Fan-Linear Maps and Fan Algebras

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FAN LINEAR MAPS AND FAN ALGEBRAS

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Abstract. Fan algebras arise from fan-linear maps, a special class of functions defined on partitions of the nonnegative integer lattice in the plane. These algebras are natural objects to study in commutative algebra as they include many classical examples commutative rings. Additionally, the ubiquity of this structure has only been recently identified, therefore little is known regarding the properties of these algebras. We begin our study by classifying all fan-linear maps via the conditions imposed on them by their domains. This classification includes a general result regarding all semigroup homomorphisms from finitely generated subsemigroups of the nonnegative integer lattice into the integers. We then go on to show that the set of all fan-linear maps on any fixed partition is necessarily a finitely-generated affine semigroup. Finally, this leads to the conclusion that the set of fan algebras corresponding to a fixed partition and a fixed set of ideals forms a finitely generated semigroup. This is accomplished through the identification of generating maps in the semigroup of all fan-linear maps with generating algebras and the description of a natural additive operation.

1. Introduction

Let $I_1, \dots, I_n$ be a collection of ideals in a domain R, let $f = \{f_1, \dots, f_n\}$ be a collection of special functions $f_i \colon \mathbb{N}^2 \to \mathbb{N}$ called *fan-linear maps*, and let u and v be indexing variables. Fan algebras are bi-graded rings of the following form:

$$\mathcal{B}(\Sigma_{\mathbf{a},\mathbf{b}}, f) = \bigoplus_{r,s} I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} u^r v^s \text{ as } (r,s) \text{ ranges over } \mathbb{N}^2$$

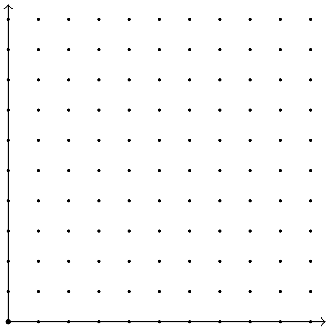
Fan-algebras first appeared in the work of Sara Malec and we cite here an important property of these objects.

Theorem 1.1. *([1], Theorem 2.3.3) Fan algebras are finitely generated as R-algebras.*

In what follows, we extend Malec's work by presenting a classification of all fan-linear maps. The study begins by examining the nonnegative integer lattice in the plane in order to deduce the general properties that it imposes on fan-linear maps (which have this lattice as their domain). Subsequently, we use these properties to describe all such maps and derive a correspondence between the set of all such maps and other semigroups similar to $\mathbb{N}^2$. Finally, we examine fan algebras in this new context.

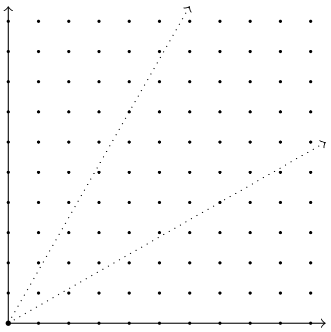
2. Preliminaries

To begin, we will develop an intuitive grasp of what a fan-linear map is and then discuss some general elements of semigroup theory necessary for the further study of the objects before us. The first treatment of fan-linear maps will be concrete - we will avoid, for the time being, general and abstract definitions. An *affine semigroup* is most simply described as a semigroup that can be identified as subsemigroup of $\mathbb{Z}^d$. The affine semigroup $\mathbb{N}^2$, where addition is defined componentwise, is best visualized as the integer coordinate pairs in the northeastern quadrant of the real plane adjoined with the integer coordinates on the positive axes:



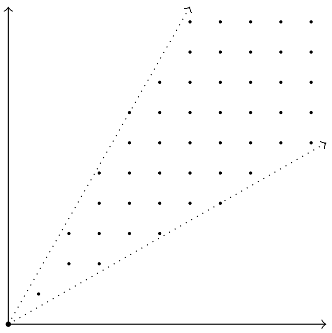
$\mathbb{N}^2$

A *fan of cones* in $\mathbb{N}^2$ is somewhat of a partition of $\mathbb{N}^2$ (although certainly not in the strictest sense); to construct a fan, we first n-sect the entire quadrant with positively oriented rays of rational slope meeting at the origin:



$\Sigma_{\mathbf{a},\mathbf{b}}$

Above we see that the entirety of $\mathbb{N}^2$ is a union of these slices, and this segmenting is exactly what we mean when we say a *fan*. We use the notation $\Sigma_{\mathbf{a},\mathbf{b}}$, the details of which we will explain later, to denote one of these fans. By convention, we order and index each of the slices in a clockwise manner and the collection of the coordinates that reside within or on the defining rays of any one of these slices is what we call a *cone* in $\mathbb{N}^2$. Here, we have a *fan of* 3 *cones* which we may label Q_0, Q_1 and Q_2. The integer coordinates that fall on either ray of the two rays (including the axes when necessary) that define any of these cones Q_i comprise a *face* of that cone. Note that this implies that a non-axis face can be seen as the intersection of two cones (hence why a fan is not a true partition of $\mathbb{N}^2$). For example, below we show the cone Q_1 alone:



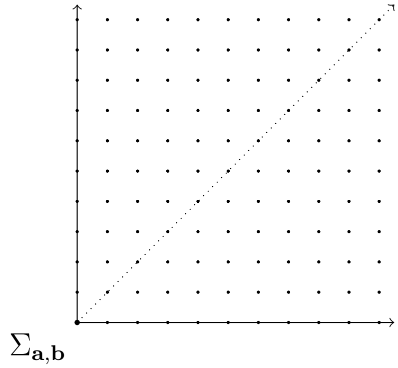
Q_1

With this picture in mind, we may construct a more formal criterion for cone membership. In the case above, we see that the integer coordinates in $\mathbb{N}^2$ which lie in Q_1 can be described entirely by the integer coordinates between (inclusive) the lines of rational slope $y = \frac{7}{4}x$ and $y = \frac{4}{7}x$. For this to occur at the point (r, s), we must have $\frac{4}{7} \leq \frac{s}{r} \leq \frac{7}{4}$. Consequently, we may say that in this case, the faces of Q_1 are *defined by* $(4, 7)$ *and* $(7, 4)$. Note then that the faces of Q_0 are defined by $(0, 1)$ and $(4, 7)$, while the faces of Q_2 are defined by $(1, 0)$ and $(7, 4)$. This leads to the conclusion that we may define cone membership as follows:

Definition 2.1. (Cone Membership Criterion) For any cone Q_i in an arbitrary fan $\Sigma_{\mathbf{a},\mathbf{b}}$ where the faces of Q_i are defined by (p_i, q_i) and (p_{i+1}, q_{i+1}), the element $(r, s) \in \mathbb{N}^2$ is a member of Q_i if and only if $\frac{q_{i+1}}{p_{i+1}} \leq \frac{s}{r} \leq \frac{q_i}{p_i}$ where we take the convention that $\frac{a}{b} < \frac{1}{0}$ for all $(a, b) \in \mathbb{N}^2$.

2.1. **Fan-Linear Maps.** A fan-linear map is a function $f \colon \mathbb{N}^2 \to \mathbb{N}$ defined with two specific requirements corresponding to a given fan. The first requirement is that for any $(r_1, s_1), (r_2, s_2)$ in a given cone Q_i, we must have that $f[(r_1, s_1)+(r_2, s_2)] = f(r_1, s_1)+f(r_2, s_2)$. In this case, we may say that f is *additive on* Q_i, but since Q_i is a subsemigroup of $\mathbb{N}^2$ (closure under addition is easily verified) and $\mathbb{N}$ is a semigroup as well, this is the same as saying that f is a semigroup homomorphism when restricted to Q_i. The second requirement is that for $(r_1, s_1) \in Q_i$ and $(r_1, s_2) \in Q_j$ (where i may or may not be equal to j), $f[(r_1, s_1) + (r_2, s_2)] \leq f(r_1, s_1) + f(r_2, s_2)$. We call this property subadditivity.

Example 2.2. Consider the function $\max(r, s) \colon \mathbb{N}^2 \to \mathbb{N}$. We may view this function as fan-linear map on the fan described as $\mathbb{N}^2$ segmented by the ray $y = x$:



As the convention dictates, we take Q_0 to be the coordinates on or above the line $y = x$ and Q_1 to be the coordinates on or below it. When we restrict the $\max(r, s)$ function to say Q_1 and take the convention that $\max(r, s) = r$ when $r \geq s$, we have that $\max(r + r', s + s') = r + r' = \max(r, s) + \max(r', s')$, so $\max(r, s)$ is a semigroup homomorphism when restricted to Q_1. It follows similarly that this is the case when $\max(r, s)$ is restricted to Q_0. To verify subadditivity, assume that $(r, s) \in Q_0$ and $(r', s') \in Q_1$ and $(r + r', s + s') \in Q_0$. Then $\max(r + r', s + s') = s + s' \leq s + r' = \max(r, s) + \max(r', s')$. If $(r + r', s + s') \in Q_1$, then $\max(r + r', s + s') = r + r' \leq s + r' = \max(r, s) + \max(r', s')$. Then $\max(r, s)$ is a semigroup homomorphism on each cone and is subadditive on all of $\mathbb{N}^2$, hence $\max(r, s)$ is a fan-linear function.

Upon closer examination, we see that we may regard the function $\max(r, s)$ as a piecewise function where each piece is a semigroup homomorphism:

$$\max(r,s) = \begin{cases} s & \text{if } (r,s) \in Q_0 \\ r & \text{if } (r,s) \in Q_1 \end{cases}$$

Indeed, for any fan linear map $f\colon \mathbb{N}^2 \to \mathbb{N}$ defined on a fan of any number of cones n, we may write f in such a way. Where $g_i\colon Q_i \to \mathbb{N}$ is a semigroup homomorphism for $i = 1, \ldots, n$, we may write f in the following way:

$$f(r,s) = \begin{cases} g_1(r,s) & \text{if } (r,s) \in Q_0 \\ g_2(r,s) & \text{if } (r,s) \in Q_1 \\ \vdots & \\ g_n(r,s) & \text{if } (r,s) \in Q_n \end{cases}$$

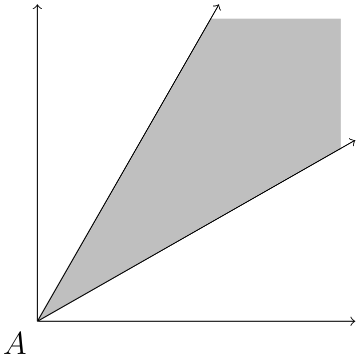
These semigroup homomorphisms must meet other requirements, but because they are the essential building blocks of fan-linear maps and our task is to classify every such map, the first step is to deduce what form each of the $g_i\colon Q_i \to \mathbb{N}$ can take. This is handled in section 3, but before we move on to that development, it is essential that we gather some tools to examine the concerned semigroups more rigorously.

2.2. **Cone Semigroup Properties.** Much of the theory of affine semigroups that applies to cones as we know them from our previous discussion is stated in terms of the properties in linear systems of equations in $\mathbb{R}^2$. For this reason, we will reconcile our understanding of a cone in $\mathbb{N}^2$ with this other view. We illustrate the concept with an example.

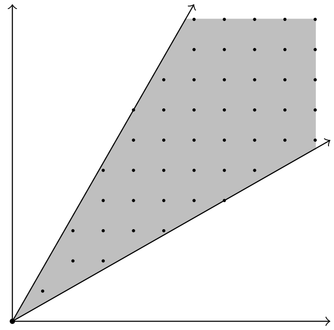
Example 2.3. Consider the following set:

$$A = \left\{ (x,y) \in \mathbb{R}^2 \mid y - \frac{4}{7}x \geq 0 \right\} \cap \left\{ (x,y) \in \mathbb{R}^2 \mid y - \frac{7}{4}x \leq 0 \right\}$$

This is exactly the set of real coordinates above and below (inclusive) the lines $y = \frac{4}{7}x$ and $y = \frac{7}{4}x$ respectively in the first quadrant of $\mathbb{R}^2$.



Now consider the integer coordinates that lie within and on the boundaries of A.



It is clear then that the cone Q_1 that we previously examined can be viewed as the set $A \cap \mathbb{N}^2$. Considering this construction of Q_1 has some advantages that we will exploit in the classification of all fan-linear maps. In order to facilitate our handling of affine semigroups that are not necessarily contained within $\mathbb{N}^2$, we will present all of the following semigroup theory in full generality. First, we give some important definitions.

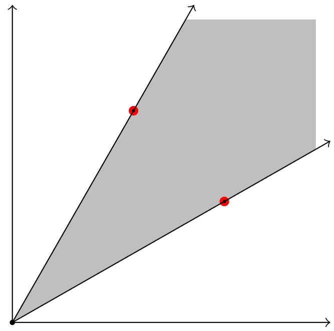
Definition 2.4. For fixed $x_1, \ldots, x_k, b \in \mathbb{R}$, a *hyperplane* in $\mathbb{R}^k$ is a set $A = \big\{(a_1, \ldots, a_k) \in \mathbb{R}^k \mid a_1x_1 + \cdots + a_kx_k = b\big\}$. A *half-space determined by* A, denoted $A^\diamond$, is the set $\big\{(a_1, \ldots, a_k) \in \mathbb{R}^k \mid a_1x_1 + \cdots + a_kx_k \geq b\big\}$ or the set $\big\{(a_1, \ldots, a_k) \in \mathbb{R}^k \mid a_1x_1 + \cdots + a_kx_k \leq b\big\}$.

Definition 2.5. A *polyhedral cone* C in $\mathbb{R}^d$ is the intersection of finitely many closed half-spaces in $\mathbb{R}^d$, each of whose determining hyperplanes contains the origin. A hyperplane $H \subset \mathbb{R}^d$ containing the origin is called a supporting hyperplane if $H \cap C \neq \{\mathbf{0}\}$ and C is a subset of a closed half-space determined by H. In this case, $H \cap C$ is called a *face* of C.

Note that the sets $\big\{(x, y) \in \mathbb{R}^2 \mid y - \frac{4}{7}x \geq 0\big\}$ and $\big\{(x, y) \in \mathbb{R}^2 \mid y - \frac{7}{4}x \leq 0\big\}$ form half spaces, and the supporting hyperplane of each is $y - \frac{4}{7}x = 0$ and $y - \frac{7}{4}x = 0$ respectively. By the above definitions, the set A in example 2.3 is a polyhedral cone in $\mathbb{R}^2$.

Theorem 2.6. *([3], Corollary 7.1a) (Farkas-Minkowski-Weyl Theorem) A cone C is polyhedral if and only if it is finitely generated, i.e. $C = \{\lambda_1 c_1 + \cdots + \lambda_r c_r | \lambda_i \in \mathbb{R}_{\geq 0}\}$ for $c_1, \ldots, c_r \in \mathbb{R}^d$.*

For the set A in example 2.3, we may choose the integer pairs $(4, 7)$ and $(7, 4)$ as generators so that $A = \{\lambda_1(4, 7) + \lambda_2(7, 4) \mid \lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}\}$. We may then further classify the polyhedral cone A.



Definition 2.7. A polyhedral cone $C = \{\lambda_1 c_1 + \cdots + \lambda_r c_r | \lambda_i \in \mathbb{R}_{\geq 0}\}$ is said to be *rational* if we may choose all $c_1, \ldots, c_r \in \mathbb{Q}^d \subset \mathbb{R}^d$. We say that C is *trivial* when all $\mathbf{c}_i = \mathbf{0}$ and that C is *nontrivial* otherwise. We say that C is a *ray* if $C = \{\lambda \mathbf{c} \mid \lambda \in \mathbb{R}_{\geq 0}\}$ for some $\mathbf{c} \in \mathbb{R}^d$ and we say C is a *line* if $C = \{\lambda_1 \mathbf{c} + \lambda_2(-\mathbf{c}) \mid \lambda_i \in \mathbb{R}_{\geq 0}\}$.

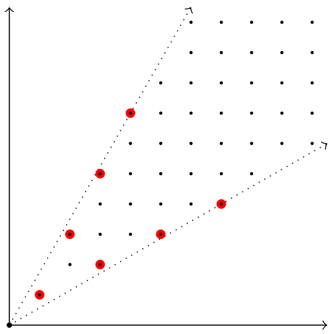
We may then say that A is a rational cone since both $(4, 7)$ and $(7, 4)$ are elements of $\mathbb{Z}^2 \subset \mathbb{Q}^2$. It is already clear that Q_1 is an affine semigroup, but the following theorem gives us the ability to readily identify affine semigroups embeddable in $\mathbb{Z}^d$ where $d > 2$.

Theorem 2.8. *([2], Proposition 7.16) (Gordan's Lemma) If $C \subset \mathbb{R}^d$ is a rational cone and A is any subgroup of $\mathbb{Z}^d$, then $C \cap A$ is an affine semigroup.*

We say that a commutative semigroup with identity Q is *finitely generated* if Q is trivial or there exists $q_1, \ldots, q_k \in Q$ such that $Q = \{n_1 q_1 + \cdots + n_k q_k \mid n_i \in \mathbb{N}\}$. Finding such generators greatly simplifies the study of such semigroups, especially if there exists a unique minimal set of such generators. The next theorem implies that most of the semigroups encountered in this study have this property.

Theorem 2.9. *([2], Proposition 7.15) A semigroup C is said to be* pointed *if it contains the identity and the identity is the only invertible element in C, i.e. $C \cap (-C) = \{\mathbf{0}\}$. Any pointed affine semigroup C has a unique finite minimal generating set.*

Consider $Q_1 = A \cap \mathbb{N}^2$ from before. We see that as $(0, 0) \in Q_1$ and that $Q_1 \cap (-Q_1) = \{(0, 0)\}$ (where $-Q_1$ denotes the reflection of Q_1 about the origin). This shows that Q_1 is a pointed affine semigroup and in fact, this is true for any cone in $\mathbb{N}^2$ [as $\mathbb{N}^2 \cap (-\mathbb{N}^2) = \{(0, 0)\}$]. We may then extract a unique minimal generating set for Q_1 which we highlight below:



Q_1

For any finitely generated subsemigroup of $\mathbb{N}^2$, we call its minimal generating set a *Hilbert basis*. There is a symmetry in the Hilbert basis for $Q_1 = A \cap \mathbb{N}^2$, and it is of a manageable size, however, in general this is not to be expected. Depending on the slope of the bounding hyperplanes defining a cone in $\mathbb{N}^2$, finding a unique minimal generating set can be very cumbersome, and once it is found, it is even more difficult to verify that it is indeed the unique minimal generating set. We have chosen to rely on the **Normaliz** package executable in **Macaulay2** to extract minimal generating sets of pointed affine semigroups when necessary; this tool saves a great deal of time.

In familiar territory, finite generation can be a simplifying property of extreme value. As an example, if B is a linearly independent generating set for a free module F, every R-module homomorphism $\phi \colon F \to M$ is completely determined the images of the basis elements $\phi(b)$. The converse of this is also true; that is, every map $b \to m \in M$ defined for all $b \in B$ determines a *unique* R-module homomorphism $\lambda\colon R^n \to M$. This property of free R-modules not only makes their study more accessible; since every R-module is the quotient of a free R-module, this property can be leveraged in the study of R-modules that are not necessarily free.

In the unfamiliar territory of cones $Q \subset \mathbb{N}^2$, it is true that every semigroup homomorphism defined on Q can be completely determined by the images of the basis elements; however, the converse property that every set map on the unique minimal set of generators induces a semigroup homomorphism is not true in general (and specifically not true in the case of homomorphisms $g\colon Q \to \mathbb{N}$ for nontrivial cones Q in $\mathbb{N}^2$). In the following section, however, we prove a theorem that gives us the ability to classify all fan-linear maps with relative ease.

3. Homomorphisms of Cones in $\mathbb{N}^2$

Consider a semigroup homomorphism $g : Q \to \mathbb{N}$ where Q is a cone in $\mathbb{N}^2$ and assume that $\{b_1, \ldots, b_k\}$ is the Hilbert basis for Q. We know that the image of g is determined by the images of the generators for Q; that is, if $q \in Q$, $q = n_1 b_1 + \cdots + n_k b_k$, then $g(q) = n_1 g(b_1) + \cdots + n_k g(b_k) \in \mathbb{N}$. The issue arises in that the minimal generating set for Q will rarely be linearly independent. There may be several ways to represent q in terms of the b_i, and even if there were a dependable way to choose such a representation, it would do little to simplify the piecewise form of fan-linear maps:

$$f(r,s) = \begin{cases} g_1(r,s) & \text{if } (r,s) \in Q_0 \\ g_2(r,s) & \text{if } (r,s) \in Q_1 \\ \vdots & \\ g_n(r,s) & \text{if } (r,s) \in Q_n \end{cases}$$

If we chose to study a fan-linear map by its mapping of the generators for the cones on which it is defined, we would immediately encounter problems. First and foremost, it is unlikely that the cones of a given fan would even have the same number of generators between them, so studying the interactive properties of the map between each cone could be exceedingly difficult. Secondly, we do not have a universal mapping property that allows us to define arbitrary maps in terms of the generators and the ability to do so is essential to the study and utility of fan algebras. It would therefore be most beneficial to escape the classification of cone homomorphisms in terms of Hilbert bases and somehow state $g(r,s)$ in terms of r and s. This problem is what motivates the main result of this section, over which we will show that any cone homomorphism $g : Q \to \mathbb{N}$ *must* have the form $g(r,s) = ar + bs$ for some $a, b \in \mathbb{Z}$ and that this a and b are determined by *only* the images of certain generators. This will allow us to classify all fan-linear maps accordingly for any number of cones.

One might conclude that showing $g(r,s) = ar + bs$ for some $a, b \in \mathbb{Z}$ would not be so difficult. It is well known that every semigroup is contained in a minimal group. It would be enough then to show that for any cone $Q \subset \mathbb{N}^2$, the minimal group containing Q is in fact $\mathbb{Z}^2$. This would imply that $\mathbb{Z}^2$ has the property that all semigroup homomorphisms defined $Q \to \mathbb{Z}$ factor through a unique group homomorphism $\mathbb{Z}^2 \to \mathbb{Z}$, which immediately gives us that $g(r, s) = ar + bs$ for some $a, b \in \mathbb{Z}$. But then a problem arises in that we must, in the context of our problem, describe all semigroup homomorphisms $Q \to \mathbb{N}$. This requires us to examine how the particular map acts on the generators of Q in order to assure that when we restrict to Q, the image lies entirely within $\mathbb{N}$. The ability to do so is lost in the generality of the above consideration. For this reason, we require a stronger criteria which we will provide in Theorem 3.6.

To begin, we will formalize some rather intuitive results in the interest of rigor. The following proposition shows that in any interesting case of a polyhedral cone in $\mathbb{R}^2$, i.e. a nontrivial cone that is not a ray and covers less than two quadrants, we may choose exactly two generators for the cone. Additionally, we show that these two generators must generate the entire vector space when taken as elements of $\mathbb{R}^2$.

Proposition 3.1. *Let $C \subset \mathbb{R}^2$ be a nontrivial polyhedral cone. If C is neither a half-space, a line, nor a ray, then C has exactly two supporting hyperplanes and is generated by two elements, i.e. $C = \{\lambda_1 \mathbf{c}_1 + \lambda_2 \mathbf{c}_2 | \lambda_i \in \mathbb{R}_{\geq 0}\}$ for some $\mathbf{c}_1, \mathbf{c}_2 \in \mathbb{R}^2$. Furthermore, $\mathbf{c}_1$ and $\mathbf{c}_2$ generate the vector space $\mathbb{R}^2$ lie on the two distinct faces of C.*

Proof. Since C is the intersection of a finite number of half-spaces in $\mathbb{R}^2$, assume that C is a subset of the first and fourth quadrants of $\mathbb{R}^2$ (otherwise we could perform a rotation of axes and make it so). Note that if H is a nontrivial hyperplane in $\mathbb{R}^2$ that passes through the origin, then $H = \{(x, y) \in \mathbb{R}^2 \mid x \cdot \lambda_1 + y \cdot \lambda_2 = 0,\ \lambda_i \in \mathbb{R}\}$ so that at least one of λ_1, λ_2 is nonzero. Then either $H = \{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$ or $H = \{(x, y) \in \mathbb{R}^2 \mid y = \lambda x,\ \lambda \in \mathbb{R}\}$. It also follows that if $H^\diamond$ is a half-space determined by H and $H = \{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$, then either $H^\diamond = \{(x, y) \in \mathbb{R}^2 \mid y \in \mathbb{R} \text{ and } x \leq 0\}$ or $H^\diamond = \{(x, y) \in \mathbb{R}^2 \mid y \in \mathbb{R} \text{ and } x \geq 0\}$. If $H = \{(x, y) \in \mathbb{R}^2 \mid y = \lambda x,\ \lambda \in \mathbb{R}\}$, then either $H^\diamond = \{(x, y) \in \mathbb{R}^2 \mid y \geq \lambda x\}$ or $H^\diamond = \{(x, y) \in \mathbb{R}^2 \mid y \leq \lambda x\}$.

Let H_1, H_2, and H_3 be distinct, nontrivial hyperplanes in $\mathbb{R}^2$ and assume that $H_i \cap C \neq \{\mathbf{0}\}$ for all $i = 1, 2, 3$. Further assume without loss of generality that H_1 and H_2 are supporting hyperplanes for C. We claim that H_3 is not a supporting hyperplane of C.

Since C is not a half-space, C is the intersection of more than one half-space. Let $H_1^\diamond$ and $H_2^\diamond$ be the half-spaces containing C determined by H_1 and H_2 respectively. Then $C \subseteq H_1^\diamond \cap H_2^\diamond$. Assume that $H_1 = \{(x, y) \in \mathbb{R}^2 \mid y = \lambda_1 x\}$, $H_2 = \{(x, y) \in \mathbb{R}^2 \mid y = \lambda_2 x\}$, and $H_3 = \{(x, y) \in \mathbb{R}^2 \mid y = \lambda_3 x\}$. Without loss of generality, further assume that $\lambda_1 < \lambda_2$ and that $H_1^\diamond = \{(x, y) \in \mathbb{R}^2 \mid y \geq \lambda_1 x\}$ and $H_2^\diamond = \{(x, y) \in \mathbb{R}^2 \mid y \leq \lambda_2 x\}$ so that $H_1^\diamond \cap H_2^\diamond = \{(x, y) \in \mathbb{R}^2 \mid \lambda_1 x \leq y \leq \lambda_2 x \text{ with } x \geq 0\}$ ($x \geq 0$ by the assumption that C is a subset of the eastern quadrants). If $\lambda_3 < \lambda_1$ or $\lambda_2 < \lambda_3$, then $H_3 \cap C \subseteq H_3 \cap H_1^\diamond \cap H_2^\diamond = \{\mathbf{0}\}$, false by $H_3 \cap C \neq \{\mathbf{0}\}$. Then $\lambda_1 < \lambda_3 < \lambda_2$. Assume that H_3 is a supporting hyperplane for C. Let $H_3^\diamond$ be the half-space determined by H_3 containing C. Then (1) $H_3^\diamond = \{(x, y) \in \mathbb{R} \mid y \geq \lambda_3 x\}$ or (2) $H_3^\diamond = \{(x, y) \in \mathbb{R} \mid y \leq \lambda_3 x\}$. If (1), then $C \subset H_1^\diamond \cap H_2^\diamond \cap H_3^\diamond = H_2^\diamond \cap H_3^\diamond$, hence $H_1 \cap C = \{\mathbf{0}\}$, false. Similarly, if (2), then $H_2 \cap C = \{\mathbf{0}\}$, false again. Conclude that if H_1 and H_2 are supporting hyperplanes, H_3 is not. The proof follows similarly in the case that some $H_i = \{(0, y) \in \mathbb{R}^2 \mid y \in \mathbb{R}\}$, hence C has at most two supporting hyperplanes.

It is clear that if C has exactly one distinct supporting hyperplane, then C is a half-space or a line. If C has two supporting hyperplanes and C is the intersection of three or more half-spaces, C must be trivial or a ray. Then if C is nontrivial and is not a half-space, a line, or a ray, C has exactly two supporting hyperplanes and C is the intersection of two half-spaces determined by those hyperplanes. By choice of axes (avoiding a hyperplane that intersects the vertical axis), we may set $H_1 = \{(x, y) \in \mathbb{R}^2 \mid y = \lambda_1 x\}$, $H_2 = \{(x, y) \in \mathbb{R}^2 \mid y = \lambda_2 x\}$ to be hyperplanes for C so that $C = H_1^\diamond \cap H_2^\diamond = \{(x, y) \in \mathbb{R} \mid \lambda_1 x \leq y \leq \lambda_2 x \text{ with } x \geq 0\}$. Then $C = \{(x, y) \in \mathbb{R}^2 \mid (x, y) = c_1(1, \lambda_1) + c_2(1, \lambda_2),\ c_i \in \mathbb{R}_{\geq 0}\}$, which shows that C is generated by exactly two elements of $\mathbb{R}^2$.

If $(1, \lambda_1) = a(1, \lambda_2)$ for $a \in \mathbb{R}$, then we have two cases. If $a \geq 0$, $C = \{(x, y) \in \mathbb{R}^2 \mid (x, y) = c(1, \lambda_2),\ c \in \mathbb{R}_{\geq 0}\}$, false by the hypothesis that C is not a ray. If $a < 0$, then $C = \{(x, y) \in \mathbb{R}^2 \mid (x, y) = c_1(1, \lambda_2) + c_2(-1, -\lambda_2),\ c_i \in \mathbb{R}_{\geq 0}\}$, which is impossible as C is not a line. It follows then that if $c_1(1, \lambda_1) + c_2(1, \lambda_2) = (0, 0)$ for $c_1, c_2 \in \mathbb{R}$, $c_1 = c_2 = 0$, i.e. $(1, \lambda_1)$ and $(1, \lambda_2)$ are linearly dependent. Consequently, $\overline{C} = \{(x, y) \in \mathbb{R}^2 \mid (x, y) = c_1(1, \lambda_1) + c_2(1, \lambda_2),\ c_i \in \mathbb{R}\} = \mathbb{R}^2$, hence C is maximal. It is also immediate that $H_1 \cap C = \{c \cdot (1, \lambda_1) \mid c \in \mathbb{R}\}$ and $H_2 \cap C = \{c \cdot (1, \lambda_2) \mid c \in \mathbb{R}\}$ so that $(1, \lambda_1)$ and $(1, \lambda_2)$ lie on the distinct faces of C. This completes the proof. □

The following definition and proposition refine the idea of a polyhedral cone and its relationship to cones in $\mathbb{N}^2$ to the vector space $\mathbb{Q}^2$. We make this refinement as in all practicality, a cone in $\mathbb{N}^2$ is much "closer" to $\mathbb{Q}^2$ than it is to $\mathbb{R}^2$, just as any field containing $\mathbb{Z}$ must also contain $\mathbb{Q}$.

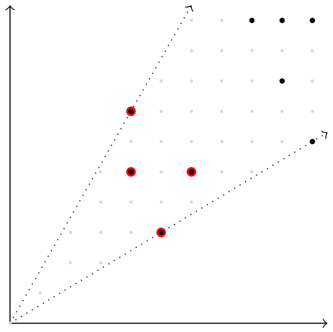
Definition 3.2. Let $v_1 = (r_1, s_1)$ and $v_2 = (r_2, s_2)$ be elements in $\mathbb{N}^2$ so that v_1 and v_2 are linearly independent in $\mathbb{Q}^2$. Define Q_{v_1,v_2} to be the set $\mathbb{N}^2 \cap \{q_1 v_1 + q_2 v_2 \mid q_i \in \mathbb{Q}_{\geq 0}\}$ so that $Q_{v_1,v_2} = \{(r, s) \in \mathbb{N}^2 \mid (r, s) = q_1 v_1 + q_2 v_2 \text{ for some } q_1, q_2 \in \mathbb{Q}_{\geq 0}\}$. Furthermore, if r_i and s_i are relatively prime for $i = 1, 2$, we call v_1, v_2 the *edges* of Q_{v_1,v_2}.

Proposition 3.3. *Let $C = \{\lambda_1 \mathbf{c}_1 + \lambda_2 \mathbf{c}_2 | \lambda_i \in \mathbb{R}_{\geq 0}\}$ be a rational cone entirely within the first quadrant of $\mathbb{R}^2$ so that $\mathbf{c}_1$ and $\mathbf{c}_2$ are linearly independent in $\mathbb{R}^2$. Let $Q = C \cap \mathbb{N}^2$ be a cone in $\mathbb{N}^2$ and assume that $v_1, v_2 \in Q$ lie on the faces of C. Then $Q = Q_{v_1,v_2}$.*

Proof. Since C is in the first quadrant of $\mathbb{R}^2$, C is not a half-space or a line, and since $\mathbf{c}_1$ and $\mathbf{c}_2$ are linearly independent in $\mathbb{R}^2$, C is not a ray. Let H_1, H_2 be the distinct supporting hyperplanes of C so that $v_1 \in H_1 \cap C$ and $v_2 \in H_2 \cap C$. Then $v_1 = \lambda_1 \mathbf{c}_1$ and $v_2 = \lambda_2 \mathbf{c}_2$ for some $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$. As $\mathbb{R}$ is a field, we may write $C = \{\lambda_1 v_1 + \lambda_2 v_2 \mid \lambda_i \in \mathbb{R}_{\geq 0}\}$. As $Q = C \cap \mathbb{N}^2$ and $Q_{v_1,v_n} = \{(r, s) \in \mathbb{N}^2 \mid (r, s) = q_1 v_1 + q_2 v_2 \text{ for some } q_1, q_2 \in \mathbb{Q}_{\geq 0}\}$, it is clear that $Q_{v_1,v_n} \subseteq C \cap \mathbb{N}^2 = Q$.

For the reverse containment, we claim that for $(r, s) \in Q = C \cap \mathbb{N}^2$, $(r, s) = q_1 v_1 + q_2 v_2$ for $q_1, q_2 \in \mathbb{Q}$. Since $(r, s) \in C$, $(r, s) = \lambda_1 v_1 + \lambda_2 v_2$ for $\lambda_1, \lambda_2 \in \mathbb{R}_{\geq 0}$. If $v_1 = (r_1, s_1)$ and $v_2 = (r_2, s_2)$, then $r = \lambda_1 r_1 + \lambda_2 r_2$ and $s = \lambda_1 s_1 + \lambda_2 s_2$. This is a system two of linear equations in two unknowns, λ_1 and λ_2. As v_1, v_2 are linearly independent in $\mathbb{R}^2$, they are linearly independent in $\mathbb{Q}^2$ which implies that there exists a unique solution $\lambda_1 = q_1 \in \mathbb{Q}$ and $\lambda_2 = q_2 \in \mathbb{Q}$. As $\mathbb{Q} \subset \mathbb{R}$, this solution is also unique in $\mathbb{R}$, so we may take $q_1, q_2 \geq 0$. This proves the claim. It follows then that $Q \subset \mathbb{N}^2 \cap \{q_1 v_1 + q_2 v_2 \mid q_i \in \mathbb{Q}_{\geq 0}\} = Q_{v_1,v_2}$, hence $Q = Q_{v_1,v_2}$. □

In the interest of generalizing the main theorem of the section, we now show that every finitely generated subsemigroup S of $\mathbb{N}^2$ is contained in some cone $Q \subset \mathbb{N}^2$. This is fairly easy to see; given any set of generators, we may draw two supporting hyperplanes that pass through the origin and generating coordinates in $\mathbb{N}^2$ with the greatest possible interior angle (which is equivalent to choosing two generators (r, s) with the greatest and least ratios $\frac{s}{r}$). A cone can then be constructed to contain all of the elements of S. We provide an illustration below where $S = \langle (4, 7), (4, 5), (6, 5), (5, 3) \rangle \subset \mathbb{N}^2$:



$S \subset Q_{(4,7),(5,3)}$

Definition 3.4. Let S be a finitely generated subsemigroup of $\mathbb{N}^2$. The *cone generated by* S (denoted $\langle S\rangle$) is the cone Q in $\mathbb{N}^2$ such that for any cone $N \subset \mathbb{N}^2$ containing S, $Q \subseteq N$.

Proposition 3.5. *For any finitely generated nonzero subsemigroup S of $\mathbb{N}^2$, either $\langle S\rangle$ is a ray or $\langle S\rangle = Q_{h_1,h_2}$ where $h_1 = (r_1, s_1)$ and $h_2 = (r_2, s_2)$ are generators for S and for any generator (r', s') for S, $\frac{s_1}{r_1} \leq \frac{s'}{r'} \leq \frac{s_2}{r_2}$.*

Proof. Let $H = \{h_1, \dots, h_n\}$ be a generating set for S. If S is generated by one element h, then take $Q = C \cap \mathbb{N}^2$ where $C = \{ch \mid c \in \mathbb{R}\}$. Then S is a subsemigroup of the cone Q in $\mathbb{N}^2$, and this ray Q is clearly contained in any cone containing S.

Assume that 2 bounds the number of possible generators in H below and renumber the h_i so that for $h_1 = (r_1, s_1)$ and $h_2 = (r_2, s_2)$, $\frac{s_2}{r_2} \leq \frac{s_k}{r_k} \leq \frac{s_1}{r_1}$ for all $h_k = (r_k, s_k) \in H$. We claim that S is a subsemigroup of Q_{h_1,h_2}. Consider $C = \{c_1h_1 + c_2h_2 \mid c_i \in \mathbb{R}_{\geq 0}\}$. Since $n \geq 2$, $\frac{s_1}{r_1} < \frac{s_2}{r_2}$ which implies that $r_1s_2 - r_2s_1 > 0$, so for any $h_k \in H$ we have that:

$$(r_k, s_k) = (\frac{s_2r_k - r_2s_k}{r_1s_2 - r_2s_1})(r_1, s_1) + (\frac{r_1s_k - s_1r_k}{r_1s_2 - r_2s_1})(r_2, s_2)$$

Note that both coordinates are positive by the previous conditions on (r_k, s_k), hence S is a subsemigroup of $Q = C \cap \mathbb{N}^2 = Q_{h_1,h_2}$, as claimed. Now suppose $N = C' \cap \mathbb{N}^2$ is another cone in $\mathbb{N}^2$ containing S. Then $h_1, h_2 \in N$ which implies that $h_1, h_2 \in C'$ and hence $c_1h_1 + c_2h_2 \in C'$ for all $c_1, c_2 \in \mathbb{R}_{\geq 0}$, so it follows that $Q \subseteq N$. Uniqueness follows easily; if Q and Q' have the property above, then $Q \subseteq Q'$ and $Q' \subseteq Q$, hence $Q' = Q$. ☐

We now present the main theorem of the section, which implies that we may use the properties of linear transformations $\mathbb{Q}^2 \to \mathbb{Q}$ to examine homomorphisms of finitely generated subsemigroups of $\mathbb{N}^2$.

Theorem 3.6. *Let S be a finitely generated subsemigroup of $\mathbb{N}^2$ and $\varphi : S \to \mathbb{Z}$ a homomorphism of semigroups. There exists a linear transformation $T^\varphi : \mathbb{Q}^2 \to \mathbb{Q}$ such that the restriction of T^φ to S gives φ. If S is generated by at least two elements, then this linear transformation is unique.*

Proof. Let $H = \{h_1, \dots, h_n\}$ be a set of generators for S. If S is generated by one nonzero element $h_1 \in H$, choose some element $h_2 \in \mathbb{Q}^2$ so that h_1 and h_2 are linearly independent over $\mathbb{Q}$, and hence $\{h_1, h_2\}$ is a vector space basis for $\mathbb{Q}^2$. Define $T^\varphi \colon \mathbb{Q}^2 \to \mathbb{Q}$ by $q_1h_1 + q_2h_2 \mapsto q_1\varphi(h_1)$. It is immediate that T^φ is a linear transformation and that $T^\varphi_{|S} = \varphi$. If $S = 0$, then define T^φ as the zero map, which has the desired properties as $\varphi(0) + \varphi(0) = \varphi(0+0) = \varphi(0)$ must be 0 in $\mathbb{Z}$.

Now suppose that the number of generators in H is bounded below by 2 and choose $v_1, v_2 \in H$ so that $Q = Q_{v_1,v_2} = \langle S \rangle$ is the cone in $\mathbb{N}^2$ generated by S. For each generator $h_i \in H$ such that $h_i \neq v_1$ and $h_i \neq v_2$, let $\varphi(h_i) = E_i \in \mathbb{Z}$ and let $\varphi(v_1) = V_1 \in \mathbb{Z}$ and $\varphi(v_2) = V_2 \in \mathbb{Z}$. Note that since S is generated by at least two elements, v_1 and v_2 are linearly independent vectors in $\mathbb{Q}^2$, so $\{v_1, v_2\}$ is a vector space basis for $\mathbb{Q}^2$.

Since $S \subseteq Q$, for any $h_i \in H$, there exists $q_{1,i}, q_{2,i} \in \mathbb{Q}_{\geq 0}$ such that $h_i = q_{1,i}v_1 + q_{2,i}v_2$. It follows that we may choose some positive integer ω that clears the denominators of $q_{1,i}$ and $q_{n,i}$ so that $\omega \cdot h_i = \omega \cdot q_{1,i}v_1 + \omega \cdot q_{n,i}v_n$ is another element in S that is both a linear combination of h_i and also a linear combination of v_1 and v_2 with positive-integer coefficients (i.e. a non-uniquely represented element in S).

As φ is a semigroup homomorphism on S, for every $n \in \mathbb{N}$ and $\alpha \in S$, $\varphi(n \cdot \alpha) = \underbrace{\varphi(\alpha) + \cdots + \varphi(\alpha)}_{n\ times} = n \cdot \varphi(\alpha)$. By the implicit hypothesis that φ is well-defined, we must have that:

$$\begin{gathered}\varphi(\omega \cdot h_i) = \varphi(\omega \cdot q_{1,i}v_1) + \varphi(\omega \cdot q_{n,i}v_2)\\ \omega \cdot \varphi(h_i) = \omega q_{1,i} \cdot \varphi(v_1) + \omega q_{n,i} \cdot \varphi(v_2)\\ \omega \cdot E_i = \omega q_{1,i} \cdot V_1 + \omega q_{n,i} \cdot V_2\end{gathered}$$

As $\omega \in \mathbb{Z}^+$, we may divide through and obtain the identity:

$$h_i = q_{1,i} \cdot v_1 + q_{n,i} \cdot v_2 \Longrightarrow E_i = q_{1,i} \cdot V_1 + q_{2,i} \cdot V_2$$

Note that this identity holds for each h_i, $i = 1, \ldots, n$, and consider $\alpha \in Q$ so that $\alpha = \sum_{i=1}^{n} d_i h_i$ for $d_i \in \mathbb{N}$. Then:

$$\begin{array}{rcl}\alpha & = & d_1h_1 + \cdots + d_nh_n\\ & = & d_1(q_{1,1} \cdot v_1 + q_{2,1} \cdot v_2) + \cdots + d_n(q_{1,n} \cdot v_1 + q_{2,n} \cdot v_2)\\ & = & (d_1q_{1,1} + \cdots + d_nq_{1,n})v_1 + (d_1q_{2,1} + \cdots + d_nq_{2,n})v_2\\ & = & q_{1,\alpha}v_1 + q_{2,\alpha}v_2 \text{ for some } q_{1,\alpha}, q_{2,\alpha} \in \mathbb{Q}_{\geq 0}.\end{array}$$

Accordingly, we have that:

$$\begin{array}{rcl}\varphi(\alpha) & = & d_1\varphi(h_1) + \cdots + d_n\varphi(h_n)\\ & = & d_1(q_{1,1} \cdot V_1 + q_{2,1} \cdot V_2) + \cdots + d_n(q_{1,n} \cdot V_1 + q_{2,n} \cdot V_2)\\ & = & (d_1q_{1,1} + \cdots + d_nq_{1,n})V_1 + (d_1q_{2,1} + \cdots + d_nq_{2,n})V_2\\ & = & q_{1,\alpha}V_1 + q_{2,\alpha}V_2\end{array}$$

This shows that if $\alpha = q_{1,\alpha}v_1 + q_{2,\alpha}v_2$, then $\varphi(\alpha) = q_{1,\alpha}V_1 + q_{2,\alpha}V_2$ for all $\alpha \in S$. Note here that $q_{\alpha,1}, q_{\alpha,n} \in \mathbb{Q}_{\geq 0}$ are also the unique coordinates of α in $\mathbb{Q}^2$ with respect to the basis $\{v_1, v_2\}$. We now extend the map φ to all of $\mathbb{Q}^2$. Define the map $T^{\varphi} : \mathbb{Q}^2 \to \mathbb{Q}$ by the rule $x = c_1v_1 + c_2v_2 \longmapsto c_1V_1 + c_nV_n$ allowing the $c_i \in \mathbb{Q}$ to be negative and thereby providing coordinates for all of $\mathbb{Q}^2$. Then for all $x, y \in \mathbb{Q}^2$, $x = b_1v_1 + b_2v_2$ and $y = c_1v_1 + c_2v_2$, and for all scalars $q, s \in \mathbb{Q}$:

$$\begin{aligned} T^{\varphi}(qx+sy) &= T^{\varphi}(qb_1v_1+qb_2v_2+sc_1v_1+sc_2v_2) \\ &= T^{\varphi}[(qb_1+sc_1)v_1+(qb_2+sc_2)v_2] \\ &= (qb_1+sc_1)V_1+(qb_2+sc_2)V_2 \\ &= q(b_1V_1+b_2V_2)+s(c_1V_1+c_2V_2) \\ &= q[T^{\varphi}(x)]+s[T^{\varphi}(y)] \end{aligned}$$

This shows that T^{φ} is a linear transformation from $\mathbb{Q}^2$ to $\mathbb{Q}$. Since $\alpha \in S$ implies that if $\alpha = q_{1,\alpha}v_1 + q_{2,\alpha}v_2$ then $\varphi(\alpha) = q_{1,\alpha}V_1 + q_{2,\alpha}V_2$, we clearly have that $T^{\varphi}_{|S} = \varphi$. Uniqueness follows in that T^{φ} is determined by the images of the basis vectors v_1 and v_2. This completes the proof. □

The above theorem is important mainly due to the the following two corollaries.

Corollary 3.7. *If S is a finitely generated subsemigroup of $\mathbb{N}^2$ and $\varphi\colon S \to \mathbb{Z}$ is a semigroup homomorphism, then for all $(r,s) \in S$, $\varphi(r,s) = q_1r + q_2s$ for some $q_1, q_2 \in \mathbb{Q}$.*

Proof. Since $T^{\varphi}\colon \mathbb{Q}^2 \to \mathbb{Q}$ is a linear transformation, there exists a unique 1×2 matrix $A_{T^{\varphi}} = [q_1\ q_2]$ with entries in $\mathbb{Q}$ so that $T^{\varphi}[(r,s)^T] = A_{T^{\varphi}} \cdot (r,s)^T = q_1r + q_2s$ for all $(r,s)^T \in \mathbb{Q}^2$. In particular, $T^{\varphi}[(r,s)^T] = \varphi(r,s) = q_1r + q_2s$ for all $(r,s)^T$ such that $(r,s) \in S$. □

The next corollary to Theorem 3.6 shows that when S is a cone in $\mathbb{N}^2$, the above can be stated in even stronger terms.

Corollary 3.8. *Let C be a rational cone in the first quadrant of $\mathbb{R}^2$ so that C is not a ray. If $Q = C \cap \mathbb{N}^2$ and $f : Q \to \mathbb{Z}$ is a homomorphism of semigroups, then $f(r,s) = ar + bs$ for $a, b \in \mathbb{Z}$.*

Proof. Note that by Theorem 2.9, Q is finitely generated. By Corollary 3.7, we may write $f(r,s) = ar + bs$ for $a, b \in \mathbb{Q}$. We claim that there exists some $(r,s) \in Q$ such that $(r, s+1)$ and $(r+1, s)$ are also in Q. Choose $v_1 = (x,y)$, $v_n = (x',y') \in Q$ so that these elements lie on the two distinct faces of C and $Q = Q_{v_1,v_2}$. Suppose also, without loss of generality, that $\frac{y'}{x'} < \frac{y}{x}$. Let $\varepsilon = \frac{y}{x} - \frac{y'}{x'} > 0$ and note that the sequence $\{n \cdot \varepsilon\}_{n=1}^{\infty}$ diverges. Each term $n \cdot \varepsilon$ in the sequence is simply the measure of distance between the rational numbers $n \cdot \frac{y'}{x'}$ and $n \cdot \frac{y}{x}$. As $\{n \cdot \varepsilon\}_{n=1}^{\infty}$ diverges, the distance between these rational multiples can be made arbitrarily large so that we may find, for a sufficiently large $r \in \mathbb{N}$, some $s \in \mathbb{N}$ so that $r\frac{y'}{x'} < (r+1)\frac{y'}{x'} < s < s+1 < r\frac{y}{x} < (r+1)\frac{y}{x}$. Then $\frac{y'}{x'} < \frac{s}{r+1} < \frac{s}{r} < \frac{s+1}{r} < \frac{y}{x}$, which implies that $(r, s+1), (r,s)$ and $(r+1, s)$ are all elements in Q.

Now suppose that $a = \frac{p_1}{q_1}$ and $b = \frac{p_2}{q_2}$ with each pair p_i, q_i relatively prime. As $f(r,s) = \frac{p_1}{q_1}r + \frac{p_2}{q_2}s = \frac{p_1q_2r + p_2q_1s}{q_1q_2} \in \mathbb{Z}$, we must have that q_1q_2 divides $p_1q_2r + p_2q_1s$ for every $(r,s) \in Q$. By the claim above, choose $(r^*, s^*) \in Q$ so that (r^*+1, s^*) and (r^*, s^*+1) are also elements in Q. We must first have that $p_1q_2r^* + p_2q_1s^*$ is divisible by q_1q_2. Then $p_1q_2r^* + p_2q_1s^* \equiv 0(mod\ q_1q_2)$. But then, by $(r^*+1, s^*) \in Q$, we must have that $p_1q_2 \equiv 0(mod\ q_1q_2)$ or equivalently $p_1 \equiv 0(mod\ q_1)$. As we took p_1 and q_1 to be relatively prime, it must be true that $q_1 = 1$. Similarly, considering $(r^*, s^*+1) \in Q$ forces $q_2 = 1$, so that for $f(r,s) = ar+bs$ we may take $a, b \in \mathbb{Z}$. ☐

We may now collect our observations in a theorem which validates the claim made at the beginning of this section. This gives us a means to both classify assumed maps and define arbitrary maps, which was our goal.

Theorem 3.9. *Let Q be a cone in a fan $\Sigma_{\mathbf{a},\mathbf{b}}$. A mapping $g\colon Q \to \mathbb{Z}$ is a semigroup homomorphism if and only if $g(r,s) = ar+bs$ for some $a, b \in \mathbb{Z}$.*

Proof. The forward implication is simply a collection of the previous results. For the converse, it is easily verified that g defined as such is a semigroup homomorphism. ☐

Remark 3.10. Note that the above theorem does not force a and b to be nonnegative integers when we require that f is nonnegative (i.e. $f : Q \to \mathbb{N} \subset \mathbb{Z}$). The map $f(r,s) = 2r+(-1)s$ is an example of a homomorphism on the cone $Q = Q_{(1,2),(1,0)}$. Since $2r \geq s$ for all $(r,s) \in Q$, this function still maps from Q to $\mathbb{N}$. In general, it is not always possible to state a homomorphism $\varphi\colon S \to \mathbb{Z}$ as $\varphi(r,s) = ar+bs$ for $a, b \in \mathbb{Z}$. As a counter example, take $\{(2,0),(4,0),(6,0)\ldots\} = \langle(2,0)\rangle \subset Q = C \cap \mathbb{N}^2$ where $C = \{c(0,2) \mid c \in \mathbb{R}_{\geq 0}\}$. We may map $(2,0) \mapsto 1 \in \mathbb{Z}$ which would determine all of φ, however there are no integers $a, b \in \mathbb{Z}$ such that $a2 + b0 = 1$.

Since by Theorem 3.6, any semigroup homomorphism $f\colon Q \to \mathbb{Z}$ is determined by the image of the edges v_1, v_2 when $Q = Q_{v_1,v_2}$, it is tempting to think that one might be able to define arbitrary maps $Q \to \mathbb{Z}$. The next proposition shows that this is false.

Proposition 3.11. (Counterexample to Arbitrary Maps $S \to \mathbb{Z}$) *Let S be a finitely generated affine semigroup such that $S \subset C \cap \mathbb{N}^d$ where $C \subset \mathbb{R}^d$ is a positive rational cone. It is not always possible to define arbitrary maps $S \to \mathbb{Z}$ by the images of the generators of S.*

Proof. Consider the cone $Q_{(4,3),(2,5)}$ in $\mathbb{N}^2$ and consider a map $\varphi(4,3) = 2$ and $\varphi(2,5) = 1$. We claim that either φ is *not* a semigroup homomorphism or φ does *not* send Q to $\mathbb{Z}$. Assume the contrary and let c_1, c_2 be the integers such that $\varphi(r,s) = c_1 r + c_2 s$ for all $(r,s) \in Q$ as guaranteed by Corollary 3.8. Then we have the following matrix equation:

$$\begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix} \cdot \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Solving by the inverse of the matrix $[(4,3)^T, (2,5)^T]$, we have:

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{14} & -\frac{1}{7} \\ -\frac{3}{14} & \frac{2}{7} \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{4}{7} \\ -\frac{1}{7} \end{bmatrix}$$

Here we see that neither c_1 nor c_2 is an integer, a contradiction, and hence there is no semigroup homomorphism $Q \to \mathbb{Z}$ such that $\varphi(4,3) = 2$ and $\varphi(2,5) = 1$. ☐

We close this section with the remark that Theorem 3.6 may be extended to $\mathbb{N}^d$ for any d under certain circumstance. Let $C \subset \mathbb{R}^d$ be a rational cone and consider the resulting affine semigroup $Q = C \cap \mathbb{N}^d$. If Q has a Hilbert basis such that it contains a subset of generators that generate the entire rational cone C (in the proper context) and these generators are a linearly independent set of generators for $\mathbb{R}^d$, then any homomorphism $g : Q \to \mathbb{N}$ will be a restriction of some linear transformation $\mathbb{Q}^d \to \mathbb{Q}$. The proof of this fact follows exactly as in Theorem 3.6. In the case of cones in a fan $\Sigma_{\mathbf{a},\mathbf{b}}$, it happens that the above hypothesis is always satisfied. This is why Theorem 3.6 has a universal utility in the classification of fan-linear maps.

4. A Complete Classification of Fan Linear Maps

We now move on to the main task of this study, the classification of all functions from which fan-algebras arise. First, we give formal definitions to some objects with which we have already dealt intuitively.

Definition 4.1. For a finite indexing set I, a *fan* is a collection $\Sigma \subset \mathcal{P}(\mathbb{R}^d)$ of cones $\{C_i\}_{i\in I}$ so that the following conditions are satisfied:

(1) No C_i is a ray.
(2) The faces of each C_i are also in Σ.
(3) The intersection of any pair of cones, when the intersection is not $\{\mathbf{0}\}$, is a shared face of the two cones.

Definition 4.2. Let $\Sigma = \{C_i\}_{i=0}^n \subset \mathcal{P}(\mathbb{R}^2)$ be a fan such that $\bigcup_{i=0}^n C_i$ gives exactly the first quadrant of $\mathbb{R}^2$ and each C_i is rational. Let $\mathbf{a} = \{p_0, \dots, p_{n+1}\}$ and $\mathbf{b} = \{q_0, \dots, q_{n+1}\}$ so that each (p_i, q_i) and (p_{i+1}, q_{i+1}) define the faces of C_i, each p_i and q_i are relatively prime, and $\frac{q_{i+1}}{p_{i+1}} < \frac{q_i}{p_i}$ for $i = 0, \dots, n$. Define a *fan in* $\mathbb{N}^2$ *of* $n+1$ *cones* to be the collection $\Sigma_{\mathbf{a},\mathbf{b}} = \{Q_i\}_{i=0}^n$ where each $Q_i = C_i \cap \mathbb{N}^2$ is a cone in $\mathbb{N}^2$, noting that $\bigcup_{i=0}^n Q_i = \mathbb{N}^2$. For $Q_i \in \Sigma_{\mathbf{a},\mathbf{b}}$, $Q_i \cap Q_{i+1}$ is a *face* of $\Sigma_{\mathbf{a},\mathbf{b}}$ and we say that each (p_i, q_i) and (p_{i+1}, q_{i+1}) *define a face* of Q_i.

Herein, we will be discussing only fans in $\mathbb{N}^2$, therefore when we say a fan of $n+1$ cones, we mean fan in $\mathbb{N}^2$ of $n+1$ cones. Then next few definitions and propositions give a formal definition of fan-linear maps and show that for any fixed fan $\Sigma_{\mathbf{a},\mathbf{b}}$, the set of all fan-linear maps on $\Sigma_{\mathbf{a},\mathbf{b}}$ is itself a semigroup.

Definition 4.3. Let f be a function so that addition is an associative binary operation defined on both the domain and range of f. Further assume that Range(f) is totally ordered. We say that f is *subadditive* if $f(x+y) \le f(x) + f(y)$ for all $x, y \in$ Domain(f).

Definition 4.4. Let $\Sigma_{\mathbf{a},\mathbf{b}}$ be a fan of $n+1$ cones. A *fan-linear map on* $\Sigma_{\mathbf{a},\mathbf{b}}$ is a map $f : \mathbb{N}^2 \to \mathbb{N}$ such that for cones $Q_0, Q_1, \dots, Q_n \in \Sigma_{\mathbf{a},\mathbf{b}}$, f is a semigroup homomorphism when restricted to each Q_i and f is subadditive on $\mathbb{N}^2 = \bigcup_{i=1}^n Q_i$.

As a result of the discussion in the previous section, we may write any fan-linear maps in a much simplified form.

Proposition 4.5. *Let* $f : \mathbb{N}^2 \to \mathbb{N}$ *be a fan linear map on* $\Sigma_{\mathbf{a},\mathbf{b}} = \{Q_0, Q_1, \dots, Q_n\}$. *Then* f *can be written as a piecewise function:*

$$f(r,s) = \begin{cases} a_0 r + b_0 s & \textit{if } (r,s) \in Q_0 \\ a_1 r + b_1 s & \textit{if } (r,s) \in Q_1 \\ \vdots & \\ a_n r + b_n s & \textit{if } (r,s) \in Q_n \end{cases}$$

where each $a_i, b_i \in \mathbb{Z}$.

Proof. This follows immediately from Corollary 3.8 and the definition requiring that f is a semigroup homomorphism when restricted to each cone Q_i. ☐

Proposition 4.6. *If* f *and* g *are subadditive on* $\mathbb{N}^2$, *then* $f + g$ *is subadditive on* $\mathbb{N}^2$.

Proof. If f and g are subadditive on $\mathbb{N}^2$ then for all $x, y \in \mathbb{N}^2$, $(f+g)(x+y) = f(x+y) + g(x+y) \le f(x) + f(y) + g(x) + g(y) = (f+g)(x) + (f+g)(y)$. ☐

Definition 4.7. Let $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ denote the collection of all fan linear maps on a fixed fan $\Sigma_{\mathbf{a},\mathbf{b}}$.

Proposition 4.8. *The collection* $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ *is a commutative, additive semigroup with identity under the operation of function addition.*

Proof. Let $\Sigma_{\mathbf{a},\mathbf{b}} = \{Q_0, Q_1, \dots, Q_n\}$. If $f, g \in Fan(\Sigma_{\mathbf{a},\mathbf{b}})$, then for each $x, y \in Q_i$, $(f+g)(x+y) = f(x+y) + g(x+y) = f(x) + f(y) + g(x) + g(y) = (f+g)(x) + (f+g)(y)$. By Proposition 4.6, $f + g$ is subadditive on $\mathbb{N}^2$ and as function addition is associative, $(\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}}), +)$ is a semigroup. Since the zero function $\mathbf{0} : \mathbb{N}^2 \to \mathbb{N}$ is linear on each Q_i and subadditive on $\mathbb{N}^2$, $\mathbf{0} \in \mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$, and as addition commutes in $(\mathbb{N}, +)$, $(\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}}), +)$ is commutative. ☐

The following map is essential to the classification of all fan-linear maps and will give us the ability late to deduce that the set $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ of all fan-linear maps is indeed an *affine* semigroup.

Definition 4.9. Let p and q be relatively prime positive integers. Define $f_{p,q} : \mathbb{N}^2 \to \mathbb{N}$ so that:

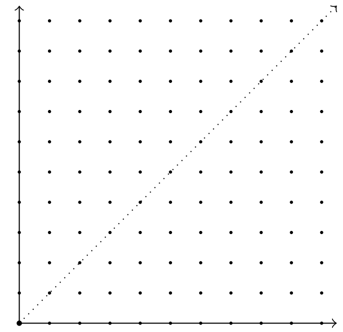
$$f_{p,q}(r,s) = \begin{cases} 0 & \text{if } \frac{s}{r} > \frac{q}{p} \\ qr - ps & \text{if } \frac{s}{r} \le \frac{q}{p} \end{cases}$$

We call $f_{p,q}$ the *determining map* of (p,q) and note here that as $qr \ge ps$ for all $(r,s) \in \mathbb{N}^2$ such that $\frac{s}{r} \le \frac{q}{p}$, $f_{p,q}$ is nonnegative on $\mathbb{N}^2$.

Proposition 4.10. *For any relatively prime pair* $(p,q) \in \mathbb{N}^2$, *the determining map* $f_{p,q}$ *is subadditive on* $\mathbb{N}^2$.

Proof. Choose (r,s) and (r',s') in $\mathbb{N}^2$ so that $\frac{s}{r} > \frac{q}{p}$ and $\frac{s'}{r'} \leq \frac{q}{p}$. If $\frac{s+s'}{r+r'} > \frac{q}{p}$, then $f_{p,q}(r+r', s+s') = 0 \leq f_{p,q}(r',s') = 0 + f_{p,q}(r',s') = f_{p,q}(r,s) + f_{p,q}(r',s')$. If $\frac{s+s'}{r+r'} \leq \frac{q}{p}$, then $f_{p,q}(r+r', s+s') = q(r+r') - p(s+s') = qr + qr' - ps - ps'$. Since $\frac{s}{r} > \frac{q}{p}$, it follows that $ps > qr$ and $qr + qr' - ps - ps' - (qr' - ps') = qr - ps < 0$ which implies that $f_{p,q}(r+r', s+s') \leq 0 + f_{p,q}(r',s') = f_{p,q}(r,s) + f_{p,q}(r',s')$. It follows that $f_{p,q}$ is subadditive on $\mathbb{N}^2$. $\square$

Example 4.11. Recall the fan given in example 2.2 which has faces determined by the coordinates $(0,1),(1,1),(1,0)$.



$\Sigma_{\mathbf{a},\mathbf{b}}$

Consider the max(r,s) function from example which we wrote in piecewise form.

$$\max(r,s) = \begin{cases} s & \text{if } (r,s) \in Q_0 \\ r & \text{if } (r,s) \in Q_1 \end{cases}$$

The determining map gives us the ability to rewrite this function as the sum of a semigroup homomorphism mapping $Q_0 \to \mathbb{Z}$ and $f_{1,1}$. The determining map $f_{1,1}$ is given by:

$$f_{1,1} = \begin{cases} 0 & \text{if} \frac{s}{r} > 1 \\ r - s & \text{if} \frac{s}{r} \leq 1 \end{cases}$$

Consider the function $f(r,s) = s + f_{1,1}(r,s)$. If $(r,s) \in Q_1$, then $f(r,s) = s + r - s = r$. If $(r,s) \in Q_0$, then $f(r,s) = s + 0 = s$. This gives us that $f(r,s) = \max(r,s)$. In this manner, we may use the determining maps of coordinates defining the faces $\Sigma_{\mathbf{a},\mathbf{b}}$ of to give necessary and sufficient conditions on all fan-linear maps defined on any fan $\Sigma_{\mathbf{a},\mathbf{b}}$.

Theorem 4.12. *Fix a fan* $\Sigma_{\mathbf{a},\mathbf{b}}$ *of* $n+1$ *cones. Assume that for* $\Sigma_{\mathbf{a},\mathbf{b}} = \{Q_0, Q_1, \ldots, Q_n\}$, *the faces of* $\Sigma_{\mathbf{a},\mathbf{b}}$ *are defined by* $\{(p_i,q_i) \in \mathbb{N}^2\}_{i=0}^{n+1}$. *A function* f *on* $\Sigma_{\mathbf{a},\mathbf{b}}$ *is fan linear if and only if:*

$$f(r,s) = a_0 r + b_0 s + k_1 \cdot f_{p_1,q_1}(r,s) + \cdots + k_n \cdot f_{p_n,q_n}(r,s)$$

With the following conditions:

(1) $k_i \geq 0$ *for* $i = 1, \dots, n$

(2) $f(\alpha) \geq 0$ *for all* $\alpha \in \{(p_i, q_i) \in \mathbb{N}^2\}_{i=0}^{n+1}$

Proof. Let f be a map that is fan linear on $\Sigma_{\mathbf{a},\mathbf{b}}$. Then by Proposition 4.5 we may write f:

$$f(r,s) = \begin{cases} a_0 r + b_0 s & if\ (r,s) \in Q_0 \\ a_1 r + b_1 s & if\ (r,s) \in Q_1 \\ \vdots & \\ a_n r + b_n s & if\ (r,s) \in Q_n \end{cases}$$

For some $i \in \{1, \dots, n\}$, choose two cones Q_{i-1} and Q_i so that these cones share the face (p_i, q_i). By the assumption that $\Sigma_{\mathbf{a},\mathbf{b}}$ is well defined, we must have $a_{i-1}p_i + b_{i-1}q_i = a_i p_i + b_i q_i$ which implies that $(a_{i-1} - a_i)p_i = (b_i - b_{i-1})q_i$. As p_i and q_i are relatively prime, we must have that $a_{i-1} - a_i = dq$ and $b_i - b_{i-1} = dp$ so that $a_i = a_{i-1} - dq_i$ and $b_i = b_{i-1} + dp_i$ for some $d \in \mathbb{Z}$. Then we may rewrite f:

$$f(r,s) = \begin{cases} a_0 r + b_0 s & if\ (r,s) \in Q_0 \\ \vdots & \\ a_{i-1} r + b_{i-1} s & if\ (r,s) \in Q_{i-1} \\ (a_{i-1} - dq_i)r + (b_{i-1} + dp_i)s & if\ (r,s) \in Q_i \\ \vdots & \\ a_n r + b_n s & if\ (r,s) \in Q_n \end{cases}$$

Define a *connected subfan of* $\Sigma_{\mathbf{a},\mathbf{b}}$ to be the sequential union $Q_j \cup Q_{j+1} \cup \cdots \cup Q_k$ for $j \leq k$, $j, k \in \{0, 1, \dots, n\}$. Let f_{Q_{i-1},Q_i} denote the restriction of f to the connected subfan $Q_{i-1} \cup Q_i$. It is clear that $f_{Q_{i-1},Q_i}(r,s) = a_{i-1}r + b_{i-1}s + (-d)f_{p_i,q_i}(r,s)$ where f_{p_i,q_i} is the determining map of (p_i, q_i).

Now choose some $(r, s) \in Q_{i-1}$ and $(r', s') \in Q_i$ so that the sum $(r + r', s + s')$ lies in Q_i. Then $f(r + r', s + s') \leq f(r, s) + f(r' + s')$ implies that:

$$\begin{aligned}
f(r+r', s+s') &\leq f(r, s) + f(r' + s') \\
a_{i-1}(r+r') + b_{i-1}(s+s') + (-d)f_{p_i,q_i}(r+r', s+s') &\leq a_{i-1}(r+r') + b_{i-1}(s+s') + (-d)f_{p_i,q_i} + (-j)f_{p,q}(r', s') \\
-d \cdot f_{p_i,q_i}(r+r', s+s') &\leq -d \cdot f_{p_i,q_i}(r', s') \\
-d[q_i(r+r') - p_i(s+s')] &\leq -d(q_i r' - p_i s') \\
-d(q_i r - p_i s) &\leq 0 \\
-d q_i r &\leq -d p_i s
\end{aligned}$$

Since $(r, s) \in Q_{i-1} \subset \mathbb{N}^2$, r and s must satisfy the inequality $q_i r \leq p_i s$. It follows then that for $-dq_i r \leq -dp_i s$ to hold, we must have $-d \geq 0$ which implies that $d \leq 0$. Let $k_i = -d \geq 0$ so that we may now write $f_{Q_{i-1},Q_i}(r, s) = a_{i-1}r + b_{i-1}s + k_i \cdot f_{p_i,q_i}(r, s)$ for $a_{i-1}, b_{i-1} \in \mathbb{Z}$, $k_i \in \mathbb{N}$.

It follows then that if $a_{i-1} = a_0 + k_1 q_1 + \cdots + k_{i-1}q_{i-1}$ and $b_{i-1} = b_0 - k_1 p_1 - \cdots - k_{i-1}p_{i-1}$, the restriction of f to the connected subfan $\displaystyle\bigcup_{j=0}^{i} Q_j$ is given by $f_{Q_0,\ldots,Q_i} = a_0 r + b_0 s + k_1 \cdot f_{p_1,q_1}(r, s) + \cdots + k_{i-1} \cdot f_{p_{i-1},q_{i-1}}(r, s) + k_i \cdot f_{p_i,q_i}(r, s)$. Consequently, we see that for all $(r, s) \in \mathbb{N}^2$, $f(r, s) = a_0 r + b_0 s + k_1 \cdot f_{p_1,q_1}(r, s) + \cdots + k_n \cdot f_{p_n,q_n}(r, s)$ for $k_1, \ldots, k_n \in \mathbb{N}$. Note that this implies condition (1). Since f is fan linear and this implies that $f(Q_i) \subseteq \mathbb{N}$ for each $i = 0, 1, \ldots, n$, condition (2) follows immediately.

For the converse, assume that f is a map satisfying the given condition that $f(r, s) = a_0 r + b_0 s + k_1 \cdot f_{p_1,q_1}(r, s) + \cdots + k_n \cdot f_{p_n,q_n}(r, s)$ for all $(r, s) \in \mathbb{N}^2$. Assume further that $k_i \geq 0$ for all $i = 1, \ldots, n$ and $f(\alpha) \geq 0$ for all $\alpha \in \{(0, 1) = (p_0, q_0), (p_1, q_1), \ldots, (p_n, q_n), (1, 0) = (p_{n+1}, q_{n+1})\}$. In order to show that f is fan linear, we must show that f is a semigroup homomorphism that maps into $\mathbb{N}$ when restricted to each cone Q_i and also that f is subadditive on all of $\mathbb{N}^2$.

Let $(r, s) \in Q_i$ for any $i \in \{0, 1, \ldots, n\}$. As f_{p_j,q_j} vanishes when $\frac{s}{r} > \frac{p_j}{q_j}$, we see that $f(r, s) = a_0 r + b_0 s + k_1 \cdot f_{p_1,q_1}(r, s) + \cdots + k_i \cdot f_{p_i,q_i}(r, s) = (a_0 + k_1 q_1 + \cdots + k_i q_i)r + (b_0 - k_1 p_1 - \cdots - k_i p_i)s$ for all $(r, s) \in Q_i$. Let $(r, s), (r', s') \in Q_i$ and let $a' = a_0 + k_1 q_1 + \cdots + k_i q_i$ and $b' = b_0 - k_1 p_1 - \cdots - k_i p_i$. Then $f(r + r', s + s') = a'(r + r') + b'(s + s') = a'r + b's + a'r' + b's' = f(r, s) + f(r', s')$ which shows that f is a semigroup homomorphism when restricted to Q_i.

Since $f_{|Q_i}$ is a semigroup homomorphism into $\mathbb{Z}$, by Theorem 3.6 we may extend $f_{|Q_i}$ to a linear transformation $T^{f_{|Q_i}} : \mathbb{Q}^2 \to \mathbb{Q}$. Note that by condition (2), for the edges $\alpha = (p_i, q_i)$ and $\beta = (p_{i+1}, q_{i+1})$, both $f(\alpha) \geq 0$ and $f(\beta) \geq 0$. Then for all $(r, s) \in Q_i$ where $(r, s) = c_1\alpha + c_2\beta$ for $c_1, c_2 \in \mathbb{Q}_{\geq 0}$, $f(r, s) = T^{f_{|Q_i}}(r, s) = c_1 f(\alpha) + c_2 f(\beta) \geq 0$ which shows that $f(Q_i) \subseteq \mathbb{N}$ for each $i = 0, 1, \ldots, n$.

Note that the map $(r, s) \mapsto a_0 r + b_0 s$ is linear on $\mathbb{N}^2$, which implies that it is trivially subadditive on $\mathbb{N}^2$. By Proposition 4.10, f_{p_i,q_i} is subadditive for each $i = 1, \ldots, n$. Since the sum of subadditive functions is subadditive by Proposition 4.6, $k_i \cdot f_{p_i,q_i}$ is subadditive for each $i = 1, \ldots, n$. It follows then that since f is the sum of subadditive functions,

$f(r,s) = a_0 r + b_0 s + k_1 \cdot f_{p_1,q_1}(r,s) + \cdots + k_n \cdot f_{p_n,q_n}(r,s)$ is subadditive on $\mathbb{N}^2$. This completes the proof. ☐

4.1. **Representative Semigroup and Correspondence with $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$.** We now show that $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ is naturally isomorphic to an affine semigroup that embeds in $\mathbb{Z}^{n+2}$ when $\Sigma_{\mathbf{a},\mathbf{b}}$ has $n+1$ cones.

Definition 4.13. Fix a fan $\Sigma_{\mathbf{a},\mathbf{b}} = \{Q_0, Q_1, \ldots, Q_n\}$ of $n+1$ cones with faces defined by $\{(p_i, q_i) \in \mathbb{N}^2\}_{i=0}^{n+1}$. Let $C_{\mathcal{F}(A)} = \{(a, b, k_1, \ldots, k_n) \in \mathbb{Z}^{n+2}\}$ so that:

1. $k_i \geq 0$ for $i = 1, \ldots, n$
2. $ap_i + bp_i + k_1(q_1 p_i - q_i p_1) + \cdots + k_{i-1}(q_{i-1} p_i - q_i p_{i-1}) \geq 0$ for all $i = 0, 1, \ldots, n$.

We call $C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}$ the *correspondence semigroup* of $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$.

Proposition 4.14. *$C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}$ is an affine semigroup and has a finite and unique minimal set of generators.*

Proof. Let B be the set of coordinates $(a, b, k_1, \ldots, k_n) \in \mathbb{R}^{n+2}$ that satisfy the following inequalities:

$$\begin{aligned}
&a \cdot 1 + b \cdot 0 + k_1 \cdot q_1 + \cdots + k_n \cdot q_n \geq 0\\
&a \cdot p_n + b \cdot q_n + k_1 \cdot (q_1 p_n - q_n p_1) + \cdots + k_{n-1} \cdot (q_{n-1} p_n - q_n p_{n-1}) + k_n \cdot 0 \geq 0\\
&a \cdot p_{n-1} + b \cdot q_{n-1} + k_1 \cdot (q_1 p_{n-1} - q_{n-1} p_1) + \cdots + k_{n-2} \cdot (q_{n-1} p_n - q_n p_{n-1}) + +k_{n-1} \cdot 0 + k_n \cdot 0 \geq 0\\
&\vdots\\
&a \cdot p_2 + b \cdot q_2 + k_1 \cdot (q_1 p_2 - q_2 p_1) + k_2 \cdot 0 + \cdots + k_n \cdot 0 \geq 0\\
&a \cdot p_1 + b \cdot q_1 + k_1 \cdot 0 + \cdots + k_n \cdot 0 \geq 0\\
&a \cdot 0 + b \cdot 1 + k_1 \cdot 0 + \cdots + k_n \cdot 0 \geq 0
\end{aligned}$$

Then B is the intersection of finitely many closed linear half-spaces in $\mathbb{R}^{n+2}$. Considering the resulting equations by setting the inequalties equal to 0, it follows that each bounding hyperplane of these half-spaces contains the origin. B is therefore a polyhedral cone in $\mathbb{R}^{n+2}$, and by Theorem 2.6, $B = \{\lambda_1 b_1 + \cdots + \lambda_r b_r | \lambda_i \in \mathbb{R}_{\geq 0}\}$.

Since the bounding hyperplanes of B are all solutions to linear equations with integer coefficients, it is clear that each has a solution set in $\mathbb{Q}^{n+2}$, so we may take $b_1, \ldots, b_r \in \mathbb{Q}^{n+2}$ which implies that B is a rational cone. Since $C_{\mathcal{F}(A)} \subset B$ consists of all coordinates in B with integer entries, it follows that $C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})} = B \cap \mathbb{Z}^{n+2}$ and by Theorem 2.8, $C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}$ is an affine semigroup.

Consider $(a, b, k_1, \ldots, k_n) \in C_{\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})}$. With the possibility that $a < 0$, two cases arise. First, if not all $b, k_1, \ldots, k_n$ are zero, then as $b \geq 0$ and $k_i \geq 0$ for all $i = 1, \ldots, n$,

$(-a,-b,-k_1,\dots,-k_n) \notin C_{\mathcal{F}\left(\Sigma_{\mathbf{a},\mathbf{b}}\right)}$. Second, if all $b = k_1 = \cdots = k_n = 0$ and $a \neq 0$, then we must have that $a > 0$ by condition (3) of definition 4.13. This implies that $(-a,0,0,\dots,0) \notin C_{\mathcal{F}(A)}$. It follows then that $C_{\mathcal{F}\left(\Sigma_{\mathbf{a},\mathbf{b}}\right)} \cap (-C_{\mathcal{F}\left(\Sigma_{\mathbf{a},\mathbf{b}}\right)}) = \{\mathbf{0}\}$ and by Theorem 2.9, $C_{\mathcal{F}\left(\Sigma_{\mathbf{a},\mathbf{b}}\right)}$ has a unique finite minimal generating set. □

Now we establish the natural isomorphism.

Theorem 4.15. *There is a an isomorphism of semigroups between the semigroup of fan linear maps $\mathcal{F}\left(\Sigma_{\mathbf{a},\mathbf{b}}\right)$ and the affine semigroup $C_{\mathcal{F}\left(\Sigma_{\mathbf{a},\mathbf{b}}\right)}$. Note that in particular, this shows that $\mathcal{F}\left(\Sigma_{\mathbf{a},\mathbf{b}}\right)$ is an affine semigroup with a finite and unique minimal set of generators.*

Proof. Define the set map Φ from $C_{\mathcal{F}\left(\Sigma_{\mathbf{a},\mathbf{b}}\right)}$ to $\mathcal{F}\left(\Sigma_{\mathbf{a},\mathbf{b}}\right)$ by $(a,b,k_1,\dots,k_n) \mapsto f(r,s) = ar + bs + k_1 \cdot f_{p_1,q_1}(r,s) + \cdots + k_n \cdot f_{p_n,q_n}(r,s)$. This map is clearly well defined and the conditions of definition 4.13 imply that $k_i \geq 0$ for $i = 1,\dots,n$ and $f(\alpha) \geq 0$ for all $\alpha \in \{(0,1) = (p_0,q_0),(p_1,q_1),\dots,(p_n,q_n),(1,0) = (p_{n+1},q_{n+1})\}$. Then by Theorem 4.12, Φ is surjective.

Suppose that $ar + bs + k_1 \cdot f_{p_1,q_1}(r,s) + \cdots + k_n \cdot f_{p_n,q_n}(r,s) = a'r + b's + k_1' \cdot f_{p_1,q_1}(r,s) + \cdots + k_n' \cdot f_{p_n,q_n}(r,s)$ for all $(r,s) \in \mathbb{N}^2$. Then evaluating the given fan linear maps at $(0,1)$, we see that $b = b'$. Next, evaluating at (p_1,q_1), we have that $ap_1 + bq_1 = a'p_1 + b'q_1$. As $b = b'$, it follows that $a = a'$. Similarly, by successively evaluating at (p_i,q_i) for each $i = 2,\dots,n+1$, we have that $k_i = k_i'$ for each $i = 1,\dots,n$. This shows that Φ is injective.

Since $\Phi[(a,b,k_1,\dots,k_n) + (a',b',k_1',\dots,k_n')] = \Phi[(a+a',b+b',k_1+k_1',\dots,k_n+k_n')] = (a+a')r + (b+b')s + (k_1+k_1') \cdot f_{p_1,q_1}(r,s) + \cdots + (k_n+k_n') \cdot f_{p_n,q_n}(r,s) = ar + bs + k_1 \cdot f_{p_1,q_1}(r,s) + \cdots + k_n \cdot f_{p_n,q_n}(r,s) + a'r + b's + k_1' \cdot f_{p_1,q_1}(r,s) + \cdots + k_n' \cdot f_{p_n,q_n}(r,s) = \Phi[(a,b,k_1,\dots,k_n)] + \Phi[(a',b',k_1',\dots,k_n')]$, we see that Φ is a semigroup homomorphism and therefore an isomorphism, which completes the proof. □

Because $\mathcal{F}\left(\Sigma_{\mathbf{a},\mathbf{b}}\right)$ is a pointed affine semigroup, we now can assume that for any fixed fan $\Sigma_{\mathbf{a},\mathbf{b}}$, there exists a unique, minimal set of fan-linear maps $g_1,\dots,g_k \in \mathcal{F}\left(\Sigma_{\mathbf{a},\mathbf{b}}\right)$ such that *every* fan linear map can be written as $f(r,s) = n_1g_1(r,s) + \cdots + n_kg_k(r,s)$. In fact, the above correspondence gives us a method to find such generating functions; we simply need to find the generators for $C_{\mathcal{F}\left(\Sigma_{\mathbf{a},\mathbf{b}}\right)}$. Unlike more loosely defined semigroups, pointed affine semigroups are reasonably well-structured. Now we know that the properties of these semigroups will be imposed on our objects of study whenever we can establish an isomorphism of semigroups, so this correspondence is inherently valuable.

5. Fan Algebras

In this final section, we draw some brief conclusions which apply directly to fan-algebras. For this section, assume that we are given a fan $\Sigma_{\mathbf{a},\mathbf{b}}$. First, we recall the definition of a fan algebra.

Definition 5.1. Given ideals $I_1,\dots,I_n$ in a domain R, a fan $\Sigma_{\mathbf{a},\mathbf{b}}$ of cones in $\mathbb{N}^2$, and fan-linear maps $f_1,\dots,f_n$, define a fan-algebra of $f = (f_1,\dots,f_n)$ on $\Sigma_{\mathbf{a},\mathbf{b}}$ as:

$$\mathcal{B}(\Sigma_{\mathbf{a},\mathbf{b}}, f) = \oplus_{r,s} I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} u^r v^s$$

In order to study fan-algebras in the context of semigroups, we need an associative binary operation that allows to classify certain sets of fan algebras as such.

Proposition 5.2. *Let $\mathcal{B}_{\mathbf{a},\mathbf{b}}$ be the set of all fan algebras on a fixed fan $\Sigma_{\mathbf{a},\mathbf{b}}$. For $A, B \in \mathcal{B}_{\mathbf{a},\mathbf{b}}$, $A = \bigoplus_{r,s} I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} u^r v^s$ and $B = \bigoplus_{r,s} J_1^{g_1(r,s)} \cdots J_m^{g_m(r,s)} u^r v^s$, define $G(A,B) = C = \bigoplus_{r,s} I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} J_1^{g_1(r,s)} \cdots J_m^{g_m(r,s)} u^r v^s$. Then $G\colon \mathcal{B}_{\mathbf{a},\mathbf{b}} \times \mathcal{B}_{\mathbf{a},\mathbf{b}} \to \mathcal{B}_{\mathbf{a},\mathbf{b}}$ is an associative, commutative binary operation with identity so that $\mathcal{B}_{\mathbf{a},\mathbf{b}}$ forms a semigroup.*

Proof. First, we must show that the operation above is well defined. Assume that $\bigoplus_{r,s} I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} u^r v^s = \bigoplus_{r,s} I_1'^{f_1'(r,s)} \cdots I_n'^{f_n'(r,s)} u^r v^s$ and $\bigoplus_{r,s} J_1^{g_1(r,s)} \cdots J_m^{g_m(r,s)} u^r v^s = \bigoplus_{r,s} J_1'^{g_1'(r,s)} \cdots J_m'^{g_m'(r,s)} u^r v^s$. Because u and v are indexing variables, it follows that $\bigoplus_{r,s} I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} u^r v^s = \bigoplus_{r,s} I_1'^{f_1'(r,s)} \cdots I_n'^{f_n'(r,s)} u^r v^s$ if and only if $I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} = I_1'^{f_1'(r,s)} \cdots I_n'^{f_n'(r,s)}$ for all $(r,s) \in \mathbb{N}^2$. It follows that to show well definition, it is enough to show that under the assumptions above $I_1^{f_1(r,s)} \cdots I_n^{f_n(r,s)} J_1^{g_1(r,s)} \cdots J_m^{g_m(r,s)} = I_1'^{f_1'(r,s)} \cdots I_n'^{f_n'(r,s)} J_1'^{g_1'(r,s)} \cdots J_m'^{g_m'(r,s)}$ for any fixed $(r,s) \in \mathbb{N}^2$. But for any fixed $(r,s) \in \mathbb{N}^2$, both the left and the right side of this equation are ideals in R, so it is clear then that the equation holds. Associativity follows similarly since $A(BC) = (AB)C$ for any ideals $A, B, C \subseteq R$, and commutativity follows from the assumption that R is commutative. We may also take the polynomial ring in u and v, that is $R[u,v] = \oplus_{r,s} R u^r v^s$, to be the identity element of this operation. Then under this operation, $\mathcal{B}_{\mathbf{a},\mathbf{b}}$ forms a semigroup. ☐

To highlight the fact that the above operation is a commutative semigroup operation, we write $G(A,B)$ as $A + B$. For $n \in \mathbb{N}$, $n \cdot A = \underbrace{A + \cdots + A}_{n \text{ times}}$. For convenience herein, we will use the notation $\alpha \in \mathbb{N}^2$ meaning that $\alpha = (r,s)$, and we write $\mathbf{u}^\alpha = u^r v^s$. Next, we use the semigroup properties of the affine semigroup $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ to show that fixing a set of ideals and considering all of the fan algebras constructed on those ideals establishes a finitely generated subsemigroup of $\mathcal{B}_{\mathbf{a},\mathbf{b}}$.

Proposition 5.3. *Let $\mathcal{I} = \{I_1, \ldots, I_n\}$ a finite family of ideals in R and let $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ denote the set of all fan algebras of the form $A = \oplus_\alpha I_1^{f_1(\alpha)} \cdots I_n^{f_n(\alpha)} \mathbf{u}^\alpha$. Then $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ is a finitely generated subsemigroup of $\mathcal{B}_{\mathbf{a},\mathbf{b}}$.*

Proof. Let $A = \oplus_\alpha I_1^{f_1(\alpha)} \cdots I_n^{f_n(\alpha)} u^\alpha$ and $B = \oplus_\alpha I_1^{g_1(\alpha)} \cdots I_n^{g_n(\alpha)} u^\alpha$. Then $A + B = \oplus_\alpha I_1^{f_1(\alpha)} \cdots I_n^{f_n(\alpha)} I_1^{g_1(\alpha)} \cdots I_n^{g_n(\alpha)} u^\alpha = I_1^{(f_1+g_1)(\alpha)} \cdots I_n^{(f_n+g_n)(\alpha)} u^\alpha$, and as each $f_i + g_i$ is fan-linear, $A + B \in \mathcal{B}_{I_1,\ldots,I_n}$.

Now consider $A \in \mathcal{B}_{I_1,\ldots,I_n}$, $A = \oplus_\alpha I_1^{f_1(\alpha)} \cdots I_n^{f_n(\alpha)} u^\alpha$. As $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ is a pointed affine semigroup by Theorem 4.15, there exists a unique minimal set of fan-linear maps $\{b_1, \ldots, b_m\}$ such that $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}}) = \{c_1 b_1 + \cdots + c_m b_m \mid c_i \in \mathbb{N}\}$. Then we may rewrite $f_1, \ldots, f_n$:

$$\begin{aligned} f_1(\alpha) &= c_{1,1} b_1(\alpha) + \cdots + c_{1,m} b_m(\alpha) \\ &\vdots \\ f_n(\alpha) &= c_{n,1} b_1(\alpha) + \cdots + c_{n,m} b_m(\alpha) \end{aligned}$$

Let $\mathcal{B}_{i,j} = \oplus_\alpha I_i^{b_j(\alpha)} u^\alpha$ for $i = 1, \ldots, n$, $j = 1, \ldots, m$, noting that since 0 is a fan-linear map, $\mathcal{B}_{i,j} \in \mathcal{B}_{I_1,\ldots,I_n}$. We may write A:

$$\begin{array}{rcl} A & = & \oplus_\alpha I_1^{f_1(\alpha)} \cdots I_n^{f_n(\alpha)} u^\alpha \\ & = & \oplus_\alpha I_1^{c_{1,1}b_1(\alpha)+\cdots+c_{1,m}b_m(\alpha)} \cdots I_n^{c_{n,1}b_1(\alpha)+\cdots+c_{n,m}b_m(\alpha)} u^\alpha \\ & = & \oplus_\alpha I_1^{c_{1,1}b_1(\alpha)} \cdots I_1^{c_{1,m}b_m(\alpha)} \cdots I_n^{c_{n,1}b_1(\alpha)} \cdots I_n^{c_{n,m}b_m(\alpha)} u^\alpha \\ & = & \oplus_\alpha I_1^{c_{1,1}b_1(\alpha)} \cdots I_n^{c_{n,1}b_1(\alpha)} I_1^{c_{1,2}b_{1,2}(\alpha)} \cdots I_1^{c_{1,m}b_m(\alpha)} \cdots I_n^{c_{n,m}b_m(\alpha)} u^\alpha \\ & = & \oplus_\alpha I_1^{c_{1,1}b_1(\alpha)} u^\alpha + \cdots + \oplus_\alpha I_n^{c_{n,1}b_1(\alpha)} u^\alpha + \cdots + \oplus_\alpha I_1^{c_{1,m}b_m(\alpha)} u^\alpha + \cdots + \oplus_\alpha I_n^{c_{n,m}b_m(\alpha)} u^\alpha \\ & = & \oplus_\alpha (I_1^{b_1(\alpha)})^{c_{1,1}} u^\alpha + \cdots + \oplus_\alpha (I_n^{b_1(\alpha)})^{c_{n,1}} u^\alpha + \cdots + \oplus_\alpha (I_1^{b_m(\alpha)})^{c_{1,m}} u^\alpha + \cdots + \oplus_\alpha (I_n^{b_m(\alpha)})^{c_{n,m}} u^\alpha \\ & = & c_{1,1} \cdot \mathcal{B}_{1,1} + \cdots + c_{n,1} \cdot \mathcal{B}_{n,1} + c_{1,2} \cdot \mathcal{B}_{1,2} + \cdots + c_{1,m} \cdot \mathcal{B}_{1,m} + \cdots + c_{n,m} \cdot \mathcal{B}_{n,m} \end{array}$$

It follows that the set of $\mathcal{B}_{i,j} = \oplus_\alpha I_i^{b_j(\alpha)} u^\alpha$ generate $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$. $\square$

As we have previously mentioned, affine semigroups have desirable properties and for this reason, we wish to answer when the subsemigroups of $\mathcal{B}_{\mathbf{a},\mathbf{b}}$ constructed as above are in fact affine semigroups. It could be difficult to show that a class of these subsemigroups is *not* an affine semigroup, however we may put some conditions on the ring R and the family of ideals $\mathcal{I}$ so that $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ is indeed an affine semigroup.

Proposition 5.4. *Let R be a Noetherian domain and I a proper, nontrivial ideal in R. For $\mathcal{I} = \{I\}$, the collection of fan algebras $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ is a pointed affine semigroup.*

Proof. It is clear that there is a surjective semigroup homomorphism $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}}) \to \mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ given by $f \mapsto \bigoplus_\alpha I^{f(\alpha)} \mathbf{u}^\alpha$. Recall that in a Noetherian domain, $J^2 = J$ for any ideal $J \subseteq R$ implies that $J = \langle 0 \rangle$ or $J = R$. We proceed by contrapositive and assume that the map described above is not injective.

This assumption is equivalent to the assumption that for some $f \neq g \in \mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$, $\bigoplus_\alpha I^{f(\alpha)} \mathbf{u}^\alpha = \bigoplus_\alpha I^{g(\alpha)} \mathbf{u}^\alpha$ and $f \neq g$. Then for some $\alpha \in \mathbb{N}^2$, $I^{f(\alpha)} = I^{g(\alpha)}$ and $f(\alpha) \neq g(\alpha)$. Assume that $f(\alpha) > g(\alpha)$ so that $f(\alpha) = j + k$ and $g(\alpha) = k$.

Then $I^{j+k} = I^j I^k = I^k$ which implies that $(I^k)^{j+1} = I^{jk+k} = I^{jk} I^k = I^j \cdots I^j I^k = I^k$. It follows that $I^k \subseteq (I^k)^{j+1} \subseteq (I^k)^j \subseteq \cdots \subseteq (I^k)^3 \subseteq (I^k)^2$, so I^k is idempotent and hence $I^k = R$ or $I^k = 0$. It follows then that either I contains a unit or $i^k = 0$ for all $i \in I$, in which case $I = 0$. By contrapositive, we may conclude that if $\langle 0 \rangle \subsetneq I \subsetneq R$, then the map $f \mapsto \bigoplus_\alpha I^{f(\alpha)} \mathbf{u}^\alpha$ is an injection and therefore is an isomorphism of semigroups. Since $\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ is an affine semigroup, we may conclude that $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ is as well. $\square$

Next, we prove the affine semigroup property for any finite family of ideals $\mathcal{I}$ in a principal ideal domain. We first show that any such family generates a subsemigroup of fan algebras in a semigroup taken over a family of relatively co-prime ideals.

Proposition 5.5. *Let R be a principal ideal domain and $\mathcal{I} = \{I_1, \ldots, I_n\}$ a finite family of proper nontrivial ideals in R. There exists a finite family of ideals $\mathcal{J} = \{J_1, \ldots, J_m\}$ in R such that for $J_i = \langle d_i \rangle$, $\gcd(d_i, d_k) = 1$ for all $i \neq k$ and $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ is a subsemigroup of $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$.*

Proof. To prove the proposition, it is enough to show that the statement is true for $n = 2$, then the case for n arbitrary follows naturally. Let $I, J \subsetneq R$ be ideals and write $I = \langle p_1^{r_1} \cdots p_k^{r_k} p_{k+1}^{r_{k+1}} \cdots p_n^{r_n} \rangle$ and $J = \langle p_{k+1}^{r_{k+1}} \cdots p_n^{r_n} q_{l_1}^{l_1} \cdots q_{l_m}^{l_m} \rangle$. Then for any $f, g \in \mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$ we have the following:

$$\begin{aligned}\textstyle\bigoplus_\alpha I^{f(\alpha)}J^{g(\alpha)}\mathbf{u}^\alpha &= \textstyle\bigoplus_\alpha \langle p_1^{r_1}\cdots p_k^{r_k}p_{k+1}^{r_{k+1}}\cdots p_n^{r_n}\rangle^{f(\alpha)}\langle p_{k+1}^{r_{k+1}}\cdots p_n^{r_n}q_{l_1}^{l_1}\cdots q_{l_m}^{l_m}\rangle^{g(\alpha)}\mathbf{u}^\alpha\\ &= \textstyle\bigoplus_\alpha \langle p_1^{r_1}\cdots p_k^{r_k}\rangle^{f(\alpha)}\langle p_{k+1}^{r_{k+1}}\cdots p_n^{r_n}\rangle^{f(\alpha)}\langle p_{k+1}^{r_{k+1}}\cdots p_n^{r_n}\rangle^{g(\alpha)}\langle q_{l_1}^{l_1}\cdots q_{l_m}^{l_m}\rangle^{g(\alpha)}\mathbf{u}^\alpha\\ &= \textstyle\bigoplus_\alpha \langle p_1^{r_1}\cdots p_k^{r_k}\rangle^{f(\alpha)}\langle p_{k+1}^{r_{k+1}}\rangle^{f(\alpha)}\langle p_{s_{k+1}}^{s_{k+1}}\rangle^{g(\alpha)}\cdots\langle p_n^{r_n}\rangle^{f(\alpha)}\langle p_{s_n}^{s_n}\rangle^{g(\alpha)}\langle q_{l_1}^{l_1}\cdots q_{l_m}^{l_m}\rangle^{g(\alpha)}\mathbf{u}^\alpha\\ &= \textstyle\bigoplus_\alpha \langle p_1^{r_1}\cdots p_k^{r_k}\rangle^{f(\alpha)}\langle p_{k+1}\rangle^{r_{k+1}f(\alpha)+s_{k+1}g(\alpha)}\cdots\langle p_n\rangle^{r_nf(\alpha)+s_ng(\alpha)}\langle q_{l_1}^{l_1}\cdots q_{l_m}^{l_m}\rangle^{g(\alpha)}\mathbf{u}^\alpha\end{aligned}$$

Set $J_0 = \langle p_1^{r_1}\cdots p_k^{r_k}\rangle$, $J_i = \langle p_{k+i}\rangle$ for $i = 1,\dots,n-k$, and $J_{n-k+1} = \langle q_{l_1}^{l_1}\cdots q_{l_m}^{l_m}\rangle$ and let $\mathcal{J} = \{J_0, J_1,\dots,J_{n-k+1}\}$. Noting that the $r_{k+i}f+s_{k+i}g$ are fan-linear for all $i = 1,\dots,n-k$, it follows that $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ is a subsemigroup of $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$. The case where $n > 2$ follows easily. □

We now show that for any finite family $\mathcal{I}$ of ideals in a principal ideal domain R, $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ is an affine semigroup.

Proposition 5.6. *Let R be a principal ideal domain and $\mathcal{I} = \{I_1,\dots,I_n\}$ a finite family of proper nontrivial ideals in R. $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ is an affine semigroup.*

Proof. Constructing the family $\mathcal{J}$ as in the above proposition, it is sufficient to show that $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ is an affine semigroup, in which case there is a natural embedding $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I}) \hookrightarrow \mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ so that $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ embeds also in $\mathbb{Z}^d$ for some $d \in \mathbb{N}$. Assume $\mathcal{J} = \{\langle d_1\rangle,\cdots,\langle d_k\rangle\}$ and that $\gcd(d_i, d_k) = 1$ whenever $i \neq k$.

We may proceed directly and consider the surjective semigroup homomorphism $(\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}}))^k \to \mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ so that for $f_1,\dots,f_k \in \mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$, $(f_1,\dots,f_k) \mapsto \bigoplus_\alpha J_1^{f_1(\alpha)}\cdots J_k^{f_k(\alpha)}\mathbf{u}^\alpha$. If $\bigoplus_\alpha J_1^{f_1(\alpha)}\cdots J_k^{f_k(\alpha)}\mathbf{u}^\alpha = \bigoplus_\alpha J_1^{g_1(\alpha)}\cdots J_k^{g_k(\alpha)}\mathbf{u}^\alpha$, then $J_1^{f_1(\alpha)}\cdots J_k^{f_k(\alpha)} = J_1^{g_1(\alpha)}\cdots J_k^{g_k(\alpha)}$ for all $\alpha \in \mathbb{N}^2$. Then for any α, $\langle d_1^{f_1(\alpha)}\cdots d_k^{f_k(\alpha)}\rangle = \langle d_1^{g_1(\alpha)}\cdots d_k^{g_k(\alpha)}\rangle$ which implies that $d_1^{f_1(\alpha)}\cdots d_k^{f_k(\alpha)} = ud_1^{g_1(\alpha)}\cdots d_k^{g_k(\alpha)}$ for some unit $u \in R$. It follows then that since $\gcd(d_i, d_k) = 1$ when $i \neq k$, $f_i(\alpha) = g_i(\alpha)$ for all $\alpha \in \mathbb{N}^2$ and hence $f_i = g_i \in \mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$. It follows that $(\mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}}))^k \cong \mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{J})$ as semigroups and hence $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ embeds in $\mathbb{Z}^d$ for some $d \in \mathbb{N}$. This implies that $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ is an affine semigroup. □

Note that the above proposition does not prove that $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ is a pointed affine semigroup, but this is largely unnecessary. Recall that proposition 5.3 already shows that $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\mathcal{I})$ is finitely generated, so whether or not we may consider it pointed is irrelevant to our purposes. We conclude this section (and consequently the paper) with an example of how semigroup properties might be used to simplify the study of fan algebras. We could show the next proposition using the fact that for nonzero ideals $I, A, B \leq R$ where R is a principal ideal domain, $IA = IB$ if and only if $A = B$. Instead, we use the cancellative property of affine semigroups.

Proposition 5.7. *Let R be a principal ideal domain and let $I, J_1,\dots,J_k,N_1,\dots N_m$ be ideals in R. For any fan-linear maps $f, g_1,\dots,g_k,h_1,\dots,h_m \in \mathcal{F}(\Sigma_{\mathbf{a},\mathbf{b}})$, $\bigoplus_\alpha I^{f(\alpha)}J_1^{g_1(\alpha)}\cdots J_k^{g_k(\alpha)}\mathbf{u}^\alpha = \bigoplus_\alpha I^{f(\alpha)}N_1^{h_1(\alpha)}\cdots N_m^{h_m(\alpha)}\mathbf{u}^\alpha$ if and only if $\bigoplus_\alpha J_1^{g_1(\alpha)}\cdots J_k^{g_k(\alpha)}\mathbf{u}^\alpha = \bigoplus_\alpha N_1^{h_1(\alpha)}\cdots N_m^{h_m(\alpha)}\mathbf{u}^\alpha$.*

Proof. The converse is trivial. For the forward implication, we note that affine semigroups enjoy the cancellative property; that is, $A+B = A+C$ implies that $B = C$. Regarding both the left and right sides of the equation $\bigoplus_\alpha I^{f(\alpha)}J_1^{g_1(\alpha)}\cdots J_k^{g_k(\alpha)}\mathbf{u}^\alpha = \bigoplus_\alpha I^{f(\alpha)}N_1^{h_1(\alpha)}\cdots N_m^{h_m(\alpha)}\mathbf{u}^\alpha$ as members of $\mathcal{B}_{\mathbf{a},\mathbf{b}}(\{I, J_1,\dots,J_k,N_1,\dots,N_m\})$, we have that both sides are members of an affine semigroup by proposition 5.6. With this observation, the result follows easily. □

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