Extensions of Vizing fans and Vizing's Theorem in graph edge coloring

EXTENSIONS OF VIZING FANS AND VIZING'S THEOREM IN GRAPH EDGE COLORING

by

XULI QI

Under the Direction of Guantao Chen, PhD

ABSTRACT

Graph edge coloring is a well established subject in the field of graph theory. It is one of the basic combinatorial optimization problem: Color the edges of a graph G with as few colors as possible such that each edge receives a color and adjacent edges receive different colors. The minimum number of colors needed for such a coloring of G is called the chromatic index, denoted by $\chi'(G)$. Let $\Delta(G)$ and $\mu(G)$ be maximum degree and maximum multiplicity of G, respectively. Vizing and Gupta, independently, proved in the 1960s that

$\chi'(G) \leq \Delta(G) + \mu(G)$, by using the Vizing fan as main tool. Vizing fans and Vizing's Theorem play an important role in graph edge coloring. In this dissertation, we introduce two new generalizations of Vizing fans and obtain their structural properties for simple graphs, and partly comfirm one conjecture on the precoloring extension of Vizing's Theorem for multigraphs.

INDEX WORDS: Edge coloring, Vizing fans, Vizing's Theorem, Critical graph, Precoloring extension.

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DEDICATION

This dissertation is dedicated to my family and friends.

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CHAPTER 1

INTRODUCTION

1.1 Basic concepts and notation

In this dissertation, we generally follow the book [30] of Stiebitz et al. for notation and terminologies. Graphs in this dissertation are finite, undirected, and without loops, but may have multiple edges. In particular, simple graphs are always graphs with maximum multiplicity at most one and multigraphs are always graphs with maximum multiplicity at least two. Let $G = (V(G), E(G))$ be a graph, where $V(G)$ and $E(G)$ are the vertex set and the edge set of the graph G, respectively. Let $\Delta(G)$ and $\mu(G)$ be the maximum degree and the maximum multiplicity of graph G, respectively. For a vertex set $N \subseteq V(G)$, let $G - N$ be the graph obtained from G by deleting all the vertices in N and edges incident with them. For an edge set $F \subseteq E(G)$, let $G - F$ be the graph obtained from G by deleting all the edges in F but keeping their endvertices. If $F = \{e\}$, we simply write $G - e$. Similarly, we let $G + e$ be the graph obtained from G by adding the edge e to $E(G)$. For disjoint $X, Y \subseteq V(G)$, $E_G(X, Y)$ is the set of edges of G with one endvertex in X and the other in Y. If $X = \{x\}$ and $Y = \{y\}$, we simply write $E_G(x, y)$. For two disjoint subgraphs H_1 and H_2 of G, we simply write $E_G(H_1, H_2)$ for $E_G(V(H_1), V(H_2))$. For $X \subseteq V(G)$, the edge set $\partial_G(X) := E_G(X, V(G)\backslash X)$ is called the *boundary* of X in G. For a subgraph H of G, we simply write $\partial_G(H)$ for $\partial_G(V(H))$. For $u \in V(G)$, let $d_G(u)$ or $d(u)$ denote the *degree* of vertex u in G.

Let $[k] := \{1, \dots, k\}$ be a palette of k available colors (whose elements are called colors). A k*-edge-coloring* of G is a map φ that assigns to each edge e of G a color from the palette $[k]$ such that no two adjacent edges receive the same color (the coloring is also called proper coloring). Denote by $\mathcal{C}^k(G)$ the set of all k-edge-colorings of G. The *chromatic index* $\chi'(G)$ is the least integer k such that $\mathcal{C}^k(G) \neq \emptyset$.

An edge e of a graph G is called a *k-critical edge* if $k = \chi'(G-e) < \chi'(G) = k+1$. A graph G is called *$k$-critical* if $\chi'(H) < \chi'(G) = k+1$ for each proper subgraph H of G. It is easy to see that a connected graph G is k-critical if and only if every edge of G is k-critical.

Let G be a graph, $v \in V(G)$ and $\varphi \in \mathcal{C}^k(G)$ for some positive integer k. We define $\varphi(v) = \{\varphi(f) : f \in E(G) \text{ and } f \text{ is incident with } v\}$ and $\overline{\varphi}(v) = [k] \setminus \varphi(v)$. We call $\varphi(v)$ the set of colors *present* at v and $\overline{\varphi}(v)$ the set of colors *missing* at v. For a vertex set $X \subseteq V(G)$, define $\overline{\varphi}(X) = \bigcup_{v \in X} \overline{\varphi}(v)$. A vertex set $X \subseteq V(G)$ is called *φ-elementary* if $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$ for every two distinct vertices $u, v \in X$. The set X is called *φ-closed* if each color on edges from $\partial_G(X)$ is present at each vertex of X. Moreover, the set X is called *strongly φ-closed* if X is φ-closed and colors on edges from $\partial_G(X)$ are pairwise distinct. For a subgraph H of G, let φ_H or $(\varphi)_H$ be the edge coloring of G restricted on H. We say a subgraph H of G is φ-elementary, φ-closed and strongly φ-closed, if $V(H)$ is φ-elementary, φ-closed and strongly φ-closed, respectively. Clearly, if H is φ_H-elementary then H is φ-elementary, but the converse is not true as the edges in $\partial_G(H)$ are removed when we consider φ_H.

Let φ be a k-edge-coloring of G using the palette $[k]$. For a color α, let $E_\alpha := E_{\alpha,\varphi}(G)$ denote the set of edges assigned the color α, which is commonly referred to as a color class. Given two distinct colors α, β, an *(α, β)-chain* is a component of the subgraph induced by edges assigned color α or β, which is either an even cycle or a path. We call the operation that swaps the colors α and β on an (α, β)-chain C the *Kempe change* on C and denote it by φ/C. Clearly, the resulting coloring after a Kempe change is still a proper k-edge-coloring. Furthermore, we say that a chain C has *endvertices* u and v if C is a path connecting vertices u and v. For a vertex v of G, we denote by $P_v(\alpha, \beta, \varphi)$ or $P_v(\alpha, \beta)$ the unique (α, β)-chain containing the vertex v. For two vertices u, $v \in V(G)$, the two chains $P_u(\alpha, \beta, \varphi)$ and $P_v(\alpha, \beta, \varphi)$ are either identical or disjoint. More generally, for an (α, β)-chain, if it is a path and it contains two vertices a and b, we let $P_{[a,b]}(\alpha, \beta, \varphi)$ or $P_{[a,b]}(\alpha, \beta)$ be its subchain with endvertices a and b. The operation of swapping colors α and β on the subchain $P_{[a,b]}(\alpha, \beta, \varphi)$ is still called a Kempe change, but the resulting coloring may no longer be a proper edge coloring.

The (proper) *r-vertex-coloring* of a graph G is a mapping from $V(G)$ to $[r]$ such that distinct adjacent vertices receive different values. The minimum integer r, denoted by $\chi(G)$, such that G has a (proper) r-vertex-coloring is called the *chromatic number* of G. For a graph G, the *line graph* of G, denoted by $L(G)$, is the graph whose vertex set corresponds to the edge set of G and in which two vertices are adjacent if the corresponding edges of G have a common endvertex.

1.2 Research background and results

Graph theory originated from the famous Seven Bridge of Königsberg problem in 1700s. Graph edge coloring is a well established and very active subject in the field of graph theory. It has a rich theory, many applications and beautiful conjectures, and is studied not only by mathematicians, but also by computer scientists.

It is one of the basic combinatorial optimization problem: Color the edges of a graph G with as few colors as possible such that each edge receives a color and adjacent edges receive different colors. The chromatic index $\chi'(G)$ is just minimum number of colors needed for such a coloring of G. As proved by Holyer [24] in 1981 the determination of the chromatic index is an NP-hard optimization problem, even when restricted to a simple cubic graph. Note that maximum degree of G, is a natural lower bound of the chromatic index. Researchers are interested in upper bounds for the chromatic index that can be efficiently realized by a coloring algorithm.

Vizing [33, 34] and Gupta [22] independently in 1960s proved that $\Delta(G) \le \chi'(G) \le \Delta(G) + \mu(G)$ (that is commonly called Vizing's Theorem), by introducing the Vizing fan as main tool. By Vizing's Theorem, for a simple graph G, we have $\chi'(G) = \Delta(G)$ or $\Delta(G) + 1$. A simple graph G is of class one if $\chi'(G) = \Delta(G)$, otherwise it is of class two.

In the study of edge colorings of graphs, critical graphs are of particular interest. On the one hand, each graph G contains a critical graphs H with $\chi'(H) = \chi'(G)$ as a subgraph. On the other hand, critical graphs have more structural properties than arbitrary graphs. Hence, by focusing on critical graphs, we can keep the relevant information and often gain a better understanding. The study of critical graphs was initiated by Vizing [35] in 1960s, and he made a number of conjectures regarding structural properties of critical class two graphs [36].

All known techniques in studying edge chromatic problems especially for critical graphs are built on the elementary property of Vizing fans and its generalizations. Recently, Chen, Jing, and Zang [14] proved the outstanding Goldberg-Seymour Conjecture [19, 28], which gives a complete characterization of elementary set for any critical multigraph G with chromatic index at least $\Delta(G) + 2$. However, characterizing elementary sets for critical simple graphs is an interesting yet challenging problem in graph edge coloring.

In the Chapter 2 of this dissertation, we will focus on critical class two graphs, and introduce two new generalizations of Vizing fans, called e-fan (**Definition 2.1.2**) and C-e-fan (**Definition 2.2.2**), and obtain their elementary properties and new fan equations (**Theorem 2.1.3** and **Theorem 2.2.3**). Our results give a common generalization of several recently developed new results on multi-fan, double fan, C-fan, Kierstead path with four vertices and broom, which not only may imply or simplify proofs of previous results, but also could be new useful tools on attacking chromatic problems and conjectures for critical graphs.

Vizing's Theorem plays an important role in graph edge coloring. Berge and Fournier [5] strengthened the classical Vizing's Theorem by showing that if M^* is a maximal matching of G, then $\chi'(G - M^*) \leq \Delta(G) + \mu(G) - 1$. There is a problem about the precoloring of extension of Vizing's Theorem: Using the palette $[\Delta(G) + \mu(G)]$, when can we extend a precolored edge set $F \subseteq E(G)$ to a proper edge coloring of G? The distance between two edges e and f in G is the length of a shortest path connecting an endvertex of e and an endvertex of f. A *distance-t matching* is a set of edges having pairwise distance at least t. Albertson and Moore [2] conjectured that if G is a simple graph, any precolored distance-3 matching in G can be extended to a proper edge coloring of G using the palette $[\Delta(G) + 1]$. Edwards et al. [17] proposed a much more general and stronger conjecture: For any graph G, using the palette $[\Delta(G) + \mu(G)]$, any precolored distance-2 matching can be extended to a proper edge coloring of G. Girão and Kang [18] verified the conjecture of Edwards et al. for distance-9 matchings.

In the Chapter 3 of this dissertation, we will consider multigraph G with $\mu(G) \geq 2$, and improve the required distance from 9 to 3 (**Theorem 3.1.1**), using the result of Goldberg-Seymour Conjecture and properties of dense subgraphs. Our result not only greatly improves the distance condition, but also offers more hope to solve this conjecture completely.

Actually, there was a famous vertex-precoloring extension problem. In 1997, Thomassen [32] posed the following problem: Suppose that G is a planar graph and $W \subseteq V(G)$ such that the distance between any two vertices in W is at least 100. Can a (proper) 5-vertex-coloring of W be extended to a (proper) 5-vertex-coloring of G? In 1998, Albertson [1] provided a best possible solution to Thomassen's problem, and he obtained as well as one general result for graphs: If G is any graph with $\chi(G) = r$ and $W \subseteq V(G)$ such that the distance between any two vertices in W is at least 4, then any $(r+1)$-vertex-coloring of W can be extended to a $(r+1)$-vertex-coloring of G. The obvious relationship betwween edge-precoloring and vertex counterpart is the line graph. Every edge coloring of G is a vertex coloring of $L(G)$ and $\chi'(G) = \chi(L(G))$. Notice that if $\chi'(G) \leq \Delta(G) + \mu(G) - 1$, then we can apply the result or proof idear of Albertson's theorem on the edge-precoloring extension problem. Hence we mainly consider the edge-precoloring extension problem under the condition $\chi'(G) = \Delta(G) + \mu(G)$.

In the Chapter 4 of this dissertation, we will list some future work, such as extensions of Vizing fans with diameter at least four and further confirming the conjecture of Edwards et al., as well as some problems about total colorings for multigraphs.

CHAPTER 2

GENERALIZATIONS OF VIZING FANS

In this chapter, we mainly introduce two new generalizations of Vizing fans. In Section 2.1, we define one new fan structure called e-fan based on the definition of Vizing fan and multi-fan, and obtain its elementary property and new fan equation, i.e., Theorem 2.1.3. In Section 2.2, we define another new fan structure called C-e-fan based on the definition of C-fan, and obtain its elementary property and new fan equation, i.e., Theorem 2.2.3. In Section 2.3 and Section 2.4, we respectively present the proofs of Theorem 2.1.3 and Theorem 2.2.3.

The main approachs of the two proofs are similar and roughly as follows. We first prove the conclusions on the vertex set of some special subsequence (called linear sequence), then generalize to any two such special subsequences and finally to the entire fan. Actually, a Vizing fan is a special such subsequence, so one can also use our method to prove properties of multi-fans.

2.1 Introduction

Our results in this chapter are on simple graphs, but we will mention some definitions and results on multigraphs. Recently, Chen et al. [14] proved the outstanding Goldberg-Seymour Conjecture [19, 28] that if G is a k-critical multigraph with $k \geq \Delta(G) + 1$, then for every edge e and every coloring $\varphi \in \mathcal{C}^k(G-e)$, $V(G)$ is φ-elementary. Consequently, $V(G)$ is elementary for every k-edge-coloring of $G - e$. This result gives a complete characterization for critical multigraphs G with chromatic index at least $\Delta(G) + 2$. However, characterizing elementary sets for k-critical graphs with $k = \Delta(G)$, in particular, Δ-critical simple graphs (i.e., critical class two graphs with maximum degree Δ), is an interesting yet challenging problem in graph edge coloring.

Let G be a Δ-critical simple graph, $e \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G-e)$. We in general do not know much about the largest φ-elementary sets except the following three outstanding conjectures. Hilton's overfull conjecture [15,16]: $V(G)$ is φ-elementary if $\Delta(G) > |V(G)|/3$; Seymour's exact conjecture [29]: $V(G)$ is φ-elementary if G is a planar graph; and Hilton and Zhao's core conjecture [23]: $V(G)$ is φ-elementary if the core G_{Δ} has maximum degree at most 2, where G_{Δ}, named the *core* of G, is the subgraph of G induced by all maximum degree vertices. Cao et al. [11] recently confirmed Hilton and Zhao's core conjecture. The other two of these three conjectures are remaining wild open. Vizing [33,34] showed that the vertex set of every Vizing fan is elementary. Almost all known techniques in studying edge chromatic problems are built on the elementary properties of Vizing fans and its generalizations. In [30], Stiebitz et al. gave a survey, up to that time, of the work in this direction. We will give a common generalization of these results.

Definition 2.1.1 (Tashkinov Tree)**.** Let G be a k-critical graph, $e \in E_G(x,y)$ and $\varphi \in \mathcal{C}^k(G-e)$ for some integer $k \geq 0$. A sequence $T = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$ of alternating distinct vertices and distinct edges is called a *Tashkinov tree* if for each $i \in [p]$, e_i is incident with z_i and satisfies the following two conditions.

T1. The other endvertex of e_i is in $\{x, y, z_1, \ldots, z_{i-1}\}$ for $i \in [p]$.

T2. $\varphi(e_1) \in \overline{\varphi}(x) \cup \overline{\varphi}(y)$ and $\varphi(e_i) \in \overline{\varphi}(x) \cup \overline{\varphi}(y) \cup \overline{\varphi}(z_h)$ for some $h \in [i-1]$ if $2 \leq i \leq p$.

Tashkinov trees are given by Tashkinov in [31], where he proved that if G is a k-critical multigraph with $k \geq \Delta(G)+1$, $e \in E(G)$ and $\varphi \in \mathcal{C}^k(G-e)$, then the vertex set of every Tashkinov tree is φ-elementary. Clearly, each Tashkinov tree is indeed a tree. We in the following notice that Vizing fans and some other well-studied subgraphs are special classes of Tashkinov trees.

(1) If for every i we restrict in **T1**, each e_i is incident with x and in **T2** $h = i-1$, then T is a *Vizing fan.*

(2) If we only impose the above restriction to **T1**, then T is a *multi-fan* introduced by Stiebitz et al. [30].

(3) If we restrict in **T1**, e_1 is incident with y and e_i is incident with z_{i-1} for each $i \geq 2$, then T is a *Kierstead path* [25].

(4) If we restrict in **T1**, $p \geq 2$ and each e_i with $i \geq 2$ is incident with z_1, then T is a *broom* defined in [12, 13].

We notice that not every vertex set of Tashkinov tree is elementary. Let P^* be obtained from the Petersen graph by deleting a vertex. It is not difficult to verify that P^* is a 3-critical graph, but there exist an edge e and a coloring $\varphi \in \mathcal{C}^3(P^* - e)$, such that the vertex set of a Kierstead path with 4 vertices is not elementary. By imposing degree condition $\min\{d(y), d(z_1)\} \leq \Delta(G) - 1$, Stiebitz and Kostachka [26] and Luo and Zhao [27] showed that the vertex set of each Kierstead path $(x, e, y, e_1, z_1, e_2, z_2)$ is elementary. The result has been extended to brooms [12, 13]. We generalize these results to a much broader class of Tashkinov trees in this chapter.

Definition 2.1.2 (e-fan)**.** Let G be a Δ-critical simple graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G - e)$. A Tashkinov tree $F^e = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$ is a *simple e-fan* if in **T1** we additionally require each e_i is only incident with x or y, i.e., $e_i = xz_i$ or $e_i = yz_i$ for each $1 \leq i \leq p$. Furthermore, in the above definition of simple e-fan if we relax the condition that each z_i is distinct by allowing it with possibility to be repeated one more time, say $z_i = z_j = z$ with $i \neq j$, i.e., edges xz and yz can appear in F^e, then F^e is called an *e-fan.*

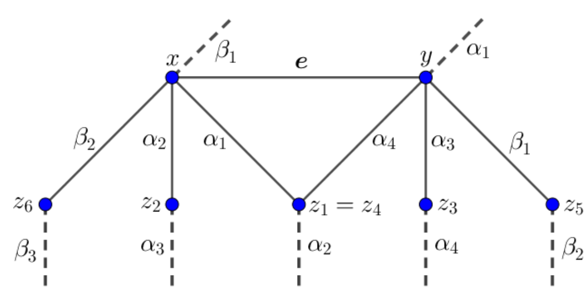

Figure 2.1. An e-fan $F^e = (x, e, y, e_1, z_1, \ldots, e_6, z_6)$.

(See Figure 2.1 for a depiction that shows an e-fan $F^e = (x, e, y, e_1, z_1, \ldots, e_6, z_6)$, where a dashed line at a vertex represents a color missing at the vertex.) Clearly, a multi-fan is an e-fan in simple graphs. Moreover, if F_x and F_y are two multi-fans centered at x and y, respectively, then $F_x \cup F_y$, named a *double fan*, is also an e-fan. The below Theorem 2.1.3 shows that the vertex set of every e-fan provided $\min\{d(x), d(y)\} \le \Delta(G) - 1$ is elementary, which is one of the two main results of this chapter. We will give its proof in Section 2.4, in which it is worth mentioning that we first prove the vertex set of some special subsequence (will be called linear e-sequence) is elementary, then generalize to any two special subsequences and finally to the entire e-fan. Actually, a Vizing fan is such a special subsequence centered at one vertex in a multi-fan, so one can also use our above method to prove the vertex set of every multi-fan is elementary.

Theorem 2.1.3. *Let G be a Δ-critical simple graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G - e)$. If $\min\{d(x), d(y)\} \le \Delta(G) - 1$, then $V(F^e)$ is φ-elementary for every e-fan F^e. Furthermore, if F^e is maximal, i.e., there is no e-fan containing F^e as a proper subsequence, then*

$$d(x) + d(y) - 2\Delta + \sum_{z \in V(F^e)\setminus\{x,y\}} (2d(z) + e_{F^e}(x, z) + e_{F^e}(y, z) - 2\Delta) = 2,$$

where $e_{F^e}(x, z)$ and $e_{F^e}(y, z)$ taking value 0 or 1 are the number of edges between x and z and between y and z in F^e, respectively.

We notice that Theorem 2.1.3 immediately gives that all vertex sets of Vizing fans, multi-fans, and double fans provided $\min\{d(x), d(y)\} \le \Delta(G) - 1$ are respectively elementary. We also notice a few applications below.

Corollary 2.1.4. [26, 27] *Let G be a Δ-critical simple graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G-e)$. For any Kierstead path $K = (x, e, y, e_1, z_1, e_2, z_2)$, if $\min\{d(y), d(z_1)\} \le \Delta(G)-1$, then $V(K)$ is φ-elementary.*

Proof. Let φ' be obtained from $\varphi \in \mathcal{C}^\Delta(G - e)$ by uncoloring e_1 and coloring e with color $\varphi(e_1)$. Since $\varphi(e_1) \in \overline{\varphi}(x)$, φ' is an edge $\Delta(G)$-coloring of $G - e_1$. Moreover, since $\varphi'(e) \in$

$\overline{\varphi}(z_1)$ and $\varphi'(e_2) \in \overline{\varphi}'(x) \cup \overline{\varphi}'(y)$, $F^e = (y, e_1, z_1, e, x, e_2, z_2)$ is an e-fan with respect to e_1 and φ'. By Theorem 2.1.3, $V(F^e) = V(K)$ is φ'-elementary, and so φ-elementary. □

Using the same trick in the above proof, we get the following more general result.

Corollary 2.1.5. [12] *Let G be a Δ-critical simple graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G-e)$. For any broom $B = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$, if $\min\{d(y), d(z_1)\} \leq \Delta(G) - 1$, then $V(B)$ is φ-elementary.*

2.2 Extensions of missing color sets

In this section we will consider some extensions of the missing color set at a vertex and some more generally elementary properties and structures. Starting with Vizing's classic results [33, 34], missing colors have played a crucial role in revealing properties of critical graphs. Let G be a Δ-critical graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G - e)$. Woodall [37, 38] treated colors $\varphi(yz)$ of edge yz as a missing color in $\overline{\varphi}(y)$ if $d(z)$ is "small". This technique was used in [6, 7, 8, 9] in their work on Vizing's average degree conjecture and hamiltonian property of Δ-critical simple graphs. For a vertex $v \in V(G)$, let

$$\begin{aligned} \varphi_x^s(v) &= \{\varphi(vw) : w \neq x \text{ and } d(w) \leq \frac{1}{2}(\Delta(G) - d(x))\}, \text{ and} \\ C_{\varphi,x}(v) &= \overline{\varphi}(v) \cup \varphi_x^s(v). \end{aligned}$$

Similarly, we define $\varphi_y^s(v)$ and $C_{\varphi,y}(v)$. Since $d(x) + d(w) \geq \Delta(G) + 2$ for every neighbor w of x [30], we have $\varphi_x^s(x) = \emptyset$, i.e., $C_{\varphi,x}(x) = \overline{\varphi}(x)$. Similarly, $\varphi_y^s(y) = \emptyset$, i.e., $C_{\varphi,y}(y) = \overline{\varphi}(y)$. Incorporating this idea, Kostochka and Stiebitz [26] extended multi-fan as follows. A sequence $F^c = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$ of alternating distinct vertices and distinct edges is called a *C-fan* if for each e_i with $i \in [p]$, $e_i \in E_G(x, z_i)$ and there exists a h with $0 \leq h \leq i-1$ such that $\varphi(e_i) \in C_{\varphi,x}(z_h)$, where $z_0 = y$. The vertex set $V(F^c)$ is called *φ^c-elementary* if $C_{\varphi,x}(z_i) \cap C_{\varphi,x}(z_j) = \emptyset$ for every two distinct vertices z_i, z_j in $V(F^c)$, where $0 \leq i < j \leq p$ and $z_0 \in \{x, y\}$.

Theorem 2.2.1. [26,30] *Let G be a Δ-critical graph, $e \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G-e)$. Then $V(F^c)$ is φ^c-elementary for every C-fan F^c.*

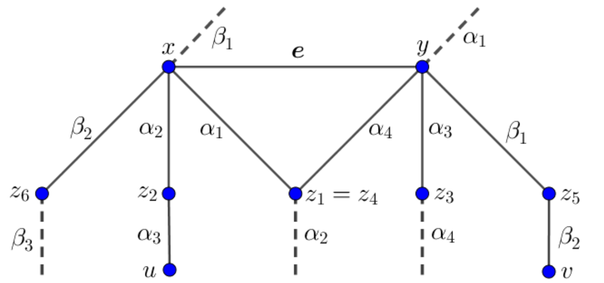

Figure 2.2. A C-e-fan $F^{ce} = (x, e, y, e_1, z_1, \ldots, e_6, z_6)$.

Definition 2.2.2 (C-e-fan)**.** Let G be a Δ-critical simple graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^\Delta(G-e)$. A *$C$-$e$-fan* at x and y is a sequence $F^{ce} = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$ of alternating vertices and edges satisfying the following two conditions.

C1. The edges $e, e_1, \ldots, e_p$ are distinct with $e_i = xz_i$ or $e_i = yz_i$ for $i \in [p]$.

C2. $\varphi(e_1) \in C_{\varphi,y}(x) \cup C_{\varphi,x}(y)$ and $\varphi(e_i) \in C_{\varphi,y}(x) \cup C_{\varphi,x}(y) \cup C_{\varphi,w(e_h)}(z_h)$ for some $h \in [i-1]$ if $2 \le i \le p$, where $w(e_h)$ is the endvertex of e_h in $\{x, y\}$.

(See Figure 2.2 for a depiction that shows a C-e-fan $F^{ce} = (x, e, y, e_1, z_1, \ldots, e_6, z_6)$ with $d(u) \le \frac{1}{2}(\Delta(G) - d(x))$ and $d(v) \le \frac{1}{2}(\Delta(G) - d(y))$ in a graph G, where a dashed line at a vertex represents a color missing at the vertex.) Since each edge e_i with $i \in [p]$ is incident with x or y, let $w(e_i)$ denote this vertex. Note that some vertices of $z_1, \ldots, z_p$ may appear twice, say $z_i = z_j = z$ with $i \neq j$, i.e., edges xz and yz appear in F^{ce}. In C-e-fan F^{ce}, we define $C_\varphi(x) = C_{\varphi,y}(x)$, $C_\varphi(y) = C_{\varphi,x}(y)$, $C_\varphi(z_i) = C_{\varphi,w(e_i)}(z_i)$ for single z_i, and $C_\varphi(z) = C_{\varphi,w(e_i)}(z_i) \cup C_{\varphi,w(e_j)}(z_j)$ for repeated z_i and z_j with $z_i = z_j = z$. The set $V(F^{ce})$ is called *φ^{ce}-elementary* if $C_\varphi(u) \cap C_\varphi(v) = \emptyset$ for every two distinct vertices u, v in $V(F^{ce})$. The below Theorem 2.2.3 is the other of the two main results of this chapter, whose proof will be given in Section 2.5 and has the similar main idea of Theorem 2.1.3 but much more complicated.

Theorem 2.2.3. *Let G be a Δ-critical simple graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G-e)$. For a C-e-fan $F^{ce} = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$, if $\max\{d(x), d(y)\} \leq \Delta(G) - 1$ and the following condition holds, then $V(F^{ce})$ is φ^{ce}-elementary.*

C3. *For any two distinct colors α, β with $\alpha \in \varphi^s_{w(e_i)}(z_i)$ and $\beta \in \varphi^s_{w(e_j)}(z_j)$ for $1 \leq i < j \leq p$, denote by u, v the two vertices, and $e' = z_i u$ and $e'' = z_j v$ the two edges such that $\varphi(e') = \alpha$ and $\varphi(e'') = \beta$, then we have $u \neq v$.*

Furthermore, if F^{ce} is maximal, i.e., there is no C-e-fan containing F^{ce} as a proper subsequence, then the following equation holds.

$$|C_\varphi(x)| + |C_\varphi(y)| = \sum_{z \in V(F^{ce}) \setminus \{x,y\}} (e_{F^{ce}}(x, z) + e_{F^{ce}}(y, z) - 2|C_\varphi(z)|),$$

where $e_{F^{ce}}(x, z)$ and $e_{F^{ce}}(y, z)$ taking value 0 or 1 are the number of edges between x and z and between y and z in F^{ce}, respectively.

2.3 Preliminary Lemmas

Lemma 2.3.1. [30] *Let G be a Δ-critical simple graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G-e)$. And let $F = (x, e, y_0, e_1, y_1, \ldots, e_p, y_p)$ be a multi-fan at x, where $y_0 = y$. Then the following statements hold.*

(a) $V(F)$ is φ-elementary.

(b) If $\alpha \in \overline{\varphi}(x)$ and $\beta \in \overline{\varphi}(y_i)$ for $0 \leq i \leq p$, then $P_x(\alpha, \beta, \varphi) = P_{y_i}(\alpha, \beta, \varphi)$.

The following lemma is a simple corollary of Lemma 2.3.1.

Lemma 2.3.2. [30] *Let G be a Δ-critical simple graph. Then for any edge $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G - e)$, we have $d(x) + d(y) \geq \Delta + 2$.*

Lemma 2.3.3. [26, 30] *Let G be a Δ-critical simple graph, $e = xy \in E(G)$ and $\varphi \in \mathcal{C}^{\Delta}(G - e)$. And let $F^c = (x, e, y_0, e_1, y_1, \ldots, e_p, y_p)$ be a C-fan at x, where $y_0 = y$. Then the following statements hold.*

(a) $V(F^c)$ is φ^c-elementary, i.e., $C_{\varphi,x}(x) \cap C_{\varphi,x}(y_i) = \emptyset$ for $i = 0, 1, \ldots, p$, and $C_{\varphi,x}(y_i) \cap C_{\varphi,x}(y_j) = \emptyset$ for $0 \le i < j \le p$.

(b) If $\alpha \in C_{\varphi,x}(x)$ and $\beta \in C_{\varphi,x}(y_i)$ for $0 \le i \le p$, then $P_x(\alpha, \beta, \varphi) = P_{y_i}(\alpha, \beta, \varphi)$.

In a Δ-critical simple graph G with $e = xy \in E(G)$, a vertex u is called a *small vertex* with respect to x (with respect to y, respectively) if $d(u) \le \frac{\Delta - d(x)}{2}$ ($d(u) \le \frac{\Delta - d(y)}{2}$, respectively). We list the following simple facts [26, 30].

Lemma 2.3.4. *In a Δ-critical simple graph G with $e = xy \in E(G)$, for any small vertices u, v with respect to x (with respect to y, respectively), we have $|\overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)| \ge 1$ ($|\overline{\varphi}(y) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)| \ge 1$, respectively). In particular, provided $d(x) \le d(y)$, no matter u and v are small vertices with respect to x or y, then we have $|\overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)| \ge 1$. Furthermore, if $d(x) \le \Delta(G) - 1$ and u is a small vertex with respect to x ($d(y) \le \Delta(G) - 1$ and u is a small vertex with respect to y, respectively), then we have $|\overline{\varphi}(x) \cap \overline{\varphi}(u)| \ge 2$ ($|\overline{\varphi}(y) \cap \overline{\varphi}(u)| \ge 2$, respectively).*

2.4 Proof of Theorem 2.1.3

In a simple e-fan $F^e = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$, a *linear e-sequence* is a subsequence $(x, e, y, e_{l_1}, z_{l_1}, \ldots, e_{l_s}, z_{l_s})$ with $1 \le l_1 < l_2 < \cdots < l_s \le p$ such that $\varphi(e_{l_1}) \in \overline{\varphi}(x) \cup \overline{\varphi}(y)$ and $\varphi(e_{l_i}) \in \overline{\varphi}(z_{l_{i-1}})$ for $2 \le i \le s$. Specifically, a linear e-sequence is a *x-generated e-sequence* if $\varphi(e_{l_1}) \in \overline{\varphi}(x)$, or a *$y$-generated e-sequence* if $\varphi(e_{l_1}) \in \overline{\varphi}(y)$.

Proof. In the e-fan $F^e = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$, if $z_i = z_j$ with $1 \le i < j \le p$, we delete the edge e_j and the vertex z_j from F^e. By the definition of e-fan, one can easily check that the remaining sequence is still an e-fan. Repeat the above operation. Finally, we get a simple e-fan F'^e with respect to the e-fan F^e. Obviously, $V(F^e) = V(F'^e)$. Hence, we may assume that the original e-fan F^e is a simple e-fan. We show the following two claims.

Claim 1: The vertex set of any linear e-sequence is elementary.

Proof. Suppose that Claim 1 is false. Without loss of generality, we choose φ such that there exists a y-generated e-sequence $S_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1}, \ldots, e_{l_s}, z_{l_s})$ with $e_{l_0} = e$, whose vertex set is not elementary with s as small as possible. Note that $e_{l_1} = xz_{l_1}$. Let $\varphi(e_{l_1}) = \beta_{l_1} \in \overline{\varphi}(y)$ and $\varphi(e_{l_i}) = \beta_{l_i} \in \overline{\varphi}(z_{l_{i-1}})$ for $2 \le i \le s$.

If $s \le 1$, then S_y is a Vizing fan at x, which has elementary vertex set by Lemma 2.3.1. We assume $s \ge 2$. By the minimality of s, $V(S_y)\backslash\{z_{l_s}\}$ is elementary. Together with the definition of y-generated e-sequence, we have that for any color $\gamma_1 \in \overline{\varphi}(x)$, no edge in $E(S_y)$ is colored with γ_1; for any color γ_2, if $\gamma_2 \in \overline{\varphi}(y)$ or $\gamma_2 \in \overline{\varphi}(z_{l_i})$ for $1 \le i \le s-1$, then only the edge e_{l_1} or $e_{l_{i+1}}$ in $E(S_y)$ may be colored with γ_2. We will use above facts about S_y without explicit mention. The following observation will also be used very often.

I. For any two colors $\gamma_1 \in \overline{\varphi}(x)$ and $\gamma_2 \in \overline{\varphi}(z_{l_i})$ with $1 \le i \le s-1$, we have $\gamma_1 \ne \gamma_2$ and $P_x(\gamma_1, \gamma_2, \varphi) = P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$.

Proof. Recall that $V(S_y)\backslash\{z_{l_s}\}$ is elementary. We easily have $\gamma_1 \ne \gamma_2$. Suppose $P_x(\gamma_1, \gamma_2, \varphi) \ne P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$. For the path $P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$, one endvertex is z_{l_i} and the other endvertex is some vertex $z' \ne x$. Note that $z' \notin \{y, z_{l_1}, \ldots, z_{l_{i-1}}\}$ and none of $e_{l_1}, \ldots, e_{l_i}$ is colored with γ_1 or γ_2. Hence, the coloring $\varphi' = \varphi / P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ satisfies $\varphi'(e_{l_j}) = \varphi(e_{l_j})$ for each $j \in [i]$, $\overline{\varphi}'(x) = \overline{\varphi}(x)$, $\overline{\varphi}'(y) = \overline{\varphi}(y)$, $\overline{\varphi}'(z_{l_j}) = \overline{\varphi}(z_{l_j})$ for each $j \in [i-1]$ and $\overline{\varphi}'(z_{l_i}) = (\overline{\varphi}(z_{l_i})\backslash\{\gamma_2\}) \cup \{\gamma_1\}$. Consequently, the coloring φ' results in a new y-generated e-sequence $S'_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1}, \ldots, e_{l_i}, z_{l_i})$ with $\gamma_1 \in \overline{\varphi}'(z_{l_i}) \cap \overline{\varphi}'(x)$, contradicting the minimality of s. This completes the proof of the observation I. □

Subclaim 1.1: We may assume that $\overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(x) \ne \emptyset$.

Proof. Since $V(S_y)$ is not elementary, and by the minimality of s, there exists a color $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(\{x, y, z_{l_1}, \ldots, z_{l_{s-1}}\})$. If $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(x)$, then we are done. Otherwise, we have $\overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(x) = \emptyset$ and $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(\{y, z_{l_1}, \ldots, z_{l_{s-1}}\})$, i.e., $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(y)$ or $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(\{z_{l_1}, \ldots, z_{l_{s-1}}\})$. By the definition of S_y, we have $\eta \ne \beta_{l_s} \in \overline{\varphi}(z_{l_{s-1}})$. Let $\alpha \in \overline{\varphi}(x)$. Since $\overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(x) = \emptyset$, we have $\alpha \ne \eta$ and $\alpha \in \varphi(z_{l_s})$. Note that if $\eta \in \overline{\varphi}(y)$, then $P_x(\alpha,\eta,\varphi) = P_y(\alpha,\eta,\varphi)$ by Lemma 2.3.1 since Vizing fan (x, e_{l_0}, y). Also if $\eta \in \overline{\varphi}(z_{l_i})$, then $P_x(\alpha,\eta,\varphi) = P_{z_{l_i}}(\alpha,\eta,\varphi)$ for $1 \le i \le s-1$ by the observation I. Therefore, $P_x(\alpha,\eta,\varphi)$ and $P_{z_{l_s}}(\alpha,\eta,\varphi)$ are disjoint. For the path $P = P_{z_{l_s}}(\alpha,\eta,\varphi)$, one endvertex is z_{l_s} and the other endvertex $z' \notin V(S_y)$, and we have $E_{\varphi,\alpha}(P) \cap E(S_y) = \emptyset$. Note that if $\eta = \beta_{l_1} \in \overline{\varphi}(y)$, then e_{l_1} is on $P_x(\alpha,\eta,\varphi)$. To further discuss $E_{\varphi,\eta}(P) \cap E(S_y)$, we consider the following two cases.

If $\eta = \beta_{l_{i+1}}$ and $e_{l_{i+1}}$ is on P for $\eta \in \overline{\varphi}(z_{l_i})$ and $1 \le i \le s-2$, then we have $E_{\varphi,\eta}(P) \cap E(S_y) = \{e_{l_{i+1}}\}$. Hence, the coloring $\varphi_1 = \varphi/P$ satisfies $\varphi_1(e_{l_j}) = \varphi(e_{l_j})$ for $j \neq i$, $\varphi_1(e_{l_{i+1}}) = \alpha$, $\overline{\varphi}_1(x) = \overline{\varphi}(x)$, $\overline{\varphi}_1(y) = \overline{\varphi}(y)$, $\overline{\varphi}_1(z_{l_j}) = \overline{\varphi}(z_{l_j})$ for each $j \in [s-1]$ and $\overline{\varphi}_1(z_{l_s}) = (\overline{\varphi}(z_{l_s}) \backslash \{\eta\}) \cup \{\alpha\}$. Consequently, the coloring φ_1 results in a smaller x-generated e-sequence $(x, e_{l_0}, y, e_{l_{i+1}}, z_{l_{i+1}}, \ldots, e_{l_s}, z_{l_s})$ with $\alpha \in \overline{\varphi}_1(z_{l_s}) \cap \overline{\varphi}_1(x)$, contradicting the minimality of s.

If $\eta \in \overline{\varphi}(y)$, or $\eta \neq \beta_{l_{i+1}}$ for $\eta \in \overline{\varphi}(z_{l_i})$ and $1 \le i \le s-1$, or $\eta = \beta_{l_{i+1}}$ and $e_{l_{i+1}}$ is not on P for $\eta \in \overline{\varphi}(z_{l_i})$ and $1 \le i \le s-2$, then we have $E_{\varphi,\eta}(P) \cap E(S_y) = \emptyset$. Hence, the coloring $\varphi_1 = \varphi/P$ satisfies $\varphi_1(e_{l_j}) = \varphi(e_{l_j})$ for each $j \in [s]$, $\overline{\varphi}_1(x) = \overline{\varphi}(x)$, $\overline{\varphi}_1(y) = \overline{\varphi}(y)$, $\overline{\varphi}_1(z_{l_j}) = \overline{\varphi}(z_{l_j})$ for each $j \in [s-1]$ and $\overline{\varphi}_1(z_{l_s}) = (\overline{\varphi}(z_{l_s}) \backslash \{\eta\}) \cup \{\alpha\}$. Consequently, S_y is still a y-generated e-sequence with $\alpha \in \overline{\varphi}_1(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired. This completes the proof of Subclaim 1.1. □

By the subclaim above, we assume that there exists a color $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(x)$. To reach contradictions, we consider the following two cases.

Case 1: $e_{l_s} = xz_{l_s}$.

Note that $\varphi(e_{l_s}) = \beta_{l_s} \in \overline{\varphi}(z_{l_{s-1}})$ and none of $e_{l_1}, \ldots, e_{l_{s-1}}$ is colored with β_{l_s} or η. Recolor e_{l_s} with η to obtain a new coloring φ_1. Thus $S'_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1}, \ldots, e_{l_{s-1}}, z_{l_{s-1}})$ is a new y-generated e-sequence under φ_1 such that $\beta_{l_s} \in \overline{\varphi}_1(z_{l_{s-1}}) \cap \overline{\varphi}_1(x)$, contradicting the minimality of s.

Case 2: $e_{l_s} = yz_{l_s}$.

By the observation I, we have $P_x(\eta,\beta_{l_s},\varphi) = P_{z_{l_{s-1}}}(\eta,\beta_{l_s},\varphi)$. For the path $P = P_{z_{l_s}}(\eta,\beta_{l_s},\varphi)$, one endvertex is z_{l_s} and the other endvertex $z' \notin V(S_y)$, and we have

$E(P) \cap E(S_y) = \{e_{l_s}\}$. Let $\varphi_1 = \varphi/P$. Hence $(x, e_{l_0}, y, e_{l_1}, z_{l_1}, \ldots, e_{l_{s-1}}, z_{l_{s-1}})$ is still a y-generated e-sequence under φ_1 whose vertex set is still elementary, and $(y, e_{l_0}, x, e_{l_s}, z_{l_s})$ is a Vizing fan at y under φ_1 since $\varphi_1(e_{l_s}) = \eta \in \overline{\varphi}_1(x)$. Since $min\{d(x), d(y)\} \leq \Delta - 1$, there exists a missing color $\delta \in \overline{\varphi}_1(x) \cup \overline{\varphi}_1(y)$ such that $\delta \neq \eta, \beta_{l_1}$. Suppose $\delta \in \overline{\varphi}_1(x)$. We have $P_x(\delta, \beta_{l_s}, \varphi_1) = P_{z_{l_s}}(\delta, \beta_{l_s}, \varphi_1)$ by Lemma 2.3.1, since otherwise, the coloring $\varphi' = \varphi_1/P_{z_{l_s}}(\delta, \beta_{l_s}, \varphi_1)$ results in $\delta \in \overline{\varphi}'(z_{l_s}) \cap \overline{\varphi}'(x)$, which is a contradiction. But we have $P_x(\delta, \beta_{l_s}, \varphi_1) = P_{z_{l_{s-1}}}(\delta, \beta_{l_s}, \varphi_1)$ by the observation I, giving a contradiction. Similarly, if $\delta \in \overline{\varphi}_1(y)$, then $P_y(\delta, \beta_{l_s}, \varphi_1) = P_{z_{l_s}}(\delta, \beta_{l_s}, \varphi_1)$ by Lemma 2.3.1. But $P_y(\delta, \beta_{l_s}, \varphi_1) = P_{z_{l_{s-1}}}(\delta, \beta_{l_s}, \varphi_1)$, also giving a contradiction. This completes the proof of Claim 1. □

Claim 2: The union of vertex sets of any two linear e-sequences is elementary.

Proof. Suppose that Claim 2 is false. Without loss of generality, we choose φ such that there exist two linear e-sequences $S_1 = (x, e, y, e_{l_1}, z_{l_1}, \ldots, e_{l_s}, z_{l_s})$ and $S_2 = (x, e, y, e_{l'_1}, z_{l'_1}, \ldots, e_{l'_s}, z_{l'_t})$ whose vertex sets have common missing color with $s + t$ as small as possible, where $s, t \geq 1$. Note that $V(S_1)$ and $V(S_2)$ are elementary by Claim 1. By the minimality of $s + t$, we have $z_{l_s} \neq z_{l'_t}$ and there exists a color $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(z_{l'_t})$. Since $\min\{d(x), d(y)\} \leq \Delta - 1$, there exists a missing color $\delta \in \overline{\varphi}(x) \cup \overline{\varphi}(y)$ such that δ is different from the colors $\varphi(e_{l_1})$ and $\varphi(e_{l'_1})$ which are also in $\overline{\varphi}(x) \cup \overline{\varphi}(y)$. ($\varphi(e_{l_1})$ and $\varphi(e_{l'_1})$ could be the same color.) Assume $\delta \in \overline{\varphi}(z_0)$, where $z_0 \in \{x, y\}$. Then $P_{z_0}(\delta, \eta, \varphi) = P_{z_{l_s}}(\delta, \eta, \varphi)$, since otherwise, for the coloring $\varphi' = \varphi/P_{z_{l_s}}(\delta, \eta, \varphi)$, we have S_1 is still a linear e-sequence under φ', but $\delta \in \overline{\varphi}'(z_0) \cap \overline{\varphi}'(z_{l_s})$, giving a contradiction to Claim 1. Similarly, we have $P_{z_0}(\delta, \eta, \varphi) = P_{z_{l'_t}}(\delta, \eta, \varphi)$. Hence z_0, z_{l_s} and $z_{l'_t}$ are endvertices of one (δ, η)-chain, which is a contradiction. This completes the proof of Claim 2. □

Now we are ready to show that $V(F^e)$ is elementary. Suppose not. Note that $\{x, y\}$ is elementary and each linear e-sequence in F^e contains vertices x and y. There exist one color η and two distinct vertices z_i and z_j in $V(F^e)$ such that $\eta \in \overline{\varphi}(z_i) \cap \overline{\varphi}(z_j)$, where $0 \leq i < j \leq p$ and $z_0 \in \{x, y\}$. By the definition of simple e-fan, there exist two linear e-sequences (may not be disjoint) with z_i and z_j respectively as the last vertex, which is a contradiction to Claim 1 for $i = 0$ or a contradiction to Claim 2 for $1 \leq i \leq p - 1$. This proves that $V(F^e)$ is elementary.

Now we show the "furthermore" part. We assume that F^e is maximal. Let the edge set $\Gamma = \{e_1, \dots, e_p\}$ and the color set $\Gamma' = \bigcup_{z \in V(F^e)} \overline{\varphi}(z)$. Note that $\overline{\varphi}(x)$, $\overline{\varphi}(y)$ and $\overline{\varphi}(z_i)$ for each $i \in [p]$ are disjoint since $V(F^e)$ is elementary. Let $\Gamma^* = \{\varphi(e_1), \dots, \varphi(e_p)\}$ be a multiset. We have

$$p = |\Gamma| = \sum_{z \in V(F^e) \setminus \{x,y\}} (e_{F^e}(x,z) + e_{F^e}(y,z)) = |\Gamma^*|. \tag{1}$$

Now we calculate $|\Gamma^*|$ in another way. By the definition of e-fan, $\varphi(e_i) \in \Gamma'$ for each $i \in [p]$. By the maximality of F^e, for any color $\alpha \in \Gamma'$, α appears exactly once in Γ^* if $\alpha \in \overline{\varphi}(x) \cup \overline{\varphi}(y)$. Otherwise, α appears exactly twice in Γ^*. Thus we have

$$|\Gamma^*| = |\overline{\varphi}(x)| + |\overline{\varphi}(y)| + \sum_{z \in V(F^e) \setminus \{x,y\}} 2|\overline{\varphi}(z)|. \tag{2}$$

Combining equations (1) and (2), we prove that

$$d(x) + d(y) - 2\Delta + \sum_{z \in V(F^e) \setminus \{x,y\}} (2d(z) + e_{F^e}(x,z) + e_{F^e}(y,z) - 2\Delta) = 2$$

since $\overline{\varphi}(x) = \Delta - d(x) + 1$, $\overline{\varphi}(y) = \Delta - d(y) + 1$ and $\overline{\varphi}(z) = \Delta - d(z)$. The proof is now finished. □

2.5 Proof of Theorem 2.2.3

Note that when $d(x) \neq d(y)$ the values of $|C_{\varphi, w(e_i)}(z_i)|$ and $|C_{\varphi, w(e_j)}(z_j)|$ may not be equal for repeated vertices $z_i = z_j$ with $i \neq j$ in C-e-fan F^{ce}. We define *simple C-e-fan* if we further require that vertices $x, y, z_1, \dots, z_p$ are distinct except the repeated vertices $z_i = z_j$ with $1 \leq i < j \leq p$ such that $C_{\varphi, w(e_i)}(z_i) \subset C_{\varphi, w(e_j)}(z_j)$ in the definition of C-e-fan. In a simple C-e-fan $F^{ce} = (x, e, y, e_1, z_1, \dots, e_p, z_p)$, a *linear ce-sequence* is a subsequence $(x, e, y, e_{l_1}, z_{l_1}, \dots, e_{l_s}, z_{l_s})$ with $1 \leq l_1 < l_2 < \cdots < l_s \leq p$ such that $\varphi(e_{l_1}) \in C_{\varphi,y}(x) \cup C_{\varphi,x}(y)$

and $\varphi(e_{l_i}) \in C_{\varphi,w(e_{l_{i-1}})}(z_{l_{i-1}})$ for $2 \le i \le s$. Specifically, a linear ce-sequence is a *x-generated ce-sequence* if $\varphi(e_{l_1}) \in C_{\varphi,y}(x)$, or a *$y$-generated ce-sequence* if $\varphi(e_{l_1}) \in C_{\varphi,x}(y)$.

Proof. In the C-e-fan $F^{ce} = (x, e, y, e_1, z_1, \ldots, e_p, z_p)$, if $z_i = z_j$ with $1 \le i < j \le p$ and $C_{\varphi,w(e_i)}(z_i) \supseteq C_{\varphi,w(e_j)}(z_j)$, we delete the edge e_j and the vertex z_j from F^{ce}. By the definition of C-e-fan, one can easily check that the remaining sequence is still a C-e-fan. Repeat the above operation. Finally, we get a simple C-e-fan F'^{ce} with respect to the C-e-fan F^{ce}. Obviously, $V(F^{ce}) = V(F'^{ce})$ and the $C_\varphi(u)$ in F^{ce} is the same as the $C_\varphi(u)$ in F'^{ce} for each vertex u. Hence, we may assume that the original C-e-fan F^{ce} is a simple C-e-fan. We show the following two claims.

Claim 1: The vertex set of any linear ce-sequence is φ^{ce}-elementary.

Proof. Suppose that Claim 1 is false. Without loss of generality, we choose φ such that there exists a y-generated ce-sequence $S_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1}, \ldots, e_{l_s}, z_{l_s})$ with $e_{l_0} = e$, whose vertex set is not φ^{ce}-elementary with s as small as possible. Note that $e_{l_1} = xz_{l_1}$. Let $\varphi(e_{l_1}) = \beta_{l_1} \in C_{\varphi,x}(y)$ and $\varphi(e_{l_i}) = \beta_{l_i} \in C_{\varphi,w(e_{l_{i-1}})}(z_{l_{i-1}})$ for $2 \le i \le s$. We consider the following two cases of s.

First we consider the case $s \le 1$. It is easy to see that S_y is a C-fan at x. By the statement (a) of Lemma 2.3.3, we have $C_{\varphi,x}(x) \cap C_{\varphi,x}(y) = \emptyset$, $C_{\varphi,x}(x) \cap C_{\varphi,x}(z_{l_1}) = \emptyset$ and $C_{\varphi,x}(y) \cap C_{\varphi,x}(z_{l_1}) = \emptyset$. Recall that $C_{\varphi,x}(x) = \overline{\varphi}(x)$. Since we suppose that Claim 1 is false, there are four subcases left to consider.

If there exists $\eta \in \varphi^s_y(x) \cap \overline{\varphi}(y)$, then it contradicts Lemma 2.3.3 since C-fan (y, e_{l_0}, x) at y. If there exists $\eta \in \varphi^s_y(x) \cap \varphi^s_x(y)$, then there is an edge $e' = xu$ such that $u \ne y$, $\varphi(e') = \eta$ and $d(u) \le \frac{\Delta - d(y)}{2}$, and there is an edge $e'' = yv$ such that $v \ne x$, $\varphi(e'') = \eta$ and $d(v) \le \frac{\Delta - d(x)}{2}$. Obviously, $u \ne v$. Recall that $\max\{d(x), d(y)\} \le \Delta - 1$. It follows from Lemma 2.3.4 that there are two colors $\delta_1 \in \overline{\varphi}(x) \cap \overline{\varphi}(v)$ and $\delta_2 \in \overline{\varphi}(y) \cap \overline{\varphi}(u)$ with $\delta_2 \ne \beta_{l_1}$. We have $\delta_1 \ne \delta_2$ and $P_x(\delta_1, \delta_2, \varphi) = P_y(\delta_1, \delta_2, \varphi)$ by Lemma 2.3.1 since Vizing fan (x, e_{l_0}, y). Apply Kempe changes on $P_u(\delta_1, \delta_2, \varphi)$ and $P_v(\delta_1, \delta_2, \varphi)$ to get a new coloring φ_1 such that $\delta_1 \in \overline{\varphi}_1(x) \cap \overline{\varphi}_1(u)$ and $\delta_2 \in \overline{\varphi}_1(y) \cap \overline{\varphi}_1(v)$. Recolor the edge e' with δ_1 and the edge e'' with δ_2 to get a new coloring φ_2 such that $\eta \in \overline{\varphi}_2(x) \cap \overline{\varphi}_2(y)$. Now by coloring the edge e with η, we color the entire graph G with Δ colors, which contradicts the fact that $\chi'(G) = \Delta + 1$.

If there exists $\eta \in \varphi^s_y(x) \cap \overline{\varphi}(z_{l_1})$, then there is an edge $e' = xu$ such that $u \neq y$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(y)}{2}$. Since $\max\{d(x), d(y)\} \leq \Delta - 1$, it follows from Lemma 2.3.4 that there is a color $\delta \in \overline{\varphi}(y) \cap \overline{\varphi}(u)$ with $\delta \neq \beta_{l_1}$. We have $x \in P_y(\eta, \delta, \varphi) = P_u(\eta, \delta, \varphi)$ by Lemma 2.3.3 since C-fan (y, e_{l_0}, x) at y. Recall that $S_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1})$ is a C-fan at x. The coloring $\varphi_1 = \varphi/P_{z_{l_1}}(\eta, \delta, \varphi)$ results in $\delta \in \overline{\varphi}_1(z_{l_1}) \cap \overline{\varphi}_1(y)$, which contradicts Lemma 2.3.3 because S_y is still a C-fan at x under φ_1.

If there exists $\eta \in \varphi^s_y(x) \cap \varphi^s_x(z_{l_1})$, then there is an edge $e' = xu$ such that $u \neq y$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(y)}{2}$, and there is an edge $e'' = z_{l_1}v$ such that $v \neq x$, $\varphi(e'') = \eta$ and $d(v) \leq \frac{\Delta - d(x)}{2}$. Obviously, $u \neq v$, and we have $v \neq y$ by Lemma 2.3.2. By Lemma 2.3.4, there are two colors $\delta_1 \in \overline{\varphi}(x) \cap \overline{\varphi}(v)$ and $\delta_2 \in \overline{\varphi}(y) \cap \overline{\varphi}(u)$ with $\delta_2 \neq \beta_{l_1}$. We have $P_x(\delta_1, \delta_2, \varphi) = P_y(\delta_1, \delta_2, \varphi)$ by Lemma 2.3.1 since Vizing fan (x, e_{l_0}, y). Apply Kempe changes on $P_u(\delta_1, \delta_2, \varphi)$ and $P_v(\delta_1, \delta_2, \varphi)$ to get a new coloring φ_1 such that $\delta_1 \in \overline{\varphi}_1(x) \cap \overline{\varphi}_1(u)$ and $\delta_2 \in \overline{\varphi}_1(y) \cap \overline{\varphi}_1(v)$. Note that $S_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1})$ is still a C-fan at x under φ_1. Recolor the edge e' with δ_1 to get a new coloring φ_2. Thus $\eta \in \overline{\varphi}_2(x) \cap C_{\varphi_2,x}(z_{l_1})$, which contradicts Lemma 2.3.3 because S_y is still a C-fan at x under φ_2. This completes the proof of Claim 1 for $s \leq 1$.

Now we consider the case $s \geq 2$. By the minimality of s, $V(S_y \backslash \{e_{l_s}, z_{l_s}\})$ is φ^{ce}-elementary. Together with the definition of y-generated ce-sequence, we have that for any color $\gamma_1 \in C_{\varphi,y}(x)$, no edge in $E(S_y)$ is colored with γ_1; for any color γ_2, if $\gamma_2 \in C_{\varphi,x}(y)$ or $\gamma_2 \in C_{\varphi,w(e_{l_i})}(z_{l_i})$ with $1 \leq i \leq s-1$, where z_{l_i} is not a repeated vertex, then only the edge e_{l_1} or $e_{l_{i+1}}$ in $E(S_y)$ may be colored with γ_2; for any color $\gamma_3 \in C_\varphi(z)$, where z is a repeated vertex with $z = z_{l_i} = z_{l_j}$ and $1 \leq i < j \leq s-1$, only the edge $e_{l_{i+1}}$ or $e_{l_{j+1}}$ in $E(S_y)$ may be colored with γ_3. We will use above facts about S_y without explicit mention. The following observation will also be used very often.

II. For any color γ_1 with $\gamma_1 \in \overline{\varphi}(x) \cup \overline{\varphi}(y)$ and $\gamma_1 \neq \beta_{l_1}$, if color $\gamma_2 \in \overline{\varphi}(z_{l_i})$ with $1 \leq i \leq s-1$, then we have $\gamma_1 \neq \gamma_2$ and $P_x(\gamma_1, \gamma_2, \varphi) = P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ or $P_y(\gamma_1, \gamma_2, \varphi) = P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$; if $\gamma_2 \in \varphi^s_{w(e_{l_i})}(z_{l_i})$ with $1 \leq i \leq s-1$, denote by u the vertex and

$e' = z_{l_i}u$ the edge such that $\varphi(e') = \gamma_2$, and further provide $\gamma_1 \in \overline{\varphi}(u)$, then we have $z_{l_i} \in P_x(\gamma_1, \gamma_2, \varphi) = P_u(\gamma_1, \gamma_2, \varphi)$ or $z_{l_i} \in P_y(\gamma_1, \gamma_2, \varphi) = P_u(\gamma_1, \gamma_2, \varphi)$.

Proof. We first assume $\gamma_1 \in \overline{\varphi}(x)$. Recall that $V(S_y \backslash \{e_{l_s}, z_{l_s}\})$ is φ^{ce}-elementary. We easily have $\gamma_1 \neq \gamma_2$ and $\gamma_2 \in \varphi(x)$. Suppose $P_x(\gamma_1, \gamma_2, \varphi) \neq P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($P_x(\gamma_1, \gamma_2, \varphi) \neq P_u(\gamma_1, \gamma_2, \varphi)$, respectively). For the path $P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($P_u(\gamma_1, \gamma_2, \varphi)$, respectively), one endvertex is z_{l_i} (u, respectively) and the other endvertex is some vertex $z' \neq x$. Note that $z' \notin \{y, z_{l_1}, \ldots, z_{l_{i-1}}\}$ and none of $e_{l_1}, \ldots, e_{l_i}$ is colored with γ_1. Since z_{l_i} may be a repeated vertex in S_y, we consider the following two cases. If z_{l_i} is not a repeated vertex or z_{l_i} is a repeated vertex with $z_{l_i} = z_{l_k}$ and $1 \leq i < k \leq s-1$, then none of $e_{l_1}, \ldots, e_{l_i}$ is colored with γ_2. Hence, the coloring $\varphi_1 = \varphi/P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($\varphi_1 = \varphi/P_u(\gamma_1, \gamma_2, \varphi)$, respectively) results in a new y-generated ce-sequence $S'_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1}, \ldots, e_{l_i}, z_{l_i})$ with $\gamma_1 \in \overline{\varphi}_1(z_{l_i}) \cap \overline{\varphi}_1(x)$ ($\gamma_1 \in C_{\varphi_1, w(e_{l_i})}(z_{l_i}) \cap \overline{\varphi}_1(x)$, respectively), contradicting the minimality of s.

If z_{l_i} is a repeated vertex with $z_{l_k} = z_{l_i}$ and $1 \leq k < i \leq s-1$, then only the edge $e_{l_{k+1}}$ of $e_{l_1}, \ldots, e_{l_i}$ may be colored with γ_2. We claim that $e_{l_{k+1}}$ is not on $P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($P_u(\gamma_1, \gamma_2, \varphi)$, respectively). If $\varphi(e_{l_{k+1}}) \neq \gamma_2$, then we are done. If $\varphi(e_{l_{k+1}}) = \gamma_2$ and $e_{l_{k+1}} = xz_{l_{k+1}}$, then $e_{l_{k+1}}$ is on $P_x(\gamma_1, \gamma_2, \varphi)$, and we are also done. If $\varphi(e_{l_{k+1}}) = \gamma_2$, $e_{l_{k+1}} = yz_{l_{k+1}}$ and $e_{l_{k+1}}$ is on $P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($P_u(\gamma_1, \gamma_2, \varphi)$, respectively), then the coloring $\varphi' = \varphi/P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($\varphi' = \varphi/P_u(\gamma_1, \gamma_2, \varphi)$, respectively) results in a smaller x-generated ce-sequence $(x, e_{l_0}, y, e_{l_{k+1}}, z_{l_{k+1}}, \ldots, e_{l_i}, z_{l_i})$ since $\varphi'(e_{l_{k+1}}) = \gamma_1 \in \overline{\varphi}'(x)$ such that $\gamma_1 \in \overline{\varphi}'(z_{l_i}) \cap \overline{\varphi}'(x)$ ($\gamma_1 \in C_{\varphi', w(e_{l_i})}(z_{l_i}) \cap \overline{\varphi}'(x)$, respectively), contradicting the minimality of s. Now we have that $e_{l_{k+1}}$ is not on $P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($P_u(\gamma_1, \gamma_2, \varphi)$, respectively). Let the coloring $\varphi_1 = \varphi/P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($\varphi_1 = \varphi/P_u(\gamma_1, \gamma_2, \varphi)$, respectively), which results in a new y-generated ce-sequence $S'_y = (x, e_{l_0}, y, e_{l_1}, z_{l_1}, \ldots, e_{l_i}, z_{l_i})$ with $\gamma_1 \in \overline{\varphi}_1(z_{l_i}) \cap \overline{\varphi}_1(x)$ ($\gamma_1 \in C_{\varphi_1, w(e_{l_i})}(z_{l_i}) \cap \overline{\varphi}_1(x)$, respectively), also contradicting the minimality of s. This completes the proof of $P_x(\gamma_1, \gamma_2, \varphi) = P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($P_x(\gamma_1, \gamma_2, \varphi) = P_u(\gamma_1, \gamma_2, \varphi)$, respectively). Similarly, we have $P_y(\gamma_1, \gamma_2, \varphi) = P_{z_{l_i}}(\gamma_1, \gamma_2, \varphi)$ ($P_y(\gamma_1, \gamma_2, \varphi) = P_u(\gamma_1, \gamma_2, \varphi)$, respectively) for $\gamma_1 \in \overline{\varphi}(y)$ and $\gamma_1 \neq \beta_{l_1}$. □

By the minimality of s, we have that either z_{l_s} is not a repeated vertex or z_{l_s} is a repeated vertex with $z_{l_k} = z_{l_s}$ and $C_{\varphi,w(e_{l_k})}(z_{l_k}) \subset C_{\varphi,w(e_{l_s})}(z_{l_s})$, where $1 \le k < s$. By the minimality of s again, there exists a color $\eta \in C_{\varphi,w(e_{l_s})}(z_{l_s}) \cap (C_{\varphi,y}(x) \cup C_{\varphi,x}(y) \cup C_{\varphi,w(e_{l_i})}(z_{l_i}))$ with $1 \le i \le s-1$. And if z_{l_s} is a repeated vertex with $z_{l_k} = z_{l_s}$ and $1 \le k < s$, then we have $\eta \in C_{\varphi,w(e_{l_s})}(z_{l_s}) \backslash C_{\varphi,w(e_{l_k})}(z_{l_k}) = \varphi^s_{w(e_{l_s})}(z_{l_s}) \backslash \varphi^s_{w(e_{l_k})}(z_{l_k})$. Let $\alpha \in \overline{\varphi}(x)$.

Subclaim 1.1: We may assume that $C_{\varphi,w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}(x) \neq \emptyset$.

Proof. In order to prove the above subclaim, we consider the following three cases.

Case 1. $\eta \in C_{\varphi,w(e_{l_s})}(z_{l_s}) \cap C_{\varphi,y}(x)$.

If $\eta \in C_{\varphi,w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}(x)$, then we are done. Otherwise, first suppose $\eta \in \overline{\varphi}(z_{l_s}) \cap \varphi^s_y(x)$, then there is an edge $e' = xu$ such that $u \neq y$, $\varphi(e') = \eta$ and $d(u) \le \frac{\Delta - d(y)}{2}$. It follows from Lemma 2.3.4 that there is a color $\delta \in \overline{\varphi}(y) \cap \overline{\varphi}(u)$ with $\delta \neq \beta_{l_1}$. We have $x \in P_y(\eta, \delta, \varphi) = P_u(\eta, \delta, \varphi)$ by Lemma 2.3.3 since C-fan (y, e_{l_0}, x) at y. The coloring $\varphi_1 = \varphi / P_{z_{l_s}}(\eta, \delta, \varphi)$ results in $\delta \in \overline{\varphi}_1(z_{l_s})$ and S_y is still a y-generated ce-sequence under φ_1. We have $P_x(\alpha, \delta, \varphi_1) = P_y(\alpha, \delta, \varphi_1)$ by Lemma 2.3.1 since Vizing fan (x, e_{l_0}, y) under φ_1. Then the coloring $\varphi_2 = \varphi_1 / P_{z_{l_s}}(\alpha, \delta, \varphi_1)$ results in $\alpha \in \overline{\varphi}_2(z_{l_s}) \cap \overline{\varphi}_2(x)$, which is as desired because S_y is still a y-generated ce-sequence under φ_2 and $C_{\varphi_2,w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}_2(x) \neq \emptyset$.

Now suppose $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \varphi^s_y(x)$. Thus there is an edge $e' = xu$ such that $u \neq y$, $\varphi(e') = \eta$ and $d(u) \le \frac{\Delta - d(y)}{2}$, and there is an edge $e'' = z_{l_s}v$ such that $v \neq w(e_{l_s})$, $\varphi(e'') = \eta$ and $d(v) \le \frac{\Delta - d(w(e_{l_s}))}{2}$. Obviously, $u \neq v$. We consider the following two subcases. If $d(x) \le d(y)$, then by Lemma 2.3.4, there is a color $\delta \in \overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. Recolor the edge e' with δ to get a new coloring φ_1 such that $\eta \in (\varphi_1)^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}_1(x)$. Then we are done because S_y is still a y-generated ce-sequence under φ_1 and $C_{\varphi_1,w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}_1(x) \neq \emptyset$. If $d(x) > d(y)$, then by Lemma 2.3.4, there is a color $\delta \in \overline{\varphi}(y) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. We have $P_x(\alpha, \delta, \varphi) = P_y(\alpha, \delta, \varphi)$ by Lemma 2.3.1 since Vizing fan (x, e_{l_0}, y). Note that e_{l_1} is on $P_x(\alpha, \delta, \varphi)$ if $\delta = \beta_{l_1}$. Apply Kempe changes on $P_u(\alpha, \delta, \varphi)$ and $P_v(\alpha, \delta, \varphi)$ to get a new coloring φ_2 such that $\alpha \in \overline{\varphi}_2(x) \cap \overline{\varphi}_2(u) \cap \overline{\varphi}_2(v)$. Since S_y is still a y-generated ce-sequence under φ_2, we are in the previous subcase in this paragraph with α in place of δ.

Case 2: $\eta \in C_{\varphi,w(e_{l_s})}(z_{l_s}) \cap C_{\varphi,x}(y)$.

If $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(y)$, then we have $P_x(\alpha, \eta, \varphi) = P_y(\alpha, \eta, \varphi)$ by Lemma 2.3.1 since Vizing fan (x, e_{l_0}, y). Note that e_{l_1} is on $P_x(\alpha, \eta, \varphi)$ if $\eta = \beta_{l_1}$. Then the coloring $\varphi_1 = \varphi/P_{z_{l_s}}(\alpha, \eta, \varphi)$ results in $\alpha \in \overline{\varphi}_1(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired because S_y is still a y-generated ce-sequence under φ_1 and $C_{\varphi_1, w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}_1(x) \neq \emptyset$.

If $\eta \in \overline{\varphi}(z_{l_s}) \cap \varphi_x^s(y)$, then there is an edge $e' = yu$ such that $u \neq x$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(x)}{2}$. By Lemma 2.3.4, there is a color $\delta \in \overline{\varphi}(x) \cap \overline{\varphi}(u)$. We have $y \in P_x(\eta, \delta, \varphi) = P_u(\eta, \delta, \varphi)$ by Lemma 2.3.3 since C-fan $(x, e_{l_0}, y, e_{l_1}, z_{l_1})$ at x. Note that e_{l_1} is on $P_x(\eta, \delta, \varphi)$ if $\eta = \beta_{l_1}$. Then the coloring $\varphi_1 = \varphi/P_{z_{l_s}}(\eta, \delta, \varphi)$ results in $\delta \in \overline{\varphi}_1(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired.

If $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}(y)$, then there is an edge $e' = z_{l_s}u$ such that $u \neq w(e_{l_s})$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(w(e_{l_s}))}{2}$. It follows from Lemma 2.3.4 that there is a color $\delta \in \overline{\varphi}(w(e_{l_s})) \cap \overline{\varphi}(u)$ with $\delta \neq \eta$. We consider the following two subcases. If $w(e_{l_s}) = x$, then we have $P_x(\eta, \delta, \varphi) = P_y(\eta, \delta, \varphi)$ by Lemma 2.3.1 since Vizing fan (x, e_{l_0}, y). Note that e_{l_1} is on $P_x(\eta, \delta, \varphi)$ if $\eta = \beta_{l_1}$. Then the coloring $\varphi_1 = \varphi/P_u(\eta, \delta, \varphi)$ results in $\delta \in (\varphi_1)^s_x(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired. If $w(e_{l_s}) = y$, then we have $P_x(\alpha, \delta, \varphi) = P_y(\alpha, \delta, \varphi)$ by Lemma 2.3.1. Note that e_{l_1} is on $P_x(\alpha, \delta, \varphi)$ if $\delta = \beta_{l_1}$. Then the coloring $\varphi_2 = \varphi/P_u(\alpha, \delta, \varphi)$ results in $\alpha \in \overline{\varphi}_2(u)$. We have $P_x(\alpha, \eta, \varphi_2) = P_y(\alpha, \eta, \varphi_2)$ by Lemma 2.3.1 since Vizing fan (x, e_{l_0}, y) under φ_2. Then the coloring $\varphi_3 = \varphi_2/P_u(\alpha, \eta, \varphi_2)$ results in $\alpha \in (\varphi_3)^s_y(z_{l_s}) \cap \overline{\varphi}_3(x)$, as desired.

If $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \varphi^s_x(y)$, then there is an edge $e' = yu$ such that $u \neq x$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(x)}{2}$, and there is an edge $e'' = z_{l_s}v$ such that $v \neq w(e_{l_s})$, $\varphi(e'') = \eta$ and $d(v) \leq \frac{\Delta - d(w(e_{l_s}))}{2}$. Obviously, $u \neq v$. We consider the following two subcases. If $d(x) \leq d(y)$, then by Lemma 2.3.4, there is a color $\delta \in \overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. We have $y \in P_x(\eta, \delta, \varphi) = P_u(\eta, \delta, \varphi)$ by Lemma 2.3.3 since C-fan $(x, e_{l_0}, y, e_{l_1}, z_{l_1})$ at x. Note that e_{l_1} is on $P_x(\eta, \delta, \varphi)$ if $\eta = \beta_{l_1}$. Then the coloring $\varphi_1 = \varphi/P_v(\eta, \delta, \varphi)$ results in $\delta \in (\varphi_1)^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired. If $d(x) > d(y)$, then by Lemma 2.3.4, there is a color $\delta \in \overline{\varphi}(y) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. We have $P_x(\alpha, \delta, \varphi) = P_y(\alpha, \delta, \varphi)$ by Lemma 2.3.1. Note that e_{l_1} is on $P_x(\alpha, \delta, \varphi)$ if $\delta = \beta_{l_1}$. Apply Kempe changes on $P_u(\alpha, \delta, \varphi)$ and $P_v(\alpha, \delta, \varphi)$ to get a new coloring φ_2 such that $\alpha \in \overline{\varphi}_2(x) \cap \overline{\varphi}_2(u) \cap \overline{\varphi}_2(v)$. Thus we are in the previous subcase in this paragraph with α in place of δ.

Case 3: $\eta \in C_{\varphi, w(e_{l_s})}(z_{l_s}) \cap C_{\varphi, w(e_{l_i})}(z_{l_i})$ for $1 \leq i \leq s-1$.

By the minimality of s, we have $z_{l_s} \neq z_{l_i}$. If $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(z_{l_i})$, then $P_x(\alpha, \eta, \varphi) = P_{z_{l_i}}(\alpha, \eta, \varphi)$ by the observation II. For the path $P = P_{z_{l_s}}(\alpha, \eta, \varphi)$, one endvertex is z_{l_s}, the other endvertex is $z' \notin V(S_y)$ and $E_{\varphi,\alpha}(P) \cap E(S_y) = \emptyset$. In order to apply a Kempe change on P, we should discuss the following $E_{\varphi,\eta}(P) \cap E(S_y)$. Let $z_{l_i} = z_{l_j}$ with $1 \leq i \neq j \leq s-1$ if z_{l_i} is a repeated vertex in S_y. Note that only one of $e_{l_{i+1}}, e_{l_{j+1}}$ may be colored with η. We consider the following two subcases. If $\eta = \beta_{l_{i+1}}$ and $e_{l_{i+1}}$ is on P (or $\eta = \beta_{l_{j+1}}$ and $e_{l_{j+1}}$ is on P by symmetry), then $E_{\varphi,\eta}(P) \cap E(S_y) = \{e_{l_{i+1}}\}$ and the coloring $\varphi_1 = \varphi/P$ results in a smaller x-generated ce-sequence $(x, e_{l_0}, y, e_{l_{i+1}}, z_{l_{i+1}}, \ldots, e_{l_s}, z_{l_s})$ since $\varphi_1(e_{l_{i+1}}) = \alpha \in \overline{\varphi}_1(x)$ such that $\alpha \in \overline{\varphi}_1(z_{l_s}) \cap \overline{\varphi}_1(x)$, contradicting the minimality of s. If $\eta \neq \beta_{l_{i+1}}, \beta_{l_{j+1}}$, or $\eta = \beta_{l_{i+1}}$ and $e_{l_{i+1}}$ is not on P, then $E_{\varphi,\eta}(P) \cap E(S_y) = \emptyset$ and the coloring $\varphi_1 = \varphi/P$ results in $\alpha \in \overline{\varphi}_1(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired because S_y is still a y-generated ce-sequence under φ_1.

If $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}(z_{l_i})$, then there is an edge $e' = z_{l_s}u$ such that $u \neq w(e_{l_s})$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(w(e_{l_s}))}{2}$. It follows from Lemma 2.3.4 that there is a color $\delta \in \overline{\varphi}(w(e_{l_s})) \cap \overline{\varphi}(u)$ and $\delta \neq \beta_{l_1}$. We claim that we may assume $\overline{\varphi}(x) \cap \overline{\varphi}(u) \neq \emptyset$. If $w(e_{l_s}) = x$, then we are done. Otherwise, consider the case $w(e_{l_s}) = y$. We have $P_x(\alpha, \delta, \varphi) = P_y(\alpha, \delta, \varphi)$ by Lemma 2.3.1. Then the coloring $\varphi' = \varphi/P_u(\alpha, \delta, \varphi)$ results in $\alpha \in \overline{\varphi}'(x) \cap \overline{\varphi}'(u)$, as desired. Now let $\gamma \in \overline{\varphi}(x) \cap \overline{\varphi}(u)$. By the observation II, we have $P_x(\gamma, \eta, \varphi) = P_{z_{l_i}}(\gamma, \eta, \varphi)$. By the similar proof of the first subcase of Case 3 (i.e., the case $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(z_{l_i})$) with $P_u(\gamma, \eta, \varphi)$ in place of P and γ in place of α, we can obtain the coloring $\varphi_1 = \varphi/P_u(\gamma, \eta, \varphi)$ such that $\gamma \in (\varphi_1)^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired.

If $\eta \in \overline{\varphi}(z_{l_s}) \cap \varphi^s_{w(e_{l_i})}(z_{l_i})$, then there is an edge $e' = z_{l_i}u$ such that $u \neq w(e_{l_i})$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(w(e_{l_i}))}{2}$. It follows from Lemma 2.3.4 that there is a color $\delta \in \overline{\varphi}(w(e_{l_i})) \cap \overline{\varphi}(u)$ with $\delta \neq \beta_{l_1}$. We claim that we may assume $\overline{\varphi}(x) \cap \overline{\varphi}(u) \neq \emptyset$. If $w(e_{l_i}) = x$, then we are done. Otherwise, consider the case $w(e_{l_i}) = y$. We have $P_x(\alpha, \delta, \varphi) = P_y(\alpha, \delta, \varphi)$ by Lemma 2.3.1. Then the coloring $\varphi' = \varphi/P_u(\alpha, \delta, \varphi)$ results in $\alpha \in \overline{\varphi}'(x) \cap \overline{\varphi}'(u)$, as desired. Now let $\gamma \in \overline{\varphi}(x) \cap \overline{\varphi}(u)$. By the observation II, we have $P_x(\gamma, \eta, \varphi) = P_u(\gamma, \eta, \varphi)$. By the similar proof of the first subcase of Case 3 with γ in place of α, we can obtain the coloring $\varphi_1 = \varphi/P_{z_{l_s}}(\gamma, \eta, \varphi)$ such that $\gamma \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired.

If $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \varphi^s_{w(e_{l_i})}(z_{l_i})$, then there is an edge $e' = z_{l_s}u$ such that $u \neq w(e_{l_s})$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(w(e_{l_s}))}{2}$, and there is an edge $e'' = z_{l_i}v$ such that $v \neq w(e_{l_i})$, $\varphi(e'') = \eta$ and $d(v) \leq \frac{\Delta - d(w(e_{l_i}))}{2}$. Obviously, $u \neq v$. We claim that we may assume $\overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v) \neq \emptyset$. If $d(x) \leq d(y)$, then it follows from Lemma 2.3.4 that there is a color $\delta \in \overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$, and so we are done. If $d(x) > d(y)$, then it follows from Lemma 2.3.4 that there is a color $\delta \in \overline{\varphi}(y) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. We have $P_x(\alpha, \delta, \varphi) = P_y(\alpha, \delta, \varphi)$ by Lemma 2.3.1. Apply Kempe changes on $P_u(\alpha, \delta, \varphi)$ and $P_v(\alpha, \delta, \varphi)$, and get a new coloring φ' such that $\alpha \in \overline{\varphi}'(x) \cap \overline{\varphi}'(u) \cap \overline{\varphi}'(v)$, as desired. Now let $\gamma \in \overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. By the observation II, we have $P_x(\gamma, \eta, \varphi) = P_u(\gamma, \eta, \varphi)$. By the similar proof of the first subcase of Case 3 with $P_u(\gamma, \eta, \varphi)$ in place of P and γ in place of α, we can obtain the coloring $\varphi_1 = \varphi / P_u(\gamma, \eta, \varphi)$ such that $\gamma \in (\varphi_1)^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}_1(x)$, as desired.

Combining the above Cases 1, 2 and 3, we complete the proof of Subclaim 1.1. □

Thus we assume that there exists a color $\eta \in C_{\varphi, w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}(x)$. We consider the following two cases.

Case 1: $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(x)$.

Suppose $w(e_{l_s}) = x$. Recolor the edge e_{l_s} with η to get a new coloring φ_1. Thus $\beta_{l_s} \in \overline{\varphi}_1(x) \cap C_{\varphi_1, w(e_{l_{s-1}})}(z_{l_{s-1}})$, which contradicts the minimality of s. So we assume $w(e_{l_s}) = y$. Since $d(y) \leq \Delta - 1$, there exists a missing color γ with $\gamma \neq \beta_{l_1}$. We have $P_x(\eta, \gamma, \varphi) = P_y(\eta, \gamma, \varphi)$ by Lemma 2.3.1. Let $\varphi_2 = \varphi / P_{z_{l_s}}(\eta, \gamma, \varphi)$, and we have $\gamma \in \overline{\varphi}_2(y) \cap \overline{\varphi}_2(z_{l_s})$. Recolor the edge e_{l_s} with γ to get a new coloring φ_3. Thus $\beta_{l_s} \in \overline{\varphi}_3(y) \cap C_{\varphi_3, w(e_{l_{s-1}})}(z_{l_{s-1}})$, also contradicting the minimality of s.

Case 2: $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}(x)$.

Suppose $\beta_{l_s} \in \overline{\varphi}(z_{l_{s-1}})$. Since $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s})$, there is an edge $e' = z_{l_s}u$ such that $u \neq w(e_{l_s})$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(w(e_{l_s}))}{2}$. It follows from Lemma 2.3.4 that there is a color $\delta \in \overline{\varphi}(w(e_{l_s})) \cap \overline{\varphi}(u)$ with $\delta \neq \eta, \beta_{l_1}$. By the observation II, we have $P_{w(e_{l_s})}(\delta, \beta_{l_s}, \varphi) = P_{z_{l_{s-1}}}(\delta, \beta_{l_s}, \varphi)$. Note that e_{l_s} is on $P_{w(e_{l_s})}(\delta, \beta_{l_s}, \varphi)$. Let $\varphi_1 = \varphi / P_u(\delta, \beta_{l_s}, \varphi)$. Hence S_y is still a y-generated sequence under φ_1 with $\beta_{l_s} \in \overline{\varphi}_1(u)$. We claim that we may assume $\eta \in \overline{\varphi}_1(w(e_{l_s}))$. If $w(e_{l_s}) = x$, we are done. Otherwise, $w(e_{l_s}) = y$. We have $P_x(\eta, \delta, \varphi_1) =$

$P_y(\eta, \delta, \varphi_1)$ by Lemma 2.3.1. Recall $\delta \neq \beta_{l_1}$. The coloring $\varphi' = \varphi_1/P_x(\eta, \delta, \varphi)$ results in $\eta \in \overline{\varphi}'(y)$, as desired. Now we assume $\eta \in \overline{\varphi}_1(w(e_{l_s}))$. We have $P_{w(e_{l_s})}(\eta, \beta_{l_s}, \varphi_1) = P_u(\eta, \beta_{l_s}, \varphi_1) = w(e_{l_s})z_{l_s}u$. Then the coloring $\varphi_2 = \varphi_1/P_{w(e_{l_s})}(\eta, \beta_{l_s}, \varphi_1)$ results in $\beta_{l_s} \in \overline{\varphi}_2(w(e_{l_s})) \cap \overline{\varphi}_2(z_{l_{s-1}})$, contradicting the minimality of s.

Now we suppose $\beta_{l_s} \in \varphi^s_{w(e_{l_{s-1}})}(z_{l_{s-1}})$. In this case, there is an edge $e' = z_{l_s}u$ such that $u \neq w(e_{l_s})$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(w(e_{l_s}))}{2}$, and there is an edge $e'' = z_{l_{s-1}}v$ such that $v \neq w(e_{l_{s-1}})$, $\varphi(e'') = \beta_{l_s}$ and $d(v) \leq \frac{\Delta - d(w(e_{l_{s-1}}))}{2}$. By the condition **C3** in Section 2, we have $u \neq v$. It follows from Lemma 2.3.4 that there is a color $\delta \in (\overline{\varphi}(w(e_{l_s})) \cup \overline{\varphi}(w(e_{l_{s-1}}))) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. We first claim that we may assume that $\delta \in \overline{\varphi}(w(e_{l_s}))$ and $\delta \neq \beta_{l_1}$. Suppose $\delta \in \overline{\varphi}(w(e_{l_s}))$ but $\delta = \beta_{l_1}$. Thus $w(e_{l_s}) = y$. Recall that $\max\{d(x), d(y)\} \leq \Delta - 1$. Hence there exist $\gamma_1 \in \overline{\varphi}(x)$ with $\gamma_1 \neq \eta$ and $\gamma_2 \in \overline{\varphi}(y)$ with $\gamma_2 \neq \delta = \beta_{l_1}$. By Lemma 2.3.1, we have $P_x(\gamma_1, \delta, \varphi) = P_y(\gamma_1, \delta, \varphi)$ and $P_x(\gamma_1, \gamma_2, \varphi) = P_y(\gamma_1, \gamma_2, \varphi)$. Apply Kempe changes on $P_u(\gamma_1, \delta, \varphi)$ and $P_v(\gamma_1, \delta, \varphi)$ to get a new coloring φ'. And then apply Kempe changes on $P_u(\gamma_1, \gamma_2, \varphi')$ and $P_v(\gamma_1, \gamma_2, \varphi')$ to get a new coloring φ''. Consequently, we have $\gamma_2 \in \overline{\varphi}''(u) \cap \overline{\varphi}''(v)$, as desired because γ_2 is the desired color instead of δ. Now suppose $\delta \notin \overline{\varphi}(w(e_{l_s}))$. Thus we have $w(e_{l_s}) \neq w(e_{l_{s-1}})$ and $\delta \in \overline{\varphi}(w(e_{l_{s-1}}))$. Since $\max\{d(x), d(y)\} \leq \Delta - 1$, there exists a missing color $\gamma \in \overline{\varphi}(w(e_{l_s}))$ such that $\gamma \neq \delta, \beta_{l_1}$. We have $P_x(\gamma, \delta, \varphi) = P_y(\gamma, \delta, \varphi)$ by Lemma 2.3.1. Apply Kempe changes on $P_u(\gamma, \delta, \varphi)$ and $P_v(\gamma, \delta, \varphi)$ to get a new coloring φ'''. Thus $\gamma \in \overline{\varphi}'''(w(e_{l_s})) \cap \overline{\varphi}'''(u) \cap \overline{\varphi}'''(v)$, as desired because γ is the desired color instead of δ. Now we assume that $\delta \in \overline{\varphi}(w(e_{l_s}))$ and $\delta \neq \beta_{l_1}$. Then $P_{w(e_{l_s})}(\delta, \beta_{l_s}, \varphi) = P_v(\delta, \beta_{l_s}, \varphi)$ by the observation II. Note that e_{l_s} is on $P_{w(e_{l_s})}(\delta, \beta_{l_s}, \varphi)$. Let the coloring $\varphi_1 = \varphi/P_u(\delta, \beta_{l_s}, \varphi)$. Hence S_y is still a y-generated ce-sequence under φ_1 with $\beta_{l_s} \in \overline{\varphi}_1(u)$.

Next we show that we may assume $\eta \in \overline{\varphi}_1(w(e_{l_s}))$. If $w(e_{l_s}) = x$, we are done. Otherwise, $w(e_{l_s}) = y$. We have $P_x(\eta, \delta, \varphi_1) = P_y(\eta, \delta, \varphi_1)$ by Lemma 2.3.1. The coloring $\varphi_1' = \varphi_1/P_x(\eta, \delta, \varphi)$ results in $\eta \in \overline{\varphi}_1'(y)$, as desired. Now note that $P_{w(e_{l_s})}(\eta, \beta_{l_s}, \varphi_1) = P_u(\eta, \beta_{l_s}, \varphi_1) = w(e_{l_s})z_{l_s}u$. Then the coloring $\varphi_2 = \varphi_1/P_{w(e_{l_s})}(\eta, \beta_{l_s}, \varphi_1)$ results in $\beta_{l_s} \in \overline{\varphi}_2(w(e_{l_s})) \cap (\varphi_2)^s_{w(e_{l_{s-1}})}(z_{l_{s-1}})$, contradicting the minimality of s. This completes the proof of Case 2.

Combining the above Cases 1 and 2, we complete the proof of Claim 1 for $s \geq 2$. Together with the proof of Claim 1 for $s \leq 1$, we prove Claim 1. □

Claim 2: The union of vertex sets of any two linear ce-sequences is φ^{ce}-elementary.

Proof. Suppose that Claim 2 is false. Without loss of generality, we choose φ such that there exist two linear ce-sequences $S_1 = (x, e, y, e_{l_1}, z_{l_1}, \ldots, e_{l_s}, z_{l_s})$ and $S_2 = (x, e, y, e_{l'_1}, z_{l'_1}, \ldots, e_{l'_t}, z_{l'_t})$ whose union of vertex sets is not φ^{ce}-elementary with $s+t$ as small as possible, where $s, t \geq 1$. Note that $V(S_1)$ and $V(S_2)$ are φ^{ce}-elementary by Claim 1. By the minimality of $s+t$, $z_{l_s} \neq z_{l'_t}$ and there exists a color $\eta \in C_{\varphi,w(e_{l_s})}(z_{l_s}) \cap C_{\varphi,w(e_{l'_t})}(z_{l'_t})$. We consider the following three cases. If $\eta \in \overline{\varphi}(z_{l_s}) \cap \overline{\varphi}(z_{l'_t})$, then z_{l_s} and $z_{l'_t}$ are respectively not repeated vertices in S_1 and S_2 since the minimality of $s+t$. By the same proof of Claim 2 in Theorem 2.1.3, we can obtain three endvertices on one Kempe chain, which gives a contradiction.

If $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \overline{\varphi}(z_{l'_t})$ (or $\eta \in \overline{\varphi}(z_{l_s}) \cap \varphi^s_{w(e_{l'_t})}(z_{l'_t})$ by symmetry), then there is an edge $e' = z_{l_s}u$ such that $u \neq w(e_{l_s})$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(w(e_{l_s}))}{2}$. It follows from Lemma 2.3.4 that there is a color $\delta \in \overline{\varphi}(w(e_{l_s})) \cap \overline{\varphi}(u)$. By the definition of linear ce-sequence in C-e-fan and the minimality of $s+t$, z_{l_s} may be a repeated vertex in S_1, while $z_{l'_t}$ is not a repeated vertex in S_2. Note that $\varphi(e_{l_1})$ and $\varphi(e_{l'_1})$ are in $C_{\varphi,y}(x) \cup C_{\varphi,x}(y)$. ($\varphi(e_{l_1})$ and $\varphi(e_{l'_1})$ could be the same color.) We consider the following two subcases. If $\delta \notin \{\varphi(e_{l_1}), \varphi(e_{l'_1})\}$, then we have $P_{w(e_{l_s})}(\delta, \eta, \varphi) = P_u(\delta, \eta, \varphi)$ by the observation II since S_1 is φ^{ce}-elementary. Similarly, we have $P_{w(e_{l_s})}(\delta, \eta, \varphi) = P_{z_{l'_t}}(\delta, \eta, \varphi)$ by the observation II since S_2 is φ^{ce}-elementary. Thus $w(e_{l_s}), z_{l'_t}$ and u are three endvertices of $P_{w(e_{l_s})}(\delta, \eta, \varphi)$, which gives a contradiction. Now we consider the remaining case $\delta \in \{\varphi(e_{l_1}), \varphi(e_{l'_1})\}$. Let $w'(e_{l_s}) \in \{x, y\} \backslash \{w(e_{l_s})\}$. Recall that $\max\{d(x), d(y)\} \leq \Delta - 1$. Hence we can choose a color $\gamma \in \overline{\varphi}(w'(e_{l_s}))$ with $\gamma \notin \{\varphi(e_{l_1}), \varphi(e_{l'_1})\}$. We have $P_x(\delta, \gamma, \varphi) = P_y(\delta, \gamma, \varphi)$ by Lemma 2.3.1. Apply a Kempe change on $P_u(\delta, \gamma, \varphi)$ to get a new coloring φ_1. Thus $\gamma \in \overline{\varphi}_1(w'(e_{l_s})) \cap \overline{\varphi}_1(u)$. Similarly as the subcase above (when $\delta \notin \{\varphi(e_{l_1}), \varphi(e_{l'_1})\}$), we have $P_{w'(e_{l_s})}(\gamma, \eta, \varphi_1) = P_{z_{l'_t}}(\gamma, \eta, \varphi_1)$ and $P_{w'(e_{l_s})}(\gamma, \eta, \varphi_1) = P_u(\gamma, \eta, \varphi_1)$. Thus $w'(e_{l_s}), z_{l'_t}$ and u are three endvertices of $P_{w'(e_{l_s})}(\delta, \eta, \varphi_1)$, which also gives a contradiction.

If $\eta \in \varphi^s_{w(e_{l_s})}(z_{l_s}) \cap \varphi^s_{w(e_{l'_t})}(z_{l'_t})$, then there is an edge $e' = z_{l_s}u$ such that $u \neq w(e_{l_s})$, $\varphi(e') = \eta$ and $d(u) \leq \frac{\Delta - d(w(e_{l_s}))}{2}$, and there is an edge $e'' = z_{l'_t}v$ such that $v \neq w(e_{l'_t})$, $\varphi(e'') = \eta$ and $d(v) \leq \frac{\Delta - d(w(e_{l'_t}))}{2}$. Obviously, $u \neq v$, and z_{l_s} and $z_{l'_t}$ may be repeated vertices respectively in S_1 and S_2. Without loss of generality, we suppose that $d(x) \leq d(y)$. It follows from Lemma 2.3.4 that there is a color $\delta \in \overline{\varphi}(x) \cap \overline{\varphi}(u) \cap \overline{\varphi}(v)$. We consider the following two subcases. If $\delta \notin \{\varphi(e_{l_1}), \varphi(e_{l'_1})\}$, then we have $P_x(\delta, \eta, \varphi) = P_u(\delta, \eta, \varphi)$ by the observation II. Similarly, we have $P_x(\delta, \eta, \varphi) = P_v(\delta, \eta, \varphi)$. Thus x, u and v are three endvertices on one (δ, η)-chain, which is a contradiction. Now we consider the remaining case $\delta \in \{\varphi(e_{l_1}), \varphi(e_{l'_1})\}$. Recall that $\max\{d(x), d(y)\} \leq \Delta - 1$. Hence we can choose a color $\gamma \in \overline{\varphi}(y)$ with $\gamma \notin \{\varphi(e_{l_1}), \varphi(e_{l'_1})\}$. We have $P_x(\delta, \gamma, \varphi) = P_y(\delta, \gamma, \varphi)$ by Lemma 2.3.1. Apply Kempe changes on $P_u(\delta, \gamma, \varphi)$ and $P_v(\delta, \gamma, \varphi)$ to get a new coloring φ_1. Thus we have $\gamma \in \overline{\varphi}_1(y) \cap \overline{\varphi}_1(u) \cap \overline{\varphi}_1(v)$. Thus we are back to the previous subcase with y in place of x and γ in place of δ. This completes the proof of Claim 2. □

Now we are ready to show that $V(F^{ce})$ is φ^{ce}-elementary. Suppose not. Note that $\{x, y\}$ is φ^{ce}-elementary and each linear ce-sequence in F^{ce} contains vertices x and y. There exist one color η and two distinct vertices z_i and z_j such that $\eta \in C_{\varphi, w(e_i)}(z_i) \cap C_{\varphi, w(e_j)}(z_j)$, where $0 \leq i < j \leq p$ and $z_0 \in \{x, y\}$. By the definition of simple C-e-fan, there exist two linear ce-sequences with z_i and z_j respectively as the last vertex, which is a contradiction to Claim 1 for $i = 0$ or a contradiction to Claim 2 for $1 \leq i \leq p-1$. This proves that $V(F^{ce})$ is φ^{ce}-elementary.

Now we show the "furthermore" part. We assume that F^{ce} is maximal. Let the edge set $\Gamma = \{e_1, \ldots, e_p\}$ and the color set $\Gamma' = \bigcup_{z \in V(F^{ce})} C_\varphi(z)$. Note that $C_\varphi(x)$, $C_\varphi(y)$ and $C_\varphi(z)$, where $z \in V(F^{ce}) \backslash \{x, y\}$, are disjoint since $V(F^{ce})$ is φ^{ce}-elementary. We have

$$p = |\Gamma| = \sum_{z \in V(F^{ce}) \backslash \{x,y\}} (e_{F^{ce}}(x, z) + e_{F^{ce}}(y, z)) = |\Gamma^*|. \qquad \textbf{(3)}$$

Now we calculate $|\Gamma^*|$ in another way. By the definition of C-e-fan, $\varphi(e_i) \in \Gamma'$ for each $i \in [p]$. By the maximality of F^{ce}, for any $\alpha \in \Gamma'$, α appears exactly once in Γ^* if $\alpha \in C_\varphi(x) \cup C_\varphi(y)$.

Otherwise, α appears exactly twice in Γ^*. Thus we have

$$|\Gamma^*| = |C_\varphi(x)| + |C_\varphi(y)| + \sum_{z \in V(F^{ce}) \setminus \{x,y\}} 2|C_\varphi(z)|. \qquad \mathbf{(4)}$$

Combining equations (3) and (4), we prove that

$$|C_\varphi(x)| + |C_\varphi(y)| = \sum_{z \in V(F^{ce}) \setminus \{x,y\}} (e_{F^{ce}}(x,z) + e_{F^{ce}}(y,z) - 2|C_\varphi(z)|).$$

The proof is now finished. □

CHAPTER 3

PRECOLORING EXTENSION OF VIZING'S THEOREM

In this chapter, we focus on one conjecture of Edwards et al. about precoloring extension of Vizing's Theorem, and partly comfirm it for multigraphs with distance-3 matching, i.e., Theorem 3.1.1. In Section 3.2, we introduce some new structural properties of dense subgraphs. In Section 3.3, we define a general multi-fan and obtain some generalizations of Vizing's Theorem. In Section 3.4, we present the proof of Theorem 3.1.1.

The proof is based on the assumption of Goldberg-Seymour Conjecture and relies on dense subgraphs and refinements of multi-fans as tools. Its main strategy is roughly as follows. We first define a feasible tripe containing some matching and coloring, which can achieve our result easily, then we fix one initial prefeasible tripe and modify it step by step into a desired feasible tripe by some recoloring technologies.

3.1 Introduction

Recall that the distance between two edges e and f in G is the length of a shortest path connecting an endvertex of e and an endvertex of f, and a distance-t matching is a set of edges having pairwise distance at least t. Following this definition, a matching is a distance-1 matching and an induced matching is a distance-2 matching. For a matching M, we use $V(M)$ to denote the set of vertices saturated by M.

In the 1960s, Vizing [33,34] and, independently, Gupta [22] proved that $\Delta(G) \leq \chi'(G) \leq \Delta(G)+\mu(G)$, which is commonly called Vizing's Theorem. Using the palette $[\Delta(G)+\mu(G)]$, when can we extend a precoloring on a given edge set $F \subseteq E(G)$ to a proper edge coloring of G? Albertson and Moore [2] conjectured that if G is a simple graph, using the palette $[\Delta(G)+1]$, any precoloring on a distance-3 matching can be extended to a proper edge coloring of G. Edwards et al. [17] proposed a stronger conjecture: *For any graph G, using* *the palette* $[\Delta(G)+\mu(G)]$*, any precoloring on a distance-*2 *matching can be extended to a proper edge coloring of* G*.* Girão and Kang [18] verified the conjecture of Edwards et al. for distance-9 matchings. In this chapter, we improve the required distance from 9 to 3 for multigraphs with the maximum multiplicity at least 2 as follows.

Theorem 3.1.1. *Let* G *be a multigraph and* M *be a distance-*3 *matching of* G*. If* $\mu(G) \geq 2$ *and* M *is arbitrarily precolored from the palette* $[\Delta(G)+\mu(G)]$*, then there is a proper edge coloring of* G *using colors from* $[\Delta(G)+\mu(G)]$ *that agrees with the precoloring on* M*.*

The *density* of a graph G, denoted $\Gamma(G)$, is defined as

$$\Gamma(G) = \max\left\{\frac{2|E(H)|}{|V(H)|-1} : H \subseteq G, |V(H)| \geq 3 \text{ and } |V(H)| \text{ is odd}\right\}$$

if $|V(G)| \geq 3$ and $\Gamma(G) = 0$ otherwise. Note that for any $X \subseteq V(G)$ with odd $|X| \geq 3$, we have $\chi'(G[X]) \geq \frac{2|E(G[X])|}{|X|-1}$, where $G[X]$ is the subgraph of G induced by X. Therefore, $\chi'(G) \geq \lceil \Gamma(G) \rceil$. So, besides the maximum degree, the density provides another lower bound on the chromatic index of a graph. In the 1970s, Goldberg [19] and Seymour [28] independently conjectured that actually $\chi'(G) = \lceil \Gamma(G) \rceil$ provided $\chi'(G) \geq \Delta(G)+2$. The conjecture was commonly referred to as one of the most challenging problems in graph chromatic theory [30]. Chen et al. gave a proof of the Goldberg-Seymour Conjecture recently [14]. We assume that the Goldberg-Seymour Conjecture is true in this chapter.

3.2 Dense subgraphs

Throughout the rest of this chapter, we reserve the notation Δ and μ for the maximum degree and the maximum multiplicity of the graph G, respectively. A subgraph H of G is *k-dense* if $|V(H)|$ is odd and $|E(H)| = (|V(H)|-1)k/2$. Moreover, H is a *maximal k-dense subgraph* if there does not exist a k-dense subgraph H' containing H as a proper subgraph.

Lemma 3.2.1. [10] *Given a graph* G*, if* $\chi'(G) = k \geq \Delta(G)+1$*, then distinct maximal* k*-dense subgraphs of* G *are pairwise vertex-disjoint.*

Lemma 3.2.2. *Let G be a graph with $\chi'(G) = k$ and H be a k-dense subgraph of G. Then H is an induced subgraph of G with $\chi'(H) = \Gamma(H) = k$. Furthermore, for any coloring $\varphi \in \mathcal{C}^k(G)$, H is φ_H-elementary and strongly φ-closed.*

Proof. Since H is k-dense, by the definition, $|E(H)| = \frac{|V(H)|-1}{2}k$. Thus $k \leq \Gamma(H) \leq \chi'(H) \leq \chi'(G) = k$ implying $\chi'(H) = \Gamma(H) = k$. Thus H is an induced subgraph of G, since otherwise there exists a subgraph H' of G with $V(H') = V(H)$ such that $\chi'(H') \geq \Gamma(H') > k$, a contradiction to $\chi'(H') \leq \chi'(G) = k$. Since H has an odd order, the size of a maximum matching in H has size at most $(|V(H)| - 1)/2$. Therefore, under any k-edge-coloring φ of G, each color class in H is a matching of size exactly $(|V(H)| - 1)/2$. Thus every color in $[k]$ is missing at exactly one vertex of H or it appears exactly once in $\partial_G(H)$. Consequently, H is φ_H-elementary and strongly φ-closed. □

For $e \in E(G)$, let $V(e)$ denote the set of the two endvertices of e. The following lemma is a consequent of the Goldberg-Seymour Conjecture.

Lemma 3.2.3. *Let G be a multigraph and $e \in E(G)$. If e is a k-critical edge of G and $k \geq \Delta(G) + 1$, then $G - e$ has a k-dense subgraph H containing $V(e)$ such that e is also a k-critical edge of $H + e$.*

Proof. Clearly, $\chi'(G) = k+1$ and $\chi'(G-e) = k$. By the assumption of the Goldberg-Seymour Conjecture, $\chi'(G) = \lceil\Gamma(G)\rceil = k + 1$. So, there exists a subgraph H^* of odd order containing e such that $|E(H^*)| > (|V(H^*)|-1)k/2$. On the other hand, we have $\frac{2|E(H^*-e)|}{|V(H^*-e)|-1} \leq \lceil\Gamma(H^*-e)\rceil \leq \chi'(H^*-e) \leq \chi'(G-e) = k$, which in turn gives $|E(H^*-e)| \leq (|V(H^*)|-1)k/2$. Thus $|E(H^* - e)| = (|V(H^*)| - 1)k/2$. Then $k \leq \lceil\Gamma(H^* - e)\rceil \leq \chi'(H^* - e) \leq \chi'(G - e) = k$ and $k + 1 \leq \lceil\Gamma(H^*)\rceil \leq \chi'(H^*) \leq \chi'(G) = k + 1$, which implies that $k = \chi'(H^* - e) < \chi'(H^*) = k + 1$. Thus $H := H^* - e$ is a k-dense subgraph containing $V(e)$, and e is also a k-critical edge of $H + e$. □

The *diameter* of a graph G, denoted $\text{diam}(G)$, is the greatest distance between any pair of vertices in $V(G)$.

Lemma 3.2.4. *Let G be a multigraph with $\chi'(G) = k+1 \geq \Delta(G)+2$ and e be a k-critical edge of G. We have the following statements.*

(a) $G-e$ has a unique maximal k-dense subgraph H containing $V(e)$, and e is also a k-critical edge of $H+e$.

(b) For any $\varphi \in \mathcal{C}^k(G-e)$, H is φ_H-elementary and strongly φ-closed.

(c) If $\chi'(G) = \Delta(G)+\mu(G)$, then $\Delta(H+e) = \Delta(G)$, $\mu(H+e) = \mu(G)$ and $\mathrm{diam}(H+e) \leq \mathrm{diam}(H) \leq 2$.

Proof. By Lemma 3.2.3, $G-e$ contains a k-dense subgraph H containing $V(e)$ and e is also a k-critical edge of $H+e$. We may assume that H is a maximal k-dense subgraph, and the uniqueness of H is a direct consequence of Lemma 3.2.1. This proves (a). By applying Lemma 3.2.2 on $G-e$, we immediately have statement (b).

For (c), by (a) and Vizing's Theorem, $\Delta(G)+\mu(G) = \chi'(G) = \chi'(H+e) \leq \Delta(H+e)+\mu(H+e) \leq \Delta(G)+\mu(G)$ implying that $\Delta(H+e) = \Delta(G) = \Delta$ and $\mu(H+e) = \mu(G) = \mu$. For any $\varphi \in \mathcal{C}^k(G-e)$, H is φ_H-elementary by (b). For any $x \in V(H)$, with respect to φ_H, all the colors missing at other vertices of H present at x. Note that $k = \Delta+\mu-1$. For each vertex $v \in V(H)$, we have that $|\overline{\varphi}_H(v)| = k - d_H(v) \geq k-\Delta = \mu-1$ if $v \notin V(e)$, and $|\overline{\varphi}_H(v)| = k-d_H(v)+1 \geq k-\Delta+1 \geq (\mu-1)+1$ if $v \in V(e)$. Denote $|V(H)|$ by n. We then have $d_H(x) \geq |\bigcup_{v\in V(H), v\neq x} \overline{\varphi}_H(v)| \geq (k-\Delta)(n-1)+1 = (\mu-1)(n-1)+1$.

Since $\mu(H) \leq \mu(G) = \mu$, we get $|N_H(x)| \geq \frac{d_H(x)}{\mu} \geq \frac{(\mu-1)(n-1)+1}{\mu}$, where $N_H(x)$ is the neighbor set of x in H. Since $\mu \geq 2$, we have $\frac{(\mu-1)(n-1)+1}{\mu} \geq \frac{n}{2}$. Hence, every vertex in H is adjacent to at least half vertices in H. Consequently, every two vertices of H share a common neighbor, which in turn gives $\mathrm{diam}(H) \leq 2$. This proves (c). □

An *i-edge* is an edge colored with the color i. The following technical lemma will be used several times in our proof.

Lemma 3.2.5. *Let G be a graph with $\chi'(G) = k$ and H be a k-dense subgraph of G. Let ψ and φ respectively be k-edge-colorings of H and $G-E(H)$ such that colors on edges in $\partial_G(H)$ are pairwise distinct under φ. The following two statements hold.*

(a) If $k \geq \Delta(G)$, then by renaming color classes of ψ on $E(H)$, we can obtain a (proper) k-edge-coloring of G by combining φ and the modified coloring based on ψ.

(b) For any fixed color $i \in [k]$, if $k \geq \Delta(G)+1$, then by renaming other color classes of ψ on $E(H)$ we can obtain a coloring of G such that all color classes are matchings except the i-edges. The only exception is as follows: exactly one i-edge from $E(H)$ and exactly one i-edge from $\partial_G(H)$ share an endvertex.

Proof. Since $\chi'(G) = k$ and H is k-dense, $\chi'(H) = k$ and H is ψ-elementary by Lemma 3.2.2. This following fact will be used to combine an edge coloring of H and an edge coloring of $G - E(H)$ into an edge coloring of G: for any distinct $u, v \in V(H)$, $\overline{\psi}(u) \cap \overline{\psi}(v) = \emptyset$, and no two colors on edges in $\partial_G(H)$ under φ are the same.

For (a), we have $|\overline{\psi}(v)| = k - d_H(v) \geq \Delta(G) - d_H(v) \geq d_{G-E(H)}(v) = |\varphi(v)|$ for each $v \in V(H)$. So, by renaming color classes of ψ on $E(H)$, we may assume that $\varphi(v) \subseteq \overline{\psi}(v)$ for each $v \in V(H)$. The combination of φ and the modified coloring based on ψ gives a desired proper edge coloring of G.

For (b), under the condition $k \geq \Delta(G)+1$, we have $|\overline{\psi}(v)| = k - d_H(v) \geq \Delta(G)+1-d_H(v) \geq d_{G-E(H)}(v)+1 = |\varphi(v)|+1$ for each $v \in V(H)$. So $|\overline{\psi}(v)\backslash\{i\}| \geq |\varphi(v)\backslash\{i\}|$. Notice that when $i \in \overline{\psi}(v) \cap \overline{\varphi}(v)$, we need $|\overline{\psi}(v)| - 1 \geq |\varphi(v)|$ to ensure the inequality above, where the condition $k \geq \Delta(G)+1$ is applied. By renaming color classes of ψ on $E(H)$ except the i-edges (keeping all i-edges unchanged and other color classes not renamed by i), we may assume that $\varphi(v)\backslash\{i\} \subseteq \overline{\psi}(v)$ for each $v \in V(H)$. Again, the combination of φ and the modified coloring based on ψ gives a desired coloring of G. The only case that the set of i-edges is not a matching is when exactly one i-edge from $E(H)$ and exactly one i-edge from $\partial_G(H)$ share an endvertex, since colors on edges in $\partial_G(H)$ are pairwise distinct under φ. $\square$

3.3 Refinements of multi-fans and some consequences

Let G be a graph with an edge $e \in E_G(x, y)$, and φ be a proper edge coloring of G or $G - e$. A sequence $F = (x, e_0, y_0, e_1, y_1, \ldots, e_p, y_p)$ with integer $p \geq 0$ consisting of vertices and distinct edges is called a (general) *multi-fan* at x with respect to e and φ if $e_0 = e$,

$y_0 = y$, for each $i \in [p]$, $e_i \in E_G(x, y_i)$ and there is a vertex y_j with $0 \le j \le i-1$ such that $\varphi(e_i) \in \overline{\varphi}(y_j)$. Notice that the definition of a (general) multi-fan in this chapter is slightly general than the one in Chapter 2 since the edge e may be colored in G. We say a multi-fan F is *maximal* if there is no multi-fan containing F as a proper subsequence. Similarly, we say a multi-fan F is *maximal without any i-edge* if F does not contain any i-edge and there is no multi-fan without any i-edge containing F as a proper subsequence. The set of vertices and edges contained in F are denoted by $V(F)$ and $E(F)$, respectively. Let $e_G(x, y) = |E_G(x, y)|$ for $x, y \in V(G)$. Note that a multi-fan may have repeated vertices. By $e_F(x, y_i)$ for some $y_i \in V(F)$ we mean the number of edges joining x and y_i in F.

Let $s \ge 0$ be an integer. A *linear sequence* $S = (y_0, e_1, y_1, \ldots, e_s, y_s)$ at x from y_0 to y_s in G is a sequence consisting of distinct vertices and distinct edges such that $e_i \in E_G(x, y_i)$ for $i \in [s]$ and $\varphi(e_i) \in \overline{\varphi}(y_{i-1})$ for $i \in [s]$. Clearly for any $y_j \in V(F)$, the multi-fan F contains a linear sequence at x from y_0 to y_j (take a shortest sequence $(y_0, e_1, y_1, \ldots, e_j, y_j)$ of vertices and edges with the property that $e_i \in E_G(x, y_i) \cap E(F)$ for $i \in [j]$ and $\varphi(e_i) \in \overline{\varphi}(y_{i-1})$ for $i \in [j]$). The following local edge recoloring operation will be used in our proof. A *shifting* from y_i to y_j in the linear sequence S is an operation that replaces the current color of e_t by the color of e_{t+1} for each $i \le t \le j-1$ with $1 \le i < j \le s$. Note that the shifting does not change the color of e_j, where e_j joins x and y_j, so the resulting coloring after a shifting is not a proper coloring. In our proof we will uncolor or recolor the edge e_j to make the resulting coloring proper. We also denote by $V(S)$ and $E(S)$ the set of vertices and the set of edges contained in the linear sequence S, respectively.

The following lemma is a genelization of Lemma 2.3.1 for graphs with at least one critical edge.

Lemma 3.3.1. [20,30] *Let G be a graph, $e \in E_G(x, y)$ be a k-critical edge and $\varphi \in \mathcal{C}^k(G-e)$ with $k \ge \Delta(G)$. Let $F = (x, e, y_0, e_1, y_1, \ldots, e_p, y_p)$ be a multi-fan at x with respect to e and φ, where $y_0 = y$. Then the following statements hold.*

(a) $V(F)$ is φ-elementary, and each edge in $E(F)$ is a k-critical edge of G.

(b) If $\alpha \in \overline{\varphi}(x)$ and $\beta \in \overline{\varphi}(y_i)$ for $0 \le i \le p$, then $P_x(\alpha, \beta) = P_{y_i}(\alpha, \beta)$.

(c) If F is a maximal multi-fan at x with respect to e and φ, then x is adjacent in G to at least $\chi'(G)-d_G(y)-e_G(x,y)+1$ vertices z in $V(F)\backslash\{x,y\}$ such that $d_G(z)+e_G(x,z)=\chi'(G)$.

A Δ-*vertex* in G is a vertex with degree exactly Δ in G. A Δ-*neighbor* of a vertex v in G is a neighbor of v that is a Δ-vertex in G.

Lemma 3.3.2. *Let G be a multigraph with maximum degree Δ and maximum multiplicity $\mu \geq 2$. Let $e \in E_G(x,y)$ and $k = \Delta + \mu - 1$.*

Assume that $\chi'(G) = k+1$, e is k-critical and $\varphi \in \mathcal{C}^k(G-e)$. Let $F = (x, e, y_0, e_1, y_1, \ldots, e_p, y_p)$ be a multi-fan at x with respect to e and φ, where $y_0 = y$. Then the following statements hold.

(a) If F is maximal, then x is adjacent in G to at least $\Delta + \mu - d_G(y) - e_G(x,y) + 1$ vertices z in $V(F)\backslash\{x,y\}$ such that $d_G(z) = \Delta$ and $e_G(x,z) = \mu$.

(b) If F is maximal, $d_G(y) = \Delta$ and x has only one Δ-neighbor z' in G from $V(F)\backslash\{x,y\}$, then $e_F(x,z) = e_G(x,z) = \mu$ for all $z \in V(F)\backslash\{x\}$ and $d_G(z) = \Delta - 1$ for all $z \in V(F)\backslash\{x,y,z'\}$.

(c) For $i \in [k]$ and $i \notin \overline{\varphi}(y)$, if F is maximal without any i-edge, then F not containing any Δ-vertex of G from $V(F)\backslash\{x,y\}$ implies that $d_G(y) = \Delta$, and there exists a vertex $z^* \in V(F)\backslash\{x,y\}$ with $i \in \overline{\varphi}(z^*)$ such that $d_G(z^*) = \Delta - 1$.

Assume that $\chi'(G) = k$, $\varphi \in \mathcal{C}^k(G)$ and $V(G)$ is φ-elementary. Then the following statement holds.

(d) If a multi-fan F' is maximal at x with respect to e and φ in G, then x having no Δ-neighbor in G from $V(F')$ implies that $d_G(z) = \Delta - 1$ for all $z \in V(F')\backslash\{x\}$ and every edge in F' is colored by a missing color at some vertex in $V(F')$. Furthermore, for $i \in [k]$ and $\varphi(e) \notin \overline{\varphi}(V(F'))$, if F' is maximal without any i-edge, then F' not containing any Δ-vertex in G from $V(F')\backslash\{x\}$ implies that there exists a vertex $z^* \in V(F')\backslash\{x\}$ with $i \in \overline{\varphi}(z^*)$ such that $d_G(z^*) = \Delta - 1$.

Proof. For statements (a), (b) and (c), $V(F)$ is φ-elementary by Lemma 3.3.1(a). Statement (a) holds easily by Lemma 3.3.1(c). Assume that there are q distinct vertices in $V(F)\backslash\{x\}$.

For (b), we have

$$\begin{aligned} q\mu &\geq \sum_{z\in V(F)\backslash\{x\}} e_G(x,z) \geq \sum_{z\in V(F)\backslash\{x\}} e_F(x,z) = 1 + \sum_{z\in V(F)\backslash\{x\}} |\overline{\varphi}(z)| \\ &\geq 1 + (k-\Delta+1) + (k-\Delta) + (q-2)(k-\Delta+1) = q(k-\Delta+1) = q\mu, \end{aligned}$$

as $|\overline{\varphi}(y)| = k-\Delta+1$, $|\overline{\varphi}(z')| = k-\Delta$ and $|\overline{\varphi}(z)| \geq k-\Delta+1$ for $z \in V(F)\backslash\{x,y,z'\}$. Therefore, $e_F(x,z) = e_G(x,z) = \mu$ for each $z \in V(F)\backslash\{x\}$ and $d_G(z) = \Delta - 1$ for each $z \in V(F)\backslash\{x,y,z'\}$. This proves (b).

Now for (c), we must have that there exists a vertex $z^* \in V(F)\backslash\{x,y\}$ with $i \in \overline{\varphi}(z^*)$, since otherwise by (a), x has at least one Δ-neighbor in G from $V(F)\backslash\{x,y\}$, a contradiction. Since $V(F)$ is φ-elementary, x must be incident with an i-edge. Since now there is no i-edge in F and $i \in \overline{\varphi}(z^*)$, we have

$$\begin{aligned} q\mu &\geq \sum_{z\in V(F)\backslash\{x\}} e_G(x,z) \geq \sum_{z\in V(F)\backslash\{x\}} e_F(x,z) = 1 + (|\overline{\varphi}(z^*)| - 1) + \sum_{z\in V(F)\backslash\{x,z^*\}} |\overline{\varphi}(z)| \\ &\geq 1 + k - \Delta + (q-1)(k-\Delta+1) = q(k-\Delta+1) = q\mu. \end{aligned}$$

Therefore, $d_G(y) = \Delta$ and $d_G(z) = \Delta - 1$ for each $z \in V(F)\backslash\{x,y\}$. This proves (c).

Statement (d) follows from similar calculations as in the proof of (b) and (c). □

Let G be a graph with maximum degree Δ and maximum multiplicity μ. Berge and Fournier [5] strengthened the classical Vizing's Theorem by showing that if M^* is a maximal matching of G, then $\chi'(G - M^*) \leq \Delta + \mu - 1$. An edge $e \in E_G(x,y)$ is *fully G-saturated* if $d_G(x) = d_G(y) = \Delta$ and $e_G(x,y) = \mu$. For every graph G with $\chi'(G) = \Delta + \mu$, observe that G contains a $(\Delta + \mu - 1)$-critical subgraph H with $\chi'(H) = \Delta + \mu$ and $\Delta(H) = \Delta$ by Lemma 3.2.4(c), and G contains at least two fully G-saturated edges by Lemma 3.3.2(a).

Stiebitz et al.[Page 41 Statement (a), [30]] obtained the following generalization of Vizing's Theorem with an elegant short proof: *Let G be a graph and let $k \geq \Delta + \mu$ be an integer. Then there is a k-edge-coloring φ of G such that every edge e with $\varphi(e) = k$ is fully G-saturated.* We observe that their proof actually gives a slightly stronger result which also generalizes the Berge-Fournier theorem as follows.

Lemma 3.3.3. *Let G be a graph, and M and M' be two vertex-disjoint matchings of G. If every edge of M' is fully G-saturated and M' is maximal subject to this property, then* $\chi'(G-(M\cup M'))\le \Delta(G)+\mu(G)-1$.

Proof. Let $G' = G-(M\cup M')$. Note that every vertex $v\in V(M\cup M')$ has $d_{G'}(v)\le \Delta-1$. By the maximality of M', $G-V(M\cup M')$ contains no fully G-saturated edges. So, G' does not have a fully G-saturated edge. By the observation of graphs with chromatic index $\Delta+\mu$ and Lemma 3.3.2(a), $\chi'(G')\le \Delta+\mu-1$, since otherwise $\Delta(G')=\Delta$ and there exist at least two fully G-saturated edges in one multi-fan centered at a Δ-vertex, a contradiction. □

Lemma 3.3.3 has the following consequence.

Corollary 3.3.4. *Let G be a graph. If M is a matching such that every edge in M is fully G-saturated and M is maximal subject to this property, then* $\chi'(G-M)\le \Delta(G)+\mu(G)-1$.

We strengthen Lemma 3.3.3 for multigraphs G with $\mu(G)\ge 2$ as follows.

Lemma 3.3.5. *For a fixed matching M of a graph G, if $\mu(G)\ge 2$ and $\chi'(G-M)=\Delta(G)+\mu(G)$, then there exists a matching M^* of $G-V(M)$ such that $\chi'(G-(M\cup M^*))=\Delta(G)+\mu(G)-1=:k$ and every edge $e\in M^*$ is k-critical and fully G-saturated in the graph H_e+e, where H_e is the unique maximal k-dense subgraph of $G-(M\cup M^*)$ containing $V(e)$.*

Proof. Let M^* be a matching of $G-V(M)$ consisting of fully G-saturated edges. We further choose M^* such that M^* is maximal. By Lemma 3.3.3, $\chi'(G-(M\cup M^*))=k$. If there exists $e\in M^*$ such that $\chi'(G-(M\cup M^*\backslash\{e\}))=k$, we remove e out of M^*. Thus we may assume that for each $e\in M^*$, $\chi'(G-(M\cup M^*\backslash\{e\}))=k+1$, i.e., each e is a k-critical edge of $G-(M\cup M^*\backslash\{e\})$. By Lemma 3.2.4($a$), there exists a unique maximal k-dense subgraph H_e of $G-(M\cup M^*)$ such that $V(e)\subseteq V(H_e)$ and e is also a k-critical edge of H_e+e. Notice that $\Delta(H_e+e)=\Delta$ and $\mu(H_e+e)=\mu$ by Lemma 3.2.4(c). It is now only left to show that each $e\in M^*$ is full G-saturated in the graph H_e+e. Suppose on the contrary that there exists $e\in M^*$ such that e is not fully G-saturated in H_e+e.

Since e is a k-critical edge of $G-(M\cup M^*\backslash\{e\})$, we let $\varphi\in\mathcal{C}^k(G-(M\cup M^*))$. By Lemma 3.2.2, H_e is φ_{H_e}-elementary and strongly φ-closed. Let $V(e)=\{x,y\}$ and F_x be a maximum multi-fan at x with respect to e and φ_{H_e}. By Lemma 3.3.2(a), x has a Δ-neighbor, say x_1, in H_e from $V(F_x)\backslash\{x,y\}$. By Lemma 3.3.1(a), the edge $e_{xx_1}\in E_G(x,x_1)$ in F_x is also a k-critical edge of H_e+e. By Lemma 3.3.2(a) again, in a maximum multi-fan at x_1 there exists a fully G-saturated edge e'. Let $M'=(M^*\backslash\{e\})\cup\{e'\}$. Since every vertex of $V(M\cup M^*)$ has degree less than Δ in $G-(M\cup M^*)$, it follows that $M\cup M'$ is a matching of G. Let $H_{e'}=H_e+e-e'$. Clearly, $H_{e'}$ is also k-dense. Applying Lemma 3.3.1(a), we see that e' is also a k-critical edge of H_e+e. Thus $\chi'(H_{e'})=k$ and $H_{e'}$ is also an induced subgraph of $G-(M\cup M')$ by Lemma 3.2.2. Moreover, $H_{e'}$ is a maximal k-dense subgraph of $G-(M\cup M')$, since otherwise there exists a k-dense subgraph H' containing $H_{e'}$ as a proper subgraph which implies that the k-dense subgraph $H'+e'-e$ is also a k-dense subgraph containing H_e as a proper subgraph in $G-(M\cup M^*)$, a contradiction to the maximality of H_e. As H_e is strongly φ-closed, colors on edges of $\partial_{G-(M\cup M')}(H_{e'})=\partial_{G-(M\cup M^*)}(H_e)$ are pairwise distinct. Applying Lemma 3.2.5(a) on any k-edge-coloring of $H_{e'}$ and the k-edge-coloring of $G-(M\cup M'\cup E(H_{e'}))$, we have $\chi'(G-(M\cup M'))=k$. In order to claim that we can replace e by e' in M^*, and so repeat the same process for every edge f of M^* that is not fully G-saturated in H_f+f (H_f is the maximal k-dense subgraph of $G-(M\cup M^*)$ with $V(f)\subseteq V(H_f)$), we discuss that this replacement will not affect the properties of other edges in M^* as follows.

By Lemmas 3.2.1 and 3.2.2, maximal k-dense subgraphs of $G-(M\cup M^*)$ are induced and vertex-disjoint. Thus for any $f\in M^*\backslash\{e\}$, either $V(H_f)\cap V(H_e)=\emptyset$ or $H_f=H_e$. If $V(H_f)\cap V(H_e)=\emptyset$, then H_f is still the induced maximal k-dense subgraph of $G-(M\cup M')$ containing $V(f)$ and f is k-critical in H_f+f. If $H_f=H_e$, then as $H_{e'}$ is an induced maximal k-dense subgraph of $G-(M\cup M')$ with $V(H_e)=V(H_{e'})$, it follows that $H_f+e-e'=H_{e'}$ is the maximal k-dense subgraph of $G-(M\cup M')$ containing $V(f)$ and f is k-critical in $H_f+e-e'+f$ by Lemma 3.2.4(a). As $V(f)\cap V(e)=\emptyset$ and $V(f)\cap V(e')=\emptyset$, the property that whether or not f is fully G-saturated in H_f+f is not changed after replacing e by e' in

M^*. Therefore, by repeating the replacement process as for the edge e above for every edge f of M^* that is not fully G-saturated in $H_f + f$, we may assume that each edge $e \in M^*$ is fully G-saturated in $H_e + e$. The proof is completed. □

3.4 Proof of Theorem 3.1.1

Proof. Let $k = \Delta + \mu - 1$ and $\Phi : M \to [\Delta + \mu]$ be a given precoloring on M. Note that $\chi'(G - M) \leq k + 1$ by Vizing's Theorem. The conclusion of Theorem 3.1.1 holds easily if $\chi'(G - M) \leq k$ with the reason as follows. For any k-edge-coloring ψ of $G - M$, if there exists $e \in E(G - M)$ such that e is adjacent in G to an edge $f \in M$ and $\psi(e) = \Phi(f)$, we recolor each such e with the color $\Delta + \mu$ and get a new coloring ψ' of $G - M$. Under ψ', the edges colored by $\Delta + \mu$ form a matching in G since M is a distance-3 matching. Thus the combination of Φ and ψ' is a $(k + 1)$-edge-coloring of G. Therefore, in the remainder of the proof, we assume $\chi'(G - M) = k + 1$.

Let $M_{\Delta+\mu}$ be the set of edges precolored with $\Delta + \mu$ in M under Φ. For any matching $M^* \subseteq G - V(M)$ and any $(k+1)$-edge-coloring or k-edge-coloring φ of $G - (M \cup M^*)$, denote the $\Delta + \mu$ color class of φ by $E^{\varphi}_{M^*}$. In particular, $E^{\varphi}_{M^*} = \emptyset$ if φ is a k-edge-coloring. We introduce the following notation. For $f \in E_G(u, v) \cap M$, if there exists $f_1 \in E(G - (M \cup M^*))$ such that $V(f_1) \cap V(f) = \{u\}$ and $\varphi(f_1) = \Phi(f)$, we call f **T1-improper** (Type 1 improper) at u if $V(f_1) \cap V(M^*) = \emptyset$, and **T2-improper** (Type 2 improper) at u if $V(f_1) \cap V(M^*) \neq \emptyset$. If f is T1-improper or T2-improper at u, we say that f is **improper** at u. Define

$$
\begin{aligned}
E_1(M^*, \varphi) &= \{f_1 \in E(G - (M \cup M^*)) : f_1 \text{ is adjacent in } G \text{ to a T1-improper edge}\}, \\
E_2(M^*, \varphi) &= \{f_1 \in E(G - (M \cup M^*)) : f_1 \text{ is adjacent in } G \text{ to a T2-improper edge}\}.
\end{aligned}
$$

Observe that $E_1(M^*, \varphi) \cup E_2(M^*, \varphi)$ is a matching since M is a distance-3 matching in G. We call the triple $(M^*, E^{\varphi}_{M^*}, \varphi)$ **prefeasible** if the following conditions are satisfied:

(a) $M_{\Delta+\mu} \cup M^* \cup E^{\varphi}_{M^*}$ is a matching;

(b) each $e \in M^*$ such that e is adjacent in G to an edge of $E_2(M^*, \varphi)$, e is k-critical and fully G-saturated in the graph $H_e + e$, where H_e is the unique maximal k-dense subgraph of $G - (M \cup M^*)$ containing $V(e)$;

(c) the colors on edges of $\partial_{G-(M\cup M^*)}(H_e)$ are all distinct under φ.

Let $(M^*, E^{\varphi}_{M^*}, \varphi)$ be a prefeasible triple. Since $M \cup M^*$ is a matching in G, if $(M^*, E^{\varphi}_{M^*}, \varphi)$ also satisfies *Condition* (d): $|E_1(M^*, \varphi)| = |E_2(M^*, \varphi)| = 0$, then by assigning the color $\Delta + \mu$ to all edges of M^*, we obtain a (proper) $(k+1)$-edge-coloring of G, where the $(k+1)$-edge-coloring is the combination of the precoloring Φ on M, the coloring using the color $\Delta + \mu$ on M^*, and the coloring φ of $G - (M \cup M^*)$. Thus we define a **feasible** triple $(M^*, E^{\varphi}_{M^*}, \varphi)$ as one that satisfies Conditions (a)-(d).

The rest of the proof is devoted to showing the existence of a feasible triple $(M^*, E^{\varphi}_{M^*}, \varphi)$ of G. Our main strategy is to first fix a particular prefeasible triple $(M_0^*, E^{\varphi_0}_{M_0^*}, \varphi_0)$, then modify it step by step into a feasible triple $(M^*, E^{\varphi}_{M^*}, \varphi)$. In particular, we will choose M_0^* and φ_0 such that $E^{\varphi_0}_{M_0^*} = \emptyset$. At the end, when we modify φ_0 into φ, we will ensure that the $\Delta + \mu$ color class of G is $M_{\Delta+\mu} \cup M^* \cup E_1(M_0^*, \varphi_0) \cup E_2(M_0^*, \varphi_0)$. The process is first to modify M_0^* and φ_0 at the same time to deduce the number of T2-improper edges.

By Lemma 3.3.5, there exists a matching M_0^* of $G - V(M)$ such that $\chi'(G - (M \cup M_0^*)) = k$ and each edge $e \in M_0^*$ is k-critical and fully G-saturated in $H_e + e$, where H_e is the unique maximal k-dense subgraph of $G - (M \cup M_0^*)$ containing $V(e)$. By Lemmas 3.2.1 and 3.2.2, H_e is induced in $G - (M \cup M_0^*)$ with $\chi'(H_e) = k$, and H_e and $H_{e'}$ are either identical or vertex-disjoint for any $e' \in M_0^* \setminus \{e\}$. Moreover, by Lemma 3.2.4, $\mathrm{diam}(H_e + e) \le \mathrm{diam}(H_e) \le 2$, and H_e is $(\varphi_0)_{H_e}$-elementary and strongly φ_0-closed in $G - (M \cup M_0^*)$. As $\chi'(G - M) = k+1$, we have $|M_0^*| \ge 1$. Let φ_0 be a k-edge-coloring of $G - (M \cup M_0^*)$. Thus $E^{\varphi_0}_{M_0^*} = \emptyset$. Obviously, the triple $(M_0^*, \emptyset, \varphi_0)$ is prefeasible, which we take as our initial triple.

For $(M_0^*, \emptyset, \varphi_0)$, if $|E_1(M_0^*, \varphi_0)| = |E_2(M_0^*, \varphi_0)| = 0$, then we are done. If $|E_1(M_0^*, \varphi_0)| \ge 1$ and $|E_2(M_0^*, \varphi_0)| = 0$, then we recolor each edge in $E_1(M_0^*, \varphi_0)$ with the color $\Delta + \mu$ to produce a $(k+1)$-edge-coloring φ_1 of $G - (M \cup M_0^*)$, since $E_1(M_0^*, \varphi_0)$ is a matching. Then as $|E_1(M_0^*, \varphi_1)| = |E_2(M_0^*, \varphi_1)| = 0$ and $M_{\Delta+\mu} \cup M_0^* \cup E_{M_0^*}^{\varphi_1}$ is a matching, it follows that the new triple $(M_0^*, E_1(M_0^*, \varphi_0), \varphi_1)$ is feasible. Then we are also done.

Therefore, we assume that $|E_1(M_0^*, \varphi_0)| \geq 0$ and $|E_2(M_0^*, \varphi_0)| \geq 1$. Recall that for each $e \in M_0^*$, e is fully G-saturated in $H_e + e$. Thus we have the following observation: for an edge $f_{uv} \in M$ with $V(f_{uv}) = \{u, v\}$, if $\{u, v\} \cap V(H_e) = \emptyset$ for any $e \in M_0^*$, then f_{uv} cannot be a T2-improper edge.

Since $|E_2(M_0^*, \varphi_0)| \geq 1$, we consider one T2-improper edge in M, say f_{uv} with $V(f_{uv}) = \{u, v\}$. Suppose that f_{uv} is T2-improper at u and $\Phi(f_{uv}) = i \in [k]$ (as φ_0 is a k-edge-coloring, $i \neq k+1 = \Delta + \mu$). Then there exist $e_{xy} \in E_G(x, y) \cap M_0^*$ and a maximal k-dense subgraph H of $G - (M \cup M_0^*)$ such that $V(e_{xy}) \subseteq V(H)$ and f_{uv} and e_{xy} are both adjacent in G to an i-edge $e_{yu} \in E_H(y, u)$. Since M is a distance-3 matching and $\text{diam}(H) \leq 2$, we have $V(H) \cap V(M \setminus \{f_{uv}\}) = \emptyset$. We will modify φ_0 into a new coloring such that f_{uv} is not T2-improper at u under this new coloring and that no other edge of M_0^* is changed into a new T2-improper edge. We consider the three cases below regarding the location of f_{uv} with respect to H.

Case 1: f_{uv} is not improper at v, or f_{uv} is T1-improper at v but $v \notin V(H)$.

Let F_x be a maximal multi-fan at x with respect to e_{xy} and $(\varphi_0)_H$ in $H + e_{xy}$. There exist at least one Δ-vertex in $V(F_x) \setminus \{x, y\}$ by Lemma 3.3.2(a) and a linear sequence at x from y to this Δ-vertex in F_x. We consider two subcases as follows.

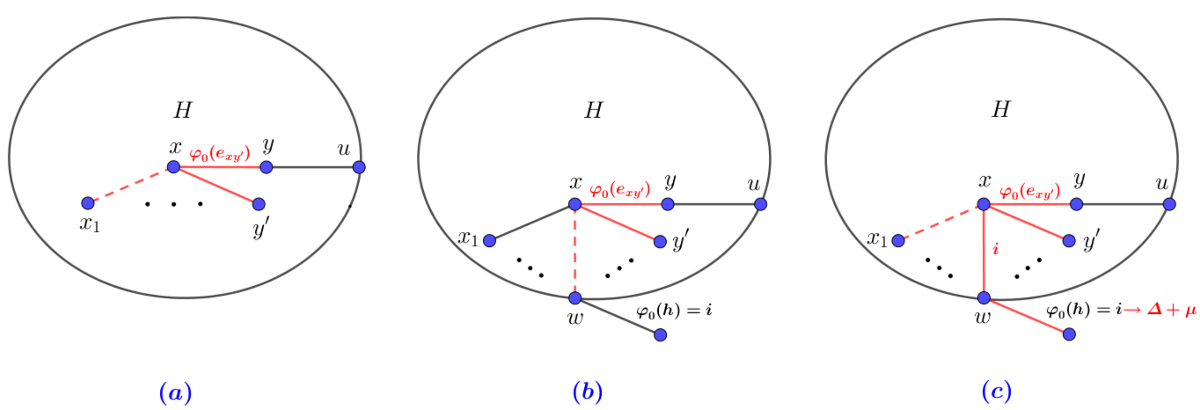

Figure 3.1. Operations I, II and III in Case 1.

Subcase 1.1: $V(F_x) \setminus \{x, y\}$ has a Δ-vertex x_1 and there is a linear sequence S at x from y to x_1 such that S contains no i-edge or S contains no vertex w such that w is incident with an i-edge of $\partial_{G-(M\cup M_0^*)}(H)$.

Let $S = (y, e_{xy'}, y', \ldots, e_{xx_1}, x_1)$ be the linear sequence (where $y' = x_1$ is possible). We apply Operation I as follows: apply a shifting in S from y to x_1, color e_{xy} with $\varphi_0(e_{xy'})$, uncolor e_{xx_1}, and replace e_{xy} by e_{xx_1} in M_0^*. (See Figure 3.1(a), where the edge of the dashed line represents the uncolored edge.) Since x_1 is not incident with any edge in $M \cup M_0^*$, $M_1^* := (M_0^* \setminus \{e_{xy}\}) \cup \{e_{xx_1}\}$ is a matching. Denote $H_1 := H + e_{xy} - e_{xx_1}$. Let ψ be the k-edge coloring of H_1 after Operation I. Note that for any vertex $z \in V(H_1)$ that is incident with an edge of $\partial_{G-(M\cup M_1^*)}(H_1)$, if $\overline{\psi}(z) \neq \overline{(\varphi_0)}_H(z)$, then $z \in V(S)$. By the condition of Subcase 1.1 and Operation I, there is no such vertex w such that w is incident with both an i-edge of $E(S)$ and an i-edge of $\partial_{G-(M\cup M_1^*)}(H_1)$. Thus we can rename some color classes of ψ but keep the color i unchanged to match all colors on edges of $\partial_{G-(M\cup M_1^*)}(H_1)$. In this way we obtain a (proper) k-edge-coloring φ_1 of $G - (M \cup M_1^*)$ by Lemma 3.2.5(b).

We claim that $(M_1^*, \emptyset, \varphi_1)$ is a prefeasible triple. As $M_{\Delta+\mu} \cup M_1^*$ is a matching, we verify that M_1^* and φ_1 satisfy the corresponding conditions. Clearly H_1 is k-dense with $V(H_1) = V(H)$ and $\partial_{G-(M\cup M_1^*)}(H_1) = \partial_{G-(M\cup M_0^*)}(H)$ and $\chi'(H_1) = \chi'(H) = k$, and e_{xx_1} is k-critical and fully G-saturated in $H_1 + e_{xx_1}$. Furthermore, as distinct maximal k-dense subgraphs are vertex-disjoint we know that each edge $e \in M_1^* \setminus \{e_{xx_1}\}$ is still contained in a k-dense subgraph of $G - (M \cup M_1^*)$ such that e is k-critical and fully G-saturated in the graph $H_e + e$ if e is adjacent in G to an edge of $E_2(M_1^*, \varphi_1)$, where H_e is the unique maximal k-dense subgraph of $G - (M \cup M_0^*)$ containing $V(e)$ if H_e and H_1 are vertex-disjoint, and $H_e = H_1$ otherwise. Since φ_1 is a k-edge-coloring of $G - (M \cup M_1^*)$, H_e is strongly φ_1-closed for each $e \in M_1^*$. Therefore, $(M_1^*, \emptyset, \varphi_1)$ is a prefeasible triple.

Next, we claim that $|E_2(M_1^*, \varphi_1)| = |E_2(M_0^*, \varphi_0)| - 1$. Note that under φ_1, we still have $\varphi_1(e_{yu}) = i$. Since $e_{xy}, e_{yu} \in E(H_1)$, $e_{xx_1} \in M_1^*$ and e_{xx_1} is not adjacent to e_{yu} in $G - (M \cup M_1^*)$, we see that now f_{uv} is no longer T2-improper at u but T1-improper at u with respect to M_1^* and φ_1. For any edge $f \in M \setminus \{f_{uv}\}$, since both x and x_1 are Δ-

vertices of $H + e_{xy}$ and $V(H_1) \cap V(M \setminus \{f_{uv}\}) = \emptyset$, we see that the distance between f and e_{xx_1} in $G - (M \cup M_1^*)$ is at least 2. Thus the property of f being T1-improper or T2-improper is not changed under M_1^* and φ_1. Thus the new triple $(M_1^*, \emptyset, \varphi_1)$ is prefeasible with $|E_1(M_1^*, \varphi_1)| = |E_1(M_0^*, \varphi_0)| + 1$ and $|E_2(M_1^*, \varphi_1)| = |E_2(M_0^*, \varphi_0)| - 1$, and so we can consider $(M_1^*, \emptyset, \varphi_1)$ instead.

Subcase 1.2: For any Δ-vertex in $V(F_x) \setminus \{x, y\}$, any linear sequence from y to this Δ-vertex contains both an i-edge h_i and a vertex w such that w is incident with an i-edge h of $\partial_{G-(M\cup M_0^*)}(H)$.

Let $F \subseteq F_x$ be the maximal multi-fan at x without any i-edge with respect to e_{xy} and $(\varphi_0)_H$. By the condition of Subcase 1.2, F does not contain any Δ-vertex from $V(F)\backslash\{x, y\}$ in H. By Lemma 3.3.2(c), there exists a vertex $z^* \in V(F)\backslash\{x, y\}$ with $i \in \overline{(\varphi_0)}_H(z^*)$ and $d_H(z^*) = \Delta - 1$. Since $V(F_x)$ is $(\varphi_0)_H$-elementary by Lemma 3.3.1(a) and every color on edges of $\partial_{G-(M\cup M_0^*)}(H)$ under φ_0 is a missing color at some vertex of H under $(\varphi_0)_H$, it follows that $z^* = w$, i.e., $d_H(w) = \Delta - 1$ and $d_{G-(M\cup M_0^*)}(w) = \Delta$. Thus the i-edge h is the only edge incident with w from $\partial_{G-(M\cup M_0^*)}(H)$, and w is not adjacent in G to any edge from $M \cup M_0^*$. Let $S = (y, e_{xy'}, y', \ldots, e_{xx_1}, x_1)$ be a linear sequence at x from y to x_1. Notice that w is in S by the condition of Subcase 1.2. We consider the following two subcases according whether the boundary i-edge h belongs to $E_1(M_0^*, \varphi_0)$.

Subcase 1.2.1: $h \notin E_1(M_0^*, \varphi_0)$, i.e., h is not adjacent in G to any precolored i-edge in M.

Let $e_{xw} \in E_H(x, w)$ be an edge in S. We apply Operation II as follows: apply a shifting in S from y to w, color e_{xy} with $\varphi_0(e_{xy'})$, uncolor e_{xw}, and replace e_{xy} by e_{xw} in M_0^*. (See Figure 3.1(b), where the edge of the dashed line represents the uncolored edge.) Since $d_{G-(M\cup M_0^*)}(w) = \Delta$, $M_1^* := (M_0^*\backslash\{e_{xy}\}) \cup \{e_{xw}\}$ is a matching. Denote $H_1 := H + e_{xy} - e_{xw}$. Let ψ be the k-edge coloring of H_1 after Operation II. Note that for any vertex $z \in V(H_1)$ that is incident with an edge of $\partial_{G-(M\cup M_1^*)}(H_1)$, if $\overline{\psi}(z) \neq \overline{(\varphi_0)}_H(z)$, then z is contained in the subsequence of S from y to w. Since h is the only i-edge of $\partial_{G-(M\cup M_1^*)}(H_1)$, there is no such vertex w such that w is incident with both an i-edge contained in the subsequence of S

from y to w and an i-edge of $\partial_{G-(M\cup M_1^*)}(H_1)$ after Operation II. Thus we can rename some color classes of ψ but keep the color i unchanged to match all colors on boundary edges of $\partial_{G-(M\cup M_1^*)}(H_1)$. In this way we obtain a (proper) k-edge-coloring φ_1 of $G-(M\cup M_1^*)$ by Lemma 3.2.5(b).

By the similar argument in the proof of Subcase 1.1, it can be verified that $(M_1^*, \emptyset, \varphi_1)$ is prefeasible, and that f_{uv} is no longer T2-improper at u but T1-improper at u with respect to M_1^* and φ_1. For any edge $f \in M \setminus \{f_{uv}\}$, we see that the distance between f and e_{xw} is at least 2 or just 1 when h is adjacent in G to f with $\Phi(f) \neq i$. Thus the property of f being T1-improper or T2-improper is not changed under M_1^* and φ_1. Thus the new triple $(M_1^*, \emptyset, \varphi_1)$ is prefeasible with $|E_1(M_1^*, \varphi_1)| = |E_1(M_0^*, \varphi_0)| + 1$ and $|E_2(M_1^*, \varphi_1)| = |E_2(M_0^*, \varphi_0)| - 1$, and so we can consider $(M_1^*, \emptyset, \varphi_1)$ instead.

Subcase 1.2.2: $h \in E_1(M_0^*, \varphi_0)$, i.e., h is adjacent in G to some precolored i-edge f_i in M.

We apply Operation III as follows: recolor the i-edge h with the color $\Delta+\mu$, apply a shifting in S from y to x_1, color e_{xy} with $\varphi_0(e_{xy'})$, uncolor e_{xx_1}, and replace e_{xy} by e_{xx_1} in M_0^*. (See Figure 3.1(c), where the edge of the dashed line represents the uncolored edge.) By the same argument as in the proof of Subcase 1.1, we know that $M_1^* := (M_0^* \backslash \{e_{xy}\}) \cup \{e_{xx_1}\}$ is a matching. Denote $H_1 := H + e_{xy} - e_{xx_1}$. Let ψ be the k-edge coloring of H_1 after Operation III. Note that there is no i-edge in $\partial_{G-(M\cup M_1^*)}(H_1)$ after Operation III. By the similar argument as in the proof of Subcase 1.1, we can rename some color classes of ψ but keep the color i unchanged to match all colors on edges of $\partial_{G-(M\cup M_1^*)}(H_1)$. In this way we obtain a (proper) $(k+1)$-edge-coloring φ_1 of $G-(M\cup M_1^*)$ by Lemma 3.2.5(b).

We claim that $(M_1^*, \emptyset, \varphi_1)$ is a prefeasible triple. As $M \cup M_1^*$ is a matching and h is adjacent to f_i and $\Phi(f_i) = i \in [k]$, it follows that h is not adjacent to any edge from $M_{\Delta+\mu} \cup M_1^*$, which implies that $M_{\Delta+\mu} \cup M_1^* \cup \{h\}$ is a matching. By the same argument as in the proof of Subcase 1.1, we know that e_{xx_1} is k-critical and fully G-saturated in $H_1 + e_{xx_1}$, and each edge $e \in M_1^* \setminus \{e_{xx_1}\}$ is still contained in a k-dense subgraph of $G-(M\cup M_1^*)$ such that e is k-critical and fully G-saturated in the graph $H_e + e$ if e is adjacent in G to an edge of

$E_2(M_1^*, \varphi_1)$, where H_e is the unique maximal k-dense subgraph of $G-(M\cup M_0^*)$ containing $V(e)$ if H_e and H_1 are vertex-disjoint, and $H_e = H_1$ otherwise. If the color $\Delta+\mu$ is not used on edges of $\partial_{G-(M\cup M_1^*)}(H_e)$, then colors on edges of $\partial_{G-(M\cup M_1^*)}(H_e)$ are all distinct by the fact that H_e is strongly φ_1-closed. If the color $\Delta+\mu$ is used on edges of $\partial_{G-(M\cup M_1^*)}(H_e)$, then it was used on exactly one edge of $\partial_{G-(M\cup M_1^*)}(H_e)$. This, together with the fact that H_e is $(\varphi_1)_{H_e}$-elementary, implies that colors on edges of $\partial_{G-(M\cup M_1^*)}(H_e)$ are all distinct. Therefore, $(M_1^*, \emptyset, \varphi_1)$ is a prefeasible triple.

By the same argument as in the proof of Subcase 1.1, we know that now f_{uv} is no longer T2-improper at u but T1-improper at u with respect to M_1^* and φ_1, and that for any edge $f\in M\setminus\{f_{uv}\}$, the distance between f and e_{xx_1} in $G-(M\cup M_1^*)$ is at least 2. Except the i-edge f_i of M that is adjacent in G to h, the property of f being T1-improper or T2-improper is not changed under M_1^* and φ_1. The edge f_i is originally T1-improper at w_i, and now is no longer improper at w_i with respect to φ_1, where we assume $h\in E_G(w,w_i)$. Thus $|E_1(M_1^*,\varphi_1)| = |E_1(M_0^*,\varphi_0)|+1-1$ and $|E_2(M_1^*,\varphi_1)| = |E_2(M_0^*,\varphi_0)|-1$, and so we can consider $(M_1^*, \{h\}, \varphi_1)$ instead. Note that assigning the color $\Delta+\mu$ to h will not affect the modification of φ_0 into φ and M_0^* into M^*, since $h\in E_1(M_0^*,\varphi_0)$ and we will assign the color $\Delta+\mu$ to all edges in $E_1(M_0^*,\varphi_0)$ in the final process.

Case 2: f_{uv} is T2-improper at v with $v\in V(H')$ for a maximal k-dense subgraph H' other than H.

For this case, we apply the same operations as we did in Case 1 first with respect to the vertex u in H and then with respect to the vertex v in H'. Recall that $V(H)\cap V(H')=\emptyset$ and $E_1(M_0^*,\varphi_0)$ is a matching. By Case 1, the operations applied within $G[V(H)]$ or $G[V(H)]+h_u$ do not affect the operations applied within $G[V(H')]$ or $G[V(H')]+h_v$, where h_u and h_v are the two possible i-edges with $h_u\in\partial_{G-(M\cup M_0^*)}(H)\cap E_1(M_0^*,\varphi_0)$ and $h_v\in\partial_{G-(M\cup M_0^*)}(H')\cap E_1(M_0^*,\varphi_0)$. Furthermore, if h_u and h_v exist at the same time, then $V(h_u)\cap V(h_v)=\emptyset$ and there is no maximal k-dense subgraph H'' other than H and H' such that $V(H'')\cap V(h_u)\neq\emptyset$ and $V(H'')\cap V(h_v)\neq\emptyset$. Denote the matching resulting from M_0^* by M_1^*, and the coloring resulting from φ_0 by φ_1. By Case 1, $E_{M_1^*}^{\varphi_1}\subseteq\{h_u,h_v\}$, $M_{\Delta+\mu}\cup M_1^*\cup\{h_u,h_v\}$ is a matching, and

$(M_1^*, E^{\varphi_1}_{M_1^*}, \varphi_1)$ also satisfies Conditions (b) and (c). Thus $(M_1^*, E^{\varphi_1}_{M_1^*}, \varphi_1)$ is a prefeasible triple. With respect to M_1^* and φ_1, f_{uv} is no longer T2-improper but is T1-improper at both u and v. Furthermore, we have $|E_1(M_1^*, \varphi_1)| \geq |E_1(M_0^*, \varphi_0)|$ and $|E_2(M_1^*, \varphi_1)| = |E_2(M_0^*, \varphi_0)| - 2$. Thus we can consider $(M_1^*, E^{\varphi_1}_{M_1^*}, \varphi_1)$ instead.

Case 3: f_{uv} is T1-improper or T2-improper at v with $v \in V(H)$.

Assume first that $d_H(b) < \Delta$. Let $e_{bv} \in E_H(b, v)$ with $\varphi_0(e_{bv}) = i$. If f_{uv} is T1-improper at v, then we apply the same operations with respect to u as we did in Case 1. Denote the new matching resulting from M_0^* by M_1^*, and the new coloring resulting from φ_0 by φ_1. Then the vertex b is not incident in G with any edge of M_1^* by Operations I-III in Case 1. Thus f_{uv} is no longer T2-improper at u but T1-improper at u with respect to M_1^* and φ_1. Furthermore, we have $|E_1(M_1^*, \varphi_1)| \geq |E_1(M_0^*, \varphi_0)|$ and $|E_2(M_1^*, \varphi_1)| = |E_2(M_0^*, \varphi_0)| - 1$. Thus we can consider $(M_1^*, E^{\varphi_1}_{M_1^*}, \varphi_1)$ instead.

If f_{uv} is T2-improper at v, let $e_{ab} \in M_0^*$ with $V(e_{ab}) = \{a, b\}$. We apply the same operations with respect to u as we did in Case 1. Denote the resulting matching by M_1^*, and the resulting coloring by φ_1. With respect to M_1^* and φ_1, the edge f_{uv} is still T2-improper at v as $d_H(a) < \Delta$ and $d_H(b) < \Delta$. By Case 1, now f_{uv} is no longer T2-improper at u but T1-improper at u with respect to the prefeasible triple $(M_1^*, E^{\varphi_1}_{M_1^*}, \varphi_1)$, where $E^{\varphi_1}_{M_1^*} = \emptyset$ or $\{h\}$ with some vertex w and its incident i-edge $h \in \partial_{G-(M \cup M_0^*)}(H) \cap E_1(M_0^*, \varphi_0)$. Denote by H_1 the new k-dense subgraph after the operations with respect to u in $H + e_{xy}$. In particular, the situation under $(M_1^*, \emptyset, \varphi_1)$ is actually the same as the case $d_H(b) = \Delta$ in the previous paragraph since now $d_{H_1}(y) = \Delta$.

Thus we consider only the case that f_{uv} is T2-improper at v, T1-improper at u and $d_{H_1}(y) = \Delta$. Consider a maximal multi-fan F_a at a with respect to e_{ab} and $(\varphi_1)_{H_1}$ in $H_1 + e_{ab}$. Clearly we can apply the same operations in Case 1 for v so that f_{uv} is no longer T2-improper at v with respect to the resulting matching M_2^* and coloring φ_2, unless these operations would have to put one edge $e_{ay} \in E_{H_1}(a, y)$ into M_2^*. Then f_{uv} would become T2-improper at u again with respect to M_2^* and φ_2. The only operations that have to uncolor an edge of H_1 incident with y are Operations I and III. Therefore, we make the following two assumptions on F_a in the rest of our proof.

(1) y is the only Δ-vertex in $V(F_a)\backslash\{a,b\}$.

(2) If a linear sequence in F_a at a from b to y contains a vertex w' such that $d_{H_1}(w') = \Delta - 1$ and w' is incident with an i-edge $h' \in \partial_{G-(M\cup M_1^*)}(H_1)$, then $h' \in E_1(M_1^*, \varphi_1)$.

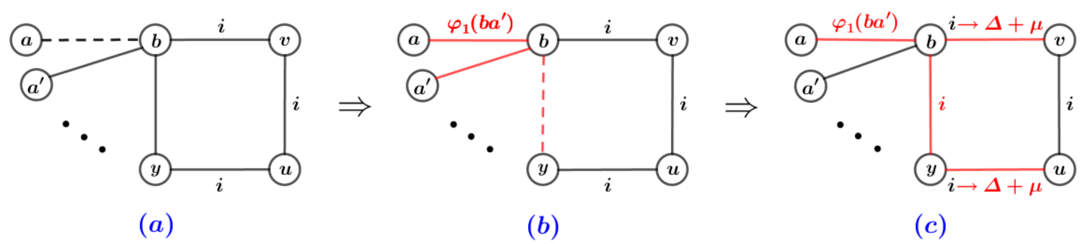

Figure 3.2. Operation in Subcase 3.1.

Let F_b be a maximal multi-fan at b with respect to e_{ab} and $(\varphi_1)_{H_1}$ in $H_1 + e_{ab}$. We consider the following three subcases.

Subcase 3.1: F_b contains a linear sequence S at b from a to y such that S does not contain any i-edge.

Let $S = (a, e_{ba'}, a', \ldots, e_{by}, y)$ be the linear sequence (where $a' = y$ is possible). We apply a shifting in S from a to y, color e_{ab} with $\varphi_1(e_{ba'})$, uncolor e_{by}. (See Figure 3.2(a)-(b), where the edges of the dashed line represent uncolored edges.) Note that $M_2^* := (M_1^* \backslash \{e_{ab}\}) \cup \{e_{by}\}$ is a matching, and $H_2 := H_1 + e_{ab} - e_{by}$ is a k-dense subgraph of $G - (M \cup M_2^*)$. As S does not contain any i-edge, by Lemma 3.2.5(b), we obtain a k-edge-coloring φ_2 of $G - (M \cup M_2^*)$. Note that f_{uv} is T2-improper at both u and v with respect to M_2^* and φ_2. However, we have $\Phi(f_{uv}) = i$, $\varphi_2(e_{bv}) = \varphi_2(e_{yu}) = i$, and $e_{by} \in M_2^*$ ($bvuyb$ is a cycle with length 4 in G). By assigning the color i to e_{by} and recoloring e_{bv} and e_{yu} with the color $\Delta + \mu$, we obtain a new matching $M_3^* := M_2^* \backslash \{e_{by}\} = M_1^* \backslash \{e_{ab}\}$ of $G - V(M)$ and a new $(k+1)$-edge-coloring φ_3 of $G - (M \cup M_3^*)$. (See Figure 3.2(c).) The edge f_{uv} is now not improper at neither of its endvertices. Note that $E_{M_3^*}^{\varphi_3} = \{e_{bv}, e_{yu}\}$ if $E_{M_1^*}^{\varphi_1} = \emptyset$ and $E_{M_3^*}^{\varphi_3} = \{h, e_{bv}, e_{yu}\}$ if $E_{M_1^*}^{\varphi_1} = \{h\}$.

Since $E^{\varphi_3}_{M_3^*} \subseteq (E_1(M_0^*, \varphi_0) \cup E_2(M_0^*, \varphi_0))$ is a matching, and those edges in $E^{\varphi_3}_{M_3^*}$ do not share any endvertex with edges in $M_{\Delta+\mu} \cup M_3^*$, it follows that $M_{\Delta+\mu} \cup M_3^* \cup E^{\varphi_3}_{M_3^*}$ is a matching. Note that $V(H_2) \cap V(M \setminus \{f_{uv}\}) = \emptyset$. For each $e \in M_3^*$ such that e is adjacent in G to an edge of $E_2(M_3^*, \varphi_3)$, e is still k-critical and fully G-saturated in the graph $H_e + e$, where H_e is still the unique maximal k-dense subgraph of $G - (M \cup M_0^*)$ containing $V(e)$ and H_e is also strongly φ_3-closed. Thus the new triple $(M_3^*, E^{\varphi_3}_{M_3^*}, \varphi_3)$ is prefeasible. Furthermore, $|E_1(M_3^*, \varphi_3)| = |E_1(M_1^*, \varphi_1)| - 1 \geq |E_1(M_0^*, \varphi_0)| - 1$ and $|E_2(M_3^*, \varphi_3)| = |E_2(M_1^*, \varphi_1)| - 1 = |E_2(M_0^*, \varphi_0)| - 2$. Thus we can consider $(M_3^*, E^{\varphi_3}_{M_3^*}, \varphi_3)$ instead.

Subcase 3.2: F_b contains a vertex w'' with $d_{H_1}(w'') = \Delta - 1$ and $i \in \overline{(\varphi_1)}_{H_1}(w'')$.

The i-edge e_{bv} is in F_b by the maximality of F_b. Let $S = (a, e_{ba'}, a', \ldots, e_{bw''}, w'', e_{bv}, v)$ be a linear sequence at b from a to v in F_b (where $a = a'$ and $a' = w''$ are possible). Since $i \in \overline{(\varphi_1)}_{H_1}(w'')$, we have that either $i \in \overline{\varphi}_1(w'')$ or w'' is incident with an i-edge $h'' \in \partial_{G-(M\cup M_1^*)}(H_1)$.

Assume first that $i \in \overline{\varphi}_1(w'')$ or w'' is incident with an i-edge $h'' \in \partial_{G-(M\cup M_1^*)}(H_1)$ such that $h'' \in E_1(M_1^*, \varphi_1)$. We apply a shifting in S from a to v, color e_{ab} with $\varphi_1(e_{ba'})$, and uncolor e_{bv}. Note that $e_{bw''}$ was recolored by the color i in the shifting operation. We then recolor the i-edge h'' with the color $\Delta + \mu$ if h'' exists, and rename some color classes of $H_2 := H_1 + e_{ab} - e_{bv}$ but keep the color i unchanged without producing any improper i-edge by Lemma 3.2.5(b). Finally we assign the color $\Delta + \mu$ to e_{bv}. Note that $h \neq h''$ since $\varphi_1(h) = \Delta + \mu \neq i = \varphi_1(h'')$, and h and h'' cannot both exist in $\partial_{G-(M\cup M_0^*)}(H) = \partial_{G-(M\cup M_1^*)}(H_1)$ since otherwise $\varphi_0(h) = \varphi_0(h'') = i$ contradicting that H is strongly φ_0-closed. Now we obtain a new matching $M_2^* := M_1^* \setminus \{e_{ab}\}$ of $G - V(M)$ and a new (proper) $(k+1)$-edge-coloring φ_2 of $G - (M \cup M_2^*)$ such that f_{uv} is no longer T2-improper at v or even T1-improper at v with respect to a new triple $(M_2^*, E^{\varphi_2}_{M_2^*}, \varphi_2)$, where $E^{\varphi_2}_{M_2^*} = \{e_{bv}\}$ if $E^{\varphi_1}_{M_1^*} = \emptyset$ but h'' does not exist, $E^{\varphi_2}_{M_2^*} = \{e_{bv}, h''\}$ if $E^{\varphi_1}_{M_1^*} = \emptyset$ and h'' exists, and $E^{\varphi_2}_{M_2^*} = \{e_{bv}, h\}$ if $E^{\varphi_1}_{M_1^*} = \{h\}$. Since $E^{\varphi_2}_{M_2^*} \subseteq (E_1(M_0^*, \varphi_0) \cup E_2(M_0^*, \varphi_0))$ is a matching, and those edges in $E^{\varphi_2}_{M_2^*}$ do not share any endvertex with edges in $M_{\Delta+\mu} \cup M_2^*$, it follows that $M_{\Delta+\mu} \cup M_2^* \cup E^{\varphi_2}_{M_2^*}$ is a matching. Note that $V(H_2) \cap V(M \setminus \{f_{uv}\}) = \emptyset$. By the similar argument as in the proof of Subcase 3.1, the new triple $(M_2^*, E_{M_2^*}^{\varphi_2}, \varphi_2)$ is prefeasible. Furthermore, $|E_1(M_2^*, \varphi_2)| \geq |E_1(M_0^*, \varphi_0)|$ and $|E_2(M_2^*, \varphi_2)| = |E_2(M_0^*, \varphi_0)| - 2$. Thus we can consider $(M_2^*, E_{M_2^*}^{\varphi_2}, \varphi_2)$ instead.

Now we may assume that the i-edge $h'' \notin E_1(M_1^*, \varphi_1)$. Since h and h'' cannot both exist, we have $E_{M_1^*}^{\varphi_1} = \emptyset$. Note that the vertex $w'' \notin V(F_a)$ by Assumption (2) prior to Subcase 3.1. Moreover, w'' is not incident with any edge in $M \cup M_1^*$ and w'' is only incident with the i-edge h'' in $\partial_{G-(M\cup M_1^*)}(H_1)$. Since $d_{G-(M\cup M_1^*)}(w'') = \Delta$ and φ_1 is a k-edge-coloring of $G - (M \cup M_1^*)$ with $k \geq \Delta + 1$, there exists a color $\alpha \in \overline{\varphi}_1(w'')$ with $\alpha \neq i$. Since $V(H_1)$ is $(\varphi_1)_{H_1}$-elementary, there exists an α-edge e_1 incident with the vertex a. Thus we can define a maximal multi-fan at a, denoted by F_a', with respect to e_1 and $(\varphi_1)_{H_1}$ in $H_1 + e_1$. (Notice that e_1 is colored by the color α in F_a'.) Moreover, $V(F_a')$ is $(\varphi_1)_{H_1}$-elementary since $V(H_1)$ is $(\varphi_1)_{H_1}$-elementary. By Lemma 3.3.2(b) and Assumption (1) prior to Subcase 3.1, we have $e_{F_a}(a, b') = e_{H_1+e_{ab}}(a, b') = \mu$ for any vertex b' in $V(F_a)\backslash\{a\}$. Therefore, $V(F_a')\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are disjoint, since otherwise we have $V(F_a') \subseteq V(F_a)$ and $\alpha \in \overline{(\varphi_1)}_{H_1}(b')$ for some $b' \in V(F_a)$ implying $b' = w'' \in V(F_a)$, a contradiction. Note that if $w'' \notin V(F_a')$, then $V(F_a')\backslash\{a\}$ must contain a Δ-vertex in H_1, since otherwise Lemma 3.3.2(d) and the fact $(\varphi_1)_{H_1}(e_1) = \alpha \in \overline{\varphi}_1(w'')$ imply that $w'' \in V(F_a')$, a contradiction. Thus F_a' contains a linear sequence $S' = (b_1, e_2, b_2, \ldots, e_t, b_t)$ at a, where $b_1 \in V(e_1)$, b_t (with $t \geq 1$) is a Δ-vertex if $w'' \notin V(F_a')$, and b_t is w'' if $w'' \in V(F_a')$. Notice that b_t is not incident with any edge in $M \cup M_1^*$ by our choice of b_t. Moreover, $b_t \neq y$ since $V(F_a')\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are disjoint. Let β ($\beta \neq i$) be a color in $\overline{\varphi}_1(b)$. By Lemma 3.3.1(b), we have $P_b(\beta, \alpha) = P_{w''}(\beta, \alpha)$. We then consider the following two subcases according the set $(V(S')\backslash\{a\}) \cap (V(S)\backslash\{a\})$.

We first assume that $(V(S')\backslash\{a\}) \cap (V(S)\backslash\{a\}) \subseteq \{b_t\}$. If $e_1 \notin P_b(\beta, \alpha)$, then we apply a Kempe change on $P_{[b,w'']}(\beta, \alpha)$, uncolor e_1 and color e_{ab} with α. If $e_1 \in P_b(\beta, \alpha)$ and $P_b(\beta, \alpha)$ meets b_1 before a, then we apply a Kempe change on $P_{[b,b_1]}(\beta, \alpha)$, uncolor e_1 and color e_{ab} with α. If $e_1 \in P_b(\beta, \alpha)$ and $P_{w''}(\beta, \alpha)$ meets b_1 before a, then we uncolor e_1, apply a Kempe change on $P_{[w'',b_1]}(\beta, \alpha)$, apply a shifting in S from a to w'', color e_{ab} with $\varphi_1(e_{ba'})$, and recolor $e_{bw''}$ with β. In all three cases above, e_{ab} is colored with a color in $[k]$ and e_1 is uncolored. Finally we apply a shifting in S' from b_1 to b_t, color e_1 with $\varphi_1(e_2)$, and uncolor e_t. Notice that the above shifting in S' does nothing if $t = 1$. Denote $H_2 := H_1 + e_{ab} - e_t$. Since H_2 is also k-dense and $\chi'(H_2) = k$, we can rename some color classes of $E(H_2)$ but keep the color i unchanged to match all colors on boundary edges without producing any improper i-edge by Lemma 3.2.5(b). Now we obtain a new matching $M_2^* := (M_1^* \backslash \{e_{ab}\}) \cup \{e_t\}$ and a new (proper) k-edge-coloring φ_2 of $G - (M \cup M_2^*)$ such that f_{uv} is no longer T2-improper at v but T1-improper at v with respect to the new prefeasible triple $(M_2^*, \emptyset, \varphi_2)$. Furthermore, $|E_1(M_2^*, \varphi_2)| = |E_1(M_0^*, \varphi_0)| + 2$ and $|E_2(M_2^*, \varphi_2)| = |E_2(M_0^*, \varphi_0)| - 2$. Thus we can consider $(M_2^*, \emptyset, \varphi_2)$ instead.

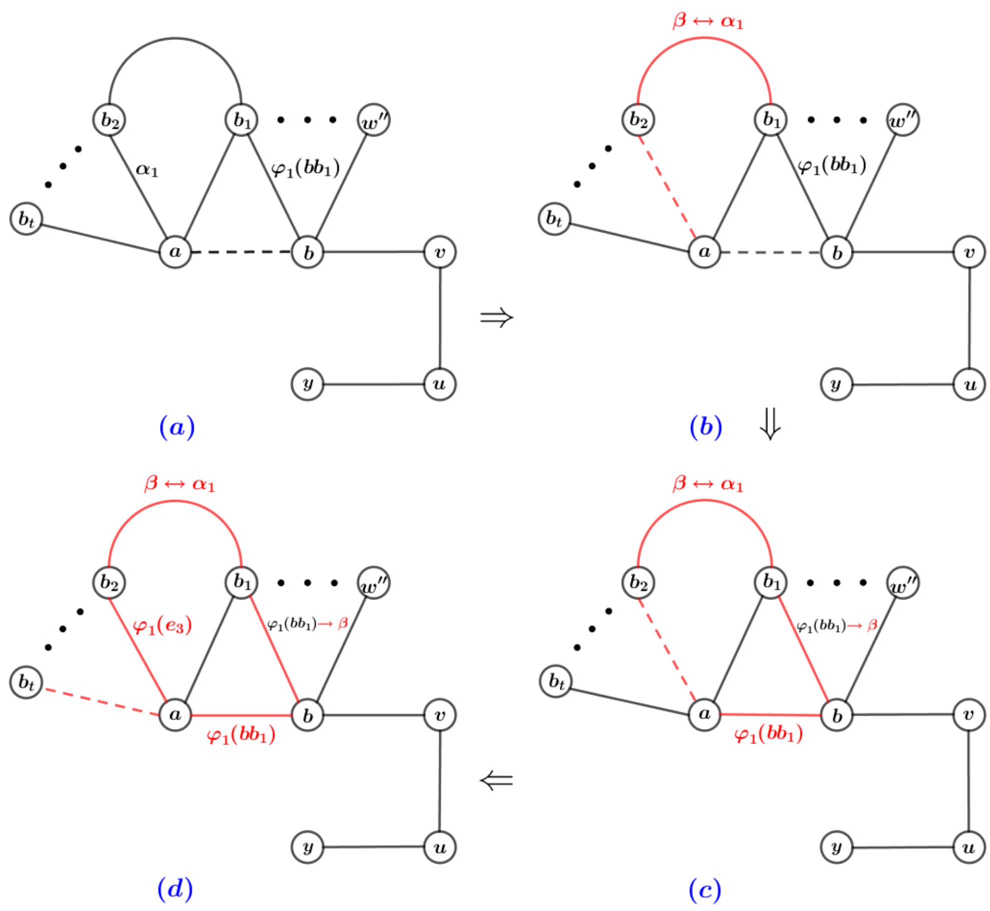

Figure 3.3. One possible operation for $b_j = a^* \in (V(S')\backslash\{a\}) \cap (V(S)\backslash\{a\})$ in Subcase 3.2, where $b_1 = b_j = a^* = a'$.

Then we assume that there exists $b_j = a^* \in (V(S')\backslash\{a\}) \cap (V(S)\backslash\{a\})$ for some $j \in [t-1]$

and $a^* \in V(S)$. (See Figure 3.3 for a depiction when $b_1 = b_j = a^* = a'$, where the edges of the dashed line represent uncolored edges). In this case we assume a^* is the closest vertex to the vertex a along S. Note that $b_j \neq b$ as $V(F'_a)\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are disjoint. Let $\alpha_j = \varphi_1(e_{j+1}) \in \overline{(\varphi_1)}_{H_1}(b_j)$. By Lemma 3.3.1($b$), we have $P_b(\beta, \alpha_j) = P_{b_j}(\beta, \alpha_j)$. If $e_{j+1} \notin P_b(\beta, \alpha_j)$, then we apply a Kempe change on $P_{[b,b_j]}(\beta, \alpha_j)$, uncolor e_{j+1} and color e_{ab} with α_j. If $e_{j+1} \in P_b(\beta, \alpha_j)$ and $P_b(\beta, \alpha_j)$ meets b_{j+1} before a, then we apply a Kempe change on $P_{[b,b_{j+1}]}(\beta, \alpha_j)$, uncolor e_{j+1} and color e_{ab} with α_j. If $e_{j+1} \in P_b(\beta, \alpha_j)$ and $P_{b_j}(\beta, \alpha_j)$ meets b_{j+1} before a, then we uncolor e_{j+1}, apply a Kempe change on $P_{[b_i,b_{j+1}]}(\beta, \alpha_j)$, apply a shifting in S from a to b_j (i.e., a^*), color e_{ab} with $\varphi_1(e_{ba'})$, and recolor the edge $e_{bb_j} \in E_{H_1}(b, b_j)$ with β. (See Figure 3.3(a)-(c).) In all three cases above, e_{ab} is colored with a color in $[k]$ and e_{j+1} is uncolored. Finally we apply a shifting in S' from b_{j+1} to b_t, color e_{j+1} with $\varphi_1(e_{j+2})$, and uncolor e_t. (See Figure 3.3(d).) Notice that the above shifting in S' does nothing if $b_{j+1} = b_t$. Denote $H_2 := H_1 + e_{ab} - e_t$. Since H_2 is also k-dense and $\chi'(H_2) = k$, we can rename some color classes of $E(H_2)$ but keep the color i unchanged to match all colors on boundary edges without producing any improper i-edge by Lemma 3.2.5(b). Now we obtain a new matching $M_2^* := (M_1^*\backslash\{e_{ab}\}) \cup \{e_t\}$ of $G - V(M)$ and a new (proper) k-edge-coloring φ_2 of $G-(M \cup M_2^*)$ such that f_{uv} is no longer T2-improper at v but T1-improper at v with respect to the new prefeasible triple $(M_2^*, \emptyset, \varphi_2)$. Furthermore, $|E_1(M_2^*, \varphi_2)| = |E_1(M_0^*, \varphi_0)| + 2$ and $E_2(M_2^*, \varphi_2) = |E_2(M_0^*, \varphi_0)| - 2$. Thus we can consider $(M_2^*, \emptyset, \varphi_2)$ instead.

Subcase 3.3: F_b does not contain a linear sequence at b from a to y without i-edge, and F_b does not contain a vertex w'' with $d_{H_1}(w'') = \Delta - 1$ and $i \in \overline{(\varphi_1)}_{H_1}(w'')$.

We claim that F_b contains a linear sequence S^* at b from a to a Δ-vertex y^* such that $y^* \neq y$ and there is no i-edge in S^*. By Lemma 3.3.2(a), the multi-fan F_b contains at least one Δ-vertex in H_1. Now if F_b does not contain any linear sequence without i-edges from a to any Δ-vertex in H_1, then by Lemma 3.3.2(c), the multi-fan F_b contains a vertex w'' with $d_{H_1}(w'') = \Delta - 1$ and $i \in \overline{(\varphi_1)}_{H_1}(w'')$, contradicting the condition of Subcase 3.3. So F_b contains a linear sequence S^* from a to a vertex y^* such that $d_{H_1}(y^*) = \Delta$ and there is no i-edge in S^*. Note that $y^* \neq y$, since otherwise we also have a contradiction to the condition of Subcase 3.3. Thus the claim is proved.

Assume that $S^* = (a, e_{ba'}, a', \ldots, e_{by^*}, y^*)$ at b from a to y^* (where $a' = y^*$ is possible), and S^* contains no i-edge. Let $\theta \in \overline{\varphi}_1(y^*)$.

Subcase 3.3.1: $\theta = i$.

Since S^* contains no i-edge, we apply a shifting in S^* from a to y^*, color e_{ab} with $\varphi_1(e_{ba'})$, uncolor e_{by^*}, and rename some color classes of $E(H_1 + e_{ab} - e_{by^*})$ but keep the color i unchanged to match all colors on boundary edges without producing any improper i-edge by Lemma 3.2.5(b). By coloring e_{by^*} with i and recoloring e_{bv} from i to $\Delta+\mu$, we obtain a new matching $M_2^* := M_1^* \backslash \{e_{ab}\}$ of $G - V(M)$ and a new (proper) $(k+1)$-edge-coloring φ_2 of $G - (M \cup M_2^*)$. Then f_{uv} is no longer T2-improper at v or even T1-improper at v with respect to the new prefeasible triple $(M_2^*, E^{\varphi_2}_{M_2^*}, \varphi_2)$ with $E^{\varphi_2}_{M_2^*} = \{e_{bv}\}$ if $E^{\varphi_1}_{M_1^*} = \emptyset$, and $E^{\varphi_2}_{M_2^*} = \{e_{bv}, h\}$ if $E^{\varphi_1}_{M_1^*} = \{h\}$ (when $y^* \in V(F_x) \cap V(F_b)$). Furthermore, $E^{\varphi_2}_{M_2^*} \subseteq (E_1(M_0^*, \varphi_0) \cup E_2(M_0^*, \varphi_0))$, $|E_1(M_2^*, \varphi_2)| \geq |E_1(M_0^*, \varphi_0)|$ and $|E_2(M_2^*, \varphi_2)| = |E_2(M_0^*, \varphi_0)| - 2$. Thus we can consider $(M_2^*, E^{\varphi_2}_{M_2^*}, \varphi_2)$ instead.

Subcase 3.3.2: $\theta \neq i$.

Since $V(H_1)$ is $(\varphi_1)_{H_1}$-elementary, there exists a θ-edge e_1 incident with the vertex a. Thus by the similar argument as in the proof of Subcase 3.2, we define a maximal multi-fan at a, denoted by F_a', with respect to e_1 and $(\varphi_1)_{H_1}$ in $H_1 + e_1$, and we have $e_{F_a}(a, b') = e_{H_1+e_{ab}}(a, b') = \mu$ for any vertex b' in $V(F_a) \backslash \{a\}$. Therefore, $V(F_a') \backslash \{a\}$ and $V(F_a) \backslash \{a\}$ are disjoint, since otherwise we have $V(F_a') \subseteq V(F_a)$ and $\varphi_1(e_1) = \theta \in \overline{(\varphi_1)}_{H_1}(b')$ for some $b' \in V(F_a)$ implying $y^* = b' \in V(F_a)$, which contradicts Assumption (1). Note that $V(F_a') \backslash \{a\}$ must contain a Δ-vertex in H_1, since otherwise Lemma 3.3.2(d) and the fact $(\varphi_1)_{H_1}(e_1) = \theta \in \overline{\varphi}_1(y^*)$ imply that $y^* \in V(F_a')$, which contradicts $d_{H_1}(y^*) = \Delta$. If F_a' contains a vertex of $V(H_1)$ that is incident with an i-edge of $\partial_{G-(M \cup M_1^*)}(H_1)$ in $G - (M \cup M_1^*)$, then we denote the vertex by w^* and the i-edge by h^*. If F_a' does not contain any linear sequence to a Δ-vertex in H_1 without i-edge and boundary vertex w^*, then by Lemma 3.3.2(d), the multi-fan F_a' contains a vertex z^* with $i \in \overline{(\varphi_1)}_{H_1}(z^*)$ and $d_H(z^*) = \Delta - 1$. Since H_1 is $(\varphi_1)_{H_1}$-elementary, we have $z^* = w^*$ and $d_{H_1}(w^*) = \Delta - 1$. Thus F_a' contains a linear sequence $S' = (b_1, e_2, b_2, \dots, e_t, b_t)$ at a, where $b_1 \in V(e_1)$, b_t (with $t \geq 1$) is w^* if there exists w^* with $d_{H_1}(w^*) = \Delta - 1$ such that $h^* \in \partial_{G-(M\cup M_1^*)}(H_1)$ but $h^* \notin E_1(M_0^*, \varphi_0)$, and b_t is a Δ-vertex in H_1 otherwise. Notice that b_t is not incident with any edge in $M \cup M_1^*$ by our choice of b_t. Moreover, if $b_t = w^*$ as defined above, then $b_t = w^*$ is not a vertex in $V(F_b)$ by the condition of Subcase 3.3. And $b_t \neq y$ since $V(F'_a)\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are disjoint. Let β ($\beta \neq i$) be a color in $\overline{\varphi}_1(b)$. By Lemma 3.3.1(b), we have $P_b(\beta, \theta) = P_{y^*}(\beta, \theta)$. We then consider the following two subcases according the set $(V(S')\backslash\{a\}) \cap (V(S^*)\backslash\{a\})$.

We first assume that $(V(S')\backslash\{a\}) \cap (V(S^*)\backslash\{a\}) \subseteq \{b_t\}$. If $e_1 \notin P_b(\beta, \theta)$, then we apply a Kempe change on $P_{[b,y^*]}(\beta, \theta)$, uncolor e_1 and color e_{ab} with θ. If $e_1 \in P_b(\beta, \theta)$ and $P_b(\beta, \theta)$ meets b_1 before a, then we apply a Kempe change on $P_{[b,b_1]}(\beta, \theta)$, uncolor e_1 and color e_{ab} with θ. If $e_1 \in P_b(\beta, \theta)$ and $P_{y^*}(\beta, \theta)$ meets b_1 before a, then we uncolor e_1, apply a Kempe change on $P_{[y^*,b_1]}(\beta, \theta)$, apply a shifting in S^* from a to y^*, color e_{ab} with $\varphi_1(e_{ba'})$, and recolor e_{by^*} with β. In all three cases above, e_{ab} is colored with a color in $[k]$ and e_1 is uncolored. Then we apply a shifting in S' from b_1 to b_t, color e_1 with $\varphi_1(e_2)$, and uncolor e_t. Denote $H_2 := H_1 + e_{ab} - e_t$. Since H_2 is also k-dense and $\chi'(H_2) = k$, we can rename some color classes of $E(H_2)$ but keep the color i unchanged to match colors on boundary edges except i-edges by Lemma 3.2.5(b). Finally recolor h^* with the color $\Delta + \mu$ if $h^* \in \partial_{G-(M\cup M_0^*)}(H) \cap E_1(M_0^*, \varphi_0)$. Now we obtain a new matching $M_2^* := (M_1^*\backslash\{e_{ab}\}) \cup \{e_t\}$ of $G - V(M)$ and a new (proper) $(k+1)$-edge-coloring φ_2 of $G - (M \cup M_2^*)$ such that f_{uv} is no longer T2-improper at v but T1-improper at v with respect to the new prefeasible triple $(M_2^*, E_{M_2^*}^{\varphi_2}, \varphi_2)$, where $\emptyset$ or $\{h\}$ or $\{h^*\} = E_{M_2^*}^{\varphi_2} \subseteq E_1(M_0^*, \varphi_0)$. Furthermore, $|E_1(M_2^*, \varphi_2)| \geq |E_1(M_0^*, \varphi_0)|$ and $|E_2(M_2^*, \varphi_2)| = |E_2(M_0^*, \varphi_0)| - 2$. Thus we can consider $(M_2^*, E_{M_2^*}^{\varphi_2}, \varphi_2)$ instead.

Then we assume that there exists $b_j = a^* \in (V(S')\backslash\{a\}) \cap (V(S^*)\backslash\{a\})$ for some $j \in [t-1]$ and $a^* \in V(S^*)$. (See Figure 3.4 for a depiction when $b_1 = b_j = a^* = a'$, where the edges of the dashed line represent uncolored edges.) In this case we assume a^* is the closest vertex to a along S^*. Note that $b_j \neq b$ as $V(F'_a)\backslash\{a\}$ and $V(F_a)\backslash\{a\}$ are disjoint. Let $\theta_j = \varphi_1(e_{j+1}) \in \overline{(\varphi_1)}_{H_1}(b_j)$. By Lemma 3.3.1($b$), $P_b(\beta, \theta_j) = P_{b_j}(\beta, \theta_j)$. If $e_{j+1} \notin P_b(\beta, \theta_j)$, then we apply a Kempe change on $P_{[b,b_j]}(\beta, \theta_j)$, uncolor e_{j+1} and color e_{ab}

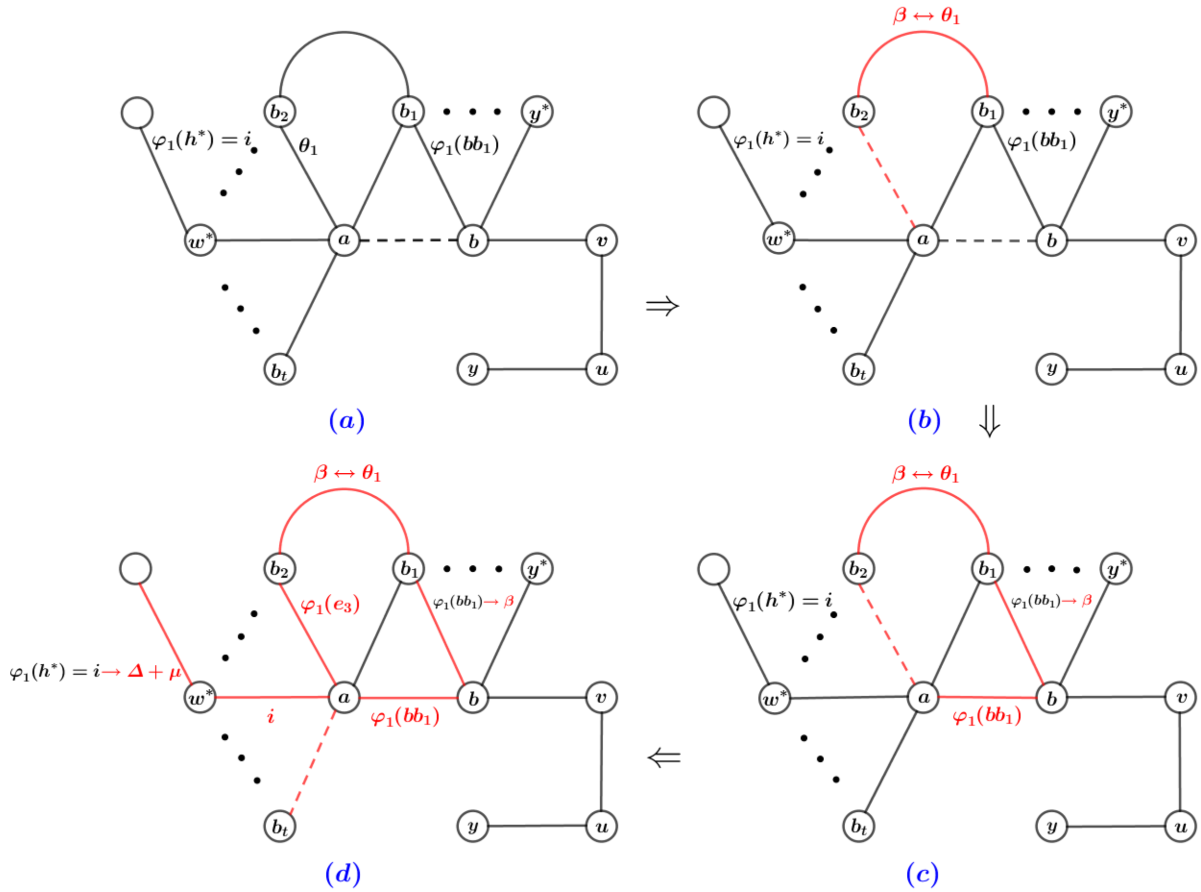

Figure 3.4. One possible operation for $b_j = a^* \in (V(S')\backslash\{a\}) \cap (V(S)\backslash\{a\})$ in Subcase 3.3, where $b_1 = b_j = a^* = a'$.

with θ_j. If $e_{j+1} \in P_b(\beta, \theta_j)$ and $P_b(\beta, \theta_j)$ meets b_{j+1} before a, then we apply a Kempe change on $P_{[b,b_{j+1}]}(\beta, \theta_j)$, uncolor e_{j+1} and color e_{ab} with θ_j. If $e_{j+1} \in P_b(\beta, \theta_j)$ and $P_{b_j}(\beta, \theta_j)$ meets b_{j+1} before a, then we uncolor e_{j+1}, apply a Kempe change on $P_{[b_j,b_{j+1}]}(\beta, \theta_j)$, apply a shifting in S^* from a to b_j (i.e., a^*), color e_{ab} with $\varphi_1(e_{ba'})$, and recolor the edge $e_{bb_j} \in E_{H_1}(b, b_j)$ with β. (See Figure 3.4(a)-(c).) In all three cases above, e_{ab} is colored with a color in $[k]$ and e_{j+1} is uncolored. Denote $H_2 := H_1 + e_{ab} - e_t$. Then we apply a shifting in S' from b_{j+1} to b_t, color e_{j+1} with $\varphi_1(e_{j+2})$, and uncolor the edge e_t, and rename some color classes of $E(H_2)$ but keep the color i unchanged to match all colors on boundary edges except i-edges by Lemma 3.2.5(b). Finally recolor h^* with $\Delta + \mu$ if $h^* \in \partial_{G-(M \cup M_0^*)}(H) \cap E_1(M_0^*, \varphi_0)$. (See Figure

3.4(d).) Now we obtain a new matching $M_2^* = (M_1^* \backslash \{e_{ab}\}) \cup \{e_t\}$ of $G - V(M)$ and a new (proper) $(k+1)$-edge-coloring φ_2 of $G - (M \cup M_2^*)$ such that f_{uv} is no longer T2-improper at v but T1-improper at v with respect to the new prefeasible triple $(M_2^*, E_{M_2^*}^{\varphi_2}, \varphi_2)$, where $\emptyset$ or $\{h\}$ or $\{h^*\} = E_{M_2^*}^{\varphi_2} \subseteq E_1(M_0^*, \varphi_0)$. Furthermore, $|E_1(M_2^*, \varphi_2)| \geq |E_1(M_0^*, \varphi_0)|$ and $|E_2(M_2^*, \varphi_2)| = |E_2(M_0^*, \varphi_0)| - 2$. Thus we can consider $(M_2^*, E_{M_2^*}^{\varphi_2}, \varphi_2)$ instead. The proof is now finished. $\square$

CHAPTER 4

FUTURE WORK

Vizing fans and Vizing's Theorem have played a crucial role in revealing properties of graphs and soloving problems and conjectures in graph edge coloring. Our main results in this dissertation are new extensions of Vizing fans and Vizing's Theorem, which not only have valuable theoretical significance, but also could be usefull on attacking graph chromatic problems and conjectures.

Based on our results in this dissertation , we will continue the following related work.

(1) We will continue to consider other general extensions of Vizing fans with diameter at least four.

(2) We will continue to consider the conjecture of Edwards et al. with precolored distance-3 matchings for simple graphs, and precolored distance-2 matchings for (multi)graphs.

(3) Besides the above two topics, we will consider some related problems about total coloring for multigraphs.

A *k-total-coloring* is an assignment of k colors to the vertices and edges of a graph G such that no two adjacent or incident elements of $V(G) \cup E(G)$ receive the same color. The *total chromatic number*, denoted by $\chi''(G)$, is the minimum integer k such that G admits a k-total-coloring. Behzad (1965) in his Ph.D. dissertation [3] conjectured that $\chi''(G) \leq \Delta(G) + 2$ if G is simple. This conjecture is known as the total coloring conjecture. Clearly, $\chi''(G) \geq \Delta(G) + 1$. So in other words, the total coloring conjecture is equivalent to saying that $\chi''(G) = \Delta(G) + 1$ or $\Delta(G) + 2$. More generally, Vizing [36] in 1968 conjectured that $\chi''(G) \leq \Delta(G) + \mu(G) + 1$. Goldberg [21] in 1984 proposed the following conjecture, which gives a further strengthening when $\chi'(G) \geq \Delta(G) + 3$, if $\chi'(G) \geq \Delta(G) + 3$, then $\chi''(G) = \chi'(G)$. In the survey paper entitled "Unsolved graph colouring problems" in [4], Jensen and Toft asked, as analogue of the Goldberg-Seymour Conjecture, whether every graph satisfies $\chi''(G) \leq \max\{\Delta(G) + 2, \lceil \Gamma(G) \rceil\}$.

There are many fascinating problems and conjectures in graph coloring and other fields of graph theory. I will keep moving forward on the road of my research work with my gratitude to my teachers, my family and friends.

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