Evaluating Students' Understanding of the Matrix Representation of Linear Transformations Using the Onto-Semiotic Approach

LINEAR TRANSFORMATIONS, COORDINATE VECTORS, AND MATRIX REPRESENTATION FOR ABSTRACT VECTOR SPACES AND ARBITRARY BASES: EVALUATING STUDENTS' UNDERSTANDING OF THE MATRIX REPRESENTATION OF LINEAR TRANSFORMATIONS USING THE ONTO-SEMIOTIC APPROACH

by

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Abstract

This study investigates how college students and first-year graduate students understand the matrix representation of linear transformations, expressed as $A[v]_E = [L(v)]_F$, where A is the matrix of the transformation L with respect to bases E and F. While many can perform the computations, they often struggle to interpret what the matrix means and how it represents the transformation, especially in abstract vector spaces and nonstandard bases. These challenges reveal not only procedural gaps but also conceptual difficulties.

To examine these issues, the study draws on the Onto-Semiotic Approach (OSA) as its theoretical foundation. OSA has been widely used in mathematics education research and teaching, especially in Europe and South America, over the past three decades. Its growing adoption in the context of college mathematics in the United States reflects a broader interest in frameworks that go beyond correctness and focus on meaning-making. OSA analyzes how students interact with mathematical objects, language, representations, and processes during learning.

To complement this, Statistical Implicative Analysis (SIA), implemented via the CHIC algorithm, was also applied to uncover quasi-implicative relationships among students' conceptual competencies based on written tasks and interviews. Data were collected from upper-level undergraduates and first-year graduate students in Fall 2024 and Spring 2025.

The analysis identified common obstacles, such as confusion between vectors and their coordinates, difficulty transitioning between symbolic and structural views, and limited understanding of how matrices represent transformations between bases. By integrating OSA and SIA, this research offers insight into how students make sense of linear algebra and suggests strategies for promoting deeper conceptual understanding.

INDEX WORDS: Matrix representation, coordinate vectors, linear transformations, onto-semiotic approach, proof comprehension

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Dedication

To my grandparents and parents:

Whenever I was around, you were always busy preparing the best seasonal food and making sure I began with a healthy diet—and, in turn, a healthy mind. Through the care you put into every detail, I came to understand the meaning of diligence, endurance, patience, wisdom, and true love—from our dining table in China to my office desk in the United States.

Without your love, care, and support, I would not have had the energy or the peace of mind to reflect on philosophy, first in life and then in mathematics, which eventually became my path.

This work is rooted in the strength you gave me.

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To my friends and fellow graduate students: thank you for sharing your thoughts, offering moral support, and walking beside me on this path.

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Completing this dissertation is more than an academic milestone; it is a personal testament to resilience, growth, and the invisible hands of those who believed in me along the way.

With the passion I have for mathematics, I will carry this spirit - and all your expectations for me - into the next chapter of my journey.

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Introduction

1.1 Background and Context

The study of matrix representations of linear transformations occupies a foundational role in college linear algebra. Yet, despite its structural elegance and widespread application in fields ranging from computer science to physics, students often struggle to grasp the deeper conceptual underpinnings, especially when transformations are defined on arbitrary bases. These difficulties persist even among advanced students who have otherwise demonstrated procedural fluency. The challenge lies not in the mechanics of matrix computation alone, but in students' ability to coordinate multiple *semiotic registers*—that is, different systems of representation through which mathematical meaning is expressed and interpreted (Duval, 2006). These include the *symbolic register*, which involves the use of algebraic notation and formulas (e.g., representing a transformation); the *graphical register*, which includes visualizations such as coordinate axes, vector diagrams, or transformation effects on geometric objects; and the *tabular register*, which organizes data in structured tables to represent relationships between inputs and outputs or basis transformations. Each register provides unique insights and situates these within the abstract frameworks of vector spaces and linear mappings. This perspective is further elaborated within the Onto-Semiotic Approach (OSA), where registers of representation are integrated into a broader framework of semiotic and epistemic processes (Godino, 2024).

The matrix representation of a linear transformation serves as a critical bridge between theoretical and applied linear algebra. It encapsulates the effect of a transformation by expressing it in terms of basis vectors from the domain and codomain, allowing for computational efficiency and structural insight. However, this representation requires a chain of reasoning that integrates

knowledge of abstract vector spaces, basis changes, and coordinate vectors. Students must not only understand what a transformation does, but also how that action is represented with respect to chosen bases. Often, students can perform algorithmic steps without fully understanding the rationale behind them. This often reveals a gap between procedural performance and conceptual understanding, particularly when students rely on memorized algorithms without grasping their underlying logic.

In this dissertation, the matrix representation theorem is positioned as a focal concept through which to investigate students' conceptual development. The study draws on historical, epistemological, and cognitive perspectives to analyze the obstacles and learning trajectories encountered by students in linear algebra. Of particular interest is the way students navigate the symbolic construction of a matrix representation when faced with non-standard bases and abstract polynomial spaces, such as transformations from P_3 to P_2. The transformation $L(p(x)) = p'(x) + p(0)$ exemplifies this complexity: it blends symbolic manipulation with conceptual abstraction, requiring students to connect the transformation rule to its matrix form through basis-specific representations.

This investigation is situated within a broader shift in mathematics education research, which emphasizes the importance of representational fluency and epistemic coherence in students' mathematical reasoning. Recent studies highlight the limitations of traditional assessment methods in revealing the layered structure of student understanding. As such, this study employs a mixed-methods approach that integrates qualitative analysis via the Onto-Semiotic Approach (OSA) and quantitative analysis through Statistical Implicative Analysis (SIA) using the CHIC algorithm. These methods enable a fine-grained analysis of students' knowledge structures and support the development of targeted instructional strategies to scaffold conceptual understanding.

Through this work, the dissertation aims to illuminate the semiotic and conceptual demands of matrix representation and contribute to the design of pedagogical interventions that address persistent learning barriers in linear algebra.

1.2 Statement of the Problem

Despite completing prerequisite coursework in linear algebra, many college students continue to experience significant challenges when asked to construct matrix representations of linear transformations, particularly when these transformations are defined on abstract vector spaces and involve non-standard bases. While some students demonstrate computational fluency, their ability to reason conceptually about basis selection, transformation direction, and coordinate representation often remains underdeveloped. This disconnect hinders their ability to transition between symbolic definitions of transformations and their matrix formulations, a process that is foundational to further study in linear algebra and its applications.

Existing research has identified various conceptual and procedural difficulties in learning linear algebra, including confusion between vectors and coordinate vectors, inadequate understanding of function spaces, and symbolic manipulation errors. However, few studies have systematically mapped the dependencies among these conceptual components or examined how such dependencies influence students' performance on matrix-related tasks. Moreover, traditional assessments fail to capture the layered nature of students' reasoning or the directional dependencies that exist among prerequisite skills.

This dissertation addresses these gaps by investigating the specific knowledge structures and learning obstacles that influence students' success in constructing matrix representations. Through an integrated analysis of qualitative and quantitative data, the study seeks to reveal both the semiotic conflicts students encounter and the logical dependencies that underlie their reasoning processes. By identifying the conceptual bottlenecks and directional knowledge chains, this research aims to inform instructional design and improve learning outcomes in advanced linear algebra settings.

1.3 Purpose of the Study

The primary purpose of this study is to explore how undergraduate students understand and construct the matrix representation of linear transformations in the context of abstract vector

spaces and non-standard bases. Specifically, the research aims to examine students' conceptual frameworks, symbolic fluency, and representational reasoning as they work through tasks involving transformations such as $L(p(x)) = p'(x) + p(0)$ from P_3 to P_2. By analyzing both qualitative data (e.g., student interviews, written work) and quantitative data (e.g., binary-coded diagnostic indicators analyzed via Statistical Implicative Analysis), the study seeks to:

- Identify key conceptual and procedural components required for successful matrix representation;
- Map the semiotic and logical dependencies among these components;
- Understand common patterns of misunderstanding and semiotic conflict;
- Provide empirically grounded recommendations for instructional design and scaffolding.

1.4 Research Questions

To fulfill this purpose, the study is guided by the following research questions:

1. What conceptual and procedural obstacles do students encounter when constructing matrix representations of linear transformations defined in abstract vector spaces?
2. How do students navigate the transition between symbolic transformation rules and matrix-based representations?
3. What semiotic conflicts arise in students' reasoning about basis, coordinate vectors, and transformation expressions?
4. What implicative knowledge structures can be identified through Statistical Implicative Analysis (SIA) of student responses?
5. How can insights from Onto-Semiotic Analysis and SIA inform targeted teaching strategies to support conceptual understanding?

1.5 Significance of the Study

This study contributes to the field of mathematics education by offering a detailed and empirically grounded analysis of students' cognitive and semiotic processes in linear algebra. By focusing on the matrix representation of linear transformations, a concept known for its pedagogical complexity, the research addresses a critical area of student difficulty that is both underexplored and foundational for advanced mathematical thinking. The significance of this study lies in its dual methodological approach: combining Onto-Semiotic Analysis with Statistical Implicative Analysis enables a nuanced investigation of how students build, relate, and misinterpret key ideas in symbolic construction tasks.

Additionally, the findings have implications for curriculum design and pedagogical practice in undergraduate mathematics. By identifying specific knowledge dependencies and representational challenges, the study offers targeted recommendations for instructional scaffolding that can enhance students' conceptual understanding and fluency in matrix representation. These contributions support broader efforts to align mathematics instruction with students' cognitive development and to bridge the gap between procedural success and conceptual insight.

1.6 Theoretical and Analytical Framework

The theoretical framework for this study is grounded in the Onto-Semiotic Approach (OSA) on mathematics education, developed by Godino and colleagues. OSA conceptualizes mathematical knowledge as a system of interrelated practices, representations, and objects that emerge from students' engagement with mathematical tasks. It provides a structure for analyzing learning in terms of semiotic processes (e.g., representation, generalization, synthesis), epistemic configurations (e.g., objects, processes, arguments), and institutional constraints.

In the context of this study, the OSA is used to examine how students interpret and construct matrix representations through the coordination of semiotic registers—symbolic expressions, coordinate vectors, and matrices. This lens enables the identification of semiotic conflicts and

conceptual obstacles that arise in the translation between different representational systems.

Complementing the OSA is Statistical Implicative Analysis (SIA), a logic-based method that models directional knowledge dependencies among binary-coded indicators. SIA reveals the hierarchical and asymmetrical nature of students' conceptual development, offering a quantitative structure for understanding how mastery of one skill implies success in another. The integration of OSA and SIA thus provides a robust framework for investigating both the content and structure of student understanding.

1.7 Definition of Terms

Matrix Representation A rectangular array of numbers that represents a linear transformation with respect to specified bases for the domain and codomain.

Linear Transformation A function between vector spaces that preserves vector addition and scalar multiplication.

Coordinate Vector A vector that represents the weights of the linear combinations of a given vector with respect to a specific basis.

Basis A set of linearly independent vectors that spans a vector space.

Semiotic Register A system of representation, such as symbolic, numerical, or graphical, through which mathematical objects are expressed and understood.

Onto-Semiotic Approach (OSA) A theoretical framework in mathematics education that analyzes mathematical learning in terms of semiotic processes and epistemic configurations.

Statistical Implicative Analysis (SIA) A quantitative method that identifies directional logical relationships between binary-coded variables to reveal knowledge dependencies.

1.8 Delimitations and Limitations

This study focuses on undergraduate and graduate students enrolled in Linear Algebra II at a public southern university in the United States. The scope is limited to students' understanding of a particular type of linear transformation from P_3 to P_2, specifically $L(p(x)) = p'(x) + p(0)$ as an example, and its matrix representation with respect to non-standard bases. The selection of this task reflects the research aim of examining both conceptual and representational challenges within a well-defined and cognitively demanding setting.

Data were collected through diagnostic tasks, questionnaires, and semi-structured interviews and analyzed using methods aligned with the Onto-Semiotic Approach and Statistical Implicative Analysis. As such, the findings pertain specifically to the selected population and instructional context and may not generalize across all linear algebra topics or student populations.

Several limitations must be acknowledged in this study. First, the sample size is relatively small and restricted to students from a single institution, which may limit the generalizability of the findings. Second, while the binary coding of student responses facilitates the use of SIA, it may oversimplify nuanced reasoning patterns, particularly in open-ended written or verbal explanations. Third, the analysis is constrained by the nature of the diagnostic task. The findings pertain to a specific transformation and basis system and may not fully capture students' reasoning in other transformation contexts. Finally, while the study employs a mixed-methods design, it remains interpretive and exploratory in nature; causal inferences regarding learning processes are not warranted.

1.9 Organization of the Study

The present chapter serves as the introduction to the dissertation. It outlines the background of the study, presents the research problem and questions, and introduces the theoretical framework. The remainder of the dissertation is organized into five additional chapters. Chapter 2 provides a historical review of relevant literature, including studies on student understanding of linear

transformations, semiotic reasoning in mathematics, and prior uses of SIA and OSA. Chapter 3 focuses on the theoretical framework, the OSA, in detail. Chapter 4 outlines the methodology, detailing the research design, participant selection, data collection instruments, and analytical procedures. Chapter 5 presents the qualitative and quantitative findings, including thematic patterns identified through OSA and Statistical Implicative Analysis, focusing on the CHIC-generated implicative graph and its instructional implications. Chapter 6 synthesizes the findings, discusses their implications for teaching and curriculum development, and offers recommendations for future research. Together, these chapters provide a comprehensive examination of students' understanding of matrix representation in the context of abstract linear transformations and coordinate systems.

Chapter 2

Historical and Epistemological Perspectives

2.1 Understanding Mathematical Objectives

Mathematical objects—encompassing ideas, procedures, and concepts, or anything that supports mathematical activity—often emerge from practical needs such as counting, measuring, and recognizing patterns. These objects evolve through the use of symbols and ontological development, shaped by a complex interplay between human cognition and cultural context. Over time, they undergo historical evolution and formalization, ultimately forming the foundation of mathematical representation. The emergence of a personal mathematical object is a gradual process influenced by individual experience and learning, whereas the development of an institutional object unfolds collectively over extended periods (Pino-Fan, Guzmán, Duval, & Font, 2015, p. 4–35).

One of the most compelling aspects of teaching and learning matrix representation lies not only in mastering modern mathematical concepts and applying them to problem-solving, but also in understanding the historical emergence and evolution of these concepts as mathematical objects. For example, as early as 200 BCE, Chinese mathematicians developed elimination methods for solving systems of linear equations using counting rods, as recorded in *The Nine Chapters on the Mathematical Art* (Martzloff, 1997). These methods predate similar Gaussian elimination techniques in Europe by over a thousand years. Such explorations span both personal and institutional dimensions, encompassing historical roots, cognitive development, and pedagogical perspectives—dimensions that are often overlooked yet essential in today's fast-paced instructional environments. Mathematical representations are dynamic, evolving through iterative refinement and deeper conceptual understanding.

When students approach mathematics only through formulas and modern techniques, their

understanding can stay on the surface. They may learn to perform procedures, but miss the bigger picture, the *why* behind the ideas. Of course, in a single semester, we cannot expect students to become experts in the history or philosophy of mathematics. But when instructors themselves have a sense of the historical roots and philosophical depth of the subject, they can bring that perspective into the classroom. Even small things, such as mentioning where a concept came from or why a definition matters, can help students make deeper connections and see math as more than just a set of tools.

In response, this chapter revisits key pedagogical ideas essential for understanding matrix representation as a mathematical object. It traces the historical and cognitive development of matrix representation and extends these insights to contemporary educational contexts. The approach proposed here introduces students first to the matrix representation theorem through intuitive construction and then to the broader theoretical framework. In doing so, students come to understand matrix representation not merely as a computational tool, but as an evolving conceptual structure that facilitates problem-solving from both historical and theoretical perspectives.

Section 1 investigates common student challenges in learning matrix representation, beginning with foundational concepts such as vector spaces, linear combinations, linear independence, coordinate vectors, change of basis, and linear transformations. These are not merely abstract notions; rather, they constitute the essential groundwork for understanding matrix representation. They also function as cognitive bridges, helping students transition to higher-level concepts within abstract vector spaces defined over arbitrary bases.

Section 2 considers these challenges through a historical lens, highlighting the contributions of pivotal mathematicians and the epistemological development of key linear algebraic ideas. By examining the evolution of these concepts over time, we introduce matrix representation in a logical sequence that culminates in the study of the matrix representation theorem and its theoretical underpinnings.

Section 3 emphasizes the central role of mathematical notation and symbolic language in shaping both understanding and communication. By linking notation to broader social and

anthropological contexts, we show that mathematical symbols are not arbitrary but have evolved through complex historical and cultural processes. This focus on symbolic development reflects the view that genuine mathematical understanding involves more than fluency in conventional notation; it requires recognition of the deeper meanings that such symbols carry, an idea that will be explored further in Chapter 3 through an ontological lens.

In summary, this chapter highlights conceptual discontinuities across different historical periods in the development of matrix representation, focusing on the contributions of key mathematicians. It also includes examples that demonstrate how these theoretical developments connect to the learning processes of modern students.

2.2 Conceptual and Pedagogical Challenges in Matrix Representation

The teaching and learning of linear algebra continue to present significant cognitive challenges for students at all levels (Dorier, Artigue, Dreyfus, Robert, & Rogalski, 2000; Montiel, Wilhelmi, Vidakovic, & Elstak, 2011; Sierpińska, 2000; S. Stewart, Andrews-Larson, Berman, & Zandieh, 2019; Uhlig, 2023). Although foundational concepts are commonly introduced in early courses, they are often presented in a fragmented manner—one definition following another—without adequate emphasis on the relationships among those ideas. This sequential but disconnected approach can obscure the deep interconnections and underlying logic of linear algebra, making it difficult for students to build a coherent understanding of the subject.

Matrix representation, in particular, demands mastery of several abstract concepts that are deeply interrelated. These include vectors and vector spaces, coordinate systems, bases (both standard and non-standard), linear combinations, linear independence and dependence, and the role of matrices in representing linear transformations. Students often struggle with these concepts due to their high level of abstraction and symbolic complexity, especially if they lack a historical or philosophical perspective on their development. The absence of such perspectives may also limit their engagement with formal reasoning and proof-based learning, further deepening the conceptual gap.

Dorier et al. (2000) emphasizes that even historically, concepts like linear dependence posed difficulties for mathematicians such as Euler, reinforcing the idea that these ideas are nontrivial. As Dorier notes, "The basic idea of linear dependence was not so easy to formalize, even by great mathematicians like Euler" (p. 187). Similarly, Alves-Dias (2001) identifies critical obstacles students face, including the sheer number of new terms and the unfamiliar structure of mathematical exposition and argumentation, which can feel akin to learning a foreign language (p. 256) (as cited in Montiel et al., 2011, p.12). These linguistic and conceptual hurdles complicate students' attempts to bridge foundational knowledge with more sophisticated applications. This dual linguistic-conceptual challenge requires students to both interpret symbolic language and apply abstract structures. On the linguistic side, students must navigate precise symbolic expressions and definitions; on the conceptual side, they must reinterpret familiar mathematical operations in new and abstract contexts. M. P. Carlson (1993) and Rensaa (2019) address this tension, with Rensaa introducing the "LA-split" framework to capture the divide between algebraic versus geometric thinking, and between conceptual understanding and procedural proficiency. When teaching overemphasizes step-by-step procedures, it can reinforce rote learning at the expense of deeper insight.

For instance, students must learn to express vectors semiotically—as meaningful combinations of other vectors—rather than simply as columns of numbers. This foundational skill supports an understanding of vector spaces and the role of coordinate systems in representing abstract quantities concretely. Representing vectors with respect to different bases enables students to bridge theory with computation and explore how abstract structures manifest in various contexts.

These conceptual obstacles are mirrored in the historical development of the subject. Historically, mathematicians such as Richard Dedekind played a pivotal role in advancing the algebraic treatment of structures, including number systems, which informed the development of coordinate systems and the formalization of vector spaces. The capacity to switch between bases—such as from a standard basis to one adapted to a particular problem—reveals the flexibility and universality of matrix representations. This concept is not limited to Euclidean vectors; it extends to

polynomials, function spaces, and matrices themselves.

The formal development of vector spaces accelerated in the 20th century with the work of the Nicolas Bourbaki group, whose axiomatic approach provided a rigorous foundation for modern linear algebra. Their systematic emphasis on structure and abstraction remains central to how matrix representation is taught and understood today.

In sum, the challenges students face in learning matrix representation are deeply rooted in both the historical complexity of the subject and the cognitive demands it places on learners. Addressing these challenges requires instructional strategies that go beyond algorithmic manipulation to foster conceptual clarity, historical awareness, and symbolic fluency. Integrating these elements into the pedagogy helps students engage more meaningfully with linear algebra and develop a more profound understanding of the role that matrix representation plays within it.

2.3 Foundational Contributions to the Development of Matrix Representation

This section explores the historical emergence of key ideas that underpin matrix representation, tracing the evolution of foundational concepts in both Euclidean and abstract vector spaces. By examining the work of influential mathematicians, we highlight how early developments laid the groundwork for the formal structures and pedagogical frameworks used in contemporary linear algebra. Understanding these origins allows us to draw connections between the evolution of mathematical thought and the cognitive tools necessary for student comprehension.

2.3.1 Early Chinese Contributions to Matrix Representation

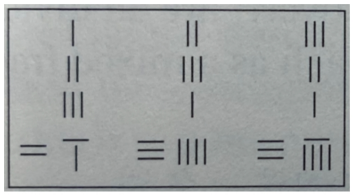
The origins of matrix representation can be traced back over two millennia to ancient China. Around 200 BCE, Chinese mathematicians documented advanced techniques for solving systems of linear equations in the canonical text, *The Nine Chapters on the Mathematical Art* (*Jiuzhang Suanshu*). Chapter 8 of this work, titled *Fangcheng*, presents procedures that are analogous to

modern elimination methods used in linear algebra (Martzloff, 1997).

These early algorithms employed counting rods on a counting board to manipulate tabular arrays of coefficients, effectively anticipating the principles behind Gaussian elimination. Although not expressed in the formal language of matrices as used today, the systematic elimination of variables demonstrated an intuitive grasp of linear structure and transformations.

Given 3 bundles[7] of superior paddy [unhusked rice],[8] 2 bundles of ordinary paddy, and 1 bundle of inferior paddy, [together they yield] 39 *dou*[9] of grain; 2 bundles of superior paddy, 3 bundles of ordinary paddy, and 1 bundle of inferior paddy [together yield] 34 *dou* of grain; 1 bundle of superior paddy, 2 bundles of ordinary paddy, and 3 bundles of inferior paddy [together yield] 26 *dou* of grain. Problem: 1 bundle of superior, ordinary, and inferior paddy each yield how much grain?

今有上禾三秉，中禾二秉，下禾一秉，實三十九斗；上禾二秉，中禾三秉，下禾一秉，實三十四斗；上禾一秉，中禾二秉，下禾三秉，實二十六斗。問上、中、下禾實一秉各幾何？



$$\begin{bmatrix} 3 & 2 & 1 & 39 \\ 2 & 3 & 1 & 34 \\ 1 & 2 & 3 & 26 \end{bmatrix}$$

Figure 2.1: A reconstruction of a traditional counting board setup using counting rods, based on descriptions in *The Nine Chapters* and adapted from Martzloff's historical account (Martzloff, 1997; Urschel, 2025). Each vertical group of rods represents a numerical coefficient in a linear equation, arranged row by row.

Figure 2.2: A numerical layout of counting rods showing the initial coefficient setup for a three-variable system:

$$\begin{cases} 3x + 2y + z = 39 \\ 2x + 3y + z = 34 \\ x + 2y + 3z = 26 \end{cases}.$$

Adapted from Martzloff and guest lecture visuals (Martzloff, 1997; Urschel, 2025).

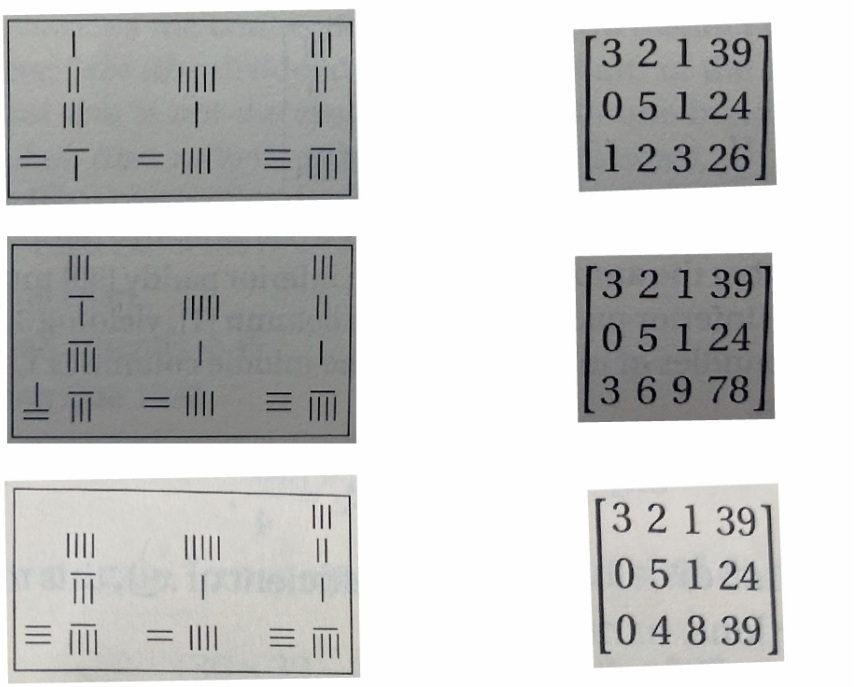
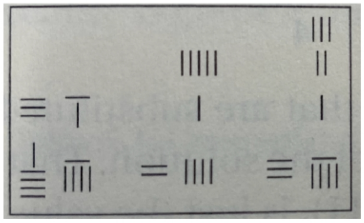


Figure 2.3: Transformation of the rod arrangement as elimination proceeds. This figure shows the intermediate steps in the reduction of the system from Figure 2.2. Each row operation simplifies the system toward triangular form.

By physically manipulating the rods to zero out coefficients below the leading entry in each column, Chinese mathematicians effectively performed early row operations—an intuitive precursor to Gaussian elimination. This method, known historically as *fangcheng*, represents one of the earliest systematic techniques for solving linear systems.

While the core technique remained consistent, historical records also show variations across time and region. Different manuscripts and commentaries reveal regional differences in formatting and procedure, but all retained the shared logic of coefficient manipulation through elimination.



$$\begin{bmatrix} 3 & 2 & 1 & 39 \\ 0 & 5 & 1 & 24 \\ 0 & 0 & 36 & 99 \end{bmatrix}$$

Answer: 1 bundle of superior paddy [yields] $9\frac{1}{4}$ *dou*,[10] 1 bundle of ordinary paddy [yields] $4\frac{1}{4}$ *dou*, and 1 bundle of inferior paddy [yields] $2\frac{3}{4}$ *dou*.
答曰：上禾一秉，九斗四分斗之一；中禾一秉，四斗四分斗之一；下禾一秉，二斗四分斗之三。

Figure 2.4: An alternative elimination method from a later manuscript showing different rod orientations and spacing. The numerical content still follows the same problem structure shown earlier, demonstrating the universality of the procedural logic (Martzloff, 1997; Urschel, 2025).

These procedures demonstrate that even without the modern algebraic formalism, ancient mathematicians engaged in systematic methods that embody the core principles of matrix representation and row reduction techniques. Making these historical insights explicit not only reveals the continuity of mathematical ideas but also enriches students' conceptual frameworks when learning about systems of equations and matrix transformations.

2.3.2 Euler's Perspective on Inclusive Versus Linear Dependence

Leonhard Euler (1707–1783) played a crucial role in the early conceptualization of linear systems. In his 1750 work, *Sur une Contradiction Apparente dans la Doctrine des Lignes Courbes*, Euler challenged the assumption that a system of n linear equations with n unknowns necessarily yields a unique solution (Dorier et al., 2000).

Euler illustrated this by analyzing systems that, despite appearing determined, yielded undetermined variables. In one example involving three equations:

$$x + y + z = 3 \tag{2.1}$$

$$2x + 2y + 2z = 6 \tag{2.2}$$

$$x - y + z = 1 \tag{2.3}$$

Euler observed that one equation might be expressible as a combination of others, indicating redundancy. Without the formal notion of linear dependence, Euler described such situations using expressions like "an equation is contained in the others," a qualitative observation now interpreted as linear dependence.

To distinguish Euler's approach from contemporary terminology, the term *inclusive dependence* can be used. It reflects how Euler perceived embedded relationships among equations without explicitly invoking vector space theory.

Euler's insights were not limited to algebraic systems. His exploration of Latin and Greco-Latin squares—such as the *36 officers problem*—demonstrates how combinatorial constraints also imply a form of structural dependence. For instance, in a 3×3 Graeco-Latin square:

Aα	Bβ	Cγ
Bγ	Cα	Aβ
Cβ	Aγ	Bα

Table 2.1: Example of a 3×3 Graeco-Latin square, illustrating Euler's concept of structural pairing in combinatorial design. Adapted from Euler's 1782 study on Latin squares (Encyclopædia Britannica, 2025).

Each row and column contains all Latin and Greek letters without repetition of ordered pairs. This combinatorial consistency echoes the kind of structural embedding Euler considered in algebraic contexts.

Although Euler lacked the symbolic tools of modern linear algebra, his conceptual analysis of equation interdependence laid critical groundwork for future definitions of linear dependence, coordinate systems, and matrix representations.

2.3.3 Gauss's Contributions to Linear Systems and Coordinate Representations

Carl Friedrich Gauss (1777–1855) made pivotal contributions to the development of methods for solving systems of linear equations and introduced fundamental concepts related to coordinate systems and transformations. Although Gauss did not use the formal language of matrices, his work significantly influenced the development of linear algebra.

One of Gauss's key innovations was the systematic use of what is now known as Gaussian elimination—a method for reducing a system of linear equations to a simpler, equivalent system in row echelon form. This process streamlines the solution of linear systems by eliminating variables step-by-step. While similar techniques appeared in earlier Chinese mathematics, which we introduced in section 2.2.1, Gauss formalized the approach and applied it rigorously across various scientific domains, including astronomy and geodesy (Hawkins, 1977; Katz, 2009).

Gauss also advanced the method of least squares, which is used to find the best-fit solution to overdetermined systems. This technique relies on projecting data vectors onto subspaces, a concept closely related to orthogonality and inner product spaces. Though not expressed in matrix terms, Gauss's formulation prefigured modern ideas of vector projection and orthogonal decomposition (G. W. Stewart, 1998).

Example: Gaussian Elimination in a Simple System

Consider the system:

$$2x + 3y - z = 5 \tag{2.4}$$

$$4x + 4y - 3z = 3 \tag{2.5}$$

$$-2x + 3y + 2z = 4 \tag{2.6}$$

This system can be written as an augmented matrix:

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 5 \\ 4 & 4 & -3 & 3 \\ -2 & 3 & 2 & 4 \end{array}\right].$$

Applying Gaussian elimination, one can reduce this matrix to row echelon form and solve the system. This example reflects Gauss's methodical approach to solving equations and anticipating linear transformations.

In addition to his algebraic contributions, Gauss's work in differential geometry introduced intrinsic coordinate systems on curved surfaces. His 1827 work, *Disquisitiones Generales Circa Superficies Curvas*, laid the foundation for local coordinate frames—concepts essential to modern notions of basis and change of basis (Gauss, 1827).

Although Gauss did not explicitly define vector spaces or matrices, his insights bridged arithmetic procedures and abstract representation. His methods shaped later developments in matrix theory, basis transformations, and coordinate geometry. The cognitive structures and problem-solving techniques he developed remain foundational in both theory and pedagogy.

Lagrange's Algebraic Legacy in Linear Transformation

Joseph-Louis Lagrange (1736–1813) significantly influenced the development of ideas foundational to matrix representation, particularly in the context of algebraic structures and linear

transformations. Although his work predated the formal establishment of matrix theory, Lagrange laid the groundwork for key concepts such as substitution, diagonalization, and coordinate systems.

One of Lagrange's notable contributions was his analysis of polynomial equations and their roots. In his 1770–1771 work on the solution of algebraic equations, Lagrange examined how roots behaved under permutations and introduced what would now be described as linear substitutions. These substitutions involved replacing variables with linear combinations of others, conceptually similar to the application of linear transformations in vector spaces (Katz, 2009).

Lagrange also pioneered methods for simplifying expressions through rational invariants and symmetric functions. His techniques often involved changing variables to achieve simpler or more symmetric forms, an early analogue of what modern linear algebra terms a *change of basis*. Though he did not have access to matrix notation, his approach demonstrated a structural understanding of how coordinate systems can clarify mathematical relationships (Wussing, 2007).

In the study of quadratic forms, Lagrange demonstrated that such expressions could be reduced to sums of squares through appropriate transformations. This process anticipated the diagonalization of symmetric matrices and the computation of eigenvalues—central procedures in linear algebra. His insight that transformations could simplify the structure of a problem foreshadowed the use of orthogonal bases and matrix representations to analyze linear systems (Stillwell, 2010).

Furthermore, in his treatise *Mécanique Analytique* (1788), Lagrange reformulated Newtonian mechanics using generalized coordinates. These coordinates enabled the expression of mechanical systems as linear combinations of fundamental components. This methodology, grounded in abstract representation, aligns closely with the modern use of coordinate vectors and transformation matrices in both physics and mathematics (Parshall, 1988).

Although Lagrange did not develop the language of matrices or vector spaces, his analytical strategies and structural thinking directly influenced the evolution of linear algebra. His techniques of substitution, reduction, and generalization laid the conceptual foundation for later formalizations by mathematicians like Grassmann, Cayley, and Peano.

Lagrange's legacy endures in contemporary instruction. When students perform basis changes, simplify quadratic forms, or represent transformations using matrices, they are engaging with ideas rooted in Lagrange's pioneering work. His contributions serve as a bridge between classical algebra and modern linear theory, enriching both the historical context and the pedagogical understanding of matrix representation.

Taken together, the contributions of Chinese mathematicians, Euler, Gauss, and Lagrange trace a developmental arc from practical algorithms to increasingly abstract and structural formulations of linear relationships. These figures introduced key ideas—such as elimination, dependence, transformation, and substitution—that laid the groundwork for more formal definitions of matrices, vector spaces, and coordinate systems. The next section continues this trajectory by examining how later mathematicians formalized solution structures and criteria for consistency in linear systems.

2.3.4 Smith and Dodgson: From Linear Combination to Independence

The intellectual contributions of Henry J. S. Smith and Charles L. Dodgson have had a lasting impact on the theory and pedagogy of matrix representation. While Smith's work focused on linear combinations and the structure of solution spaces, Dodgson extended this foundation by developing determinant-based approaches for evaluating the consistency of linear systems.

Smith on Homogeneous and Non-Homogeneous Systems

According to Katz (2009), Henry J. S. Smith (1826–1883) made fundamental advances in understanding both homogeneous and non-homogeneous linear systems. Central to his work was the concept of the *index of indeterminates*, defined as the difference $n - m$ between the number of unknowns (n) and the number of linearly independent equations (m) (Smith, 1984). This index quantifies the degrees of freedom in a system and directly connects to the dimension of the solution space.

For homogeneous systems of the form $Ax = 0$, Smith introduced the notion of a *complete set of independent solutions*. These $n - m$ linearly independent solutions form a basis for the solution space, enabling every solution to be expressed as a linear combination of basis vectors. He further demonstrated that determinants of order m associated with the coefficient matrix must be nonzero to guarantee the independence of the solution set.

For non-homogeneous systems $Ax = b$, where $b \neq 0$, Smith proposed a two-stage solution process. First, a particular solution x^* is identified. Then, the general solution is written as $x = x^* + x_h$, where x_h is a solution to the corresponding homogeneous system. This decomposition separates the fixed and free components of the solution.

Example: Solving a Homogeneous System Consider the system:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 1 \end{bmatrix}, \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Solving $Ax = 0$, we obtain:

$$x_1 + 2x_2 + 3x_3 = 0$$
$$x_2 + x_3 = 0$$

From the second equation, $x_2 = -x_3$. Substituting into the first:

$$x_1 + 2(-x_3) + 3x_3 = x_1 - 2x_3 + 3x_3 = x_1 + x_3 = 0 \quad \Rightarrow \quad x_1 = -x_3$$

Letting $x_3 = t$, the solution becomes:

$$x = t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

The set of all solutions to $Ax = 0$ forms the null space (or kernel) of A, which consists of all vectors that are mapped to the zero vector by A. This single vector spans the null space of A, and since $\text{rank}(A) = 2$, the dimension of the solution space—i.e., the number of free variables—is $3 - 2 = 1$, confirming Smith's framework.

Limitations and Further Development

While Smith addressed systems with full-rank coefficient matrices, he did not provide a comprehensive analysis of systems with dependencies among the equations—cases where $\text{rank}(A) < m$. This analytical gap was later filled by Charles L. Dodgson.

Dodgson's General Theorem for Linear Systems

Charles L. Dodgson (1832–1898), better known as Lewis Carroll, expanded upon Smith's work by examining systems in which not all equations are linearly independent. In his 1867 treatise *An Elementary Treatise on Determinants*, Dodgson introduced determinant-based techniques for diagnosing both consistent and inconsistent systems (Dodgson, 2023; Nature, 2015).

Central to Dodgson's analysis was the concept of *rank*, which he defined as the highest order of a nonzero minor in a matrix—a definition consistent with the notion of the largest square submatrix with a nonzero determinant. While this differs from the more common modern definition of rank as "The *rank* of a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is defined as the *dimension of the range* (or image) of T. That is,

$$\text{rank}(T) = \dim(\text{Range}(T)) = \dim\left(\{T(x) \mid x \in \mathbb{R}^n\}\right).$$

(Friedberg & Insel, 2019, p.151) This definition reflects the number of linearly independent vectors in the image of T, and is equivalent to the dimension of the column space of any matrix that represents T. " the dimension of the row or column space, the two are mathematically equivalent. He applied this concept to the augmented matrix $(A \mid b)$, where A is the coefficient matrix and

b is the column of constants. If the ranks of A and $(A \mid b)$ are equal, the system is consistent; otherwise, the system is inconsistent.

Dodgson's key result—his *general theorem*—unified the treatment of determined, underdetermined, and inconsistent systems. He recognized that when the system is consistent but underdetermined (i.e., $\text{rank}(A) < n$, where n is the number of variables), the solution set contains infinitely many solutions. In such cases, one or more variables are *free*, and their arbitrary assignment determines the rest of the solution. This foreshadowed the modern concept of solution spaces, nullity, and parametric vector forms.

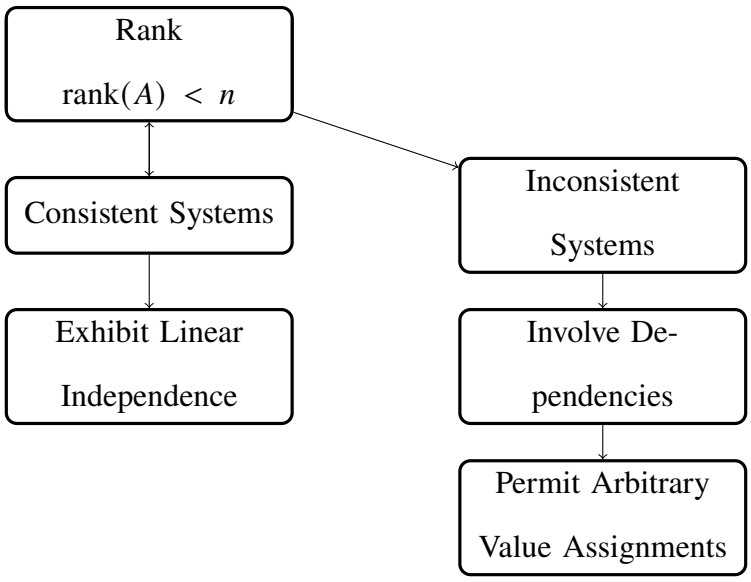


Figure 2.5: Visualization of Dodgson's General Theorem: Relationships in Linear Systems

Dodgson's insight also clarified the role of free variables in systems with zero determinants. When the rank of the coefficient matrix is less than the number of variables, certain variables are not determined by the system and can assume arbitrary values. These degrees of freedom allow for infinitely many solutions expressible as linear combinations of basis vectors in the null space.

In sum, the work of Smith and Dodgson forms a critical link between algebraic theory and pedagogical clarity. Their respective emphases on solution structure and consistency criteria continue to inform modern instruction in linear algebra, particularly in the teaching of matrix rank, null spaces, and general solution strategies.

2.3.5 The Formal Development of Vector Spaces: Grassmann and Peano

Hermann Grassmann (1809–1877) and Giuseppe Peano (1858–1932) made transformative contributions to the conceptual and formal development of vector spaces. While Grassmann introduced the symbolic and geometric underpinnings of linear structure, Peano developed rigorous axioms that abstracted these ideas into algebraic form, allowing vector spaces to be applied broadly across mathematics.

Grassmann's Contribution: From Parallelograms to Modern Vector Spaces

Grassmann's foundational work, *Die lineale Ausdehnungslehre* (The Theory of Linear Extension), published in 1844 and revised in 1862, presented a symbolic framework for representing spatial magnitudes in arbitrary dimensions (Grassmann, 1862). His theory included the concepts of linear independence, linear combinations, and dimensionality, predating the formal definition of a vector space by several decades.

Grassmann introduced linearly independent units $\epsilon_1, \epsilon_2, \ldots, \epsilon_n$, and expressed quantities as linear combinations:

$$\sum \alpha_i \epsilon_i + \sum \beta_i \epsilon_i = \sum (\alpha_i + \beta_i) \epsilon_i,$$

where α_i and β_i are scalars. This formulation captures both vector addition and scalar multiplication, anticipating the vector space operations formalized later.

His framework allowed extensive quantities to be expressed as algebraic objects, independent of any specific geometric intuition. This abstraction enabled Grassmann to reason about high-dimensional spaces long before such ideas were accepted in mainstream mathematics.

Modern Formalization of Vector Spaces

A vector space is a mathematical structure that formalizes the notion of linear combination,

generalizing the properties of Euclidean spaces. Formally, a *vector space* V over a field $\mathbb{F}$ (such as $\mathbb{R}$ or $\mathbb{C}$) is a set equipped with two operations:

- **Vector addition**, defined as a function $+ : V \times V \to V$, which takes any two vectors in V and returns their sum;
- **Scalar multiplication**, defined as a function $\cdot : \mathbb{F} \times V \to V$, which takes a scalar from $\mathbb{F}$ and a vector from V, and returns their product.

These operations must satisfy ten axioms, including associativity and commutativity of vector addition, distributivity of scalar multiplication over both field addition and vector addition, and the existence of additive identity and additive inverses.

This formal structure extends beyond the familiar coordinate space $\mathbb{R}^n$, encompassing function spaces, polynomial spaces, and spaces of matrices—each of which can be treated as a vector space under appropriate definitions of addition and scalar multiplication.

Semiotic and Educational Implications

Grassmann's innovations hold pedagogical value. His use of symbolic language to describe geometric and algebraic structures supports the development of students' conceptual understanding. By interpreting vectors as structured combinations of fundamental elements, students gain insight into dimensionality, representation, and operations across various mathematical registers.

Grassmann's System–Solution Duality in Linear Algebra

Through the application of refined symbolic language, Grassmann's work reveals a dual perspective on linear equation systems: the algebraic structure encoded in the coefficient matrix and the geometric nature of the set of solutions.

Consider the general system:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned} \tag{2.7}$$

The corresponding coefficient matrix is:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

Key Structural Properties Through the Refined Symbols

- **Solvability**: The rank of A determines whether solutions exist and how many.
- **Uniqueness**: If A is square and invertible ($\det(A) \neq 0$), the system has a unique solution.
- **Consistency**: The system $Ax = b$ is consistent if $\text{rank}(A) = \text{rank}([A|b])$.

This matrix-based perspective, foreshadowed by Grassmann, emphasizes how symbolic algebraic form and geometric interpretation work in tandem. Instructionally, this duality helps students move beyond procedures to reasoning about the structure and meaning of solutions.

Peano's Formalization of Vector Spaces

Building on Grassmann's work, Giuseppe Peano (1858-1932) introduced a formal set of axioms that defined vector spaces in purely algebraic terms. His axiomatization established vector spaces as abstract algebraic structures, decoupled from their geometric origins (Katz, 2009).

Peano's formulation clarified properties such as associativity, distributivity, and scalar multiplication, creating a rigorous foundation that enabled the expansion of vector space theory into new domains—including function spaces and infinite-dimensional spaces.

For instance, when students work in the space of polynomials and use the basis $\{1, x, x^2\}$, they are operating within a vector space as formalized by Peano. These spaces obey the same axioms as $\mathbb{R}^n$, despite the different nature of the elements.

Conclusion

Together, Grassmann and Peano provided the conceptual and formal foundations of modern linear algebra. Grassmann introduced symbolic representations and intuitive operations in arbitrary dimensions, while Peano's axioms systematized these ideas into a coherent algebraic framework. Their contributions continue to shape how linear algebra is taught and understood, linking historical innovation with contemporary mathematical education.

2.3.6 Formalizing Matrices and Linear Operators: Sylvester and Cayley

James Joseph Sylvester (1814–1897) and Arthur Cayley (1821–1895) played central roles in establishing the formal language and algebraic structure of matrices. Their work marked a transition from intuitive manipulation of linear systems to the development of a systematic matrix theory that underpins modern linear algebra.

Sylvester coined the term *matrix* to describe a rectangular array of elements from which determinants could be derived. He also contributed to the development of concepts such as minors, determinants of submatrices formed by deleting specific rows and columns from a larger matrix, and invariant theory, both of which are foundational in understanding matrix structure and rank (Hawkins, 1977). By identifying patterns within arrays of coefficients, Sylvester developed a more structured and symbolic approach to linear systems.

Arthur Cayley expanded upon Sylvester's ideas by developing the algebra of matrices. In

his landmark 1858 paper, Cayley defined matrix addition, scalar multiplication, and matrix multiplication—operations now fundamental in all linear algebra curricula. He introduced the *Cayley-Hamilton Theorem*, which states that every square matrix satisfies its own characteristic polynomial (Cayley, 1858). This theorem links matrix theory to polynomial algebra and provides tools for diagonalization and finding eigenvalues.

Matrix Algebra and Transformations

Cayley's formalization of matrix operations allowed mathematicians to represent linear transformations as algebraic objects. Given a linear transformation $T : V \to V$ on a finite-dimensional vector space V, and a basis B, the transformation can be uniquely represented by a matrix $[T]_B$. The entries of this matrix describe how the basis vectors are mapped under T.

$$[T]_B = \begin{bmatrix} | & | & & | \\ T(\mathbf{v}_1) & T(\mathbf{v}_2) & \cdots & T(\mathbf{v}_n) \\ | & | & & | \end{bmatrix}$$

This representation connects abstract mappings to concrete numerical procedures, enabling the use of algorithms for computation and analysis.

Educational Relevance

Students encounter Cayley's and Sylvester's ideas when learning about matrix multiplication, determinant properties, and eigenvalue computation. When solving systems of equations, applying linear transformations, or computing powers of matrices using the Cayley-Hamilton Theorem, students are drawing on the legacy of this formalization.

Moreover, these symbolic representations allow learners to recognize patterns, make generalizations, and connect algebraic structure with geometric intuition. The formal language developed by Sylvester and Cayley supports both abstract reasoning and computational fluency.

Historical Significance

The collaborative and complementary work of Sylvester and Cayley marked the beginning of matrix theory as a distinct and powerful field. By shifting the emphasis from individual equations to structured arrays and algebraic systems, they laid the groundwork for future developments in spectral theory, canonical forms, and operator theory.

Their contributions form a critical chapter in the evolution of linear algebra, transforming it from a collection of ad hoc techniques into a unified, symbolic discipline with wide-ranging applications across mathematics, science, and engineering.

2.3.7 David Hilbert's Abstraction of Inner Product Spaces and Functional Representations

David Hilbert (1862–1943) revolutionized linear algebra by extending its concepts into infinite-dimensional settings. His introduction of *Hilbert spaces* provided a rigorous framework for understanding orthogonality, projection, and spectral decomposition in spaces of functions. These abstractions are fundamental in modern functional analysis, quantum mechanics, and signal processing (Reid, 1970).

A Hilbert space is a complete inner product space. It generalizes the geometric notions of length and angle from finite-dimensional Euclidean spaces to spaces of functions, ensuring convergence and stability in infinite-dimensional analysis. This extension allows mathematicians and scientists to interpret functions as vectors and operations such as integration and differentiation as linear transformations.

Inner Product and Projection in Hilbert Space

The defining features of a Hilbert space are the presence of an inner product $\langle f, g \rangle$, which induces a norm and supports geometric interpretation, and completeness concerning the resulting norm.

For example, consider the Hilbert space $H = L^2([0, 1])$, the space of square-integrable

functions on the interval $[0, 1]$. The inner product is defined as:

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx.$$

To orthogonally project a function $f(x) = x$ onto the subspace spanned by $g(x) = 1$, we compute:

$$P(f) = \frac{\langle f, g \rangle}{\langle g, g \rangle} \cdot g = \frac{\int_0^1 x \cdot 1\,dx}{\int_0^1 1 \cdot 1\,dx} = \frac{1}{2}.$$

Thus, the projection of $f(x) = x$ onto the constant function space is $P(f) = \frac{1}{2}$, demonstrating orthogonal projection in an infinite-dimensional space.

From Finite to Infinite Dimensions

Hilbert's abstraction bridged finite-dimensional linear algebra with broader mathematical structures. Just as a vector in $\mathbb{R}^n$ can be expressed as a linear combination of basis vectors, a function in L^2 can be expressed as a sum of orthonormal basis functions, such as those from a Fourier series.

Hilbert spaces also enabled the formulation of spectral theory, where operators (generalized matrices) are studied through their eigenfunctions and eigenvalues. This theory underpins quantum mechanics, where observables are represented by self-adjoint operators on Hilbert spaces.

Pedagogical Relevance

For students, Hilbert's contributions provide a conceptual bridge from linear transformations in $\mathbb{R}^n$ to functional transformations in L^2 or other infinite-dimensional spaces. When learning about orthogonality, least squares, or eigenfunction expansions, students are enacting Hilbert's vision of linear structure in abstract spaces.

Hilbert's ideas continue to influence modern curricula in applied mathematics, physics, and engineering. His abstraction of inner product spaces and emphasis on orthogonality, completeness, and projection remain foundational for understanding both the theory and application of linear

transformations.

Further, Hilbert's extension of linear algebra into the infinite-dimensional realm reshaped the field and broadened its reach. His work laid the foundation for functional analysis and inspired entire branches of mathematical research. By formalizing the notion of inner product spaces and enabling analysis in abstract settings, Hilbert ensured that linear algebra would remain central to modern mathematics.

2.3.8 Basis Change and Matrix Symmetry: Frobenius and Weyl

The work of Ferdinand Frobenius (1849–1917) and Hermann Weyl (1885–1955) further developed the structural understanding of matrices, particularly with respect to basis change and symmetry. Their contributions influenced not only the theoretical progression of linear algebra but also the pedagogical strategies for teaching transformation and representation in abstract vector spaces.

Frobenius: Canonical Forms and Invariant Subspaces

Frobenius is well known for his work on canonical forms of matrices and the theory of invariant subspaces. His introduction of the *Frobenius normal form* provided a systematic way to classify linear transformations up to similarity (Marchisotto & Zakeri, 1994). This form reveals the internal structure of a matrix by transforming it into a block matrix, capturing essential features such as minimal polynomials and invariant factors.

One of Frobenius's key insights was the identification of invariant subspaces, subspaces that remain unchanged under the action of a linear transformation. These subspaces provide the foundation for decomposing vector spaces and understanding how matrices act relative to different bases. His work clarified how the structure of a matrix is preserved under similarity transformations, laying the groundwork for the study of diagonalization and the Jordan canonical form.

When students diagonalize a matrix or reduce it to its Jordan form, they are applying Frobenius's ideas. These canonical forms offer powerful tools for simplifying computations and understanding the behavior of linear systems.

Weyl: Symmetry, Group Representations, and Basis Dependence

Hermann Weyl extended the notion of matrix symmetry by connecting it to group theory and the representation of abstract algebraic structures. His application of matrices to continuous transformation groups, particularly in quantum mechanics and theoretical physics, helped establish the importance of *basis-dependent representations* (Weyl, 1952).

Weyl demonstrated that while a linear transformation remains fixed, its matrix representation varies with the choice of basis. This insight is critical in understanding concepts such as matrix similarity and equivalence. Weyl's work showed that the choice of basis is not merely a matter of convenience but a reflection of the underlying structure of the space and the transformation itself. In physics, Weyl's insights supported the development of quantum theory, where operators representing observables are expressed differently depending on the chosen basis (e.g., position versus momentum space). In mathematics, his group-theoretic approach to symmetry laid the foundation for modern representation theory. Frobenius and Weyl's work underscores the importance of basis choice and matrix transformation in both theoretical and applied contexts. Students learn these ideas when they explore the diagonalization of matrices, the role of eigenvalues and eigenvectors, and the geometric interpretation of similarity. By understanding that matrix representations depend on bases while the transformation itself does not, students gain a deeper appreciation of abstraction and generalization. This perspective also helps students navigate between concrete computations and abstract structural reasoning.

2.4 Summary Table

Table 2.2: Matrix Representation: From History to Modern Concepts

Mathematician (Years)	**Contribution**	**Modern Concept**
Ancient Chinese Mathematicians (c. 200 BCE)	Elimination of variables using counting rods as recorded in *The Nine Chapters*	Gaussian elimination; systems of equations
Leonhard Euler (1707–1783)	Inclusive dependence in linear systems; Latin and Graeco-Latin squares	Linear dependence; combinatorial structure
Joseph-Louis Lagrange (1736–1813)	Linear substitutions; diagonalization of quadratic forms; generalized coordinates	Linear transformations; change of basis; diagonalization
Carl Friedrich Gauss (1777–1855)	Gaussian elimination; least squares method; intrinsic coordinates in geometry	Row operations; orthogonal projection; local coordinates
Henry J. S. Smith (1826–1883)	Index of indeterminates; complete basis of solutions for homogeneous systems	Rank-nullity theorem; solution space dimension
Charles L. Dodgson (1832–1898)	General theorem of system consistency; determinant simplification method	Rank; solvability of linear systems; Dodgson condensation
James J. Sylvester (1814–1897)	Coined the term "matrix"; minors and determinants	Matrix theory; structural matrix properties
Arthur Cayley (1821–1895)	Matrix operations; Cayley-Hamilton theorem; matrix algebra	Matrix multiplication; characteristic polynomial; eigenvalues
Hermann Grassmann (1809–1877)	Exterior algebra; concept of n-dimensional space; symbolic representation	Vector spaces; linear combinations; dimension
Giuseppe Peano (1858–1932)	Formal axioms of vector spaces; abstraction from geometry	Axiomatic vector space theory; structural definition
Ferdinand Frobenius (1849–1917)	Canonical forms; invariant subspaces; matrix similarity	Frobenius normal form; matrix diagonalization; eigenstructure
David Hilbert (1862–1943)	Hilbert spaces; orthogonal projection in function spaces; spectral theory	Functional analysis; infinite-dimensional vector spaces
Hermann Weyl (1885–1955)	Basis-dependent representation; matrix groups; transformation symmetry	Change of basis; group representations; linear symmetry

2.5 Historical Development of Matrix Notation and Symbolic Conventions

The notational systems we now take for granted have emerged from the collective contributions of many mathematicians, each of whom refined how we write, interpret, and teach matrix-related ideas.

Hermann Grassmann (1809–1877) was one of the earliest to formalize vector operations in higher-dimensional spaces. His 1844 work introduced the exterior product and advanced the use of symbols to express linear combinations, setting the stage for modern vector space theory (Flament, 2001). Grassmann demonstrated that notation was not merely a communication device—it was a tool for abstraction and generalization.

Arthur Cayley (1821–1895) made a major contribution to symbolic mathematics by intro-

ducing matrix multiplication and defining the characteristic equation. He standardized the use of capital letters for matrices and helped distinguish matrices from vectors and scalars in a systematic way, shaping the conventions still used today (Katz, 2009).

James Joseph Sylvester (1814–1897), who coined the term "matrix," emphasized the need for symbolic structure to articulate linear transformations. His introduction of determinant-related terminology helped establish matrices as central tools in algebra and geometry.

Henry John Stephen Smith (1826–1883) advanced canonical forms and clarified symbolic distinctions in determinant theory. His notation helped differentiate between matrix types and made abstract properties more accessible to learners.

Charles Dodgson (Lewis Carroll, 1832–1898), although more widely known as an author, also made significant contributions to simplifying determinant calculations. His innovations, such as Dodgson condensation, demonstrate how thoughtful notation can streamline calculations and clarify underlying mathematical ideas.

Ferdinand Frobenius (1849–1917) formalized key ideas in matrix theory, including the definition of rank and matrix similarity. His symbolic contributions supported the standardization of expressions used in eigenvalue theory and diagonalization—crucial topics in linear transformation and vector space analysis (Frobenius, 1879).

Together, these mathematicians crafted a symbolic language that enhanced clarity, supported generalization, and enabled more efficient problem-solving. Their cumulative contributions shaped both the content and pedagogy of matrix theory.

2.6 Conclusion

This chapter has traced the historical and epistemological development of matrix representation from ancient Chinese elimination methods to the abstract formalism of vector spaces and transformations. By examining the contributions of pivotal mathematicians—such as Euler, Gauss, Lagrange, Smith, Grassmann, Peano, Cayley, and Hilbert—we have seen how ideas of linear

dependence, basis transformation, and symbolic representation evolved over time.

These developments are not only significant in the history of mathematics but also shape how students engage with matrix concepts today. Understanding the historical context and conceptual foundations of matrix representation allows for a more nuanced and pedagogically effective approach to teaching linear algebra.

Chapter 3

The Onto-Semiotic Approach (OSA)

Building on the historical development outlined in Chapter 2, this chapter examines philosophical and epistemological perspectives, particularly how the OSA sheds light on student interactions with these evolving mathematical ideas. Here, we introduce the OSA as a theoretical framework in mathematics education, focusing on its ability to bridge philosophical foundations with the real challenges students face when learning complex concepts. By examining how students develop an understanding of matrix representation of linear transformations, we demonstrate how the OSA provides meaningful insights into cognitive processes and fosters a deeper comprehension of abstract mathematical structures.

Section 1 examines six major philosophical perspectives that have shaped mathematical logic in education. These include Platonism, Materialism, Intuitionism, Nominalism, Empiricism, and Anthropology. Each philosophy provides distinct logic, strengths, and boundaries on the nature and existence of mathematical entCities, as well as how mathematical concepts are perceived, developed, and applied, particularly in the educational context. We will analyze the strengths and limitations of these philosophies in the context of studying vector spaces, coordinate vectors, and linear transformations, highlighting their implications for mathematical reasoning and teaching approaches related to matrix representation.

Section 2 exposes some typical epistemological obstacles and dimensions that students face in their learning journey, highlighting the complexities inherent in understanding abstract mathematical concepts. It will delve into how misconceptions and rigid beliefs about knowledge can impede students' ability to engage with material deeply, thereby fostering a fragmented view of their educational experiences. Additionally, we will address the various levels of analysis that can be employed to confront these learning challenges, which range from understanding individual learner

needs to examining institutional influences and systemic factors. By integrating these analytical perspectives, educators can develop targeted strategies that promote a more cohesive and effective learning environment for all students. Sections 1 and 2 will serve as a foundation and demand for the OSA, which we will explore in Section 3.

Section 3 delves into the OSA. This theoretical framework integrates various philosophical perspectives and emphasizes the relationship between ontological mathematical objects and the semiotic processes involved in their representation. This approach addresses the complexities of mathematical cognition and instruction, incorporating the strengths of all six philosophical perspectives discussed in Section 1. The OSA is particularly relevant for understanding how students and educators interact with mathematical concepts and reasoning, such as those found in matrix theory. Furthermore, we will outline the five paired dualities of OSA, explaining how they operate in tandem to enhance teaching and learning within mathematical contexts. This examination will highlight the OSA's considerable value in forming a comprehensive model that elucidates how students engage with mathematical knowledge.

In conclusion, this chapter hopes to highlight the pivotal role of the OSA in enriching mathematics education. By offering a robust framework that clarifies student understanding and informs effective teaching practices, the OSA reveals significant pathways for enhancing the learning experience. By integrating philosophical perspectives, theoretical foundations, and practical applications, we aim to illustrate OSA's transformative potential in shaping the teaching and learning of complex mathematical concepts. This approach will improve educational outcomes in mathematics and ultimately address the evolving demands of contemporary mathematics education.

3.1 Philosophical Foundations of Mathematical Objects

Mathematics, often described as the language of the universe, raises profound questions about the nature of mathematical objects. The six major philosophies of mathematics offer diverse perspectives on this question, each contributing to our understanding of mathematical truth by addressing different aspects of existence, abstraction, and application. Engaging with these

viewpoints can deepen our appreciation for the complexities of mathematical inquiry and its philosophical foundations. However, each philosophy carries inherent limitations, which are challenged by evolving educational needs, advancements in cognitive science, and the increasing role of technology in mathematical instruction. For instance, in linear algebra, do concepts like vector spaces, coordinate vectors, linear transformations, and matrix representations exist independently of human cognition, or are they construct-shaped through abstraction, symbolic representation, and logical reasoning? Based on the research, these concepts emerge from axiomatic foundations, evolve through historical developments, and are reinforced through formal education, where definitions, theorems, and proofs serve as vehicles for their transmission.

Understanding how mathematical objects are constructed, shaped, and taught from different perspectives is essential for refining pedagogical approaches and fostering deeper comprehension in learners.

3.1.1 Platonism: Abstract Existence of Absolute Mathematical Objects

Platonism asserts that mathematical objects exist independently of human cognition; they are discovered rather than created. From this perspective, mathematical truths are absolute and unchanging, existing in a non-physical, timeless realm (Study.com, 2023). This viewpoint provides an authoritative foundation for mathematics, aligning with its objectivity and universality. In the context of matrix representation, Platonism suggests that all fundamental concepts, such as vector spaces, coordinate vectors, and linear transformations, exist as abstract entities independent of our perception.

While this perspective reinforces the logical structure and consistency of mathematical theory, it also presents pedagogical challenges. The fundamental epistemological challenge in Platonism—namely, how humans can access non-empirical mathematical entities—bears a striking resemblance to the cognitive difficulties students encounter when learning abstract mathematical concepts. As Benacerraf (1973) famously argued, the tension between the abstract nature of mathematical truth and our means of knowing it poses deep philosophical concerns. Similarly,

educational researchers such as Sfard (2008) and Radford (2021) have emphasized that students' difficulties in grasping abstract concepts often stem from challenges in meaning-making through semiotic and discursive means rather than empirical experience. Without clear physical representations of vector spaces, coordinate vectors, and linear transformations, students may struggle to develop an intuitive understanding of these structures.

One way to address this challenge lies in the need for using coordinate systems as cognitive bridges between abstract vector spaces and their concrete representations. By introducing basis in familiar spaces such as $\mathbb{R}^2$ and $\mathbb{R}^3$, students gain an intuitive foundation that can later be extended to more abstract settings (Feldman, 2012). This approach aligns with the pedagogical strategy of scaffolding, wherein learners progress from concrete, empirical examples to purely theoretical constructs. Emphasizing the role of coordinate vectors in structuring vector spaces helps mitigate difficulties associated with grasping non-empirical mathematical entities (Li & Schoenfeld, 2019).

However, the concept of linear transformations further illustrates the challenges of interacting with abstract structures. These transformations, which define mappings between vector spaces, are fundamental to matrix representation. Yet, their significance is often obscured by the lack of direct real-world referents, the actual objects, ideals, or entities. For instance, when vector spaces are defined in terms of polynomial functions or matrices, their transformations become even more detached from the tangible experience. This difficulty parallels a key critique of Platonism: if mathematical objects exist independently of human cognition, how do we meaningfully interact with them?

By acknowledging both the strengths and limitations of Platonism in mathematical education, educators can develop more effective teaching strategies. While Platonism provides a rigorous framework for understanding the objectivity of mathematical structures, a purely Platonic approach may hinder students' ability to internalize abstract concepts. Therefore, the demand for bridging the gap between abstraction and intuition and experience through coordinate systems, visual representations, and scaffolding techniques is necessary. While Platonism provides a robust theoretical framework, Materialism - which we will bring in the following subsection - offers a contrasting

perspective grounded in empirical experience (Study.com, 2023).

3.1.2 Materialism: Mathematics Objects as Products of the Physical World

As opposed to Platonism, which holds that mathematical objects exist in an abstract, non-physical space, Materialism rejects the independent existence of mathematical abstraction. Instead, it asserts that all phenomena, including ideas and consciousness, originate from physical interactions and empirical observations. According to this perspective, mathematical concepts, such as vector spaces, coordinate vectors, and linear transformations, consist solely of physical objects (Wikipedia contributors, n.d.). Materialism underlines that mathematical ideas arise from human experience with the physical world, implying that mathematical objects do not exist apart from human consciousness; rather, they are tools created to describe and model real-world phenomena (Fu, n.d.) From an educational standpoint, materialism advocates an empirical and hands-on approach to learning mathematics. It emphasizes sensory experiences, concrete representations, and practical applications as essential components of mathematical learning. When applied to the study of vector spaces, coordinate vectors and linear transformations, materialism supports integrating visual, physical, and applied settings to enhance student comprehension. Although this approach offers a grounded and easily accessible foundation for learning, it faces restrictions, especially when addressing abstract mathematical structures that lack direct physical counterparts.

By connecting mathematical ideas to observable real-world phenomena, a materialist perspective on matrix representation learning fosters intuitive understanding. For example, vector spaces can be introduced through physical contexts such as forces in physics, displacement vectors in mechanics, or velocity fields in fluid dynamics. Before engaging with abstract formalism, students can develop an intuitive grasp of vector space properties by recognizing how vectors possess magnitude and direction within tangible systems. Materialism promotes the use of interactive tools, physical models, and graphical representations to reinforce learning. For instance, grids, coordinate planes, and three-dimensional models can help students visualize the role of basis and coordinate representations. Software tools like GeoGebra, MATLAB, and Python further support

this approach by allowing students to manipulate and explore vector spaces dynamically. Materialism aligns well with applied mathematics and physics disciplines, where practical experience plays a crucial role. Linear transformations, which may seem abstract when introduced purely through algebraic mappings, become more meaningful when presented through applications such as image transformations in computer graphics, rotations, and reflections in physics, or signal processing in engineering. By emphasizing real-world relevance, materialist teaching strategies can make mathematical concepts more engaging and accessible. However, this perspective faces challenges when addressing topics that lack clear physical interpretations. Concepts such as vector spaces over function fields, infinite-dimensional spaces, and abstract algebraic structures do not always lend themselves to immediate real-world analogies (Sepúlveda Obreque & Lezama Andalon, 2023). A strict materialist approach may lead students to over-rely on physical intuition, making it harder for them to transition from specific cases to more general, abstract reasoning.

Furthermore, while materialism facilitates concrete understanding, its emphasis on direct experience can sometimes hinder engagement with purely symbolic and formal reasoning. For example, while coordinate vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$ can be easily visualized, however, the transition from lower-dimensional vector spaces to higher-dimensional vector spaces challenges direct geometric representation, requiring students to rely on algebraic and logical reasoning. Similarly, while linear transformations are fundamentally defined by their algebraic properties, students who depend too much on physical intuition may struggle to grasp abstract function spaces or general transformations.

A rigid materialist view could also lead to the belief that mathematical structures are merely human constructs rather than objective truths. While this perspective fosters adaptability and application-based learning, it may undermine an appreciation for mathematical rigor and the intrinsic structural properties that define vector spaces and transformations, regardless of their physical applications. Students who see mathematics purely as a tool for modeling physical reality may fail to recognize its broader theoretical coherence and logical necessity (Sepúlveda Obreque & Lezama Andalon, 2023).

3.1.3 Intuitionism: Mathematics as an Intuitive Sense of the Mind

Intuitionism, founded by L.E.J. Brouwer, offers a distinct perspective on mathematics, positing that mathematical constructs are products of the human mind rather than external existing entities (*Intuitionism in the Philosophy of Mathematics*, 2008). This philosophy underscores the importance of human intuition in shaping mathematical knowledge, challenging the traditional view that mathematical objects exist independently of our understanding (L. E. J. Brouwer, n.d.). An essential tenet of intuitionism is its rejection of the law of excluded middle, which asserts that every mathematical statement is either true or false. In contrast, intuitionists argue that the truth of a statement is established only when a proof is explicitly constructed.

This perspective reshapes how mathematical objects are conceptualized, particularly in their application. For instance, the matrix representation of a linear transformation may be problematic under an intuitionistic framework, especially when the basis is defined in a non-constructive or non-standard way. Since intuitionistic mathematics requires explicit constructions, widely accepted results—such as *every linear transformation has a matrix representation*—may require reevaluation or reinterpretation when the existence of a basis cannot be explicitly verified.

Expanding upon this notion, the interplay between coordinate vectors and intuitionistic mathematics warrants consideration. For a coordinate vector to be deemed valid, intuitionism mandates a concrete method for its construction. This emphasis aligns with Brouwer's assertion that a mathematical object is considered nonexistent until it can be explicitly constructed.

Similarly, when examining linear transformations, intuitionism requires that these transformations must be expressible through constructive methods. An illustrative case involves asserting that a linear map $L : V \to W$ (with V and W as vector spaces) holds validity only if there exists a procedure to demonstrate the transformation of elements from V to W. Should such explicit methods be absent, intuitionism finds the transformation untenable.

The difficulties associated with learning matrix representations show what intuitionism indicates. In classical mathematics, the idea that any linear transformation can be described by a matrix

relies on unrestricted freedom in defining dimensions and entries. However, from an intuitionistic viewpoint, we need to show how matrix representation can be explicitly derived from the linear transformation. These factors lead to a major rethinking of linear algebra problems, especially in higher-dimensional or complex vector spaces, where building examples can become complicated.

Relying too much on personal feelings and limited teaching methods might result in rejecting widely accepted conclusions in traditional mathematics, which could question the reliability of some key ideas. Intuitionism rejects traditional reasoning, which leads to giving up many well-known math results. This creates a system that works differently from standard math methods. Focusing on constructivism might challenge the basic rules that mathematics aims to follow, possibly changing how we understand what math is and what is considered true.

In conclusion, while intuitionism fosters a thought-provoking discussion on the foundations of mathematics, it compels mathematicians to reconsider fundamental notions of proof, existence, and mathematical objects through the lens of human intuition and constructive methods. Its challenges to classical results, particularly in linear algebra, illustrate its profound influence on mathematical philosophy, enriching the ongoing discourse on the nature and epistemology of mathematics.

3.1.4 Nominalism: Mathematics as a Symbolic System

Nominalism presents a pragmatic outlook on the philosophy of mathematics, arguing that mathematical statements do not describe physical realities but instead serve as expressions of relationships among symbols. Unlike Platonism, which posits the existence of abstract mathematical objects, nominalism rejects such entities in favor of a convention-based interpretation, framing mathematics as a language-based system designed primarily for effective communication and problem-solving rather than the discovery of inherent truths(Internet Encyclopedia of Philosophy, 2023).

A significant strength of nominalism lies in its compatibility with the principles of scientific empiricism. By framing mathematics as a functional tool instead of a descriptor of an independent abstract reality, nominalism aligns closely with the idea that knowledge should be firmly grounded

in observable phenomena (Stanford Encyclopedia of Philosophy, 2011).

However, nominalism faces a formidable challenge: if mathematics is merely a system of symbols devoid of ontological commitment, why is it so effective in describing and predicting real-world phenomena? The physicist Wigner (1960) explored this question in his seminal essay, *The Unreasonable Effectiveness of Mathematics in the Natural Sciences*. Wigner questioned how a system primarily composed of symbols manages to exhibit profound explanatory power within domains such as physics and engineering. Critics of nominalism argue that this effectiveness challenges the position by suggesting that mathematical structures may reveal a deeper, ontologically significant relationship with empirical reality beyond mere human constructs.

Take the matrix representation in linear algebra as an example. From a nominalist perspective, a matrix isn't an inherent property of a transformation—it's just a convenient way to express how one set of coordinates changes into another. The same transformation can have different matrix representations depending on the choice of basis, which reinforces the idea that these representations are human-made conventions rather than objective truths.

In this sense, a matrix is like a map—it helps us navigate a mathematical landscape, but it doesn't *exist* in any deeper sense than the symbols we assign to it. We use it because it works, not because it reveals some fundamental reality.

This line of thought inevitably raises a compelling question: If mathematical objects are solely symbolic conventions, how do they correlate so harmoniously with empirical observations? One plausible explanation posits that mathematics has been intentionally designed and meticulously refined over time to correspond with the underlying structures and relationships we encounter in the natural world. Rather than residing in an abstract realm, vector spaces, transformations, and matrices may well be human-engineered frameworks that fortuitously merge with empirical reality due to their evolutionary adaptation to it.

In conclusion, nominalism provides valuable perspectives on the essence of mathematics and its function as a means for problem-solving and communication. It simultaneously prompts critical examination of the persistent inquiry regarding the connection between mathematical constructs

and the empirical realities they aim to represent. This interaction is a crucial aspect of the investigation into the philosophy of mathematics, exposing the intricate complexity embedded in human comprehension and the natural world.

3.1.5 Empiricism: Mathematics as a Generalization of Experience

Empiricism holds that all knowledge, including mathematical knowledge, arises from sensory experience. This perspective posits that mathematical concepts develop through interaction with the physical world, making them fundamentally tied to observation and experimentation (Kollosche, 2021).

While this approach addresses the practical applications of mathematics and its connections to empirical science, it encounters significant challenges in addressing purely abstract notions that elude direct observation, such as infinity or imaginary numbers (MBMA Blog, n.d.). Critics argue that by grounding mathematics entirely in experience, empiricism risks reducing mathematical inquiry to a mere tool for describing physical phenomena while neglecting its broader theoretical significance.

Understanding abstract mathematical topics, such as vector spaces, coordinate vectors, linear transformations, and matrix representation, reveals both benefits and challenges posed by empiricism. This approach, emphasizing experiential learning, can indeed help students forge robust connections between mathematical concepts and their practical applications in real-world scenarios (Intellectual Mathematics, 2021).

One substantial advantage of an empiricist approach is its ability to make abstract ideas more relatable through concrete examples and applications (Intellectual Mathematics, 2021). For instance, vector spaces and coordinate vectors become much easier to comprehend when they are linked to real-world fields like physics, engineering, and computer graphics. In these disciplines, vectors are utilized to represent phenomena such as forces, motion, and transformations. Similarly, the matrix representation of linear transformations is equally crucial in areas such as robotics and image processing, demonstrating the tangible benefits of applied mathematics.

However, there are some significant drawbacks to relying solely on an empiricist approach, particularly when it comes to more abstract aspects of these mathematical structures. Although it is possible to visualize coordinate vectors and transformations in two and three dimensions, higher-dimensional vector spaces and complex transformations do not have direct physical counterparts. Due to the focus on experience-based learning, students may encounter difficulties in grasping purely theoretical concepts, such as polynomial function spaces or differential equation-based coordinate vectors, which are not easily observed or experimented upon in a tangible context.

Another concern with an empiricist perspective is that it may limit students' ability to appreciate the broader conceptual nature of mathematics. By focusing primarily on observable and applied aspects, there is a risk that students will see mathematics as merely a tool for describing physical phenomena rather than recognizing its deep logical and structural foundations. This issue can be particularly problematic in linear algebra, where axiomatic reasoning and abstract generalizations play a crucial role. Without exposure to the formal reasoning that underlies these mathematical structures, students may find it difficult to construct and evaluate proofs, which are an essential component of higher-level mathematical thinking (Hjelte, Schindler, & Nilsson, 2020). In conclusion, while empiricism offers valuable insights in making mathematical concepts more accessible and engaging through real-world applications, it alone is insufficient for mastering abstract mathematical ideas. Students must take a comprehensive approach that combines empirical insights with formal deductive reasoning to fully grasp topics like vector spaces, coordinate vectors, linear transformations, and matrix representation. By integrating both practical experience and theoretical understanding, educators can help students develop a more complete and versatile mathematical foundation.

Empiricism and materialism, while related, differ in focus. Empiricism is an epistemological stance that emphasizes sensory experience as the basis for knowledge, often influencing educational approaches that prioritize hands-on learning and observation. Materialism, on the other hand, is an ontological view that asserts all phenomena, including thought, arise from physical matter. While both stress the importance of the observable world, empiricism concerns how we know, whereas

materialism concerns what exists. Building on these foundations, anthropology offers a broader perspective by examining how knowledge is shaped through cultural, social, and institutional contexts. In mathematics education, this shift highlights that understanding is not only individual or sensory but also socially constructed and mediated through language and practice.

3.1.6 Anthropology: Mathematics as a Product of Human History

Anthropology examines mathematics as a cultural and historical construct rather than a purely abstract, universal truth (Read, 2019). This perspective highlights how mathematical knowledge evolves through social contexts, cultural practices, and historical developments, leading to diverse conceptualizations across societies.

Applying this lens to mathematics education, particularly in topics like vector spaces, linear transformations, and matrix representation, enhances students' appreciation for mathematics as a dynamic discipline. The historical use of matrices in solving linear equations, as seen in the Chinese *Nine Chapters on the Mathematical Art* (circa 200 BCE), exemplifies the long-standing development of mathematical ideas (Read, 2019). Recognizing these historical roots can make abstract concepts more meaningful.

Moreover, exploring the real-world applications of vector spaces and transformations in fields like physics, computer science, and economics reinforces their relevance. In quantum mechanics, state vectors describe physical systems, while high-dimensional vector spaces underpin machine learning algorithms. Such connections help bridge abstract mathematical theory with practical implementation.

However, an anthropological perspective also presents challenges, particularly regarding mathematical relativism (Kay, 2024). Viewing mathematics solely as a cultural construct risks undermining its rigorous, universal foundations. While cultural influences shape mathematical thought, fundamental principles—such as the axiomatic structure of vector spaces and the properties of linear transformations—remain consistent across different contexts.

Additionally, overemphasizing cultural relativity may hinder students' engagement with for-

mal proofs and abstract reasoning. Without exposure to rigorous mathematical structures, students may struggle with concepts like infinite-dimensional vector spaces or coordinate transformations beyond three dimensions (Read, 2019).

Integrating an anthropological perspective with formal mathematical reasoning fosters a more comprehensive understanding. Recognizing mathematics as both a human endeavor and a logically structured discipline enables students to appreciate its historical, cultural, and theoretical dimensions. By balancing historical awareness, cultural context, and formal abstraction, educators can provide a deeper and more versatile mathematical education (National Council of Teachers of Mathematics, 2024)

In examining mathematical objects such as vector spaces, coordinate vectors, and linear transformations, the six major philosophical perspectives—Platonism, Materialism, Intuitionism, Nominalism, Empiricism, and Anthropology—each offer distinct insights and reveal inherent limitations that are crucial for understanding mathematics education.

3.2 Epistemological Barriers in Matrix Representation

Having explored how different philosophies frame mathematical existence, we now turn to the epistemological challenges that arise when students engage with these abstract objects, particularly in the context of matrix representation.

Understanding essential concepts of matrix representation through different philosophical lenses can help identify the epistemological obstacles students face. One common challenge arises from students' reliance on geometric intuition, for example, visualizing $\mathbb{R}^n$ using familiar spatial or physical ideas. While this kind of intuition is not the same as philosophical intuitionism, it shares a constructive flavor in that students often seek concrete, visual representations to make sense of abstract concepts. While these representations provide a starting point, they frequently hinder the exploration of more abstract concepts, such as the role of a basis, the implications of basis transformation, and the nuanced interpretations of coordinate vectors across different contexts. Research indicates that such challenges are not isolated but rather part of a wider epistemological

struggle prevalent in mathematical education (Trigueros, Cabrera, & Aguilar, 2024).

Difficulties in learning matrix representation frequently stem from an overemphasis on algebraic manipulation, which can detract from developing a robust conceptual understanding. Many students become fixated on procedural elements, leading to a fragmented grasp of fundamental theorems like the Rank-Nullity Theorem. This disconnection underscores the necessity of intertwining procedural fluency with conceptual insights to foster deeper learning. Moreover, the transition between coordinate systems presents additional hurdles; students must assimilate conceptual shifts—such as those occurring between Cartesian and polar coordinates—before applying these ideas to higher-dimensional function spaces. Such transitions can be perplexing and often plunge students into confusion, as they grapple with varying representations of mathematical objects (Fitriana, Kusmayadi, & Sahara, 2024).

The epistemological dimension plays a crucial role in understanding how knowledge is constructed, perceived, and utilized within educational contexts. Specifically, this dimension explores the nature of knowledge, its sources, and the means by which learners acquire, evaluate, and apply this knowledge in various settings. Within the framework of learning matrix representation, the epistemological perspective can illuminate how different knowledge beliefs influence learning outcomes.

For instance, learners' epistemological beliefs impact their engagement with complex information and their strategies for problem-solving. Research suggests that students who perceive knowledge as fixed and absolute are more inclined to adopt surface-level learning approaches. In contrast, those who view knowledge as evolving and shaped by context are more likely to engage in deeper learning strategies that promote critical thinking and meaningful application. The work of Mete (2023); Rezaei (2016); Schommer (1990, 1993) highlights the value of fostering adaptive epistemological beliefs to enhance student learning outcomes.

Within learning matrix representations, incorporating diverse epistemological perspectives enhances the understanding of how knowledge is organized and contextualized. This approach allows educators to design learning experiences that reflect the complexity of knowledge in real-

world scenarios, fostering higher-order thinking skills among learners. For example, visual tools such as concept matrices or semantic maps—where rows and columns correspond to key ideas—can help illustrate the interconnections between definitions, theorems, and operations. By making these relationships explicit, such representations support a more integrated and relational understanding of the subject matter.

Moreover, acknowledging the epistemological dimension enables researchers to examine the cognitive and affective dimensions of learning in more detail. By understanding how learners perceive knowledge and the processes by which they engage with learning tasks, educators can develop instructional strategies that are more aligned with learners' needs, ultimately improving the quality of education and fostering genuine learning experiences.

In conclusion, understanding levels of analysis and the epistemological dimension offers valuable frameworks for examining educational complexities. By utilizing a multi-level approach to analyze individual, institutional, and systemic factors, researchers can gain deeper insights into the dynamics of learning processes. Additionally, exploring the epistemological beliefs held by students enhances the understanding of how knowledge is constructed and applied within educational contexts. Together, these frameworks contribute to advancing educational research and practice, thus promoting richer learning experiences for students across diverse educational settings.

3.3 Level of Analysis in Learning Matrix Representation

Analyzing educational phenomena requires a robust framework that incorporates various analytical levels and epistemological insights. Levels of analysis are critical for systematically examining educational issues at multiple layers of complexity. In the context of education, three primary levels can be identified: the individual level, the institutional level, and the systemic level.

At the individual level, analysis focuses on learners themselves, emphasizing their thinking processes, motivations, and prior knowledge. Understanding how students approach problem-solving and monitor their own progress offers valuable insight into how they interact with ed-

ucational materials. Learners who are more aware of their thinking and actively adjust their strategies tend to engage more effectively and perform better in complex subjects such as matrix representation.

The institutional level of analysis examines the role of educational institutions, such as schools and universities, in shaping learning experiences. This level emphasizes the policies, teaching styles, and curriculum designs that influence student engagement and learning outcomes. Institutions can disseminate best practices that enhance student experiences, such as implementing technology-enhanced learning environments to cater to diverse learning needs. This level also examines how factors like school culture and teacher professional development impact teaching effectiveness and student success.

Finally, the systemic level focuses on broader factors that influence education, such as socioeconomic conditions, governmental policies, and cultural norms. Systematic analysis allows researchers to understand how these external factors shape educational opportunities and outcomes. For instance, disparities in funding and resources across different regions can lead to unequal educational opportunities for students, as documented in various studies on educational equity.

By applying levels of analysis and epistemology dimensions from the OSA to educational research, which we will investigate in the following section, researchers can gain a comprehensive understanding of the intricacies involved in learning processes and the factors that affect student achievement.

3.4 The OSA: Theory and Application

While each of the six philosophical perspectives provides valuable insight into the nature of mathematical objects and learning, they also reveal distinct limitations when considered in isolation. The OSA offers a unifying framework that does not merely align with a single philosophy but synthesizes its strengths to provide a multifaceted view of mathematical knowledge. In this sense, OSA transcends individual philosophical positions by contextualizing mathematical meaning as culturally constructed and cognitively experienced. This integrative potential allows OSA

to address specific learning challenges in linear algebra—such as abstraction, symbolic manipulation, and conceptual generalization—in ways that traditional philosophical lenses alone may not. Highlighting this synthesis clarifies OSA's unique value as both a theoretical and practical tool in mathematics education.

Emerging in the late 1990s and early 2000s, the OSA arose from the growing demand for a comprehensive understanding of mathematical learning. This framework integrates two fundamental perspectives: semiotic and ontological. The semiotic dimension concentrates on how signs and symbols shape meaning, while the ontological aspect seeks to elucidate the nature of mathematical objects themselves. Scholars like Juan D. Godino, Carmen Batanero, and Vicenç Font have been instrumental in refining this approach, augmenting its relevance within mathematics education (Godino, 2024).

At its essence, the OSA challenges the notion that mathematical knowledge is stagnant. Instead, it posits that mathematics is a dynamic, evolving system influenced by a myriad of interpretations, symbols, and practices. Rather than merely being fixed entities, mathematical objects are perceived as entities whose meanings change in accordance with their context of use. This dynamic perspective aligns well with both ontological and semiotic principles, creating a holistic lens through which to analyze how students engage with mathematical concepts.

3.5 The Role of Ontology and Semiotics

Ontology, derived from the Greek word *onto*, meaning *being*, is the study of existence and the categorization of entities. It explores the essence of reality by addressing questions about what types of things exist (e.g., physical objects, abstract concepts, mathematical entities), their structures, and the relationships among them. In essence, ontology seeks to understand the fundamental nature of being. In the context of mathematics, ontology plays a crucial role in clarifying the existence of mathematical objects. It emphasizes that entities like numbers, geometric shapes, and algebraic structures exist within defined frameworks established by mathematical theories (Godino, 2024).

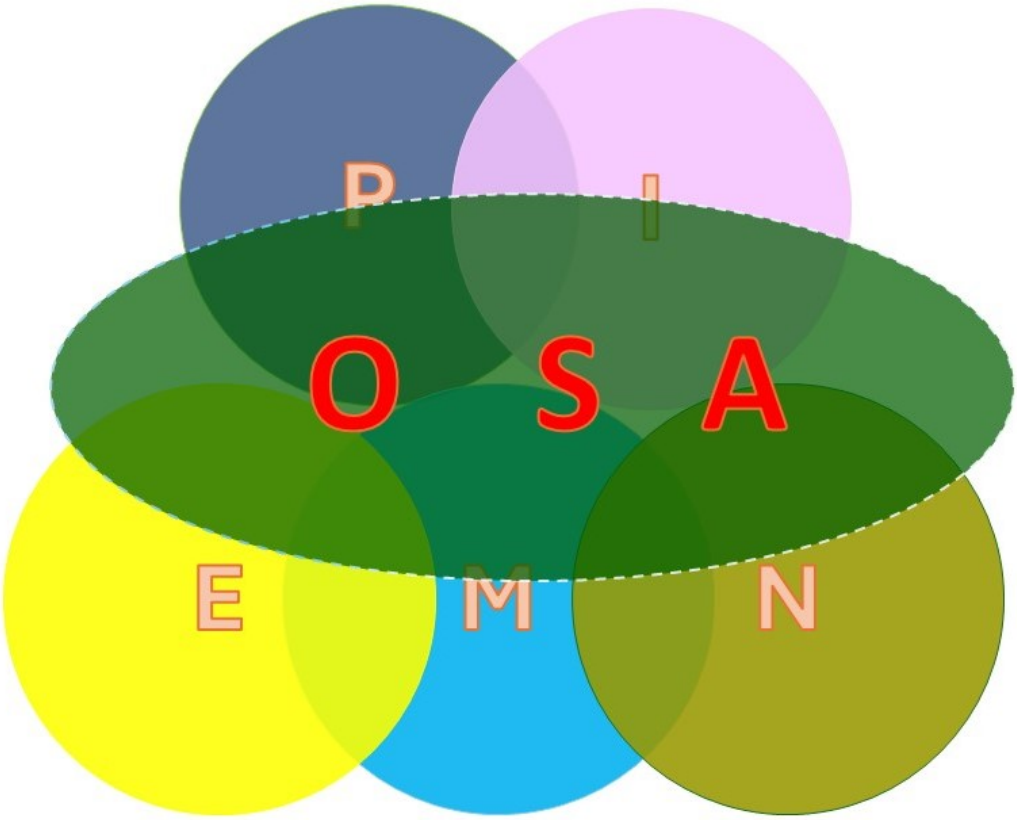


Figure 3.1: OSA: Comprehensive Lens Over Six Philosophies in Mathematics Education

Ontology significantly influences mathematical reasoning by shaping how both students and educators conceptualize mathematical entities. Varying ontological perspectives, such as Platonism, Nominalism, and Empiricism, offer distinct explanations for the existence of mathematical objects, affecting how students argue for proofs, perceive axioms, and grapple with the inherently abstract nature of mathematics.

Complementing ontology, semiotics enhances comprehension by focusing on the representation of mathematical objects. The language of mathematics is often laden with symbols, diagrams, and formal notations that are essential for effective communication. A student's ability to interpret these representations significantly impacts their capacity to construct and validate mathematical concepts. For instance, comprehending linear transformations relies not only on a student's understanding of vector spaces but also on their skill in interpreting matrix representation with logic.

In contemporary research, understanding the foundational elements of ontology and semiotics profoundly impacts the way knowledge is generated, structured, and communicated. Ontology examines the nature of existence and reality, categorizing entities within the universe and detailing how these entities interrelate. This branch of philosophy goes beyond superficial definitions to probe critical questions concerning what it means for something to exist. Conversely, semiotics, the study of signs and symbols, investigates how meaning is constructed and communicated through various forms such as language, images, and gestures (Godino, Batanero, and Font (2019)). This integrative perspective proves especially relevant in subjects such as matrix representation, where concepts necessitate both conceptual insight and symbolic manipulation skills. By employing OSA, educators gain valuable insights into how students navigate various representations and reasoning strategies, ultimately leading to enhanced teaching methods.A clear example of how ontology is applied in mathematics can be found in the use of matrix representation.

For instance, consider the linear transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ defined by:

$$L(x, y) = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x - y \\ 3x + 4y \end{bmatrix}. \tag{3.1}$$

This transformation maps vectors in $\mathbb{R}^2$ to new positions according to the structural relationships encoded within the matrix. The existence of such transformations illustrates the ontological foundation of mathematical entities—here, the matrix is not merely a computational tool but a concrete representation of how linear mappings function within a structured mathematical framework. The ontological significance lies in how these objects are categorized, visualized, and manipulated within mathematical theories.

In addition to ontology, semiotics plays a pivotal role in understanding how knowledge is communicated and interpreted. This field focuses on the processes involved in the creation and interpretation of signs, addressing questions about the meanings of signs and symbols, their origins, and their ontological status. Semiotics explores inquiries such as: What do mathematical symbols represent? Where do these concepts originate? How can we construct or access their meanings?

The foundations of semiotic theory are illustrated through the models proposed by Ferdinand de Saussure and Charles Sanders Peirce. Saussure's dyadic model of the sign consists of the signifier (the physical form) and the signified (the concept it represents), establishing a connection between language and thought. This relationship is crucial for understanding how symbols in mathematics convey specific meanings. For example, consider a matrix equation in the context of linear transformations:

$$A\mathbf{x} = \mathbf{b}$$

Here, A represents a matrix that acts as a linear transformation, $\mathbf{x}$ is the vector of variables (the input), and $\mathbf{b}$ is the resulting vector (the output). In this equation, the matrix A is the signifier (the physical form), while the relationship it describes between the vectors is the signified (the concept it represents). This semiotic framework allows us to understand how the equation models the transformation of one vector space into another. Through this lens, semiotics helps analyze the meanings of symbols in linear transformations and how these meanings influence comprehension and problem-solving approaches for learners. As data complexity increases, matrices provide a systematic approach to organizing and analyzing information, ensuring that meanings remain intact

while exploring relationships across vast datasets. This is crucial in domains like social sciences, natural language processing, and cognitive informatics, where understanding complex connections profoundly influences comprehension and application of knowledge.

The interplay between ontology and semiotics shapes the methodological frameworks researchers adopt to investigate their subjects. By applying ontological principles, researchers can define the entities they wish to study. From this foundation, semiotic analysis enables them to communicate findings effectively using the appropriate signs and symbols necessary for comprehension. The combination of these disciplines fosters a more comprehensive understanding of knowledge construction.

3.6 The Six Types of Primary Mathematical Objects

The OSA provides a robust framework for understanding mathematical activity through the interrelation of various mathematical objects. Within this framework, six primary types of mathematical objects are distinguished, which serve as essential components in the analysis of mathematical practices. These objects are classified as problems, definitions, propositions, procedures, languages, and arguments, and their interconnectedness plays a vital role in enhancing mathematical comprehension and problem-solving capabilities.

Problems

The first type of mathematical object, **problems**, represents the challenges or tasks that prompt mathematical inquiry. These problems serve as the starting point for mathematical activities, guiding the direction of investigations and the processes that follow. They are often accompanied by a set of conditions or constraints that learners must navigate to arrive at a solution.

Definitions

The second mathematical object, **definitions**, articulates the meanings and descriptions necessary to understand the problems at hand. Definitions are crucial as they establish the foundational

terminology and concepts that students and mathematicians must grasp to effectively communicate and engage with mathematical ideas.

Propositions

The third type, **propositions**, serves as the statements or claims about the relationships between mathematical entities. These propositions often emerge from the problems and definitions, as they encapsulate the truths that can be derived from the established definitions. In this sense, propositions serve as a bridge between theoretical concepts and practical applications.

Procedures

The fourth object, **procedures**, encompasses the methods and techniques used to manipulate mathematical entities and derive solutions to problems. Procedures are essential not only for finding answers but also for fostering deeper understanding, as they reveal the systematic ways in which mathematical operations can be performed. They often require the application of definitions and propositions to successfully navigate through mathematical tasks.

Languages

The fifth mathematical object, **languages**, consists of the various forms of expression used in mathematics, including symbolic, verbal, graphic, and other formats of representation. The use of language is paramount in mathematics as it allows for efficient communication of ideas and solutions, facilitating collaboration and knowledge sharing among mathematicians and learners.

Arguments

Finally, the sixth object, **arguments**, refers to the reasoning processes employed to justify solutions or validate propositions. These arguments are integral to mathematical discourse, as they provide the logical foundations that allow individuals to present their reasoning and conclusions in a coherent manner. The effectiveness of an argument relies heavily on the clarity and precision of the preceding mathematical objects.

The interconnections among these six types of mathematical objects within the OSA illuminate the complexity of mathematical activity. For example, a problem may prompt the identification of key definitions that frame the context, leading to the formulation of relevant propositions. Subsequently, these propositions guide the selection of appropriate procedures, which are expressed through various languages, ultimately supported by robust arguments that justify the solutions obtained. This network of interconnected objects enhances learners' ability to grasp abstract concepts, fosters critical thinking, and promotes a more comprehensive understanding of mathematics as a coherent field of study.

In conclusion, the OSA's emphasis on the six types of mathematical objects exemplifies the intrinsic connections within mathematical activity. By analyzing how these objects support and interact with one another, educators and researchers can gain insights into the processes through which students construct mathematical knowledge and develop their problem-solving skills.

3.7 The Five Paired Attributes

The OSA contributes to the understanding of teaching and learning mathematics by providing a comprehensive framework for analyzing the relationships between different mathematical objects At its core, the OSA emphasizes that primary objects of study can be interpreted through five pairs of contextual attributes, or dualities, which facilitate diverse perspectives on mathematical concepts. This essay details these five pairs: personal vs. institutional, expression vs. content, example vs. type, unitary vs. systemic, and ostensive vs. non-ostensive. Each pair reveals not only the depth of mathematical cognition but also the intricate social and cognitive frameworks upon which mathematical understanding is constructed.

Personal vs. Institutional

The distinction between personal and institutional attributes lies in the nature of knowledge as either individually experienced or collectively shared within an institutional context. Personal

attributes refer to the individual subjects who engage with mathematical objects, reflecting their unique interpretations and experiences. In contrast, institutional attributes encompass how knowledge is constructed and disseminated within a community of practice, often influenced by shared norms, standards, and curricula. This duality underscores the importance of recognizing that personal insights can be enriched by institutional frameworks, while institutional knowledge must also be accessible and relatable to individual learners.

Expression vs. Content

Another critical dichotomy is the distinction between expression and content. In a semiotic function, the expression refers to the symbolic, verbal, or visual representation of a mathematical object, while the content is the underlying meaning or concept it conveys. A semiotic function maps from expression to content, although what is content as the consequent of one semiotic function can be the antecedent (expression) in another, it depends on the context (Montiel, Wilhelmi, Vidakovic, & Elstak, 2009, p.144). For example, the matrix A in the expression $[L(v)]_F = A[v]_E$ can be seen as the expression of the linear transformation L with respect to bases E and F. In this semiotic function, A serves as the expression, while the content is the action of the transformation L applied to the vector space. In a subsequent context—such as reasoning about similarity or change of basis—the matrix A itself may now become the content of a new semiotic function, with a different expression (e.g., $P^{-1}AP$) representing its transformed form. This layered relationship illustrates how mathematical understanding is constructed through a dynamic interplay of expressions and meanings across different levels of abstraction.

Example vs. Type

The third pair, example vs. type, differentiates between specific instances and general classifications. An example is a concrete representation of a particular instance within a mathematical domain, while a type refers to the general category or concept that encompasses these instances. For instance, within the context of geometric shapes, a triangle could serve as an example, whereas the notion of polygons represents a broader type that includes numerous shapes. Recognizing

this duality is essential for developing a more nuanced understanding of mathematical concepts, enabling learners to navigate between specific cases and abstract classifications effectively.

Unitary vs. Systemic

Unitary involves viewing an object as a single, coherent entity that represents a broader mathematical idea or phenomenon. For example, the matrix representation of a linear transformation can be understood as a complete model of how vectors are mapped from one basis to another. In contrast, a systemic perspective looks inside the model, focusing on the relationships between its components. This includes how the entries of the matrix relate to the choice of basis, how matrix operations reflect algebraic structure, and how transformations behave under composition. Together, both support a more comprehensive understanding: one that values both the overall meaning of the mathematical object and the internal structure that makes that meaning possible.

Ostensive vs. Non-Ostensive

Finally, the ostensive vs. non-ostensive pair contrasts material, perceptible objects with abstract, ideal notions. Ostensive attributes are tied to physical representations of mathematical ideas that can be directly observed or manipulated, such as shapes or graphs. On the other hand, non-ostensive attributes refer to abstract concepts such as theories, axioms, or mathematical relations that exist beyond tangible forms. This duality illustrates the diversity of mathematical knowledge and indicates that effective learning often requires bridging the gap between concrete experiences and abstract reasoning.

Conclusion

In conclusion, the OSA, through its delineation of five pairs of attributes, enriches our understanding of mathematical objects by presenting them from multiple perspectives. Each duality—personal vs. institutional, expression vs. content, example vs. type, unitary vs. systemic, and ostensive vs. non-ostensive—highlights the complexity of mathematical cognition and the socio-cultural dimensions of learning in mathematics education. As educators and researchers

Figure 3.2: The six objects cooperate with each other.

continue to explore these attributes, they will further illuminate the dynamic interactions that shape how mathematical knowledge is constructed, shared, and understood in diverse contexts.

3.8 Analyzing Five Pairs through a Worked Example

The OSA provides a comprehensive framework for understanding how mathematical objects —such as the matrix representation of linear transformations—are constructed, interpreted, and communicated in educational settings. By analyzing five key dualities—personal vs. institutional, expression vs. content, example vs. type, unitary vs. systemic, and ostensive vs. non-ostensive—we can gain deeper insights into how students interact with and make meaning of mathematical representations.

To illustrate the application of these five pairs in context, we analyze the following intra-mathematical problem as an example. This task serves as a basis for exploring how each pair contributes to the understanding of linear transformations and their matrix representations.

Problem Statement

Let L be a linear transformation defined by

$$L(p(x)) = p'(x) + p(0),$$

which maps the polynomial space P_3 into P_2.

(a) Find the matrix representation of L with respect to the ordered bases $\{x^2, x, 1\}$ for P_3 and $\{2, 1 - x\}$ for P_2.

(b) Given $p(x) = x^2+2x-3$, compute $L(p(x))$, and express the result in coordinate form relative to the basis $\{2, 1 - x\}$.

Personal vs. Institutional

The personal vs. institutional pair emphasizes the distinction between individual experiences and collective knowledge within mathematical practices. Personal attributes refer to the ways in which learners conceptualize matrix representations based on their unique backgrounds and prior knowledge of linear algebra. For instance, a student who has extensively used matrix operations in practical scenarios may interpret a transformation's impact differently than a peer who approaches the subject more theoretically. Conversely, institutional attributes describe how matrix representation is framed within educational curricula, classroom practices, and community standards.

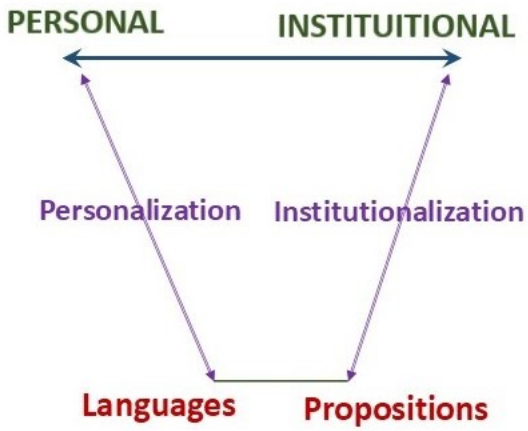


Figure 3.3: Illustration of personal and institutional meaning in the context of linear transformation

From a personal perspective, students may demonstrate limited prior experience with linear transformations involving derivative operations and evaluation maps. These learners often struggle to connect transformations to bases and coordinate systems conceptually. Their understanding may be grounded in isolated procedures without a clear grasp of how these operations interrelate. As a result, the meaning they construct is often idiosyncratic, intuitive, and tied to informal or incomplete mental models of linear algebraic structures.

In contrast, the institutional perspective reflects how mathematical knowledge is structured, communicated, and validated within the academic and disciplinary community. For example, instructors and textbooks typically employ standard basis manipulations, change-of-basis procedures, and formal constructions of matrix representations. In this view, polynomials are treated systematically as vectors in function spaces, and the transformation is formalized through conventional symbolic techniques. The institutional meaning of the content includes accepted norms, practices, and representations that define how the mathematical community understands and uses these concepts.

Expression vs. Content

The distinction between expression and content plays a critical role in understanding how matrices symbolize linear transformations. An expression—such as a matrix—is the concrete

representation of a mathematical operation. This might include various types of matrices such as transformation matrices or augmented matrices used in systems of linear equations. On the other hand, the content refers to the mathematical concepts that these expressions signify, such as the geometrical implications of a transformation or the relationships between the vectors they map. Recognizing this distinction allows students to appreciate that the symbols they manipulate are more than just characters; they represent substantial mathematical ideas that govern how transformations operate within vector spaces.

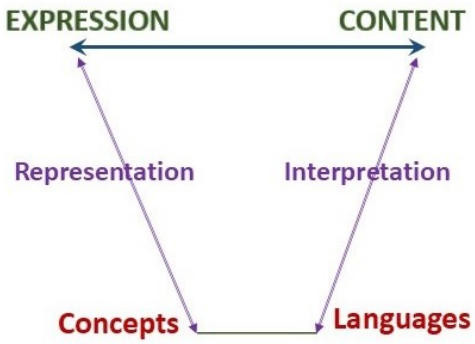


Figure 3.4: Visual comparison of expression and content in polynomial transformation

In the OSA, the expression refers to the symbolic, verbal, or visual form used to represent mathematical objects or operations. In this example, we consider the linear transformation $L : P_3 \to P_2$ defined on polynomial spaces. The standard basis $E = \{x^2, x, 1\}$ is used for P_3, and a non-standard basis $F = \{2, 1 - x\}$ is used for P_2. The transformation is explicitly defined by the rules:

$$L(x^2) = 2x, \quad L(x) = 1, \quad L(1) = 1.$$

These symbolic expressions represent the operational structure of the transformation in terms of basis elements and coordinate rules.

While the expression captures the surface structure, the content reflects the underlying mathematical meaning conveyed by the transformation. In this case, $L(p(x))$ represents a combination of the derivative of a polynomial and its evaluation at zero—capturing both a rate of change

and a fixed value. Conceptually, the transformation connects the polynomial space P_3 with the lower-dimensional space P_2, thereby encoding a mathematical relationship that reduces dimension while preserving key information. The distinction between expression and content is critical for understanding how students interpret symbols in relation to mathematical meaning.

Example vs. Type

Within the context of matrix representation of linear transformations, the example vs. type dichotomy helps differentiate between specific operations (examples) and overarching categories (types) of transformations. For instance, a transformation matrix representing a rotation in a two-dimensional Euclidean space serves as a concrete example, while the concept of linear transformations encompasses a broader type that includes other transformations like scaling or reflections. Understanding this duality encourages learners to see specific instances of matrix use as part of a larger mathematical landscape, thereby enhancing their ability to generalize or transfer knowledge across different types of linear transformations and enrich their conceptual grasp of linear algebra.

Unitary vs. Systemic

The unitary refers to interpreting a mathematical object as a single, cohesive entity that conveys a broader mathematical meaning or models a specific phenomenon. For example, the matrix representation of a linear transformation can be viewed as a complete model that illustrates how vectors are mapped from one coordinate system to another.

In contrast, the systemic approach shifts the focus to the internal structure and relationships among the components of the object. This includes examining how the entries of the matrix are determined by the chosen bases, how operations such as multiplication or inversion connect to broader algebraic principles, and how different matrices interact in processes like composition or change of basis.

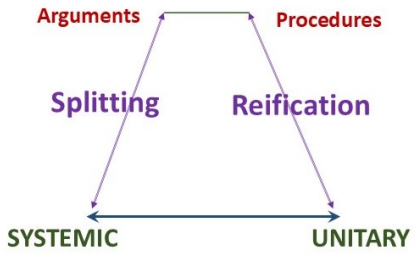


Figure 3.5: Visual summary of systemic and unitary perspectives

The systemic perspective emphasizes decomposing a transformation into intermediate steps based on basis elements. For the linear transformation defined by $L(p(x)) = p'(x) + p(0)$, the process involves:

- Computing the derivative $p'(x)$.
- Evaluating $p(0)$.
- Adding the results and expressing the outcome in the target basis $\{2, 1 - x\}$.
- Breaking the expression into parts that align with the target basis to derive coordinates.

For example, applying L to the standard basis $\{x^2, x, 1\}$:

$$\begin{aligned} L(x^2) &= 1 \cdot 2 + (-2)(1 - x), \\ L(x) &= \frac{1}{2} \cdot 2 + 0 \cdot (1 - x), \\ L(1) &= \frac{1}{2} \cdot 2 + 0 \cdot (1 - x). \end{aligned}$$

The resulting matrix of L in the basis $\{2, 1 - x\}$ is:

$$[L(p(x))]_F = \begin{bmatrix} 1 & 1/2 & 1/2 \\ -2 & 0 & 0 \end{bmatrix}$$

The columns represent the coordinates of the image of each basis element under L in the basis $\{2, 1 - x\}$.

The unitary perspective treats the transformation as a whole, focusing on the overall input-output relationship rather than breaking the process into components. In this view:

- The entire polynomial $p(x)$ is considered as a single input.
- Its coordinate vector is found in the domain basis $\{x^2, x, 1\}$.
- The transformation matrix of L is applied directly to this vector.
- The resulting vector gives the coordinates of $L(p(x))$ in the codomain basis $\{2, 1 - x\}$.

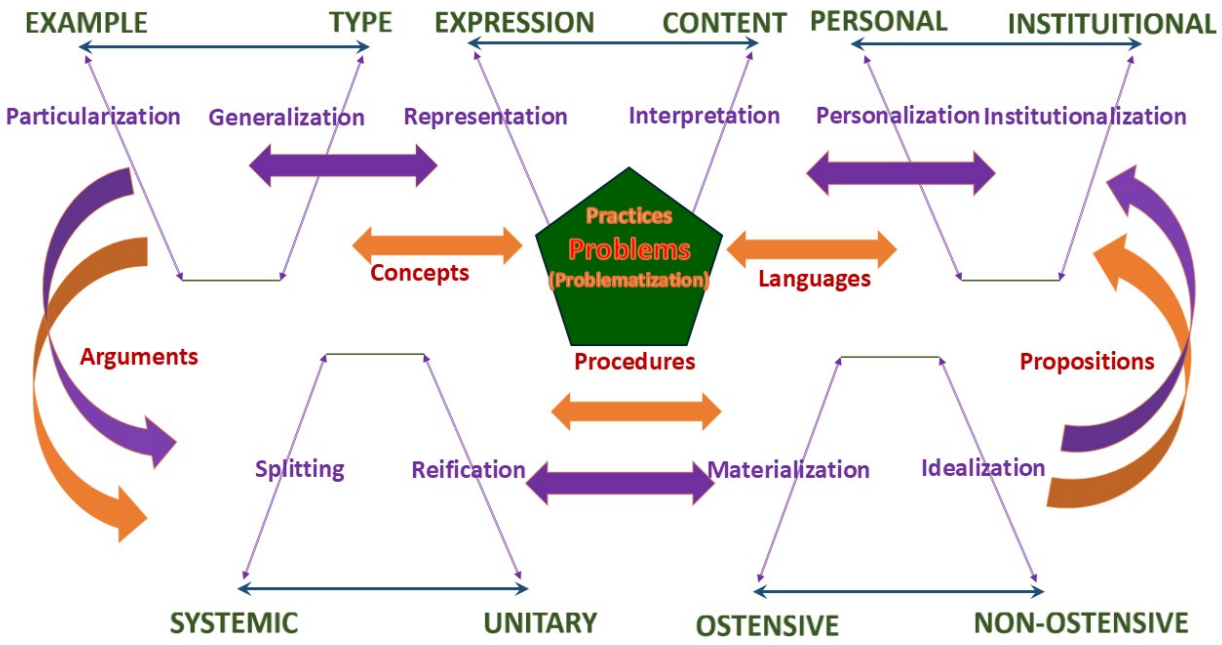
For instance, if $p(x) = x^2 + 2x - 3$, then:

$$L(p(x)) = 2x - 1.$$

From this result, the coordinate vector of $L(p(x))$ can be directly determined in the basis $\{2, 1 - x\}$, without explicitly computing the intermediate derivative or evaluation.

Ostensive vs. Non-Ostensive

The final duality, ostensive vs. non-ostensive, highlights the relationship between tangible representations of matrices and the abstract concepts they signify. Ostensive attributes include the physical or visual representations of matrices, such as their graphical interpretations or the rows and columns that one can visually manipulate. Non-ostensive attributes, however, pertain to the underlying principles of linear transformations—such as their theoretical implications regarding dimensionality and vector spaces—which may not be immediately observable. For effective learning, educators must create experiences that bridge the gap between these two realms, helping students to conceptualize both the physical representation of matrices and the abstract concepts they encapsulate. This engagement fosters a comprehensive understanding of how transformations enact changes within mathematical structures.



3.9 Conclusion

Applying the OSA to matrix representation of linear transformations affords a multi-faceted view of how mathematical understanding is constructed. The five pairs of attributes not only illuminate the cognitive processes involved but also underscore the social dynamics underpinning mathematics education. By integrating personal experiences with institutional knowledge, differentiating between expressions and their meanings, recognizing specific examples within broader types, acknowledging both unitary and systemic views, and navigating between ostensive and non-ostensive knowledge, educators can enrich their instructional practices and promote a deeper comprehension of linear transformations. This approach not only fosters mathematical insight but also empowers learners to engage meaningfully with the complexities of mathematical concepts in diverse contexts.

Chapter 4

Methodology

4.1 Methodology

Qualitative and quantitative research approaches serve complementary purposes in understanding educational processes. Qualitative research explores experiences, discourses, and narratives in depth, while quantitative research examines reliability, validity, and generalizability to predict trends through statistical analysis (Bhattacharya, 2017, p.19). Recent scholarship has emphasized that combining qualitative and quantitative methods—often referred to as mixed methods research—can offer a more holistic perspective on complex educational phenomena (Johnson & Onwuegbuzie, 2004). Journals such as the *International Journal of Educational Qualitative Quantitative Research* underscore the growing academic consensus that methodological pluralism enhances both the rigor and relevance of educational studies (International Journal of Educational Qualitative Quantitative Research, 2024).

To capture the multifaceted nature of students' learning in matrix representation, this study employs an explanatory sequential mixed methods research design (Creswell & Plano Clark, 2018). This design integrates qualitative and quantitative approaches in a two-phase structure. In the first phase, qualitative data are collected through classroom and participant observations, semi-structured interviews, and students' written responses to explore their meaning-making processes in depth. In the second phase, quantitative instruments—developed in light of the qualitative findings—are used to measure patterns of understanding across a broader student population. This integration enhances both the interpretive depth and the generalizability of the study's findings.

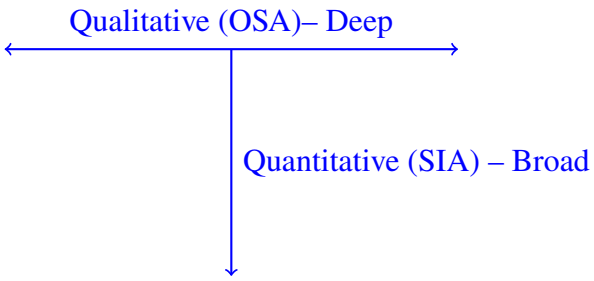


Figure 4.1: Schematic representation of qualitative and quantitative research orientations

4.2 Overview of Mixed Methods Design: OSA + SIA

The use of mixed methods in this study is justified on both theoretical and practical grounds. Theoretically, the combination of qualitative and quantitative perspectives aligns with the emphasis of the OSA on multilevel qualitative analysis—linking semiotic activity from personal to institutional and epistemological structures. This alignment is further enriched by anthropological considerations of the cultural and institutional practices surrounding mathematical meaning-making, constructivist views that emphasize knowledge as actively constructed through learners' experiences, and nominalist perspectives that foreground the role of language and symbolic systems in defining mathematical objects.

Quantitatively, this study employs Statistical Implicative Analysis (SIA), a data-driven method designed to detect logical and implicative patterns within categorical data. SIA complements the qualitative findings by uncovering regularities and directional associations in students' responses, thereby supporting the identification of common learning pathways and conceptual dependencies that may not be immediately evident through thematic analysis alone.

In alignment with the explanatory sequential design, this chapter is organized into two main parts reflecting the qualitative and quantitative components of the study.

Part one focuses on the qualitative research design and is organized into four sections. Section 4.2 introduces the OSA, which serves as the theoretical framework guiding this study. It outlines the key components of OSA—such as mathematical objects, processes, configurations, and ontological and semiotic representations—and explains how these elements inform the interpreta-

tion of students' meaning-making in the context of matrix representation. Section 4.3 describes the data collection methods, including the design and sequencing of conceptually rich tasks, classroom and participant observations, and semi-structured interviews used to elicit students' reasoning and representational strategies. Section 4.4 presents the procedures for analyzing the qualitative data, detailing the coding process, thematic categorization, and interpretive techniques aligned with the OSA framework. Section 4.4 presents the findings from the qualitative analysis, emphasizing recurring patterns in students' reasoning and conceptual challenges related to matrix representation. Together, these sections provide a thorough account of how the qualitative dimension of the study was theoretically grounded, implemented, and analyzed.

Part two focuses on the quantitative strand of the research and is organized into three sections. Section 4.5 describes the design of the assessment instrument, which was developed based on insights from the qualitative analysis. This instrument was structured to probe students' understanding of key concepts such as coordinate vectors, change of basis, and matrix representation in abstract vector spaces. Section 4.6 outlines the procedures for data collection and analysis, including the use of Statistical Implicative Analysis (SIA) and complementary descriptive and inferential statistical methods. These procedures were chosen to reveal implicative relationships, performance trends, and conceptual dependencies across a broader sample of students. Together, these sections articulate the rationale, implementation, and outcomes of the quantitative methodology, providing empirical support that complements and extends the qualitative findings introduced in Part One.

4.3 Part One: Qualitative Research

4.3.1 Theoretical Framework: The OSA Approach

This study is grounded in the OSA, a framework theory of educational design in mathematics. As Godino (2025) explains, the theory of educational design in mathematics offers a comprehensive framework built upon the OSA. It integrates theoretical assumptions and analytical tools to design

and interpret teaching–learning processes, emphasizing the onto-semiotic nature of mathematical activity, emergent objects, and cognition. This framework models the structural and dynamic features of educational processes, accounting for the multifaceted nature of didactic interactions (Godino, 2025, p. 79).

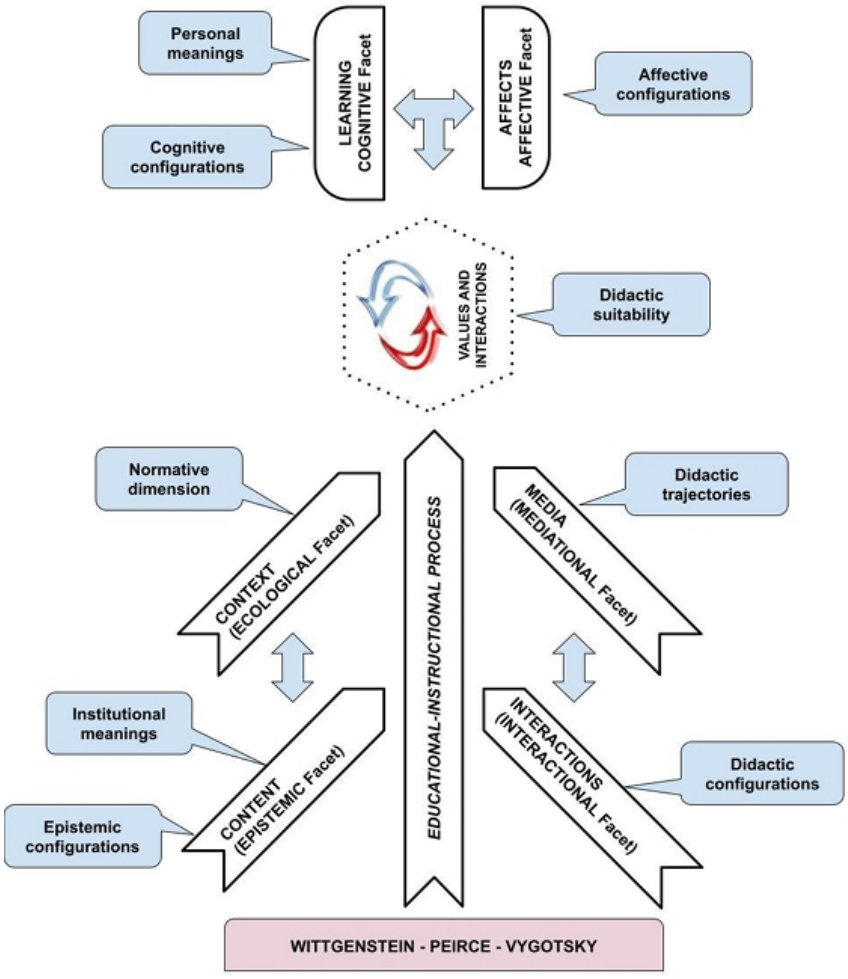


Figure 4.2: OSA tools and connections (Godino, 2025, p.83).

As our chapter three explains, the OSA includes a model of categories of norms and metanorms- how norms themselves should be followed, evaluated, or changed- along with criteria for evaluating didactical suitability, enabling the explanation of didactical phenomena and offering guidelines for optimizing educational practices. Central constructs such as didactical configuration and didactical trajectory facilitate detailed descriptive and explanatory analyses of both

the design and implementation of educational processes.

Furthermore, when combined with the postulate of the ontosemiotic complexity of mathematical content and the construct of didactical suitability, these tools support the development of a mixed didactical model. This model aims to reconcile the traditional objectivist (transmissive) models with constructivist (inquiry-based) approaches, providing a balanced pathway toward optimizing students' mathematical learning outcomes (Godino, 2024; Godino, Batanero, & Font, 2014).

In general, the OSA provides analytical tools to examine how students develop understanding through their engagement with mathematical objects, language, and representations.

4.3.2 Application to Matrix Representation Learning

For this study, we apply the analytical tool and find the connection as the OSA framework aligns with the research objectives by focusing on both the mathematical objects involved in matrix representation and the meaning-making processes students engage in while learning.

The learning processes targeted in this study—such as interpreting semiotic representations, transitioning between forms, and resolving conceptual conflicts—can be understood through the OSA. As illustrated in *Figure 2*, the *Mediational*, *Interactional*, and *Epistemic* facets of the educational-instructional process (Godino, 2025; Godino et al., 2014) directly relate to how students engage with matrix representation tasks as following.

1. **Interpret and manipulate semiotic representations of matrices, vectors, changing bases, coordinate vectors, and linear transformations.**

   This action engages the *Media (Mediational Facet)* and *Semiotic Configurations* under the Epistemic Content facet. It reflects how students use symbols and notations to understand abstract concepts.

2. **Navigate transitions between different representations (algebraic, geometric, ab-**

stract).

This requires coordination across the *Media*, *Interactional*, and *Didactic Configuration* facets (Font, Godino, & D'Amore, 2010). It also involves *Affective-Cognitive* dimensions, as students respond emotionally to representational shifts.

3. **Develop epistemic configurations related to matrix representation.**

   This is situated within the *Content (Epistemic Facet)* and is influenced by *Institutional Meanings* and the *Context (Ecological Facet)*. These configurations reflect how students interrelate concepts like basis, transformation, and matrix.

4. **Resolve semiotic conflicts that arise during learning (D'Amore, 2006).**

   Conflict resolution is mediated through *Values and Interactions*, shaped by *Media*, *Interactional* support, and *Affective Configurations*. These moments reveal conceptual tensions that impact both understanding and affective engagement.

The OSA framework allows for a layered analysis of how these student actions are situated within broader didactic and institutional structures, offering a comprehensive lens to study learning dynamics in abstract vector spaces with arbitrary basis.

4.3.3 Research Setting and Participants

Part One: Second-Level Linear Algebra Classes

This study was conducted at a large public university in the southern United States and targeted undergraduate students enrolled in a second-level Linear Algebra course (Linear Algebra II). The research aimed to examine students' learning processes and conceptual development across two academic semesters: Fall 2024 and Spring 2025.

In Fall 2024, the first phase of the study involved 15 students who participated in the first six-week instructional intervention. Each week, students engaged in two 20-minute, concept-focused sessions aimed at promoting critical thinking and deepening their conceptual understanding of matrix representation. Student learning was assessed through a midterm and a final exam, which offered valuable insights into their evolving grasp of core topics such as basis change, coordinate systems, and linear transformations. Data collection during this semester included analysis of students' written work, classroom and participant observations, and informal post-class conversations about their study experiences. This exploratory phase served as a pilot and groundwork for the more comprehensive and formal study conducted in Spring 2025.

Building on insights and data collection experience gained from the exploratory phase in Fall 2024, the second phase of the study, conducted in Spring 2025, constituted the formal investigation. This phase involved 16 students observed over a comparable six-week period and featured a more structured and comprehensive data collection protocol. Students completed a midterm and final assessment, and an open-ended questionnaire was administered two weeks after the matrix representation unit. The questionnaire was designed to elicit students' reflective feedback on their learning processes, representational reasoning, and conceptual interpretations.

In mid-April, a formative review test was administered to assess students' understanding of key ideas related to vector spaces, basis change, coordinate systems, and matrix representation of linear transformation to an abstract level. Thirteen students participated in this diagnostic assessment, which served to identify persistent misconceptions and reinforce essential concepts. Data collection continued through a series of individual online interviews, with six students voluntarily participating in semi-structured sessions conducted via Webex. Additionally, open Q&A sessions were held one week prior to the final exam, offering further opportunities to probe students' reasoning strategies, clarify conceptual ambiguities, and document their evolving understanding of linear transformations and coordinate vectors, etc.

The six-week instructional sequence was deliberately aligned with foundational topics in Linear Algebra II, including vector spaces, coordinate systems, linear transformations, and change

of basis. These topics build on knowledge acquired in Linear Algebra I and require students to synthesize procedural fluency with conceptual reasoning. By targeting these concepts early in the semester, the study allowed for a deep examination of how students construct meaning, reconcile symbolic representations with abstract mathematical structures, and negotiate ontological and epistemological perspectives on matrix representation.

Table 4.1: Summary of data collection during the exploratory phase

Method	**Purpose**	**Participants**
20-minute exit sheets (2×/week, 6 weeks)	Capture student reflections and conceptual difficulties immediately after each session	18 students
Two Midterm Exams	Assess conceptual understanding during and across instructional phases	13 students
Final Exam	Evaluate overall learning outcomes	8 students
Student Written Work	Track written reasoning and use of representations	15 students
Observation & Informal Conversations	Gain insights into student attitudes and experiences	13 students

*Note.*This table summarizes the data collected during the exploratory phase in Fall 2024

Part Two: First-Level Linear Algebra Class

We also decided to administer an on-site review test to a large group because, over the past few years, instructors have consistently encountered challenges in teaching Linear Algebra II. As a result of a comment from one of our committee members, the goal was to trace and evaluate students' foundational understanding at the end of their Linear Algebra I course, particularly to see how well they were prepared to engage with matrix representation. When students, on average, struggle with core concepts such as vectors with respect to different bases, coordinate systems vs. coordinate vectors, and linear transformations, both geometrically and algebraically, it becomes difficult to advance to more abstract topics.

Although it is not usual to present results in the methodology section, in this case, we consider the findings as part of a prior analysis based on diagnostic assessment. The review test

was therefore a necessary step to gain a clearer picture of students' conceptual readiness and to better identify the gaps that need to be addressed through targeted instructional support. Therefore, we also incorporated data from a broader group of 50 undergraduate students enrolled in a Linear Algebra I course. These students provided a representative snapshot of student understanding prior to advancing to Linear Algebra II. A 30-minute diagnostic test was administered by the end of the spring semester of 2025 to assess their foundational knowledge of matrix representation—a key concept bridging both course levels.

Table 4.2: Summary of data collection during the formal phase

Method	**Purpose**	**Participants**
Midterm Exam	Assess conceptual understanding during intervention	13 students
Final Exam	Evaluate overall learning outcomes	8 students
Open-Ended Questionnaire	Elicit reflective feedback on learning experiences and conceptual interpretations	13 students
Formative Review Test	Diagnose misconceptions and reinforce key concepts	13 students
Webex Semi-Structured Interviews	Probe reasoning strategies and clarify conceptual gaps	6 students
Open Q&A Sessions	Address final questions and document evolving understanding	1 student
Diagnostic Test (Linear Algebra I)	Assess pre-knowledge of matrix representation among a broader population before enrollment in Linear Algebra II	50 students

Note. This table presents data gathered during the formal phase in Spring 2025, representing the study's second stage.

The test included eight multiple-choice questions that target conceptual understanding, two reasoning items to elicit deductive thinking, and two open-ended problem-solving tasks that require procedural and representational fluency. An item specifically asked students to sketch a graph that illustrates the transition from a geometric vector to its coordinate vector representation. This task was intended to reveal students' understanding of the distinction between a vector and its representation relative to a given basis, thereby assessing semiotic flexibility and structural understanding of vector spaces. Student responses were systematically coded and entered into an Excel database

for analysis. This quantitative data set provides information on student preparation and serves as a baseline for comparison with qualitative findings targeted at Linear Algebra II.

4.3.4 Qualitative Data Collection Methods

Data collection was not a separate, isolated phase but an ongoing and dynamic part of our research process. The way we collect data is influenced by the methodological choices and the questions we are trying to explore. In our case, the OSA provided a lens through which we viewed each classroom and participant observation, interview, and written response, not just as standalone data points, but as part of a larger effort to understand how students construct mathematical meaning. Our epistemological stance also played a role: we approached the study from a flexible and eclectic perspective, which meant being open to the idea that students' understanding could shift throughout the process and that our interpretations would evolve as well. Because of this, data collection and analysis often happened simultaneously. For instance, something a student said in an interview would lead us to revisit the classroom video or rethink how we framed a task. This back-and-forth between gathering and making sense of the data was not only inevitable but essential to how the research unfolded. Therefore, multiple qualitative instruments were employed to capture the depth and variation in students' cognitive and epistemological responses as the following:

Concept-Based Tasks for Linear Algebra II

Students enrolled in Linear Algebra II completed structured concept-based questions designed to probe their understanding of the following key topics:

- **Vector spaces and subspaces:** including recognition of subspace criteria, linear independence, and span relationships;
- **Linear transformations between vector spaces:** with a focus on properties such as linearity, kernel, image, and dimension changes;

- **Coordinate vectors relative to different bases:** examining how students transition between bases and interpret coordinate representations;
- **Matrix representation of linear transformations:** assessing their ability to derive and interpret matrices that represent transformations across specified domain and codomain bases to an abstract level.

These tasks were tailored to the level and content expectations of students in a second-level undergraduate linear algebra course. To capture a multifaceted view of student understanding, questions were presented in a variety of formats, including fill-in-the-blank items, identification and correction of symbolic errors, multiple-choice conceptual checks, and applied mathematical problems requiring interpretation of real-world scenarios.

Each item required students to explain their reasoning, construct examples, or analyze transformations across different bases. Tasks were intentionally developed to reveal students' epistemic configurations and to identify potential semiotic conflicts—such as confusion between mathematical objects and their symbolic representations (Font et al., 2010). This variety in task design enabled a deeper exploration of students' representational fluency, conceptual flexibility, and problem-solving strategies.

Sample Task 1: Problem Solving in Linear Algebra II (Fall 2024 & Spring 2025)

The linear transformation L defined by $L(p(x)) = p'(x) + p(0)$ *maps* P_3 *into* P_2. *Find the matrix representation of L with respect to the ordered bases* $\{x^2, x, 1\}$ *and* $\{2, 1 - x\}$. *Then, for* $p(x) = x^2 + 2x - 3 \in P_3$*:*

(a) Find the matrix representation with respect to the ordered bases.

(b) Find the coordinates of $L(p(x))$ *with respect to the ordered basis* $\{2, 1 - x\}$ *(Leon, 2015, p.198).*

In addition to the tasks administered in Linear Algebra II, a separate 30-minute review test was conducted during the final class session of Spring 2025 for 50 students enrolled in a Linear

Algebra I course. This review assessment included multiple-choice questions, fill-in-the-blank items, error-identification tasks, and structured problem-solving exercises. The purpose of this assessment was to evaluate students' foundational understanding of coordinate systems and basis representations and to prepare them for the more abstract material introduced in Linear Algebra II.

Concept-Based Tasks for Linear Algebra I

- Assess students' ability to recognize and apply a non-standard basis in $\mathbb{R}^2$;
- Evaluate procedural fluency in computing coordinate vectors using both equation-based and matrix-based approaches;
- Reinforce understanding of the distinction between a vector and its coordinate representation;
- Identify semiotic conflicts, such as conflating mathematical objects with symbolic expressions;
- Diagnose common conceptual and symbolic errors in basis-related reasoning;
- Promote representational flexibility and foundational understanding in preparation for Linear Algebra II.

Sample Task 2: Problem Solving in Linear Algebra I (Spring 2025)

Let $S = \{(2,3),(1,-1)\}$ *be a basis for* $\mathbb{R}^2$*, and let* $x = (8,7)$ *be a vector in* $\mathbb{R}^2$*. Find the coordinate vector* $[x]_S$ *relative to the basis S. Use both methods below:*

- **Method 1**: Solve the system of linear equations.
- **Method 2**: Use the P^{-1} method (Change of Basis via Inverse Matrix).

Coding Procedures and Analytical Approach

The data were analyzed using both theory-driven and inductive coding strategies, grounded in the OSA. The theory-to-data component of the analysis involved generating initial codes based on key constructs from the OSA framework, such as *semiotic conflict*, *ontological object*, and *epistemic configuration*. A data-to-theory approach was used to allow new patterns to emerge inductively from the students' written responses.

This dual strategy enabled the identification of both theoretically anticipated constructs and novel, unanticipated insights—such as students' use of metaphors, spatial or visual language, and embodied reasoning when articulating their understanding of vectors, coordinate vectors, basis, change of basis, and the matrix representation of linear transformations.

To support the coding process, Excel spreadsheets were used to organize and categorize emerging codes and themes. This facilitated the identification of recurring patterns in how students reasoned about key linear algebra concepts. Some patterns aligned closely with the OSA framework—for instance, instances of semiotic conflict or references to specific mathematical objects. Others emerged more organically from the data and reflected students' intuitive or informal reasoning. These included reliance on spatial descriptions, hand-drawn representations, or physically grounded metaphors. While not directly anticipated by the theoretical framework, these emergent themes offered valuable insights into the diverse semiotic and cognitive strategies students employed to make sense of abstract mathematical structures.

Participant Observations

According to DeWalt and DeWalt (2010), participant observation is widely recognized as the central and defining method of research in cultural anthropology. In this study, participant observation was adapted from ethnographic traditions and applied within the context of matrix

representation to better understand students' learning processes as they unfolded in real time. Rather than maintaining a detached role, the researcher engaged actively and consistently in the classroom environment during both the Fall 2024 and Spring 2025 semesters. Drawing on the guidelines outlined by DeWalt and Dewalt, the following practices were central to the implementation of participant observation:

- Regular engagement in the classroom over an extended six-week instructional sequence;
- Close observation of how students communicated mathematically, including both formal terminology and informal language used to describe abstract concepts;
- Participation in instructional activities such as group discussions, formative assessments, and in-class problem-solving tasks;
- Use of informal conversations before, during, and after class to elicit spontaneous reflections on learning experiences;
- Observation of non-instructional moments (e.g., peer interactions during transitions or breaks) to gain insight into authentic learning behaviors;
- Maintenance of detailed, chronological field notes that documented verbal responses, gestures, classroom interactions, and significant patterns of engagement;
- Integration of both tacit knowledge (e.g., classroom norms and behavioral cues) and explicit student reasoning into the analytical process.

These strategies provided a rich, contextualized view of students' meaning-making processes and supported triangulation with interview, written, and assessment data.

Semi-Structured Interviews

Interview prompts were designed in part based on patterns observed in students' written responses and classroom behavior. The semi-structured format ensured consistency across participants while allowing flexibility to pursue emergent themes (Kvale & Brinkmann, 2015).

One-on-one interviews were conducted following completion of exams and questionnaires. These interviews, lasting one hour, were held either virtually via Webex or in-person based on participant preference. Each interview was audio-recorded with consent and later transcribed.

The interview protocol included:

- Think-aloud problems related to matrix representation
- Reflective questions about students' solution strategies
- Questions designed to elicit students' conceptual understanding
- Probes to clarify reasoning observed in written work

The semi-structured format allowed for consistent data collection while providing flexibility to explore emerging themes (Kvale & Brinkmann, 2015).

4.3.5 Data Analysis: Thematic Coding and Onto-Semiotic Interpretation

To analyze the qualitative data gathered from interviews, participant observations, written tasks, and pre- and post-assessments, this study employed a thematic analysis following the six-phase model proposed by Braun and Clarke (2006). This method was selected for its flexibility and rigor in identifying patterns across diverse data sources, while allowing for a theoretically grounded interpretation aligned with the OSA.

Analysis of qualitative data employed thematic analysis (Braun & Clarke, 2006) guided by the OSA framework. The analysis followed these steps:

1. **Familiarization with data**: Reading transcripts, field notes, and student work multiple times
2. **Initial coding**: Identifying segments related to OSA elements (practices, objects, processes)

3. **Theme development**: Grouping codes into potential themes related to students' understanding
4. **Theme review**: Checking themes against the data and theoretical framework
5. **Theme definition**: Refining and naming themes to capture key aspects of student understanding
6. **Report production**: Integrating themes into a coherent narrative with supporting evidence

Following the initial thematic coding, the emergent themes were interpreted through the lens of the OSA, which provides an analytical framework for identifying the mathematical objects, processes, semiotic representations, and configurations involved in student activity, as well as the underlying ontological and epistemological obstacles. By overlaying OSA's constructs onto the thematically coded data, the analysis moved beyond descriptive categorization and toward a deeper explanation of students' meaning-making processes in the context of matrix representation, coordinate systems, and change of basis.

This dual-layered analysis ensured methodological transparency and theoretical coherence. The thematic structure provided a systematic approach to data coding, while the OSA interpretation enabled a rich, multidimensional understanding of students' learning trajectories, semiotic conflicts, and representational strategies in linear algebra.

Analysis Procedure

To analyze the qualitative data collected from interviews, written responses, and classroom observations, we initially considered several qualitative analysis tools such as MaxQDA, NVivo, ATLAS.ti, and Dedoose. However, we ultimately decided to use Microsoft Excel as our primary tool for data coding and categorization. We chose Excel due to its flexibility, accessibility, and ability to support an organic and transparent analytic process without imposing rigid structural constraints.

Each participant's response was systematically entered into a structured spreadsheet with dedicated columns for participant ID, data source, raw excerpts, applied codes, emerging themes, and analytic memos. This approach facilitated clear documentation of our coding decisions and supported the iterative refinement of thematic categories across different data sources.

The coding categories were developed using the OSA, aligning each excerpt with key theoretical constructs, including *semiotic conflicts*, *epistemic configurations*, *ontological barriers*, and *representational fluency*. *Semiotic conflicts* were identified when students experienced difficulty coordinating different representational systems—such as symbolic notation, graphical illustrations, or verbal explanations—resulting in misunderstandings or incomplete interpretations. *Epistemic configurations* captured how students engaged with mathematical objects, processes, and discourse to construct meaning within a specific learning context. *Ontological barriers* were observed when students struggled to distinguish the nature or existence of mathematical entities, for instance, by conflating vectors with coordinate vectors or failing to differentiate between transformations as dynamic processes and their matrix representations. *Representational fluency* refers to students' ability to interpret and translate across various semiotic registers, a skill that is particularly essential for understanding abstract concepts like matrix representations in the context of changing bases.

Using Excel allowed for transparent and flexible coding while providing a clear audit trail that mapped qualitative insights to the theoretical framework. A sample coding table is included in the appendix.

A Priori Analysis of Diagnostic Test Design (Linear Algebra I)

In the last class of Spring 2025, just before the final exam, a 30-minute in-class review test was administered to 50 students enrolled in Linear Algebra I. This assessment aimed to evaluate students' foundational understanding before they transitioned into Linear Algebra II, which introduces more advanced topics. The test comprised 12 items covering key concepts such as subspace, basis selection and change, linear independence, coordinate vectors, and linear

transformations in Euclidean space. Thematic analysis was applied to identify recurring reasoning patterns and highlight specific areas of conceptual difficulty, including semiotic misalignment, limited generalization, and weak integration of symbolic and geometric representations—factors that may hinder students' progression into more abstract levels of linear algebraic thinking.

The following table presents the levels of abstraction associated with each question, providing insight into how students engaged with the test content across symbolic, algebraic, and geometric registers.

Table 4.3: OSA based abstraction levels in student responses

Abstraction Level	**OSA-Based Characteristics in Student Responses**	**Category**
Level 0	Relies on informal definitions or visual cues; misinterprets subspace properties without structured reasoning.	Low Level
Level 1	Identifies basis properties like independence and span but reasoning is often procedural or lacks depth.	Mid Level
Level 1–2	Shows understanding of coordinate vectors and basis relations with some integration across representations.	Mid–High Level
Level 2+	Connects coordinate systems, matrix operations, and change-of-basis concepts with flexible reasoning across symbolic, algorithmic, and geometric registers.	High Level

Table 4.4: Conceptual focus of test questions Q1–Q12

Question	Description
Q1	Assesses conceptual understanding of subspace definitions, particularly the distinction between algebraic closure and geometric conditions.
Q2	Evaluates the student's ability to recognize a valid basis set in $\mathbb{R}^2$, focusing on linear independence and span.
Q3	Tests interpretation of coordinate vectors relative to the standard basis.
Q4	Assesses understanding of the change-of-basis matrix and its role in coordinate transformation.
Q5	Targets recognition of the two essential properties of a basis: linear independence and spanning.
Q6	A fill-in-the-blank item that directly probes the student's understanding of linear independence in their own words.
Q7	Evaluates students' ability to articulate the definition of a coordinate vector and identify its role in expressing vectors with respect to a given basis.
Q8	Assesses symbolic fluency and conceptual understanding of the change-of-basis formula, specifically the use of $P^{-1}[v]_B$ to compute $[v]_C$.
Q9	Evaluates students' ability to critique an incorrect definition of a subspace, focusing on their understanding of closure under addition and scalar multiplication.
Q10	Assesses conceptual reasoning and identification of flawed logic in determining subspace validity, specifically targeting misinterpretation of symmetry and origin containment as sufficient conditions.
Q11	Assesses students' ability to interpret and visualize basis transformations through a combination of graphical and algebraic tasks. It evaluates semiotic fluency in switching between standard and non-standard bases, computing coordinate vectors, and representing vectors geometrically.
Q12	Tests students' procedural and conceptual understanding of coordinate vector computation relative to a non-standard basis using two distinct methods: solving a linear system and applying the inverse matrix approach. The item emphasizes flexibility in representation and reasoning across symbolic and algorithmic forms.

The eight review test questions targeted four core conceptual domains in linear algebra: vector spaces, basis, basis change, and coordinate vectors.

Vector Space (Q1 & Q9)

Students who failed to correctly identify subspaces (e.g., Q1) or critique flawed definitions (Q9) often operated at **OSA Level 0** or **early Level 1**. Their reasoning tended to rely on physical intuition (e.g., symmetry, presence of origin) rather than formal criteria such as closure under scalar multiplication and vector addition.

- *OSA Limitation:* These students exhibited ontological confusion, failing to align the mathematical object (subspace) with its institutional meaning. Their semiotic reasoning lacked formal justification or abstraction.

Basis (Q2, Q5, Q6, Q10)

Students performed relatively well on items assessing recognition of valid basis sets and linear independence. However, explanations were often surface-level or memorized. For instance:

- Q6 responses correctly identified linear independence but lacked supporting rationale.
- In Q10, higher-performing students critiqued flawed reasoning about subspaces, indicating a shift toward **OSA Level 2**.
- *OSA Observation:* Students at higher levels demonstrated semiotic generalization (e.g., distinguishing symbolic from visual representations), while lower-level responses relied on isolated examples (materialization).

Coordinate Vectors (Q3 & Q7)

Several students conflated coordinate vectors with standard vectors. For example:

- In Q3, many treated $(2, 3)$ as both the vector and its coordinates without referencing a basis.
- Q7 elicited more precise thinking due to its definitional nature, but misconceptions persisted.
- *Common OSA Barrier:* Many students lacked **signification**—they failed to interpret coordinate vector notation contextually, relying instead on symbolic matching or intuition without grounding in meaning.

Basis Change & Transformations (Q4 & Q8)

Q4 and Q8 revealed students' procedural familiarity with basis change but highlighted conceptual weaknesses.

- While many recalled or applied the formula $[v]_C = P^{-1}[v]_B$, few explained why the inverse was necessary.
- *OSA Profile:* Responses indicated engagement with idealized rules (e.g., matrix formulas) but weak argumentation and limited ontological awareness regarding how coordinate systems transform.

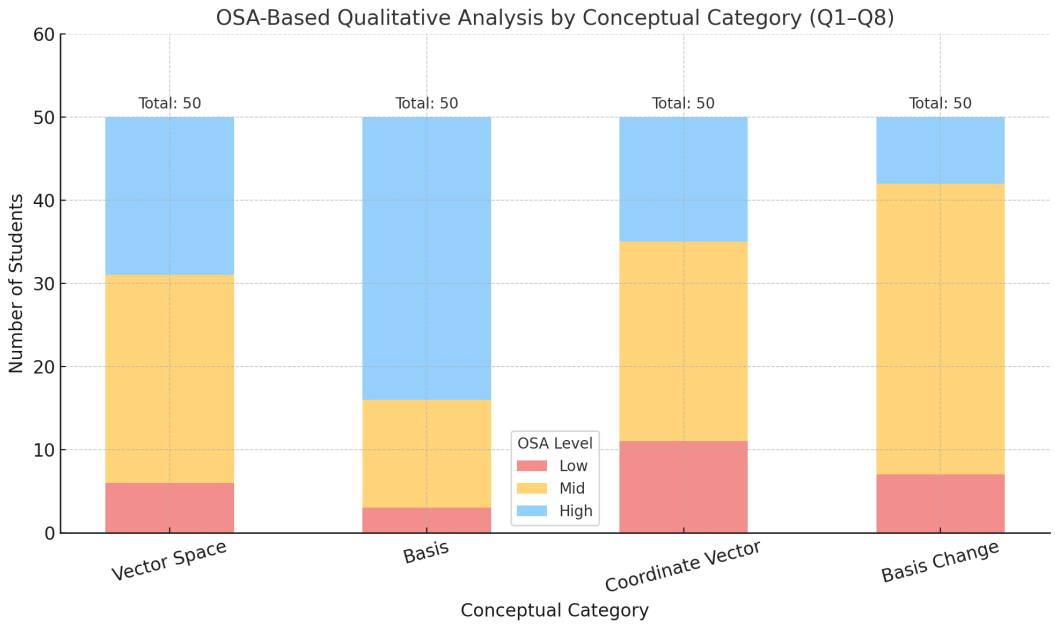


Figure 4.3: Analysis by conceptual category (Q1–Q8, Linear Algebra I, Spring 2025)

Subspace Concepts, Basis Representations, and Coordinate Transformations: Q9 - Q12

We categorized the scores of Questions 9 through 12 into three ontosemiotic levels: Low, Mid, and High. This classification was based on each student's average score across the four tasks, which included correcting definitions of subspaces, evaluating whether a set is a subspace, interpreting basis transformations graphically, and computing coordinate vectors.

25 students (50%) fell into the **Low-level** category (average score ≤ 0.3), **20 students (40%)** were classified as **Mid-level** (average score between 0.3 and 0.7), **5 students (10%)** reached the **High-level** configuration (average score ≥ 0.7).

Low-Level OSA Configuration. Students in this group demonstrated minimal activation of meaningful mathematical structures. In Q9 and Q10, they tended to accept incorrect or overly vague definitions of subspaces (e.g., focusing solely on symmetry or the presence of the origin). In Q11 and Q12, these students often failed to represent basis change conceptually or computationally. OSA interpretation suggests a breakdown in linking mathematical objects (e.g., subspace, basis) with processes (e.g., validation, transformation) and semiotic systems (e.g., symbolic vs. graphical representations).

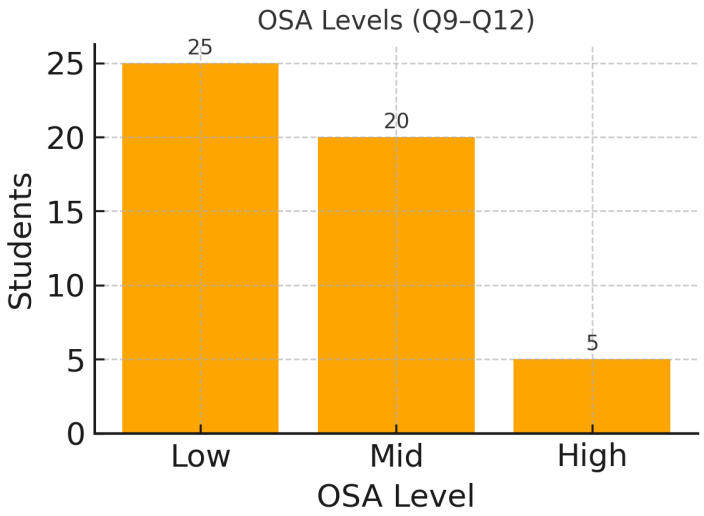


Figure 4.4: [

Analysis by conceptual category (Q9–Q12, Linear Algebra I, Spring 2025)]Distribution of Student Responses by OSA Level for Q9–Q12. The classification is based on average performance across subspace and basis transformation tasks, revealing three tiers of semiotic configuration: Low (50%), Mid (40%), and High (10%).

Mid-Level OSA Configuration. Students in the mid-level range showed partial but incomplete semiotic configurations. They might have correctly identified or critiqued flawed reasoning in subspace definitions and applied procedural methods to compute coordinate vectors. However, they often lacked conceptual coherence between representations. According to OSA, this group reflects isolated or underdeveloped links between mathematical processes and the underlying institutional or conceptual norms that give them meaning.

High-Level OSA Configuration. Students in this group demonstrated fully coordinated semiotic configurations. They accurately critiqued incorrect subspace reasoning and provided valid arguments rooted in definitions and logical necessity. They successfully applied both symbolic and graphical tools for basis transformation and understood the conceptual meaning of coordinate vectors. These students exemplified the kind of mathematical activity that OSA describes as a coherent interaction among objects, processes, representations, language, and arguments.

Conclusion: Patterns of Reasoning Across Core Concepts

The qualitative findings from Questions 1–12 revealed important insights into how students reason about core concepts in linear algebra. The high proportion of responses reflected early stage understanding, characterized by intuitive reasoning, symbolic mimicry, and reliance on surface features rather than formal mathematical structures. This was especially evident in tasks involving subspace identification and basis change, where students often applied procedures without grasping their conceptual meaning.

Students at the mid-level displayed partial understanding. While they could often carry out procedures correctly, their reasoning lacked coherence across representations. For example, they might perform a correct basis change algebraically but be unable to explain why an inverse matrix is used or how it relates to a transformation of coordinate systems.

Only a small group of students demonstrated high-level reasoning. These students consistently connected definitions, symbolic procedures, and visual representations and provided logically grounded justifications. Their responses showed a more integrated understanding aligned with the OSA, linking mathematical objects, processes, representations, and arguments in a coherent and meaningful way.

Overall, this analysis highlights a need for instruction that goes beyond procedural fluency. Helping students bridge symbolic manipulation with conceptual understanding—especially through coordinated use of definitions, diagrams, and discourse—can support deeper mathematical thinking in linear algebra.

4.4 Part Two: Quantitative Study

To evaluate students' understanding of linear transformations, coordinate vectors, and matrix representations in abstract vector spaces, this study employed a *quantitative methodology* anchored in *Statistical Implicative Analysis (SIA)*. This methodology rests on three epistemological assumptions: (1) that learning is nonlinear and context-dependent, (2) that structured patterns of reasoning can be inferred from student responses, and (3) that conceptual understanding often emerges from

complex interactions among foundational ideas. A diagnostic assessment tool was developed to elicit students' abilities in transitioning between coordinate systems, applying matrix transformations, and interpreting the algebraic structures underlying these processes. The responses collected served as a foundation for both descriptive and implicative statistical analysis, enabling deeper insight into student reasoning.

4.4.1 Statistical Implicative Analysis (SIA) as a Quasi-implicative approach

SIA was selected for its capacity to uncover latent implicational structures within response data. Unlike conventional statistical techniques, SIA is *quasi-quantitative*—a hybrid analytic approach that combines statistical rigor with conceptual interpretation. It is particularly well-suited for educational research, where students' learning often involves multiple, interdependent pathways. For instance, understanding one concept may depend on the integration of several prior ideas learned in different ways, or students may revisit earlier topics after encountering difficulties, resulting in relationships that are not strictly linear or sequential. As noted in the foundational literature, "The criterion is statistical. It is nonlinear, robust to noise (i.e., not very influenced by the first counter-examples), and it becomes very low if the counter-examples often reappear." (p.12), and "The SIA provides a complete framework to evaluate the interestingness of the rules and to structure them in order to discover relationships at different granularity levels. The underlying objective is to highlight the emerging properties of the whole system, which cannot be deduced from a simple decomposition into sub-parts." (Gras, Suzuki, Guillet, & Spagnolo, 2008, p.13). These features allow SIA to detect nonlinear and probabilistic relationships that reflect the emergent conceptual organization of students' mathematical thinking. Pairwise similarities were reviewed during preliminary screening to support interpretation of emerging patterns; however, all reported directional implications were established strictly using CHIC's implicative logic, which emphasizes rule-based, asymmetrical dependencies rather than mutual association.

4.4.2 Data Processing and Analysis

Following data collection, student responses were systematically entered into the R programming environment. The dataset was cleaned and standardized to remove incomplete responses and ensure consistency. *Descriptive statistics*, including means and standard deviations, were computed to summarize general trends and performance levels.

To examine internal relationships among conceptual domains, *correlation analyses* were conducted using the `cor.test()` function. These analyses explored the strength of association between students' proficiency in coordinate vectors and their accuracy on matrix transformation tasks. Subsequently, *linear regression* was used to determine whether skill in coordinate representation could significantly predict success in transformation-based problem-solving.

To enhance interpretability, *data visualizations* were generated using the `ggplot2` package. Scatter plots with fitted regression lines provided visual representations of performance trends, highlighting clusters of students who struggled with transformations or coordinate changes. These visual tools reinforced and extended the statistical insights, offering a clearer picture of learning difficulties and conceptual variation.

4.5 Summary

This chapter presented the mixed-methods approach used in the study, which combined the OSA and SIA to investigate how students understand matrix representation, coordinate vectors, and basis transformations. The methodology incorporated qualitative techniques—such as classroom observations, interviews, and concept-based tasks—as well as quantitative tools like diagnostic testing and implicative statistical modeling. Together, these methods allowed for a comprehensive view of students' reasoning, capturing both the depth of individual thinking and broader patterns across the group.

To provide a foundational context, the study first examined students from a Linear Algebra I course. This early analysis, guided by OSA, revealed that many students struggled with core

ideas such as distinguishing vectors from coordinate vectors, understanding basis changes, and applying definitions related to subspaces. Most responses reflected early-stage reasoning—often relying on intuition or symbolic familiarity rather than formal understanding. These conceptual gaps suggest potential obstacles as students transition to more abstract content in advanced linear algebra courses.

The SIA-based analysis uncovered several implicational chains in students' reasoning. For instance, understanding the geometric meaning of a basis often implied an ability to perform change-of-basis computations. While these relationships are not strictly causal, they highlight conceptual dependencies that can inform instructional design. Overall, the quantitative results from SIA provided empirical support for identifying common challenges and patterns in student understanding. These insights offer meaningful guidance for developing teaching strategies that help students engage more deeply with the structure of linear transformations and abstract vector spaces.

With that foundation established, the next chapter shifts focus to students in Linear Algebra II. It presents both qualitative and quantitative findings to explore how their thinking develops when engaging with more complex topics.

Chapter 5

Results

This chapter presents the integrated findings from both the qualitative and quantitative components of the study. Grounded in the mixed-methods framework outlined in Chapter 4, the results are organized to reflect students' conceptual understanding of matrix representation of linear transformations during Fall 2024 and Spring 2025. Focusing on the underlying structures of vector spaces, coordinate systems, and basis transformations, this analysis draws on student performance scores, written responses, and interviews to examine the depth and flexibility of student reasoning in Linear Algebra II.

The chapter is structured into three main sections. Section 5.1 outlines theoretical findings interpreted through the OSA, focusing on the semiotic and epistemological dimensions of students' responses. Section 5.2 presents thematic patterns and diagnostic test outcomes that reveal common trends and recurring obstacles. Section 5.3 offers detailed insights from selected individual case studies, illustrating how specific students navigate the concepts of abstract vector spaces and matrix representation in problem-solving contexts. Together, these findings provide a comprehensive view of how students internalize and operationalize coordinate-based representations and matrix forms of linear transformations.

Figure 5.1 provides a visual overview of this dual structure and serves as a conceptual framework for analyzing the data collected in this study (Godino, 2025). The upper portion of the figure illustrates a hierarchical model of abstraction, ranging from Level 0—where reasoning is intuitive, context-dependent, and often fragmented—to progressively higher levels of conceptual sophistication. These higher levels are characterized by students' ability to integrate formal definitions, symbolic representations, and abstract generalizations. This vertical structure was used to evaluate the depth of student reasoning as reflected in both their written responses and interview data. For

example, students exhibiting low-level configurations often relied on surface features or procedural recall, while those at higher levels demonstrated more coherent and flexible reasoning aligned with formal mathematical practices.

The lower portion of the framework complements the vertical hierarchy by modeling epistemic and semiotic processes in dualistic pairs that reflect the dynamic tensions inherent in mathematical activity. These include *representation vs. signification*, *generalization vs. particularization*, *analysis vs. synthesis*, and *materialization vs. idealization*—each positioned on opposing sides of the octagon to emphasize their conceptual interplay. These dualities informed both the thematic coding of qualitative data and the interpretation of students' conceptual performance. For instance, frequent confusion between vectors and coordinate vectors pointed to difficulty navigating between *materialized* (e.g., coordinate representations) and *idealized* (abstract vector space) interpretations. Similarly, errors in constructing transformation matrices often revealed misalignment between *symbolic representations* and their intended *significations*.

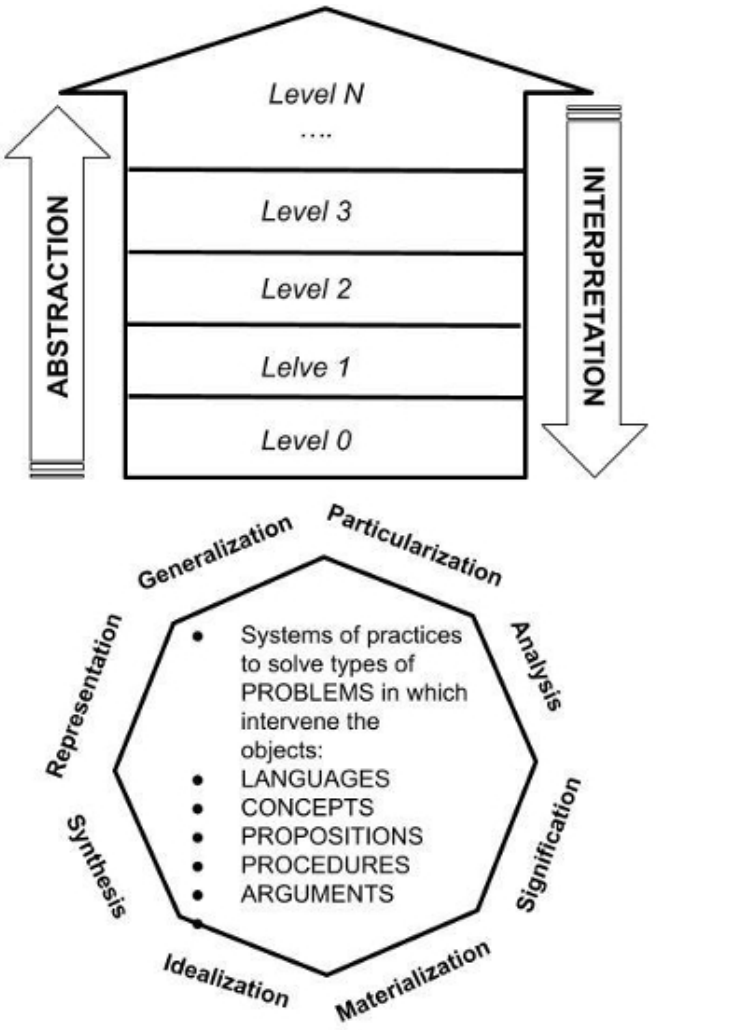


Figure 5.1: [

OSA framework for abstraction and semiotic dualities]The OSA framework used to interpret qualitative findings. The top portion represents levels of abstraction in mathematical reasoning, while the bottom portion illustrates cyclic semiotic and epistemic practices involved in problem-solving (Godino, 2025, p.83).

5.1 Theoretical Facets of Results

The responses of each student were analyzed using a rubric based on the OSA framework. The rubric operationalized the framework's semiotic practices and abstraction levels, enabling classification of observed reasoning across both dimensions.

Table 5.1: Epistemic-Semiotic Practices in Student Responses Based on the OSA Framework

Practice	**Example Indicators**	**Presence in Response**
Representation	Uses symbolic notation, graphs, or coordinate vectors appropriately.	51 / 55
Materialization	Works concretely with examples or computations.	51 / 55
Idealization	Generalizes from examples or removes irrelevant details.	51 / 55
Signification	Interprets symbols within context (e.g., understands what $[v]_B$ means).	51 / 55
Generalization	Infers beyond the task (e.g., applies reasoning to arbitrary bases).	51 / 55
Analysis	Explains the meaning of steps or connections between steps.	51 / 55
Synthesis	Integrates multiple representations (symbolic, visual, verbal).	51 / 55
Argumentation	Justifies steps logically or mathematically.	51 / 55

These semiotic and epistemic practices provide a lens to evaluate not only correctness in student work but also the quality of reasoning and representational coordination. The following table presents the abstraction levels used to categorize students' depth of understanding.

Together, these two components offer a means of linking observable patterns in student work—both correct and incorrect—to underlying semiotic, conceptual, and epistemological structures. As such, the framework helps to uncover not only what students know but also how they are constructing and negotiating meaning across different representations and levels of abstraction.

Table 5.2: Levels of Abstraction in Student Responses Based on the OSA Framework

Abstraction	Characteristics in Student Work	Category
Level 0	Uses informal language or physical intuition (e.g., "a vector is an arrow from the origin").	**Low Level**
Level 1	Uses correct semiotic forms (e.g., symbolic vectors, basis sets) but reasoning is procedural or lacks integration.	**Mid Level**
Level 2+	Connects concepts across representations (symbolic, geometric, algebraic), justifies reasoning, and uses matrix transformations meaningfully.	**High Level**

5.2 Qualitative Analysis Results

This section presents the qualitative findings based on student-written work and diagnostic assessments. We aimed to uncover how students reason through key concepts in matrix representation, vector decomposition, and basis transformation. Rather than focusing solely on correctness, this analysis sought to understand how students interpret and coordinate symbolic, structural, and procedural knowledge in linear algebra.

Analysis of Beginning of Students' Written Work (Fall, 2024)

Table 5.3 summarizes the key observations from each student's work. The responses ranged from highly fragmented and informal to coherent and well-integrated.

Table 5.3: OSA-Based Summary of Student Initial Responses (Students 1–13, Fall 2024)

Student ID	Key OSA Observations	Level
1	Demonstrates correct symbolic use and basis setup, but lacks interpretation of coordinate meaning; procedural reasoning dominates.	Middle
2	Clear structure with basis vectors and vector decomposition; interpretation is limited but well-formed symbolically.	Middle
3	Sparse structure and incomplete representation; lacks both signification and justification.	Low
4	Demonstrates basis decomposition and layout; transitions between symbolic and structural meaning are fragmented.	Middle
5	High-level abstraction with symbolic clarity, justified decomposition, and conceptual understanding of basis representation.	High
6	Mostly correct structure; lacks synthesis and argumentation; representation is symbolic but not interpreted.	Middle
7	Minimal symbolic structure; appears pre-conceptual with missing components and no evidence of synthesis.	Low
8	Starts strong symbolically but lacks matrix formation and coordinate vector extraction.	Middle
9	Incomplete and misaligned notation; suggests conflict in semiotic interpretation and symbolic overload.	Low
10	Correct representation and symbolic use; coordinate extraction lacks validation; mid-level abstraction.	Middle
11	Procedural steps shown without connection to conceptual structure; relies on examples without justification.	Low
12	Strong verbal and symbolic balance; well-integrated representation and clear epistemic coordination.	High
13	Recognizes symbolic process; limited generalization; lacks interpretive or justificatory layers.	Middle

To understand how students were thinking before they formally learned the preconcepts, we analyzed 13 written responses collected just before the first midterm in Fall 2024. These early responses provided insight into the initial ideas of the students about vector spaces, bases, and coordinate vectors. We examined key aspects, including their use of symbols, reasoning processes, and signs of conceptual confusion, and then categorized their understanding into three levels: low, middle, and high (Table 9).

Thematic Insights

Based on the coding of students' initial responses, the distribution across conceptual levels indicates that the majority of students (7 out of 13) demonstrated a middle level of understanding. These students generally displayed procedural competence and partial structural reasoning but struggled to fully integrate symbolic interpretation with conceptual justification. Four students were classified at a low level, exhibiting fragmented reasoning, limited representational coordination, and signs of semiotic confusion or pre-conceptual thinking. Only two students (Students 5 and 12) reached a high level of abstraction, characterized by coherent use of symbolic and structural representations, logical justification, and a clear understanding of basis selection and transformation mechanics. Overall, while many students showed symbolic fluency, their ability to synthesize representations and reason conceptually remained limited.

Across the responses of these 13 students, several consistent themes emerged. First, most students were able to carry out symbolic procedures correctly—such as expressing vectors in terms of a basis or initiating matrix formation. However, many struggled to move beyond procedural steps to articulate meaningful conceptual interpretations. This gap between symbolic fluency and structural understanding was especially evident among students functioning at what the OSA framework classifies as Level 1 abstraction.

Only a small subset of students demonstrated the capacity to coordinate multiple semiotic registers, justify their reasoning, and generalize across cases. These students, notably Students 5 and 12, exhibited epistemic control and flexibility in their use of representations. In contrast,

most students relied heavily on familiar templates or procedural recall, with limited evidence of argumentation or reflective explanation. This pattern suggests a need for instructional strategies that emphasize conceptual integration and promote deeper semiotic coordination in students' reasoning about matrix representations.

Figure 5.2 illustrates the distribution of abstraction levels across the student responses, with green representing high-level reasoning, orange representing mid-level reasoning, and red representing low-level reasoning.

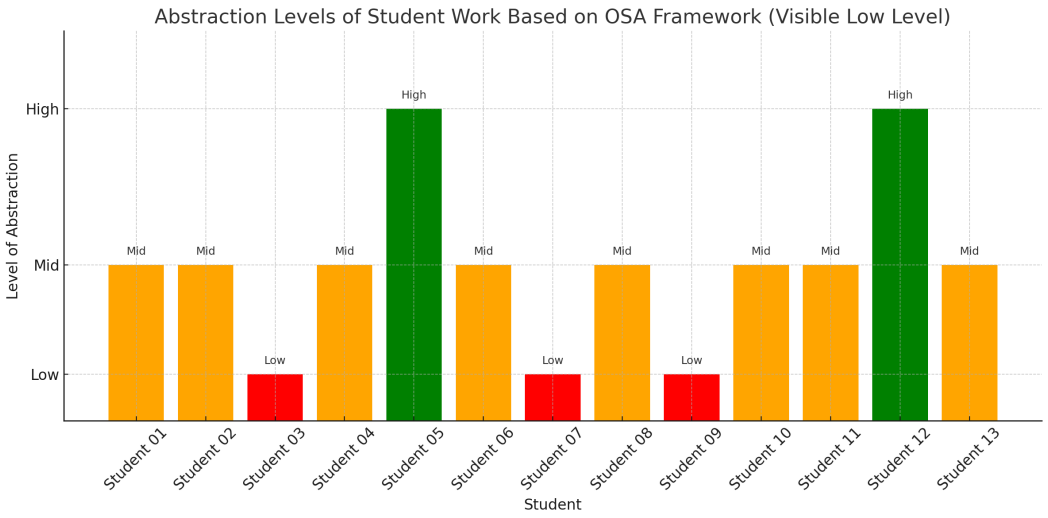


Figure 5.2: Student Abstraction Levels Interpreted via the OSA Framework. Green = high-level reasoning, orange = mid-level, red = low-level.

Diagnostic Test Performance and Thematic Insights of Students' Responses after Mid-term, Spring 2025

To gain a clearer picture of students' conceptual progress later in the semester, we administered a diagnostic review test in April 2025. The test consisted of 11 questions, including seven (Q1–Q7) that focused on key concepts such as coordinate vectors, transformation matrices, and changes of basis. Additionally, the test included one item requiring students to identify and correct an error in a proof of the matrix representation theorem, one problem-solving task involving specified bases

in $\mathbb{R}^2$, and two abstract problem-solving tasks with arbitrary bases. The questions varied in format, encompassing multiple-choice items, fill-in-the-blank prompts, error identification and correction, short proofs, and open-ended problem-solving questions. Student responses were analyzed using both basic statistical methods and thematic coding, guided by the OSA, to better understand how students reasoned through the material and where common difficulties emerged.

Test Administration Conditions: The review test was given during a regular class session in the Spring 2025 semester, about two weeks after the midterm. Thirteen students completed the test individually within a 75-minute time limit. It was a closed-book exam, with no access to notes, textbooks. The goal was to assess how well students could apply their understanding of matrix representation of linear transformations and interpretation in the later part of the semester.

Overall, students showed stronger performance on items related to symbolic fluency (Q1 and Q5) and struggled more with tasks that required a conceptual understanding of matrix structure and similarity transformations (Q2 and Q7). For instance, many students incorrectly used $[v]_B$ in place of $[v]_C$ when applying transformation matrices, indicating a fundamental misunderstanding of input/output bases.

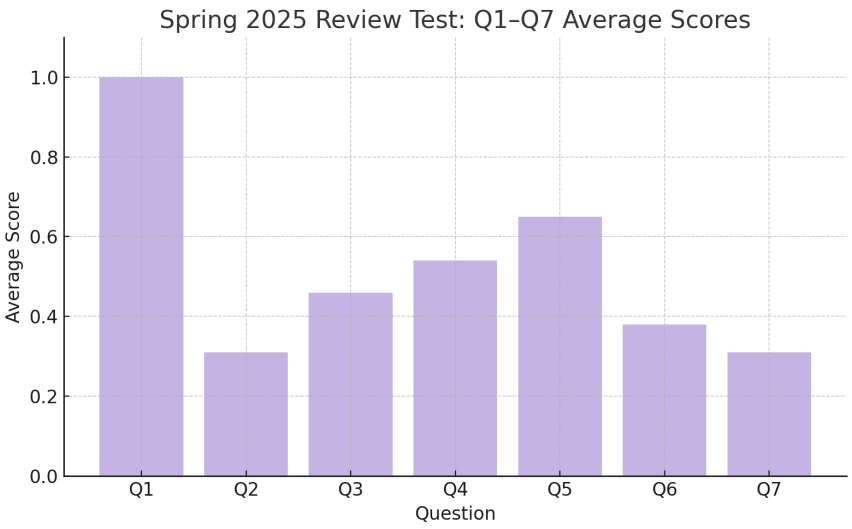


Figure 5.3: Spring 2025 Review Test: Average Scores by Conceptual Question (Q1–Q7).

Observed Reasoning Profiles

- **Q1** ($[v]_B$): Students generally demonstrated strong understanding of coordinate vectors.
- **Q2** ($[T]_C^B \cdot [v]_B$): Most common errors; revealed confusion about transformation inputs and symbolic alignment.
- **Q3/Q4** ($[T]_C^B \cdot [v]_C$): Moderate accuracy; students often followed procedure but showed limited understanding of input basis.
- **Q5 (Identity Matrix)**: High accuracy; students recognized the identity transformation but occasionally misapplied it across bases.
- **Q6 (Inverse Matrix)**: Mixed results; students could recall inverse procedures but often failed to link them conceptually.
- **Q7 (Similarity Transformation)**: Lowest performance; students struggled with the conceptual layering involved in changing between coordinate systems.

Thematic Coding by Abstraction Level Based on both statistical and qualitative analysis, we grouped student responses into three categories:

- **High-Level (Q1, Q5)**: Demonstrated symbolic fluency with interpretive alignment.
- **Mid-Level (Q3, Q4, Q6)**: Procedural knowledge evident but lacking deeper abstraction.
- **Low-Level (Q2, Q7)**: Frequent symbolic–conceptual conflicts and representational errors.

Figure 5.4 shows how these levels were distributed across the 13 students. Mid-level reasoning was the most prevalent, indicating that many students had begun to master procedures but still struggled with abstraction and synthesis.

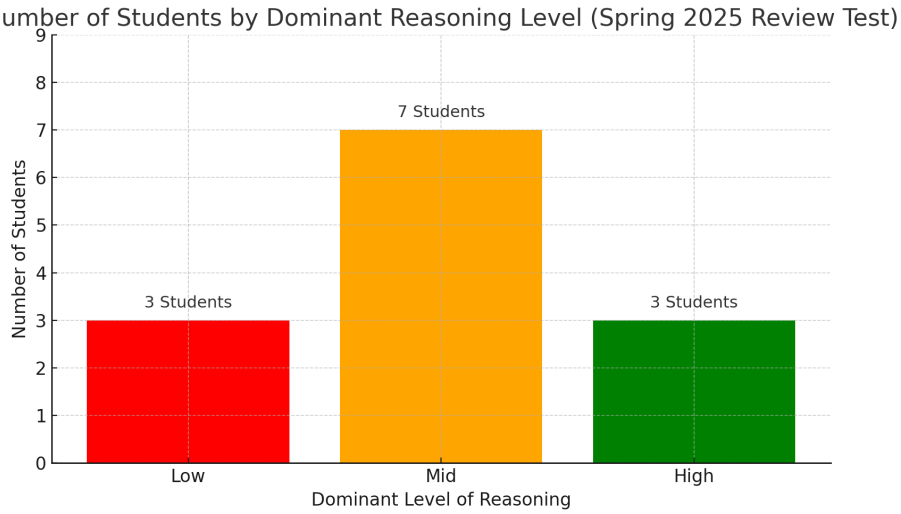


Figure 5.4: Dominant Reasoning Levels Across Students. Mid-level reasoning was most prevalent.

Together, these findings set the stage for the in-depth interview analyses in later Section 5.3, which explore how students articulate their thinking and navigate representational challenges in real-time discussions.

Progress and Persistent Challenges: A Comparative Student Understanding

Although the students in Fall 2024 and Spring 2025 were from different cohorts, the instructional content and delivery were consistent across both semesters. While the Fall 2024 data served as a pilot study and the Spring 2025 data was part of the main study, examining the student responses side by side still offers useful insights into how their understanding of the matrix representation of linear transformation developed over time. The pilot study in the Fall helped refine some of the interview prompts and instructional materials used in the Spring semester.

In Fall 2024, most students fell somewhere in the middle range of understanding (the data were collected before the first midterm). They were able to carry out basic procedures—like expressing vectors in a given basis or identifying matrix forms—but often had trouble explaining what those steps meant. A few students showed more advanced reasoning early on, but many leaned heavily on memorized methods and didn't yet make strong connections between the symbols they

were using and the underlying structure of vector spaces.

By Spring 2025, since these students had more time to digest the learning materials (the data was collected after the second midterm), some growth became evident. Students generally performed better on symbolic tasks. They were more comfortable with notions like coordinate vectors and identity transformations. However, problems that required deeper conceptual thinking—like understanding which basis a transformation was acting on, or making sense of similarity transformations—continued to be difficult for some students. From the results of the review test, most students were still in that middle range: they could do the steps but didn't always have a clear picture of why those steps worked or how different representations were connected. However, those students have potential: the students who had previously relied mostly on procedural thinking began to show glimpses of deeper reasoning. They showed better alignment between symbolic expressions and what they represented. Their responses pointed to a move toward more flexible and meaningful ways of thinking.

In the big picture, the comparison between two different groups of students between the two semesters suggests that abstract ideas like matrix representation, basis change, and coordinate systems take time to fully develop. These challenges are well documented in the research on linear algebra learning and reflect the complexity of the content—not necessarily gaps in instruction (Biza, Jaworski, & Weber, 2022; Dorier et al., 2000; Harel, 2020; Montiel et al., 2011; Wawro, 2017). What this shows is the importance of giving students ongoing opportunities to connect procedures with meaning, to reflect on their reasoning, and to work across different representations as they continue to build a deeper understanding.

Case Study and Interview: Student Reasoning on Similarity and Matrix Representations in Abstract Settings

To better understand how students reason about matrix representation by the end of the course, this section presents a focused case study based on both midterm review interviews and

final exam responses. The analysis draws on three one-hour, semi-structured interviews conducted via Webex during the latter part of the semester, in which students discussed specific diagnostic questions, explained their problem-solving approaches, and clarified their interpretations of key concepts. These interviews are complemented by an examination of students' written responses on the Spring 2025 final exam, which provides additional insight into their cognitive development and reveals whether they were able to overcome earlier difficulties and move toward more abstract and integrated reasoning. The semi-structured interview questions are provided in Appendix C. While each interview covered a range of topics, only selected excerpts related to triangular similarity and problem-solving strategies are included in this analysis. During the sessions, students were guided step-by-step through targeted problems from the midterm review test, allowing for in-depth exploration of their conceptual understanding and reasoning processes.

Summary of Emerging Themes from All Six Students

(a) Misidentification of Domain and Codomain Bases (4 students): Most students attempted to represent the outputs of L using the domain basis $\{x^2, x, 1\}$ rather than the codomain basis $\{2, 1-x\}$. This suggests a semiotic limitation and ontological confusion about the structural role of bases in transformation matrices.

(b) Symbolic–Procedural Disconnection (3 students): While some students correctly computed $p'(x)$ and $p(0)$, they struggled to express the result in terms of the codomain basis. This reflects an incomplete epistemic configuration: symbolic manipulation was evident, but it was not meaningfully connected to coordinate representation.

(c) Misapplication of Basis Elements (2 students): A few students incorrectly applied L to elements of the codomain basis (e.g., $L(2)$, $L(1-x)$), indicating confusion over the directionality and functional domain of the transformation.

(d) Emerging Contextual Awareness (1 student): One student acknowledged the need to express outputs of L in terms of the codomain basis. Though their computational work

was incomplete, their verbal reasoning showed promising coordination among polynomials, differentiation, and coordinate decomposition.

Interpreting Interview Data Through the Onto-Semiotic Lens

These interviews revealed persistent conceptual difficulties with non-standard basis representations in linear transformations. Students showed limited semiotic coordination, particularly in associating symbolic expressions with coordinate systems. Confusion between symbolic results and coordinate decomposition highlights both ontological and institutional gaps in understanding.

The next section highlights three student profiles—ST1, ST2, and ST3—that illustrate varying levels of abstraction and cognitive development. To trace their learning progression, the analysis also examines their responses to particular problems on the Spring 2025 final exam. Because the exam was conducted under formal assessment conditions, these responses provide valuable insight into how well students consolidated their understanding of matrix representation, coordinate systems, and basis transformations. Moreover, the analysis offers reflective feedback on the effectiveness of instructional strategies implemented throughout the course. Taken together, these findings reveal patterns in student reasoning and offer implications for instructional design aimed at bridging the gap between symbolic fluency and conceptual understanding.

ST1: Procedural Fluency with Emerging Conceptual Awareness

ST1 is an undergraduate student concurrently enrolled in the university's athletic program, which imposes significant time constraints, particularly on weekends. Despite these challenges, she actively engaged with course materials through YouTube tutorials and peer-generated notes. Visual explanations—especially those using color-coded representations—were particularly effective in helping her develop an initial understanding of coordinate vectors and transformation matrices.

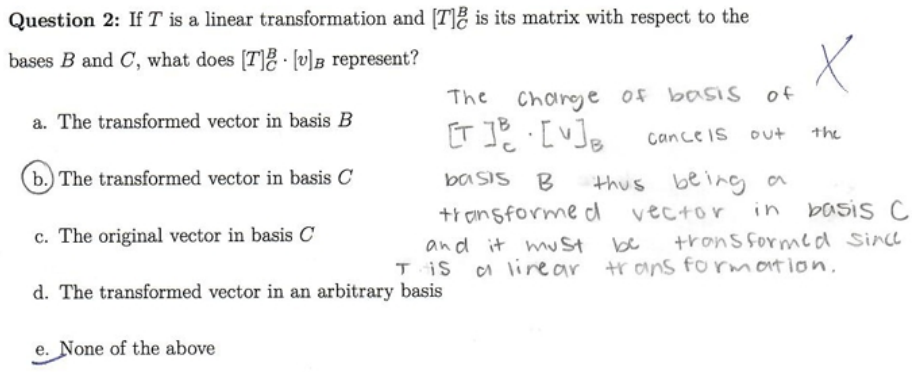


Figure 5.5: ST1's written response to Question 2: Interpretation of $[T]_C^B \cdot [v]_B$

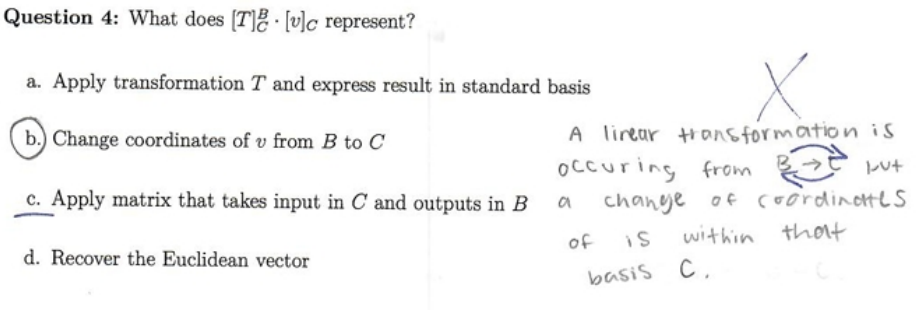


Figure 5.6: ST1's written response to Question 4: Interpretation of $[T]_C^B \cdot [v]_C$

Written Work Analysis: ST1's Responses to Questions 2 and 4 ST1's written responses to transformation expressions involving basis change reflect symbolic engagement but a partial and inconsistent structural understanding. Two items on the review test—Question 2 and Question 4—provide a window into her interpretation of matrix representation and basis interaction.

In **Question 2**, the student was asked to interpret the meaning of the expression $[T]_C^B \cdot [v]_B$. ST1 selected option (b), "The transformed vector in basis C," which is incorrect. Her written justification stated:

- *"The change of basis of $[T]_C^B \cdot [v]_B$ cancels out the basis B, thus being a transformed vector in basis C and it must be transformed since T is a linear transformation."*

This explanation reveals a symbolic misreading of the subscripts in the matrix notation. The expression $[T]_C^B$ refers to a matrix that expects input vectors expressed in basis C and outputs

vectors in basis B, whereas $[v]_B$ is a coordinate vector expressed in basis B. The mismatch makes the multiplication undefined. ST1's reference to "canceling out" basis labels suggests she is treating the basis subscripts algebraically, without regard to their functional role in transformation structure. This points to a semiotic conflict between formal notation and conceptual meaning. Her reasoning shows procedural effort but lacks clarity in distinguishing coordinate vectors from transformation outputs.

In **Question 4**, ST1 evaluated the expression $[T]_C^B \cdot [v]_C$ and selected option (b), "Change coordinates of v from B to C," which is also incorrect. She explained:

- *"A linear transformation is occurring from $B \rightarrow C$ but a change of coordinates is within that basis C."*

Here again, she confuses the roles of transformation matrices and change-of-basis matrices. In fact, the expression $[T]_C^B \cdot [v]_C$ is valid: it applies a transformation matrix that takes vectors expressed in basis C and outputs the result in basis B. The correct answer should have been (c), "Apply matrix that takes input in C and outputs in B." ST1 incorrectly interpreted the expression as a coordinate change and did not recognize the transformation as mapping from domain to codomain under specified bases. Her written notes show that she recognizes a transformation is occurring but misidentifies its direction and mechanism.

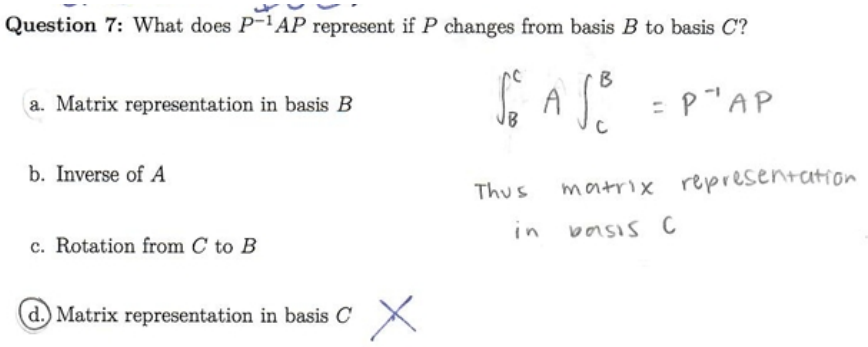
Taken together, ST1's responses suggest that she is operating at a symbolic-procedural level of reasoning with emerging awareness of structural meaning. While she engages with the notation and attempts to justify her answers, her explanations reflect a fragile understanding of how basis and coordinate systems interact with linear transformations. These difficulties align with common epistemic obstacles in linear algebra, where students often conflate change-of-basis matrices with transformation matrices and overlook the importance of input-output basis alignment.

ST1 Interview Results: Misunderstanding the Role of Codomain Basis

During the one-hour review test interview, ST1 demonstrated familiarity with symbolic procedures but revealed conceptual misunderstandings when reasoning through basis changes and

matrix operations. For example, she correctly identified $[v]_B$ as a coordinate vector but struggled with the meaning of matrix multiplication in the expression $[T]_C^B \cdot [v]_B$. She assumed that the transformation matrix could be applied to $[v]_B$ regardless of the codomain basis, indicating a misconception about how matrices representing linear transformations depend jointly on domain and codomain bases. When asked to clarify why $[T]_C^B \cdot [v]_B$ might yield incorrect results, she hesitated and admitted that

- *"I just memorize the order and multiply."*
- *"I definitely got confused there... when you're looking at the change of base from B to C and you're looking at the coordinate vector as well."*
- *"Yes, that makes sense with the P^{-1}... now I get it."*


Question 7: What does $P^{-1}AP$ represent if P changes from basis B to basis C?

a. Matrix representation in basis B

b. Inverse of A

c. Rotation from C to B

(d.) Matrix representation in basis C

$\int_B^C A \int_C^B = P^{-1}AP$

Thus matrix representation in basis C

Figure 5.7: ST1's written response to Question 7: Interpretation of $P^{-1}AP$

ST1's Reasoning on Similarity Transformation: Question 7 ST1's response to Question 7 further illustrates her ongoing difficulty distinguishing between transformation matrices and their representation under different bases. When asked to interpret the expression $P^{-1}AP$, where P represents a change of basis from B to C, ST1 selected option (d), "Matrix representation in basis C," which is incorrect. Her written work included the following remarks:

- *"Thus matrix representation in basis C"*
- *She also wrote the notation:* $\int_B^C A \int_C^B = P^{-1}AP$

This informal notation, although nonstandard, suggests that ST1 attempted to conceptually represent a basis shift from B to C, likely drawing from other external materials improperly. However, her interpretation reveals a semiotic conflict: while she correctly recalls the symbolic structure of a similarity transformation, she misidentifies the resulting matrix's basis. The expression $P^{-1}AP$ transforms the matrix representation of a linear transformation from basis B to basis C—meaning the resulting matrix is in basis C, not basis B as she claimed. Her selection of option (d) reflects this misinterpretation.

Her reflection highlights a disconnection between semiotic manipulation and conceptual grounding. ST1 appears to operate at a symbolic-procedural level, the lower level, correctly reproducing transformation rules but misunderstanding their referential meaning.

These statements reflect a tendency to rely on memorized steps rather than internalized meaning. However, as the interview progressed, ST1 began to express greater clarity and understanding, particularly regarding the role of $P^{-1}AP$ in similarity transformations. With guided questioning, she articulated the correct direction of basis change and began to recognize the structure encoded in transformation matrices.

This case illustrates a reasoning profile centered on symbolic fluency and procedural recall, with emerging conceptual awareness that remained fragile under assessment pressure. Her responses reflect an initial grasp of matrix representation and basis transformation, but also highlight the need for instructional strategies that support students in linking procedures to the underlying mathematical structures they represent.

"I just memorize the order and multiply." This statement suggests that while she was familiar with the mechanics of matrix multiplication, she had not yet internalized the meaning behind each component.

Evidence of Growth: ST1's Work on Change of Basis Matrix (Final Exam Question 2)

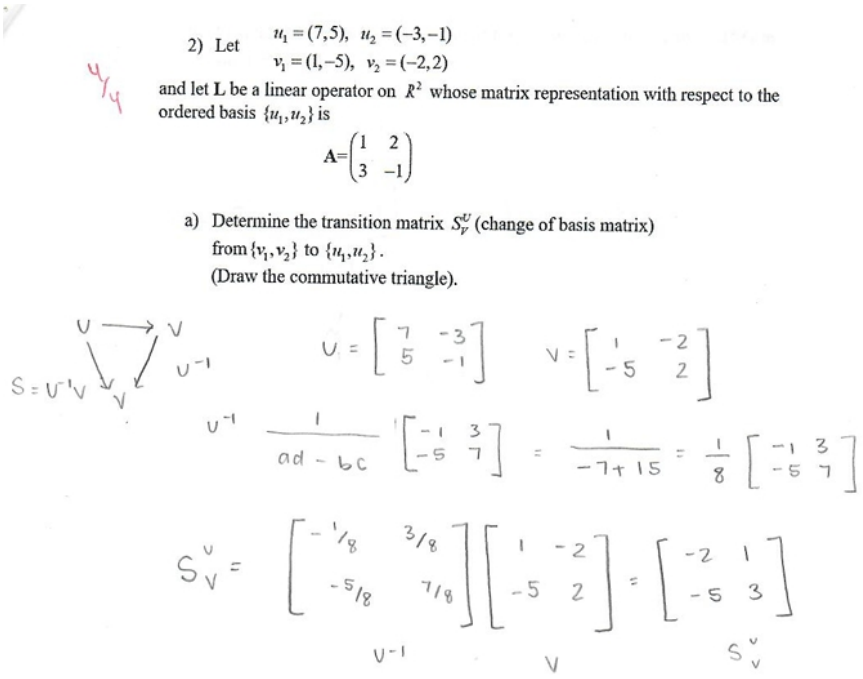


Figure 5.8: ST1's final exam work constructing the change of basis matrix S_V^U from $\{v_1, v_2\}$ to $\{u_1, u_2\}$

ST1's solution to Final Exam Question 2 provides strong evidence of conceptual progress compared to her earlier responses (*Figure 8*. In this problem, she was asked to compute the transition matrix S_V^U—that is, the change of basis matrix from the non-standard basis $\{v_1, v_2\}$ to the target basis $\{u_1, u_2\}$. Her work demonstrates a more integrated understanding of the structural relationships between coordinate systems, linear transformations, and similarity representations.

ST1 began by clearly identifying the columns of both bases and organizing them into matrices V and U. She correctly computed the inverse of matrix V, explicitly showing the use of the formula for a 2×2 inverse:

$$V^{-1} = \frac{1}{\det(V)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Her symbolic fluency improved here, especially in correctly calculating $\det(V) = -7$ and scaling the adjugate matrix appropriately. Most importantly, she correctly performed the matrix product $S_V^U = U \cdot V^{-1}$, confirming her understanding that the change of basis matrix is formed by multiplying the target basis matrix by the inverse of the source basis matrix.

She also included a well-labeled commutative triangle diagram, a sign of structural awareness. This is significant, as it demonstrates that ST1 now sees basis change as a mapping between coordinate systems rather than merely symbolic manipulation. The triangle indicates a movement between different representations of the same vector space via transition matrices, which connects the concept of change-of-basis to the broader framework of linear transformations and similarity.

Her work reflects substantial progress in conceptual integration. Her earlier responses (e.g., to Questions 2 and 7 in the review test) suggested symbolic-procedural reasoning with semiotic confusion about basis direction and transformation structure. In contrast, this final exam problem shows that ST1 is beginning to coordinate multiple representations (algebraic, geometric, and diagrammatic), resolve prior semiotic conflicts, and work flexibly with notational conventions. This shift suggests growth from local symbolic manipulation toward global structural reasoning.

While ST1's solution shows clear improvement in procedural fluency and matrix manipulation, her work remains primarily symbolic and lacks full verbal justification. She correctly computes the transition matrix S_V^U but does not explicitly explain its function as a coordinate converter from the $\{v_1, v_2\}$ basis to the $\{u_1, u_2\}$ basis. Additionally, her commutative triangle diagram, though helpful, is not fully labeled with matrix names or directional arrows. These omissions suggest that although she is now operating competently at a symbolic-procedural level, there is still room for growth in structural understanding and formal mathematical communication.

ST1's Work on the final exam: Polynomial Transformation and Basis Representation

Figure 5.9 shows ST1's written response to a matrix representation task on the Spring 2025 final exam. The problem required students to find the matrix representation of the linear transformation $L(p(x)) = p'(x) + p(0)$, which maps polynomials from P_3 into P_2. The matrix was to be computed with respect to the ordered basis $\{1, x, x^2\}$ for the domain and $\{1, 1 - x\}$ for the codomain.

ST1 began by correctly applying the transformation to each basis element of the domain:

$$L(1) = 0 + 0 = 0$$

$$L(x) = 1 + 0 = 1$$

$$L(x^2) = 2x + 0 = 2x$$

These outputs were then to be expressed as linear combinations of the codomain basis $\{1, 1 - x\}$. Only for $L(x) = 1$, ST1 correctly identified the coordinate vector as $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. For $L(1) = 0$, ST1 mistakenly wrote the coordinate vector $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, which is easily ignored by some students who treat $L(1)$ is simple as deriviating a constant, but not the real constant of this transformation. Further, a critical error occurred in attempting to represent $L(x^2) = 2x$ in terms of the basis $\{1, 1-x\}$. Instead of equating coefficients or solving symbolically, ST1 substituted values for x (e.g., $x = 1, x = 0$) and solved for the coefficients using a plug-in method. While this approach can sometimes yield correct results, ST1 incorrectly concluded that the coordinates were $\begin{bmatrix} \frac{1}{2} \\ \frac{1}{3} \end{bmatrix}$, which does not correspond to the given codomain basis and suggests a basis mismatch.

The final matrix written by ST1 was:

$$\begin{bmatrix} \frac{1}{2} & 1 & 0 \\ \frac{1}{3} & \frac{1}{2} & 0 \end{bmatrix}$$

This matrix is structurally misordered in format, where each column represents the image of a domain basis vector with the coordinate vector for $L(x^2) = 2x$ computed incorrectly with respect to the codomain basis, messed up with wrong orders in the matrix.

ST1 demonstrates symbolic fluency and procedural competence in applying the transformation and assembling the matrix. However, within a time frame to complete a problem-solving, her work still reflects a semiotic conflict between the symbolic manipulation of polynomials and the conceptual understanding of coordinate systems. The error suggests that ST1 has not fully internalized the meaning of representing vectors relative to a non-standard basis and relies heavily on procedural shortcuts rather than structural reasoning.

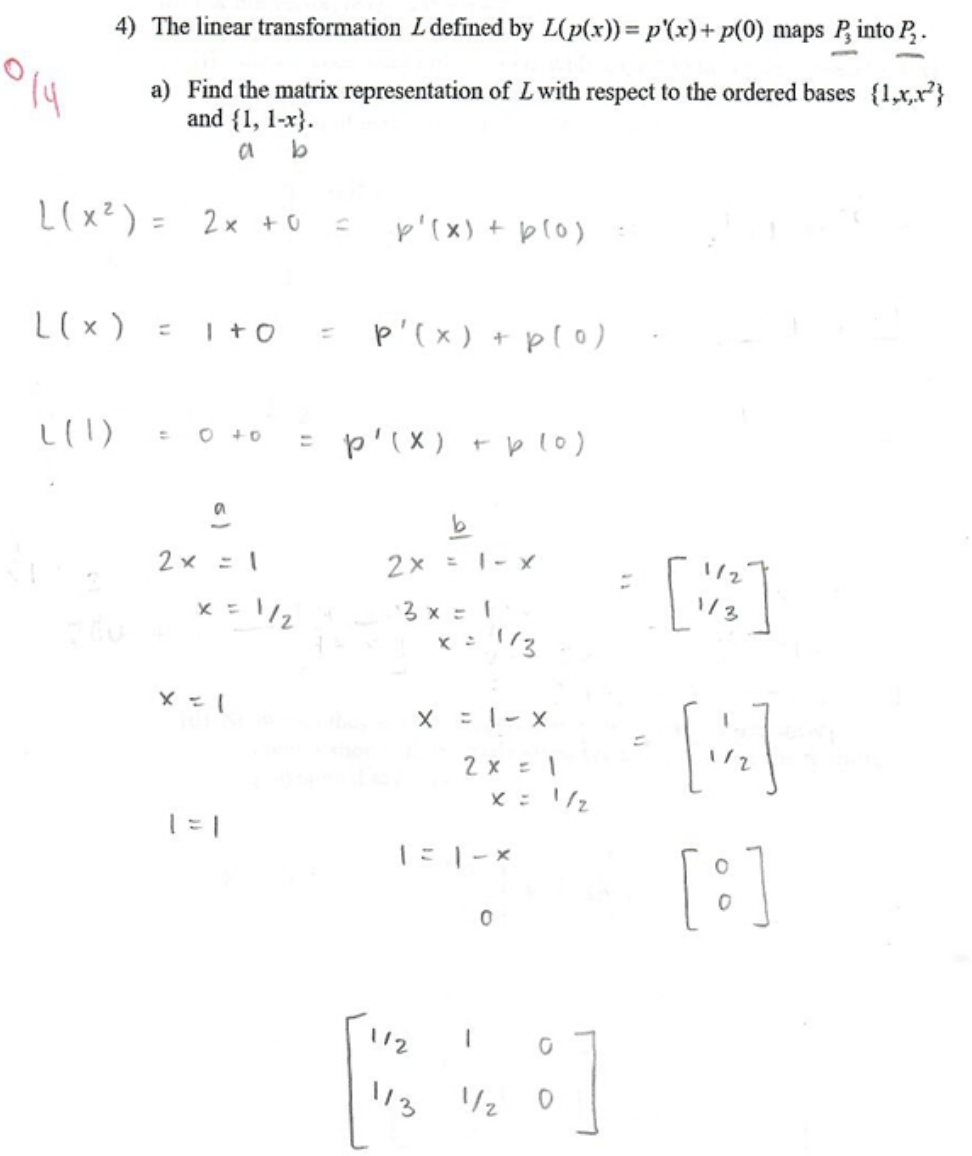


Figure 5.9: [

ST1 final exam response to matrix representation]ST1's written response to a problem-solving item on the Spring 2025 final exam.

The question asked for the matrix representation of the transformation $L(p(x)) = p'(x)+p(0)$, mapping from P_3 to P_2, with respect to the ordered bases $\{1, x, x^2\}$ in the domain and $\{1, 1-x\}$ in the codomain. While ST1 engages symbolically with the task, her work reveals conceptual misunderstandings that ultimately invalidate her final matrix.

This work illustrates a common epistemic obstacle in linear algebra: the failure to coordinate

the procedural manipulation of expressions with a conceptual understanding of vector space dimensionality and basis completeness. ST1 operates primarily at a symbolic-procedural level without fully grasping the ontological constraints of the problem. This justifies the instructor's decision to assign a score of zero, as the matrix does not faithfully represent the transformation under the given bases.

Table 5.4: ST1's Obstacles Interpreted Through the OSA

Type of Obstacle	**Simplified Description**
Semiotic Conflict	Struggled to interpret mathematical symbols and notation, especially in tasks involving change of basis, coordinate vectors, and transformation matrices.
Cognitive Overload	Struggled with inefficient problem-solving approaches during tests, resulting in confusion and errors under pressure.
Incomplete Schema	Lacked a clear understanding of how transformations, bases, and coordinate systems are connected.
Didactical Obstacle	Relied on YouTube videos and peer notes, while overlooking classroom explanations and spending insufficient time to fully address her learning needs.
Ontological Conflict	Recognized terms like "identity matrix" and "basis," but couldn't always explain them in formal or general terms.
Systemic Conflict	Experienced a disconnect between textbook explanations and classroom instruction.
Affective Barrier	Low confidence and test anxiety limited her engagement and assessment performance.

This analysis underscores the need for scaffolding that links semiotic representations with conceptual understanding, while accommodating gaps in prior knowledge and time constraints. For students struggling to catch up, retaking Linear Algebra I may be appropriate. Clearer notation,

targeted practice, and aligned classroom discourse can help bridge the gap between personal and institutional mathematical knowledge.

In sum, while ST1 displays promising procedural effort, her case highlights the importance of developing structural reasoning and a deeper semiotic alignment between symbolic operations and their abstract mathematical meaning. This case reflects the need for focusing on interpreting what it means for a set to form a basis, and on ensuring students can validate the representational adequacy of their coordinate decompositions.

ST2's Reasoning on Transformation Expression

ST2 is an undergraduate who initially majored in computer science and mathematics. She expressed both appreciation for the applied aspects of the course and frustration over the conceptual demands.

To examine students' conceptual understanding of transformation matrices and basis changes, we analyzed ST2's responses to Questions 2 and 4 from the diagnostic review test. The work of ST2 provides insight into a common semiotic conflict: the confusion between transformation matrices and change-of-basis matrices.

In Question 2, students were asked to interpret the expression $[T]_C^B \cdot [v]_B$, where T is a linear transformation and $[T]_C^B$ denotes its matrix representation with respect to bases B and C. ST2 initially selected option (a)—"The transformed vector in basis B"—but then crossed it out and selected option (b)—"The transformed vector in basis C."

Her handwritten explanation included:

- *"Using the transition matrix to change basis of the vector."*

This statement indicates a fundamental misunderstanding: she treats $[T]_C^B$ as a transition matrix between bases rather than the matrix representation of a transformation. This reflects a semiotic conflict in interpreting mathematical objects. Specifically, ST2 misidentifies the ontological nature of $[T]_C^B$—a transformation matrix—mistaking it for a matrix that performs coordinate conversions. She often referenced classroom terminology (e.g., "domain basis," "output basis")

but could not elaborate on how those terms connected to the operations she was performing. Her explanations remained procedural, and while her written solutions appeared correct, they were not fully supported by conceptual reasoning.

One telling moment came during the interview when she stated:

- *"I know it's $[v]_B$ because that's how we set up the input, but I'm not sure what would happen if we used $[v]_C$."*

This uncertainty highlights a symbolic–conceptual gap: she could reproduce procedures but had not fully developed an understanding of why certain conventions matter or how representations correspond to geometric or algebraic structures.

A similar pattern of reasoning is evident in her response to Question 4, which presents the expression $[T]_C^B \cdot [v]_C$. ST2 again chose option (b)—"Change coordinates of v from B to C." Her written explanation stated:

- *"B to C transitioned then changing coordinate vector."*

This reasoning further confirms the symbolic misalignment. She appears to associate the subscript notation $B \to C$ with the directionality of basis change, rather than understanding the operation as matrix-vector multiplication representing a transformation applied to a vector expressed in basis C.

These interpretations reveal an epistemic configuration centered on procedural familiarity with symbolic forms but lacking structural understanding. ST2 demonstrates comfort with the notation but interprets expressions through a lens shaped by surface features (e.g., the order of subscripts), rather than by conceptual distinctions between linear maps and coordinate transformations.

Semiotic and Ontological Conflicts When discussing problems involving basis transformations and coordinate vectors, she stated:

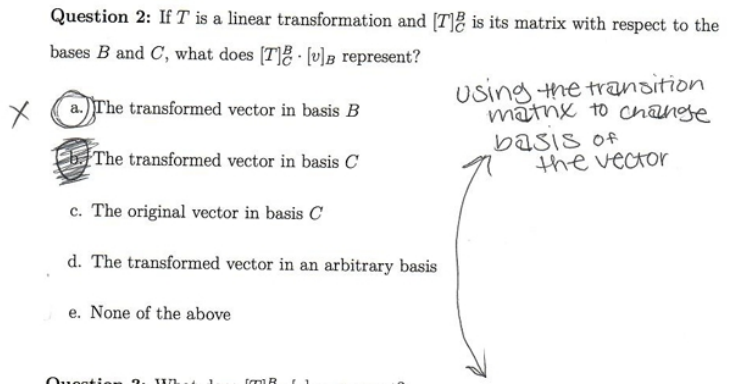


Figure 5.10: ST2's written work for Question 2, showing confusion between transformation and basis change

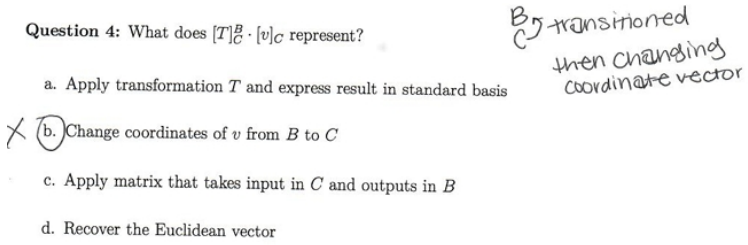


Figure 5.11: ST2's written response to Question 4, misinterpreting $[T]_C^B$ as a transition matrix

Table 5.5: OSA Summary of ST2's Reasoning

OSA Facet	**Observation**
Semiotic Conflict	Misinterprets $[T]_C^B$ as a change-of-basis matrix rather than a transformation matrix.
Ontological Misidentification	Fails to identify the object $[T]_C^B$ as representing a linear transformation.
Procedural Fluency	Displays symbolic comfort with matrix expressions.
Conceptual Limitation	Lacks structural understanding of the transformation's direction and meaning.
Level of Reasoning	Symbolic-procedural, with minimal abstraction.

"I think I actually got confused... looking at question two and question three because they're so similar."

The student's difficulties suggest a **semiotic conflict**—confusion not with the symbols themselves but with how their forms relate to conceptual meaning. Furthermore, her uncertainty about notation and transformations reflects **ontological conflict**, where she may not fully grasp what mathematical objects like coordinate vectors represent within the context of linear transformations.

Learning Preferences and Recommendations ST2 appreciated peer solutions and desired a structured review:

"I like to review a lot like I wish we had these [review problems] like before the test."

She emphasized the benefit of group work and scaffolded projects:

"Maybe like something like the reviews, so it's like you kind of know what to expect."

These responses highlight a student preference for structured, collaborative learning environments, which can help bridge the gap between procedural fluency and conceptual meaning. However, a review of her attendance records revealed that she seldom participated in review sessions and underestimated their importance. Instead, she relied heavily on self-study, which lacked the depth and context provided during classroom instruction. This limited engagement contributed to her shallow understanding of the content. She later acknowledged that by the time she received a lower evaluation, it felt "too late" to catch up—indicating a delayed recognition of the consequences of her learning approach.

Conceptual Strengths and Career Relevance She noted:

"I think the linear transformation part was probably the easiest topic for me honestly."

Her background in computer science and programming (Python) supports her confidence in understanding transformation-based reasoning. However, knowing how to program and interpret

software does not necessarily carry over to advanced mathematics. If the students have not taken the Bridge to High Mathematics course, or similar proof-based training, they would not be successful in Linear Algebra II (Dr. Montiel, personal communication, June 7, 2025).

In sum, ST2 demonstrates strengths in algorithmic thinking and applied reasoning, aided by her computing background. However, she encounters obstacles with conceptual articulation, notation interpretation, and instruction alignment. The interview indicates a need for:

- More worked examples with full solutions
- Greater conceptual scaffolding, especially in notation-heavy tasks
- Alternative mediation strategies like peer explanation and guided group projects
- In-class learning that addresses symbolic confusion and fosters ontological clarity

This case underscores how even students with computational fluency can struggle with the institutional and semiotic layers of advanced linear algebra.

ST3: Fragmented Reasoning with Representational Confusion

ST3 is an international graduate student in her first semester studying in the U.S. She acknowledged feeling overwhelmed by the notational complexity of the course and mentioned that while she could follow worked-out examples, constructing solutions independently was challenging.

Throughout the interview, when I asked about the difference between vector and coordinate vector by using her own words, ST3 displayed confusion between vectors and coordinate vectors. She often used the terms interchangeably. When referring to a coordinate vector such as $[v]_B$, she attempts to explain matrix representation involved pointing to rows and columns without clearly distinguishing between the transformation matrix and the coordinate vectors it acts on.

Although she was able to complete some basic steps correctly, her reasoning was fragmented at the beginning. For example, she could not explain why using the wrong basis in a matrix multiplication would change the meaning of the result. When prompted to visualize what the transformation matrix does, she responded:

- *"I don't know, it's just numbers changing."*

This statement suggests a disconnection between symbolic procedures and the underlying mathematical structures they represent.

Procedural Fluency vs. Conceptual Understanding

ST3 demonstrated confidence in procedural tasks such as calculating $P^{-1}AP$, but could not articulate the conceptual meaning behind the computation:

> *"I can do the calculation for P^{-1}, A, P with no problem... but it's still just a formula for me."*

This reflects reasoning situated at OSA Level 1, where students manipulate symbolic representations without fully internalizing the transformation concept. The formula is treated as a computational tool rather than a representational bridge between bases.

Coordinate Vectors vs. Original Vectors

Students showed difficulty distinguishing between coordinate vectors and the original vectors they represent. When asked whether coordinate vectors were the same as original vectors, one student paused and stated:

> *"I don't understand the meaning of the question."*

This hesitation suggests ontological confusion. In OSA terms, the student failed to establish coherent semiotic connections between the symbolic form (e.g., $\{3, 4, -5\}$) and its referent, the real-world object, idea, or entity that a word or symbol stands for in the vector space.

Language as an Epistemic Barrier

Language was a significant barrier for multilingual students. Several participants expressed that they struggled to engage with problems due to language:

"I studied a lot in my own language... but I didn't do the problem at all because there are too many English words."

This highlights how linguistic mediation influences access to mathematical meaning. The cognitive configuration remains underdeveloped not because of a lack of effort, but due to limited access to institutional discourse in a second language. For this reason, it is important, even in areas such as mathematics, that international students demonstrate fluency in the language of the country where they will study.

Symbol Awareness and Semiotic Inference

While most students could identify symbols such as $[T]_B$, they struggled to articulate their meanings:

"I know $[T]_B$ is the matrix form, but I don't remember what it stands for."

Despite procedural familiarity, the conceptual and semiotic layers of the symbol remained obscure. However, some students expressed appreciation for the format of the diagnostic tasks:

"I like this kind of test. It's not like the traditional one."

This feedback suggests openness to varied assessment designs, even though deeper semiotic reasoning is still emerging.

Summary of OSA Levels Observed for ST3 Through the Interview

Table 5.6: Summary of Semiotic Reasoning Levels Observed

Aspect	**OSA Level Observed**	**Evidence**
Symbol manipulation	Level 1	Procedural calculation of P^{-1}, A, P
Symbol-concept linkage	Level 0–1	Confusion on $[T]_B$, coordinate vectors
Generalization	Not achieved	No abstraction or justification evident
Institutional discourse	Emerging	Positive response to non-traditional test format

Analysis of ST3's Final Exam Progress

In the final exam, ST3 shows clear evidence of conceptual improvement by successfully constructing and applying the matrix representation of a linear transformation between polynomial spaces. In Question 4(a), she accurately identifies how the transformation $L(p(x)) = p'(x) + p(0)$ acts on the standard basis elements $\{1, x, x^2\}$, and decomposes each image vector with respect to the target basis $\{1, 1 - x\}$. The resulting matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & -2 \end{bmatrix}$$

is constructed correctly and reflects a solid procedural understanding of basis transformation.

In Question 4(b)(i), ST3 uses this matrix to compute $L(p(x))$ for $p(x) = 2x^2 + x - 2$. She first expresses the polynomial in the standard basis as the vector $[-2 \quad 1 \quad 2]^T$, and then applies the matrix transformation to yield the result:

$$L(p(x)) = A \cdot [p(x)]_E = \begin{bmatrix} 3 \\ -4 \end{bmatrix}_F .$$

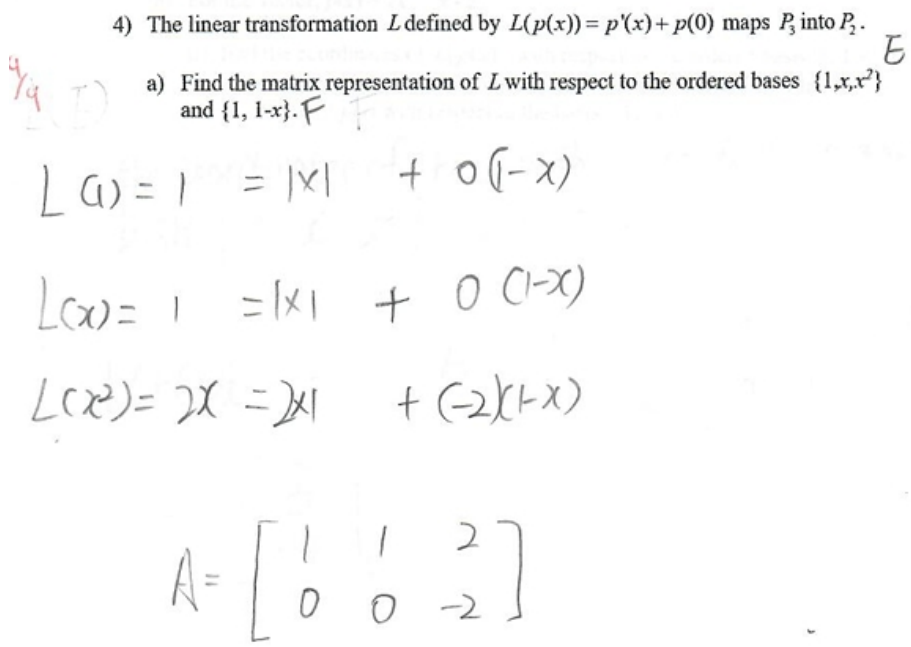


Figure 5.12: [
ST3 matrix construction (Final Exam)]ST3's matrix representation of L with respect to the bases $\{1, x, x^2\}_E$ and $\{1, 1-x\}_F$.

This result indicates an ability to coordinate multiple semiotic representations—algebraic, symbolic, and matrix-based.

Her closing remark, *"They are the same,"* reflects her confidence in the coherence of different mathematical representations.

Summary of the Progress of ST3

ST3, as a non-English-speaking background, the student faced noticeable language challenges, particularly when it came to understanding written instructions or interpreting questions in English. During the interview, ST3 mentioned enjoying the format of the review test, noting that it offered a broader and more meaningful way to approach matrix representation than simply repeating similar questions. However, the responses also revealed a reliance on procedural fluency. While the student were confident in performing calculations, such as computing $P^{-1}AP$, but found it difficult to explain the meaning behind the operations or the relationships between matrices, basis changes, and transformations. When asked to interpret expressions involving coordinate vectors or

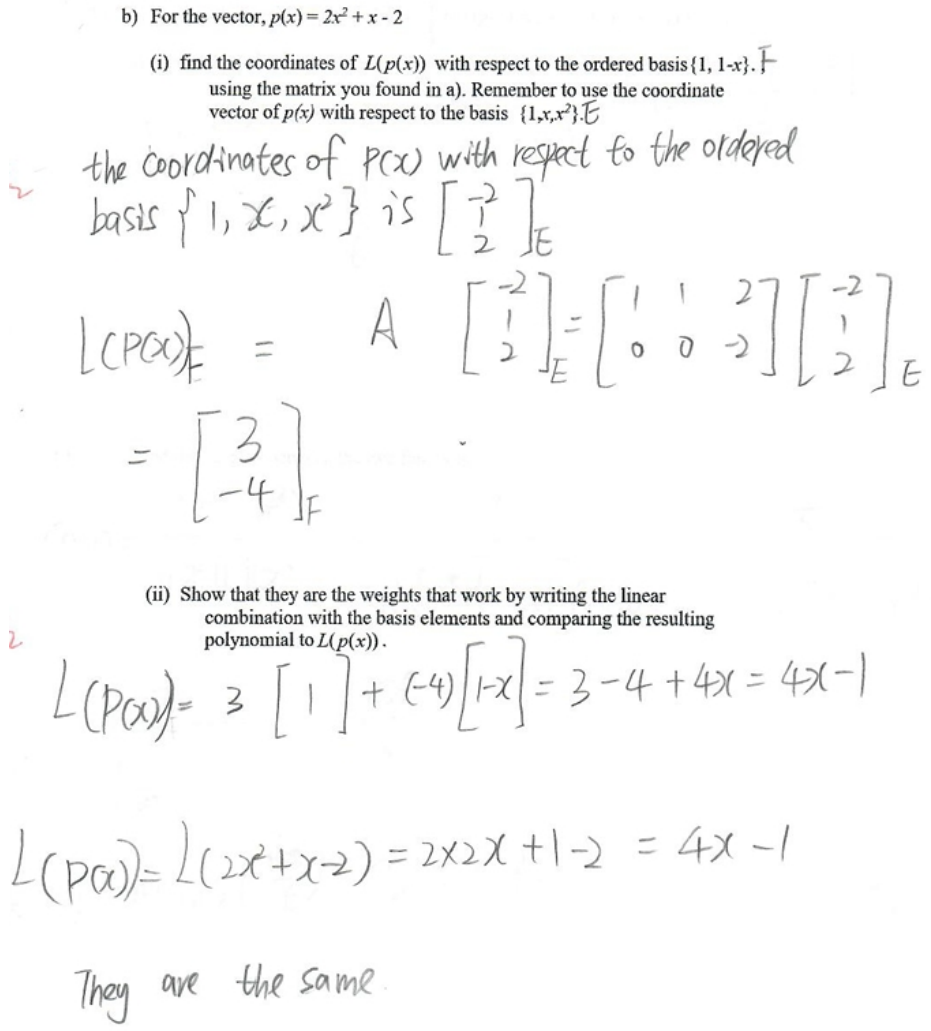


Figure 5.13: ST3's application of L to the polynomial $p(x) = 2x^2 + x - 2$, showing consistency across symbolic and algebraic representations.

symbolic representations like $[T]_B$, the student often hesitated or paused, unsure how to respond. These moments reflected a deeper struggle not just with language, but with connecting symbolic work to conceptual understanding. ST3's experience highlights how linguistic and semiotic obstacles can impact even motivated learners, especially when adjusting to a new academic and cultural environment.

Compared to earlier assessments, ST3 demonstrates substantial growth in abstract reasoning, basis transformation, and linear operator application in her final exam. Her ability to move fluently between coordinate representations, basis-specific linear mappings, and symbolic polynomial manipulation indicates strong conceptual integration. Her work reveals alignment among the epistemic configurations of object, process, and language, with no major semiotic conflicts. This represents a notable step toward structural understanding.

Cross-Case Themes

These three student profiles reflect a range of reasoning levels and challenges. ST1 and ST2 operated largely at Level 0 or Level 1 abstraction, relying heavily on memorization and symbolic manipulation. ST3, while more fluent symbolically, still lacked full conceptual integration not from the calculation patterns, but from her comprehension. Across all cases, a common theme was the difficulty students faced in coordinating symbolic representations with structural meanings—particularly when transitioning between bases or interpreting transformation matrices.

The interviews provided valuable insights into the kinds of support students might need to move beyond procedural fluency. In particular, they suggest that future instruction should explicitly address the meaning of coordinate vectors, the role of input and output bases, and the interpretive steps involved in symbolic transformation. These qualitative findings complement the broader trends identified in the written work and diagnostic tests, rounding out a comprehensive picture of student understanding in this domain.

5.3 Quantitative Results

While qualitative methods dominate research in mathematics education, recent years have seen an increase in the application of quantitative techniques to understand student thinking patterns. These include cluster analysis, regression modeling, and network-based approaches (Biehler, Hochmuth, Podworny, & Schaper, 2018; Herbst & Chazan, 2019; Inglis, Mejia-Ramos, & Simpson, 2011; Rott, 2020; Wasserman, Apkarian, & Weinberg, 2022). However, most quantitative frameworks in education emphasize correlations or classification accuracy rather than directional, logical relationships.

Statistical Implicative Analysis (SIA), developed by Gras and colleagues, offers a unique alternative by identifying directional implications between binary-coded variables (Gras et al., 2008). Unlike classical statistics, SIA focuses on "if A, then B" structures—aligning more closely with how knowledge dependencies emerge in mathematical learning. The CHIC algorithm (Classification Hiérarchique Implicative et Cohésitive) extends this model by generating implicative graphs that reveal the cognitive pathways students are likely to follow.

Rationale for Using SIA and CHIC

This study employs SIA via the CHIC method to analyze how students' mastery of prerequisite concepts influences their ability to construct matrix representations. Traditional regression or correlation models are limited in capturing asymmetrical knowledge dependencies; for example, being able to construct a matrix does not necessarily imply understanding domain basis, though the reverse may be true.

SIA provides a fine-grained, rule-based approach to uncovering logical structures in student responses. It enables researchers to detect implicative chains (e.g., understanding basis $\rightarrow$ correct symbolic differentiation $\rightarrow$ accurate matrix representation) that are otherwise obscured by aggregate scoring. This makes it especially suitable for examining layered, multi-step cognitive tasks such as those encountered in linear algebra proofs and symbolic construction.

Participants

The participants in the SIA analysis were 39 undergraduate students from the Linear Algebra II Fall 2024 and Spring 2025 semesters. As we have specified, the first part of this course (the part that is relevant to the present study) covers abstract vector spaces linear transformations, and matrix representation across arbitrary bases.

Students were selected using purposive sampling to ensure a range of mathematical competencies and exposure to proof-based instruction. Participation was voluntary, and all students provided informed consent in accordance with Institutional Review Board (IRB) guidelines.

Instrument and Coding

The diagnostic task was designed to assess students' conceptual and procedural understanding of the matrix representation of a linear transformation. The transformation was defined as

$$L(p(x)) = p'(x) + p(0), \quad L : P_3 \to P_2$$

with respect to the ordered bases $\{x^2, x, 1\}$ in P_3 and $\{2, 1 - x\}$ in P_2. Students were required to find the matrix representation of L and compute the image of the polynomial $p(x) = x^2 + 2x - 3$ under this transformation.

Student responses were analyzed using a binary coding scheme across 11 conceptual and procedural indicators. Each indicator was assigned a value of 1 (correct or present) or 0 (incorrect or absent). Table 5.7 summarizes the indicators used in this study.

Table 5.7: Binary Indicators for Student Understanding

Indicator	**Description**
Understands_P3_to_P2	Recognizes that the transformation maps from P_3 to P_2.
Understands_Transformation_Def	Understands the symbolic definition $L(p(x)) = p'(x)+p(0)$.
Understands_Domain_Basis	Correctly identifies $\{x^2, x, 1\}$ as the domain basis.
Understands_Codomain_Basis	Correctly identifies $\{2, 1-x\}$ as the codomain basis.
Correct_Derivative	Correctly computes the derivative $p'(x)$.
Correct_Evaluation_p0	Correctly evaluates $p(0)$.
Combined_Expression_Lpx	Accurately combines derivative and constant to form $L(p(x))$.
Correct_Matrix_Construction	Constructs the matrix representation of L.
Correct_Coord_Vector_Lpx	Finds the coordinate vector of $L(p(x))$ in codomain basis.
Part_A_Score	Composite score based on conceptual understanding (indicators 1–5).
Part_B_Score	Composite score based on procedural and representational construction (indicators 6–9).

Data Collection and Analysis

Student responses were collected during a structured review assessment given just before the final exam. All submissions were anonymized and scored independently to ensure fairness. The results were then entered into an Excel spreadsheet and converted into binary codes to prepare the data for statistical analysis.

To identify underlying knowledge structures, SIA was conducted using the CHIC (Classification Hiérarchique Implicative et Cohésive) algorithm, which identified directional implications among 11 binary-coded indicators to visualize conceptual dependencies among students' responses.

CHIC Method Overview

CHIC is a method for extracting directional logical relationships between binary variables based on implicative logic. For any two variables A and B, the relationship $A \Rightarrow B$ holds implicatively if all students who demonstrated understanding in A also showed success in B. Unlike correlational analysis, CHIC reveals asymmetrical, rule-like dependencies and can be represented as a directed graph.

In this study, the CHIC graph was used to identify conceptual hubs, knowledge chains, and semiotic trajectories that underpin successful matrix representation. The method enables researchers to understand not only which skills are commonly mastered but also which ones serve as gateways to higher-order understanding.

Descriptive Statistics

An initial analysis of students' responses across the 11 binary-coded indicators revealed varying levels of mastery:

- **Conceptual understanding** was relatively strong. Indicators such as *Understands_P_3 to_P_2*, *Understands_Domain_Basis*, and *Correct_Derivative* were among the most frequently demonstrated, with over 70% of students scoring 1.
- **Representational construction**, however, posed more difficulty. Indicators like *Correct_Matrix_Construction* and *Correct_Coord_Vector_$L(p(x))$* showed much lower success rates, with fewer than 40% of students answering correctly.

Figure 5.14 illustrates the frequency of student success for each indicator, based on standard statistical analysis rather than implicative modeling.

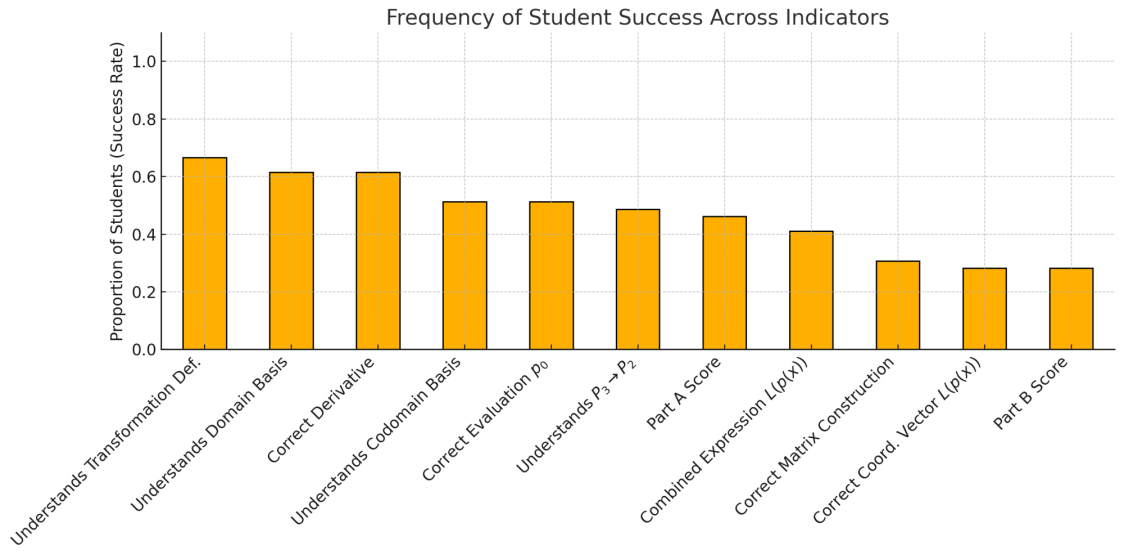


Figure 5.14: Frequencies of Student Success Across Conceptual and Procedural Indicators

This general statistical distribution highlights a clear divide between conceptual and procedural understanding. While many students successfully interpreted the function and identified the relevant bases, fewer were able to complete the procedural and representational tasks necessary to produce the matrix representation and the coordinate vector of $L(p(x))$.

CHIC Graph Construction and Implicative Relationships

To uncover directional dependencies among the 11 indicators, the CHIC algorithm was used to generate a directed implicative graph based on binary-coded student responses.

The strongest implicative relationships emerged among *Understands_P_3_to_P_2*, *Understands_Domain_Basis*, and *Correct_Derivative*, indicating that foundational understanding and symbolic fluency tend to co-occur. Moderate links (0.56–0.63) were observed between these concepts and more complex representational tasks such as matrix construction.

Implicative Graph and Knowledge Chains

Figure 5.15 displays the CHIC implicative graph based on a 0.75 similarity threshold. Arrows represent strong directional relationships where mastery of one concept is highly predictive of another.

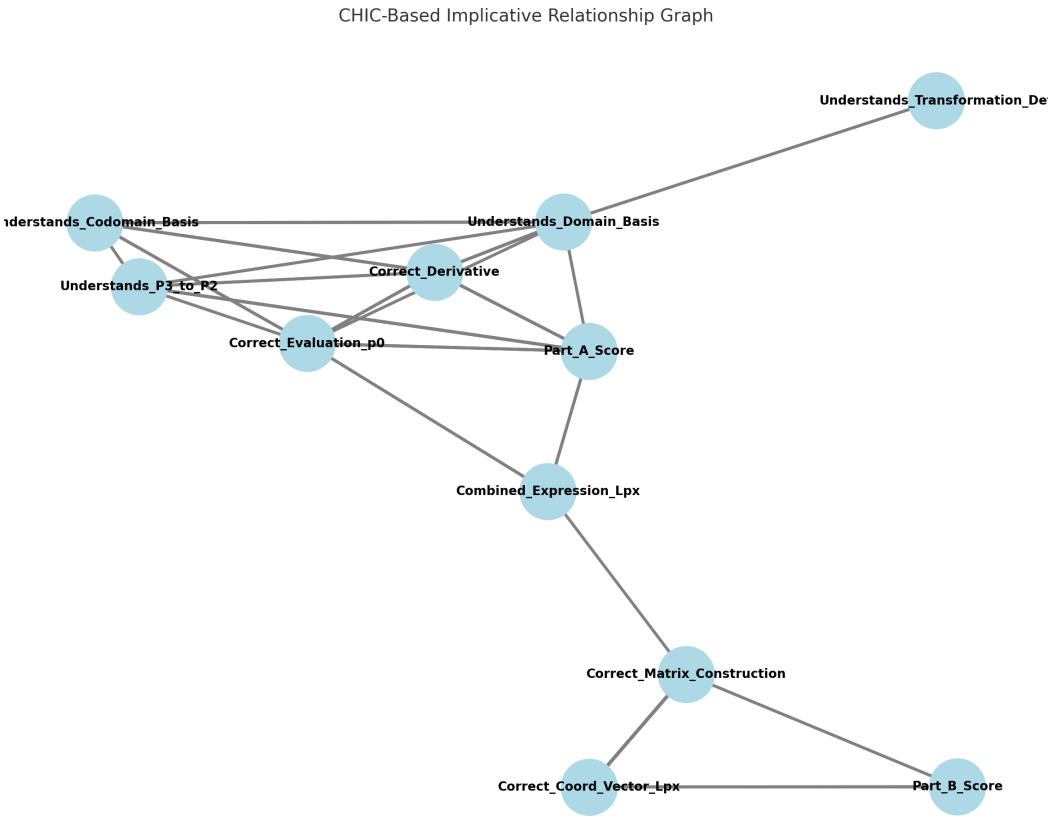


Figure 5.15: [

CHIC implicative graph of student reasoning]CHIC implicative graph with threshold ≥ 0.75.

Several key implicative chains were observed in the CHIC analysis (see Figure 5.15):

- **Conceptual Core:** $Understands\_P_3\_to\_P_2 \rightarrow Domain\_Basis \rightarrow Correct\_Derivative$
- **Symbolic to Representational Pathway:** $Correct\_Evaluation\_p(0) \rightarrow Combined\_Expression\_L(p(x)) \rightarrow Correct\_Matrix\_Construction \rightarrow Correct\_Coord\_Vector\_L(p(x))$
- **Performance Linkage:** *Conceptual indicators* $\rightarrow$ *Part_A_Score*; *Procedural indicators* $\rightarrow$ *Part_B_Score*

These implicative chains suggest that students' success in matrix representation is built upon a layered progression of conceptual understanding and symbolic synthesis. Notably, the graph

supports the hypothesis that failure to master early symbolic steps constrains representational success.

Part A vs. Part B Cognitive Structures

Part A and Part B of the task were designed to capture different cognitive dimensions of learning. Part A emphasized conceptual recognition and symbolic fluency, while Part B required multi-step synthesis across representational systems.

The CHIC graph confirms this division: Part A indicators (e.g., domain, codomain, derivative) form a tightly connected core, whereas Part B indicators (e.g., matrix and coordinate vector) form a separate chain linked to symbolic synthesis. This suggests that students may compartmentalize their knowledge, succeeding in early conceptual tasks but faltering when transitioning to coordinate or matrix-based representation.

Interpretation of Conceptual Patterns

The CHIC-based Statistical Implicative Analysis offers insight into how students' conceptual and procedural knowledge structures support or constrain their performance on tasks involving matrix representation of a linear transformation. The implicative chains observed in the graph suggest that mastery of foundational concepts, such as understanding the transformation direction($P_3 \rightarrow P_2$), basis recognition, and symbolic differentiation—is a prerequisite for success in more complex, representational tasks.

Students who demonstrated a coherent understanding of domain and codomain bases were more likely to correctly construct the transformation expression $L(p(x)) = p'(x) + p(0)$, and subsequently generate a correct matrix representation. This aligns with the findings of Weber (2012), who highlight the importance of integrating conceptual structure with semiotic flexibility in abstract linear algebra tasks.

Implications for Teaching and Curriculum

The implicative graph provides strong evidence for the sequential and hierarchical nature

of mathematical knowledge construction in this context. The separation between Part A and Part B indicators suggests that instructional strategies should explicitly scaffold the transition from symbolic manipulation to matrix and coordinate representations.

Several recommendations emerge:

- **Emphasize Conceptual Grounding:** Early instruction should ensure that students can articulate the function and target spaces of transformations and identify basis systems without reliance on memorized procedures.
- **Support Symbolic Integration:** Activities should be designed to help students combine symbolic procedures (e.g., differentiation and evaluation) into unified expressions, reinforcing the form of transformation output.
- **Bridge to Representational Tasks:** Structured practice should include guided steps that link transformation expressions to matrix construction and coordinate vectors, with visual representations when possible.

These suggestions are supported by the semiotic perspective advanced by Duval (2006) and operationalized in the Onto-Semiotic Approach (Godino, Batanero, & Font, 2007), which stresses the need to help students move between registers (e.g., symbolic, matrix, and coordinate).

Relation to Prior Studies

This study extends previous work by offering a quantitative lens through which to view knowledge dependencies in linear algebra. While studies like M. Carlson and Rasmussen (2008) and Harel (2007) describe the cognitive complexity of abstract linear algebra, few have mapped these complexities using implicative statistical logic.

By applying SIA to actual student data, this research confirms earlier qualitative insights about the layered nature of understanding, but also quantifies the strength of relationships between skills. For example, while it has long been hypothesized that understanding bases underlies matrix representation, this study demonstrates that this relationship holds across the majority of cases

in the sample, as revealed through consistent patterns of directional implications identified in the CHIC analysis

Furthermore, the use of CHIC allows for identification of directional relationships that are not visible in correlational studies. The asymmetry observed (e.g., matrix construction does not imply understanding of domain basis) is consistent with the pedagogical experience that procedural success can mask conceptual misunderstanding.

Conclusion

This study employed Statistical Implicative Analysis (SIA) using the CHIC method to investigate upper-division undergraduate students' understanding of the matrix representation of a linear transformation from P_3 to P_2. The analysis was based on a diagnostic task that required students to integrate conceptual knowledge, symbolic manipulation, and representational fluency. Student responses were coded across eleven binary indicators, and implicative relationships among these indicators were visualized using a CHIC-based directed graph.

The findings revealed clear implicational chains that illustrate how success in representational tasks depends upon mastery of conceptual and symbolic components. In particular, understanding the domain and codomain bases and correctly computing symbolic derivatives were strongly predictive of performance on matrix construction and coordinate vector identification. The distinction between the cognitive demands of Part A (conceptual) and Part B (representational) was evident, supporting prior research that emphasizes the layered and sequential nature of mathematical understanding in linear algebra.

From a methodological perspective, the use of CHIC offered a novel and informative lens to explore student reasoning. Unlike traditional correlation or regression techniques, SIA allowed for the identification of asymmetrical, directional relationships that reflect real learning dependencies. This approach contributes to the growing body of research advocating for more nuanced, logic-based tools in mathematics education.

Future research should consider expanding the dataset to include a larger and more diverse student population, incorporating additional transformation tasks, and triangulating the SIA results with qualitative data such as student interviews or think-aloud protocols. Moreover, the integration of this analytical method into formative assessment tools may provide instructors with real-time insight into student learning trajectories, allowing for more targeted and responsive teaching strategies.

In conclusion, this study demonstrates that Statistical Implicative Analysis can reveal the structural dependencies of mathematical understanding, offering both theoretical insight and practical implications for teaching linear algebra in undergraduate settings.

Research consistently shows that students struggle to transition between abstract definitions and symbolic, coordinate-based implementations of these concepts (Harel, 2007; Montiel et al., 2011; S. Stewart, 2016). In particular, mastering the construction of a matrix representation from a transformation defined on a polynomial space requires a coherent understanding of function spaces, basis systems, symbolic manipulation, and representational shifts across semiotic registers.

This research is motivated by the need to examine how students build and comprehend the conceptual and procedural knowledge necessary to construct a matrix representation of transformations. Traditional assessment methods are not enough in revealing the implicit logical structure of students' understanding, particularly when multiple layers of mathematical knowledge are involved. To address this, the present study adopts a Statistical Implicative Analysis (SIA) framework to analyze student responses involving the transformation $L(p(x)) = p'(x) + p(0)$, where $L : P_3 \to P_2$. The transformation is situated in the context of abstract vector spaces and basis change, concepts known to pose difficulties for students transitioning from computational to structural thinking (Montiel et al., 2011; Weber, 2012).

The primary objective of this study is to uncover patterns of conceptual dependency among student responses using the CHIC method, a quantitative approach that identifies directional relationships between binary-coded knowledge indicators. Specifically, the analysis seeks to determine which understandings (e.g., recognition of domain and codomain bases, symbolic differentiation,

evaluation at a point) are foundational and which are more dependent or integrative (e.g., combined expression construction, matrix formulation, and coordinate vector generation).

This implicative approach offers a fine-grained lens through which to understand student cognition in symbolic construction tasks. By identifying high-frequency chains of mastery and key nodes in the cognitive structure, the findings can inform both curriculum sequencing and targeted instructional interventions. The analysis contributes to the broader field of research on student thinking by offering a methodological bridge between qualitative semiotic studies and quantitative dependency modeling.

Chapter 6

Teaching Strategies and Directions

Building upon the historical developments discussed in Chapter 2 and the empirical findings presented in Chapters 4 and 5, this chapter explores pedagogical strategies aimed at improving students' understanding of matrix representation. The results revealed persistent conceptual challenges, particularly concerning coordinate vectors, non-standard basis transformations, and symbolic representations of linear transformations.

As the symbolic expression $L(\mathbf{x}) = A\mathbf{x}$, where L denotes a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, A is an $m \times n$ matrix, and $\mathbf{x} \in \mathbb{R}^n$, serves as a common introduction for students to the idea that matrices can represent transformations. Although this formulation is mathematically precise, students often approach it procedurally, focusing on the mechanics of matrix multiplication rather than interpreting it as a mapping between vector spaces. This procedural lens can obscure the underlying conceptual structure, echoing the concerns raised by Wawro and Serbin (2025) regarding the cognitive demands of symbolization in linear algebra. While their study focuses on the expression $A\mathbf{x} = \lambda\mathbf{x}$ in the context of eigentheory, similar challenges arise in more general matrix representations of linear transformations.

In the context of matrix representation, this symbolic form is not merely a rule for computation, but encodes the action of a linear transformation relative to a particular pair of bases. Without a clear understanding of how coordinate vectors, basis changes, and transformation matrices interact, students may struggle to recognize A as a representation of L rather than as an isolated array of numbers. This misalignment between symbol and structure presents a significant barrier to developing a deeper, more abstract understanding of linear algebra.

In response to these findings, the chapter proposes targeted instructional strategies designed to bridge the gap between procedural fluency and structural understanding. These strategies address

the conceptual difficulties observed and support the development of more integrated and meaningful understandings of matrix representation by students. The chapter concludes by outlining future directions for curriculum development and instructional design, offering practical insights for educators and researchers seeking to enhance the teaching and learning of abstract linear algebra.

6.1 Qualitative Results and the OSA

Fostering Semiotic Coordination Across Representations

One of the most prominent themes across student interviews and written responses was the difficulty in navigating between symbolic, matrix, and coordinate representations. Many students treated mathematical symbols procedurally without clear reference to the objects they represent, leading to semiotic conflicts and fragmented reasoning.

To address this, instruction should incorporate explicit activities that guide students in translating between semiotic registers. For instance, tasks that require expressing a transformation both as a symbolic rule and as a matrix, or that involve verbal explanations of expressions such as $[T]_C^B \cdot [v]_C$, can promote semiotic awareness. Visual aids, such as commutative diagrams and labeled transformation maps, may help students conceptualize the role of domain and codomain bases and understand how matrix representations encode transformations across different coordinate systems.

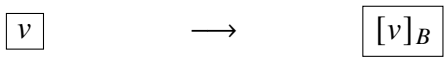
Clarifying the Ontological Status of Mathematical Objects

A second recurring issue was the ontological misidentification of key mathematical entities. Students often misinterpreted transformation matrices as change-of-basis matrices or conflated coordinate vectors with the vectors they represent. These confusions suggest a lack of clarity regarding the referents of mathematical symbols within the context of vector space structures.

Instructional design should incorporate repeated opportunities for students to identify and classify the type of object each mathematical symbol represents. For example, activities that ask students to distinguish between a transformation matrix and a transition matrix, or to explain what

$[v]_B$ denotes in relation to the original vector v, can help students internalize these distinctions. Classroom discourse should emphasize the structural role and referential meaning of matrices and vectors rather than relying on syntactic manipulation alone.

For example, instructional activities that prompt students to *distinguish between a transformation matrix and a transition (change-of-basis) matrix*, or to *explain what $[v]_B$ denotes in relation to the original vector v*, can help them internalize critical ontological distinctions. Qualitative interviews—particularly with ST1, ST2 —revealed frequent confusion between coordinate vectors and the original abstract vectors, as well as between transformation matrices and transition matrices. These misunderstandings indicate not only semiotic overload but also *epistemic obstacles in recognizing the referents of mathematical symbols*. In the CHIC implicative graph, conceptual nodes related to basis understanding and coordinate decompositions were shown to underpin successful matrix construction and symbolic synthesis. These directional dependencies suggest that *clarity in foundational distinctions is essential for higher-order reasoning* in linear algebra.



Abstract vector in V Coordinate vector in $\mathbb{R}^n$

Figure 6.1: Distinction between vector $v \in V$ and its coordinate vector $[v]_B \in \mathbb{R}^n$

When learners are asked to explicitly state what a symbol means (e.g., "What does $[v]_B$ refer to?"), they engage in *meta-representational reasoning*, reinforcing both language comprehension and mathematical abstraction. As emphasized by Duval's theory of semiotic registers (Duval, 2006) and Godino's Onto-Semiotic Approach (Godino, 2024), *meaning in mathematics emerges not from symbols alone, but through students' coordination of representational systems and institutional practices*.

Emphasizing Conceptual Justification

While many students can accurately compute expressions such as $P^{-1}AP$ or apply transformation matrices in problem settings, their explanations often reveal a superficial understanding of

the underlying concepts. This disconnect is particularly evident in tasks involving basis changes or similarity transformations, where symbolic procedures are executed correctly, but students struggle to articulate the mathematical meaning behind them.

To address this gap, instructors should incorporate justification-based prompts into both formative and summative assessments. Questions such as "Why does this matrix represent the transformation L under these specific bases?" or "How can you verify that $[v]_C$ corresponds to the correct basis?" can prompt students to connect their symbolic procedures with conceptual reasoning. Additionally, classroom strategies such as peer-led explanation, think-aloud protocols, and structured reflection tasks can foster a deeper engagement with the representational meaning of matrices and coordinate systems. By embedding these practices consistently, instructors can support students in transitioning from procedural competence to structural understanding.

Sequencing Instruction According to Levels of Abstraction

The use of the OSA framework enabled the identification of students' reasoning levels, ranging from intuitive, informal reasoning (Level 0) to structurally integrated, abstract reasoning (Level 2+). Most students operated at Level 1, demonstrating procedural competence without conceptual synthesis.

In response, instructional sequences should be explicitly structured to support transitions across these levels. Instruction should begin with concrete, example-driven representations (e.g., standard bases, low-dimensional spaces) and gradually introduce generalizations and transformations involving arbitrary bases. Scaffolding should support students in interpreting their symbolic procedures within the context of broader vector space structures and coordinate systems.

Integrating Diagnostic Assessments to Identify Obstacles

The review tests and interview data demonstrated the value of diagnostic assessments in uncovering not just correctness, but the reasoning structures underlying student responses. In particular, tasks that required interpretation of matrix expressions, justification of coordinate decompositions, or diagnosis of transformation errors yielded insights into students' semiotic and

epistemological challenges.

Instruction should incorporate periodic diagnostic assessments aligned with the abstraction levels and semiotic practices outlined in the OSA framework. These assessments should be designed to probe both procedural fluency and deeper conceptual understanding. For example, one effective diagnostic task might ask students to compute the matrix representation of a transformation from P_3 to P_2 under given non-standard bases, followed by a written justification of each step. Another might present students with an incorrect coordinate vector decomposition (e.g., an incorrect use of basis coefficients) and ask them to identify and correct the error, explaining the reasoning behind their correction.

In a different example, students could be given a transformation $L : \mathbb{R}^2 \to \mathbb{R}^2$ represented by a matrix A, and be asked to determine whether the matrix represents the transformation with respect to the standard basis or some other basis, based on contextual cues. Alternatively, an assessment may present a symbolic expression such as $P^{-1}AP$ and ask students to interpret the role of each matrix in the context of similarity transformation or change of basis, rather than merely compute the product.

Such diagnostics should be administered at key points in the instructional sequence to detect early misconceptions and prevent conceptual drift. Instructors can then use the results to tailor instruction—offering targeted remediation for students who struggle with coordinate transformations, while providing enrichment tasks for those who demonstrate strong abstraction. These assessments also serve as valuable metacognitive tools, helping students recognize gaps in their understanding and encouraging them to reflect on their own reasoning processes.

Summary

The qualitative results highlight the importance of instructional strategies that align with the layered, semiotic, and epistemic dimensions of student reasoning in linear algebra. Effective teaching must move beyond symbolic procedures to foster structural understanding, ontological clarity, and semiotic coherence. By adopting strategies that explicitly support transitions across semiotic

registers, clarify the referents of mathematical symbols, and scaffold reasoning across abstraction levels, instructors can better support student learning and facilitate meaningful engagement with complex mathematical concepts. These strategies are particularly critical in bridging the gap between procedural fluency and conceptual integration in topics such as matrix representation, basis change, and linear transformation.

6.2 Quantitative Results and CHIC

The CHIC-based implicative analysis revealed distinct directional dependencies among key conceptual and procedural knowledge indicators, particularly in relation to students' understanding of the matrix representation of linear transformations. These findings underscore the necessity of instructional scaffolding that mirrors the hierarchical and sequential nature of students' knowledge construction. Accordingly, this section outlines instructional strategies grounded in the observed implicative structures to support student learning in undergraduate linear algebra.

Scaffolding Based on Implicative Chains

The CHIC implicative graph suggests that the success of students in constructing matrix representations is based on a layered progression that begins with conceptual understanding, then with symbolic synthesis and culminates in representational fluency. This implicative sequence informs the need for instructional alignment with the actual learning trajectories of the students.

Establishing Conceptual Grounding

Given the strong implicative links from understanding the transformation direction (*Understands_P3_to_P2*) and domain basis (*Understands_Domain_Basis*) to symbolic and representational success, instruction should begin with robust conceptual anchoring. Instructors are advised to employ activities that emphasize the functional mapping between polynomial spaces. Visual tools and physical metaphors can be used to highlight how basis elements structure these spaces. Students should be asked to explicitly identify, label, and describe the roles of domain and codomain bases before engaging in any symbolic manipulation.

Supporting Symbolic Integration

The chain *Correct_Derivative* → *Correct_Evaluation_p0* → *Combined_Expression_Lpx* highlights the importance of symbolic fluency. Instructors should scaffold this progression by breaking down symbolic procedures into manageable sub-tasks, ensuring students can compute derivatives, evaluate constants, and construct expressions independently before combining them. Prompting students to explain each symbolic step in natural language may also reduce semiotic overload and facilitate deeper comprehension.

Bridging to Representational Fluency

The CHIC graph further indicates that success in matrix construction (*Correct_Matrix_Construction*) and coordinate vector generation (*Correct_Coord_Vector_Lpx*) is contingent on earlier symbolic synthesis. To bridge this gap, instructors can implement guided practice tasks that transition from symbolic expressions to their matrix equivalents. For example, exercises where students compute the image of a basis vector and convert it to coordinates in a specified basis can clarify the logic underlying matrix columns. Additionally, introducing reverse tasks—where students deduce transformation rules from given matrices—can reinforce representational understanding.

Addressing Semiotic Conflicts

Semiotic conflicts often arise when students misinterpret or fail to connect the transformation formula to its matrix or coordinate form. Drawing on Duval's (2006) theory of semiotic registers and Godino's Onto-Semiotic Approach (Godino et al., 2007), instructors should design activities that require students to transition explicitly among symbolic, coordinate, and matrix representations. For instance, students can be asked to justify how a symbolic expression corresponds to a particular matrix column or coordinate vector. Reflection prompts such as "What does this column represent?" or "Which basis are we using?" can also encourage metacognitive engagement and help resolve register conflicts.

Sequencing Curriculum by Cognitive Dependencies

Finally, curriculum sequencing should reflect the implicative structure of student knowledge. Instruction should proceed from foundational concepts to symbolic processes and then to representational constructions. Diagnostic assessments aligned with each layer can be used to monitor student readiness to progress. Students who fail to demonstrate mastery of prerequisite concepts (e.g., basis recognition or symbolic expression construction) should receive targeted remediation before engaging with matrix-level tasks.

Summary

The teaching strategies outlined above are designed to align instruction with the actual implicative dependencies observed in student data. By scaffolding learning in accordance with the CHIC-based knowledge chains, instructors can more effectively support students as they move from conceptual foundations to procedural fluency and representational.

6.3 Instructional Strategies Informed by Integrated Qualitative and Quantitative Study

The findings of this study, derived from both qualitative and quantitative data, reveal the complex and layered nature of student understanding in linear algebra, particularly concerning matrix representation, basis transformations, and coordinate systems. By integrating the OSA and SIA, this study identifies not only the specific semiotic and conceptual challenges students encounter, but also the sequential knowledge dependencies that underpin successful performance. Based on these insights, this section outlines instructional strategies that align pedagogical design with students' actual learning trajectories, addressing both procedural fluency and conceptual integration.

Aligning Instruction with Knowledge Dependencies

The CHIC implicative graph revealed a consistent progression from conceptual knowledge (e.g., understanding domain and codomain bases) to symbolic synthesis (e.g., combining derivatives and evaluations), and finally to representational construction (e.g., forming matrices and coordinate

vectors). Instruction should reflect this progression by introducing tasks in a scaffolded sequence that builds conceptual foundations before advancing to representational tasks.

Strategy: Design instructional modules that explicitly move from:

1. Conceptual grounding: identifying domain/codomain and basis structures;
2. Symbolic integration: composing transformation outputs from derivatives and constants;
3. Representational construction: translating expressions into matrix form and coordinate vectors.

This structure supports students in developing layered competencies and prevents premature engagement with matrix construction before foundational understandings are in place.

Promoting Semiotic Coordination Across Registers

Qualitative analysis revealed that students frequently struggled to coordinate symbolic expressions, matrix representations, and coordinate vectors. Semiotic conflicts—such as misinterpreting $[T]_C^B \cdot [v]_B$ or confusing transformation and transition matrices—often stemmed from treating symbols as syntactic operations rather than representations of mathematical objects.

Strategy: Incorporate tasks that require students to:

- Translate between verbal, symbolic, and matrix-based representations;
- Label components of transformation expressions with their mathematical meaning;
- Construct and interpret commutative diagrams that show input-output flows across coordinate systems.

Such activities, informed by Duval's theory of semiotic registers (Duval, 2006), can help students internalize the relationships between mathematical representations and reduce symbolic overload.

Clarifying Ontological Distinctions Between Mathematical Objects

Many students misidentified transformation matrices as change-of-basis matrices or conflated coordinate vectors with vectors in the abstract space. These ontological errors indicate that students may perform correct procedures without understanding the nature of the objects they manipulate.

Strategy: Embed frequent opportunities for students to:

- Classify symbols by type (e.g., matrix representing transformation vs. matrix for basis change);
- Identify the domain, codomain, and bases associated with each transformation;
- Use structured prompts to explain what a coordinate vector or matrix represents in the context of a transformation.

Such practices help reinforce the referential clarity necessary for understanding linear algebra beyond procedural tasks.

Encouraging Conceptual Justification and Structural Reasoning

Both the OSA analysis and the CHIC results highlighted a gap between students' ability to execute symbolic procedures and their capacity to articulate or justify their reasoning. Students functioning at Level 1 abstraction often manipulated symbols correctly but could not explain the underlying structure of their work.

Strategy: Require written or verbal justifications in problem-solving that address:

- The role of a particular basis in defining a matrix or coordinate system;
- The correctness of symbolic transformations with respect to domain and codomain;
- The logical reasoning behind basis transitions or matrix compositions.

Justification-based learning tasks can move students beyond surface-level manipulation and promote structural understanding, which is essential for high-level abstraction.

Using Diagnostic Tools to Reveal Epistemic Configurations

The CHIC-based SIA identified directional relationships such as *Domain Basis* → *Derivative* → *Matrix Construction*, which suggest that mastery of one component predicts success in others. Qualitative data further confirmed that students who failed early symbolic steps were unlikely to succeed in matrix representation.

Strategy: Implement formative diagnostic assessments that:

- Include both symbolic and representational tasks;
- Use implicative chains to identify conceptual bottlenecks;
- Provide feedback that links errors to prior concepts (e.g., missing basis identification leading to representational errors).

This approach allows for real-time pedagogical adjustments and supports students in progressing through critical knowledge dependencies.

Strategy: Differentiate instruction and support through:

- Annotated examples, visual scaffolds, and subtitled video explanations;
- Small group tasks with structured roles to encourage peer-to-peer mediation;
- Explicit discussions of notation conventions and their meaning.

These practices help reduce cognitive and linguistic load, particularly for students transitioning into advanced mathematical discourse communities.

Scaffolding by Levels of Abstraction

The abstraction levels derived from the OSA framework (Table 5.2) categorize students' reasoning as low (intuitive or fragmented), mid (symbolic-procedural), or high (structural and integrated). Most students in this study operated at mid-level, with only a few demonstrating high-level abstraction.

Strategy: Scaffold lessons and tasks to target progression across abstraction levels:

1. Begin with procedural examples grounded in familiar bases;

2. Introduce non-standard bases with conceptual interpretation;

3. Require synthesis of coordinate decompositions and basis changes through project-based learning or oral explanation.

This progression allows students to incrementally build epistemic control over increasingly abstract representations and coordinate systems.

6.4 Strategies for teaching Matrix Representation Theorem

Theorem 4.2.2 Matrix Representation Theorem

If $E = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ and $F = \{\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m\}$ are ordered bases for vector spaces V and W, respectively, then corresponding to each linear transformation $L: V \to W$, there is an $m \times n$ matrix A such that

$$[L(\mathbf{v})]_F = A[\mathbf{v}]_E \qquad \text{for each } \mathbf{v} \in V$$

A is the matrix representing L relative to the ordered bases E and F. In fact,

$$\mathbf{a}_j = \left[L(\mathbf{v}_j)\right]_F \qquad j = 1, 2, \ldots, n$$

Theorem 4.2.2 is illustrated in Figure 4.2.2. If A is the matrix representing L with respect to the bases E and F, and if

$\mathbf{x} = [\mathbf{v}]_E$ (the coordinate vector of $\mathbf{v}$ with respect to E)
$\mathbf{y} = [\mathbf{w}]_F$ (the coordinate vector of $\mathbf{w}$ with respect to F)

then L maps $\mathbf{v}$ into $\mathbf{w}$ if and only if A maps $\mathbf{x}$ into $\mathbf{y}$.

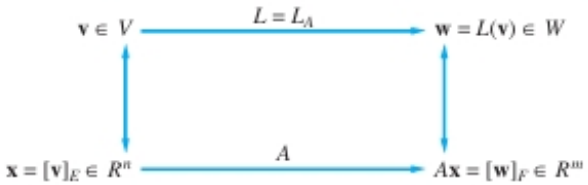


Figure 4.2.2.

Figure 6.2: Matrix representation diagram showing $[L(v)]_F = A[v]_E$ under bases E and F (Leon, 2015, p.182)

The Matrix Representation Theorem (see Theorem 4.2.2) in (Leon, 2015) states that if $E = \{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ and $F = \{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ are ordered bases for vector spaces V and W, then for each

linear transformation $L : V \to W$, there exists a matrix A such that $[L(v)]_F = A[v]_E$ for every $v \in V$.

This section uses the OSA and SIA (CHIC) to identify and address learning challenges related to this theorem.

Table 6.1: Semiotic-Epistemic Complexity of the Matrix Representation Theorem

OSA Element	**Example in Theorem**	**Student Challenge**
Mathematical Object	Matrix A, Bases E, F, Transformation L	Confusing A as a general matrix, not tied to specific bases
Semiotic Registers	Symbolic: $A[v]_E$, $[L(v)]_F$; Diagrammatic: Figure 4.2.2	Difficulty coordinating between symbols and diagram
Referential Meaning	$[v]_E, [L(v)]_F$ represent coordinates	Misunderstanding what these notations refer to
Rules of Validity	$[L(v)]_F = A[v]_E$ applies under specific bases	Using formula with incorrect or unaligned bases
Argumentation Structure	Commutative diagram as justification	Failing to interpret equivalence between mappings and matrices

Semiotic and Epistemic Complexity (OSA Perspective)

Cognitive Dependencies (CHIC Perspective) The implicative learning sequence can be summarized as follows:

`Understands_Basis` $\to$ `Finds_Coord_Vector` $\to$ `Constructs_Matrix` $\to$ `Interprets_A[v]_E` $\to$ `Explains_Diagram`

This chain reflects a progression from conceptual grounding to symbolic synthesis and representational reasoning.

Instructional Strategies

- **Promote Semiotic Coordination:** Guide students to translate between $L(v)$, $A[v]_E$, $[L(v)]_F$, and commutative diagram form.
- **Scaffold by CHIC Sequence:** Structure tasks that move from identifying bases and computing coordinates to constructing and interpreting matrix representations.
- **Diagram Reconstruction Tasks:** Ask students to recreate Figure 4.2.2 using their own example transformation. Require labeled diagrams and written interpretations.
- **Clarify Ontological Status:** Embed classification prompts distinguishing between transformation matrices and change-of-basis matrices.
- **Justification-Based Prompts:** Use written and oral questions such as: "Why does $A[v]_E = [L(v)]_F$ work only under specific bases?"
- **Visual and Linguistic Support:** Color-code domain/codomain vectors and matrices. Include bilingual annotations and simplified terminology for multilingual learners.

Example Activity (OSA & CHIC-Aligned) **Goal:** Construct and interpret the matrix representation of a transformation.

Let $V = \mathbb{R}^2$, $W = \mathbb{R}^2$. Define $L(x, y) = (x + y, x - y)$ and let $E = \{(1, 0), (0, 1)\}$, $F = \{(1, 1), (1, -1)\}$.

1. Express $L(e_1)$ and $L(e_2)$ in terms of the codomain basis F.
2. Construct the matrix A such that $[L(v)]_F = A[v]_E$.
3. Draw and label a diagram similar to Figure 4.2.2 representing the transformation and coordinate mappings.
4. Justify each step using appropriate mathematical terminology and connect symbolic to visual representations.

This activity supports conceptual reasoning, semiotic alignment, and representational fluency, bridging the gap between symbolic manipulation and structural understanding.

6.5 Concluding Remarks

The Onto-Semiotic Approach (OSA) and Statistical Implicative Analysis (SIA) have shown promising results in collegiate mathematics education in Europe and South America, but they are still not widely used in the United States. In a conversation with a senior mathematics professor, he jokingly said, "Most faculty tend to fall into one of two groups: those who are strong in doing their own research but not in teaching, and those who are active teachers but do almost no research." Although said in a light tone, this comment reflects a real issue: many instructors are not truly engaged in studying how teaching materials should be developed in relation to students' learning simultaneously. This highlights the need for educators who are not only familiar with theoretical frameworks like OSA and SIA, but also know how to apply them meaningfully in teaching practice. When instructors use these frameworks to guide their own thinking, they can model structured reasoning for students, who may then begin to learn and adopt similar approaches in their own learning.

This also underscores the importance of mathematics education research that is both practical and relevant to mathematical content. If professors find studies based on OSA and SIA interesting and useful, they may be more likely to apply these ideas to help students better understand what they are learning. In this way, such research has the potential to bridge the gap between theory and classroom practice, supporting both teaching and learning in more meaningful ways.

At the same time, there are several classical textbooks on linear algebra with applications that have been commonly used in college classrooms. However, these materials usually follow a traditional structure that emphasizes procedural formats: *Definition* → *Formula* → *Example* → *Exercises*—which prioritize algorithmic steps and routine problem-solving over conceptual exploration and flexible reasoning. From the viewpoint of OSA and SIA, there is a need for a new type of instructional resource. For example, textbooks that better reflect the layered structure of math-

ematical understanding, along with the historical development of key concepts, could encourage instructors and students to connect semiotic ontology with epistemological development across different representations. Alternatively, formal and matched videos designed to support flexible, self-paced learning could better align with how students build knowledge over time. These resources would not replace traditional materials but could help students move from surface-level procedures to deeper conceptual understanding.

The teaching strategies discussed in this section are based on both qualitative and quantitative findings about student thinking. By bringing together the semiotic and epistemic ideas from OSA and the structural learning patterns identified through SIA, these strategies aim to help students move beyond just doing procedures. Instead, the goal is to help them understand what the procedures mean and how different representations connect. The findings from this study suggest that when instruction is better aligned with students' conceptual development—especially in areas like symbolic fluency, abstraction, and interpretation of notation—it can make a real difference. Overall, combining these frameworks provides practical ways to enhance teaching in linear algebra and address the challenges students frequently encounter.

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Chapter A

Thematic Coding Scheme for Linear Algebra Review Test

Table A.1: Thematic Coding Categories for Each Section of the Review Test

Test Part	**Focus**	**Thematic Codes**
Part 1: Multiple Choice (Q1–Q5)	Conceptual recognition (e.g., subspaces, bases, coordinate vectors)	- Correct conceptual recall - Misclassification of vector properties - Confusion about basis criteria - Memorization without understanding
Part 2: Fill in the Blank (Q6–Q8)	Vocabulary and definition recall	- Use of informal language - Incomplete or vague definitions - Proper mathematical terminology - Semantic confusion (e.g., span vs. coordinates)
Part 3: Fix Definitions (Q9–Q10)	Conceptual accuracy and error correction	- Incomplete corrections - Misunderstanding vector space/subspace conditions - Conflating symmetry with linearity - Accurate rephrasing using formal language
Part 4: Graph-Based (Q11)	Visual and algebraic representation	- Graphing inconsistency - Correct visualization - Coordinate vs vector confusion - Inability to map from $[v]_B \rightarrow [v]_C$
Part 5: Problem Solving (Q12)	Coordinate vector calculation (algebraic and matrix-based)	- Linear system setup error - Matrix inversion misuse - Partial reasoning - Dual method proficiency - Symbolic fluency vs. conceptual gap

Chapter B

Linear Algebra II Review Exam Full Content

This appendix contains the complete set of questions, solutions, and explanations from the Review Exam on matrix representation, change of basis, and geometric interpretation tasks.

Section 1: Notation and Conceptual Questions

What does $[v]_B$ represent? **Answer:** (b) Coordinates of v in basis B If T is a linear transformation and $[T]_C^B$ is its matrix with respect to bases B and C, what does $[T]_C^B \cdot [v]_B$ represent? **Answer:** (e) None of the above What does $[T]_C^B \cdot [v]_C$ represent? **Answer:** (a) The transformed vector in basis B What does $[T]_C^B \cdot [v]_C$ represent? **Answer:** (c) Apply matrix that takes input in C and outputs in B What does identity matrix I mean semiotically? **Answer:** (b) Has no effect on representation

Section 2: Change-of-Basis Matrix

Which of the following is always true for a change-of-basis matrix P from B to C? **Answer:** (b) P^{-1} changes basis from C to B What does $P^{-1}AP$ represent if P changes from basis B to C? **Answer:** (a) Matrix representation in basis B

Section 3: Basis-Specific Transformation

Linear transformation: $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x+y \\ 2y \end{bmatrix}$ Bases: $B = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ $C = \left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$ Results:$[T]_B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$, $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$
$[T]_C = P^{-1}[T]_B P$

Section 4: Interpretation Tasks

Geometric interpretation of $[T] = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$: Anisotropic scaling: stretches space by 2 on the x-axis and 3 on the y-axis. Shear transformation with $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$:

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x+2y \\ y \end{bmatrix}$$

Affects x-component by $2y$; horizontal shear along x-axis.

Section 5: Abstract Polynomial Transformation

Transformation: $L(p(x)) = p'(x) + p(0)$ Domains: $P_3 \to P_2$ Bases:P_3: $\{x^2, x, 1\}$ P_2: $\{2, 1-x\}$

Tasks:

Find matrix representation of L Find coordinate vector of $L(p(x))$ for $p(x) = x^2 + 2x - 3$

Note: This appendix was constructed to provide a full, self-contained record of all review exam materials, including student-facing questions and pedagogical explanations.

Chapter C

Interview Protocol for Conceptual Understanding

This appendix provides a structured interview protocol designed to uncover students' depth of understanding regarding matrix representations of linear transformations and related concepts.

Purpose

To explore students' conceptual understanding and reasoning processes related to matrix representations, change of basis, and linear transformations.

Format

Semi-structured interviews lasting 30–45 minutes will be recorded with student consent.

Chapter D

Interview Questions

Part 1: Conceptual Clarification

- **What does it mean for a matrix to represent a linear transformation?**
  - Follow-up: **Can you give an example of a linear transformation and the corresponding matrix representation?**
- **Explain the relationship between a vector and its coordinate representation.**
  - Follow-up: **What would you need to compute the coordinates of a vector in a different basis?**
  - Follow-up: **Why can the coordinates differ even if the vector itself does not change?**
- **What does it mean to change the basis of a vector space?**
  - Follow-up: **What practical applications can you identify for changing basis?**

Part 2: Problem Solving (Think-Aloud)

- **Given $T : P_2 \to \mathbb{R}^2$, with bases $B = \{1, x, x^2\}$ and $C = \{[1\ 0]^T, [0\ 1]^T\}$, construct $[T]_C^B$.**
  - Follow-up: **Explain each step of your reasoning. Why did you choose this method?**
  - Follow-up: **How does linearity of T impact your construction?**

Part 3: Reflection

- **What aspects of matrix representation were most difficult for you to understand?**

- **How did visual or symbolic representations help or hinder your learning?**
  - Follow-up: **Which concepts remained unclear despite visual or symbolic aids?**
- **Reflect on any misconceptions you had while learning this topic.**
  - Follow-up: **How were these misconceptions resolved?**