Intersections of Longest Paths and Cycles

INTERSECTIONS OF LONGEST PATHS AND CYCLES

by

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Under the Direction of Guantao Chen

ABSTRACT

It is a well known fact in graph theory that in a *connected* graph any two longest paths must have a vertex in common. In this paper we will explore what happens when we look at $k-connected$ graphs, leading us to make a conjecture about the intersection of any two longest paths. We then look at cycles and look at what would be needed to improve on a result by Chen, Faudree and Gould about the intersection of two longest cycles.

INDEX WORDS: Longest path, longest cycle, $k-connected$ graphs

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To everyone who has helped me along the way

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CONTENTS

Chapter 1

Introduction and Notation

1.1 Notation and Original Problem

In this thesis we will generally follow Bollabas for notation and terminology. All graphs that are considered will be simple graphs, that is, graphs with a finite number of vertices, no loops and no parallel edges. Let $G = (V, E)$ be a graph with vertex set V and edge set E. Let P be a path from u to v with $u, v \in V$. We will define vertices u and v to be end vertices of the path P. We will let K_n denote the complete graph on n vertices, and let $K_{n,m}$ denote the complete bipartite graph with n vertices in one partition set and m vertices in the other. Let $\ell(P)$ or $\ell(C)$ denote the number of edges in a given path or cycle, respectively. The orientation of a cycle or path will also be important in this thesis, so we will say that a cycle or path starts at v_1 and ends with v_n where v_i comes before v_j if and only if $i < j$. Let $P_1 = v_1v_2 \cdots v_n$ and $P_2 = u_1u_2 \cdots u_n$ be longest paths such that $V(P_1) \cap V(P_2) = \{v_i = u_k\}$. Without loss of generality we will use P_1 and say the first common vertex is the smallest i such that $v_i \in V(P_1) \cap V(P_2)$. We can define similar notation for cycles. Also in this thesis we will highly scrutinize the connectivity of many graphs, so let k be a positive integer. We define the connectivity, $\kappa(G)$, to be k

if $G = K_{k+1}$ or if we need to delete at least k vertices to make the graph disconnected. G is $k-$*connected* if $\kappa(G) \geq k$. Here we note the Menger Theorem.

(1.1.1) Theorem. *(Menger) Let $G = (V, E)$ be a graph. Then G is $k-$connected if and only if there are k internally vertex-disjoint paths from u to v for all $u, v \in V$.*

Another topic that will be used often throughout this paper is the Pigeonhole Principle. Briefly stated, if we have more pigeons then pigeonholes, we must have at least one hole with more then one pigeon. In this paper we will use a stronger version of the Pigeonhole Principle. Assume we have n pigeons and m pigeonholes. If $n \geq (k-1)m+1$ then at least one pigeonhole must have at least k pigeons in it.

The inspiration for this paper came from a conjecture that was proposed by Scott Smith [2,3]. He conjectured that if G is a $k-$*connected* graph with $k \geq 2$, then every two longest cycles in G must have at least k vertices in common. Although there has been some progress toward this conjecture, it is still open. According to Grötchel [3], Smith's conjecture has been verified for all k up to 10. Burr and Zamfirescu later mentioned the that if a graph G is $k-$*connected* then every two longest cycles must have at least $\sqrt{k}-1$ vertices in common. Finally in 1998 Chen, Faudree and Gould [1] proved that if G is $k-$*connected*, then every two longest cycles in G must have at least $ck^{3/5}$ vertices in common where $c = 1/(\sqrt[3]{256}+3)^{3/5} \approx 0.2615$. Our goal in this paper is to show how this bound could be improved in order to get it closer to the conjecture stated by Smith.

To begin this thesis though, we want to take a look at a related problem. Instead of looking at longest cycles, we will look at longest paths. To do this we first will state some properties of longest paths and then these properties will help us to determine how many vertices two longest paths must have in common. We also take a look at a problem proposed by Alen Schwenk, which asks if there is a graph where the intersection of all longest paths is empty.

Chapter 2

Paths

2.1 Classical results

In chapter 2 we look a little deeper into paths, specifically longest paths. We start with some classical results which will be important to understand before moving on to the rest of this chapter. First, we make a claim about the intersection of two longest paths.

(2.1.1) Lemma. *Let G be a graph. If G is connected, then any two longest paths must share at least one vertex.*

Proof. Let these two paths be labeled $P_1 = [v_1, v_n]$ and $P_2 = [u_1, u_n]$, and furthermore assume for a moment that they don't share a common vertex. Since this graph is *connected*, we know there must be a shortest path, $Q = [v_i, u_j]$, from the set of vertices in P_1 to the set of vertices in P_2 (See figure 2.1).

If we create two new paths P_1^* and P_2^*, we see that by the Pigeonhole Principle one of these paths (or both), must be longer then P_1 and P_2, contradicting the fact that P_1 and P_2 are longest paths.

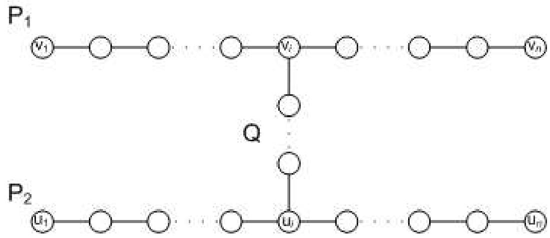

Figure 2.1:

$$P_1^* = P_1[v_1, v_i]Q[v_i, u_j]P_2[u_j, u_n]$$

$$P_2^* = P_2[u_1, u_j]Q[u_j, v_i]P_1[v_i, v_n]$$

$$\ell(p_1^*) + \ell(p_2^*) \geq \ell(p_1) + \ell(p_2) + 2\ell(Q)$$

So, from this we see that the assumption is false and so they must share at least one common vertex. □

We now know that any two longest cycles must have at least one common vertex, so can it have exactly one common vertex? In fact we can have exactly one vertex in common and in doing so we can learn more about the graphs structure. This leads us to our next statement.

(2.1.2) Lemma. *Let G be a graph. If G is connected and two longest paths meet in exactly one common vertex, then the distance to the common vertex must be the same from all 4 end vertices of the two paths.*

Proof. Suppose $Q_1[v_1, v_m]$ and $Q_2[u_1, u_m]$ are two longest paths in G such that $V(Q_1) \cap V(Q_2) = \{c\}$ and let P_1 be the path from v_1 to c, P_2 be the path from c to v_m, P_3 be the path from u_1 to c and P_4 be the path from c to u_m. Assume $\ell(P_i) \neq \ell(P_j)$ for some $i \neq j$. Without loss of generality assume that P_1 is the longest. Since

$\ell(P_1) + \ell(P_2) = m$

and

$\ell(P_3) + \ell(P_4) = m,$

then P_2 must be the shortest. By the Pigeonhole Principle we know that $\ell(P_3)$ and $\ell(P_4)$ cannot both equal $\ell(P_2)$, and that neither can be less then $\ell(P_2)$ since this would contradict that P_1 was the longest. Take the paths $P_1[v_1, c]P_3[c, u_1]$ and $P_1[v_1, c]P_4[c, u_m]$. Again, by the Pigeonhole Principle, one of these paths must be longer then m. So our assumption is false and hence $\ell(P_i) = \ell(P_j)$ for all i and j. □

This result leads us to two interesting corollaries that will now be stated.

(2.1.3) Corollary. *If G is connected and has two longest paths that meet at exactly one vertex, then the length of the longest path must be even.*

Proof. Let n be the length from the common vertex to an end vertex (which must all be the same length from above). So the length of the longest path must be $2n$, which is even. □

(2.1.4) Corollary. *If G is a connected graph and has two longest paths that meet at exactly one vertex, then that vertex must be exactly in the middle of the path.*

Proof. Let n be the length from the common vertex to an end vertex. Again the longest path must have n vertices on each side of the common vertex. So it is in the exact middle. □

So far in this chapter we have talked about the minimum number of vertices that two longest cycles must have in common, so one's next thought might be whether there is an upper bound on the number of vertices in common. Well, this gives a not so interesting result, but one that is worthy mentioning.

(2.1.5) Proposition. *Two longest paths in a connected graph can share up to every vertex in the graph.*

This can be seen by any graph that has at least two Hamiltonian paths. This will illustrated with C_5, $K_{3,3}$ and K_4.(See figure 2.2)

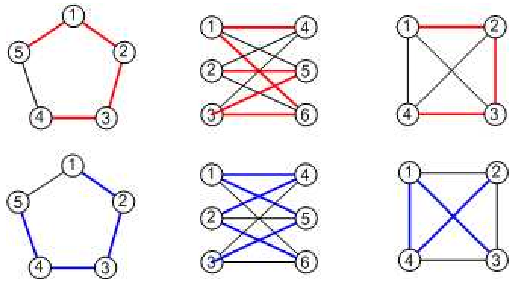

Figure 2.2:

We will state one more classical result that will be used in this paper, the proof of which is omitted.

(2.1.6) Lemma. *The end vertices in a longest path are only adjacent to other vertices in the path.*

2.2 Conjecture about $k - connected$ graphs

Everything that we have done so far has been in $1 - connected$ graphs, but what happens when we look at a $k - connected$ graph with k>1. Since a $k - connected$ graph is also a $(k - 1) - connected$ graph, the results clearly still hold, but can we improve them? In the this section we will look at some $k - connected$ graphs and show what results we have found as well as pose some interesting questions.

Since we have already looked at $1 - connected$ graphs, it is only logical to start at $2 - connected$ graphs. By doing so we get the following result.

(2.2.1) Lemma. *If G is $2-connected$, then any two longest paths in G must meet in at least two vertices.*

Proof. Let G be a $2-connected$ graph and assume that they only have one vertex in common. Since G is connected, we know that if G has two longest paths, denoted by $P_1[u_1, u_{2n+1}]$ and $P_2[v_1, v_{2n+1}]$, then they must meet in at least one vertex. Denote the common vertex $c = v_{n+1} = u_{n+1}$. We know from Corollary 2.1.4 that the common vertex must be in the middle and the length on each side must be the same, v_{n+1}. Since the graph is $2-connected$, there must be another path between our two longest paths, denoted Q, meeting P_1 at say x and meeting P_2 at say y. Now look at the two paths, (See figure 2.3)

$$L_1 = P_1[v_1, x]Q[x, y]P_2[y, u_{2n+1}]$$
$$L_2 = P_2[u_1, y]Q[y, x]P_2[x, v_{2n+1}]$$

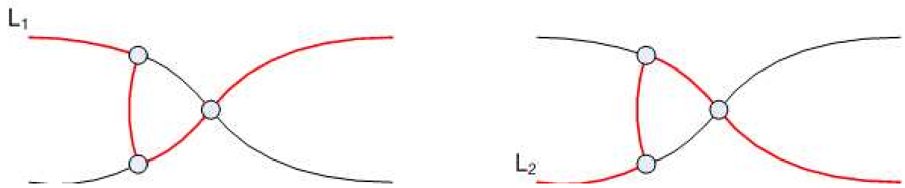

Figure 2.3:

$$\ell(L_1) + \ell(L_2) = \ell(P_1) + \ell(P_2) + 2\ell(Q)$$

So by the Pigeonhole Principle one of the paths, L_1 or L_2, must be longer than P_1 and P_2. □

Knowing that two longest paths intersect at exactly two vertices tells us something about the length of each piece.

(2.2.2) Lemma. *The distance from the start of either path to the first common vertex must be the same, the distance between the common vertices on either path must be the same, and the distance from the second common vertex to the other end vertex must be the same.*

Proof. Assume without loss of generality that one of the pieces of P_1 is longer than the corresponding piece along P_2. Let L be a new path using P_2 except using P_1 where P_1 is longer. Since P_1 is longer than P_2 in at least one section, $\ell(L) > \ell(P_2)$. This is a contradiction to the fact that P_2 is a longest path. □

Next we will look at $3-$*connected* graphs and see that we get a similar result.

(2.2.3) Lemma. *If G is $3-$connected, then any two longest paths in G must meet in at least three vertices.*

Proof. Again, clearly G is $2-$*connected*, so as above the paths must have at least two vertices in common, namely $c_1 = v_i = u_i$ and $c_2 = v_j = u_j$ where $i < j$. Once again let P_1 and P_2 be the longest paths, where P_1 goes from v_1 to v_n and P_2 goes from u_1 to u_n. Assume that i$\leq j-2$. By removing c_1 and c_2 from the graph we see that we get six disjoint pieces, and since this graph is 3-connected, there must be a path in G connecting them, particularly connecting the path $[v_{i+1}, v_{j-1}]$ to $[v_1, v_{i-1}]$ or $[v_j + 1, n_n]$ or $[u_1, u_{i-1}]$ or $[u_{j+1,u_n}]$. Let this path be Q and let it connect at vertex b in $[v_{i+1}, v_{j-1}]$ and vertex a in $[v_1, v_{i-1}]$ or $[v_j + 1, n_n]$ or $[u_1, u_{i-1}]$ or $[u_{j+1,u_n}]$. Without loss of generality assume that a is on the path from v_1 to c_1 and b is on P_1 between c_1 and c_2. If not, change the labeling as needed. Now look at the two paths,(see figure 2.4)

$$L_1 = P_1[v_1, a]Q[a, b]P_1[b, c_1]P_2[c_1, u_n]$$
$$L_2 = P_2[u_1, c_1]P_1[c_1, a]Q[a, b]P_1[b, v_n]$$

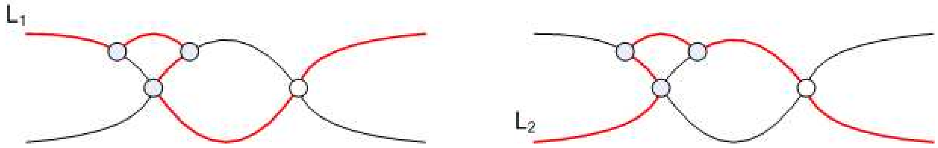

Figure 2.4:

Then the length of L_1 plus the length of $L2$ is,

$$\ell(L_1) + \ell(L_2) = \ell(P_1[v_1, a]Q[a, b]P_1[b, c_1]P_2[c_1, u_n]) + \ell(P_2[u_1, c_1]P_1[c_1, a]Q[a, b]P_1[b, v_n])$$

$$= \ell(P_1[v_1, a]P_1[c_1, a]P_1[b, c_1]P_1[b, v_n]) + \ell(P_2[u_1, c_1]P_2[c_1, u_n] + 2Q[a, b])$$
$$= \ell(P_1) + \ell(P_2) + 2\ell(Q)$$

So again, by the Pigeonhole Principle, one of the paths, L_1 or L_2, must be longer.

We make a final note on this proof that if $i + 1 = j$, then we can replace c_1 and c_2 with x such that $N(x) = \{v_{i-1}, v_{j+1}, u_{i-1}, u_{j+1}\}$. Then this is the same as the proof of lemma 2.2.1. □

This leads us to our first conjecture of the paper.

(2.2.4) Conjecture. If G is $k - \textit{connected}$, then any two paths in G must meet in at least k vertices.

It is quick and easy to prove that the result stated in this conjecture is tight; just look at the example $K_{k,2k+2}$ as seen in figure 2.5.

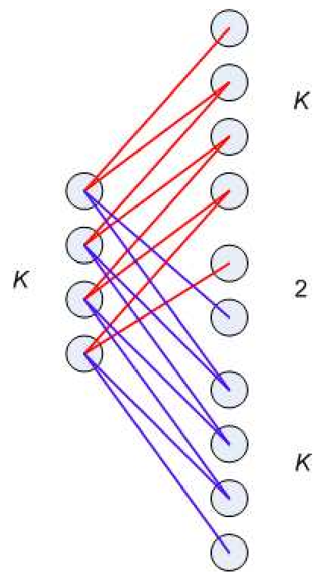

Figure 2.5:

2.3 Schwenk's Question

There are many other open problems that easily relate to longest paths. Of particular interest in this paper is a question stated be Allen Schwenk [5]. Can we find a graph where the intersection of all possible longest paths is empty? We know from the previous sections in this chapter that clearly any two longest paths cannot have an empty intersection, but using what we have learned thus far can we get a bound?

First, we would like to show that it is possible to have a graph where the intersection of all possible longest paths is empty. The example we give uses 9 longest paths.

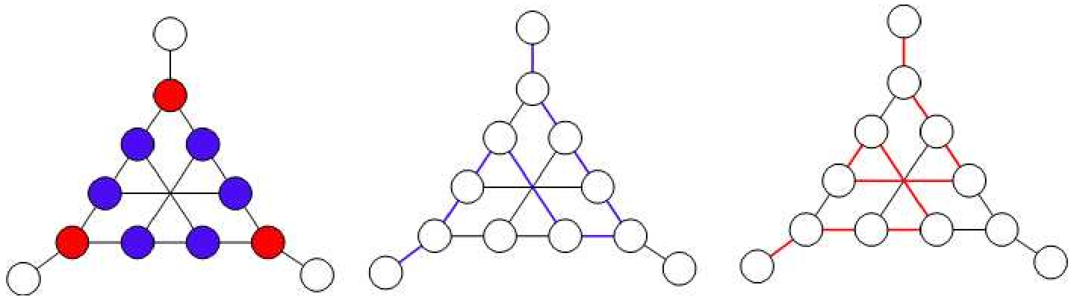

Figure 2.6:

Use the blue path 6 times rotating accordingly to not use each of the blue vertices once, then use the red path 3 times rotating accordingly to not use each of the red vertices once; after doing so, each of the 3 remaining vertices will have been left off at least once since each of them cannot be used when its only neighbor is not used.

So, how many longest cycles must a graph have in order to possibly satisfy Schwenk's question? Let's consider what vertices are not in the intersection of all longest paths. To not be in all of the longest paths, a vertex must be left out of at least one longest path. This idea gives us a lower bound for how many longest paths there must be.

(2.3.1) Lemma. *Let G be a graph with n vertices such that G does not contain a Hamiltonian path. Let m be the length of the longest path. If the intersection of all* *possible longest paths is empty, then there must be at least* $n/(n-m) \geq 3$ *distinct longest paths.*

Proof. Since m is the length of the longest path, $n-m$ must be the number of vertices not in a particular longest path. In order for the intersection to be empty, every vertex must be excluded from at least one longest path. So, since each longest path excludes $n-m$ vertices and there are m vertices, clearly we need at least $n/(n-m)$ paths. Note that since G does not contain a Hamiltonian path, $n \neq m$.

Since we have shown that any two longest paths must intersect in at least 1 vertex, then if there are only two longest paths, they must intersect, and so their intersection is non-empty. Hence, $n/(n-m) \geq 3$. □

Chapter 3

Cycles

3.1 Conjecture

In Chapter 2, we concentrated on paths, and so in Chapter 3 we will add the restriction that our path must begin and end at the same vertex, making it a cycle. The first thing to ask ourselves is whether all the results we found in Chapter 2 are still true? Unfortunately, it doesn't all hold, for example, here is a *connected* graph where the two longest cycles do not meet.

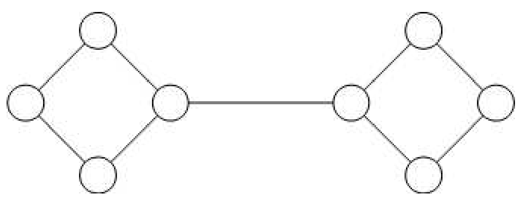

Figure 3.1:

Fortunately, many of the same ideas still work, many of the proofs are along the same lines, and many of the results are similar. With this in mind, we will state the *Conjecture* by Scott Smith, which is similar to *Conjecture* 2.2.4.

(3.1.1) Conjecture. In a $k-connected$ graph ($k \geq 2$), two longest cycles meet in at least k vertices.

Before we can move on to proving some lower bounds concerning this $Conjecture$, we must first state some lemmas that are going to be used.

(3.1.2) Lemma. *(Hylton-Cavallius). Let $G \subseteq K_{n,n}$ be a bipartite graph. Then G contains $K_{s,t}$, as a subgraph if*

$$e(G) \geq (s-1)^{1/t}(n-t+1)n^{1-1/t} + (t-1)n$$

where $e(G)$ is the number of edges in G.

(3.1.3) Corollary. *Let $G \subseteq K_{n,n}$. Then G contains $K_{3,257}$ if*

$$e(G) \geq \sqrt[3]{256}(n-2)n^{2/3} + 2n.$$

(3.1.4) Lemma. *(Erdős and Szekeres). Every sequence of n^2+1 real numbers contains a monotone subsequence of length n+1.*

In this thesis, we will use a slightly more general form of Lemma 3.1.4, stated here.

(3.1.5) Lemma. *(Chen, Faudree and Gould [1]). Let Σ be a set of n permutations of a sequence S of $2^{2^n}+1$ elements. Then there is a subsequence (a,b,c) of S on which each permutation $\sigma \in \Sigma$ is monotonic. (that is, either $\sigma(a) < \sigma(b) < \sigma(c)$ or $\sigma(a) > \sigma(b) > \sigma(c)$).*

(3.1.6) Lemma. *(Dirac). Let G be a $2-connected$ graph of minimum degree δ on n vertices, where $n \geq 3$. Then G contains either a cycle of length at least 2δ or a Hamiltonian cycle.*

3.2 $\sqrt{k} - 1$ Result

Through private communication, S.Burr and T.Zamfirescu showed that a weaker version of Smith's conjecture must hold. What they showed was the following.

(3.2.1) Theorem. *(Burr and Zamfirescu). If G is a $k-connected$ graph with $k \geq 2$, then every pair of different longest cycles meet in at least $\sqrt{k}$ vertices.*

Proof. We will prove this theorem by contradiction. Let G be a $k-connected$ graph with two longest cycles, C and D, such that $|V(C) \bigcap V(D)| < \sqrt{k}$. Let $V(C) \cap V(D) = A = \{a_1, a_2, \ldots, a_m\}$. Note that $m < \sqrt{k}$. Let $X_1, X_2, \ldots, X_m$ be the remaining segments of $C - A$, and let $Y_1, Y_2, \ldots, Y_m$ be the remaining segments of $D - A$.

What we have so far are two longest cycles with at most $\sqrt{k}$ vertices in common. We have labeled the vertices in common $a_1, a_2, \ldots, a_m$ and the sections between these vertices along C we labeled $X_1, X_2, \ldots, X_m$ and along D we labeled $Y_1, Y_2, \ldots, Y_m$. Note that any of these sections may be empty. Below is an example with $m = 4$.

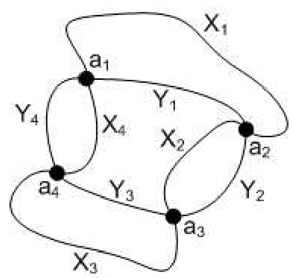

Figure 3.2:

By our assumption we know that $|A| < \sqrt{k} < k$ and so since k is always less then $|V(G)|$, we know that $|A| < |V(G)|$ hence, G is not Hamiltonian, and we also know that the minimum degree, δ, in G is at least k. So, by lemma 3.1.6, since G is not Hamiltonian, it must have a cycle of length $2\delta \geq 2k$. Hence $|V(C)| = |V(D)| \geq 2k$, and since G is not

Hamiltonian, $|V(G)| > 2k$. Since $|V(C)| = |V(D)| \geq 2k$ and $|A| < \sqrt{k}$ we know that $|C - A| = |X_1 \cup X_2 \cup \ldots \cup X_m| = |D - A| = |Y_1 \cup Y_2 \cup \ldots \cup Y_m| \geq k$.

Knowing that $m \leq k$, we have that $G - A$ must still be $(k - m) -$ *connected*. Thus there must be at least $k - m$ internally disjoint paths between any two distinct parts of G, namely $C - A$ and $D - A$. We will call these paths $P_1, P_2, \ldots, P_{k-m}$ and let $\wp = \{P_1, P_2, \ldots, P_{k-m}\}$.

Here we must note that no two paths in $\mathcal{P}$ may start with the same X_i and end with the same Y_j. If they did, then by the same argument we used repeatedly in Chapter 2, we would have a longer cycle.

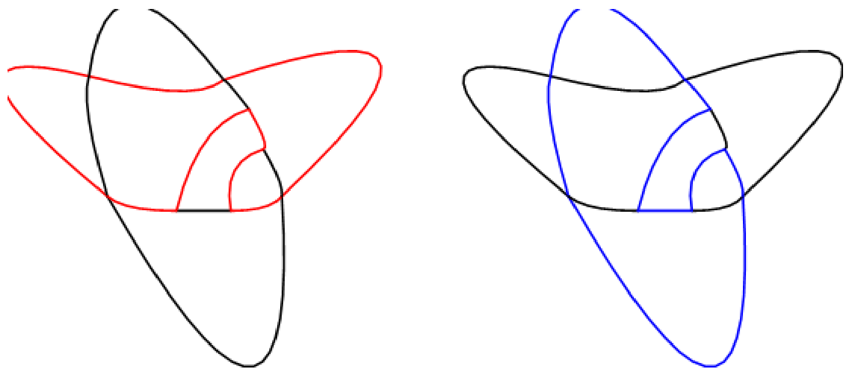

Figure 3.3:

Now we can construct an auxiliary graph H with vertex set$\{X_1, X_2, \ldots, X_m, Y_1, Y_2, \ldots, Y_m\}$ and for each $P_h \in \mathcal{P}$ we insert and edge from the corresponding X_i to Y_j. From above we see that H must be a simple bipartite graph with each partite having m vertices.

Finally, since this is a bipartite graph with m vertices in each partite, clearly there must be at most m^2 edges. This implies that $k - m \leq m^2$ which implies that $m \geq \sqrt{k} - 1$.

□

3.3 Improvement to $ck^{3/5}$

The result that we saw in the previous section can be improved upon with a refinement of our seemingly golden idea to show that there must be a longer cycle(or longer path in Chapter 2). We will begin this section where we left off excluding only the final paragraph of the previous proof.

Proof. So, we now have a simple bipartite graph with m vertices on each side. Also note that H contains at least $(\sqrt[3]{256}+2)m^{5/3}$ edges, so by Corollary 3.1.3, $K_{3,257} \subseteq H$. We will now label the vertices in H so that $X_1, X_2, X_3, Y_1, Y_2, \cdots, Y_{257}$ are the vertices that make up the $K_{3,257}$. We will also denote the path from X_i to Y_j as $P_{i,j}$ starting at $u_{i,j} \in C$ and ending at $v_{i,j} \in D$. We will also pick an arbitrary vertex a_1 to start the orientation of our cycle and will say that for any two vertices on our cycle, $x_1 \prec x_2$ if and only if x_1 is between a_1 and x_2 along the orientation of C. Note that the same can be said for two disjoint subsets of C. Also define the same notation for D using b_1 to start the orientation.

We can now say that $X_1 \prec X_2 \prec X_3$ and $Y_1 \prec Y_2 \prec \cdots \prec Y_{257}$. Next if we look at the order that the Y_i's are connected to X_1 along the orientation of C, we see that we get a permutation of the $2^{2^3}+1 = 257$ elements $Y_1, \cdots, Y_{257}$. Since we can repeat this for X_2 and X_3, we see that we have 3 permutations of the set $Y_1, \cdots, Y_{257}$. By Lemma 3.1.5 we know that there must be 3 elements of $Y_1, \cdots, Y_{257}$ that are in increasing or decreasing order; let those 3 elements be relabeled as Y_a, Y_b and Y_c also, by the Pigeonhole Principle, we loose no generality by saying that $Y_a \prec Y_b \prec Y_c$ and

$$u_{1,a} \prec u_{1,b} \prec u_{1,c}$$

and

$$u_{2,a} \prec u_{2,b} \prec u_{2,c}$$

and either

$$u_{3,a} \prec u_{3,b} \prec u_{3,c} \text{ or } u_{3,c} \prec u_{3,b} \prec u_{3,a}.$$

What is left to be shown is that for $Y_a \preceq Y_i \preceq Y_j \preceq Y_c$, either

$$v_{1,i} \prec v_{2,i} \text{ and } v_{1,j} \prec v_{2,j}$$

or

$$v_{2,i} \prec v_{1,i} \text{ and } v_{2,j} \prec v_{1,j}.$$

If we can show this then the following structure must exist:

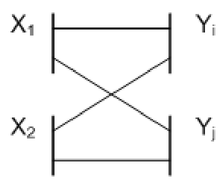

Figure 3.4:

To show this we first note that we can say $v_{1,a} \prec v_{2,a}$, for if this is not the case, reverse the labeling of X_1 and X_2. If $v_{1,b} \prec v_{2,b}$ or $v_{1,c} \prec v_{2,c}$ then the first line above would be satisfied. So assume that $v_{2,b} \prec v_{1,b}$ and $v_{2,c} \prec v_{1,c}$; but we now see that these two satisfy the second line from above, and so one of these structures must exist.

Now that we have this structure, it is easy to see that we must have a longer cycle, just look at the cycles C^* and D^*:

$C^* = a_1 - u_{1,i} - v_{1,i} - v_{2,i} - u_{2,i} - u_{1,j} - v_{1,j} - v_{2,j} - u_{2,j} - a_1$

$D^* = b_1 - v_{1,i} - u_{1,i} - u_{1,j} - v_{1,j} - v_{2,i} - u_{2,i} - u_{2,j} - v_{2,j} - b_1$

$\square$

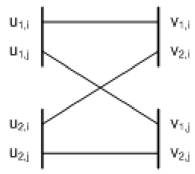

Figure 3.5:

3.4 Improvement to $ck^{2/3}$

In the last section we used a $K_{3,257}$; we note here that the use of a $K_{2,m}$ for some large m would not be enough to give us the structure that we needed. In order to improve this result to $ck^{2/3}$ we do need to use a special $K_{2,m}$ for some large m, and so we need to change the structure that we use. Again, we start with the proof given over the last two sections, noting some changes.

The first change to note is that since we only have an X_1 and X_2, we can only state that

$$u_{1,a} \prec u_{1,b}$$

and either

$$u_{2,a} \prec u_{2,b}$$

or

$$u_{2,b} \prec u_{2,a}.$$

So, instead of being able to force Figure 3.4, we can only force one of these 3 structures to be present:

Now clearly if the first structure is present then we are done by the C^* and D^* used in Section 3.3. So what if the first structure is not present. Unfortunately, there is not an easy C^* and D^* that we can create to force a longer path. We can however find a C^*,

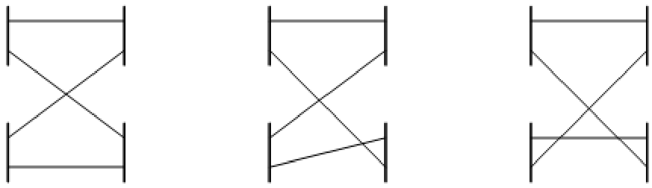

Figure 3.6:

D^*, and E^* that by the pigeonhole principle would force a longer cycle if we can force the two remaining structures to be interlaced.

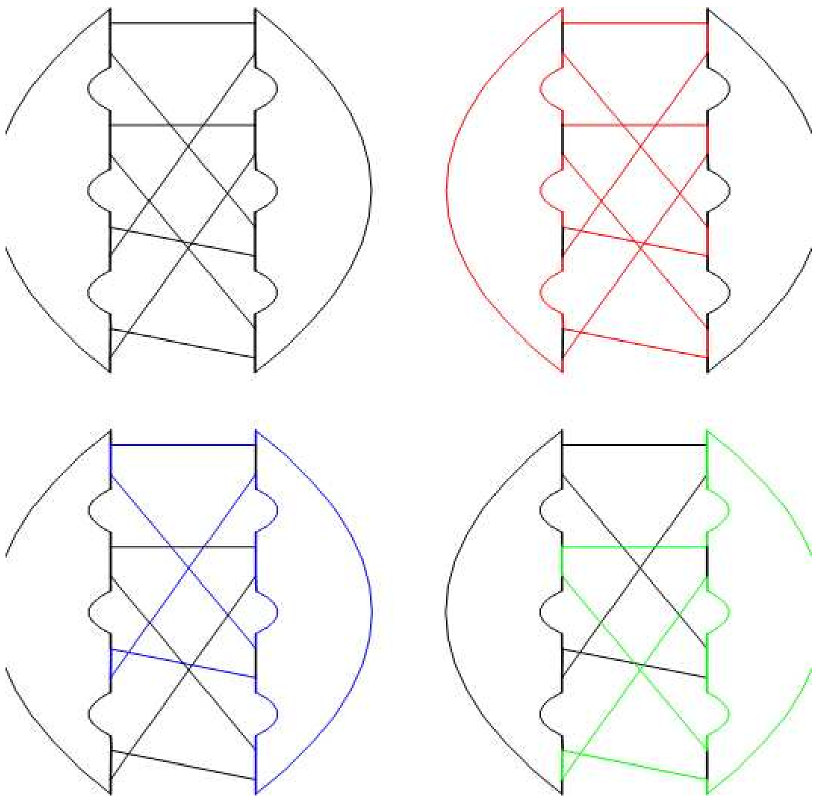

Figure 3.7:

It is the author's feeling that since $H \subseteq K_{m,m}$ and we only used 2 vertices from one side of the bipartite graph, we should be able to show that there are enough edges to force another one of these structures to be present. This is how the result could be improved to $ck^{2/3}$ for some c.

Bibliography

[1] G. Chen, R. Faudree and R Gould, Intersections of Longest Cycles in k-Connected Graphs, *Journal of Combinatorial Theory, Series B* (1998) 143-149.

[2] J.A.Bondy, Basic graph theory:paths and circuits, in "Handbook of Combinatorics" (R.L. Graham, M. Grotschel, and L.Lovasz, Eds), Vol. I, pp.3-110, Elsevier, Amsterdam, MIT Press, Cambridge, 1996.

[3] M. Grotschel, On intersectionsof longest cycles, in "Graphy Theory and Combinatorics" (B.Bollobas, Ed.), pp.171-189, Academic Press, London, 1984.

[4] Karl Menger. Zur allgemeinen kurventhoerie. *Fund. Math*, 10:96-115, 1927.

[5] Steve Hedetniemi, Open Problems in Combinatorial Optimization, http://www.cs.clemson.edu/ hedet/graphtheory.html, accessed March 21, 2008.