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Sign Patterns that Allow Diagonalizability

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SIGN PATTERNS THAT ALLOW DIAGONALIZABILITY

by

CHRISTOPHER ZAGRODNY

Under the Direction of Zhongshan Li, Ph. D.

ABSTRACT

A sign pattern matrix is a matrix whose entries in the set $\{+, -, 0\}$. These matrices are used to describe classes of real matrices with matching signs. The study of sign patterns originated with the need to solve certain problems in economics where only the signs of the entries in matrix are known. Since then applications have been found in areas such as communication complexity, neural networks, and chemistry. Currently much work has been done in identifying shared characteristics of real matrices having the same sign pattern. Of particular interest is sign patterns that allow or require particular properties. In this paper I study sign patterns that allow diagonalizability, as well as the characteristics of certain types of sign patterns.

INDEX WORDS: Sign Patterns, Matrix Theory, Diagonalizability, Minimum Rank,

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CHRISTOPHER ZAGRODNY

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DEDICATION

I would like to dedicate this to my parents, Michael and Bernadette for their encouragement over the years

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I would like to thank Dr. Li and Dr. Hall for their guidance over the past few years, as well as their help in preparing this paper. I would also like to thank Guangming Jing and Xinlei Feng and others that participated in the matrix theory research meetings.

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PART 1

INTRODUCTION

1.1 Description and Motivation

The study of sign pattern matrices comes from the need to solve certain problems when all that is known is the signs of the entries. Sign patterns were first mentioned in Paul Samuelson's text, *Foundations of Economic Behavior*. Since then there have been applications found in other areas such as biology, computer science, neural networks, oriented matroid theory, and convex polytopes theory (see [1],[2], as well as other cited papers). Sign Patterns Matrices are formed from real matrices by replacing positive entries with '+', negative entries with '-' and leaving 0's as they are. A formal definition is as follows:

Definition 1.1.1. *An $m \times n$ matrix with entries in $\{+, -, 0\}$ is called a sign pattern matrix. Alternatively, a sign pattern matrix can have entries in $\{+1, -1, 0\}$*

A sign pattern matrix with only positive and zero entries is a non-negative sign-pattern. A generalized sign pattern is a matrix with entries in $\{+, -, 0, \#\}$, with the $\#$ entry having an undetermined sign. Given a real matrix B , $A = \text{sgn } B$ is a sign pattern matrix with each entry a_{ij} equal to the sign of its corresponding entry, b_{ij} , in B .

The focus of this dissertation concerns sign patterns that allow diagonalizability. That is, sign patterns for which there is a diagonalizable real matrix with the same signs. This has been previously studied in [3], [4], [5], and [6].

1.2 Definitions and Notation

Most of the following definitions and notations can be found in [3], [6], and [2].

Definition 1.2.1 (Sign Pattern Class). *Given a real matrix $B = (b_{ij})$, let $A = \text{sgn}(B)$ be the sign pattern matrix with entries $a_{ij} = \text{sgn } b_{ij}$. For a given sign pattern, A , The sign pattern*

class $\mathcal{Q}(A)$ is the set of all real matrices B such that $\text{sgn}(B) = A$.

Definition 1.2.2. A sign pattern A allows property P if there exists $B \in \mathcal{Q}(A)$ with property P . A sign pattern A requires property P if all real matrices $B \in \mathcal{Q}(A)$ have property P .

For instance, a sign pattern A requires non-singularity if all $B \in \mathcal{Q}(A)$ are non-singular. We say that A is sign non-singular. Note that a sign pattern is sign non-singular if and only if it has at least one non-zero term and every non-zero term in the expansion of its determinant has the same sign.

Given a sign pattern matrix A

$\rho(A)$: term rank of A

$\text{mr}(A)$: minimum rank of matrices in $\mathcal{Q}(A)$

$\text{MR}(A)$: maximum rank of matrices in $\mathcal{Q}(A)$

The digraph of an $n \times n$ sign pattern $A = [a_{ij}]$, denoted by $D(A)$, is the digraph with vertex set $\{1, 2, \dots, n\}$, where (i, j) is an arc if and only if $a_{ij} \neq 0$. A sign pattern's digraph is strongly connected if and only if it is irreducible.

Example 1.2.3. For example, the SPM below has $\text{mr}(\mathcal{A}) = 2$, $\text{MR}(\mathcal{A}) = 4$, and $c(\mathcal{A}) = 4$.

$$\begin{bmatrix} + & + & 0 & 0 \\ - & - & 0 & 0 \\ 0 & 0 & + & + \\ 0 & 0 & + & + \end{bmatrix}$$

A simple cycle $\lambda = a_{i_1 i_2} a_{i_2 i_3} \dots a_{i_{k-1} i_k} a_{i_k i_1}$ in a sign pattern $A = [a_{ij}]$ is a product of non zero entries where the indices i_1, \dots, i_k are distinct. Note that a simple cycle in A corresponds to a directed cycle in the digraph $D(A)$. A composite cycle is a collection of simple cycles, all of which have distinct index sets.

Definition 1.2.4. The maximal cycle length, $c(A)$, of A is the length of the largest cycle in A .

Note that the term rank of a sign pattern is equal to the maximum rank which is greater than or equal to the maximal cycle length ($\rho(A) = \text{MR}(A) \geq c(A)$) (see [7] and [2]).

Definition 1.2.5. A signature sign pattern is a sign pattern of a non-singular diagonal matrix. A permutation sign pattern is a sign pattern of a permutation matrix.

Since several results in this paper deal with minimum and maximum rank, it can be helpful to work with equivalent and similar sign patterns.

Definition 1.2.6. Let S_1 and S_2 be signature sign patterns and A_1 and A_2 be two sign patterns. A_1 and A_2 are signature similar if $A_1 = S_1 A_2 S_1$. They are signature equivalent if $A_1 = S_1 A_2 S_2$. If P is a permutation sign pattern, then A_1 and A_2 are permutationally similar if $A_1 = P^T A_2 P$.

One concept related to diagonalization that will be referred to frequently in this paper is rank principality.

Definition 1.2.7 (Rank Principality). A real matrix B is said to be rank-principle if $\text{rank} B = k$ and B has a non-singular $k \times k$ principle submatrix. This principle submatrix of B is called a rank-principal certificate of B . For a composite cycle γ of a square sign pattern A , we say γ supports a rank-principal certificate of A if there exists a real matrix $B \in A$ that is rank principle and γ has the same row index set as a rank-principal certificate of B .

In general, a composite cycle λ supports a principle submatrix \hat{A} of A if they have the same index set.

1.3 Outline of this paper

The preceding definitions and concepts are meant to provide an background for the next several sections. Further definitions and basic concepts will be presented as needed later on. Each part of this paper represents an area of research and study under Drs. Li and Hall focusing on Sign pattern matrices allowing diagonalizability, ranks of realization

diagonalizability occurs, as well as other concepts related to diagonalizability. In part two we will discuss some known necessary and sufficient conditions for allowing diagonalizability, as well as possible ranks where diagonalizability can be achieved. In parts three and four we will cover some new results on matrices allowing diagonalizability, some of these results will be published in [8].

PART 2

NECESSARY AND SUFFICIENT CONDITIONS FOR ALLOWING
DIAGONALIZABILITY

2.1 Rank Principle

The following theorems dealing with the maximal cycle length come from [3] and [4].

Theorem 2.1.1 ([3, 4]). *If a square sign pattern \mathcal{A} satisfies $c(\mathcal{A}) = \text{MR}(\mathcal{A})$, then \mathcal{A} allows diagonalizability with rank $\text{MR}(\mathcal{A})$*

Since a combinatorially symmetric matrix's maximal cycle length is equal to its maximal rank, we have the following corollary.

Corollary 2.1.2 ([4]). *If a square sign pattern is combinatorially symmetric, then \mathcal{A} allows diagonalizability with rank $\text{MR}(\mathcal{A})$*

Theorem 2.1.3 ([3, 4]). *If a sign pattern A allows diagonalizability then $c(A) \geq \text{mr}(A)$*

We know that every diagonal matrix is rank principle [9]. It has also been shown that every rank principle matrix is diagonally equivalent to a diagonalizable matrix [6]. The following result from the same paper gives a way to generate sign pattern matrices that allow diagonalizability.

Theorem 2.1.4 ([6]). *A square sign pattern A allows diagonalizability if and only if \mathcal{A} allows rank-principality. Also, A square sign pattern A allows diagonalizability with rank k if and only if there is a rank-principle matrix $B \in Q(A)$ of rank k if and only if A has a composite cycle of length k that supports a rank principle certificate of A .*

So the sign patterns that allow diagonalizability are the sign patterns of square, rank-principal real matrices. Also, up to permutation similarity, every real rank-principal matrix

can be written as $\begin{bmatrix} B & C \\ D & DB^{-1}C \end{bmatrix}$, where B is a nonsingular matrix of order k while C and D are arbitrary real matrices of appropriate sizes. In theory, it would be possible to describe every $n \times n$ sign pattern matrix that allows diagonalizability with rank k this way. However it would be difficult to find a finite number of matrices B, C and D to generate all patterns. Instead this dissertation will focus mostly on combinatorial descriptions of sign patterns that allow diagonalizability.

The following result will be useful later.

lemma 2.1.5. *Let B be a square matrix with rank k over a field. Suppose that B has exactly k nonzero eigenvalues. Then B is rank-principal.*

Proof. Note that $S_k(B) = E_k(B)$, where $S_k(B)$ is the k th elementary symmetric function of the eigenvalues of B and $E_k(B)$ is the sum of all principal minors of order k of B (see [9]). Since B has exactly k nonzero eigenvalues, $S_k(B) \neq 0$. Thus $E_k(B) \neq 0$. It follows that B has at least one nonsingular $k \times k$ principal matrix, and hence, B is rank-principal. \square

Observe that if a square sign pattern with minimum rank 1 has a composite cycle γ of length k , then the principal submatrix of A supported by γ has no zero entries, so one can easily construct a rank-principal matrix of rank k in $Q(A)$, which ensures that $Q(A)$ contains a diagonalizable matrix with rank k by the preceding theorem. This produces the following result on sign patterns with minimum rank 1.

Theorem 2.1.6. *Let A be a square sign pattern such that $mr(A) = 1$. Then A allows diagonalizability if and only if A has at least one nonzero diagonal entry; in this case, for each integer k with $1 \leq k \leq c(A)$, A allows diagonalizability with rank k .*

A result involving chordless composite cycles found in [4] is stated below.

Theorem 2.1.7 ([4]). *If a square sign pattern A has a chordless composite cycle γ of length k such that $k \geq mr(A)$, then A allows diagonalizability with rank k .*

We note that the length k of the chordless composite cycle γ in the preceding theorem actually must be equal to $\text{mr}(A)$, as the principal submatrix of A supported by γ is sign nonsingular and hence $\text{mr}(A) \geq k$. Since every chordless composite cycle supports a principal sign nonsingular submatrix, the following result is a generalization of Theorem 2.1.7.

Theorem 2.1.8. *Suppose a square sign pattern A has minimum rank $k > 0$ and A has a sign nonsingular $k \times k$ principal submatrix. Then A allows diagonalizability with rank k .*

Proof. Every matrix $B \in Q(A)$ with rank k is clearly rank-principal due to the presence of a sign nonsingular $k \times k$ principal submatrix of A . Thus A allows diagonalizability with rank k by Theorem 2.1.4. \square

We now give a characterization of the square sign patterns that require a unique rank and allow diagonalizability.

Theorem 2.1.9. *Let A be a square sign pattern such that $\text{mr}(A) = \text{MR}(A) = k$. Then A allows diagonalizability if and only if $c(A) = k$.*

Proof. The necessity follows from Theorem 2.1.3 and the fact that $c(A) \leq \text{MR}(A)$. The sufficiency is a consequence of Theorem 2.1.1. \square

A characterization of upper triangular sign patterns that allow diagonalizability is given next.

Theorem 2.1.10. *Let A be an upper triangular square sign pattern. Then A allows diagonalizability if and only if $c(A) = \text{mr}(A)$.*

Proof. Since A is an upper triangular square sign pattern, every matrix $B \in Q(A)$ has precisely $c(A)$ nonzero eigenvalues, so $\text{mr}(A) \geq c(A)$.

Suppose that A allows diagonalizability. Then $c(A) \geq \text{mr}(A)$. In view of the opposite inequality above, we get $c(A) = \text{mr}(A)$.

Conversely, assume that $c(A) = \text{mr}(A)$. Let $B \in Q(A)$ be such that $\text{rank}(B) = \text{mr}(A)$. Clearly there is a diagonal matrix D with positive diagonal entries such that all the nonzero

diagonal entries of $DB \in Q(A)$ are distinct. Thus every nonzero eigenvalue of DB has algebraic and geometric multiplicity 1. If 0 is an eigenvalue of DB , then its algebraic and geometric multiplicities are both equal to $n - c(A) = n - \text{rank}(B) = n - \text{rank}(DB)$. Hence, $DB \in Q(A)$ is diagonalizable, so that A allows diagonalizability. \square

A square sign pattern is said to be *idempotent* if A^2 is unambiguously defined, and $A^2 = A$. More generally, we say a sign pattern is *k-potent* [10] (where k is a positive integer) if A^{1+k} is unambiguously defined and $A^{1+k} = A$. Such sign patterns always allow diagonalizability.

Theorem 2.1.11. *Every sign k-potent sign pattern A allows diagonalizability with rank $\text{mr}(A)$.*

Proof. Let A be a k -potent sign pattern and let $B \in Q(A)$ be such that $\text{rank}(B) = \text{mr}(A)$. On the one hand, clearly $\text{rank}(B^{1+k}) \leq \text{rank}(B)$. On the other hand, since $\text{rank}(B) = \text{mr}(A)$ and $B^{1+k} \in Q(A^{1+k}) = Q(A)$, we also have $\text{rank}(B^{1+k}) \geq \text{rank}(B)$. Thus, $\text{rank}(B^{1+k}) = \text{rank}(B)$. It follows that $\text{rank}(B) = \text{rank}(B^2) = \dots = \text{rank}(B^{1+k})$. By considering the Jordan canonical form of B , we see that either B is nonsingular or the eigenvalue 0 of B has index 1. Thus $\text{rank}(B)$ is equal to the number of nonzero eigenvalues of B , which ensures that B is rank-principal (see [6]). By Theorem 2.1.4, A allows diagonalizability with rank $\text{mr}(A)$. \square

2.2 Symmetrically Partitioned

A square sign pattern \mathcal{A} is in Frobenius normal form if

$$\mathcal{A} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1p} \\ 0 & A_{22} & \dots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_{pp} \end{bmatrix}$$

where each diagonal block, A_{ii} is irreducible [2]. Note that every square sign pattern is permutationally similar to one in Frobenius form, and also permutation similarity preserves

diagonalizability. Therefore we could just consider sign patterns in Frobenius normal form. However, we will look at a more general form of block upper triangular matrices, symmetrically partitioned block upper triangular sign patterns (block upper triangular matrices where the diagonal blocks are square)

Theorem 2.2.1. *A square sign pattern in symmetrically partitioned block upper triangular form,*

$$\mathcal{A} = \begin{bmatrix} A_{11} & \dots & A_{1p} \\ \vdots & \ddots & \vdots \\ 0 & \dots & A_{pp} \end{bmatrix}$$

allows diagonalizability if and only if there exists a real matrix

$$B = \begin{bmatrix} B_{11} & \dots & B_{1p} \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_{pp} \end{bmatrix}$$

where $B_{ii} \in Q(A_{ii})$ for each $i = 1, \dots, p$, such that all of $B_{11}, B_{22}, \dots, B_{pp}$ are rank-principal and $\text{rank}(B) = \text{rank}(B_{11}) + \text{rank}(B_{22}) + \dots + \text{rank}(B_{pp})$. As a result, if A allows diagonalizability, then each of $A_{11}, A_{22}, \dots, A_{pp}$ allows diagonalizability.

Proof. Suppose that $\text{rank}(B) = \text{rank}(B_{11}) + \text{rank}(B_{22}) + \dots + \text{rank}(B_{pp})$, and all of $B_{11}, B_{22}, \dots, B_{pp}$ are rank-principal. Then there is a composite cycle γ_i of A_{ii} that supports a rank-principal certificate of B_{ii} . As $\text{rank}(B) = \text{rank}(B_{11}) + \text{rank}(B_{22}) + \dots + \text{rank}(B_{pp})$, it follows that the composite cycle $\gamma_1 \dots \gamma_p$ supports a rank-principal certificate of B . By Theorem 2.1.4, A allows diagonalizability.

Suppose that A allows diagonalizability. Let $B = \begin{bmatrix} B_{11} & \dots & B_{1p} \\ \vdots & \ddots & \vdots \\ 0 & \dots & B_{pp} \end{bmatrix} \in Q(A)$ be diagonalizable, where $B_{ii} \in Q(A_{ii})$ for each $i = 1, \dots, p$. Because a square real matrix is diagonalizable if and only if its minimal polynomial is a product of distinct monic linear factors and the minimal polynomial of each diagonal block of a matrix in symmetrically

partitioned block upper triangular form is a factor of the minimal polynomial of the entire matrix, we see that each B_{ii} is diagonalizable and hence rank-principal. Since the rank of a diagonalizable matrix is equal to its total number of nonzero eigenvalues, we see that $\text{rank}(B) = \text{rank}(B_{11}) + \text{rank}(B_{22}) + \cdots + \text{rank}(B_{pp})$. \square

From the equation $\text{rank}(B) = \text{rank}(B_{11}) + \text{rank}(B_{22}) + \cdots + \text{rank}(B_{pp})$ in Theorem 2.2.1, a combinatorial necessary condition for the matrix A in symmetrically partitioned block upper triangular form as in Theorem 2.2.1 to allow diagonalizability is that $\text{mr}(A) \leq \text{MR}(A_{11}) + \cdots + \text{MR}(A_{pp})$. We are interested in identifying additional combinatorial conditions which when combined with the necessary condition that each A_{ii} allows diagonalizability would ensure that the symmetrically partitioned block upper triangular sign pattern A allows diagonalizability.

We now phrase an interesting open combinatorial sufficient condition for a symmetrically partitioned block upper triangular sign pattern to allow diagonalizability.

Problem 2.2.2. *Let A be a sign pattern in symmetrically partitioned block upper triangular form*

$$A = \begin{bmatrix} A_{11} & \cdots & A_{1p} \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A_{pp} \end{bmatrix}. \text{ Suppose that for each } i = 1, \dots, p, \text{mr}(A_{ii}) = \text{mr}([A_{ii} \ \cdots \ A_{ip}])$$

and each A_{ii} allows diagonalizability. Does it then necessarily follow that A allows diagonalizability?

A related open problem is the following.

Problem 2.2.3. *Let A_1 be a square sign pattern that allows rank-principality. Is it true that for every sign pattern $A = [A_1 \ A_2]$ such that $\text{mr}(A_1) = \text{mr}(A)$, A allows rank-principality?*

We note that an affirmative answer to Problem 2.2.3 implies an affirmative answer to Problem 2.2.2.

PART 3

**FURTHER PROPERTIES OF MATRICES THAT ALLOW
DIAGONALIZABILITY**

In this chapter some additional properties of sign pattern matrices that allow diagonalizability and rank-principality are explored. In particular we have an interest in what ranks can diagonalizability can be realized.

Note that the Kronecker product of two diagonalizable matrices is diagonalizable and signature equivalence preserves rank-principality. Thus it can be seen that the set of sign patterns that allow diagonalizability is closed under the following operations: negation, transposition, permutation similarity, signature similarity, signature equivalence, and Kronecker product.

Note that for every real matrix B there is a permutation matrix P such that BP and PB are both rank-principle this, along with our work in the previous chapter gives the following.

Theorem 3.0.1. *Let A be any square sign pattern, then there exists a permutation sign pattern P such that AP and PA both allow diagonalizability,*

For example, consider $A = \begin{bmatrix} 0 & + & + \\ 0 & 0 & + \\ 0 & 0 & 0 \end{bmatrix}$. This does not allow diagonalizability, since any

matrix in $\mathcal{Q}(A)$ has zero as an eigenvalue with algebraic multiplicity 3, but geometric multi-

plicity 1. But with permutation sign pattern $P = \begin{bmatrix} 0 & 0 & + \\ + & 0 & 0 \\ 0 & + & 0 \end{bmatrix}$, we have $PA = \begin{bmatrix} 0 & 0 & 0 \\ 0 & + & + \\ 0 & 0 & + \end{bmatrix}$

and $AP = \begin{bmatrix} + & + & 0 \\ 0 & + & 0 \\ 0 & 0 & 0 \end{bmatrix}$, both of which allow diagonalizability.

We now further explore the possible ranks of the diagonalizable matrices in the qualitative class of a sign pattern that allows diagonalizability. By Theorem 2.1.4, a sign pattern A allows diagonalizability with rank k (of course, $k \geq \text{mr}(A)$) if and only if A has a composite cycle of length k that supports a rank-principal certificate for A . A natural question is: for a sign pattern A that allows diagonalizability, can every composite cycle γ of A with length at least $\text{mr}(A)$ support a rank-principal certificate? The answer is negative, as the following two examples show.

Example 3.0.2. Consider the reducible sign pattern $A = \begin{bmatrix} + & + & 0 \\ + & + & 0 \\ 0 & 0 & + \end{bmatrix}$. Note that $\text{mr}(A) = 2$, $\text{MR}(A) = 3$. The maximum length composite cycle $\gamma_1 = a_{11}a_{22}a_{33}$ supports a rank-principal certificate. The composite cycle $\gamma_2 = a_{22}a_{33}$ supports a rank-principal certificate of order 2. But the composite cycle $\gamma_3 = a_{11}a_{22}$ cannot support a rank-principal certificate, since the third row of any matrix in $Q(A)$ cannot be in the span of the first two rows.

Example 3.0.3. Consider the irreducible sign pattern

$$A = \begin{bmatrix} 0 & + & + & 0 & + \\ 0 & + & + & 0 & + \\ 0 & 0 & 0 & + & 0 \\ 0 & + & + & 0 & + \\ + & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that $\text{mr}(A) = 3$ and $c(A) = \text{MR}(A) = 5$, so A allows diagonalizability with rank 5. Observe that A has several composite cycles of length 3 (such as $a_{23}a_{34}a_{42}$), but no composite cycle of length 3 can support a rank-principal certificate. Indeed, for every matrix in $Q(A)$, the third and fifth rows are not in the span of the other rows, so every composite cycle that supports a rank-principal certificate must contain the indices 3 and 5; similarly, by examining the columns 1 and 4, we see that every composite cycle that supports a rank-principal certificate must contain the indices 1 and 4. Thus every composite cycle that supports a

rank-principal certificate must contain the indices 1,3,4, and 5, and hence must have length at least 4. Therefore, A does not allow diagonalizability with rank $\text{mr}(A) = 3$.

As illustrated in the preceding example, a sign pattern A that allows diagonalizability may not allow diagonalizability with rank $\text{mr}(A)$, even when $\text{mr}(A)$ is the length of a composite cycle.

But for symmetric bipartite sign patterns, we have the following interesting result.

Theorem 3.0.4. *Let A be a symmetric sign pattern whose digraph is bipartite. Then $\text{mr}(A)$, $\text{MR}(A)$, and the length of every composite cycle of A are even, and for every even integer k with $\text{mr}(A) \leq k \leq \text{MR}(A)$, there is a symmetric (and hence diagonalizable) matrix $B \in Q(A)$ with rank k , and thus, there is a composite cycle of A of length k that supports a rank-principal certificate for A .*

However, even for a symmetric irreducible bipartite sign pattern A , not every composite cycle of A of length at least $\text{mr}(A)$ can support a rank-principal certificate for A , as the following example shows.

Example 3.0.5. *Consider the symmetric irreducible bipartite sign pattern*

$$A = \begin{bmatrix} 0 & 0 & 0 & + & + & 0 \\ 0 & 0 & 0 & + & + & 0 \\ 0 & 0 & 0 & + & + & + \\ + & + & + & 0 & 0 & 0 \\ + & + & + & 0 & 0 & 0 \\ 0 & 0 & + & 0 & 0 & 0 \end{bmatrix}.$$

Clearly, $\text{mr}(A) = 4$. But for every real matrix $B \in Q(A)$ with rank 4, the first two rows as well as the first two columns must be linearly dependent. Thus the composite cycle $(a_{14}a_{41})(a_{25}a_{52})$ cannot support a rank-principal certificate of A .

Hall et al. also shows that for some symmetric sign patterns A , $\text{mr}(A)$ cannot be achieved by any symmetric matrix $B \in Q(A)$. The following two natural questions arise.

Problem 3.0.6. *Does every symmetric sign pattern A allow diagonalizability with rank $\text{mr}(A)$?*

Problem 3.0.7. *Is it true that for every irreducible symmetric sign pattern A and every integer k that is the length of some composite cycle of A with $k \geq \text{mr}(A)$, there is a composite cycle of A of length k that supports a rank-principal certificate for A ?*

We point out that if symmetry is relaxed to combinatorial symmetry, the answers to the two preceding problems are negative, as the following example shows.

Example 3.0.8. *Consider the combinatorially symmetric irreducible bipartite sign pattern*

$$A = \begin{bmatrix} 0 & 0 & 0 & + & + & + & + \\ 0 & 0 & 0 & + & + & + & + \\ 0 & 0 & 0 & + & + & + & + \\ + & + & + & 0 & 0 & 0 & 0 \\ - & + & + & 0 & 0 & 0 & 0 \\ + & - & + & 0 & 0 & 0 & 0 \\ + & + & - & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Observe that the 4×3 submatrix in the lower left corner has minimum rank 3 (see Brua95), so $\text{mr}(A) = 4$. Assume that a rank 4 matrix $B \in Q(A)$ is rank-principal. Then there is a composite cycle γ of length 4 that supports a rank-principal certificate of B . Since the first three columns of B are linearly independent and are not linear combinations of the remaining columns, we see that the index set of γ must contain $\{1, 2, 3\}$. It follows that the principal submatrix supported by γ would have row indices $\{1, 2, 3, i\}$ for some $i \in \{4, 5, 6, 7\}$. Thus the principal submatrix of B supported by γ , $B[\{1, 2, 3, i\}]$, contains a 3×3 zero submatrix and has rank 2, contradicting the fact that it is a rank-principal certificate of B . Thus A does not allow diagonalizability with rank 4. Therefore, no composite cycle of length 4 can support a rank-principal certificate for A . Note, however, that A does have composite cycles of length 4, such as $(a_{34}a_{43})(a_{25}a_{52})$.

It is easy to see that the answer to Problem 3.0.7 is negative if irreducibility is dropped, as can be seen from the following example.

Example 3.0.9. *For the reducible symmetric sign pattern*

$$A = \begin{bmatrix} 0 & + & + & 0 & 0 & 0 & 0 \\ + & 0 & + & 0 & 0 & 0 & 0 \\ + & + & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & + & + \\ 0 & 0 & 0 & 0 & 0 & + & + \\ 0 & 0 & 0 & + & + & 0 & 0 \\ 0 & 0 & 0 & + & + & 0 & 0 \end{bmatrix},$$

$\text{mr}(A) = 5$, and there exist composite cycles of length 6 (such as $(a_{12}a_{21})(a_{46}a_{64})(a_{57}a_{75})$). Clearly, every composite cycle that supports a rank-principal certificate for A must contain the indices 1, 2 and 3. But there is no composite cycle of length 6 containing the indices 1, 2, and 3.

Note that if $c(A) = \text{MR}(A) = k$ for a square sign pattern A , then as shown in there is a matrix $B \in Q(A)$ such that $\text{rank}(B) = k$ and B has k distinct nonzero eigenvalues. It follows that B is diagonalizable, so A allows diagonalizability with rank $\text{MR}(A)$. Of course, it follows from Theorem 2.1.4 that if a square sign pattern A satisfies $c(A) < \text{MR}(A)$, then A does not allow diagonalizability with rank $\text{MR}(A)$. Thus we arrive at the following result.

Theorem 3.0.10. *A square sign pattern A allows diagonalizability with rank $\text{MR}(A)$ if and only if $c(A) = \text{MR}(A)$.*

But a square sign pattern A satisfying $c(A) < \text{MR}(A)$ may allow diagonalizability with some smaller rank, as illustrated by the next example.

Example 3.0.11. Let $A = \begin{bmatrix} 0 & + & + & 0 \\ + & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & + & + & 0 \end{bmatrix}$. Note that $\text{mr}(A) = 2 = c(A)$, $\text{MR}(A) = 3$, and every rank 2 matrix $B \in Q(A)$ has two distinct nonzero real eigenvalues (which are negatives of each other) and hence is diagonalizable. But there is no composite cycle of length 3 in A , so A does not allow diagonalizability with rank $\text{MR}(A) = 3$.

Concerning composite cycles that support rank-principal certificates, we have the following interesting result.

Theorem 3.0.12. Suppose that γ_1 and γ_2 are composite cycles of a square sign pattern A such that $\gamma_1 \subset \gamma_2$. If γ_1 supports a rank-principal certificate for A , then γ_2 also supports a rank-principal certificate for A .

Proof. Without loss of generality, we may assume that the sign pattern A has the following form

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}, \text{ where } A_{11} \text{ is supported by } \gamma_1 \text{ and } \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ is supported by } \gamma_2.$$

Since $\gamma_1 \subset \gamma_2$, there is a composite cycle β_2 of A such that $\gamma_2 = \gamma_1\beta_2$, where the index sets of γ_1 and β_2 are disjoint. It follows that A_{22} is supported by β_2 . Suppose that γ_1 supports a

rank-principal certificate of $B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{bmatrix} \in Q(A)$. Let k_1 and k_2 denote the lengths

of γ_1 and β_2 , respectively. By performing type III elementary row and column operations on B , we can get the following matrix of rank k_1 :

$$\begin{bmatrix} B_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $B_{22}(\beta_2)$ be the $(1, -1, 0)$ -matrix of order k_2 whose only nonzero entries occur in the positions of the entries of β_2 and have the same sign as the corresponding entries of β_2 . Clearly, $B_{22}(\beta_2)$ is nonsingular. Let \tilde{B} be the matrix obtained from B by replacing B_{22} with $B_{22} + B_{22}(\beta_2)$ while keeping the other blocks unchanged. Note that $\tilde{B} \in Q(A)$. It can be seen that via type III elementary row and column operations, \tilde{B} may be transformed to the following matrix:

$$\begin{bmatrix} B_{11} & 0 & 0 \\ 0 & B_{22}(\beta_2) & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

It follows that \tilde{B} is rank-principal, with a rank-principal certificate supported by γ_2 . \square

Repeated applications of the preceding theorem yield the following generalization.

Corollary 3.0.13. *Suppose that a square sign pattern A has k composite cycles $\gamma_1, \dots, \gamma_k$ such that $\gamma_1 \subset \gamma_2 \subset \dots \subset \gamma_k$. If γ_1 supports a rank-principal certificate for A , then each of $\gamma_2, \dots, \gamma_k$ also supports a rank-principal certificate for A .*

However, the following problem remains open.

Problem 3.0.14. *Suppose that γ_1 and γ_2 are composite cycles of a square sign pattern A such that γ_1 supports a rank-principal certificate for A , and the index set of γ_1 is a subset of that of γ_2 . Does it follow that γ_2 also supports a rank-principal certificate for A ?*

It can be seen from Theorem 3.0.12 that for every $n \times n$ sign pattern with all diagonal entries nonzero, if A allows diagonalizability with rank k , then it also allows diagonalizability with rank t for each integer t with $k \leq t \leq n$. The following intriguing question arises.

Problem 3.0.15. *Does every square sign pattern A with all diagonal entries nonzero allow diagonalizability with rank $\text{mr}(A)$?*

PART 4

SIGNPATTERNS WITH $\text{mr}(A) = 2$ AND NO ZERO LINE

In [18] it is shown that for each $k \geq 4$ there exists an irreducible sign pattern A such that $c(A) \geq \text{mr}(A) = k$ and A does not allow diagonalization.

Example 4.0.1 (18). *For example, Let*

$$A = \begin{bmatrix} 0 & + & + & 0 & + & 0 \\ 0 & + & + & 0 & + & 0 \\ 0 & 0 & 0 & + & 0 & 0 \\ + & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & + \\ + & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

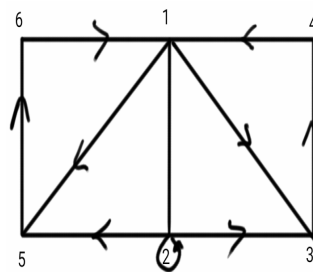


Figure (4.1) Digraph of A in Example 4.0.1

The digraph of A is strongly connected and therefore A is irreducible. Also, $\text{mr}(A) = c(A) = 4$ and $\text{MR}(A) = 5$. However A does not allow diagonalizability. Note that the entries $a_{34} = +$ and $a_{56} = +$ are the only nonzero entries in their rows and columns, so if there is a composite cycle γ that supports a rank-principal certificate for A , then γ must contain the entries $a_{34} = +$ and $a_{56} = +$, and hence the index set of γ would contain $\{3, 4, 5, 6\}$.

But a look at the digraph $D(A)$ shows that there is no composite cycle in A whose index set contains $\{3, 4, 5, 6\}$. Thus A does not allow diagonalizability.

Note that there is a way to construct irreducible sign patterns A such that $\text{mr}(A) = 2 + t \geq 4$ and $c(A) \leq 3$ (and hence A does not allow diagonalizability). For each integer $t \geq 2$, the nonnegative sign pattern $A = [a_{ij}]$ of order $2t + 1$ whose only nonzero entries are the entries of the 3-cycles $a_{1,2i}a_{2i,2i+1}a_{2i+1,1}$, $i = 1, \dots, t$ satisfies $\text{mr}(A) = 2 + t \geq 4$ and $c(A) = 3$. In contrast, the following remarkable property of irreducible sign patterns with minimum rank 3 is worth mentioning.

Observation 4.0.2. *Every irreducible sign pattern A with $\text{mr}(A) = 3$ satisfies $c(A) \geq 3$.*

Proof. If A has a simple cycle of length at least 3, then of course $c(A) \geq 3$. Now assume that the maximum simple cycle length of A is 2. Since $D(A)$ is strongly connected, we see that A is combinatorially symmetric and the underlying undirected graph of A is a tree. If this tree is not a star, then we get a composite cycle of length 4 consisting of two simple 2-cycles. If this tree is a star, then $\text{mr}(A) = 3$ ensures that there is a 1-cycle at a vertex that is not the center of the star, thus we get a composite cycle of length 3 consisting of this 1-cycle and a 2-cycle. \square

Thus we have the following problem.

Problem 4.0.3. *Does every irreducible sign pattern A with $\text{mr}(A) = 3$ allow diagonalizability?*

This is a very challenging problem that awaits further research. The irreducible sign patterns with minimum rank 3 that we have examined so far all allow diagonalizability. But note that as shown in Example 3.0.3, irreducible sign patterns with minimum rank 3 may not allow diagonalizability with rank 3.

If we relax the irreducibility to the weaker condition of having no zero line, then it is easy to find a square sign pattern with minimum rank 3 and with no zero line that does not allow diagonalizability, as the next example shows.

Example 4.0.4. *The reducible sign pattern $A = \begin{bmatrix} + & + & 0 & 0 \\ 0 & 0 & + & 0 \\ 0 & 0 & 0 & + \\ 0 & 0 & 0 & + \end{bmatrix}$ has no zero line, $\text{mr}(A) = 3$, and $c(A) = 2$. By Theorem 2.1.3, A does not allow diagonalizability.*

We now concentrate on sign patterns with minimum rank 2. If the irreducibility condition is dropped completely, it is easy to see that there exists a reducible sign pattern A such that $\text{mr}(A) = 2$ and A does not allow diagonalizability. For example, the sign pattern

$$A = \begin{bmatrix} + & + & 0 & 0 \\ + & + & 0 & 0 \\ 0 & 0 & 0 & + \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

satisfies $c(A) = \text{mr}(A) = 2$, but A does not allow diagonalizability (as its lower right 2×2 diagonal block does not allow diagonalizability).

In the following theorems, we establish that every square sign pattern A satisfying the two conditions $\text{mr}(A) = 2$ and A has no zero line (where the second condition is weaker than irreducibility) allows diagonalizability with various ranks including 2 and $\text{MR}(A)$.

Theorem 4.0.5. *Let A be a square sign pattern with $\text{mr}(A) = 2$ and no zero line. Then A allows diagonalizability with rank 2.*

Proof. Since the set of sign patterns that allow diagonalizability is closed under permutation similarity and signature equivalence, using similar methods as in [15], by replacing the sign pattern A with a sign pattern obtained from A via permutation similarity and signature equivalence if necessary, we may assume that a real matrix $B \in Q(A)$ with $\text{rank}(B) = 2$ can be written as

$$B = \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix} \begin{bmatrix} -y_1 z_1 & -y_2 z_2 & \cdots & -y_n z_n \\ z_1 & z_2 & \cdots & z_n \end{bmatrix},$$

where $x_1 \leq x_2 \leq \cdots \leq x_n$, $z_1 > 0, z_2 > 0, \dots, z_n > 0$, not all of the x_i are equal, and not all

the y_i are equal. Regard $x_1, x_2, \dots, x_n, y_1, y_2, \dots, y_n$, and z_1, z_2, \dots, z_n as real variables. For each i, j , whenever $b_{ij} = 0$, we identify y_j with x_i . Furthermore, whenever $b_{i_1j} = b_{i_2j} = 0$, we also identify x_{i_1} with x_{i_2} . Consider the 2×2 matrix

$$C = \begin{bmatrix} -y_1z_1 & -y_2z_2 & \cdots & -y_nz_n \\ z_1 & z_2 & \cdots & z_n \end{bmatrix} \begin{bmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{bmatrix},$$

the negative of whose determinant is $p = (\sum_{i=1}^n x_i z_i)(\sum_{i=1}^n y_i z_i) - (\sum_{i=1}^n z_i)(\sum_{i=1}^n x_i y_i z_i)$.

Observe that if C is nonsingular, then C has two nonzero eigenvalues, hence B has exactly two nonzero eigenvalues. It follows that the second elementary symmetric function of the eigenvalues of B is nonzero. But since this number is equal to the sum of all 2×2 principal minors of B (see [12]), we see that B is rank-principal, and hence, A allows diagonalizability. Thus it remains to show that the independent variables can be assigned suitable values so that C is nonsingular, namely, $p \neq 0$.

Since $\text{rank}(B) = 2 > 1$, at least two of the x_i are independent, and at least two of the y_i are independent. Suppose that there is no zero entry in row 1 of B . Then x_1 is distinct from x_2, \dots, x_n and no y_j is identified as x_1 . Hence, if y_1 is not identified with any x_i , then the coefficient of $x_1 y_1$ in the polynomial p is $z_1^2 - (\sum_{i=1}^n z_i)z_1 \neq 0$, so $p \neq 0$. If $y_1 = x_{i_1} = \dots = x_{i_k} = y_{j_2} = \dots = y_{j_t}$, then the coefficient of $x_1 x_{i_1}$ in p is $z_1(\sum_{j=1}^k z_{i_j}) - (\sum_{i=1}^n z_i)z_1 \neq 0$.

Now assume that row 1 of B has exactly t (> 0) zero entries, with column indices $j_1 < j_2 < \dots < j_t$. Suppose that the j_1 th column of B has k 0 entries. Since each column of B is nondecreasing (as the row indices increase), we have $b_{1j_1} = b_{2j_1} = \dots = b_{tj_1} = 0$ and the remaining entries in the column are positive. Since $\text{mr}(A) = 2 = \text{rank}(B)$ and A has no zero line, we have $1 \leq k < n$ and $1 \leq t < n$ and $B[\{1, \dots, k\}, \{j_1, \dots, j_t\}]$ is a maximal zero submatrix such that all entries of $B[\{1, \dots, k\}, \{j_1, \dots, j_t\}^c]$ are nonzero and all the entries of $B[\{1, \dots, k\}^c, \{j_1, \dots, j_t\}]$ are nonzero. Suppose that there are s elements in $\{j_1, \dots, j_t\}$ that are at most k , that is, $B[\{1, \dots, k\}, \{j_1, \dots, j_t\}]$ contains s diagonal entries of B . Note

that $x_1 = x_2 = \cdots = x_k = y_{j_1} = y_{j_2} = \cdots = y_{j_t}$. Further x_{k+1}, \dots, x_n and y_j for any $j \notin \{j_1, \dots, j_t\}$ are independent of x_1 . Thus the coefficient of x_1^2 in p is

$$\left(\sum_{i=1}^k z_i\right)\left(\sum_{i=1}^t z_{j_i}\right) - \left(\sum_{i=1}^n z_i\right)\left(\sum_{i=1}^s z_{j_i}\right) \neq 0$$

(with $\sum_{i=1}^s z_{j_i}$ understood to be 0 when $s = 0$).

Therefore, the polynomial p is never identically zero. Thus, subject to the required identifications, we can find a rational value of each free variable within a sufficiently small neighborhood of the initial value such that $p \neq 0$, so the perturbed rational matrix $\tilde{B} \in Q(A)$ satisfies $\text{rank}(\tilde{B}) = 2$ and \tilde{B} is rank-principal. It follows that A allows diagonalizability with rank 2. \square

As an immediate consequence, we get the following result.

Theorem 4.0.6. *Every irreducible sign pattern A with $\text{mr}(A) = 2$ allows diagonalizability with rank 2.*

Note that in the proof of Theorem 4.0.5, if $A[X, Y]$ is a maximal zero submatrix of the square sign pattern A such that $\text{mr}(A) = 2$ and A has no zero line, then $A[X, Y^c]$ and $A[X^c, Y]$ are full sign patterns. Hence, for any other maximal zero submatrix $A[X_1, Y_1]$ of A , we must have $X \cap X_1 = \emptyset$ and $Y \cap Y_1 = \emptyset$. Thus the maximal zero submatrices of A are strongly disjoint. It is easy to see that this holds even when A is not square. We record this fact as follows.

Observation 4.0.7. *Let A be a sign pattern such that $\text{mr}(A) = 2$ and A has no zero line. Then the maximal zero submatrices of A are strongly disjoint.*

It turns out every square sign pattern whose maximal zero submatrices (if any) are strongly disjoint allows diagonalizability with rank equal to its maximum rank, as shown below.

Theorem 4.0.8. *Let A be an $n \times n$ sign pattern whose maximal zero submatrices (if any) are strongly disjoint. Then $c(A) = \text{MR}(A)$ and A allows diagonalizability with rank $\text{MR}(A)$.*

Furthermore, for every $n \times n$ permutation sign pattern P , $c(PA) = \text{MR}(PA) = \text{MR}(AP) = c(AP) = \text{MR}(A)$.

Proof. In view of Theorem 3.0.10, it suffices to show that $c(A) = \text{MR}(A)$, as the last statement of the theorem follows from this fact applied to the matrices PA and AP (and the obvious fact that the maximum rank is invariant under permutation equivalence). Clearly, $c(A) = \text{MR}(A)$ when $\text{MR}(A) = n$.

Now assume that $\text{MR}(A) < n$. Then there are s rows, with row index set S , and t columns, with column index set T , that cover all the nonzero entries of A , where $s + t = |S| + |T| = \text{MR}(A)$. Then $A[S^c, T^c] = 0$ is a maximal zero submatrix. Since the maximal zero submatrices of A are strongly disjoint, $A[S, T^c]$ and $A[S^c, T]$ are full. Let $k = |S \cap T|$. Note that each element $z \in (S \setminus T) \cup (T \setminus S)$ gives a 1-cycle a_{zz} , so we have $|S| + |T| - 2k$ disjoint 1-cycles of A arising this way.

For each $y \in (S \cup T)^c$ and $x \in S \cap T$, we have $a_{xy} \neq 0$ and $a_{yx} \neq 0$, so $a_{xy}a_{yx}$ is a 2-cycle. Since $|(S \cup T)^c| = n - (|S| + |T| - k) = k + n - (|S| + |T|) \geq k$, we obtain k disjoint 2-cycles using k disjoint pairs of vertices $x_i \in S \cap T$ and $y_i \in (S \cup T)^c, i = 1, \dots, k$. Together with the $|S| + |T| - 2k$ disjoint 1-cycles mentioned above, we obtain a composite cycle of length $2k + (|S| + |T| - 2k) = |S| + |T| = \text{MR}(A)$. Thus $c(A) = \text{MR}(A)$. \square

As a consequence of Theorem 4.0.8 and Observation 4.0.7, we get the following result.

Theorem 4.0.9. *Let A be an $n \times n$ sign pattern such that $\text{mr}(A) = 2$ and A has no zero line. Then $c(A) = \text{MR}(A)$ and A allows diagonalizability with rank $\text{MR}(A)$. Furthermore, for every $n \times n$ permutation sign pattern P , $c(PA) = \text{MR}(PA) = \text{MR}(AP) = c(AP) = \text{MR}(A)$.*

In the proof of Theorem 4.0.5, the presence of zero entries in A imposes restrictions on some of the variables in a full rank factorization of a matrix $B \in Q(A)$. However, for any $n \times n$ full sign pattern A and any $B \in Q(A)$, there are no such restrictions on the variables arising from a full rank factorization of B . Hence, using a full rank factorization as in the proof of Theorem 4.0.5, we can show the following result.

Theorem 4.0.10. *Every $n \times n$ full sign pattern A allows diagonalizability with each rank from $\text{mr}(A)$ to n .*

Sign patterns whose maximal zero submatrices are strongly disjoint may be viewed as a generalization of full sign patterns, but it could happen that such a square sign pattern A may not allow diagonalizability with any rank less than its maximum rank, as the following example shows.

Example 4.0.11. *The maximal zero submatrices of the square sign pattern $A = \begin{bmatrix} 0 & 0 & + & + \\ 0 & 0 & + & - \\ + & + & 0 & 0 \\ + & + & 0 & 0 \end{bmatrix}$ are strongly disjoint, and A allows diagonalizability with rank $c(A) = 4$. But $\text{mr}(A) = 3$ and A does not have any composite cycle of length 3 (as $D(A)$ is bipartite), so A does not allow diagonalizability with rank 3.*

We now show another striking composite cycle property of square sign patterns whose maximal zero submatrices are strongly disjoint.

Theorem 4.0.12. *Let A be an $n \times n$ nonzero sign pattern whose maximal zero submatrices (if any) are strongly disjoint. Then A has a composite cycle of length $c(A)$ consisting of disjoint simple cycles of lengths up to 3, at most one of which is a 3-cycle.*

Proof. We proceed by induction on n .

The result is clear for $n \leq 3$.

Note that for $n = 3$, such as for the sign pattern $\begin{bmatrix} 0 & - & - \\ + & 0 & - \\ + & + & 0 \end{bmatrix}$, it is possible that the only composite cycle of length 3 is a simple 3-cycle.

Now, assume that $n \geq 4$ and suppose that the result holds for all orders less than n .

If A has no zero submatrices, then clearly A has a composite cycle of length n consisting of n 1-cycles.

Now assume that A has $m \geq 1$ strongly disjoint maximal zero submatrices and without loss of generality, suppose that the row index sets of the maximal zero submatrices of A are the pairwise disjoint subsets S_1, \dots, S_m , and their column index sets are the pairwise disjoint subsets T_1, \dots, T_m , where $|S_1| + |T_1| \geq |S_2| + |T_2| \geq \dots \geq |S_m| + |T_m|$.

Case 1. $|S_1| + |T_1| > n$.

Then fewer than n lines of A (such as rows and columns of A not intersecting $A[S_1, T_1]$) can cover all the nonzero entries of A , so $\text{MR}(A) < n$. As in the proof of Theorem 4.0.8 when $\text{MR}(A) < n$, there is a composite cycle of length $c(A)$ consisting of 1-cycles and 2-cycles.

Case 2. $|S_1| + |T_1| = n$.

Since the total size of any zero submatrix of A is at most n , we have $\text{MR}(A) = n$. Clearly, $S_1^c \neq \emptyset$ and $T_1^c \neq \emptyset$. Note that every zero submatrix of A strongly disjoint with $A[S_1, T_1]$ is a submatrix of $A[S_1^c, T_1^c]$. By avoiding using any possible nonzero entries in $A[S_1^c, T_1^c]$ in forming a composite cycle of length n , we may assume that $S_2 = S_1^c$, and $T_2 = T_1^c$ (and hence $m = 2$). Note that we then have $|S_2| + |T_2| = n - |S_1| + n - |T_1| = 2n - (|S_1| + |T_1|) = n$.

Subcase 2.1. $S_1 = T_1$.

Take $i \in S_1 = T_1, j \in S_2 = T_2$. Then $a_{ij}a_{ji}$ is a 2-cycle in A . Upon deleting i th and j th rows and columns of A , each of the two maximal zero submatrices of A loses one row and one column, and the principal submatrix $A' = A[\{i, j\}^c]$ is of order $n - 2$ and $\text{MR}(A') = n - 2$, as every zero submatrix of A' has total size at most $n - 2$. By the induction hypothesis, A' has composite cycle γ of length $c(A') = n - 2$ consisting of 1-cycles, 2-cycles, and at most one 3-cycle. Thus $(a_{ij}a_{ji})\gamma$ is a composite cycle of A of length $n = c(A)$ consisting of 1-cycles, 2-cycles, and at most one 3-cycle.

Subcase 2.2. $S_1 \neq T_1$.

Then $(S_1 \setminus T_1) \cup (T_1 \setminus S_1) \neq \emptyset$. Without loss of generality, assume that $(S_1 \setminus T_1) \neq \emptyset$ and take $k \in S_1 \setminus T_1$. Then $a_{kk} \neq 0$, as it is an element of $A[S_1, T_1^c] = A[S_1, T_2]$. Upon deleting the k th row and k th column of A , each of the two maximal zero submatrices of A loses one line, and we get a principal submatrix A' of order $n - 1$ with $\text{MR}(A') = n - 1$, since A' does not have any zero submatrix of total size greater than $n - 1$. By the induction hypothesis, A' has a composite cycle γ of length $c(A') = n - 1$ consisting of 1-cycles, 2-cycles, and at most one 3-cycle. Thus $(a_{kk})\gamma$ is a composite cycle of A of length $n = c(A)$ consisting of 1-cycles, 2-cycles, and at most one 3-cycle.

Case 3. $|S_1| + |T_1| \leq n - 1$.

Then A does not have any zero submatrix with total size greater than n , so $\text{MR}(A) = n$. By avoiding using possible nonzero entries in a suitable submatrix of $A[S_1^c, T_1^c]$ of total size less than n if necessary, we may assume that $m \geq 2$. Since the sum of the total sizes of all the maximal zero submatrices of A is at most $2n$ and $n \geq 4$, there are at most two maximal zero submatrices of A with total size $n - 1$.

Subcase 3.1. $S_1 = T_1$. Note that $|S_k| + |T_k| \leq n - 1$, for each $k = 1, \dots, m$. Take $i \in S_1$ and $j \in T_2 \subseteq T_1^c = S_1^c$. Since a_{ij} is an element of the full matrix $A[S_1, T_1^c]$ and a_{ji} is an element of the full matrix $A[S_1^c, T_1]$, we see that $a_{ij}a_{ji}$ is a 2-cycle of A . Note that A has at most two maximal zero submatrices of total size $n - 1$. Upon deleting i th and j th rows and columns of A , each of the two maximal zero submatrices $A[S_1, T_1]$ and $A[S_2, T_2]$ (with largest total sizes) loses at least one line, and the principal submatrix $A' = A[\{i, j\}^c]$ of order $n - 2$ satisfies $\text{MR}(A') = n - 2$, as every zero submatrix of A' has total size at most $n - 2$. By the induction hypothesis, A' has a composite cycle γ of length $c(A') = n - 2$ consisting of 1-cycles, 2-cycles, and at most one 3-cycle. Thus $(a_{ij}a_{ji})\gamma$ is a composite cycle of A of length $n = c(A)$ consisting of 1-cycles, 2-cycles, and at most one 3-cycle.

Subcase 3.2. $S_1 \neq T_1$. With the obvious modification that $m \geq 2$ instead of $m = 2$, the argument in Subcase 2.2 also works here .

Therefore, A has a composite cycle of length $n = c(A)$ consisting of 1-cycles, 2-cycles, and at most one 3-cycle. □

The next result follows from Theorem 4.0.12 and Observation 4.0.7

Theorem 4.0.13. *Let A be an $n \times n$ sign pattern such that $\text{mr}(A) = 2$ and A has no zero line. Then A has a composite cycle of length $c(A)$ consisting of disjoint simple cycles of lengths up to 3, at most one of which is a 3-cycle.*

Obviously, in the two preceding theorems, if $c(A)$ is odd and A has no 1-cycle, then A has a composite cycle of length $c(A)$ consisting of 2-cycles and exactly one 3-cycle.

In view of Theorem 4.0.13 and Theorem 3.0.12, we obtain the following result on the ranks achieved by diagonalizable matrices in the qualitative class of sign pattern matrix A

such that $\text{mr}(A) = 2$ and A has no zero line.

Theorem 4.0.14. *Let A be a square sign pattern.*

(a). *Suppose that $\gamma_1\gamma_2\dots\gamma_k$ ($k \geq 2$) is a composite cycle of A of length $c(A)$ such that γ_1 is a composite cycle of length 2 that supports a rank-principal certificate for A , γ_2 is a 1-cycle, and for each $2 \leq i \leq k$, γ_i is a 1-cycle or 2-cycle. Then $\{r \mid A \text{ allows diagonalizability with rank } r\} = \{2, 3, \dots, c(A)\}$.*

(b). *More generally, suppose that $\gamma_1\gamma_2\dots\gamma_k$ ($k \geq 2$) is a composite cycle of A where γ_1 is a composite cycle of length l_1 that supports a rank-principal certificate for A , and $\gamma_2\dots\gamma_k$ are simple cycles. Then*

$$\{l_1 + \sum_{j \in S} \text{length}(\gamma_j) \mid S \subseteq \{2, \dots, k\}\} \subseteq \{r \mid A \text{ allows diagonalizability with rank } r\}.$$

Example 4.0.15. *The sign pattern $A = \begin{bmatrix} 0 & - & - & - & - \\ + & 0 & - & - & - \\ + & + & 0 & + & + \\ + & + & 0 & + & + \\ + & + & 0 & + & + \end{bmatrix}$ satisfies $\text{mr}(A) = 2$ and A has no zero line. Note that the 2-cycle $a_{23}a_{32}$ supports a rank-principal certificate for A and $(a_{23}a_{32})(a_{15}a_{51})a_{44}$ is a composite cycle of length $c(A) = 5$. By the preceding theorem, the set of the ranks of the diagonalizable matrices in $Q(A)$ is equal to $\{2, 3, 4, 5\}$.*

Example 4.0.16. *The sign pattern $A = \begin{bmatrix} 0 & 0 & 0 & - & - & - \\ 0 & 0 & 0 & - & - & - \\ 0 & 0 & 0 & - & - & - \\ + & + & + & 0 & 0 & 0 \\ + & + & + & 0 & 0 & 0 \\ + & + & + & 0 & 0 & 0 \end{bmatrix}$ satisfies $\text{mr}(A) = 2$ and A has no zero line. Note that the 2-cycle $a_{34}a_{43}$ supports a rank-principal certificate for A and $(a_{34}a_{43})(a_{25}a_{52})(a_{16}a_{61})$ is a composite cycle of length $c(A) = 6$. By the preceding theorem, the set of the ranks of the diagonalizable matrices in $Q(A)$ contains $\{2, 4, 6\}$. But since the digraph of A is bipartite, every composite cycle of A has even length. Thus the set of the ranks of the diagonalizable matrices in $Q(A)$ is equal to $\{2, 4, 6\}$.*

Example 4.0.17. *The sign pattern $A = \begin{bmatrix} 0 & - & - & - & - \\ + & 0 & - & - & - \\ + & + & 0 & - & - \\ + & + & + & 0 & - \\ + & + & + & + & 0 \end{bmatrix}$ satisfies $\text{mr}(A) = 2$ since the polynomial sign change number of each row is 1. Note that the 2-cycle $a_{12}a_{21}$ supports a rank-principal certificate for A and $(a_{12}a_{21})(a_{34}a_{43})$ is a composite cycle of A . By the preceding theorem, the set of the ranks of the diagonalizable matrices in $Q(A)$ contains $\{2, 4\}$. Also, the 3-cycle $a_{12}a_{23}a_{31}$ supports a rank-principal certificate for A and $(a_{12}a_{23}a_{31})(a_{45}a_{54})$ is a composite cycle of A . By the preceding theorem, the set of the ranks of the diagonalizable*

matrices in $Q(A)$ contains $\{3, 5\}$. Thus the set of the ranks of the diagonalizable matrices in $Q(A)$ is equal to $\{2, 3, 4, 5\}$.

PART 5

CONCLUSION

In this dissertation, there is much that still could be done on the topic of sign patterns that allow and require diagonalizability. This work covered some types of sign patterns that allow diagonalizability as well as conditions for requiring diagonalizability. I will continue to work on diagonalizability problems, including rank realizations, irreducible matrices, and distinct eigenvalues.

The following are some open problems.

Problem 5.0.1. *Irreducible sign pattern A with minimum rank 3 allows diagonalizability.*

Problem 5.0.2. *Suppose that A allows diagonalizability. Is every composite cycle length that is at least equal to $mr(A)$ achievable as the rank of a diagonalizable matrix $B \in Q(A)$?*

Problem 5.0.3. *Does every symmetric sign pattern allow diagonalizability with minimum rank?*

Other areas of further study include characterizing irreducible sign patterns that allow diagonalizability.

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