong R^{a_e(R)}$, and

$$\beta_i({}^eR) = \begin{cases} a_e(R) & \text{if } i = 0 \\ 0 & \text{if } i \neq 0. \end{cases}$$

Thus, $\chi_j({}^eR) = (-1)^j a_e(R) = (-1)^j \operatorname{rank}({}^eR)$ and hence,

$$\chi_j^F(R) = \lim_{e\to\infty} \frac{(-1)^j \operatorname{rank}({}^eR)}{\operatorname{rank}({}^eR)} = (-1)^j.$$

By Corollary 3.6.4, $\operatorname{rank}(M) = \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$ and $\chi_j^F(M) = \chi_j^F(R) \operatorname{rank}(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$. □

Note that when R is regular, $\mathrm{e}_*(R) = 1$. Despite this, we retained $\mathrm{e}_*(R)$ in the proposition to highlight its significance, as we conjecture that the converse true, under mild conditions:

Conjecture 4.0.2. *Let $(R, \mathfrak{m})$ be an unmixed local ring of prime characteristic p, and let M be a finitely generated R-module such that $\operatorname{Assh}(M) = \operatorname{Assh}(R)$. If $\chi_j^F(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$ for some $j \in \mathbb{N}$, then R is regular.*

We note that the condition that R is unmixed is necessary. Indeed, consider the following ring:

$$R = k[\![x, y]\!]/((x) \cap (x^2, y)) = k[\![x, y]\!]/(x^2, xy).$$

We have that $\chi_0^F(R) = \mathrm{e}_{\mathrm{HK}}(R) = 1$, but R is not regular.

Remark 4.0.3. If there exists a ring extension $R \to S$, such as described in Remark 2.2.5, such that S is unmixed, then it suffices to study $\chi_j^F(M \otimes_R S)$. Indeed, if S is regular when $\chi_j^F(M \otimes_R S) = (-1)^j \frac{\mathrm{e}_*(M \otimes_R S)}{\mathrm{e}_*(S)}$, then R is regular when $\chi_j^F(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$. If R is unmixed and excellent, for example, then such a ring extension would exist.

4.1 Conjecture equivalency

Lemma 4.1.1. *Let $(R, \mathfrak{m})$ be an local ring, let M be a finitely generated R-module, and let $j \in \mathbb{N}$. Then*

$$\chi_j(M) = \frac{\mathrm{e}_*(\Omega_{j+1}(M)) + (-1)^j \mathrm{e}_*(M)}{\mathrm{e}_*(R)}$$

Proof. Consider an minimal free resolution of M:

$$0 \to \Omega_{j+1}(M) \to R^{\beta_j(M)} \to R^{\beta_{j-1}(M)} \to \cdots \to R^{\beta_0(M)} \to M \to 0.$$

Propositions 3.1.2 and 3.2.8 imply that

$$\begin{aligned}\mathrm{e}_*(\Omega_{j+1}(M)) &= \left(\sum_{i=0}^{j}(-1)^i\,\mathrm{e}_*(R^{\beta_{j-i}(M)})\right) - (-1)^j\,\mathrm{e}_*(M)\\ &= \left(\sum_{i=0}^{j}(-1)^i\beta_{j-i}(M)\,\mathrm{e}_*(R)\right) - (-1)^j\,\mathrm{e}_*(M)\\ &= \chi_j(M)\,\mathrm{e}_*(R) - (-1)^j\,\mathrm{e}_*(M).\end{aligned}$$

Solving for $\chi_j(M)$ yields the desired result. □

Lemma 4.1.2. *Let $(R,\mathfrak{m})$ be a local ring, M be a finitely generated R-module, and $j \in \mathbb{N}$. Then $\chi_j(M) \geq (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$.*

Proof. By Lemma 4.1.1,

$$\chi_j(M) = \frac{\mathrm{e}_*(\Omega_{j+1}(M)) + (-1)^j\,\mathrm{e}_*(M)}{\mathrm{e}_*(R)} \geq (-1)^j\frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}.$$ □

Lemma 4.1.3. *Let $(R,\mathfrak{m})$ be a local ring of prime characteristic p, let M be a finitely generated R-module, and let $j \in \mathbb{N}$. Then $\chi_j^F(M) \geq (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$.*

Proof. Without loss of generality, we may assume that R is F-finite. Applying Lemma 4.1.2 and Corollary 3.6.3, we have

$$\chi_j^F(M) = \lim_{e\to\infty}\frac{\chi_j({}^eM)}{(p^e)^{\alpha+d}} \geq \lim_{e\to\infty}\frac{(-1)^j\frac{\mathrm{e}_*({}^eM)}{\mathrm{e}_*(R)}}{(p^e)^{\alpha+d}} = (-1)^j\frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}.$$ □

Lemma 4.1.4. *Let $(R,\mathfrak{m})$ be an unmixed local ring, M be a finitely generated R-module, and $j \in \mathbb{N}$. The following are equivalent:*

(1) $\mathrm{pd}(M) > j$.

(2) $\chi_j(M) \geq \frac{1}{\mathrm{e}_*(R)} + (-1)^j\frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$.

(3) $\chi_j(M) \neq (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$.

Proof. (1) $\implies$ (2): Suppose $\mathrm{pd}(M) > j$. Since $\Omega_{j+1}(M) \subseteq R^{\beta_j(M)}$ and R is unmixed, we have

$$\mathrm{Ass}(\Omega_{j+1}(M)) \subseteq \mathrm{Ass}(R^{\beta_j(M)}) = \mathrm{Ass}(R) = \mathrm{Min}(R).$$

So, since $\Omega_{j+1}(M) \neq 0$,

$$\dim(\Omega_{j+1}(M)) = \max_{\mathfrak{p} \in \mathrm{Min}(\Omega_{j+1}(M))} \dim(R/\mathfrak{p}) = \max_{\mathfrak{p} \in \mathrm{Ass}(\Omega_{j+1}(M))} \dim(R/\mathfrak{p}) = \dim(R).$$

By Propositions 3.1.2 and 3.2.8, we have that $\mathrm{e}_*(\Omega_{j+1}(M)) \geq 1$. Therefore, applying Lemma 4.1.1, we have

$$\chi_j(M) = \frac{\mathrm{e}_*(\Omega_{j+1}(M)) + (-1)^j \, \mathrm{e}_*(M)}{\mathrm{e}_*(R)} \geq \frac{1}{\mathrm{e}_*(R)} + (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}.$$

(2) $\implies$ (3): Trivial.

(3) $\implies$ (1): Suppose $\chi_j(M) \neq (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$. Then by Lemma 4.1.1,

$$\frac{\mathrm{e}_*(\Omega_{j+1}(M)) + (-1)^j \, \mathrm{e}_*(M)}{\mathrm{e}_*(R)} \neq (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}.$$

Hence, $\mathrm{e}_*(\Omega_{j+1}(M)) \neq 0$. This implies that $\Omega_{j+1}(M) \neq 0$, so $\mathrm{pd}(M) > j$. □

Proposition 4.1.5. *Let $(R, \mathfrak{m})$ be a local ring of prime characteristic p, and let $j \in \mathbb{N}$. The following are equivalent:*

(1) $\chi_j^F(R) = (-1)^j$.

(2) $\chi_j^F(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$ *for some finitely generated R-module M with* $\operatorname{Assh} M = \operatorname{Assh} R$.

(3) $\chi_j^F(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$ *for every finitely generated R-module M.*

Proof. (1) $\implies$ (2) and (3) $\implies$ (1) are trivial.

(2) $\implies$ (3): Let M be a finitely generated R-module such that $\operatorname{Assh}(M) = \operatorname{Assh}(R)$ and $\chi_j^F(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$, and let N be an arbitrary finitely generated R-module. Since $\chi_j^F(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$, Proposition 3.6.2 implies that

$$\sum_{\mathfrak{p} \in \operatorname{Assh}(R)} \ell_{R_\mathfrak{p}}(M_\mathfrak{p}) \chi_j^F(R/\mathfrak{p}) = \sum_{\mathfrak{p} \in \operatorname{Assh}(R)} \ell_{R_\mathfrak{p}}(M_\mathfrak{p})(-1)^j \frac{\mathrm{e}_*(R/\mathfrak{p})}{\mathrm{e}_*(R)}. \tag{$*$}$$

By Lemma 4.1.3, $\chi_j^F(R/\mathfrak{p}) \geq (-1)^j \frac{\mathrm{e}_*(R/\mathfrak{p})}{\mathrm{e}_*(R)}$ for all $\mathfrak{p} \in \operatorname{Assh}(R)$, and since $\ell_{R_\mathfrak{p}}(M_\mathfrak{p}) > 0$ for all $\mathfrak{p} \in \operatorname{Assh}(M) = \operatorname{Assh}(R)$, we have that $(*)$ forces

$$\chi_j^F(R/\mathfrak{p}) = (-1)^j \frac{\mathrm{e}_*(R/\mathfrak{p})}{\mathrm{e}_*(R)}$$

for all $\mathfrak{p} \in \operatorname{Assh}(R)$. Therefore,

$$\sum_{\mathfrak{p} \in \operatorname{Assh}(R)} \ell_{R_\mathfrak{p}}(N_\mathfrak{p}) \chi_j^F(R/\mathfrak{p}) = \sum_{\mathfrak{p} \in \operatorname{Assh}(R)} \ell_{R_\mathfrak{p}}(N_\mathfrak{p})(-1)^j \frac{\mathrm{e}_*(R/\mathfrak{p})}{\mathrm{e}_*(R)}.$$

Thus, by Proposition 3.6.2, $\chi_j^F(N) = (-1)^j \frac{\mathrm{e}_*(N)}{\mathrm{e}_*(R)}$. □

In light of Proposition 4.1.5, we have the following theorem:

Theorem 4.1.6. *Let $(R, \mathfrak{m})$ be an unmixed local ring of prime characteristic p, and let $j \in \mathbb{N}$. The following implications are equivalent:*

(1) *R is regular if and only if $\chi_j^F(R) = (-1)^j$.*

(2) *R is regular if and only if $\chi_j^F(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$ for some finitely generated R-module M with $\operatorname{Assh}(M) = \operatorname{Assh}(R)$.*

(3) *R is regular if and only if $\chi_j^F(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$ for all finitely generated R-modules M.*

4.2 Sufficient F-splitting case

In this section, we define a class of modules that satisfy a certain condition, which we call "sufficient F-splitting". We then show Conjecture 4.0.2 holds for that any module that has this property.

Definition 4.2.1. Let $(R, \mathfrak{m})$ be an F-finite local ring of prime characteristic p and M be a nonzero finitely generated R-module. For each $e \in \mathbb{N}$, the *eth Frobenius indecomposable summand count of* M, denoted $g_e(M)$, is the maximum number of summands in a direct indecomposable decomposition of ${}^eM$.

We may write $g_e = g_e(M)$ if the underlying module is understood. With the setup in Definition 4.2.1, we can write ${}^eM \cong \bigoplus_{i=1}^{g_e} M_i$, where each M_i is a nonzero indecomposable R-module.

Definition 4.2.2. Let $(R, \mathfrak{m}, k)$ be an F-finite local ring of prime characteristic p and M be a nonzero finitely generated R-module. Set $\alpha = \alpha(R)$ and $d = \dim(R)$. We say that M has *sufficient F-splitting* if

$$\limsup_{e\to\infty} \frac{g_e(M)}{(p^e)^{\alpha+d}} > 0.$$

Remark 4.2.3. It is clear from the definition that $a_e(M) \leq g_e(M) \leq \mu({}^eM)$. It is not known if $\lim_{e\to\infty} \frac{g_e(M)}{(p^e)^{\alpha+d}}$ exists in general.

Many modules have sufficient F-splitting; we present the following examples:

Example 4.2.4. If R is an F-finite, local, strongly F-regular ring, then R has sufficient F-splitting. Indeed, by Theorem 3.3.4, we have that $s(R) > 0$. Therefore,

$$\limsup_{e\to\infty} \frac{g_e}{(p^e)^{\alpha+d}} \geq \lim_{e\to\infty} \frac{a_e}{(p^e)^{\alpha+d}} = s(R) > 0.$$

Thus, R has sufficient F-splitting.

Example 4.2.5. If R is an F-finite ring with FFRT, then R has sufficient F-splitting. Indeed, if R is a ring with FFRT, then there exists indecomposable R-modules $M_1, M_2, \ldots, M_k$ such that for every $e \geq 0$, ${}^eR \cong \bigoplus_{i=1}^k M_i^{n_{ei}}$, for some $n_{e1}, n_{e2}, \ldots, n_{ek} \in \mathbb{N}$. Fix e. Then,

$$\mathrm{e}_*({}^eR) = \mathrm{e}_*\left(\bigoplus_{i=1}^k M_i^{n_{ei}}\right) = \sum_{i=1}^k \mathrm{e}_*(M_i^{n_{ei}}) = \sum_{i=1}^k [n_{ei} \cdot \mathrm{e}_*(M_i)].$$

Since there are only finitely many M_i, there exists $N \in \mathbb{N}$ such that $\mathrm{e}_*(M_i) \leq N$ for all $i = 1, \ldots, k$. Moreover, since each M_i is indecomposable, we have that $g_e = \sum_{i=1}^k n_{ei}$. Therefore, $\mathrm{e}_*({}^eR) \leq g_e \cdot N$. Thus,

$$\frac{g_e}{(p^e)^{\alpha+d}} \geq \frac{\mathrm{e}_*({}^eR)}{N(p^e)^{\alpha+d}} = \frac{\mathrm{e}_*(R)(p^e)^{\alpha+d}}{N(p^e)^{\alpha+d}} = \frac{\mathrm{e}_*(R)}{N}.$$

Since $\mathrm{e}_*(R) > 0$, we have that $\limsup_{e\to\infty} \frac{g_e}{(p^e)^{\alpha+d}} > 0$, and R has sufficient F-splitting.

Example 4.2.6. If $(R, \mathfrak{m}, k)$ is an F-finite Artinian local ring and M is a nonzero finitely generated R-module, then M has sufficient F-splitting. Notice that $\mathfrak{m}\,{}^eM = \mathfrak{m}^{[p^e]}M$, and for $e \gg 0$, we have that $\mathfrak{m}^{[p^e]} = 0$. Therefore, for $e \gg 0$, ${}^eM$ is a vector space over k, and $\dim_k({}^eM) = \ell_R({}^eM) = \ell_R(M)p^{e\alpha}$. Thus,

$$\limsup_{e\to\infty} \frac{g_e(M)}{(p^e)^{\alpha+d}} = \limsup_{e\to\infty} \frac{\ell_R(M)p^{e\alpha}}{(p^e)^{\alpha+d}} = \limsup_{e\to\infty} \frac{\ell_R(M)}{p^{ed}} = \ell_R(M) > 0.$$

For the following example, we will closely follow a similar construction to that of [HY23, Lemma 5.14]. For any $s \in \mathbb{N}$, a ring is said to be $\mathbb{N}^s$-*graded local* if R is $\mathbb{N}^s$-graded and $(R_0, \mathfrak{m}_0, k)$ is local, where 0 is the zero element of the semigroup $(\mathbb{N}^s, +)$. We write $\mathbb{N}^s$-graded local rings as $(R, \mathfrak{m}, k)$, where $\mathfrak{m}$ is the maximal homogeneous ideal.

Example 4.2.7. Suppose $R = \bigoplus_{i \in \mathbb{N}^d} R_i$ is an $\mathbb{N}^d$-graded ring of prime characteristic p that is a finitely generated algebra over R_0, where $d = \dim(R)$, R_0 is a perfect field, and $R_i \neq 0$ for all $|i| \gg 0$. Then $R_{\mathfrak{m}}$ has sufficient F-splitting. To see this, fix $e \in \mathbb{N}$ and let

$$\Delta_e = \{(r_1, \ldots, r_d) \in \mathbb{N}^d \mid r_i \in \{0, 1, \ldots, p^e - 1\}\}.$$

For each $r \in \Delta_e$, denote $p^e\mathbb{N}^d + r = \{p^e n + r \mid n \in \mathbb{N}^d\}$. Then we can write

$$^eR = \bigoplus_{r \in \Delta_e} M_{e,r} \ , \quad \text{where} \quad M_{e,r} = \bigoplus_{i \in p^e\mathbb{N}^d + r} R_i. \tag{$\dagger$}$$

We note that $M_{e,r}$ is an R-submodule of ^eR for each $r \in \Delta_e$. Therefore, $^eR = \bigoplus_{r \in \Delta_e} M_{e,r}$ is a direct sum decomposition of ^eR. Since $|\Delta_e| = p^{ed}$ and $R_i \neq 0$ for all i such that $|i| \gg 0$, we have that $M_{e,r} \neq 0$ for all $e \geq 0$ and for all $r \in \Delta_e$. Thus, ^eR has p^{ed} summands.

Now, we localize at $\mathfrak{m}$. The residue field of $R_{\mathfrak{m}}$ is R_0, and since it is a perfect field, $\alpha(R_{\mathfrak{m}}) = 0$. Localizing ($\dagger$) at $\mathfrak{m}$, we get $^eR_{\mathfrak{m}} = \bigoplus_{r \in \Delta_e} (M_{e,r})_{\mathfrak{m}}$. We have that $(M_{e,r})_{\mathfrak{m}} \neq 0$ for all $e \geq 0$. Indeed, let $0 \neq x \in M_{e,r}$. Then $x = \sum_{i \in p^e\mathbb{N}^d + r} x_i$, where only finitely many x_i are nonzero. Let $t \in R \setminus \mathfrak{m}$, and write $x = x_{i_0} + x'$ and $t = t_0 + t'$, where $i_0 = \min\{i \mid x_i \neq 0\}$, so that $x_{i_0} \in R_{i_0}$, and $t_0 \in R_0$. Then $tx = t_0 x_{i_0} + t_0 x' + t' x_{i_0} + t' x'$. Since $t_0 x_{i_0} \neq 0$, $\deg(x') > i_0$, and $\deg(t') > 0$, we have that $tx \neq 0$. Thus, $x/1 \neq 0$, and $(M_{e,r})_{\mathfrak{m}} \neq 0$ for all $e \geq 0$. Thus,

${}^eR_{\mathfrak{m}}$ has p^{ed} summands. Hence,

$$\limsup_{e\to\infty} \frac{g_e(R_{\mathfrak{m}})}{(p^e)^{\alpha(R_{\mathfrak{m}})+\dim(R_{\mathfrak{m}})}} = \limsup_{e\to\infty} \frac{p^{ed}}{p^{e\dim(R_{\mathfrak{m}})}} > 0.$$

Lemma 4.2.8. *Let $(R,\mathfrak{m},k)$ be an F-finite, local ring, and let M be a finitely generated R-module. If M has sufficient F-splitting, then* $\dim(R) = \dim(M)$.

Proof. For each $e \in \mathbb{N}$, we can decompose ${}^eM$ into $g_e = g_e(M)$ nonzero R-modules. That is, ${}^eM \cong \bigoplus_{i=1}^{g_e} M_{ei}$, where each M_{ei} is a nonzero R-module. Set $d = \dim(R)$ and $\alpha = \alpha(R)$. Then

$$\begin{aligned}
\mathrm{e_{HK}}(M) = \lim_{e\to\infty} \frac{\ell_R(k\otimes_R {}^eM)}{(p^e)^{\alpha+d}} &= \lim_{e\to\infty} \frac{\ell_R(k\otimes_R \bigoplus_{i=1}^{g_e} M_{ei})}{(p^e)^{\alpha+d}} \\
&= \lim_{e\to\infty} \frac{\ell_R(\bigoplus_{i=1}^{g_e}[k\otimes_R M_{ei}])}{(p^e)^{\alpha+d}} \\
&= \lim_{e\to\infty} \frac{\sum_{i=1}^{g_e}\ell_R([k\otimes_R M_{ei}])}{(p^e)^{\alpha+d}} \\
&\geq \lim_{e\to\infty} \frac{\sum_{i=1}^{g_e} 1}{(p^e)^{\alpha+d}} \qquad \text{since } k \text{ and } M_{ei} \text{ are nonzero} \\
&= \lim_{e\to\infty} \frac{g_e}{(p^e)^{\alpha+d}} > 0.
\end{aligned}$$

Since $\mathrm{e_{HK}}(M) > 0$, we have that $\dim(M) = \dim(R)$, by Proposition 3.2.8. □

We are ready to prove the following theorem:

Theorem 4.2.9. *Let $(R,\mathfrak{m})$ be an unmixed, F-finite, local ring of prime characteristic p, and let M be a finitely generated R-module such that M has sufficient F-splitting. If $\chi_j^F(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$ for some $j \in \mathbb{N}$, then R is regular.*

Proof. Assume that R is not regular. For each $e \in \mathbb{N}$, there exists a maximal $g_e \in \mathbb{N}$ and nonzero finitely generated R-modules $M_{e_1}, M_{e_2}, \ldots, M_{e_{g_e}}$ such that ${}^eM \cong \bigoplus_{i=1}^{g_e} M_{e_i}$. So,

$$ {}^{e+1}M = {}^1({}^eM) \cong {}^1\left(\bigoplus_{i=1}^{g_e} M_{e_i}\right) \cong \bigoplus_{i=1}^{g_e} {}^1M_{e_i}.$$

By Theorem 2.2.4, we have that $\operatorname{pd}({}^1M_{e_i}) = \infty$ for all $i = 1, 2, \ldots, g_e$. Therefore, for every $j \in \mathbb{N}$,

$$\begin{aligned}
\chi_j^F(M) &= \lim_{e\to\infty} \frac{\chi_j({}^{e+1}M)}{(p^{e+1})^{\alpha+d}} \\
&= \lim_{e\to\infty} \frac{\chi_j(\bigoplus_{i=1}^{g_e} {}^1M_{e_i})}{(p^{e+1})^{\alpha+d}} \\
&= \lim_{e\to\infty} \frac{\sum_{i=1}^{g_e} \chi_j({}^1M_{e_i})}{(p^{e+1})^{\alpha+d}} \\
&\geq \limsup_{e\to\infty} \frac{\sum_{i=1}^{g_e} [1 + (-1)^j \operatorname{e}_*({}^1M_{e_i})]}{(p^{e+1})^{\alpha+d} \operatorname{e}_*(R)} && \text{(by Lemma 4.1.4)} \\
&= \limsup_{e\to\infty} \frac{(\sum_{i=1}^{g_e} 1)}{(p^{e+1})^{\alpha+d} \operatorname{e}_*(R)} + \frac{(-1)^j \sum_{i=1}^{g_e} \operatorname{e}_*({}^1M_{e_i})}{(p^{e+1})^{\alpha+d} \operatorname{e}_*(R)} \\
&= \limsup_{e\to\infty} \frac{g_e}{(p^{e+1})^{\alpha+d} \operatorname{e}_*(R)} + \lim_{e\to\infty} \frac{(-1)^j \operatorname{e}_*({}^{e+1}M)}{(p^{e+1})^{\alpha+d} \operatorname{e}_*(R)} \\
&> (-1)^j \lim_{e\to\infty} \frac{\operatorname{e}_*({}^{e+1}M)}{(p^{e+1})^{\alpha+d} \operatorname{e}_*(R)} && \text{(by Definition 4.2.2)} \\
&= (-1)^j \frac{\operatorname{e}_*(M)}{\operatorname{e}_*(R)} && \text{(by Proposition 3.6.3).}
\end{aligned}$$

Therefore, $\chi_j^F(M) \neq (-1)^j \frac{\operatorname{e}_*(M)}{\operatorname{e}_*(R)}$ for every $j \geq 0$. □

This result recovers Theorem 3.5.4, as any strongly F-regular ring is a domain that has sufficient F-splitting.

Remark 4.2.10. In Theorem 4.2.9, if R is not F-finite, then there may exist a sufficiently good ring extension $R \to S$ such that $M \otimes_R S$ has sufficient F-splitting over S. In that case,

if Theorem 4.2.9 holds for $M \otimes_R S$ over S, then it holds for M over R.

4.3 Cohen-Macaulay case

In this section, we show that Conjecture 4.0.2 holds for finitely generated modules over Cohen-Macaulay rings.

Theorem 4.3.1. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring of prime characteristic p, and let M be a finitely generated R-module such that* $\operatorname{Assh}(M) = \operatorname{Assh}(R)$. *If $\chi_j^F(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$ for some $j \in \mathbb{N}$, then R is regular.*

Proof. Fix j. We first show that $\chi_j^F(R) = (-1)^j$ implies that R is regular. Without loss of generality, we may assume that R is F-finite (see Remark 2.2.9 and Remark 3.4.2).

So, assume R is F-finite. By Lemma 4.1.1,

$$\mathrm{e}(\Omega_{j+1}({}^eR)) = \chi_j({}^eR)\,\mathrm{e}(R) - (-1)^j\,\mathrm{e}({}^eR).$$

By Proposition 2.2.11, ${}^eR$ is a maximal Cohen-Macaulay R-module, so $\Omega_{j+1}({}^eR)$ is a maximal Cohen-Macaulay R-module Proposition 2.1.30. Therefore, by Lemma 3.1.6,

$$\beta_{j+1}({}^eR) = \mu(\Omega_{j+1}({}^eR)) \leq \mathrm{e}(\Omega_{j+1}({}^eR)) = \chi_j({}^eR)\,\mathrm{e}(R) - (-1)^j\,\mathrm{e}({}^eR)$$

for all $e \geq 0$. Therefore,

$$\begin{aligned}
\beta^F_{j+1}(R) :&= \lim_{e\to\infty} \frac{\beta_{j+1}({}^eR)}{(p^e)^{\alpha+d}} \\
&\leq \lim_{e\to\infty} \frac{\chi_j({}^eR)\,\mathrm{e}(R) - (-1)^j\,\mathrm{e}({}^eR)}{(p^e)^{\alpha+d}} \\
&= \lim_{e\to\infty} \frac{\chi_j({}^eR)}{(p^e)^{\alpha+d}}\,\mathrm{e}(R) - \lim_{e\to\infty}(-1)^j \frac{\mathrm{e}({}^eR)}{(p^e)^{\alpha+d}} \\
&= \chi^F_j(R)\,\mathrm{e}(R) - (-1)^j\,\mathrm{e}(R) \\
&= 0.
\end{aligned}$$

Thus, by Theorem 3.4.5, R is regular.

We have shown that if $\chi^F_j(R) = (-1)^j$, then R is regular. Applying Theorem 4.1.6, we conclude that if $\chi^F_j(M) = (-1)^j \frac{\mathrm{e}_*(M)}{\mathrm{e}_*(R)}$ for any finitely generated R-module M such that $\mathrm{Assh}(M) = \mathrm{Assh}(R)$, then R is regular. □

This result also recovers Theorem 3.5.4, as any local strongly F-regular ring is Cohen-Macaulay.

4.4 Further results

In this section, we present some results that could aid in proving Conjecture 4.0.2. Throughout this section, $d = \dim(R)$ and $\alpha = \alpha(R)$. For simplicity, we only discuss the F-finite situation in this section.

We begin by outlining some properties under localization. Although it should be clear from context, we denote the Hilbert-Samuel or Hilbert-Kunz multiplicity of a finitely generated R-module M at a prime $\mathfrak{p} \in \mathrm{Spec}(R)$ by $\mathrm{e}_*(M_\mathfrak{p})$. Similarly, we use $\chi_j(M_\mathfrak{p})$ and $\chi^F_j(M_\mathfrak{p})$

to denote the jth Euler characteristic and the jth Frobenius Euler characteristic of $M_{\mathfrak{p}}$ as an $R_{\mathfrak{p}}$-module, respectively.

Proposition 4.4.1. *Let $(R, \mathfrak{m})$ be a local ring, M be a finitely generated R-module, and $j \in \mathbb{N}$. Then $\chi_j(M) \geq \chi_j(M_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$.*

Proof. Fix $j \in \mathbb{N}$ and let $\mathfrak{p} \in \operatorname{Spec}(R)$. By Lemma 4.1.1,

$$\chi_j(M) = \frac{\mathrm{e}_*(\Omega_{j+1}(M)_{\mathfrak{p}}) + (-1)^j \, \mathrm{e}_*(M_{\mathfrak{p}})}{\mathrm{e}_*(R_{\mathfrak{p}})}$$

and

$$\chi_j(M_{\mathfrak{p}}) = \frac{\mathrm{e}_*(\Omega_{j+1}(M_{\mathfrak{p}})) + (-1)^j \, \mathrm{e}_*(M_{\mathfrak{p}})}{\mathrm{e}_*(R_{\mathfrak{p}})}.$$

By Corollary 2.1.18, $\mathrm{e}_*(\Omega_{j+1}(M)_{\mathfrak{p}}) \geq \mathrm{e}_*(\Omega_{j+1}(M_{\mathfrak{p}}))$. Thus, $\chi_j(M) \geq \chi_j(M_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$. □

Proposition 4.4.2. *Let $(R, \mathfrak{m})$ be an F-finite local ring, M be a finitely generated R-module, and $j \in \mathbb{N}$. Then for all $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\dim(R/\mathfrak{p}) + \dim(R_{\mathfrak{p}}) = \dim(R)$, we have $\chi_j^F(M) \geq \chi_j^F(M_{\mathfrak{p}})$.*

Proof. Fix $j \geq 0$ and let $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\dim(R/\mathfrak{p}) + \dim(R_{\mathfrak{p}}) = \dim(R)$. By Proposition 4.4.1, $\chi_j({}^eM) \geq \chi_j({}^eM_{\mathfrak{p}})$. By Corollary 2.2.13, we have

$$\alpha(R_{\mathfrak{p}}) + \dim(R_{\mathfrak{p}}) = \alpha(R) + \dim(R/\mathfrak{p}) + \dim(R_{\mathfrak{p}}) = \alpha(R) + \dim(R).$$

Therefore,

$$\begin{aligned}\chi_j^F(M) := \lim_{e\to\infty} \frac{\chi_j({}^eM)}{(p^e)^{\alpha(R)+\dim(R)}} &\geq \lim_{e\to\infty} \frac{\chi_j({}^eM_{\mathfrak{p}})}{(p^e)^{\alpha(R)+\dim(R)}} && \text{(by Proposition 4.4.1)}\\ &= \lim_{e\to\infty} \frac{\chi_j({}^eM_{\mathfrak{p}})}{(p^e)^{\alpha(R_{\mathfrak{p}})+\dim(R_{\mathfrak{p}})}} && \text{(by assumption)}\\ &= \chi_j^F(M_{\mathfrak{p}}) && \square\end{aligned}$$

Now we present results relating to the parity of j.

Proposition 4.4.3. *Let $(R,\mathfrak{m},k)$ be an F-finite local ring of prime characteristic p. If $\chi_j^F(R) < 0$ for some odd $j \in \mathbb{N}$, then R is equidimensional.*

Proof. Suppose that R is not equidimensional. Then there exists $\mathfrak{p} \in \operatorname{Min}(R)$ such that $\dim(R/\mathfrak{p}) < \dim(R)$. Fix $e, j \in \mathbb{N}$ such that j is odd. Consider a minimal free resolution of ${}^eR$ over R:

$$\cdots \to R^{\beta_2({}^eR)} \to R^{\beta_1({}^eR)} \to R^{\beta_0({}^eR)} \to ({}^eR \to)\ 0.$$

We can localize this resolution at $\mathfrak{p}$ to get

$$\cdots \to R_{\mathfrak{p}}^{\beta_2({}^eR)} \to R_{\mathfrak{p}}^{\beta_1({}^eR)} \to R_{\mathfrak{p}}^{\beta_0({}^eR)} \to ({}^eR_{\mathfrak{p}} \to)\ 0.$$

Since $\mathfrak{p} \in \operatorname{Min}(R)$, we have that $\dim(R_{\mathfrak{p}}) = 0$ and $R_{\mathfrak{p}}$ is Artinian. Therefore, $\ell_{R_{\mathfrak{p}}}(\Omega_i({}^eR)_{\mathfrak{p}}) + \ell_{R_{\mathfrak{p}}}({}^eR_{\mathfrak{p}}) < \infty$. The exactness of the free resolution of ${}^eR_{\mathfrak{p}}$ yields

$$\left(\sum_{i=0}^{j}(-1)^i \ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{\beta_{j-i}({}^eR)})\right) - (-1)^j \ell_{R_{\mathfrak{p}}}({}^eR_{\mathfrak{p}}) = \ell_{R_{\mathfrak{p}}}(\Omega_{j+1}({}^eR)_{\mathfrak{p}}) \geq 0.$$

Since $\ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}^{\beta_{j-i}({}^eR)}) = \beta_{j-i}({}^eR)\ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})$, and $\ell_{R_{\mathfrak{p}}}({}^eR_{\mathfrak{p}}) = (p^e)^{\alpha+\dim(R/\mathfrak{p})}\ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})$, we have,

$$\ell_{R_{\mathfrak{p}}}(R_{\mathfrak{p}})\left(\chi_j({}^eR) - (-1)^j(p^e)^{\alpha+\dim(R/\mathfrak{p})}\right) \geq 0.$$

Since j is odd, we get $\chi_j({}^eR) + (p^e)^{\alpha+\dim(R/\mathfrak{p})} \geq 0$. Thus,

$$\begin{aligned}
\chi_j^F(R) &= \lim_{e\to\infty} \frac{\chi_j({}^eR)}{(p^e)^{\alpha+\dim(R)}} \\
&= \lim_{e\to\infty} \frac{\chi_j({}^eR)}{(p^e)^{\alpha+\dim(R)}} + \lim_{e\to\infty} (p^e)^{\dim(R/\mathfrak{p})-\dim(R)} \qquad (\text{since } \dim R/\mathfrak{p} < \dim R) \\
&= \lim_{e\to\infty} \frac{\chi_j({}^eR) + (p^e)^{\alpha+\dim(R/\mathfrak{p})}}{(p^e)^{\alpha+\dim(R)}} \\
&\geq 0. \qquad \square
\end{aligned}$$

Proposition 4.4.4. *Let $(R, \mathfrak{m})$ be a F-finite local ring of prime characteristic p. If $\chi_j^F(R) = (-1)^j$ for some even integer $j \geq 0$, then $|\operatorname{Assh}(R)| = 1$.*

Proof. Let $j \geq 0$ be an even integer. Assume, to the contrary, that $\chi_j^F(R) = (-1)^j$ and $|\operatorname{Assh}(R)| > 1$. Let $\operatorname{Assh}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r\}$ and $\operatorname{Ass}(R) = \{\mathfrak{p}_1, \mathfrak{p}_2, \ldots, \mathfrak{p}_r, \mathfrak{p}_{r+1}, \ldots, \mathfrak{p}_s\}$ for some $s \geq r \geq 2$. Let

$$0_R = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \cap \mathfrak{q}_r \cap \mathfrak{q}_{r+1} \cap \cdots \cap \mathfrak{q}_s$$

be a prime decomposition of the zero ideal, where each $\mathfrak{q}_i$ is $\mathfrak{p}_i$-primary for all $i = 1, \ldots, s$. Consider the R-linear map $h : R \to \bigoplus_{i=1}^r R/\mathfrak{q}_i$ defined by $x \mapsto (x + \mathfrak{q}_1, x + \mathfrak{q}_2, \ldots, x + \mathfrak{q}_r)$. This gives rise to the exact sequence

$$0 \to \ker h \to R \to \bigoplus_{i=1}^r R/\mathfrak{q}_i \to \operatorname{coker} h \to 0.$$

We note that for all $i = 1, \ldots, r$ and $\mathfrak{p} \in \operatorname{Assh}(R)$, if $\mathfrak{p} = \mathfrak{p}_i$, then $(\mathfrak{q}_i)_\mathfrak{p} = 0$, and if $\mathfrak{p} \neq \mathfrak{p}_i$, then $(\mathfrak{q}_i)_\mathfrak{p} \cong R_\mathfrak{p}$. Since $h/1 : R_\mathfrak{p} \to (\bigoplus_{i=1}^r R/\mathfrak{q}_i)_\mathfrak{p}$ is an isomorphism for all $\mathfrak{p} \in \operatorname{Assh}(R)$, we have $(\ker h)_\mathfrak{p} = 0 = (\operatorname{coker} h)_\mathfrak{p}$. This gives us that $\dim(\ker h) < \dim(R)$

and $\dim(\operatorname{coker} h) < \dim(R)$. Therefore,

$$\begin{aligned}
1 = \chi_j^F(R) &= \chi_j^F(\ker h) + \chi_j^F\left(\bigoplus_{i=1}^{r} R/\mathfrak{q}_i\right) - \chi_j^F(\operatorname{coker} h) && \text{(by Proposition 3.5.2)} \\
&= \sum_{i=1}^{r} \chi_j^F(R/\mathfrak{q}_i) && \text{(by Proposition 3.5.2)} \\
&\geq \sum_{i=1}^{r} \chi_j^F(R_{\mathfrak{p}_i}/(\mathfrak{q}_i)_{\mathfrak{p}_i}) && \text{(by Corollary 4.4.2)} \\
&= \sum_{i=1}^{r} \chi_j^F(R_{\mathfrak{p}_i}) \\
&\geq (-1)^j r && \text{(by Corollary 4.1.3)} \\
&= r \geq 2,
\end{aligned}$$

a contradiction. □

Corollary 4.4.5. *If $(R, \mathfrak{m})$ is an F-finite unmixed local ring and $\chi_j^F(R) = (-1)^j$ for some even integer $j \geq 0$, then R is a domain.*

Proof. By Proposition 4.4.4, we have that $|\operatorname{Ass}(R)| = |\operatorname{Assh}(R)| = 1$. Say $\operatorname{Ass}(R) = \{\mathfrak{p}\}$. Since $\chi_j^F(R) = (-1)^j$ and $R_{\mathfrak{p}}$ is Artinian, we have that $R_{\mathfrak{p}}$ is regular, and hence, $R_{\mathfrak{p}}$ is a field. That is, $\mathfrak{p}_{\mathfrak{p}} = 0$, so there exists $s \in R \setminus \mathfrak{p}$ such that $s\mathfrak{p} = 0$. Since every element in $R \setminus \mathfrak{p}$ is a nonzero divisor, $\mathfrak{p} = 0$. Hence, R is a domain. □

Discussion 4.4.6. For an unmixed local ring R, one could approach Conjecture 4.0.2 by induction on $d = \dim(R)$. In the case of $d = 0$ and $d = 1$, R is Cohen-Macaulay, so R is regular by Theorem 4.3.1. If we assume the conjecture holds for some $d \geq 1$, then $R_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. Moreover, if the $\mathfrak{m}$-adic completion of R is unmixed, then

R is a generalized Cohen-Macaulay ring (that is, $\ell_R(H^i_{\mathfrak{m}}(R)) < \infty$ for all $i < d$). Therefore, it suffices to prove Conjecture 4.0.2 under these assumptions.

Question 4.4.7. Suppose $(R, \mathfrak{m})$ is an F-finite local ring of characteristic p. We may assume that $R_{\mathfrak{p}}$ is regular for all $\mathfrak{p} \in \operatorname{Spec}(R) \setminus \{\mathfrak{m}\}$. Does there exist a constant $c > 0$, independent of e, such that $\mu(\Omega_{j+1}({}^eR)) < c \cdot \mathrm{e}_*(\Omega_{j+1}({}^eR))$ for all $e \geq 1$? If so, then we can follow a similar argument to that seen in the proof of Theorem 4.3.1 (in that proof, we have $c = 1$). That is, we would have $\beta_{j+1}({}^eR) = \mu(\Omega_{j+1}({}^eR)) \leq c \cdot \mathrm{e}_*(\Omega_{j+1}({}^eR))$. If $\chi^F_j(R) = (-1)^j$, then $\lim_{e\to\infty} \frac{\mathrm{e}_*(\Omega_{j+1}({}^eR))}{(p^e)^{\alpha+d}} = 0$, so this gives us

$$0 \leq \lim_{e\to\infty} \frac{\beta_{j+1}({}^eR)}{(p^e)^{\alpha+d}} \leq \lim_{e\to\infty} \frac{c \cdot \mathrm{e}_*(\Omega_{j+1}({}^eR))}{(p^e)^{\alpha+d}} = 0.$$

Thus, $\beta^F_{j+1}(R) = 0$, and by Theorem 3.4.5, R is regular.

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